Chapter 11

ODEs: Boundary Value Problems

Outline of Section

- Boundary Value Problems
- Shooting Method
- Finite differences
- Eigensystems

Consider the general, linear, second order equation

$$\alpha \frac{d^2y}{dx^2} + \beta \frac{dy}{dx} + \gamma\, y = k\,. \tag{11.1}$$

Normally we would require initial conditions specifying the value of the solution at some initial point $x = a$, i.e., $y(a)$ and its derivative $y'(a)$, to integrate the system to a final point.

But what if we want to find the solution $y(x)$ that matches an initial and final condition, e.g., $y(a)$ and $y(b)$? Since we are giving two conditions and we have two degrees of freedom (second order ODE) we should be able to find a consistent solution.

11.1 Shooting Method

A first method we can use is the **shooting method** where we start at a with $y(a) = A$, make a guess for $y'(a) = C_1$ and find a solution $y_1(x)$. This can be obtained via a finite difference method (from Chapter 9), or algebraically if we are lucky. In general the solution will not end at the desired point $y(b) = B$ say, but will end instead at $y_1(b) = B_1$.

We then make another guess starting from the same point $y(a) = A$ but with $y'(a) = C_2$ and find a new solution $y_2(x)$ which ends at $y_2(b) = B_2$. The system (11.1) is **linear**, so a sum of two solutions will itself be a solution of the system. In this case, we can introduce a weighted average of y_1 and y_2 as a new solution

$$\boxed{y_c(x) = c\, y_1(x) + (1-c)\, y_2(x),} \tag{11.2}$$

where c is an unknown constant which makes y_c the required solution which ends at $y(b) = B$.

We can check that

$$y_c(a) = c\, A + (1-c)\, A = A,$$

as required and the condition that

$$y_c(b) = c\, B_1 + (1-c)\, B_2 = B,$$

gives us a solution for c

$$\boxed{c = \frac{B - B_2}{B_1 - B_2}.} \tag{11.3}$$

So finding the required solution to a linear system is relatively straightforward using the shooting method. However it still requires solving the system twice to get a single solution.

In order to solve the more general problem, where the ODEs may in general constitute *non*-linear systems, we need to iterate. Specifically, we need to keep choosing new values of $y'(a)$ until we hit on a value that results in $y_2(b) = B$. This then becomes a root-finding problem: if B_1 and B_2 bracket B, then by the intermediate value theorem, at least one of the values of $y'(a)$ that solves the problem is bracketed by C_1 and C_2. 'All' that remains to do is then to refine the brackets as quickly as possible using your preferred root-finding method. This is of course much easier said than done when the underlying ODE is more complex, and e.g. multiple higher-order derivatives need to be chosen for each 'shot' to be fired from $x = a$.

Next we will look at another method which solves the system explicitly via matrix methods.

11.2 Finite Difference Method

Consider the system

$$\frac{d^2y}{dx^2} = k\,. \tag{11.4}$$

We can re-write the second derivative as a central difference and discretise the solution with m intervals of size h as in Chapter 8.1 (see also Problem Sheet)

$$\frac{y_{i-1} - 2\,y_i + y_{i+1}}{h^2} = k\,. \tag{11.5}$$

Then, since we have $y_0 = A$ and $y_m = B$, the system will look like

$$\begin{aligned} A - 2\,y_1 + y_2 &= k\,h^2, \\ y_1 - 2\,y_2 + y_3 &= k\,h^2, \\ &. \\ &. \\ &. \\ y_{m-2} - 2\,y_{m-1} + B &= k\,h^2, \end{aligned} \tag{11.6}$$

i.e. $m - 1$ linear equations. The system can be written in matrix form as

$$\begin{pmatrix} -2 & 1 & 0 & 0 & .... & 0 & 0 \\ 1 & -2 & 1 & 0 & .... & 0 & 0 \\ & & & . & & & \\ & & & . & & & \\ & & & . & & & \\ 0 & 0 & 0 & 0 & .... & 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_{m-1} \end{pmatrix} = \begin{pmatrix} k\,h^2 - A \\ k\,h^2 \\ . \\ . \\ . \\ k\,h^2 - B \end{pmatrix}, \tag{11.7}$$

i.e. $\mathbf{M} \cdot \tilde{\vec{y}} = \vec{b}$. The '$\sim$' is used to denote that the solution is approximate (compared to the true solution of eqn (11.4) at the sample points), because of the use of finite difference approximation to derivatives. The matrix $\mathbf{M}$ is close to being diagonally dominated (since it has $|m_{ii}| \geq \sum_{j \neq i} |m_{ij}|$) and is in fact suitable for solving using the Jacobi, Gauss-Seidel and SOR methods introduced in Chapters 3.5, 3.6 and 3.7. This is because the spectral radius of the associated update matrix (for each method) is less than unity. Figure 11.1 illustrates the finite difference method applied to solving a boundary value problem.

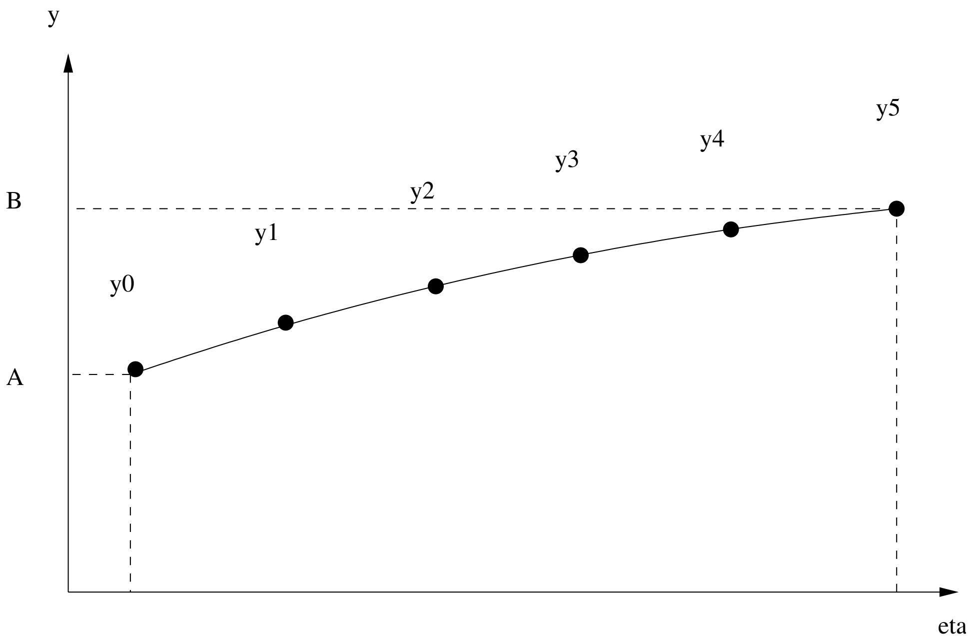

Figure 11.1: Finite difference method, for solution of boundary value problem. Example of discrete points with $m = 5$.

11.3 Derivative Boundary Conditions

On occasion we know the derivative at one of the boundaries but not the initial condition for the variable. Considering the same system as in (11.4) we could have boundary conditions

$$y'(a) = C \quad \text{and} \quad y(b) = B\,. \tag{11.8}$$

We can adapt the finite difference method by taking a central difference for the derivative at the boundary

$$y_0'(a) \approx \frac{y_1 - y_{-1}}{2h} = C, \tag{11.9}$$

and extending the system by one interval so that the first equation reads

$$y_{-1} - 2\,y_0 + y_1 = k\,h^2, \tag{11.10}$$

which, given (11.9), is

$$-2\,y_0 + 2y_1 = k\,h^2 + 2\,h\,C\,. \tag{11.11}$$

The system can then be written as m linear equations

$$\begin{pmatrix} -2 & 2 & 0 & 0 & .... & 0 & 0 \\ 1 & -2 & 1 & 0 & .... & 0 & 0 \\ & & & . & & & \\ & & & . & & & \\ & & & . & & & \\ 0 & 0 & 0 & 0 & .... & 1 & -2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ . \\ . \\ . \\ y_{m-1} \end{pmatrix} = \begin{pmatrix} k\,h^2 + 2\,h\,C \\ k\,h^2 \\ . \\ . \\ . \\ k\,h^2 - B \end{pmatrix}. \tag{11.12}$$

11.4 Eigenvalue Problems

For the particular case where the differential equations are homogeneous and linear we can view the problem as an eigensystem. For example, consider the wave equation

$$\frac{d^2y}{dx^2} + k^2 y = 0, \tag{11.13}$$

with boundary conditions $y(0) = 0$ and $y(1) = 0$. This describes the vibrations on a string of length 1 and fixed at the endpoints. The general solution to this system is easily found to be

$$y(x) = A\,\sin(k\,x) + B\,\cos(k\,x)\,. \tag{11.14}$$

The boundary conditions imply that $B = 0$ and that $k = \pm n\,\pi$. The solution describes fundamental modes of vibrations on the string where $n = 1, 2, 3, \ldots$, etc.

The system (and more complicated ones in particular) can be solved using finite differences

$$\frac{y_{i-1} - 2\,y_i + y_{i+1}}{h^2} + k^2\,y_i = 0, \tag{11.15}$$

which gives the eigensystem

$$\begin{pmatrix} +2 & -1 & 0 & 0 & .... & 0 & 0 \\ -1 & +2 & -1 & 0 & .... & 0 & 0 \\ & & & . & & & \\ & & & . & & & \\ & & & . & & & \\ 0 & 0 & 0 & 0 & .... & -1 & +2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_{m-1} \end{pmatrix} = h^2\,k^2 \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_{m-1} \end{pmatrix}, \tag{11.16}$$

i.e.

$$\mathbf{A} \cdot \tilde{\vec{y}} = \lambda \tilde{\vec{y}} \tag{11.17}$$

where $\lambda = h^2\,k^2$ are the eigenvalues of the system. One then finds $\mathbf{A}$'s eigenvalues λ_i and eigenvectors $\vec{e}_i$ using a suitable numerical eigen-solver. These should be close numerical approximations to the eigenvalues and eigenfunctions of the original ODE eigen-problem. It is often useful to find just the eigenvector corresponding to the largest (or smallest) eigenvalues, as these define the fundamental modes of the system. The power method (section 3.8) can be used to readily find these.