Optimal rotor arm pivot angles for a collapsible drone.

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Problem Description

Consider the CAD drawing of the collapsible drone shown in figure 1 below:

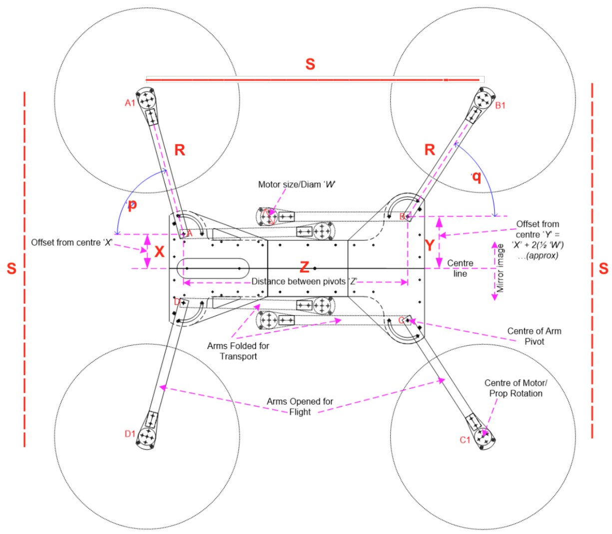

Figure 1: CAD drawing of drone with collapsible arms

In this diagram all lengths are measured in mm and the annotated lengths are design parameters for the construction of the drone:

- R the length of the rotor arm (all rotor arms have the same length)
- Z the distance between rotor arm pivot points.
- X the offset of the front pivot arm point from the horizontal axis of symmetry.
- Y the offset of the back pivot arm point from the horizontal axis of symmetry.

The problem is to determine the rotor arm angles, p and q, under the constraint that the rotor centres form a square.

Constraint Problem

Using the constraint that the rotor centres form a square with sides of some unknown length S we have:

$$\begin{aligned} S &= R\cos(p) + Z + R\cos(q) \\ S &= 2(X + R\sin(p)) \\ S &= 2(Y + R\sin(q)) \end{aligned} \tag{1}$$

The first equation in (1) is derived from an expression for the horizontal edge of length, S, whilst the last two equations are derived from expressions for the vertical front and back edges of length S.

The unknown edge length, S, can be eliminated from equations (1) yielding two non-linear equations for the angles, p and q shown in equations (2) below.

$$\begin{aligned} \cos(p) - 2\sin(p) + \cos(q) &= \frac{2X - Z}{R} \\ \sin(p) - \sin(q) &= \frac{Y - X}{R} \end{aligned} \tag{2}$$

Although it is possible to find a closed form solution for p and q the calculation requires finding the roots of a quartic polynomial which is extremely tedious as one can see from the Wikipedia entry [1].

So instead we use standard non-linear root finding technique. For example, the `scipy` python package allows us to construct the solution shown in Appendix A.

To make life easier for the reader we have also coded a `javascript` solution to the problem, this time using a numerical optimiser called *Nelder-Mead* [3]. The javascript code for method was lifted from an open source implementation by *Ben Frederickson* [**?**].

The root finding problem given in equation 2 must first be cast as a function optimisation problem. To this end we use *Nelder-Mead* to minimise the function:

$$\begin{aligned} f(p,q) &= |\cos(p) - 2\sin(p) + \cos(q) - \frac{2X - Z}{R}| + \\ & \quad |\sin(p) - \sin(q) - \frac{Y - X}{R}| \end{aligned} \tag{3}$$

This is a highly non-linear function with sudden changes in slope due to the use of the absolute value. This often results in optimisers getting stuck in local minima and we were surprised that the *Nelder-Mead* performed as well as it did.

A sketch of the function we are attempting to minimise is given in figure 2.

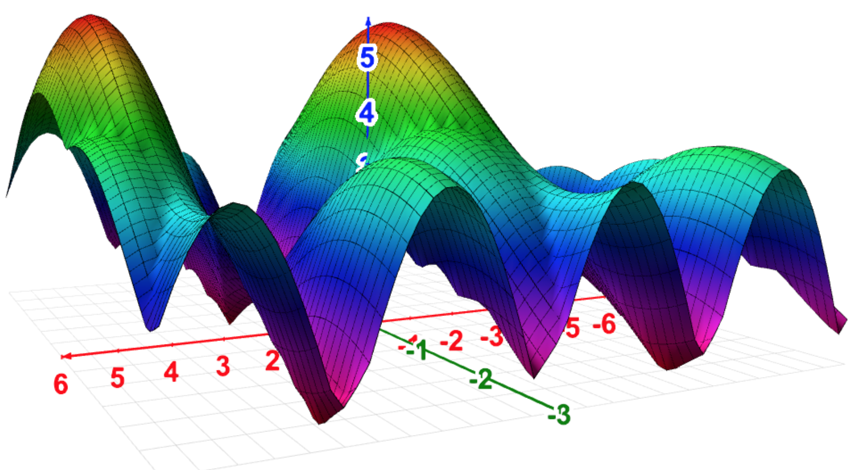

Figure 2: The surface $f(p,q)$ for which a minimum near zero must be found.

Due to the possiblity of a *local minimum* trap we needed to be confident that the values of p and q returned by *Nelder-Mead* did indeed satisfy the constraints of the drone design problem. To this end we made use of the Microsoft javascript CAD library, `Maker.js` [4], to produce drawings dependent on the user's parameter settings.

The `maker` code for the `drone` scriptis given in Appendix B, but the interested reader can access the script from:

`"hughmurrell.github.io/DroneDesign/maker/docs/playground/?script=drone"`.

References

[1] Wikipedia Article, *Quartic function* `https://en.wikipedia.org/wiki/Quartic_function` accessed January 2021.

[2] Wikipedia, *The Nelder?Mead method* , `https://en.wikipedia.org/wiki/Nelder-Mead_method`, accessed January 2021.

[3] Ben Fedrickson, *fmin, unconstrained function minimization in javascript.* , `https://github.com/benfred/fmin`, accessed January 2021.

[4] A Microsoft Garage Project, *Maker.js, a JavaScript library for creating and sharing modular line drawings for CNC and laser cutters.*, `https://maker.js.org/`, accessed January 2021.

Appendix A

pythod code for calculating rotor arm pivot angles

```
from scipy.optimize import fsolve
from math import sin, cos, pi

def equations(vars):
    p, q = vars
    R = 225
    Z = 346
    X = 56
    Y = 84
    eq1 = cos(p) - 2*sin(p) + cos(q) - (2*X-Z)/R
    eq2 = sin(p) - sin(q) - (Y-X)/R
    return [eq1, eq2]

p, q =  fsolve(equations, (1, 1))

p = (p / pi) * 180
q = (q / pi) * 180

print(p, q)
```

Appendix B

```
var makerjs = require('makerjs');


// worker for nelderMead
function weightedSum(ret, w1, v1, w2, v2) {
        for (var j = 0; j < ret.length; ++j) {
            ret[j] = w1 * v1[j] + w2 * v2[j];
        }
    }


// minimizes a function using the downhill simplex method
function nelderMead(f, x0, w, parameters) {
        parameters = parameters || {};

        var maxIterations = parameters.maxIterations || x0.length * 200,
            nonZeroDelta = parameters.nonZeroDelta || 1.05,
            zeroDelta = parameters.zeroDelta || 0.001,
            minErrorDelta = parameters.minErrorDelta || 1e-6,
            minTolerance = parameters.minErrorDelta || 1e-5,
            rho = (parameters.rho !== undefined) ? parameters.rho : 1,
            chi = (parameters.chi !== undefined) ? parameters.chi : 2,
            psi = (parameters.psi !== undefined) ? parameters.psi : -0.5,
            sigma = (parameters.sigma !== undefined) ? parameters.sigma : 0.5,
            maxDiff;

        // initialize simplex.
        var N = x0.length,
            simplex = new Array(N + 1);
        simplex[0] = x0;
        simplex[0].fx = f(x0,w);
        simplex[0].id = 0;
        for (var i = 0; i < N; ++i) {
            var point = x0.slice();
            point[i] = point[i] ? point[i] * nonZeroDelta : zeroDelta;
            simplex[i+1] = point;
            simplex[i+1].fx = f(point,w);
            simplex[i+1].id = i+1;
        }

        function updateSimplex(value) {
            for (var i = 0; i < value.length; i++) {
                simplex[N][i] = value[i];
            }
            simplex[N].fx = value.fx;
        }

        var sortOrder = function(a, b) { return a.fx - b.fx; };

        var centroid = x0.slice(),
            reflected = x0.slice(),
            contracted = x0.slice(),
            expanded = x0.slice();

        for (var iteration = 0; iteration < maxIterations; ++iteration) {
            simplex.sort(sortOrder);

            if (parameters.history) {
                // copy the simplex (since later iterations will mutate)
                // sort it to have a consistent order between iterations
                var sortedSimplex = simplex.map(function (x) {
                    var state = x.slice();
                    state.fx = x.fx;
                    state.id = x.id;
                    return state;
                });
                sortedSimplex.sort(function(a,b) { return a.id - b.id; });

                parameters.history.push({x: simplex[0].slice(),
                                         fx: simplex[0].fx,
                                         simplex: sortedSimplex});
            }
```

```
maxDiff = 0;
for (i = 0; i < N; ++i) {
    maxDiff = Math.max(maxDiff, Math.abs(simplex[0][i] -
                                         simplex[1][i]));
}

if ((Math.abs(simplex[0].fx - simplex[N].fx) < minErrorDelta) &&
    (maxDiff < minTolerance)) {
    break;
}

// compute the centroid of all but the worst point in the simplex
for (i = 0; i < N; ++i) {
    centroid[i] = 0;
    for (var j = 0; j < N; ++j) {
        centroid[i] += simplex[j][i];
    }
    centroid[i] /= N;
}

// reflect the worst point past the centroid
// and compute loss at reflected point
var worst = simplex[N];
weightedSum(reflected, 1+rho, centroid, -rho, worst);
reflected.fx = f(reflected,w);

// if the reflected point is the best seen, then possibly expand
if (reflected.fx < simplex[0].fx) {
    weightedSum(expanded, 1+chi, centroid, -chi, worst);
    expanded.fx = f(expanded,w);
    if (expanded.fx < reflected.fx) {
        updateSimplex(expanded);
    } else {
        updateSimplex(reflected);
    }
}

// if the reflected point is worse than the second worst,
// we need to contract
else if (reflected.fx >= simplex[N-1].fx) {
    var shouldReduce = false;

    if (reflected.fx > worst.fx) {
        // do an inside contraction
        weightedSum(contracted, 1+psi, centroid, -psi, worst);
        contracted.fx = f(contracted,w);
        if (contracted.fx < worst.fx) {
            updateSimplex(contracted);
        } else {
            shouldReduce = true;
        }
    } else {
        // do an outside contraction
        weightedSum(contracted, 1-psi * rho, centroid,
                    psi*rho, worst);
        contracted.fx = f(contracted,w);
        if (contracted.fx < reflected.fx) {
            updateSimplex(contracted);
        } else {
            shouldReduce = true;
        }
    }

    if (shouldReduce) {
        // if we don't contract here, we're done
        if (sigma >= 1) break;

        // do a reduction
        for (i = 1; i < simplex.length; ++i) {
            weightedSum(simplex[i], 1 - sigma, simplex[0],
                        sigma, simplex[i]);
            simplex[i].fx = f(simplex[i],w);
        }
    }
} else {
```

```
                updateSimplex(reflected);
            }
        }

        simplex.sort(sortOrder);
        return {fx : simplex[0].fx,
                x : simplex[0]};
    }

function drone(font, fontSize, R, D, Z, X, Y, P, Q){

    if (arguments.length === 0) {
        var defaultValues = makerjs.kit.getParameterValues(drone);
        font = defaultValues.shift();
        fontSize = defaultValues.shift();
        R = defaultValues.shift();
        D = defaultValues.shift();
        Z = defaultValues.shift();
        X = defaultValues.shift();
        Y = defaultValues.shift();
    }


    // compute p,q from R,Z,X,Y using nelderMead
    function loss(x,w) {
        var p = x[0], q = x[1];
        var R = w[0], Z = w[1], X = w[2], Y = w[3];
        var f1 = Math.cos(p) - 2 * Math.sin(p) + Math.cos(q) - (2*X-Z)/R;
        var f2 = Math.sin(p) - Math.sin(q) - (Y-X)/R;
        return Math.abs(f1) + Math.abs(f2);
    }

    var solution = nelderMead(loss, [0.786, 0.786], [R,Z,X,Y]);

    p = solution.x[0];
    q = solution.x[1];

    console.log("user selected:");
    console.log("R = "+R);
    console.log("D = "+D);
    console.log("X = "+X);
    console.log("Y = "+Y);
    console.log("computed:");
    console.log("P = " + p*180/Math.PI);
    console.log("Q = " + q*180/Math.PI);

    var cp=R*Math.cos(p);
    var sp=R*Math.sin(p);
    var cq=R*Math.cos(q);
    var sq=R*Math.sin(q);

    var top_dots = [
                    [0, X + sp],
                    [cp,X],
                    [cp,0],
                    [cp+Z,0],
                    [cp+Z,Y],
                    [cp+Z+cq,Y+sq]
                    ];

    var top_drone = new makerjs.models.ConnectTheDots(true, top_dots);

    var bot_dots = [
                    [0, -(X + sp)],
                    [cp,-X],
                    [cp,0],
                    [cp+Z,0],
                    [cp+Z,-Y],
                    [cp+Z+cq,-(Y+sq)]
                    ];

    var bot_drone = new makerjs.models.ConnectTheDots(true, bot_dots);


    var rotors = {
    paths: {
```

```
    "ft": new makerjs.paths.Arc([0, X + sp], D/2, 0, 360),
    "fb": new makerjs.paths.Arc([0, -(X + sp)], D/2, 0, 360),
    "bt": new makerjs.paths.Arc([cp+Z+cq,Y+sq], D/2, 0, 360),
    "bb": new makerjs.paths.Arc([cp+Z+cq,-(Y+sq)], D/2, 0, 360)
    }
};

var ovr = Math.abs((Y-X)/2);

var rot=(Math.PI-p)*180/Math.PI;
var arm_ft = new makerjs.models.Oval(R+2*ovr, 2*ovr);
makerjs.model.move(arm_ft,[cp-ovr,X-ovr]);
makerjs.model.rotate(arm_ft,rot,[cp,X]);

rot=(q)*180/Math.PI;
var arm_bt = new makerjs.models.Oval(R+2*ovr, 2*ovr);
makerjs.model.move(arm_bt,[cp+Z-ovr,Y-ovr]);
makerjs.model.rotate(arm_bt,rot,[cp+Z,Y]);

rot=(Math.PI+p)*180/Math.PI;
var arm_fb = new makerjs.models.Oval(R+2*ovr, 2*ovr);
makerjs.model.move(arm_fb,[cp-ovr,-X-ovr]);
makerjs.model.rotate(arm_fb,rot,[cp,-X]);

rot=(-q)*180/Math.PI;
var arm_bb = new makerjs.models.Oval(R+2*ovr, 2*ovr);
makerjs.model.move(arm_bb,[cp+Z-ovr,-Y-ovr]);
makerjs.model.rotate(arm_bb,rot,[cp+Z,-Y]);

var cas_tf = new makerjs.models.Oval((2*Z/3)+2*ovr, 2*ovr);
makerjs.model.move(cas_tf,[cp-ovr,X-ovr]);

var cas_tb = new makerjs.models.Oval((2*Z/3)+2*ovr, 2*ovr);
makerjs.model.move(cas_tb,[cp+Z-ovr,Y-ovr]);
makerjs.model.rotate(cas_tb,180,[cp+Z,Y]);

var cas_bf = new makerjs.models.Oval((2*Z/3)+2*ovr, 2*ovr);
makerjs.model.move(cas_bf,[cp-ovr,-X-ovr]);

var cas_bb = new makerjs.models.Oval((2*Z/3)+2*ovr, 2*ovr);
makerjs.model.move(cas_bb,[cp+Z-ovr,-Y-ovr]);
makerjs.model.rotate(cas_bb,180,[cp+Z,-Y]);

var p_annotate = new makerjs.models.Text(font,
        "P = " + (p*180/Math.PI).toFixed(1), fontSize, false, false);
makerjs.model.move(p_annotate,[cp-ovr-(7*fontSize),0]);

var q_annotate = new makerjs.models.Text(font,
        "Q = " + (q*180/Math.PI).toFixed(1), fontSize, false, false);
makerjs.model.move(q_annotate,[cp+Z+(2*fontSize),0]);

var z_annotate = new makerjs.models.Text(font,
        "Z = " + Z, fontSize/2, false, false);
makerjs.model.move(z_annotate,[cp+Z/2-(2*fontSize/2),fontSize/4]);

var r_annotate = new makerjs.models.Text(font,
        "R = " + R, fontSize/2, false, false);
makerjs.model.move(r_annotate,[cp+Z+(2*fontSize/2),Y+fontSize/4]);

var y_annotate = new makerjs.models.Text(font,
        "Y = " + Y, fontSize/2, false, false);
makerjs.model.move(y_annotate,[cp+Z+(1*fontSize/2),-Y/2]);

var x_annotate = new makerjs.models.Text(font,
        "X = " + X, fontSize/2, false, false);
makerjs.model.move(x_annotate,[cp+(1*fontSize/2),-X/2]);

var d_annotate = new makerjs.models.Text(font,
        "D = " + D, fontSize/2, false, false);
makerjs.model.move(d_annotate,[D/2+(1*fontSize/2),-Y-sp+fontSize]);

this.models = {
        "top": top_drone,
        "bottom": bot_drone,
        "rotor blades": rotors,
        "arm_ft": arm_ft,
```

```
            "arm_bt": arm_bt,
            "arm_fb": arm_fb,
            "arm_bb": arm_bb,
            "cas_tf": cas_tf,
            "cas_tb": cas_tb,
            "cas_bf": cas_bf,
            "cas_bb": cas_bb,
            "p_annotate": p_annotate,
            "q_annotate": q_annotate,
            "z_annotate": z_annotate,
            "r_annotate": r_annotate,
            "y_annotate": y_annotate,
            "x_annotate": x_annotate,
            "d_annotate": d_annotate
        };

}


    drone.metaParameters = [
        { title: "font", type: "font", value: '#stencil' },
        { title: "font size", type: "range", min: 10, max: 200, value: 48 },
        { title: "R", type: "range", min: 50, max: 400, value: 225 },
        { title: "D", type: "range", min: 10, max: 300, value: 200 },
        { title: "Z", type: "range", min: 100, max: 900, value: 346 },
        { title: "X", type: "range", min: 10, max: 100, value: 56 },
        { title: "Y", type: "range", min: 40, max: 150, value: 84 }
     ];


    module.exports = drone;
```