Applied Algorithms. Prof. Tami Tamir
Missing Proofs, lecture 7. July 9th.

An FPTAS for KNAPSACK.

Theorem: The profit achieved by the algorithm is at least (1- ε)OPT.

Proof: Let S denote the set of items packed by the PTAS, and let S* denote the set of items packed in some optimal solution. Thus, $ALG=\Sigma_{i\in S} b_i$, $OPT=\Sigma_{i\in S^*} b_i$.

The rounding could have increased each b_i by less than k, since ceiling increases the value of the fraction by less than 1.

$ALG = \sum_{i\in S} bi \geq \sum_{i\in S}(b'i - k) = \sum_{i\in S} b'i - |S|k = \sum_{i\in S} b'i - |S|\cdot \varepsilon B/n \geq \sum_{i\in S} b'i - \varepsilon B$

Since S is optimal for the rounded instance, and due to the ceiling, we have:
$\sum_{i\in S} b'i \geq \sum_{i\in S^*} b'i \sum_{i\in S^*} bi \sum_{i\in S^*} bi.$

Therefore: $ALG \geq \sum_{i\in S} b'i - \varepsilon B \geq \sum_{i\in S^*} bi - \varepsilon B$.

Finally, since $B \leq \sum_{i\in S^*} bi$ (packing only the element whose value is B is a valid solution), we have: $ALG \geq \sum_{i\in S^*} bi - \varepsilon B \geq (1-\varepsilon) \sum_{i\in S^*} bi = (1-\varepsilon)OPT$.

Any-fit Decreasing for Unit-Fraction Bin Packing

After being sorted in a non-increasing order, the input sequence has the form

$$W = \langle (\frac{1}{2})^{n2}, (\frac{1}{3})^{n3} \dots (\frac{1}{z})^{nz} \rangle$$

for some integers z ≥ 2 and ni ≥0 for 2 ≤ i ≤ z.
Assume that AFD uses h full bins (filled to capacity 1) and h' non-full bins. Thus, N_AFD(W) = h'+h.

Claim: After packing all the items of size at least 1/k, there are at most k-1 non-full bins.
Proof: For every j, items of size 1/j are packed in the following way:1. Added to already-open bins. 2. Every j items are packed together in a full bin. 3. The last items nay be olaced in a new open bin. A new non-full bin is added only in step 3 and only for j>1, and the claim follows.

Suppose the last bin that AFD opened was opened for an item of width 1/z' where z'≤ z. By the above claim, at this stage, there are less than z' non-full bins. Since this is the last opened bin, it follows that h'<z'.
Furthermore, each of the first h'-1 non-full bins must contain items whose total width is greater than 1-(1/z'), because otherwise AFD would not open a new bin for 1/z'. By definition, the last non-full bin contains at least one item of size 1/z'. It follows that H(W) ≥ h + (h'-1)(1-1/z')+1/z' = h+h'-1-(h'-2)/z' > h+h'-2 .

Since H(W) is an integer, it must be at least h'+h-1. Thus, N_AFD(W) = h' + h ≤ H(W)+1.