Fundamental Math/Physics for Computer Graphics (II)

Lecture 4

Dr. Zhigang Deng

UNIVERSITY of HOUSTON

DEPARTMENT OF COMPUTER SCIENCE

Deformations

Transformations that do not preserve shape

- Non-uniform scaling
- Shearing
- Tapering
- Twisting
- Bending

Shearing

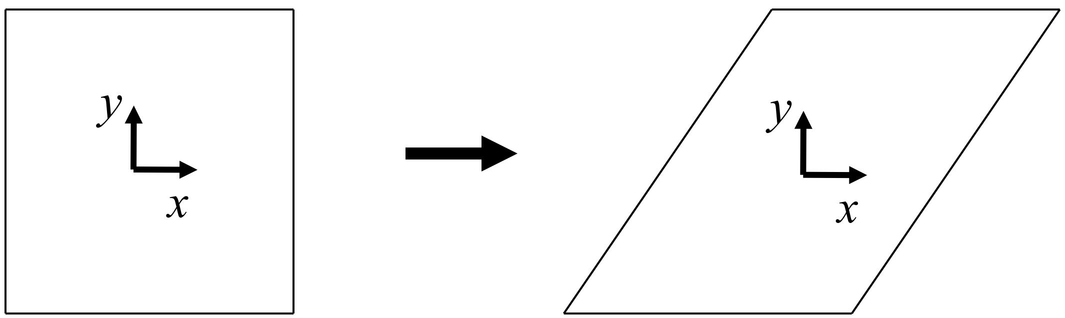
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & s_{xy} & s_{xz} & 0 \\ s_{yx} & 1 & s_{yz} & 0 \\ s_{zx} & s_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$s_{xy} = 1$$

$$s_{xz} = 0$$

$$s_{yx} = s_{yz} = 0$$

$$s_{zx} = s_{zy} = 0$$



Tapering

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f(x) & 0 & 0 \\ 0 & 0 & f(x) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

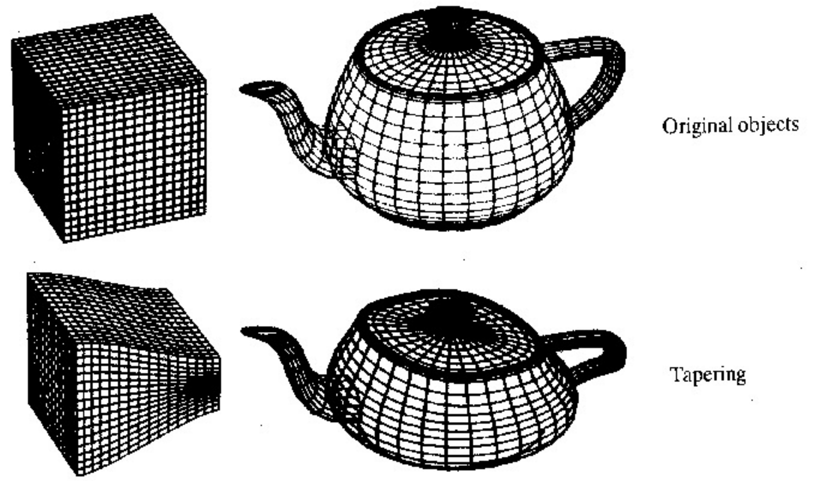


Image courtesy of Watt, 3D Computer Graphics

Twisting

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta(y)) & 0 & \sin(\theta(y)) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta(y)) & 0 & \cos(\theta(y)) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

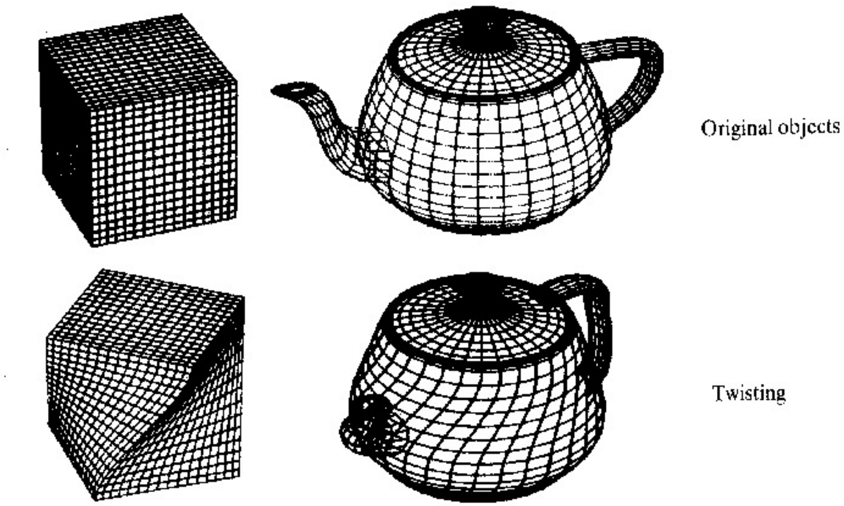


Image courtesy of Watt, 3D Computer Graphics

Bending

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f(y) & g(y) & 0 \\ 0 & h(y) & k(y) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

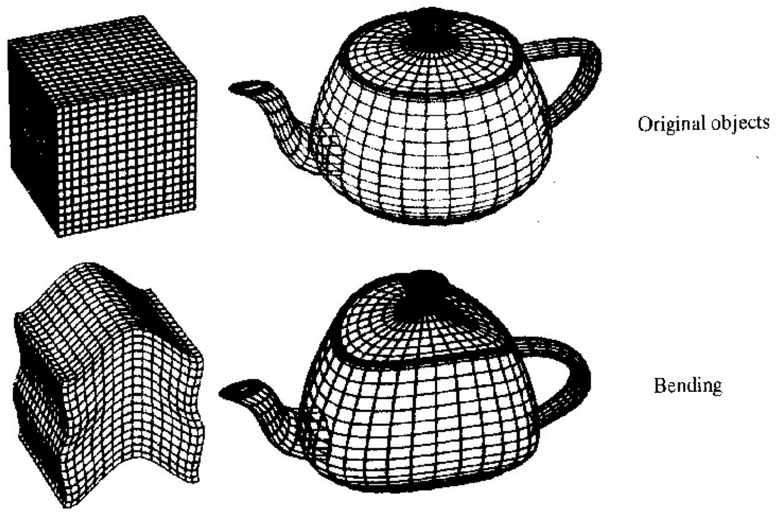
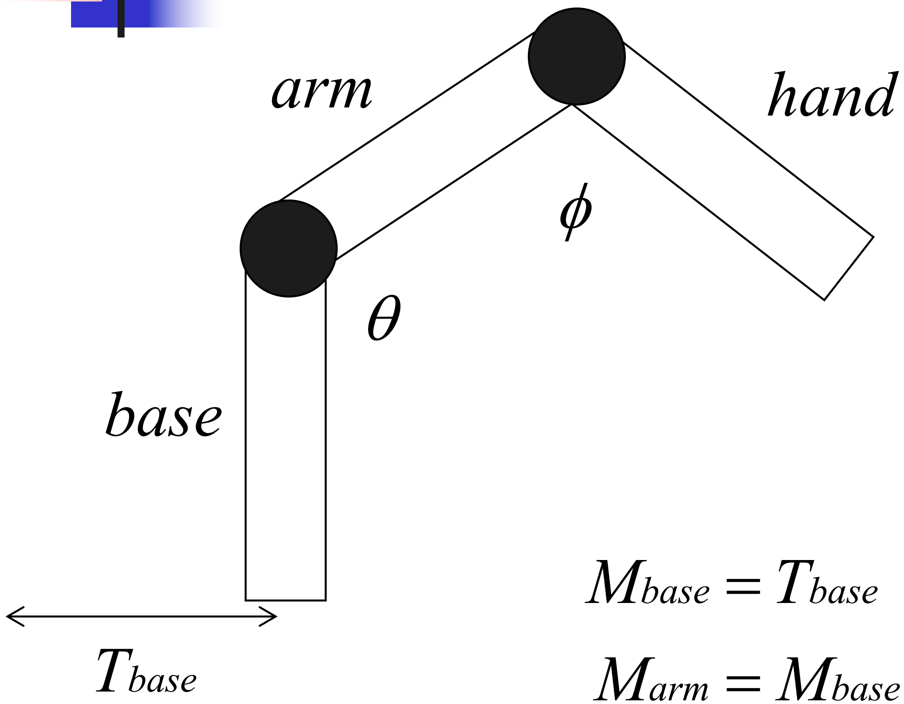


Image courtesy of Watt, 3D Computer Graphics

Hierarchical Transformations

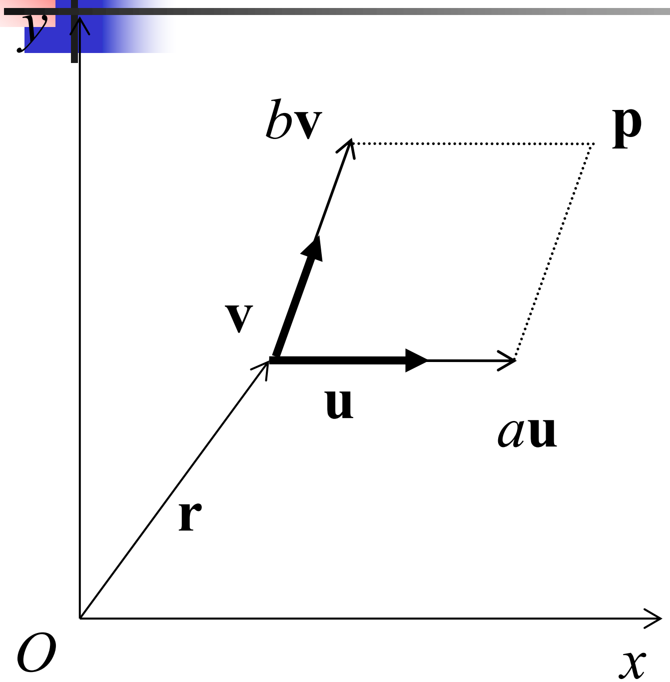


$$M_{base} = T_{base}$$

$$M_{arm} = M_{base}\ R(\theta)\ T_{arm} = T_{base}\ R(\theta)\ T_{arm}$$

$$M_{hand} = M_{arm}\ R(\phi) = T_{base}\ R(\theta)\ T_{arm}\ R(\phi)$$

Coordinate Systems



$$\mathbf{p} = \mathbf{r} + a\mathbf{u} + b\mathbf{v}$$

$$\begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = T\,M \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

$$T^{-1}\begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = M \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

$$M^{-1}T^{-1}\begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} u_x & v_x & 0 \\ u_y & v_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad and \quad T = \begin{bmatrix} 1 & 0 & r_x \\ 0 & 1 & r_y \\ 0 & 0 & 1 \end{bmatrix}$$

Orthonormal Coordinates

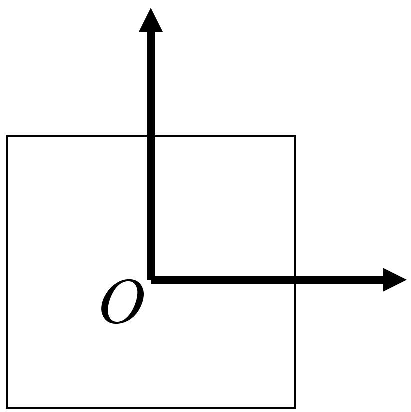
Iff **u** and **v** are orthonormal:

$$M^{-1} = \begin{bmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = M^T$$

$$M^{-1}T^{-1} = \begin{bmatrix} u_x & u_y & 0 \\ v_x & v_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -r_x \\ 0 & 1 & -r_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} u_x & u_y & -\mathbf{r}\cdot\mathbf{u} \\ v_x & v_y & -\mathbf{r}\cdot\mathbf{v} \\ 0 & 0 & 1 \end{bmatrix}$$

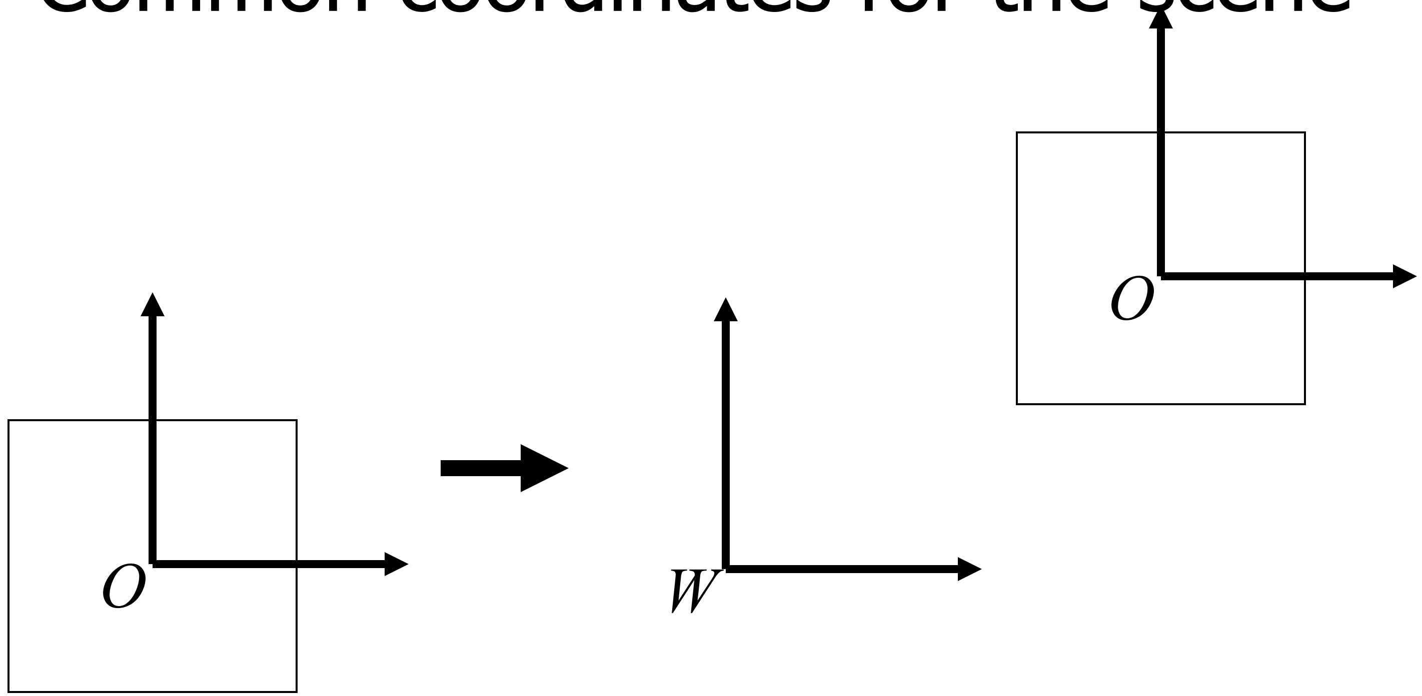
Object Coordinates

Convenient place to model the object



World Coordinates

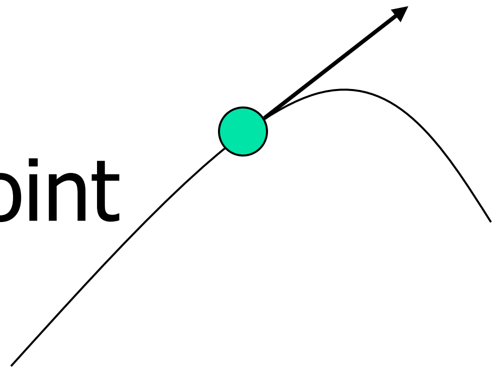
Common coordinates for the scene



Dynamics

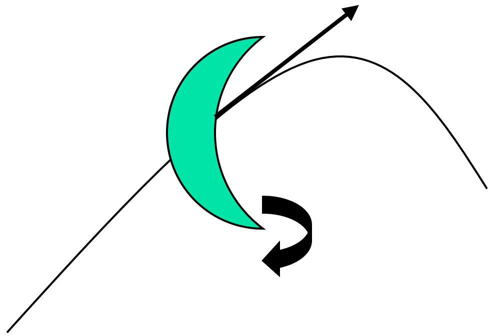
- Dynamics of a material point

  Ideal case, but often sufficient



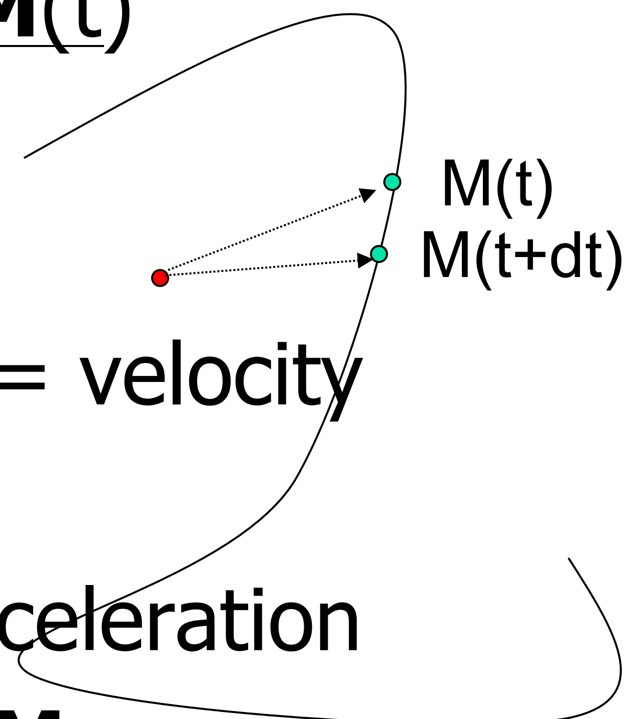
- Dynamics of a solid

  Including rotation, torques…



Position, Velocity, Acceleration

$$\mathbf{V} = \lim_{dt \to 0} \frac{\mathbf{OM}(t+dt) - \mathbf{OM}(t)}{dt}$$



$$\mathbf{V} = d\mathbf{OM}/dt = \dot{\mathbf{OM}} = \text{velocity}$$

$$||\mathbf{V}|| = \text{speed}$$

$$\mathbf{A} = d\dot{\mathbf{V}}/dt = \dot{\mathbf{V}} = \text{acceleration}$$

$$= d^2\mathbf{OM}/dt^2 = \mathbf{OM}$$

Newton Law *(material point)*

$$\mathbf{F} = m\,\mathbf{A}$$

F: sum of the forces exerted

m: mass of the material point

Exercise: Material point falling

- $\mathbf{A}(t) = \mathbf{g}$
- $\mathbf{V} = \mathbf{V_0} + \mathbf{g}\, t$
- $\mathbf{M} = \mathbf{M_0} + \mathbf{V_0}\, t + .5 * \mathbf{g}\, t^2$

Now, how can we find the motion of this point when some external forces are present?

Temporal integration

- In the previous example, we were able to explicitly integrate the motion.
- However, we must often numerically integrate the motion – For instance, Newton method:

$$\mathbf{A}(t) = \mathbf{F}(t) / m$$
$$\mathbf{V}(t) = \mathbf{V}(t\text{-}dt) + \mathbf{A}(t)\, dt$$
$$\mathbf{M}(t) = \mathbf{M}(t\text{-}dt) + \mathbf{V}(t)\, dt$$

Prb: accuracy, stability…

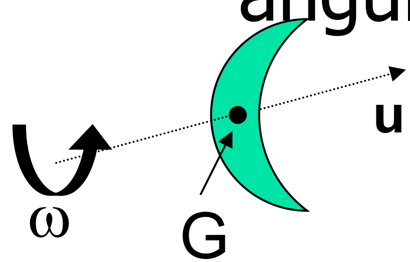
Point vs. Object

- We overlooked rotations! A real object can also spin on itself during motion.
- Notion of **angular velocity**:

$$\boldsymbol{\omega} = \omega\,\mathbf{u}$$

angular speed   axis of

rotation



(G: center of mass)

Integration

- Just a bit more complex…
- Quaternion make it easier

- We will review it later (rigid body simulation)

Other things to know

- Action/Reaction Principle:

$$\mathbf{F}_{1/2} = - \mathbf{F}_{2/1}$$

2D Triangles

- Definitions: given 2D points a, b, c
- Area:
  - Positive
  - Negative (clockwise)

$$\text{area} = \frac{1}{2}\begin{vmatrix} x_b - x_a & x_c - x_a \\ y_b - y_a & y_c - y_a \end{vmatrix}$$

$$= \frac{1}{2}\left(x_a y_b + x_b y_c + x_c y_a - x_a y_c - x_b y_a - x_c y_b\right).$$

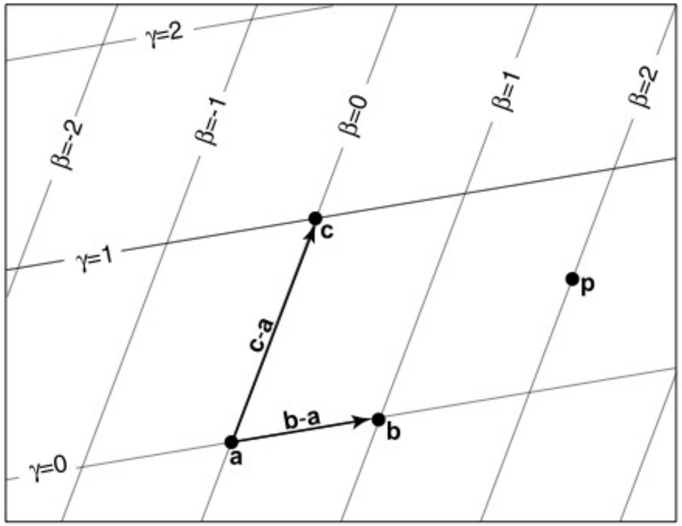
Barycentric Coordinate

$$\mathbf{p} = (1 - \beta - \gamma)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}.$$

$$\alpha \equiv 1 - \beta - \gamma,$$

$$\mathbf{p}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

$$\alpha + \beta + \gamma = 1.$$



$$\mathbf{p} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a}).$$

A point is inside the triangle if and only if

$$0 < \alpha < 1,$$

$$0 < \beta < 1,$$

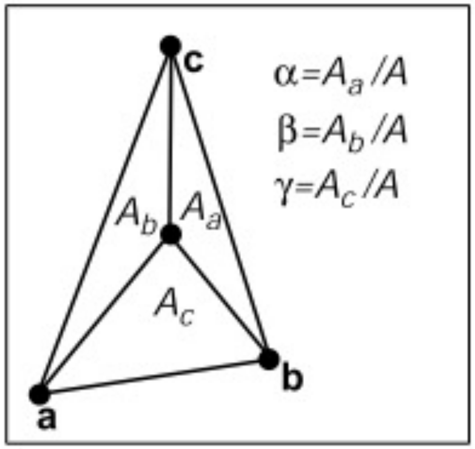
$$0 < \gamma < 1.$$

Barycentric Coordinate II

$$\gamma = \frac{(y_a - y_b)x + (x_b - x_a)y + x_a y_b - x_b y_a}{(y_a - y_b)x_c + (x_b - x_a)y_c + x_a y_b - x_b y_a}.$$

$$\beta = \frac{(y_a - y_c)x + (x_c - x_a)y + x_a y_c - x_c y_a}{(y_a - y_c)x_b + (x_c - x_a)y_b + x_a y_c - x_c y_a},$$

$$\alpha = 1 - \beta - \gamma.$$



Another way to compute Barycentric coordinates

$$\alpha = A_a/A,$$

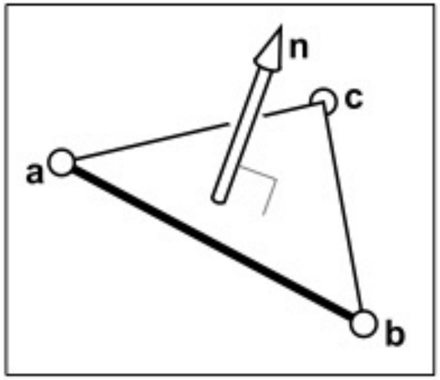
$$\beta = A_b/A,$$

$$\gamma = A_c/A,$$

3D Triangles

- Barycentric coordinates can be used from 2D from 3D.

$$\mathbf{p} = (1 - \beta - \gamma)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}.$$



$$\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}).$$

$$\text{area} = \frac{1}{2}\|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\|.$$

$$\alpha = \frac{\mathbf{n} \cdot \mathbf{n}_a}{\|\mathbf{n}\|^2},$$

$$\beta = \frac{\mathbf{n} \cdot \mathbf{n}_b}{\|\mathbf{n}\|^2},$$

$$\gamma = \frac{\mathbf{n} \cdot \mathbf{n}_c}{\|\mathbf{n}\|^2},$$

$$\mathbf{n}_a = (\mathbf{c} - \mathbf{b}) \times (\mathbf{p} - \mathbf{b}),$$
$$\mathbf{n}_b = (\mathbf{a} - \mathbf{c}) \times (\mathbf{p} - \mathbf{c}),$$
$$\mathbf{n}_c = (\mathbf{b} - \mathbf{a}) \times (\mathbf{p} - \mathbf{a}).$$