Applied Algorithms. Prof. Tami Tamir
Missing Proofs, lecture 6. July 8th.

Claim: In any graph, there is an odd number of vertices with an odd degree.

Proof: The total degree is 2|E|. The vertices with an even degree contribute together to the total degree 2k (for some integer 2k). Therefore, the vertices of odd degree contribute 2(|E|-k) to the total degree, which is an even number. Given that the sum of x odd integers is even, it must be that x is even.

Lemma 1: G has a postman cycle of length k if and only if G has an Eulerian extension of total edge weight k.
Proof: a. Suppose G has a postman cycle W of length k. Construct an extension G* of G by replacing each edge of G by p(e) parallel edges, where p(e) is the number of times e is contained in W. Then G* is Eulerian since W is an Euler tour in G*. Furthermore, length(W) = w(G*).
b. Suppose G has an Eulerian extension G* of weight k. Let R be an Euler tour of G*. Then R corresponds to a postman walk W for G with length(W) = w(G*). Thus, G has a postman walk of length k.

Missing argument in the proof of Edmonds-Johnson Algorithm: We show that the paths corresponding to the min-matching are edge disjoint. Assume by contradiction that there are two paths p1= a-----b and p2= c-----d with a common edge e=(u,v). That is, p1=a---u-v----b and p2=c-----u-v-----d, then by replacing p1 and p2 by a-----u-----c and b----v----d we get a cheaper matching of the odd-degree vertices (cheaper by 2w(e)), contradicting the optimality of the min-cost matching.

Conclusion: Every edge appears at most once in the extension and step 4 produces a valid min-cost Eulerian extension of G.

The analysis of 3/2-approximation for TSP is tight: The graph in the figure presents an instance of TSP. It has 2n+1 nodes, all solid edges have length 1, all dashed edges have weight $1+\varepsilon$.

The only MST consists of all solid lines, it has length 2n. No shortcuts are needed. The first and last nodes have odd degree. The edge connecting them has length $(1+\varepsilon)n$. All together, the TSP path detected by the algorithm has length ~3n.

An optimal tour consists of the dashed edges + the leftmost and rightmost edges. Its length is $(2n-1)(1+\varepsilon)+2$ ~2n