Fundamental Math/Physics in Computer Graphics

Lecture 3

Dr. Zhigang Deng

UNIVERSITY of HOUSTON

DEPARTMENT OF COMPUTER SCIENCE

What is Computer Graphics

Using a computer to convert a model into an image

Model $\xrightarrow{\text{computer}}$ Image

3D Geometry

Need a way to represent the model

- Geometry
- Cameras

Basic Types

- Scalars: s
- Points: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$
- Directions: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Transformations

Why use transformations?

- Create object in convenient coordinates
- Reuse basic shape multiple times
- Hierarchical modeling
- System independent
- Virtual cameras

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

$$T(t_x, t_y, t_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Properties of Translation

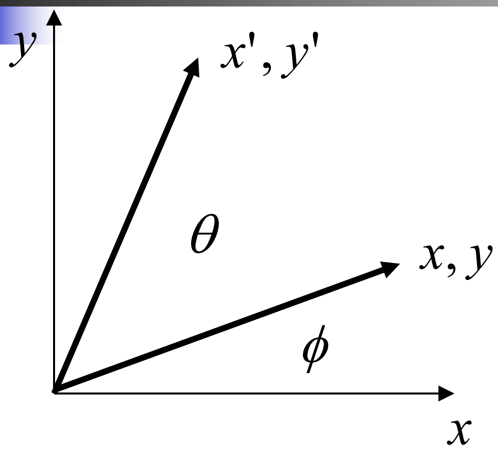
$$T(0,0,0)\,\mathbf{v} = \mathbf{v}$$

$$T(s_x, s_y, s_z)\,T(t_x, t_y, t_z)\mathbf{v} = T(s_x + t_x, s_y + t_y, s_z + t_z)\,\mathbf{v}$$

$$T(s_x, s_y, s_z)\,T(t_x, t_y, t_z)\mathbf{v} = T(t_x, t_y, t_z)\,T(s_x, s_y, s_z)\mathbf{v}$$

$$T^{-1}(t_x, t_y, t_z)\,\mathbf{v} = T(-t_x, -t_y, -t_z)\,\mathbf{v}$$

Rotations (2D)



$$\boxed{\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned}}$$

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$x' = r\cos(\phi + \theta)$$

$$y' = r\sin(\phi + \theta)$$

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$

$$\sin(\phi + \theta) = \cos\phi\sin\theta - \sin\phi\cos\theta$$

$$x' = (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta$$

$$y' = (r\cos\phi)\sin\theta + (r\sin\phi)\cos\theta$$

Rotations (3D)

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of Rotations

$$R_a(0) = I$$

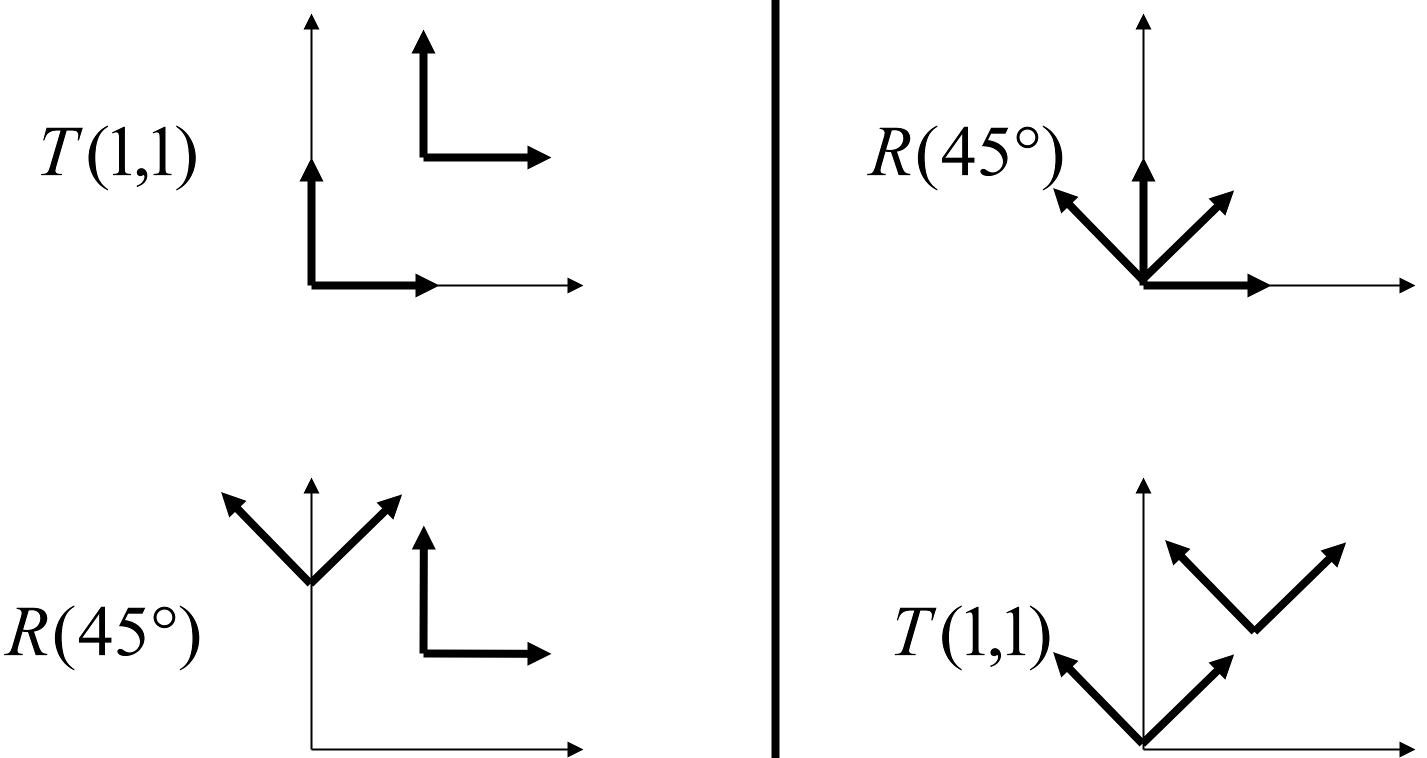
$$R_a(\theta)R_a(\phi) = R_a(\phi + \theta)$$

$$R_a(\theta)R_a(\phi) = R_a(\phi)R_a(\theta)$$

$$R_a^{-1}(\theta) = R_a(-\theta) = R_a^{T}(\theta)$$

$$R_a(\theta)R_b(\phi) \neq R_b(\phi)R_a(\theta)$$ order matters!

Combining Translation & Rotation



Combining Translation & Rotation

$$\mathbf{v}' = \mathbf{v} + T$$

$$\mathbf{v}'' = R\mathbf{v}'$$

$$\mathbf{v}'' = R(\mathbf{v} + T)$$

$$\mathbf{v}'' = R\mathbf{v} + RT$$

$$\mathbf{v}' = R\mathbf{v}$$

$$\mathbf{v}'' = \mathbf{v}' + T$$

$$\mathbf{v}'' = R\mathbf{v} + T$$

Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \\ s_z z \end{bmatrix}$$

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

Uniform scaling ***iff*** $s_x = s_y = s_z$

Homogeneous Coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ can be represented as } \begin{bmatrix} X \\ Y \\ Z \\ w \end{bmatrix}$$

where $x = \dfrac{X}{w}, \quad y = \dfrac{Y}{w}, \quad z = \dfrac{Z}{w}$

Translation Revisited

$$T(t_x, t_y, t_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Rotation & Scaling Revisited

$$R_x(\theta)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$S(s_x, s_y, s_z)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Combining Transformations

$$\mathbf{v}' = S\mathbf{v}$$

$$\mathbf{v}'' = R\mathbf{v}' = RS\mathbf{v}$$

$$\mathbf{v}''' = T\mathbf{v}'' = TR\mathbf{v}' = TRS\mathbf{v}$$

$$\mathbf{v}''' = M\mathbf{v}$$

where $M = TRS$

Rotations using Quaternions

Quaternions

$$q = q_1 + i\, q_2 + j\, q_3 + k\, q_4$$

$$\boxed{\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ i\, j\, k &= -1 \end{aligned}}$$

$$q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} s \\ v \end{pmatrix}$$

Rotations using Quaternions

Quaternion multiplication

$$q\ p = \begin{pmatrix} s_q s_p - v_q \cdot v_p \\ s_q v_p + s_p v_q + v_q \times v_p \end{pmatrix}$$

$[s_1, v_1] + [s_2, v_2] = [s_1+s_2, v_1+v_2]$

$[s, v].[1, 0, 0, 0]=[s, v]$

$$(p\ q)\ r = p\ (q\ r)$$ Associative

$$p\ q \neq q\ p$$ Not Commutative

Rotations using Quaternions

Quaternion magnitude

$$|q| = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} = \sqrt{s_q^2 + v_q \cdot v_q}$$

Quaternion inverse

$$q^{-1} = \frac{1}{|q|^2} \begin{pmatrix} s \\ -v \end{pmatrix}$$

Rotations using Quaternions

Quaternion rotation:

$$q = \begin{pmatrix} \cos(\theta / 2) \\ \sin(\theta / 2)\, r \end{pmatrix}$$

a= (x, y, z)^T

$$rot\begin{pmatrix} 0 \\ a \end{pmatrix} = q \begin{pmatrix} 0 \\ a \end{pmatrix} q^{-1}$$

Rotations using Quaternions

Combining rotations:

$$rot_1\begin{pmatrix}0\\a\end{pmatrix} = q_1 \begin{pmatrix}0\\a\end{pmatrix} q_1^{-1}$$

$$rot_2\left(rot_1\begin{pmatrix}0\\a\end{pmatrix}\right) = q_2\, q_1 \begin{pmatrix}0\\a\end{pmatrix} q_1^{-1}\, q_2^{-1}$$

$$rot_2\left(rot_1\begin{pmatrix}0\\a\end{pmatrix}\right) = (q_2\, q_1)\begin{pmatrix}0\\a\end{pmatrix} (q_2\, q_1)^{-1}$$

Rotations by Quaternions

$$ROT_q(ROT_p(v)) = ROT_{qp}(v)$$

$$ROT_{p-1}(ROT_p(v)) = v$$

$$-q = (-1).(s, v) = (s, v) = q$$

Rotations using Quaternions

Matrix Form:

$$q = w + i\,x + j\,y + k\,z$$

$$\begin{bmatrix} w^2 + x^2 - y^2 - z^2 & 2xy - 2wz & 2xz + 2wy & 0 \\ 2xy + 2wz & w^2 - x^2 + y^2 - z^2 & 2yz - 2wx & 0 \\ 2xz - 2wy & 2yz + 2wx & w^2 - x^2 - y^2 + z^2 & 0 \\ 0 & 0 & 0 & w^2 + x^2 + y^2 + z^2 \end{bmatrix}$$