Packing Algorithms

An instance of a packing problem consists of:

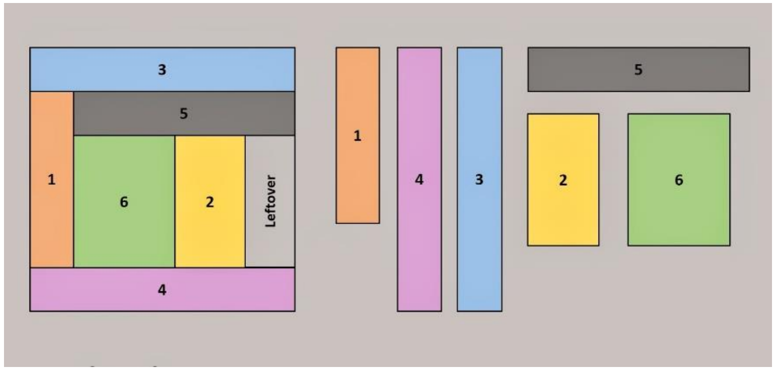
1. Items (associated with sizes, weights, profits).
2. Bins with limited capacity.
3. A set of constraints.

General Goal: Place items in the bin, items must not overlap with each other. Bins may not be filled beyond their capacity.

Packing Problems

Some Variants:

- Multi-dimensional items.
- Limited number of bins – pack as much as you can.
- Unlimited number of bins – pack all items using minimal number of bins.
- Conflicting items – cannot be placed together.
- Cardinality constraints.
- Cutting stock (minimizing wasted material)
- Class constraints.
- Online vs. offline
- Variable bins.
- Many more…



Popular Applications

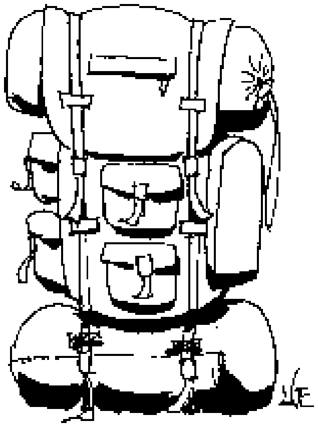
- Physical items → boxes, tracks (shipping, delivery).
- Files → disks, storage devices.
- Advertisements → commercial breaks/ magazines / web-pages.
- Orders → limited amount of material
- Jobs → processors.

The Knapsack problem

- You are about to go to a camp.
- There are many items you want to take.
- You have one knapsack. The total weight you can carry is at most W.
- Item i in your list has weight w_i, and value (benefit) b_i, that measures how much you really need it.
- You need to pack the knapsack in a way that maximizes the total value of the packed items.

The Knapsack problem

Item #	Weight	Value
1	1	8
2	3	6
3	5	5
4	4	6



Max weight =8

A possible packing: Items 2 and 3. Value: 11

An optimal packing: Items 1,2,4. Value: 20

The Knapsack problem is NP-hard.

Greedy Algorithm for Knapsack

1. Consider the items in order of non-increasing b_i/w_i ratio
   $b_1/w_1 \geq b_2/w_2 \geq \ldots \geq b_n/w_n$
2. Add items to the knapsack as long as there is space.

Time Complexity:

O(n log n) (for sorting)

O(n) for packing loop.

→ O(n log n)

Greedy Algorithm for Knapsack

Claim: The approximation ratio of Greedy is not bounded.

Proof: To get ratio c, consider the following instance:

There are two items:

$b_1 = 2,\ w_1 = 1$

$b_2 = 2c,\ w_2 = 2c$

The knapsack has volume $W = 2c$

Greedy packs only the first item, value = 2.

Optimal: Pack the second item, value=2c

Ratio = c.

Improved Algorithm for Knapsack

Take the maximum of Greedy and the most valuable item that fits by itself.

Theorem: The above algorithm is 2-approximation.

Proof: We assume w.l.o.g that no single item has weight more than W (these items can be removed in a preprocessing).

Sort the items such that $b_1/w_1 \geq b_2/w_2 \geq \ldots \geq b_n/w_n$.

Let B be the largest value of an item, and let G be the value computed by the greedy algorithm.

Let j be the first item that the greedy algorithm rejects.

Improved Algorithm for Knapsack

$ALG = \max(B, G) \geq (B + G)/2$

$G = \sum_{i=1}^{j-1} b_i$ (item j is the first to be rejected)

$B \geq b_j$ (B is the most profitable)

$G+B \geq \sum_{i=1}^{j} b_i$ $\quad opt < \sum_{i=1}^{j} b_i$ Because the first j items have the largest 'profit density'

$\rightarrow ALG > opt/2$

Exact Solution – dynamic programming

Variant 1:

Define a table M of size $(n+1)\times(W+1)$, where the (i, x) entry corresponds to the maximal profit that can be obtained from the first i items and a knapsack having capacity x.

The solution to the knapsack problem lies in $M(n,W)$.

Base cases:

If $i = 0$, then there are no items to pack: $M(0, x) = 0$.

If $x < 0$ then $M(i, x) = -\infty$.

Exact Solution - dynamic programming

The DP recursion:

$M(i, x) = \max \{ M(i-1, x) , M(i-1, x-w_i) + b_i \}.$

We take the maximum of two options:

1. $M(i-1, x)$: not packing the i-th item.
2. $M(i-1, x-w_i) + b_i$: packing the i-th item.

n								*
…								
i								
i-1								
…								
0	0	0	0	0	…	0	…	0
	1	2	3	4	…	x	…	W

The solution lies here

Knapsack DP Example

Assume n=4 and W=17.

Weights:{2, 4, 7, 10} Values: {3, 7, 9, 16}

4																	
3																	
2																	
1																	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
i/x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Knapsack DP Example

Assume n=4 and W=17.

Weights:{2, 4, 7, 10} Values: {3, 7, 9, 16}

4	0	3	3	7	7	10	10	10	12	16	16	19	19	23	23	26	26
3	0	3	3	7	7	10	10	10	12	12	12	12	19	19	19	19	19
2	0	3	3	7	7	10	10	10	10	10	10	10	10	10	10	10	10
1	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
i/x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

M(3,9)= max{M(2,9), M(2,2)+9} = max{10,3+9}=12

M(4,12)= max{M(3,12), M(3,2)+16}=max{12,3+16}=19.

Knapsack DP - Time complexity

- Table size: $(n+1)\times(W+1) = O(nW)$.
- Every entry is computed in constant time.
- Total complexity: $O(nW)$.

Is it polynomial time?

- Not really – the input size is not $O(nW)$ but $O(n \cdot \log W)$ bits.

- It is denoted "pseudo polynomial time".

Exact Solution - dynamic programming

Variant 2:
Define a table M of size $(n+1)\times(\Sigma_i b_i)$, where the (i, v) entry corresponds to the minimum weight of a combination of the first i items with value at least v.
Note: $M(i, v) = \infty$ if $b_1+\ldots+b_i < v$.

The solution is in the entry with the maximum v where $M(n, v) < W$. This entry can find by scanning all entries in the line of $i = n$.

Base cases:
If $M(0, 0) = 0$.
If $M(0, v) = \infty$ for all $v > 0$.

Exact Solution - dynamic programming

The DP recursion:

$M(i, v) = \min \{ M(i-1, v) \, , M(i-1, v-b_i) + w_i \}.$

We take the minimum of two options:

1. $M(i-1, v)$: not packing the i-th item.
2. $M(i-1, v-b_i) + w_i$: packing the i-th item.

n								
…								
i								
i-1								
…								
0	0	0	0	0	…	0	…	0
	1	2	3	4	…	v	…	$\Sigma_i b_i$

The solution lies in this line

Exact Solution - dynamic programming

The second DP variant is the basis of our next approximation algorithm.

The Knapsack problem is 'easy to approximate' - we can get as closer to an optimal solution as required.

Formally, it has a fully polynomial time approximation scheme (FPTAS)

Polynomial Time Approximation Scheme

A polynomial time approximation scheme is an algorithm which takes as input an additional parameter, ε, which determines the desired approximation ratio. This ratio can be arbitrarily close to 1, when ε approaches 0.

The time complexity of the scheme is polynomial in the input size but may be exponential in $1/\varepsilon$.
For example, the following running times are acceptable for a PTAS:

- $O(n^2/\varepsilon)$
- $O(n^{100}2^{1/\varepsilon})$
- $O(n^{2\hat{}2\hat{}1/\varepsilon})$

A Fully Polynomial Time Approximation Scheme

A fully polynomial time approximation scheme (FPTAS) is a PTAS with running time polynomial in both n and $1/\varepsilon$.

The good news: The Knapsack problem has a fully polynomial time approximation scheme.

FPTAS for Knapsack

The DP has size $n \cdot \Sigma_i b_i$

$\Sigma_i b_i$ is loosely bounded by $n \cdot B$, where $B = \max(b_i)$. Since we must compute every cell in the table, the DP running time is $O(n^2 B)$, which is pseudo-polynomial.

To construct the FPTAS, we will reduce the number of distinct value categories (columns).

We will have only $\text{poly}(n)/\varepsilon$ value categories by scaling and rounding each b_i

This way, the DP table will have size $\text{poly}(n, 1/\varepsilon)$. As a result the running time will also be $\text{poly}(n, 1/\varepsilon)$.

FPTAS for Knapsack

1. Given $\varepsilon > 0$, let $k = \varepsilon B/n$ be the scaling parameter.

2. Replace every item value b_i by $b'_i = \left\lceil \frac{b_i}{k} \right\rceil k$.

The number of columns (different values of b'_i) is now no more than $n\frac{B}{k} = \frac{n^2}{\varepsilon}$.

3. Apply the DP (variant 2) with values b'_i. The table hac size $(n+1)\times(n^2/\varepsilon)$, he running time is $O(n^3/\varepsilon) = O(poly(n,1/\varepsilon))$.

Theorem: The profit achieved by the algorithm is at least $(1- \varepsilon)OPT$.

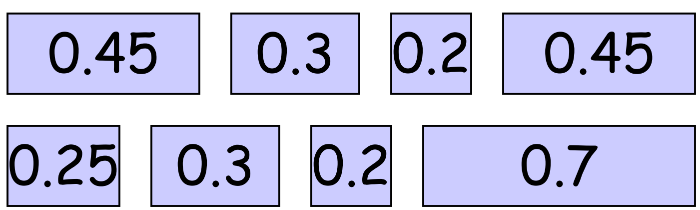
Example and Proof: In class.

The Bin Packing Problem

- Input: Items of sizes $0 < s_i < 1$
- Output: A feasible packing in bins of size 1
- Goal: minimize number of bins used.

Example:

Input:



A packing in 3 bins:

Approximating Bin Packing

Next-fit Algorithm:

1. Open an *active* bin.
2. For all $i=1,2,\ldots,n$:
   - If possible, place a_i in the current active bin;
   - Otherwise, open a new active bin and place a_i in it.

Example: The input: {0.3, 0.9, 0.2}.

Next-fit packing (three bins): (0.3), (0.9), (0.2).

Theorem: Next-fit is 2-approximation to BP

Proof: An optimal algorithm must use at least $\Sigma_i a_i$ bins (why?).

Approximating Bin Packing

Analysis of Next Fit (cont'): Assume that Next Fit uses h bins. The sum of items sizes in two consecutive bins is greater than 1 (otherwise, we can put them together).

Case 1: h is even:

$$\begin{array}{l} c(B_1) + c(B_2) > 1 \\ c(B_3) + c(B_4) > 1 \\ c(B_{h-1}) + c(B_h) > 1 \\ \hline \Sigma_i a_i \;\; > h/2 \end{array}$$

Case 1: h is odd:

$$\begin{array}{l} c(B_1) + c(B_2) > 1 \\ c(B_3) + c(B_4) > 1 \\ c(B_{h-2}) + c(B_{h-1}) > 1 \\ \hline \Sigma_i a_i \;\; > (h-1)/2 + c(B_h) \end{array}$$

In both cases, we can obtain $h \leq \lceil 2\Sigma_i a_i \rceil \leq 2opt$

Remark: it can be shown that $h \leq 2opt-1$

Approximating Bin Packing

Is the analysis tight? consider an instance with 4n items {1/2, 1/2n, 1/2, 1/2n, …}.

Next-fit will put any two consecutive items in a bin.

Total number of bin used: 2n.

An optimal packing in n+1 bins: n bins, each with 1/2+1/2, one bin for the tiny items.

The ratio: $2n/(n+1) \rightarrow 2$ as n grows.

Approximating Bin Packing

First fit algorithm: place the next item in the first open bin that can accommodate it. Open a new bin only if no open bin has enough room.

Theorem: $h_{ff} \leq 1.7opt + 2$ (proof not here)

First fit Decreasing: sort the items from largest to smallest. Run FF according to the resulting order.

Theorem: $h_{ffd} \leq 1.222opt + 3$ (proof not here)

- No additive-error approximation is known for bin packing. That is, the best known is $(1+\delta)opt$.

Unit Fractions Bin Packing

- A Unit Fraction: A fraction of the form $1/i$ for an integer i.
- Input: integers $w_1, w_2, \ldots, w_n$.
- Goal: Bin packing of the unit fractions $\{1/w_1, 1/w_2, \ldots, 1/w_n\}$.
- Let $H(W) = \left\lceil \sum_{i \in W} \frac{1}{w_i} \right\rceil$. Clearly, $OPT(W) \geq H(W)$.
- We will see: An algorithm that uses at most $H(W)+1$ bins (additive error of one for any input).

Any-fit Decreasing for UFBP

1. Sort the items such that $1/w_1 \geq 1/w_2 \geq \cdots \geq 1/w_n$
2. Pack the items in this order, each item is placed in any open bin that can accommodate it, or in a new bin, if none exists.

Theorem: The number of bins used is at most

$$1 + \left\lceil \sum_i \frac{1}{w_i} \right\rceil \leq 1 + OPT$$

Proof idea: After packing all the items of size at least $1/k$:

(i) There are at most $k-1$ non-full bins, and

(ii) Each of the non-full bins is at least $1-1/k$ full.

Details: In Class

Any-fit Decreasing for UFBP

Remark: The analysis is tight (the alg. is not optimal)

Example: $\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ - in decreasing order.

- Will be packed in three bins:

$$\left\{\frac{1}{2}, \frac{1}{3}\right\} \quad \left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{4}\right\} \quad \left\{\frac{1}{4}\right\}$$

- Can be packed in two bins:

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right\} \quad \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{4}\right\}$$

Online Bin Packing

The input: A sequence of items (numbers), $a_1,a_2,\dots,a_n$, such that for all i, $0 < a_i < 1$

The goal: 'pack' the items in bins of size 1. Use as few bins as possible.

Example: The input: 1/2, 1/3, 2/5, 1/6, 1/5, 2/5.

Optimal packing in two bins:

(1/2, 1/3, 1/6), (2/5, 2/5, 1/5).

Legal packing in three bins:

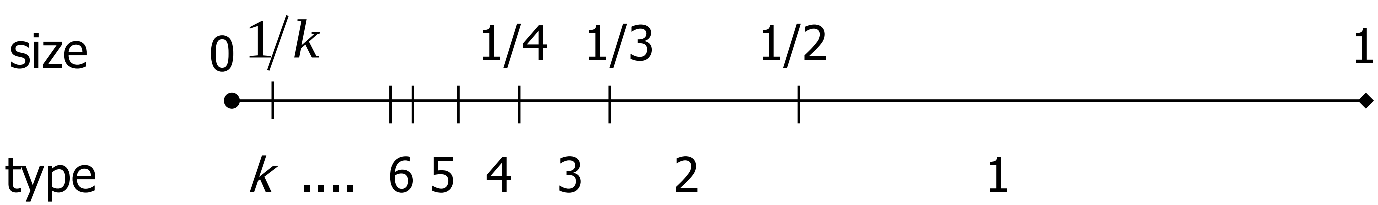
(1/2, 1/3), (2/5, 1/6, 1/5), (2/5)

Online BP: a_i must be packed before we know $a_{i+1},\dots,a_n$

The HARMONIC-k Algorithm

Classify items into k intervals according to size

(1/2,1]	one item per bin
(1/3,1/2]	two items per bin
…	
(1/k,1/(k-1)]	k-1 items per bin
(0,1/k]	use NextFit



The HARMONIC Algorithm

- Each bin contains items from only one class: i items of type i per bin
- Items of **last type** are packed using **NEXT FIT**: use one bin until next item does not fit, then start a new bin
- Keeps **k-1** bins open

Analysis of HARMONIC-3

- Let X be the number of bins for **(1/2,1]**
  - Those bins are full by more than 1/2
- Let Y be the number of bins for **(1/3,1/2]**
  - Those bins are full by more than 2/3
- Let T be the number of bins for **(0,1/3]**
  - Those bins are full by more than 2/3

Let W be the total size of all items

Then $W > X/2 + 2Y/3 + 2T/3$

Analysis of HARMONIC-3

Other bounds:

- $OPT \geq X$ (items larger than 1/2)
- $OPT \geq (X+2Y)/2$ (items larger than 1/3)

- $H3 \leq X+Y+T$ (+2) $\leq (3(W+X/6))/2$(+2)
  $\leq 3W/2+X/4$ (+2) $\leq 1.75OPT$ (+2)

Asymptotically, this is neglected.

Analysis of HARMONIC-4

- Let X be the number of bins for (1/2,1]
  - Those bins are full by more than 1/2
- Let Y be the number of bins for (1/3,1/2]
  - Those bins are full by more than 2/3
- Let Z be the number of bins for (1/4,1/3]
  - Those bins are full by more than 3/4
- Let T be the number of bins for (0,1/4]
  - Those bins are full by more than 3/4
- Let W be the total size of all items

  Then $W>X/2+2Y/3+3Z/4+3T/4$

Analysis of HARMONIC-4

Other bounds:

- $OPT \geq X$ (items larger than 1/2)
- $OPT \geq (X+2Y)/2$ (items larger than 1/3)

- $H4 \leq X+Y+Z+T\ (+3) \leq (4(W+X/4+Y/12))/3 (+3)$

  $\leq 4 \cdot W/3+X/3+Y/9\ (+3) =$

  $= 4 \cdot W/3+(x/18+Y/9)+5 \cdot X/18\ (+3)$

  $\leq 31 \cdot OPT\ /18\ (+3) \approx 1.7222 \cdot OPT\ (+3)$

Analysis of HARMONIC

- Theorem: For any k, Harmonic-k is at most 1.691 competitive.

- Proof: C. C. Lee and D. T. Lee. A simple on-line bin-packing algorithm. *Journal of the ACM* 32 (3) July 1985. (beyond our scope. Available in the course web-page).