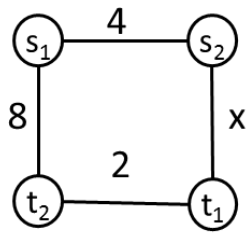
## Applied Algorithms. Prof. Tami Tamir
## Missing Proofs, lecture 8. July 10th.

**A Network Formation Game – Example.**




Consider the following undirected network.

For every value of $x \geq 0$ determine

1. The social optimum.
2. The routings that form a NE
3. The PoA and the PoS.

Solution: Every player has two strategies. The possible profiles and utilizations:

| Player 1\Player 2 | s2-t1-t2 | s2-s1-t2 |
|---|---|---|
| s1-s2-t1 | **R1** 4+x/2 , x/2+2 | **R2** 2+x ,10 |
| s1-t2-t1 | **R3** 9, x+1 | **R4** 6 , 8 |

1. The total costs are R1: x+6, R2:x+12, R3:x+10, R4: 14.
   Calculating the social optimum: The minimum total cost is either x+6 or 14.

For $x \leq 8$, SO=x+6. For $x>8$, SO=14.

2. Calculating stable profiles:

R1 is a NE if $4+x/2 \leq 9$ and $x/2+2 \leq 10$

$\underline{\mathbf{x \leq 10}}$ and $x \leq 16$.

R2 is a NE if $2+x \leq 6$ and $10 \leq x/2+2$

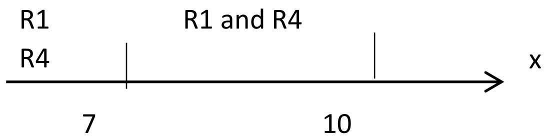
$x \leq 4$ and $x \geq 16$. **NEVER**

R3 is a NE if $9 \leq 4+x/2$ and $x+1 \leq 8$

$10 \leq x$ and $x \leq 7$. **NEVER**

R4 is a NE if $6 \leq 2+x$ and $8 \leq x+1$

$4 \leq x$ and $\underline{\mathbf{7 \leq x}}$.




3. Equilibrium inefficiency:

For $x<7$, R1 is the only one NE. PoA=PoS= (x+6)/(x+6)=1

For $x>10$, R4 is the only NE. PoA=PoS=14/14=1

For $7 \leq x \leq 8$ R1 and R4 are NE, SO=x+6

PoS=min{(x+6),14}/(x+6)=(x+6)/(x+6)=1.

PoA= max{(x+6),14}/(x+6)=14/(x+6).

For $8 \le x \le 10$    R1 and R4 are NE, SO=14

PoS=min{(x+6),14}/14=(x+6)/(x+6)=1.

PoA= max{(x+6),14)/14=x+6/14.

**Hardness proof of Min-cost NE profile -** Reduction to 3-D matching.

Claim: there exists a 3-dim matching of size n if and only if the NFG has a Nash equilibrium of cost 3n.

Proof: Let T' be a 3-dim matching of size n. For every element in $X \cup Y \cup Z$ there exists a triplet in T' to which it belongs. Consider the profile in which every element-note route to t through its corresponding triplet-node in T'. The total cost of this profile is 3|T'|=3n. Every player has cost 3/3=1. This is a NE profile, since a player can only deviate to paths in which his cost will be 3.

Let p be a NE profile of cost 3n. The social cost consists of n edges of cost 3. Therefore, exactly n triplet-nodes are included in the selected paths. Since every triplet-node can participate in the strategy of at most three players, we get that the profile induces a partition of the 3n players into n triplets, which form a 3-d matching.