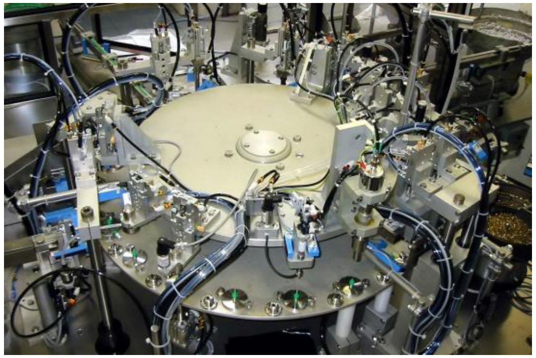
Scheduling Algorithms

Part 2



Parallel machines.
Open-shop Scheduling.
Job-shop Scheduling.

SZ1, SZ2

Parallel Machines

- n jobs need to be scheduled on m machines, $M_1, M_2, \ldots, M_m$.
- Each machine can process at most one job at any time.
- Each job can be processed by at most one machine at any time.
- Identical machines (denoted P) – all the machines have the same rate. Processing time of job j = p_j
- Uniform machines (denoted Q) – each machine has a rate, s_i, uniform for all the jobs processed on it. Processing time of job j = p_j/s_i.
- Unrelated machines (denoted R) - Specific values of $p_{i,j}$. $p_{i,j}$ =The processing time of job j on machine i.

Parallel Machines, the problem $P||\sum_j C_j$

Theorem: SPT is optimal for $P||\sum_j C_j$

Proof: Assume $n=zm$ (w.l.o.g we can add jobs with $p_j=0$, why?)

Index the jobs such that $p_1 \le p_2 \le \ldots \le p_n$.

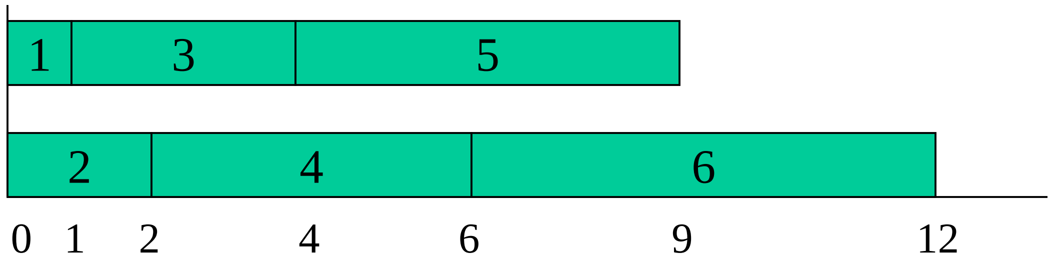
In SPT the m jobs $J_{(k-1)m+1},\ldots J_{km}$, are scheduled in the k^{th} locations on each of the m machines. Their processing time is 'counted' $z+1-k$ times in $\sum_j C_j$.

Note: We can switch jobs in the k^{th} location on different machines without affecting $\sum_j C_j$

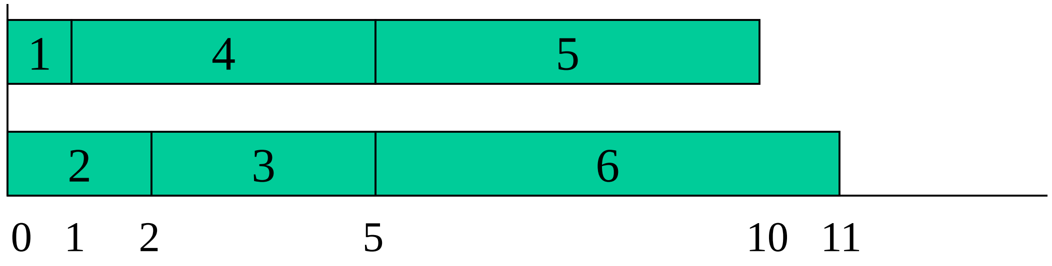
SPT for P|| $\sum_j C_j$. Example.

m=2,

6 jobs with processing times 1,2,3,4,5,6



$\sum_j C_j = 34$



$\sum_j C_j = 34$

Parallel Machines, the problem $P||\sum_j C_j$

$\sum_j C_j$ is the result of the following vector multiplication:

$$(\underbrace{z, \dots, z}_{\text{m times}}, \underbrace{z-1, \dots, z-1}_{\text{m times}}, \dots, \underbrace{1, \dots, 1}_{\text{m times}})(p_1, p_2, \dots, p_n)$$

The left vector is non-increasing, so the multiplication value is minimized if the other vector is non-decreasing, as implied by SPT.

The problem $P||\sum_j w_j C_j$

This problem is NP-hard.

It can be solved using dynamic programming or branch and bound.

In any optimal solution, the jobs scheduled on one machine are scheduled according to WSPT rule ($p_1/w_1 \leq p_2/w_2 \leq \ldots$) otherwise exchanges can improve the objective function.

However, the problem of partitioning the jobs among the machines is NP-hard.

The problem $P||C_{max}$

Theorem: The problem $P||C_{max}$ is NP-hard

Proof: Reduction from Partition.

Reminder: The partition problem:

Input: a set of n numbers, $A = \{a_1, a_2, \ldots, a_n\}$, such that $\sum_{j \in A} a_j = 2B$.

Output: Is there a subset A' of A such that $\sum_{j \in A'} a_j = B$?

Example: $A=\{5, 5, 7, 3, 1, 9, 10\}$; $B=20$

A possible partition: $A'=\{10,5,5\}$, $A-A'=\{7,3,1,9\}$

The problem $P||C_{max}$

Reduction from Partition to $P||C_{max}$:

Given an instance for partition, $A = \{a_1, a_2, \ldots, a_n\}$ such that $\sum_{j \in A} a_j = 2B$.

Build an instance for $P2||C_{max}$ such that the makespan is B if and only if there is a partition.

There are n jobs, the processing time of J_i is a_i.

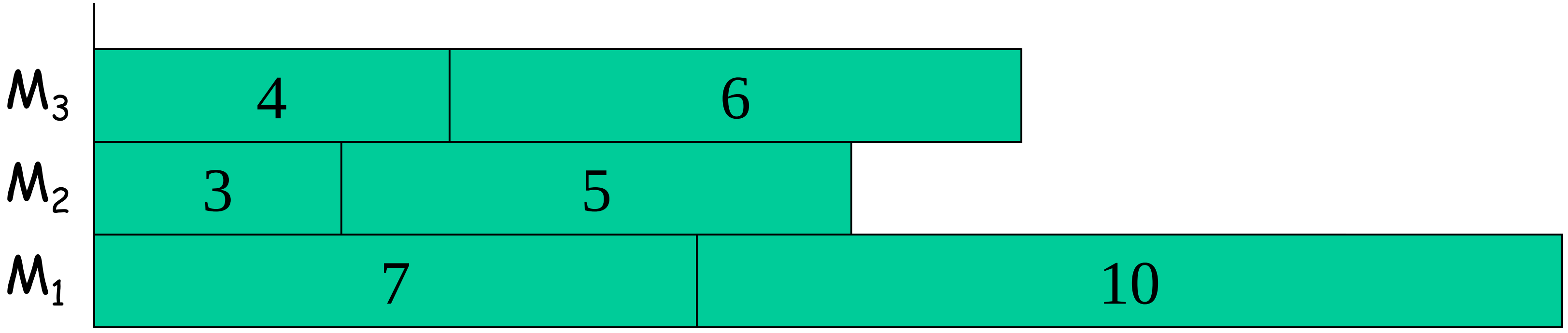
If there is a schedule with makespan =B the jobs scheduled on M_1 corresponds to items in S' – their total size must be B.

The problem $P||C_{max}$

List Scheduling [Graham 1966]:

A greedy algorithm: always schedule a job on the least loaded machine.

Example: $m=3$ σ = 7 3 4 5 6 10



Makespan = 17

List Scheduling for $P||C_{max}$

Theorem: List Scheduling provides a $(2-\frac{1}{m})$-approximation for the problem $P||C_{max}$.

Proof: Let H_i denote the last completion time on the i^{th} machine. Let k be the job that finishes last and determines C_{LS}.

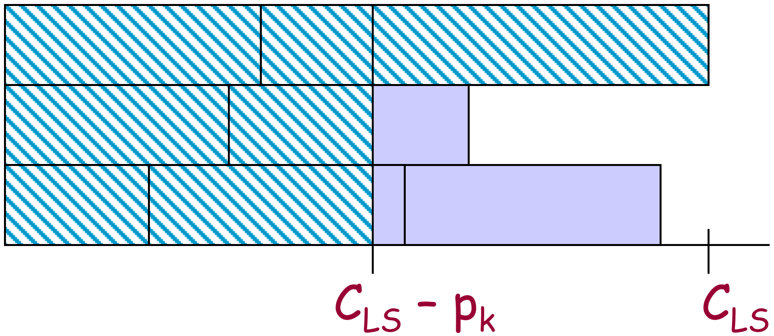
All the machines are busy when k starts its processing, thus, $\forall i$, $H_i \geq C_{LS} - p_k$.

For at least one machine (that processes k) $H_i = C_{LS}$.

→ $\sum_j p_j = \sum_i H_i \geq (m-1)(C_{LS} - p_k) + C_{LS}.$

→ $\sum_j p_j + (m-1)p_k \geq mC_{LS}.$

→ $C_{LS} \leq 1/m \sum_j p_j + p_k (m-1)/m.$



List Scheduling for $P||C_{max}$

→ $C_{LS} \leq 1/m \sum_j p_j + p_k (m-1)/m$.

Consider an optimal schedule.

$C_{opt} \geq \max_j p_j \geq p_k$ (some machine must process the longest job).

$C_{opt} \geq 1/m \sum_j p_j$ (if the load is perfectly balanced).

Therefore,

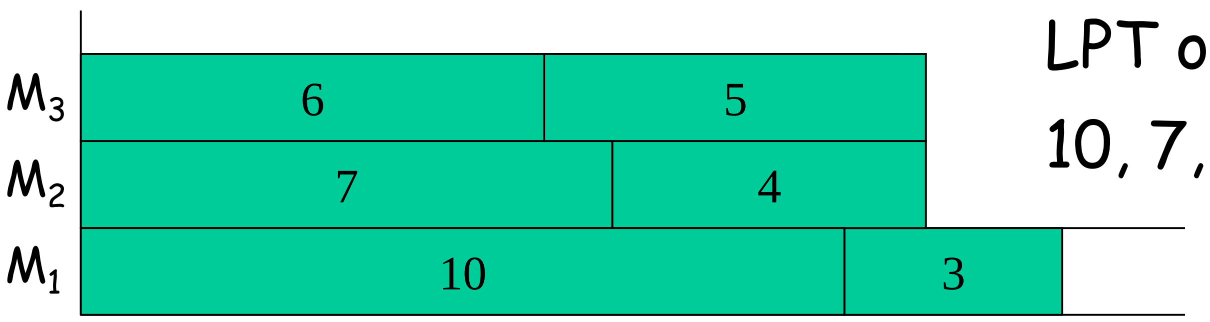
$C_{LS} \leq C_{opt} + C_{opt} (m-1)/m = (2-1/m) C_{opt}$.

Note: The analysis is tight (in class).

Longest Processing Time Rule

The $(2-1/m)$-ratio is for arbitrary order of the jobs. If the jobs are known in advance (offline problem) it is possible to determine the assignment order.

LPT algorithm: List scheduling where the jobs are arranged in non-increasing order of $p_1 \geq p_2 \geq \ldots \geq p_n$.



LPT order:

10, 7, 6, 5, 4, 3

Makespan = 13

Longest Processing Time (LPT) Rule

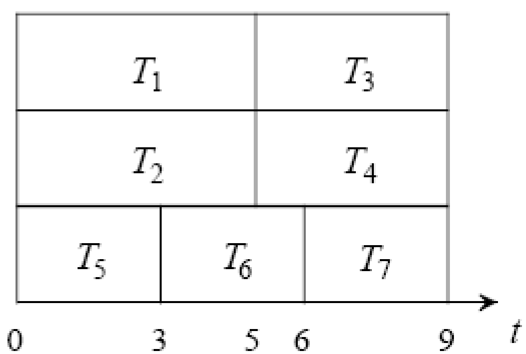
Theorem: LPT provides a $\left(\frac{4}{3}-\frac{1}{3m}\right)$ - approximation for the problem $P||C_{max}$.

Proof: In class.

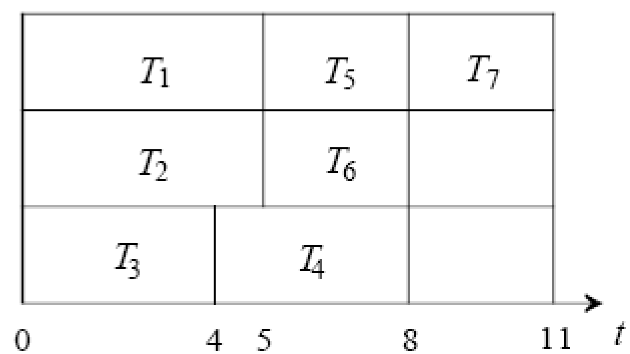
This analysis is tight: Consider $n = 2m + 1$ jobs,

$\boldsymbol{p} = [2m - 1, 2m - 1, 2m - 2, 2m - 2, \ldots, m + 1, m + 1, m, m, m]$.

An optimal schedule and an LPT schedule are (m=3):



and



The problem $P|pmtn|C_{max}$

When preemptions are allowed, the problem is optimally solvable in poly-time.

We have two lower bounds for opt:

$C_{opt} \geq \max_j p_j$ (some machine must process the longest job).

$C_{opt} \geq 1/m \sum_j p_j$ (if the load is perfectly balanced).

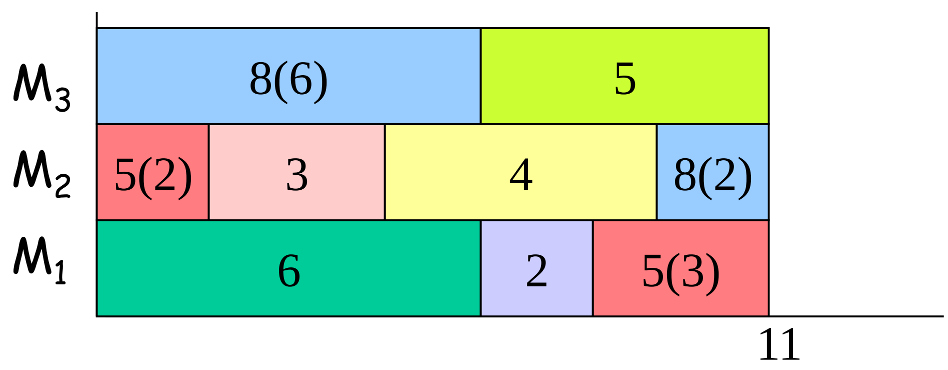
Let $w = \max(\max_j p_j \ , 1/m \sum_j p_j)$

An optimal algorithm for $P|pmtn|C_{max}$

1. Calculate $w=\max(\max_j p_j \ , 1/m \sum_j p_j)$
2. Consider the jobs in arbitrary order, schedule the jobs one after the other on the machines. Move to M_{i+1} after M_i allocated w processing units (maybe preempt the last job on M_i) .

Example: m=3, Jobs lengths are 6,2,5,3,4,8,5

w=max(8,11)=11

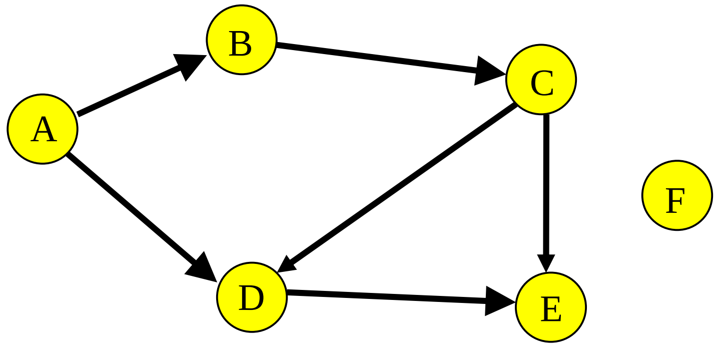


The problem $P|prec|C_{max}$

prec – There are **precedence constraints**.

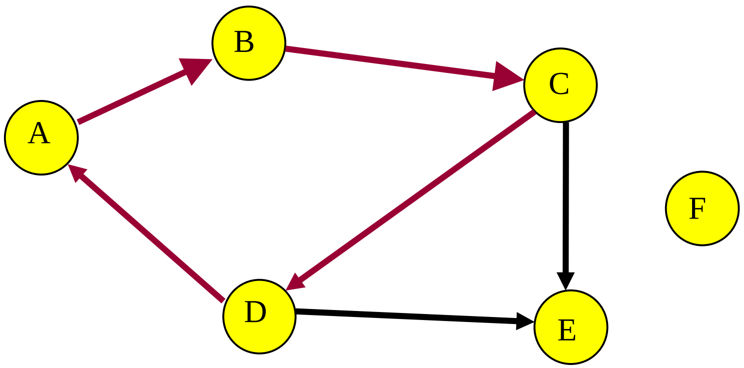
Given as a directed graph.

An edge (u,v) means that the processing of job v can start only after the completion of job u.



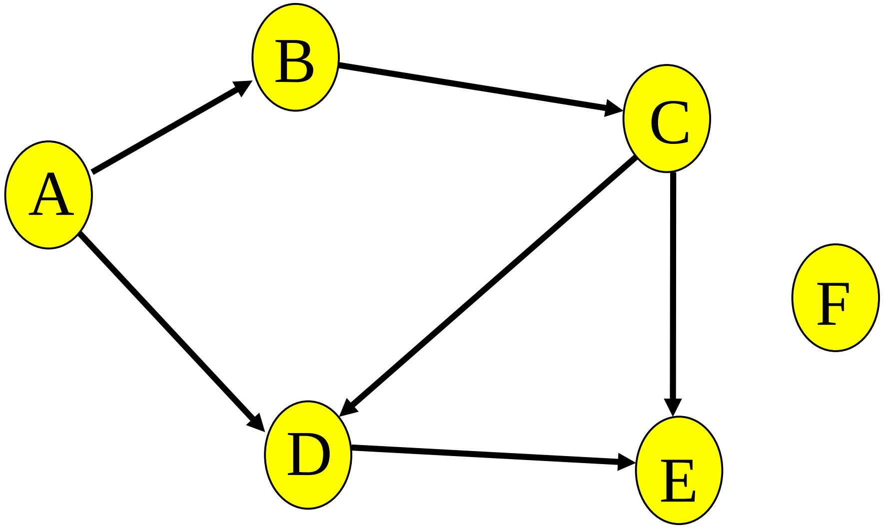
The graph must be acyclic (no directed cycles).

There is no valid ordering of A,B,C,D

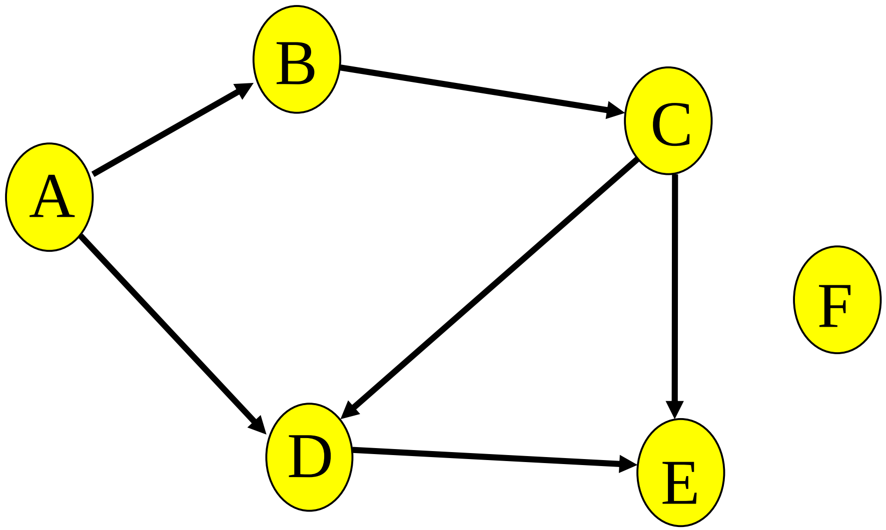


Topological Sort

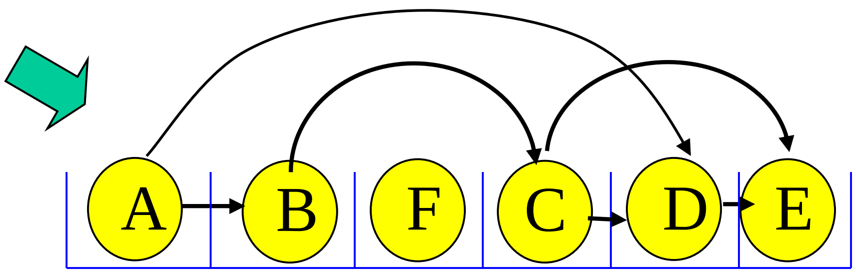
Goal: Given a directed graph $G = (V, E)$, find a linear ordering of its vertices such that for any edge (u,v) in E, u precedes v in the ordering



Topo sort - good example

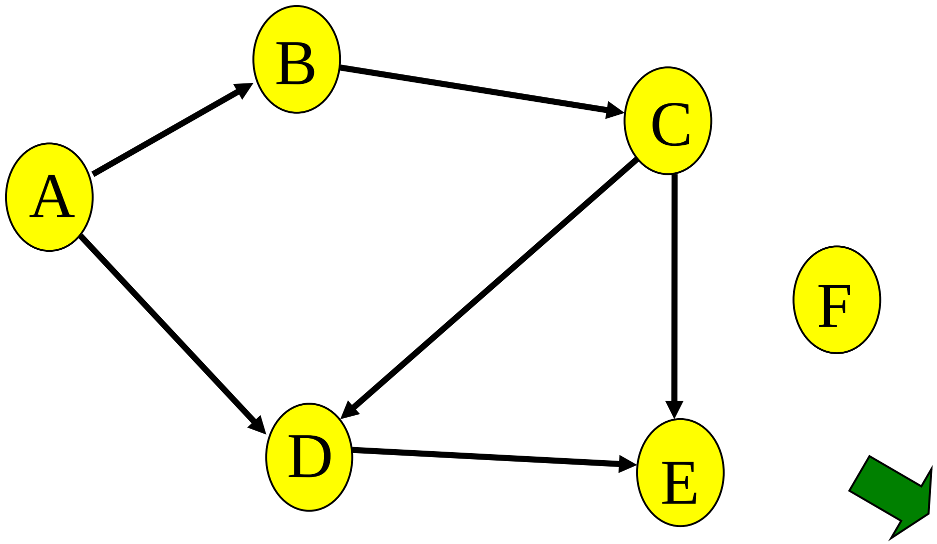


Any linear ordering in which all the arrows go to the right is a valid solution

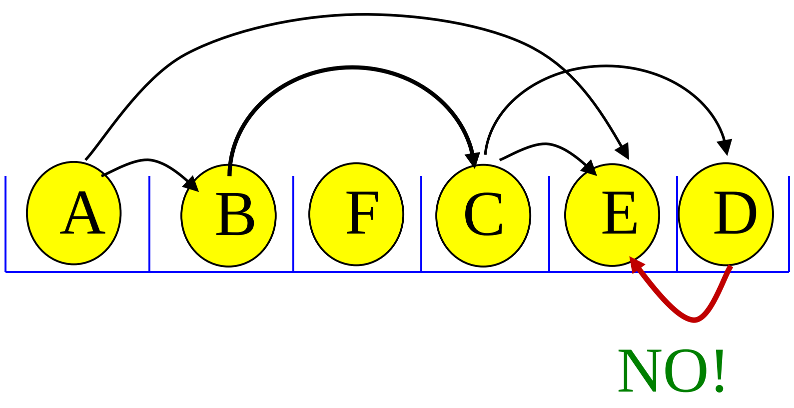


Note that F can go anywhere in this list because it is not connected. Thus, the solution is not unique.

Topo sort - bad example



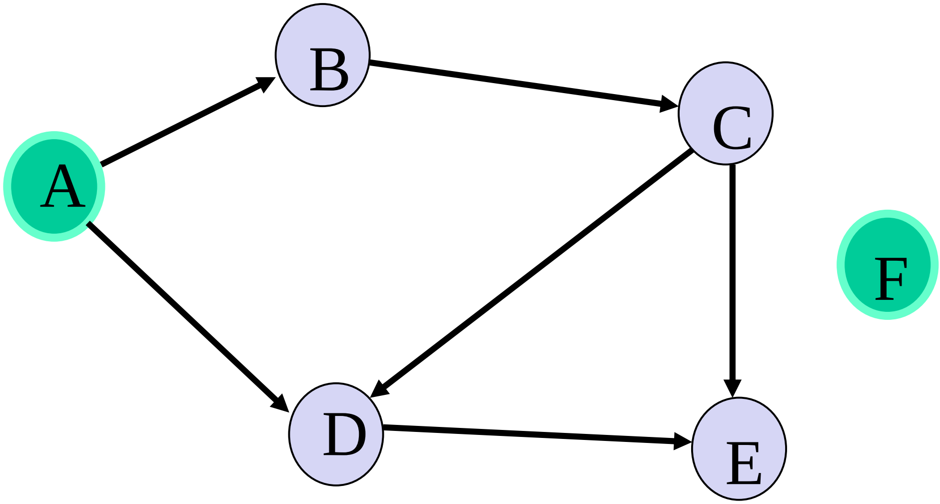
Any linear ordering in which an arrow goes to the left is not a valid solution



Topo sort algorithm

Step 1: Identify vertices that have no incoming edges

- The "in-degree" of these vertices is zero

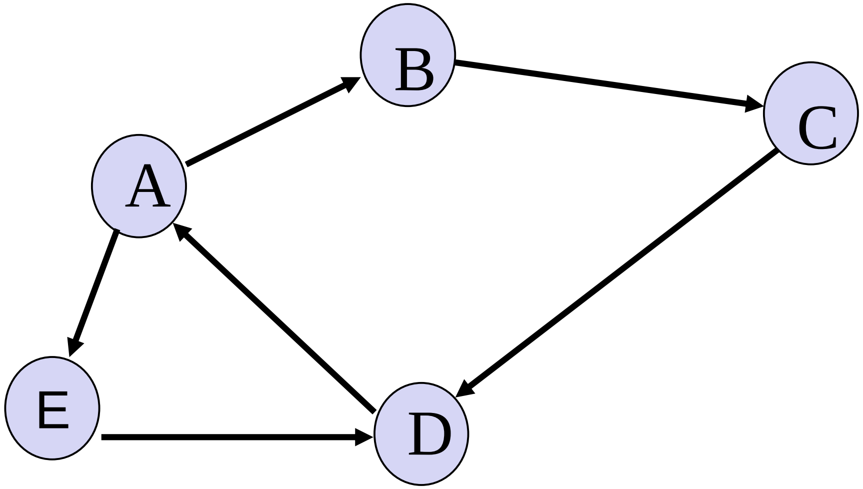


Topo sort algorithm

Step 1: Identify vertices that have no incoming edges

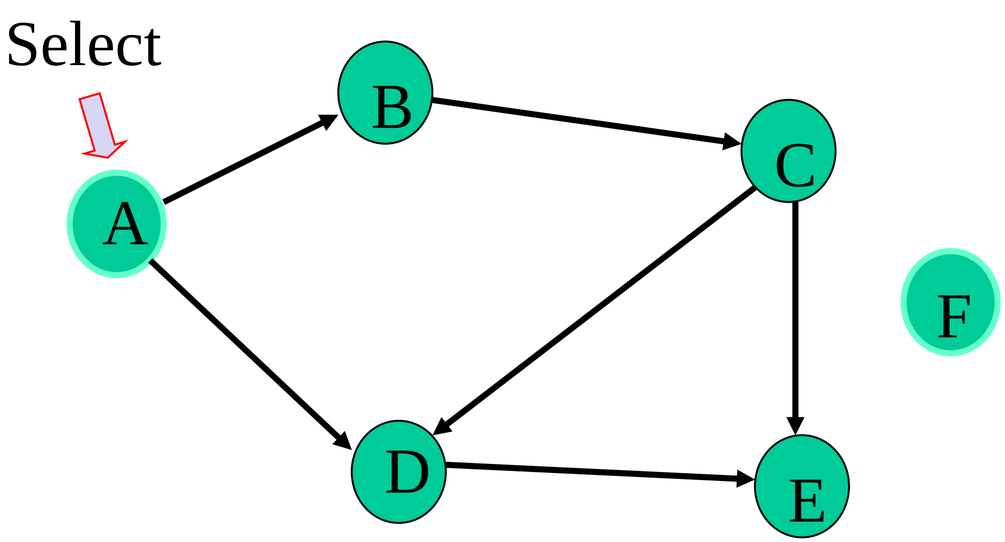
- If there are no such vertices, the graph consists of directed cycle(s).
- Topological sort is not possible – Halt.

Example of an 'only-cycles' graph



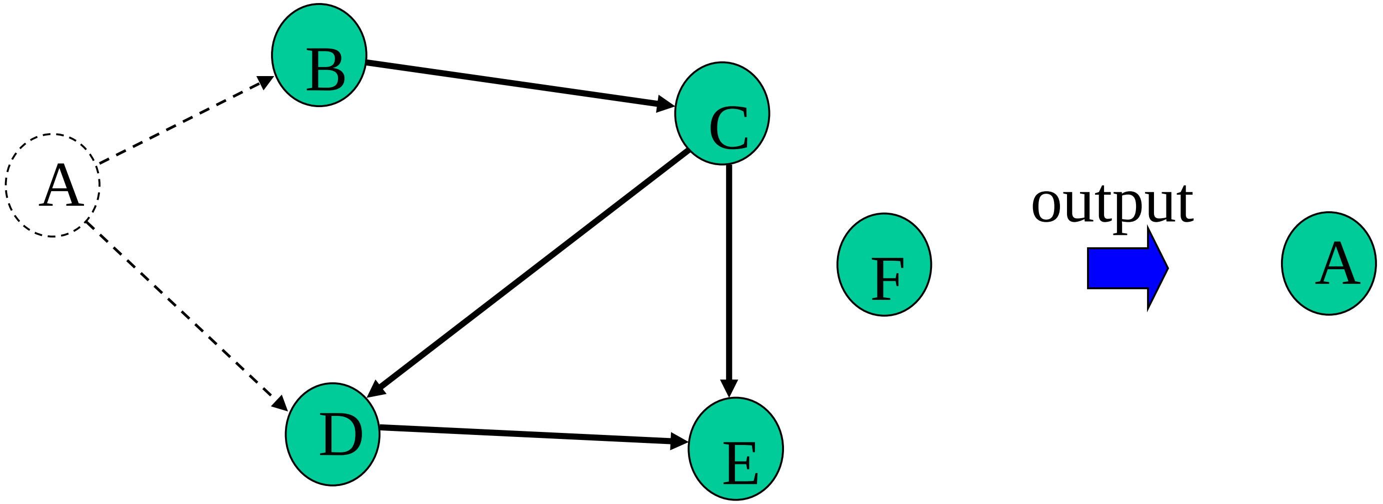
Topo sort algorithm

Step 1: Select one of the vertices that have no incoming edges



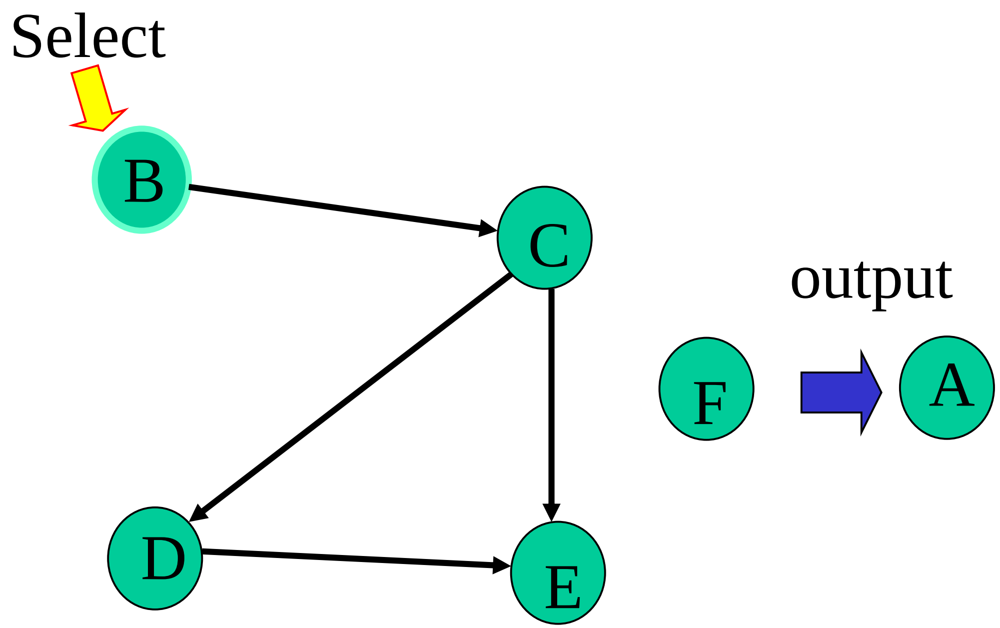
Topo sort algorithm

Step 2: Delete this vertex of in-degree 0 and all its outgoing edges from the graph. Place it in the output.



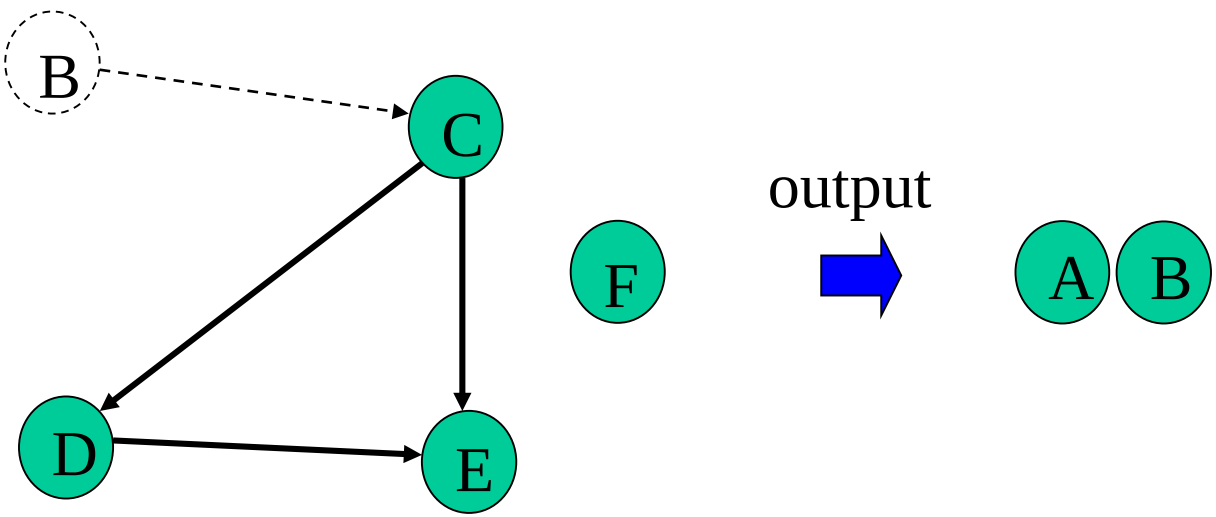
Continue until done

Repeat <u>Step 1</u> and <u>Step 2</u> until the graph is empty (or until HALT due to cycles-only').



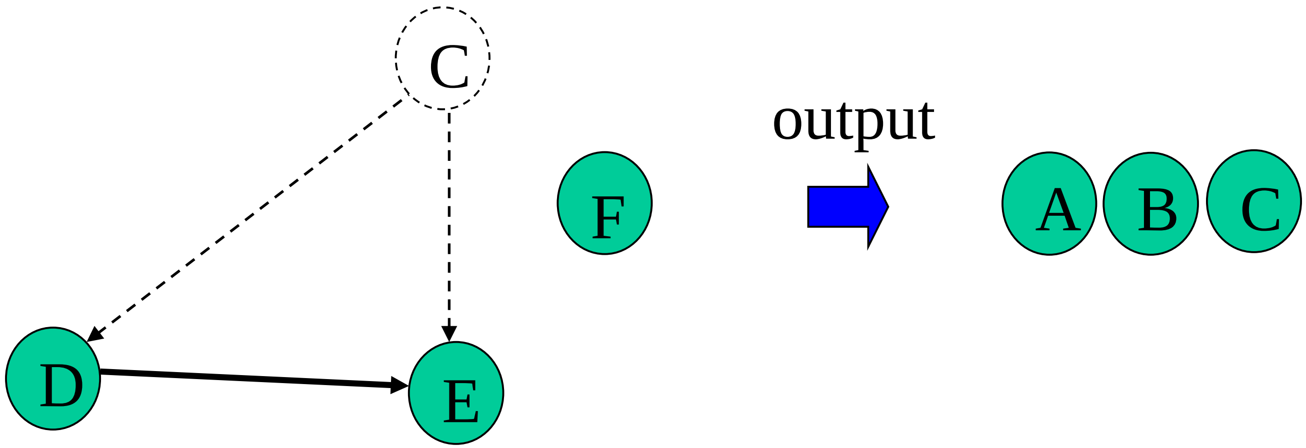
Example (cont') - B

Select B. Copy to sorted list.
Delete B and its edges.



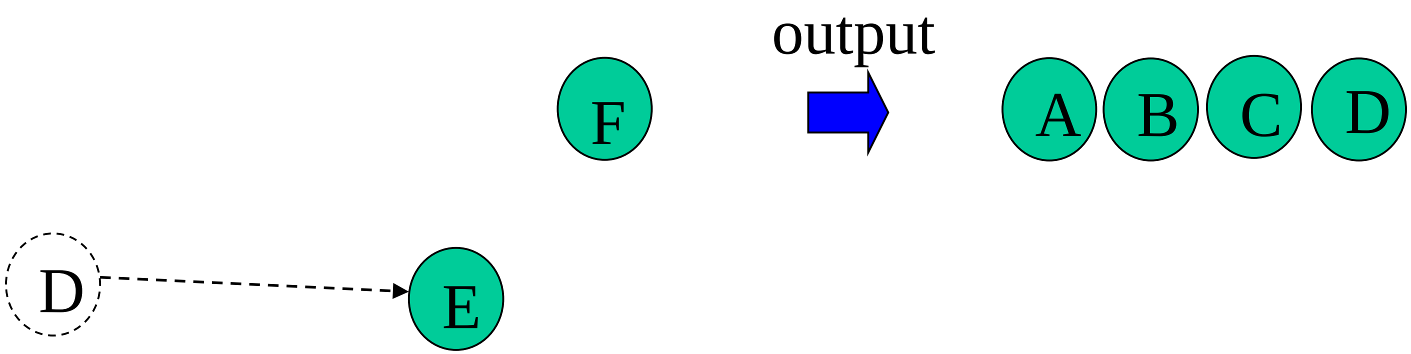
C

Select C. Copy to sorted list.
Delete C and its edges.



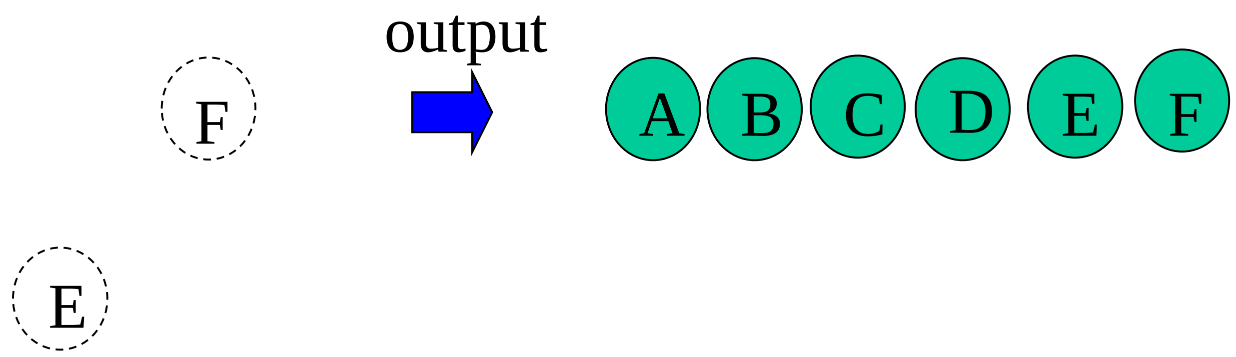
D

Select D. Copy to sorted list.
Delete D and its edges.



E, F

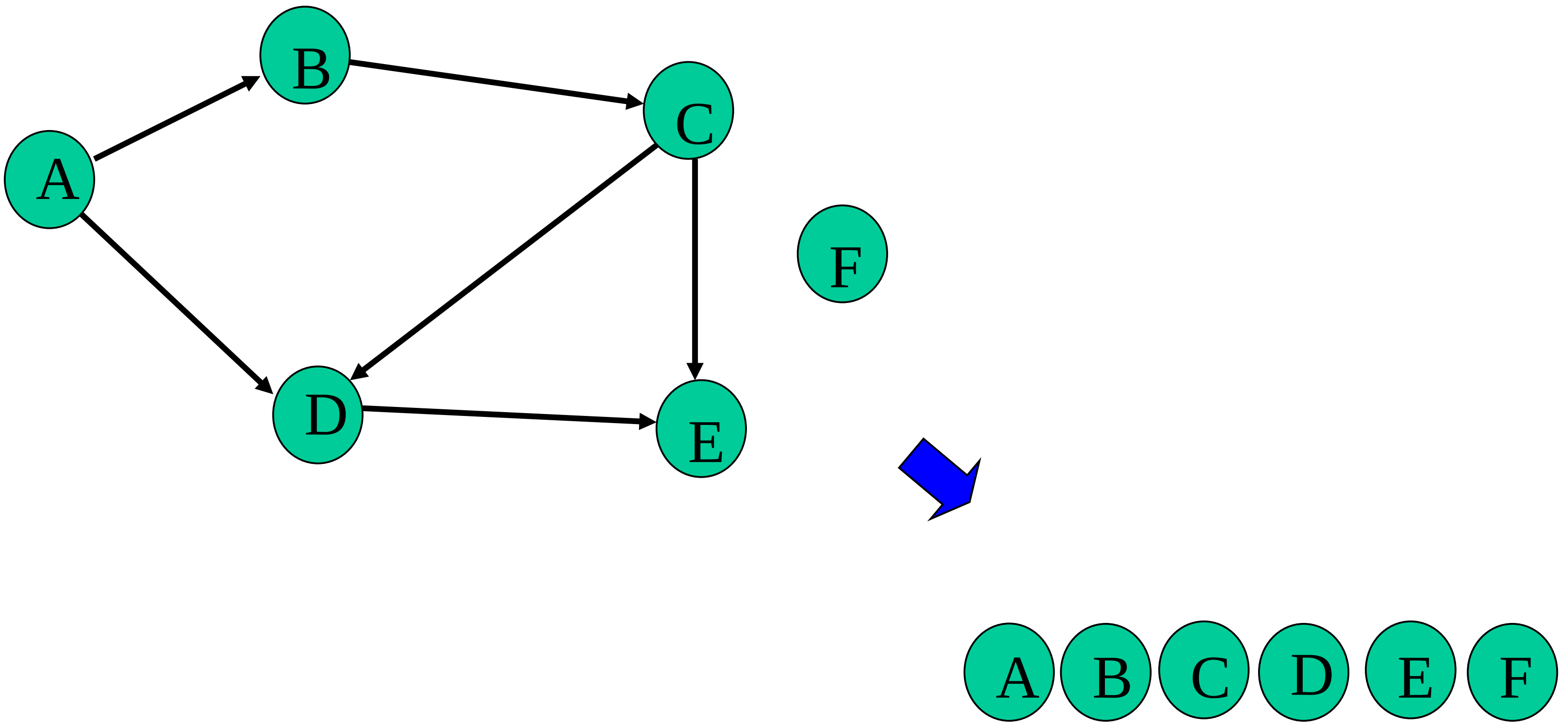
Select E. Copy to sorted list. Delete E and its edges.
Select F. Copy to sorted list. Delete F and its edges.



Yes, we could select F earlier (in any step).

The topological sort is not necessarily unique.

Done



Topological Sort Algorithm

1. Store each vertex's In-Degree in an array D
2. Initialize queue with all "in-degree=0" vertices
3. While there are vertices remaining in the queue:
   (a) Dequeue and output a vertex
   (b) Reduce In-Degree of all vertices adjacent to it by 1
   (c) Enqueue any of these vertices whose In-Degree became zero.
4. If all vertices are output then success, otherwise there is a cycle.

Time complexity: linear in $|V|+|E|$.

Back to Scheduling with precedence constraints.

A single machine:

Any topological sort is optimal for $1|prec|C_{max}$
Simply process the job according to the toposort order. $C_{max} = \sum_j p_j$ which is clearly optimal.

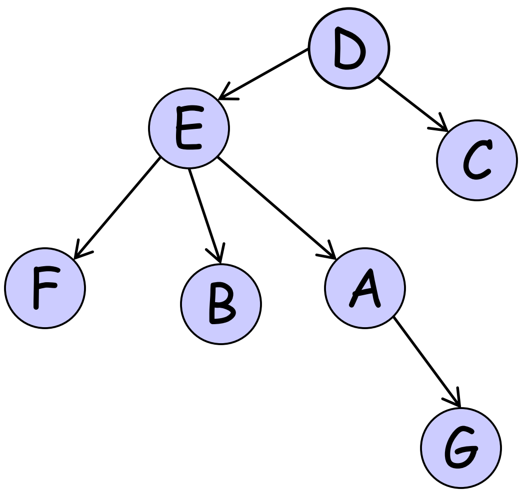
Parallel machines:

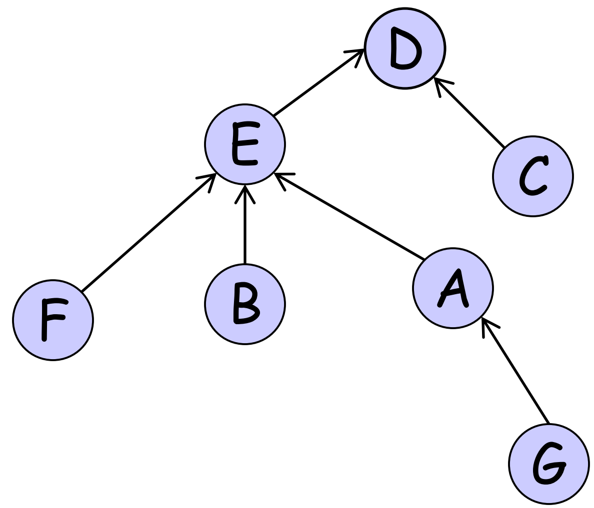
Even relaxed classes of $P|prec|C_{max}$ are known to be NP-complete.
Even with unit-length jobs, or very structured graph (collection of paths).

We are going to see optimal algorithm for two limited classes and a general approximation algorithm.

$P|tree, p_j=1|C_{max}$

The precedence constraints graph is a tree. There are two special cases:





Out-tree: each job has at most one predecessor (in-degree at most 1).

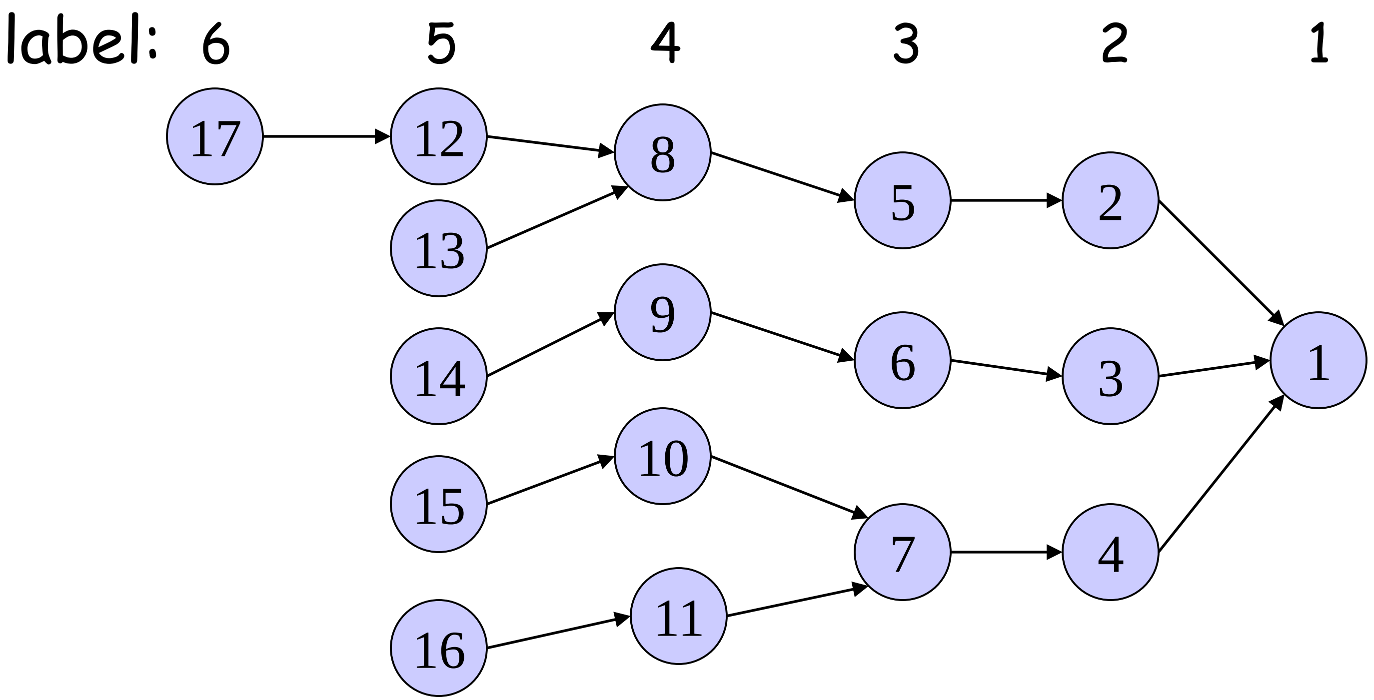
In-tree: each job has at most one consecutive (out-degree at most 1).

P|in-tree, p_j=1|C_{max}

Hu's Algorithm for in-tree.

Phase 1: Labeling

1. Label with 1 each 'sink job' (with out-degree=0)
2. For every labeled job j, find the jobs that immediately precede j and label each of them with label(j)+1.

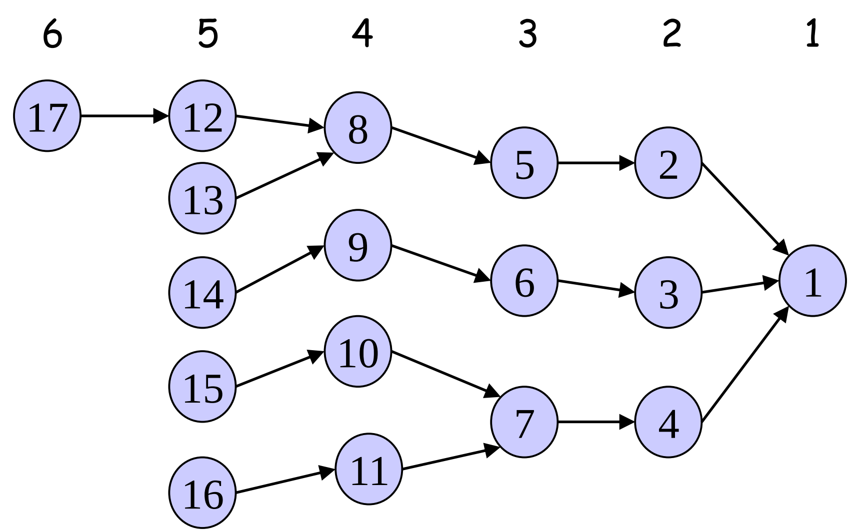


$P|in\text{-}tree, p_j=1|C_{max}$

Hu's Algorithm for in-tree.

Phase 2: scheduling

1. If the number of source-jobs is at most m, schedule them and leave the non-used machines idle.
2. Otherwise, schedule the m source-jobs with the largest labels.
3. Remove the scheduled jobs from the instance and return to step 1.



Example: m=3.

M_3	17
M_2	15
M_1	16

$P|in\text{-}tree, p_j=1|C_{max}$

Hu's Algorithm for in-tree.

Phase 2: scheduling

1. If the number of source-jobs is at most m, schedule them and leave the non-used machines idle.
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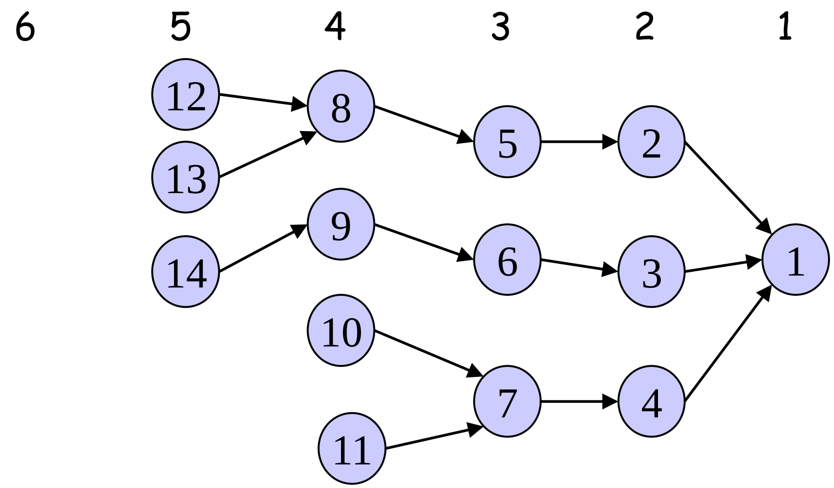

Example: m=3.

M_3	17	12
M_2	15	13
M_1	16	14

$P|\text{in-tree}, p_j=1|C_{max}$

Hu's Algorithm for in-tree.

Phase 2: scheduling

1. If the number of source-jobs is at most m, schedule them and leave the non-used machines idle.
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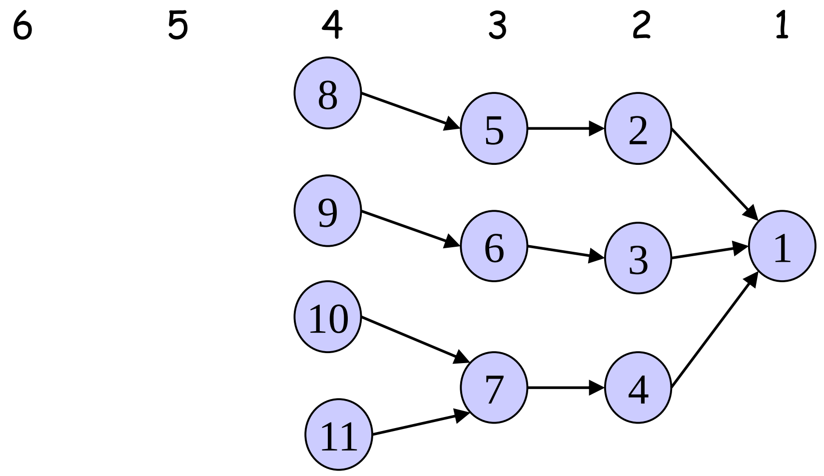


Example: m=3.

M_3	17	12	8
M_2	15	13	9
M_1	16	14	10

$P|\text{in-tree}, p_j=1|C_{max}$

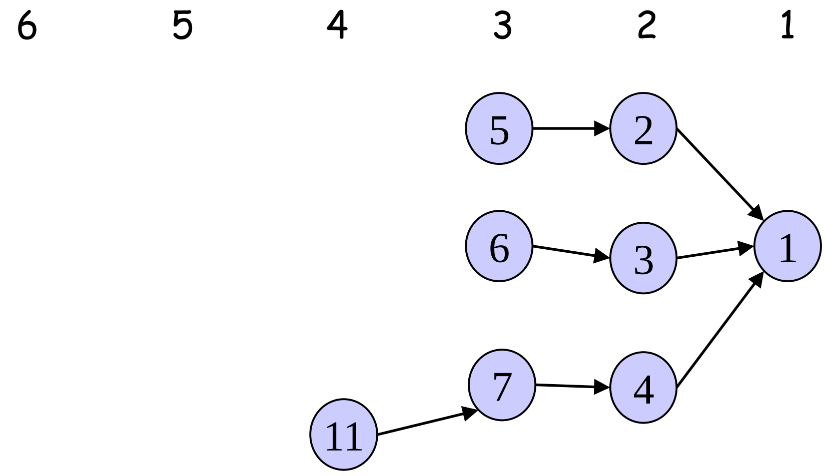
Hu's Algorithm for in-tree.

Phase 2: scheduling

1. If the number of source-jobs is at most m, schedule them and leave the non-used machines idle.
2. Otherwise, schedule the m source-jobs with the largest labels.
3. Remove the scheduled jobs from the instance and return to step 1.

Example: m=3.

M_3	17	12	8	5
M_2	15	13	9	6
M_1	16	14	10	11



P|in-tree, p_j=1|C_{max}

Hu's Algorithm for in-tree.

Phase 2: scheduling

1. If the number of source-jobs is at most m, schedule them and leave the non-used machines idle.
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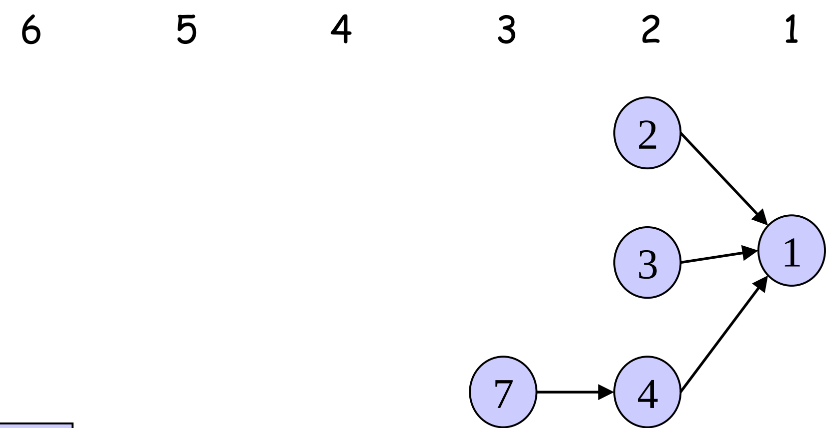


Example: m=3.

M_3	17	12	8	5	2		
M_2	15	13	9	6	3		
M_1	16	14	10	11	7	4	1

P|out-tree, $p_j=1|C_{max}$

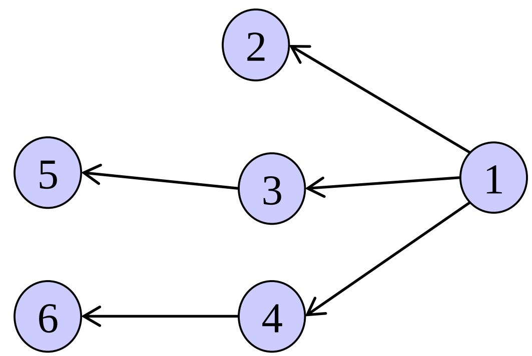
Phase 1: labeling

Label(j)=number of jobs waiting for j.

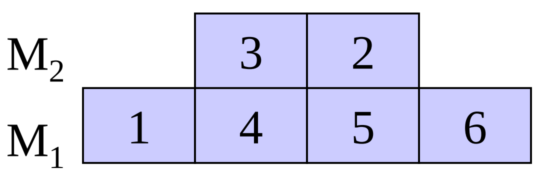
Phase 2: scheduling

same as in in-tree.

j	1	2	3	4	5	6
Label(j)	5	0	1	1	0	0



Example: m=2.



More on $P|p_j=1|C_{max}$

Scheduling forests: A forest consisting of in-trees can be scheduled by adding a dummy task that is an immediate successor of only the roots of in-trees, and then by applying Hu's Algorithm.

Scheduling out-forests: A schedule for an out-tree can be constructed by changing the orientation of edges, applying Hu's Algorithm to the obtained intree and then reading the schedule backwards, i.e. from right to left.

Remark: The problem of scheduling opposing forests (that is, combinations of in-trees and out-trees) on an arbitrary number of processors is NP-hard.

The Problem $P|prec|C_{max}$

An NP-hard problem ($P||C_{max}$ is already NP-hard)

List scheduling (LS) algorithm: Schedule the jobs greedily according to some topological sort.

LS has unexpected behavior: the schedule length may increase if:

– the number of machines increases,

– jobs' processing times decrease,

– precedence constraints are weakened, or

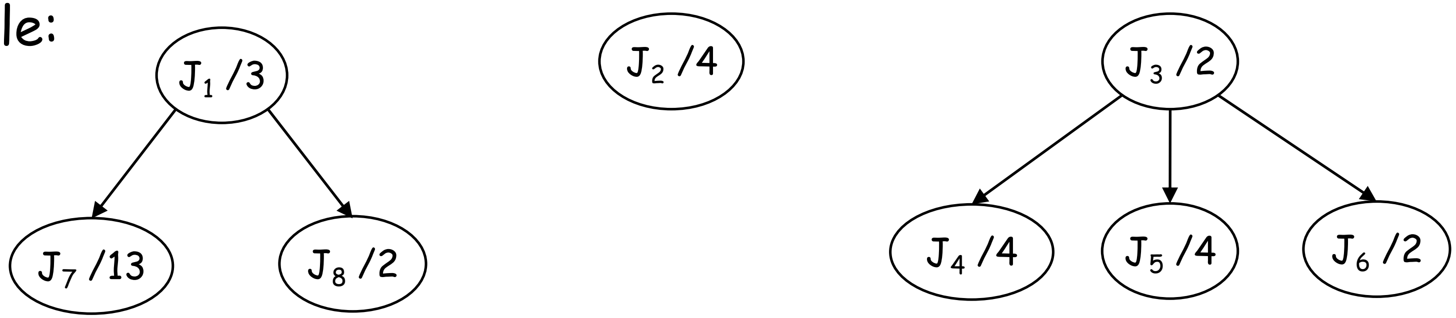
– the topological sort changes.

Whenever a machine becomes idle, schedule on it the first (in the toposort) feasible job.

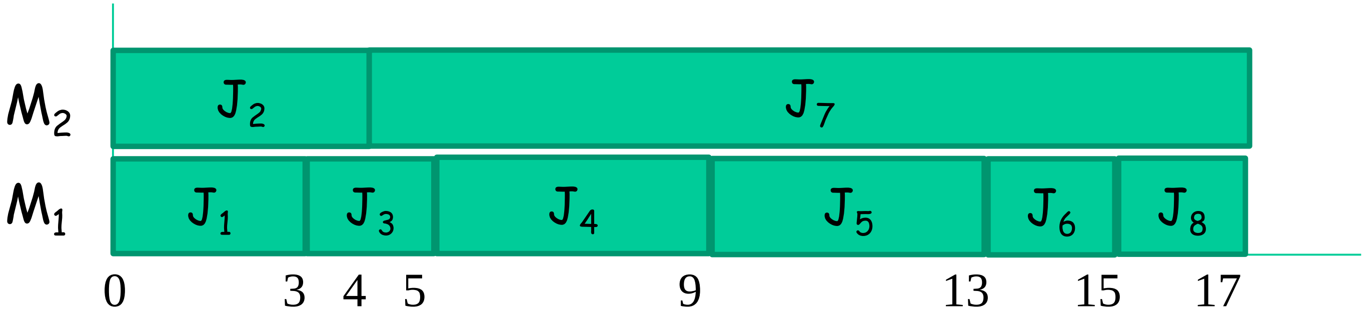
The Problem $P|prec|C_{max}$

J_j / p_j

Example:



$m=2, \; L=\{J_1,J_2,J_3,J_4,J_5,J_6,J_7,J_8\}$



$C_{max} = 17$

The Problem $P|prec|C_{max}$

Given: Job set with

- vector of processing times p
- precedence constraints θ
- job's topological sort list L
- m identical processors

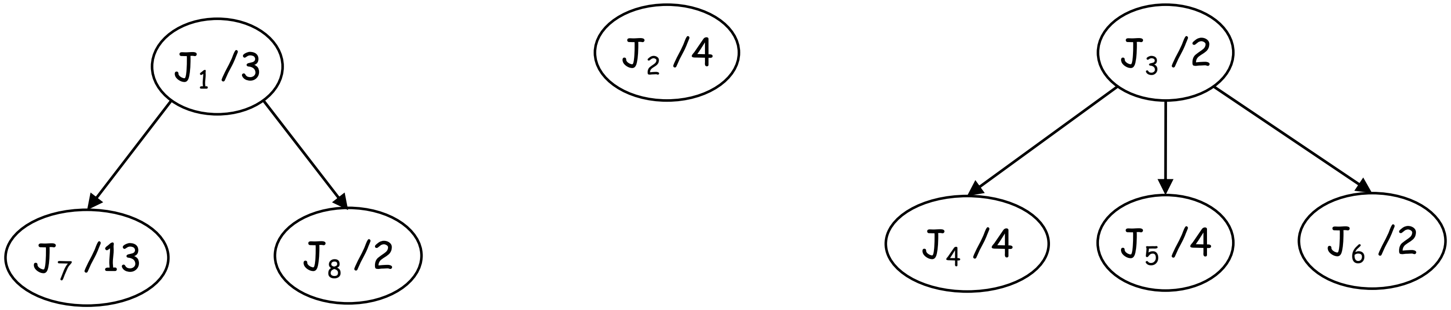
Let C_{max} be the length of the list schedule.

On the other hand, let the above parameters be changed:

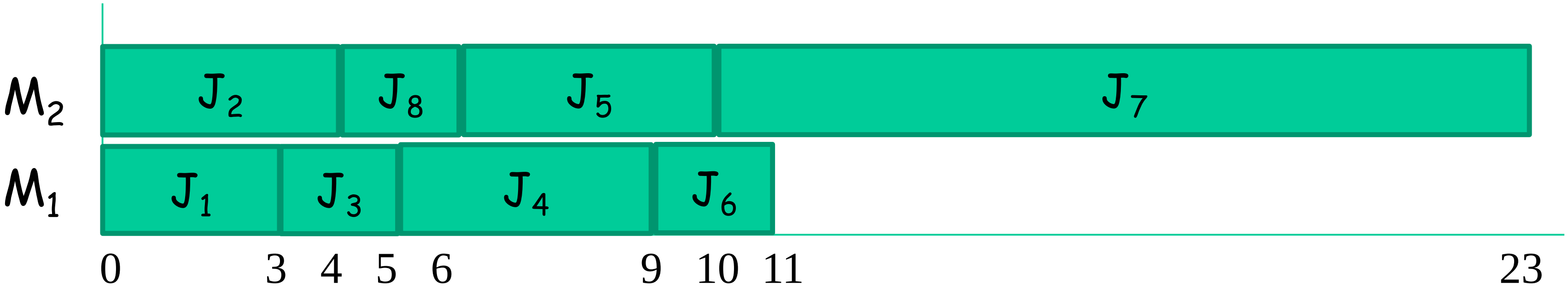
- vector of processing times $p' \leq p$ (component-wise),
- relaxed precedence constraints $\theta' \subseteq \theta$,
- list L'
- and another number of processors m'

Let the new value of schedule length be C'_{max}.

The Problem $P|prec|C_{max}$

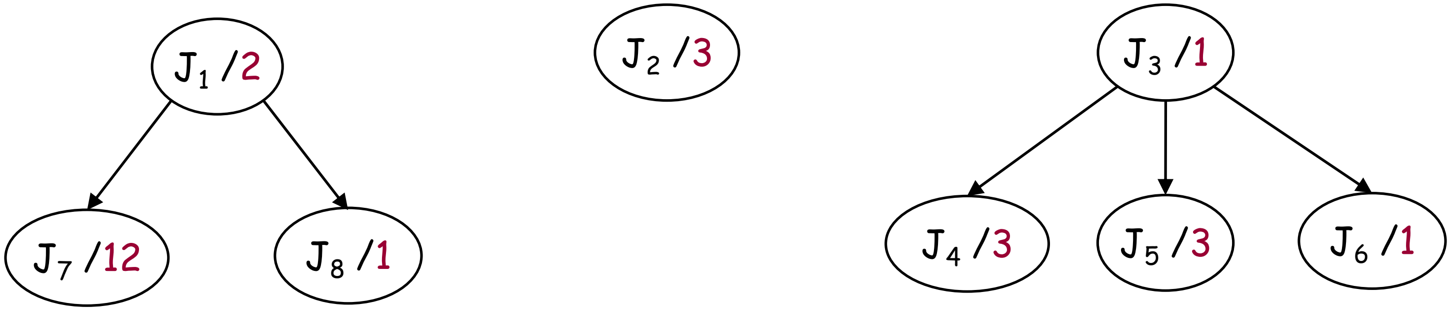


m=2, a new list: $L'=\{J_1,J_2,J_3,J_4,J_5,J_6,J_8,J_7\}$



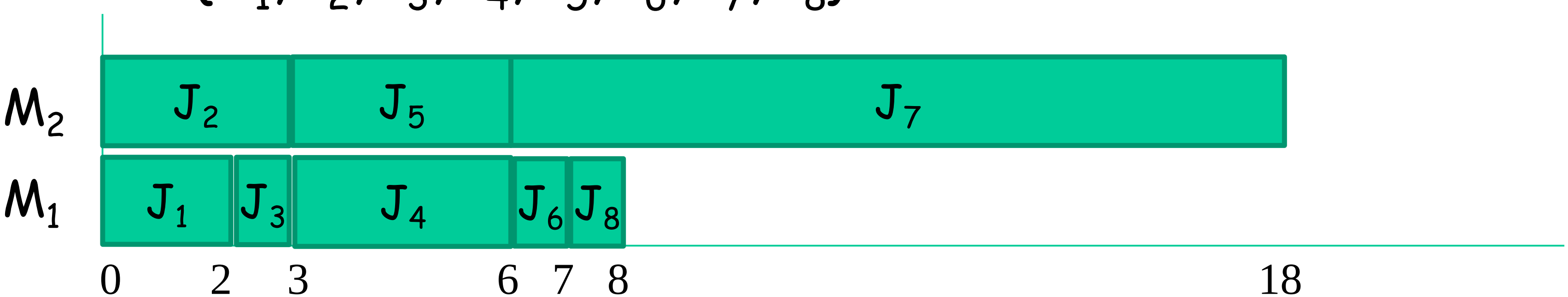
$C_{max} = 23$

The Problem $P|prec|C_{max}$



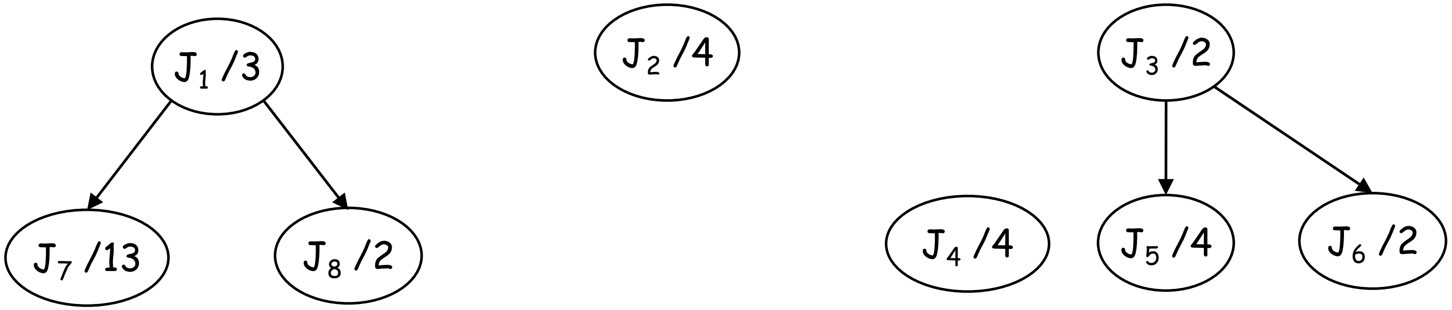
m=2, all processing times decrease by 1.

$L=\{J_1,J_2,J_3,J_4,J_5,J_6,J_7,J_8\}$



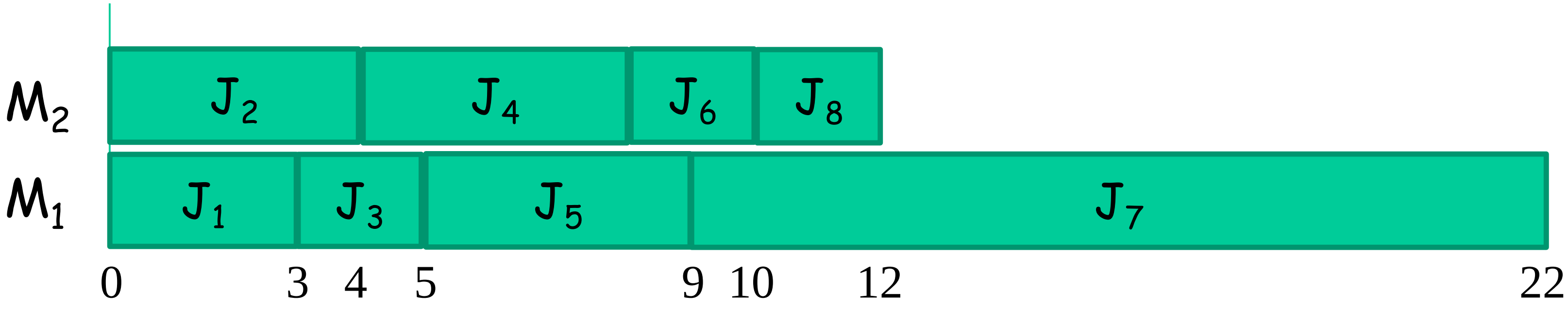
$C_{max} = 18$

The Problem $P|prec|C_{max}$



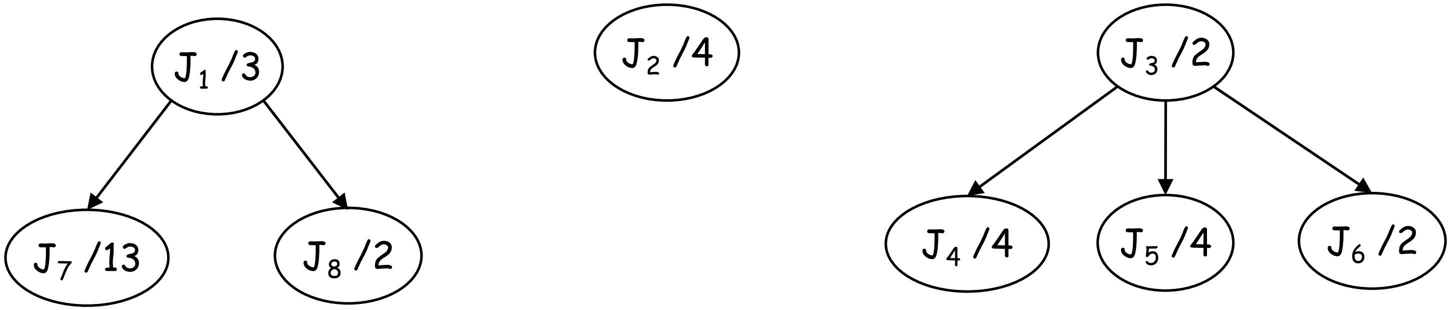
Precedence constraints weakened.

$m=2$, $L=\{J_1,J_2,J_3,J_4,J_5,J_6,J_7,J_8\}$

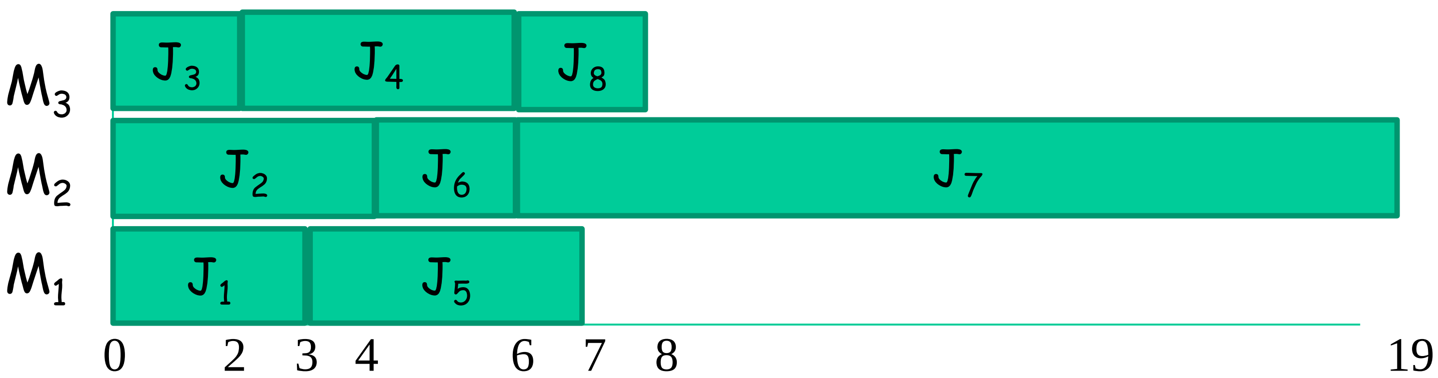


$C_{max} = 22$

The Problem $P|prec|C_{max}$



A machine is added: $m = 3$, $L=\{J_1,J_2,J_3,J_4,J_5,J_6,J_7,J_8\}$



$C_{max} = 19$

The Problem $P|prec|C_{max}$

The 'paradox' is limited:

Theorem (Graham 66): For every order L', every relaxed precedence constraints $\theta' \subseteq \theta$, every vector of processing times $p' \le p$ and new number of machines m', it holds that

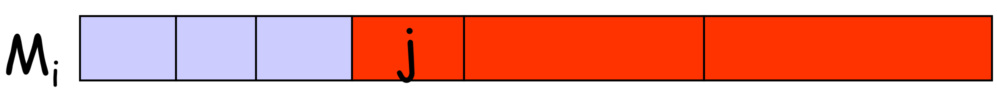
$$\frac{C'_{max}}{C_{max}} \le 1 + \frac{m-1}{m'}$$

Note: For $m=m'$ we get as a special case the $2 - \frac{1}{m}$ bound of list-scheduling with no precedence constraints.

The Problem $R||\sum_j C_j$

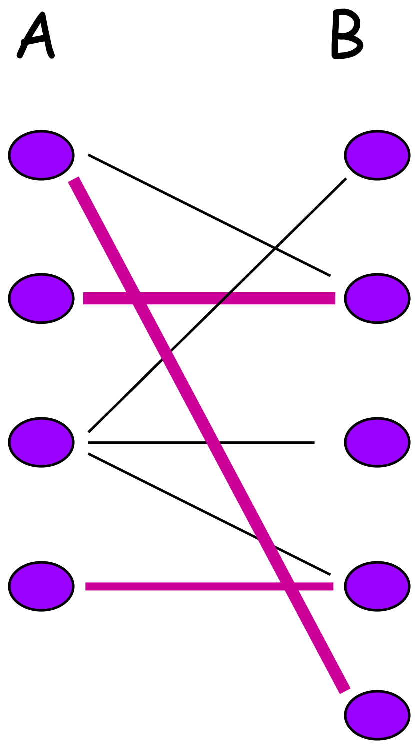
- Unrelated machines: for each machine i and job j the time p_{ij} is specified.
- For a single machine $1||\sum_j C_j$ is solvable by SPT rule.
- The problems $P||\sum_j C_j$ and $Q||\sum_j C_j$ are solvable by variants of SPT.
- For related machines, the processing times cannot be sorted.

Observation: If J_j is is the k^{th} from last job to run on a machine, it contributes exactly k times its processing time to the sum of completion times.



k=3, p_{ij} is counted 3 times in $\sum_j C_j$

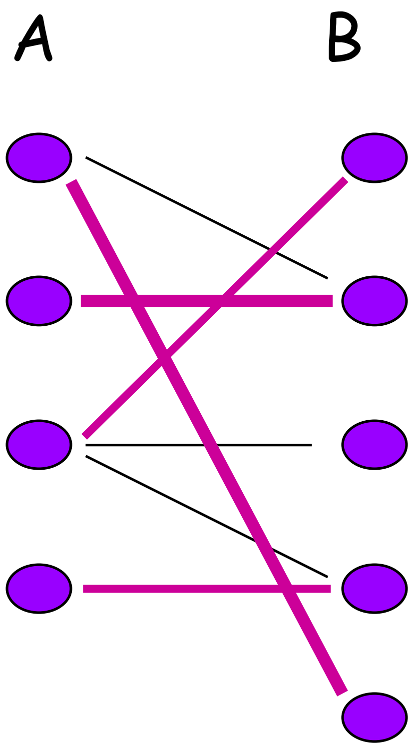
Tool: Bipartite matching



Given a bipartite graph on two sets of vertices A and B, and an edge set $E \subseteq A \times B$, a matching M is a subset of the edges, such that each vertex in A and B is an endpoint of at most one edge of M.

The optimal algorithm for $R||\ \Sigma_j C_j$ will match jobs to locations in the schedule.

Tool: Bipartite Matching



A perfect matching: $|M|=|A| \leq |B|$
(w.l.o.g $|A| \leq |B|$)

- It is possible to assign weights to the edges, and define the weight of a matching to be the sum of the weights of the matching edges.

When a perfect matching exists, it is possible to compute in polynomial time a minimum weight perfect matching.

The Problem $R||\Sigma_j C_j$

Define a bipartite with $V=A\cup B$ as follows:

A represents the set of n jobs (a single node per job).

B consists of nm nodes, w_{ik} , where vertex w_{ik} represents the k^{th} from last position on machine i, for $i=1,\ldots,m$ and $k=1,\ldots n$.

The edges set E contains an edge (v_j, w_{ik}) for every node in A and every node in B (a complete bipartite).

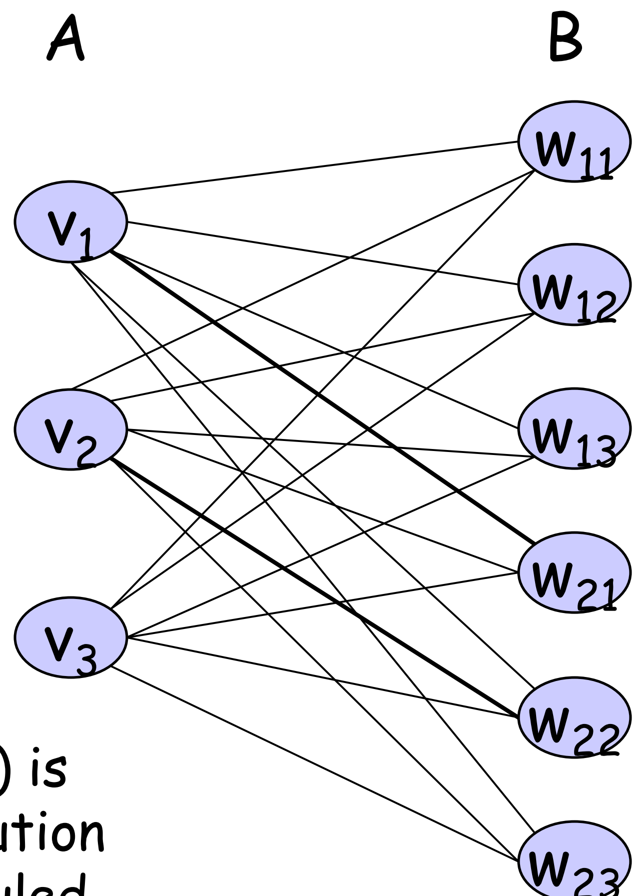
The weight of (v_j, w_{ik}) is kp_{ij}

The Problem $R||\Sigma_j C_j$

Example:

n=3, m=2

p_{ij}	J_1	J_2	J_3
M_1	4	5	7
M_2	8	6	2



The weight of (v_1,w_{21}) is 8 – the contribution of J_1 to $\Sigma_j C_j$ if scheduled as last on M_2.

← last position on M_2

← before-last position on M_2

The weight of (v_2,w_{22}) is 6*2=12 – the contribution of J_2 to $\Sigma_j C_j$ if scheduled as 2nd from last on M_2.

The Problem $R||\,\Sigma_j C_j$

Theorem: a minimum weight perfect matching corresponds to an optimal schedule.

Proof: In class

Note: With identical machines we get (as expected) SPT.

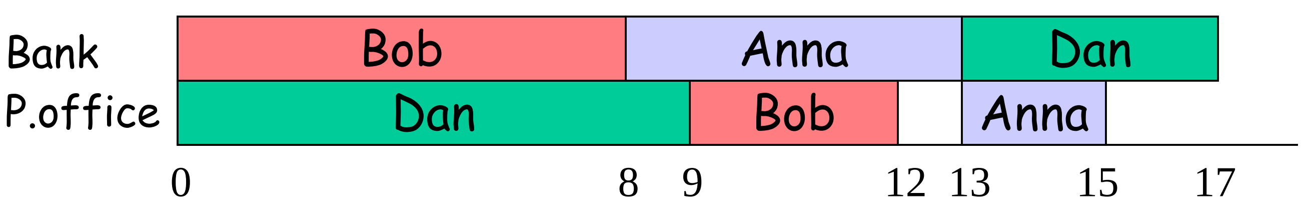
Open-shop Scheduling

- Job J_j consists of m operations, to be processed non-preemptively on each of the m machines.
- The order in which the different operations are performed is not important.
- Two operations of the same jobs cannot be processed simultaneously.
- Example:

  $m=2$

  $n=3$

	Bob	Dan	Anna
Bank	8	4	5
Post-office	3	9	2



The Problem $O||C_{max}$

- The problem $O||C_{max}$ is NP-hard for any $m>2$.
- We will see:
  1. A 2-approximation for any number of machines
  2. A hardness proof for $m>2$.
  3. An optimal algorithm for $O_2||C_{max}$
- Two basic lower bounds:
  - P_{max} – maximum total processing time of a job (total length of operations composing this job)
  - L_{max} – maximum total processing time required from a single machine
- Clearly: $C_{max}(opt)$ is at least $\max(P_{max}, L_{max})$

	Bob	Dan	Anna
Bank	8	4	5
P.office	3	9	2

P_{max} = 13 (Dan)

L_{max} = 17 (Bank)

The Problem $O||C_{max}$

A busy schedule for $O||C_{max}$: Whenever possible, schedule some operation of some job on a machine.

Can be implemented by a simple greedy algorithm.

Theorem: Any busy schedule is 2-approximation.

Proof: Consider the machine M' that finishes last. Let j' be the last job on M'. At any time during the schedule, either M' is processing a job, or j' is being processed by some other machine (why?).

The total time in which j' is processed is at most P_{max}

During the remaining time, M' is busy. But M' is busy at most L_{max} time units.

Therefore $C_{max} = C_j' \leq P_{max} + L_{max} \leq 2\ C_{max}(Opt)$

Open-shop Scheduling

- Theorem: For three or more machines, $O||C_{max}$ is NP-hard
- Proof: In class.

- We will see an optimal algorithm for $O_2||C_{max}$
- Denote $p_{j1} = a_j$, $p_{j2} = b_j$ (Time to process j on M_1 and M_2 respectively)
- Let $T_1 = \Sigma_j a_j$, $T_2 = \Sigma_j b_j$
- A lower bound for C_{max} is $\max(T_1, T_2, \max_j(a_j + b_j))$
- The algorithm achieves this lower bound.

Optimal algorithm for $O_2||C_{max}$

Let $A=\{j|a_j \geq b_j\}$ [need more time on M_1]

Let $B=\{j|a_j < b_j\}$ [need more time on M_2]

Select a job $J_r \in A$ such that $a_r \geq \max\{b_j| j \in A\}$.

Select a job $J_l \in B$ such that $b_l \geq \max\{a_j| j \in B\}$.

Let $A'=A-\{r\}$, $B'=B-\{l\}$

Example:

	1	2	3	4	5	6	T_j
a_j	10	7	3	1	12	6	39
b_j	6	9	8	2	7	6	38

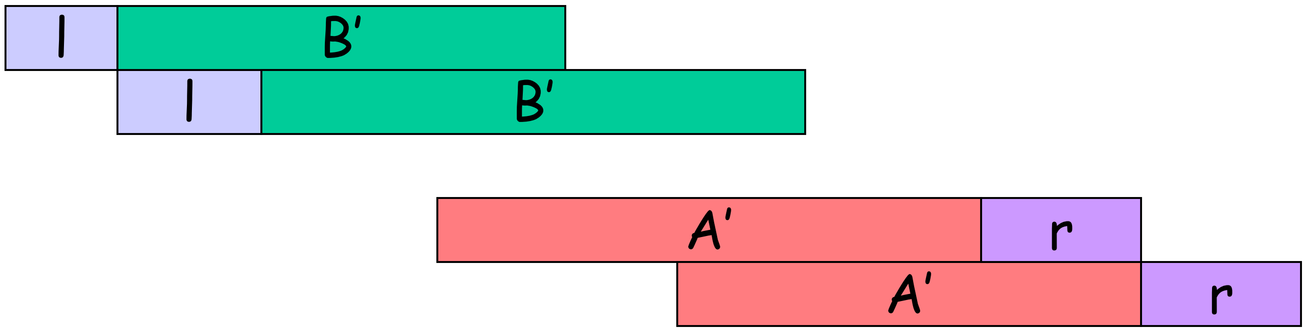
$A=\{1,5,6\}$, $B=\{2,3,4\}$

$a_r \geq \max\{b_j| j \in A\}$ is true for $r=1,5$. Select $r=1$

$b_l \geq \max\{a_j| j \in B\}$ is true for $l=2,3$. Select $l=2$

Optimal algorithm for $O_2||C_{max}$

Possible schedules for $A' \cup \{r\}$ and $B' \cup \{l\}$:

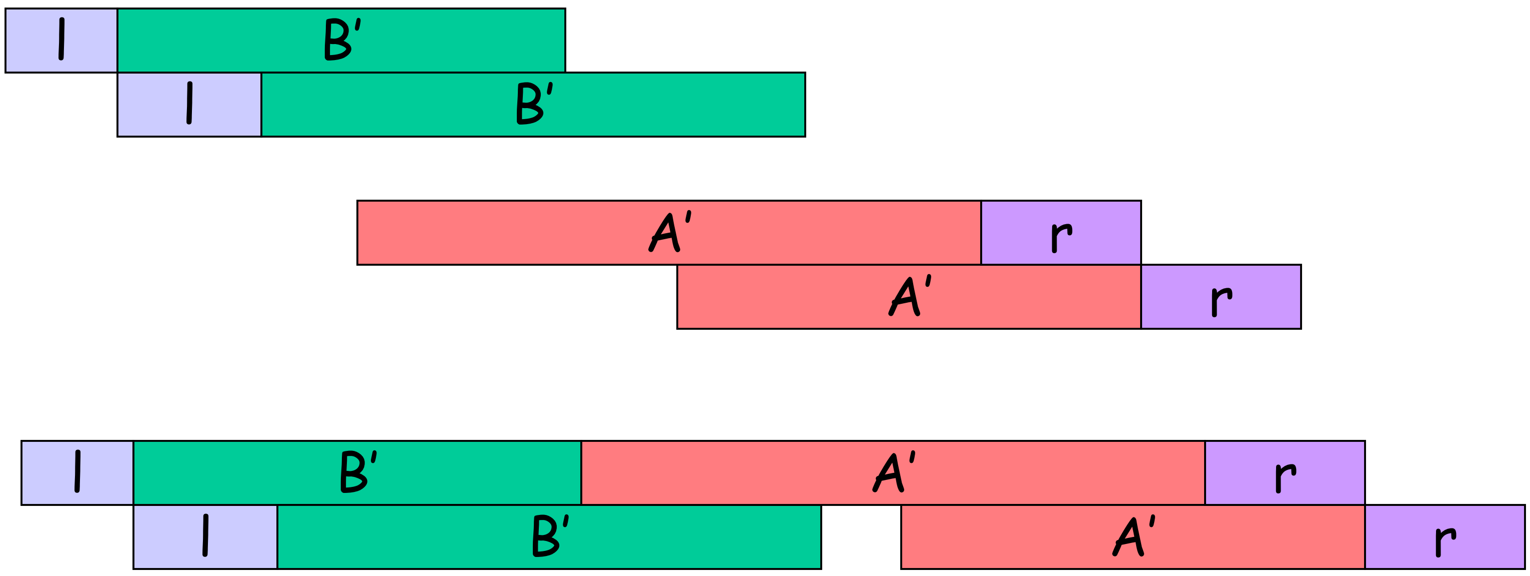


The jobs in A' and B' are in a fixed, arbitrary, order.

Note: There is no idle on any of the machines. (why?)

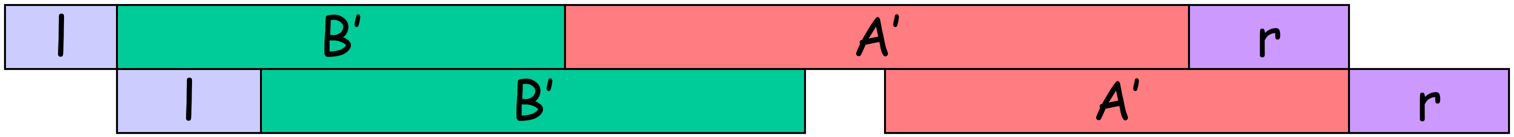
Optimal algorithm for $O_2||C_{max}$

Let us 'glue' the two schedules.

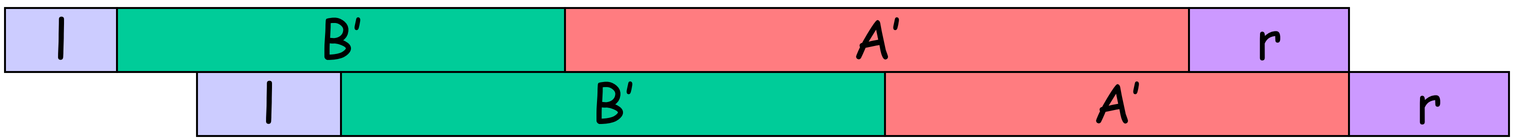


Will meet on M_1 if $T_1 - a_l > T_2 - b_r$

Optimal algorithm for $O_2||C_{max}$

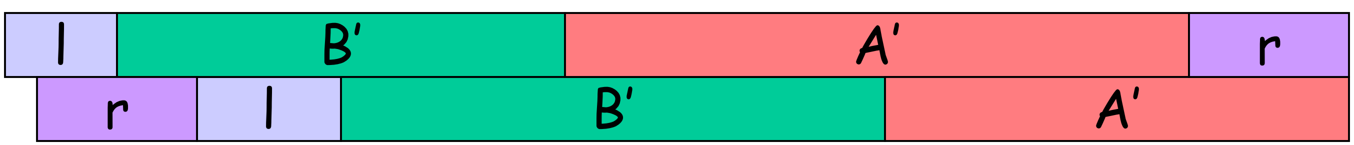


'Slide' the jobs B' and J_l on M_2 to the right.



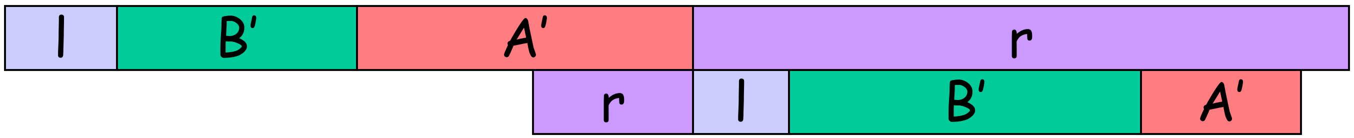
A valid schedule (why?)

Move J_r to be first on M_2



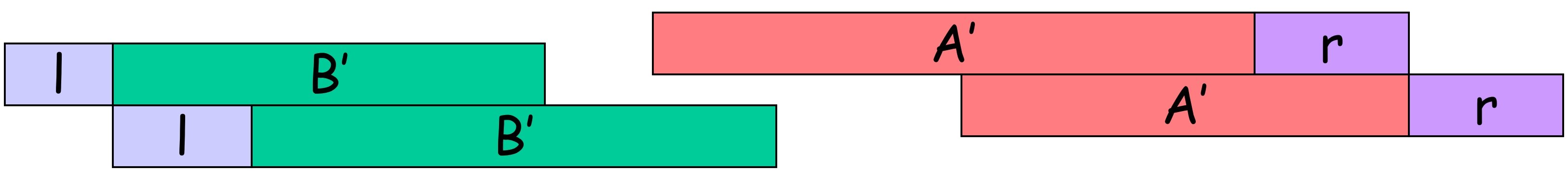
If $a_r \leq T_2 - b_r$ The schedule length is $\max(T_1, T_2)$

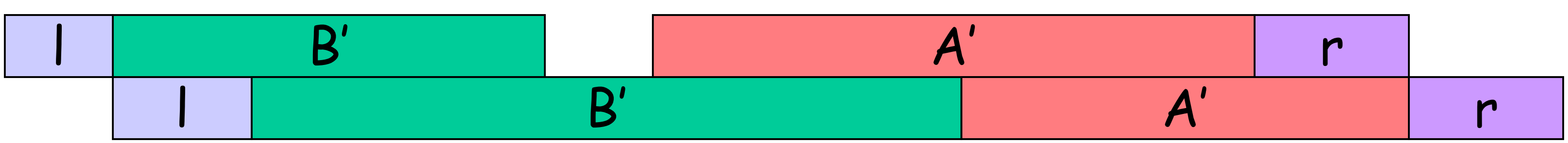
Optimal algorithm for $O_2||C_{max}$



If $a_r > T_2 - b_r$, the schedule length is $\max(T_1, a_r + b_r)$.

If $T_1 - a_l \le T_2 - b_r$, then when gluing, the schedules meet on M_2 and the analysis is symmetric.





Optimal algorithm for $O_2||C_{max}$

Let $A=\{j|a_j \geq b_j\}$ [need more time on M_1]

Let $B=\{j|a_j < b_j\}$ [need more time on M_2]

If $A=\emptyset$ or $B=\emptyset$ the problem is easy (why?)

Select a job $J_r \in A$ such that $a_r \geq \max \{b_j | j \in A\}$.

Select a job $J_l \in B$ such that $b_l \geq \max \{a_j | j \in B\}$.

Let $A'=A-\{r\}$, $B'=B-\{l\}$.

If $T_1-a_l > T_2-b_r$ then the optimal schedule is:

On M_1: (l,B',A',r), on M_2: (r,l,B',A').

Else, the optimal schedule is

On M_1: (B',A',r,l), on M_2: (l,B',A',r).

Optimal algorithm for $O_2||C_{max}$

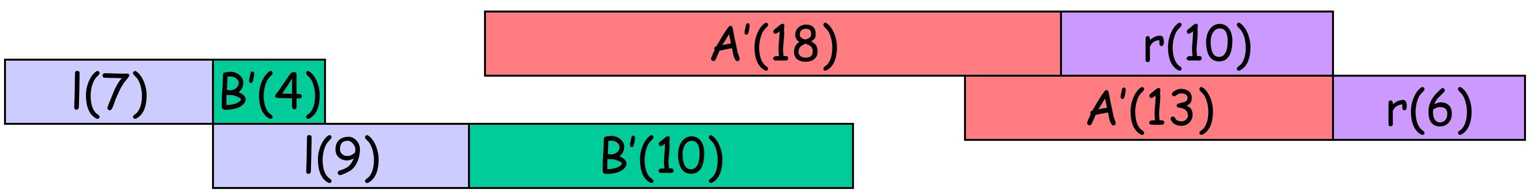
	1	2	3	4	5	6	T_j
1	10	7	3	1	12	6	39
2	6	9	8	2	7	6	38

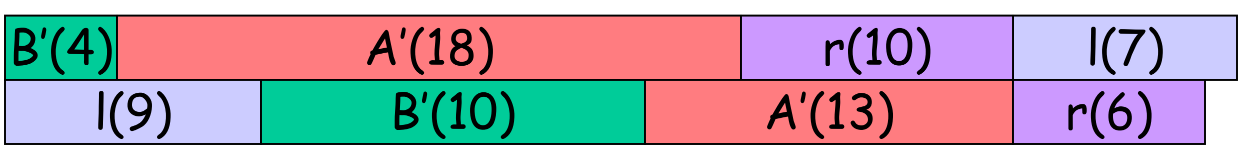
A={1,5,6}, B={2,3,4}

$a_r \geq \max\{b_j | j \in A\}$. Select r=1

$b_l \geq \max\{a_j | j \in B\}$. Select l=2

A'={5,6}, B'={3,4}





$C_{max} = T_1 = 39$

Flow-shop Scheduling

In a flow-shop schedule with m machines, $M_1, M_2, \ldots, M_m$, all the jobs must be processed by all the machines in the same order (which is, w.l.o.g., $M_1, M_2, \ldots, M_m$). For each job j and machine i, $p_{j,i}$ is the processing time required by J_j on M_i.

Example:
Two machines, three jobs.

	pizza	pie	cake
chef	8	10	4
oven	5	20	30

Flow-shop Scheduling

- Theorem: The problem $Fm||C_{max}$ is NP-hard for any $m > 2$.
- We will see a simple optimal algorithm for $m=2$

Observations for F_2:

- In any F_2-schedule, the machine M_2 is first idle, then it processes jobs, then it may be idle again, process again, and so on, depending on the flow of jobs from M_1.
- M_1 is never idle (or idles can be removed).
- Since all jobs are available at time $t=0$, our goal is to reduce the time in which M_2 is idle, waiting for the job currently processed by M_1.

Flow-shop Scheduling on Two Machines

Definition: A permutation schedule is a schedule in which the jobs are processed in the same order by M_1 and M_2.

Lemma: There exists an optimal schedule which is a permutation schedule.

Proof idea: if J_j precedes J_k on M_1, then J_j is available to M_2 before J_k and so, if J_k precedes J_j on M_2 we can swap their processing on M_2 without hurting the makespan.

$J_k\ J_j \rightarrow \boxed{M_1} \rightarrow J_k\ J_j \rightarrow \boxed{M_2}$

Flow-shop Scheduling on Two Machines

Denote $p_{j1} = a_j$, $p_{j2} = b_j$.

Let A be the set of jobs j for which $a_j \leq b_j$.

Let B be the set of jobs j for which $a_j > b_j$.

Johnson Rule: Sort the jobs in the following way: first the jobs of A in non-decreasing order of a_j, then the jobs of B in non-increasing order of b_j. Schedule the jobs on the two machines according to this order.

Example:

A = {pie, cake}

B = {pizza}

	pizza	pie	cake
chef	8	10	4
oven	5	20	30

Optimal order = {cake, pie, pizza}

Optimality of Johnson Rule for $F2||C_{max}$

- For a given permutation schedule, number the jobs according to the order they are scheduled.
- Let J_k be the first job on M_2 after its last idle section. J_k is not waiting between M_1 and M_2.
- $C_k = a_1+a_2+\ldots+a_k+b_k$.
- M_2 is not idle when the rest of the jobs are processed, thus, $C_{max}= a_1+\ldots+a_k+b_k+b_{(k+1)}+\ldots+b_n$.

➔ The makespan is determined by $n+1$ values.

➔ For any c, we can reduce c from all the p_{ij} values, without changing the relative performance of different permutation schedules.

Optimality of Johnson Rule for $F2||C_{max}$

Theorem: Johnson rule is optimal for $F_2||C_{max}$.

Proof: By induction on the number of jobs, n.

Base: For $n=1$, any schedule with no idle is optimal.

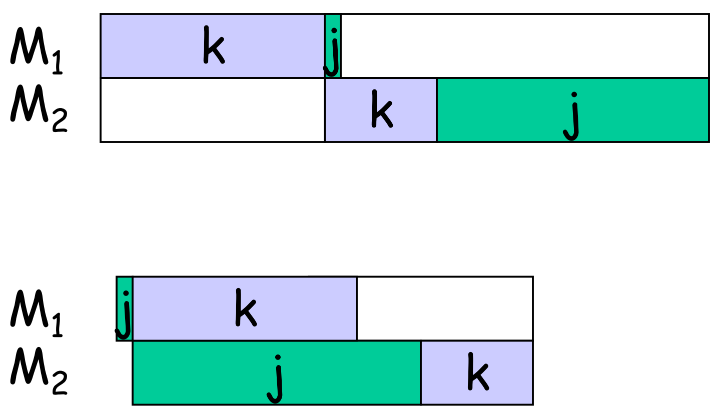
Step: Assume that Johnson rule is optimal for $n-1$ jobs, and consider an instance with n jobs.

Let $c = \min_j\{\min\{a_j, b_j\}\}$. Reduce c from all p_{ji} values. As a result, there exists a job, with $a_j=0$ or $b_j=0$. If $a_j=0$ then $j \in A$ and it is first in the Johnson-order of A. If $b_j=0$ then $j \in B$ and it is last in the Johnson-order of B.

Optimality of Johnson Rule for $F2||C_{max}$

Lemma: If $a_j=0$, then there exist an optimal schedule in which j is first, If $b_j=0$, then there exists an optimal schedule in which j is last.

Proof: If $a_j=0$ and J_j is not first, assume it is processed after J_k. The jobs J_j and J_k can be swapped without hurting the makespan.



The proof for $b_j=0$ is similar

Optimality of Johnson Rule for $F2||C_{max}$

Back to induction step: Recall that J_j is a job with $a_j=0$ or $b_j=0$.

If $a_j=0$, then there exist an optimal schedule in which j is first (and can be processed by M_2 with no delay), and if $b_j=0$, then there exists an optimal schedule in which j is last (and do not cause any delay to the makespan of M_2).

By the induction hypothesis, Johnson rule is optimal for $J-\{j\}$. By the above, Johnson rule places j optimally.

■

Johnson rule.
Example:

	1	2	3	4	5	6
a_j	7	7	3	1	12	6
b_j	4	9	8	2	7	6

A = {2,3,4}, (more time on M_2)

B = {1,5,6}, (more time on M_1).

A sorted in non-decreasing order of a_j: {4,3,2}
B sorted in non-increasing order of b_j: {5,6,1}

Optimal flow-shop schedule according to Johnson rule: