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# SIGMOID FUNCTIONS: SOME APPROXIMATION, AND MODELLING ASPECTS

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The subject of this book is cross-disciplinary. Sigmoid functions present a field of interest both for fundamental as well as application-driven research. We have tried to give the readers the flavor of both perspectives. From the perspective of fundamental science sigmoid functions are of special interest in abstract areas such as approximation theory, functional analysis and probability theory. More specifically, sigmoid function are an object of interest in Hausdorff approximations, fuzzy set theory, cumulative distribution functions, impulsive functions, etc. From the perspective of applied mathematics and modeling sigmoid functions find their place in numerous areas of life and social sciences, physics and engineering, to mention a few familiar applications: population dynamics, artificial neural networks, signal and image processing antenna feeding techniques, finances and insurance.

We consider this book to be suitable for undergraduate students attending master programs from applied mathematics and mathematical modeling. A prerequisite of elementary linear algebra and calculus is needed for easy reading. Furthermore, the book should be applicable to students at that level from essentially every science and engineering University department. Readers with some computational programming skills may find numerous ideas for writing their own programs to run graphics of various sigmoid functions (the ones presented in the book make use of the computer algebra system *Mathematica*).

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## PREFACE

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Furthermore, the book should be applicable to students at that level from essentially every science and engineering University department.

Readers with some computational programming skills may find numerous ideas for writing their own programs to run graphics of various sigmoid functions. Such readers may take advantage of the computational examples presented in the book that make use of the computer algebra system *Mathematica*). Only the case of functions of one variable is considered, but it can be expected that our results can be generalized to multivariate functions as well.

The material in the book is structured as follows.

The basic ideas are given in Chapters 1 and 2. After studying these two chapters the reader may pass to any of the remaining chapters which can be considered as independent case studies.

Chapter 1 introduces some basic notions related to sigmoid and step functions, such as Hausdorff distance and continuity, the kinetic mechanisms beyond the sigmoid models etc. The Verhulst logistic model is considered as a basic example to introduce several related mathematical problems: approximation of step and cut functions by means of logistic function, fitting a sigmoid model to time course measurement data, etc. Verhulst model is an important classical example involving a simple sigmoidal function as solution. We demonstrate that this model is induced by simple autocatalytic reactions that describe certain reproduction bio-chemical mechanisms.

Supplements containing programing code in Mathematica are provided. Supplement 2 is devoted to the relation between sigmoid functions and artificial neural networks (perceptrons).

Chapter 2 is devoted to relations between sigmoid and cut functions. More specifically we focus on the approximation of the cut and step functions by logistic functions. The approximation of the cut and step functions by squashing functions is also discussed. Computational examples are presented.

Using the ideas developed in chapters 1 and 2 in the remaining chapters we focus on several other classes of sigmoid functions. Chapter 3 is devoted to the log-logistic sigmoid functions and Chapter 4 studies the Gompertz function. In both cases we emphasize the relation between the smooth sigmoid functions and the nonsmooth step and cut functions.

Chapters 5, 6 and 7 are devoted to sigmoid functions appearing in probability theory and statistics as cumulative distribution functions. Differently to the sigmoid functions considered in the previous chapters, cumulative functions are defined for positive arguments. We demonstrate that most of our ideas (notably those on approximation using step and cut functions) can be applied also to cumulative distribution functions.

Authors

# Chapter 1

# Sigmoid Functions

We focus on some computational, modelling and approximation issues related to the logistic sigmoidal function and to Heaviside step function.

The Hausdorff approximation of the Heaviside step function by sigmoid functions is discussed from various computational and modelling aspects.

Some relations between Verhulst model and certain biochemical reaction equations are discussed and analyzed.

Numerical examples are presented using CAS *Mathematica*.

## 1.1    Introduction

Many biological dynamic processes, such as certain enzyme kinetic and population growth processes, develop almost step-wise [9], [13].

Such processes are usually described or approximated by smooth sigmoidal functions; such functions are widely used in the theory of neural networks [3], [4]. Step-wise functions are a special class of sigmoid functions; such functions are "almost" continuous, or Hausdorff continuous (H-continuous) [2].

Depending on the particular modelling situation one may decide to use either continuous or H-continous (step-wise) functions.

Moreover, in many cases both types of modelling tools can be used interchangeably.

This motivates us to study the closeness of functions of both classes.

To substitute a sigmoid function by a step function (or conversely) we need to know the approximation error between the two functions.

A natural metric used in such a situation is the Hausdorff metric between the graphs of the functions.

To this end we recall some basic results concerning the class of interval Hausdorff continuous functions and the related concept of Hausdorff approximation.

We then focus on classes of logistic sigmoid functions which are solutions of the Verhulst population model.

We demonstrate that Verhulst model arises from simple autocatalytic (bio)chemical reactions and thus can be considered as special case of a biochemical reproduction reaction mechanism.

The latter implies a more general model that permits the formulation of some important modelling and computational problems including nonautonomous, impulsive and delay DE.

In section 2 we consider sigmoid and step functions arising from biological applications.

The Hausdorff distance between the Heaviside step function and the sigmoid Verhulst function is discussed.

In section 3 we discuss certain kinetic mechanisms yielding Verhulst model via the mass action law.

We show that the Verhulst model arises from some simple autocatalytic (bio)chemical reactions.

## 1.2   Sigmoid and step functions

### 1.2.1   Hausdorff continuity

The concept of Hausdorff continuity (H-continuity) generalizes the familiar concept of continuity so that essential properties of the usual continuous real functions remain present.

It is possible to extend the algebraic operations on the set of continuous real functions $C(\Omega)$ to the set $H(\Omega)$ of H-continuous functions in such a way that the set $H(\Omega)$ becomes a commutative ring and a linear space with respect to the extended operations [2].

In this work we restrict ourselves to functions of one real variable, that is real functions defined on a subset $\Omega \subseteq R$.

### 1.2.2   The step function

For $r \in R$ denote by $h_r \in H(R)$ the (interval) Heaviside step function given by

$$h_r(t) = \left\{ \begin{array}{ll} 0, & \text{if} \quad t < r, \\ [0,1], & \text{if} \quad t = r, \\ 1, & \text{if} \quad t > r, \end{array} \right. \tag{1.1}$$

cf. Fig. 1.1.

For $r = 0$ we obtain the basic Heaviside step function

$$h_0(t) = \left\{ \begin{array}{ll} 0, & \text{if} \quad t < 0, \\ [0,1], & \text{if} \quad t = 0, \\ 1, & \text{if} \quad t > 0. \end{array} \right. \tag{1.2}$$

Functions (1.1)–(1.2) are examples sigmoid functions.

A sigmoidal function on  with a range $[a, b]$ is defined as a monotone function $s(t) :\to [a, b]$ such that $\lim_{t \to -\infty} s(t) = a$, $\lim_{t \to \infty} s(t) = b$.

### 1.2.3   The cut function

One may consider continuous (or even smooth) or discontinous sigmoid functions.

Within the class of H-continuous interval functions, the Heaviside step function is a particular case of sigmoidal function.

An example of a continuous sigmoid function is the *cut function* defined as

$$c_{[a,b]}(t) = \begin{cases} 0, & \text{if} \quad t \leq a, \\ \dfrac{t - a}{b - a}, & \text{if} \quad a < t < b, \\ 1, & \text{if} \quad t \geq b. \end{cases} \tag{1.3}$$

The cut function (2.1) is visualized on Fig. 1.2.

### 1.2.4   Sums of sigmoid functions

For a given vector $r = (r_1, r_2, ..., r_k) \in R^k$, such that $r_1 < r_2 < ... < r_k$, and a vector $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k) \in R^k$ denote

$$H(r, \alpha; t) = \sum_{i=1}^{k} \alpha_i h_{r_i}(t). \tag{1.4}$$

Function (1.4) is a step function with $k$ steps (jumps).

Using suitable values for $r_k$ and $\alpha_k$ one can represent a histogram, such as the one of Fig. 1.3, as a sum of step functions of the form (1.4); in that case we usually have $r_k = r_1 + hi, i = 1, ..., k$ and $\sum_{i=1}^{k} \alpha_i = 0$.

Similarly, one can construct sums of other suitably shifted sigmoid functions.

### 1.2.5   Hausdorff distance

Let us recall that the Hausdorff distance (H-distance) $\rho(f, g)$ between two functions $f, g \in H(\Omega)$ for $\Omega \subseteq R$ is defined as the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $R^2$ [7], [14].
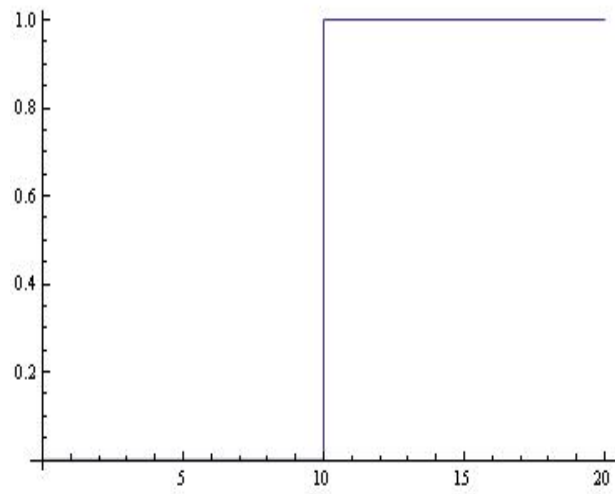
Figure 1.1: A step function (1.1) with "jump" at $r = 10$
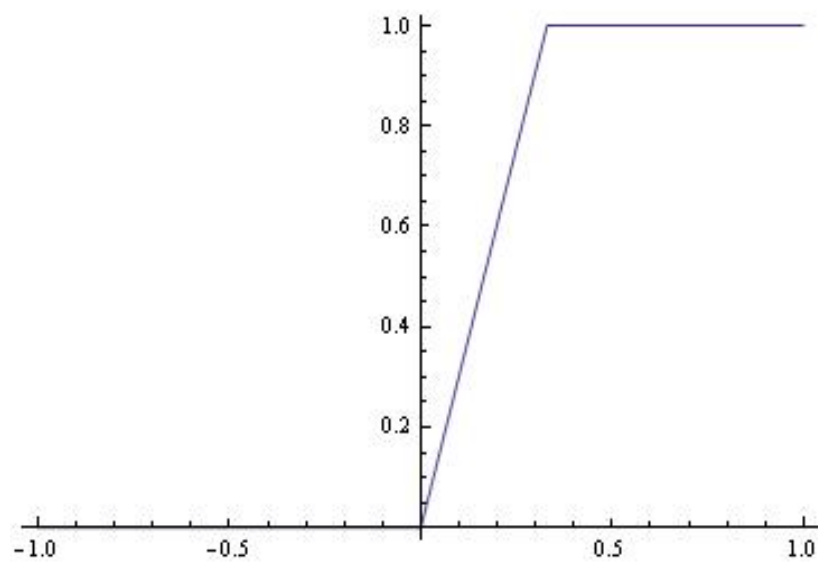


Figure 1.2: Cut function
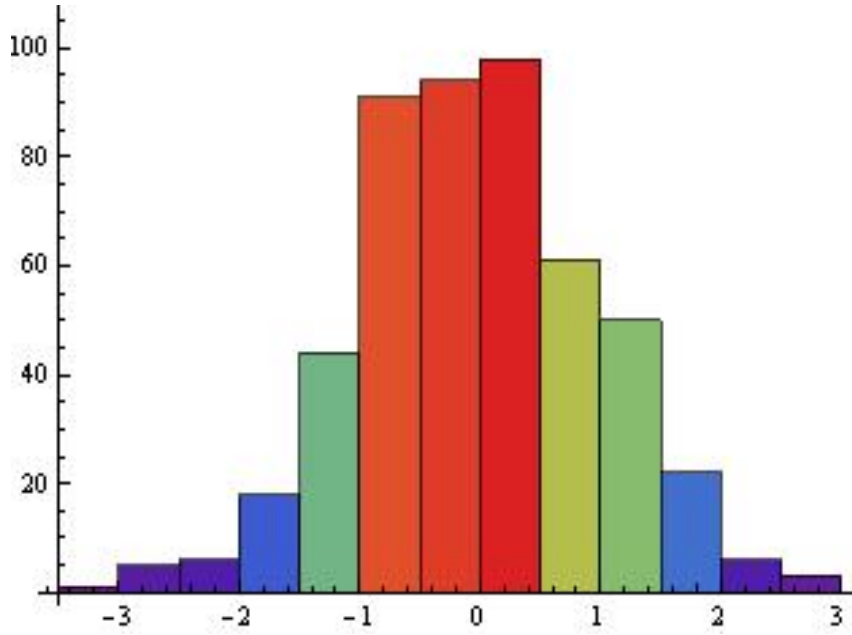
Figure 1.3: A histogram (from wikipedia)

More precisely,

$$\rho(f,g) = \max\{\sup_{A\in F(f)} \inf_{B\in F(g)} ||A-B||, \sup_{B\in F(g)} \inf_{A\in F(f)} ||A-B||\}, \qquad (1.5)$$

wherein $||.||$ is a norm in $R^2$.

According to (1.5) the H-distance $\rho(f,g)$ between two functions $f, g \in H(\Omega)$ for $\Omega \subseteq R$ makes use of the maximum norm in $R^2$ so that the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $R^2$ is given by

$$||A-B|| = \max(|t_A - t_B|, |x_A - x_B|).$$

### 1.2.6 The logistic sigmoid function

Sigmoid functions find multiple applications to neural networks and cell growth population models [4], [9].

Several practically important families of smooth sigmoid functions arise from population dynamics.

A classical example is the Verhulst population growth model to be discussed below.

Verhulst model makes an extensive use of the *logistic* sigmoid function

$$s_0(t) = \frac{a}{1 + e^{-kt}}, \qquad (1.6)$$
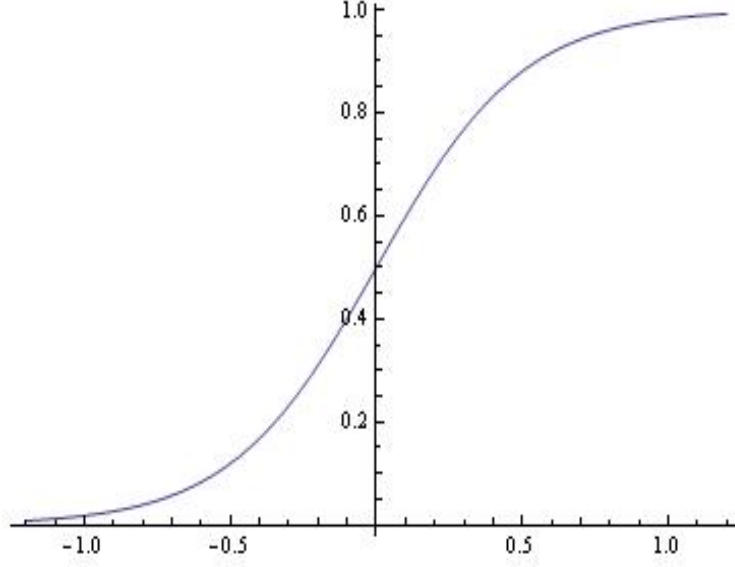
Figure 1.4: Logistic sigmoid function (1.6)

see Fig. 1.4.

We next focus on the approximation of the Heaviside step function (1.2) by logistic functions of the form (1.6) in Hausdorff distance.

### 1.2.7   Approximation issues

In what follows we shall estimate the H-distance (1.5) between a step function and a logistic sigmoid function. Without loss of generality we can consider the Heaviside step function $f = ah_0$ and the logistic sigmoid function (1.6): $g = s_0$.

As visualized on Fig. 7.5, the H-distance $d = \rho(f, g) = \rho(ah_0, s_0)$ between the step function $ah_0$ and the sigmoid function $s_0$ satisfies the relations $0 < d < \frac{a}{2}$ and $a - s_0(d) = d$, that is

$$\frac{a - d}{d} = e^{kd}, \quad (0 < d < a/2). \tag{1.7}$$

Obviously $d \to 0$ implies $k \to \infty$ (and vice versa).

From (1.7) we obtain a straightforward expression for the rate parameter $k$ as a function of $d$:

**Theorem 1.1** *The rate parameter $k$ can be expressed in terms of the H-distance $d$ as follows:*

$$k = k(d) = \frac{1}{d} \ln \frac{a - d}{d} = O(d^{-1} \ln(d^{-1})). \tag{1.8}$$

Figure 1.5: Reaction rate $k = 20$; H– distance $d = 0.106402$

Relation (1.8) gives the rate $k$ in terms of the H-distance $d$, a few examples are computed in Table 1.1.

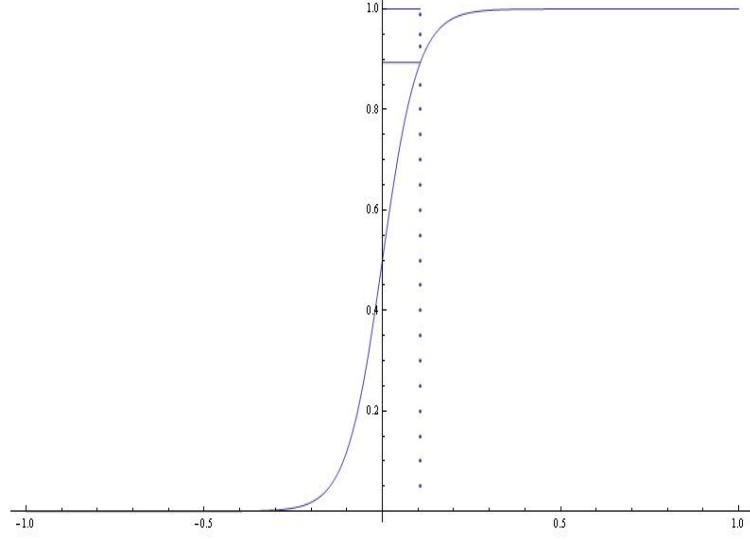| Dist. $d$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ | $10^{-5}$ | $10^{-6}$ |
|---|---|---|---|---|---|---|
| Rate $k$ | $0.22 \times 10^2$ | $0.46 \times 10^3$ | $0.69 \times 10^4$ | $0.92 \times 10^5$ | $0.12 \times 10^7$ | $0.14 \times 10^8$ |

Table 1.1: Values of the rate $k$ as function of the H-distance $d$ according to (1.8)

Conversely, a relation for the H-distance $d$ in terms of the rate parameter $k$ is given in the next proposition.

For simplicity we assume below $a = 1$, considering thus the basic logistic function as depending only on the rate $k$: $s_0(t) = (1 + e^{-kt})^{-1}$.

**Theorem 1.2** *[2] The H–distance $d = \rho(h_0, s_0)$ between the Heaviside step function $h_0$ and the sigmoid Verhulst function $s_0$ can be expressed in terms of the the reaction rate $k$ for any real $k \geq 2$ as follows:*

$$\dot{d}_l(k) = \frac{1}{k+1} < d(k) < \frac{\ln(k+1)}{k+1}, \tag{1.9}$$

$$d_l(k) = \frac{\ln(k+1)}{k+1} - \frac{\ln\ln(k+1)}{k+1} < d(k) < \frac{\ln(k+1)}{k+1} = d_r(k), \tag{1.10}$$

*or*

$$d(k) = \frac{\ln(k+1)}{k+1} \left(1 + O\left(\varepsilon(k)\right)\right), \quad \varepsilon(k) = \frac{\ln\ln(k+1)}{\ln(k+1)}. \tag{1.11}$$
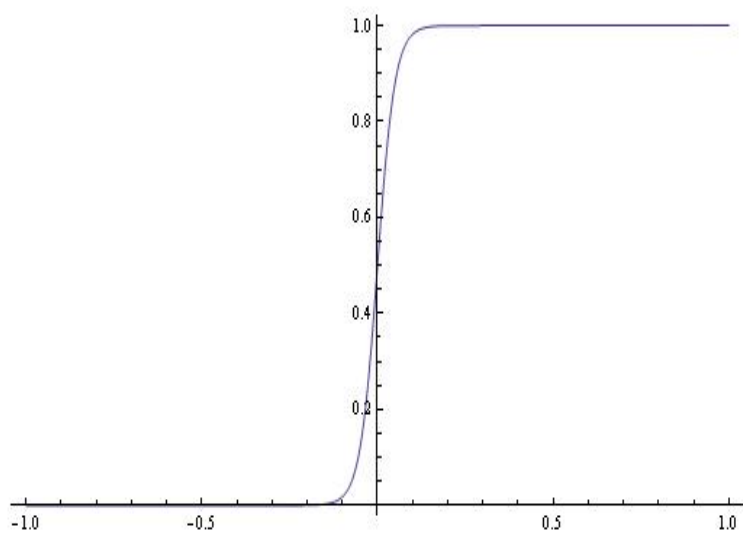
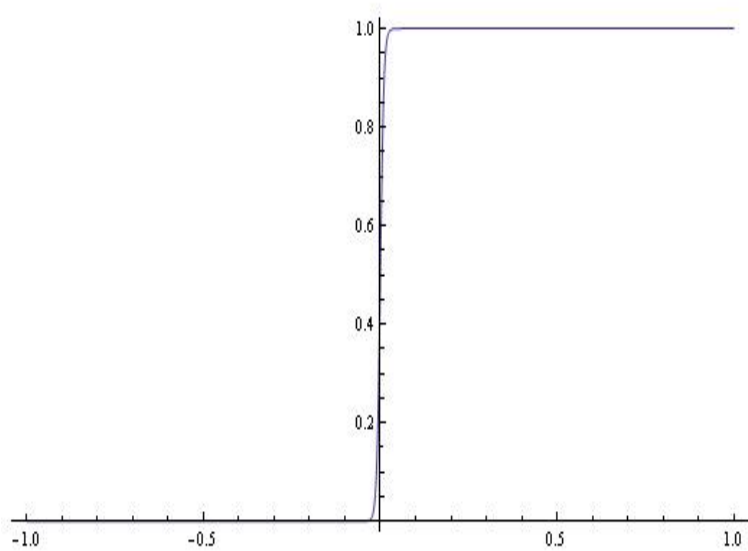Figure 1.6: Reaction rate $k = 40$; H– distance $d = 0.0661748$



Figure 1.7: Reaction rate $k = 200$; H– distance $d = 0.01957$.

A proof of relations (1.9)–(1.11) is given in [2].

In order to express $d$ in terms of $k$, let us examine the function

$$f(d) = kd - \ln\frac{1}{d} - \ln(1 - d), \ \ 0 < d < \frac{1}{2}.$$

From

$$\lim_{d \to \frac{1}{2}, d < \frac{1}{2}} f(d) = \frac{k}{2} > 0$$

we conclude that $f(d) = 0$ possesses a solution in $(0, \frac{1}{2})$.

From

$$f'(d) = k + \frac{1}{d} + \frac{1}{1 - d} > 0$$

we conclude that function $f(d)$ is strictly monotonically increasing, hence $f(d) = 0$ has an unique solution $d(k)$ in $(0, \frac{1}{2})$.

For $k \to \infty$ we have $d(k) \to 0$, hence

$$\ln(1 - d(k)) = -d(k) + O(d(k))^2).$$

Consider then the function

$$g(d) = (k + 1)d - \ln\frac{1}{d}$$

which approximates function $f$ with $d \to 0$ as $O(d^2)$; in addition $g'(d) > 0$.

So we can further denote by $d(k)$ the (unique) zero of $g$ and study $g$ instead of $f$.

We look for two reals $d_-$ and $d_+$ such that $g(d_-) < 0$ and $g(d_+) > 0$ (leading to $g(d_-) < g(d(k)) < g(d_+)$ and thus $d_- < d(k) < d_+$).

Trying $d_- = \dfrac{1}{k+1}$ and $d_+ = \dfrac{\ln(k+1)}{k+1}$ we obtain

$$g\left(\frac{1}{k+1}\right) = 1 - \ln(k+1) < 0, \ \ g\left(\frac{\ln(k+1)}{k+1}\right) = \ln\ln(k+1) > 0$$

proving the estimates (1.9).

To find a better lower bound we compute

$$g\left(\frac{\ln(k+1)}{k+1}\left(1 - \frac{\ln\ln(k+1)}{\ln(k+1)}\right)\right) = \ln\left(1 - \frac{\ln\ln(k+1)}{\ln(k+1)}\right) < 0.$$

We thus obtain

$$\frac{\ln(k+1)}{k+1} - \frac{\ln\ln(k+1)}{k+1} < d(k) < \frac{\ln(k+1)}{k+1}$$

which implies (1.11).

Some computational examples using relations (1.10)–(1.11) are presented in Table 1.2, see also Figures 1.6, 1.7, resp. Appendix 1.

The last column of Table 1.2 contains the values of $d$ for prescribed values of $k$ computed by solving the nonlinear equation (1.7).

| $k$ | $d_l(k)$ | $d_r(k)$ | $\Delta = d_r - d_l$ | $\varepsilon(k)$ | $d(k)$ by (1.7) |
|---|---|---|---|---|---|
| 2 | 0.334 | 0.366 | 0.032 | 0.0856 | 0.337416 |
| 100 | 0.0305 | 0.0456 | 0.015 | 0.3313 | 0.033592 |
| 1000 | 0.00497 | 0.00691 | 0.0019 | 0.2797 | 0.005245 |
| 10000 | 0.000698 | 0.000921 | 0.00022 | 0.2410 | 0.000723 |

Table 1.2: Bounds for $d(k)$ computed by (1.10)–(1.11) for various rates $k$

**Remarks**

a) For the general case $a \neq 1$ one should substitute everywhere in formulae (1.10)–(1.11) the expression $k + 1$ by $k + a^{-1}$.

b) An estimate similar to (1.11) in integral metric has been obtained in [6].

### 1.2.8   Shifted logistic functions

Here we are interested in arbitrary shifted (horizontally translated) logistic functions.

Both the step function and the logistic function preserve their form under horizontal translation—note that Verhulst equation possess constant isoclines.

Hence the shifted step function $h_r$ is approximated by the shifted logistic function $s_r$ in the same way as function $h_0$ is approximated by the basic logistic function $s_0$, that is for the H-distance we have $\rho(h_r, s_r) = \rho(h_0, s_0)$.

Focusing on the shifted logistic function we have

$$s_r(t) = s_0(t - r) = \frac{a}{1 + e^{-k(t-r)}}. \tag{1.12}$$

The value of the (basic) logistic function (1.6) at the point $-r$ is   $\mathbf{s}_0(-r) = \dfrac{a}{1 + e^{kr}}$. Figure 1.8 visualizes a logistic function shifted with $r = 0.5$.

## 1.3   Kinetic mechanisms yielding Verhulst model

In this section we demostrate that Verhulst model can be derived from certain (bio)chemical reaction equatiosn using mass action kinetics.

It is worth noting that Verhulst model was invented some 30-40 years before the invention of the Mass Action Law.

Let us recall the mass action law in its kinetic aspect.

The Law of Mass Action states that the rate of change in concentration of each reactant in a chemical reaction is proportional to the product of the concentrations of the reactants in that reaction.

$$\frac{1}{æ^{-40\,(t-0.5)} + 1}$$
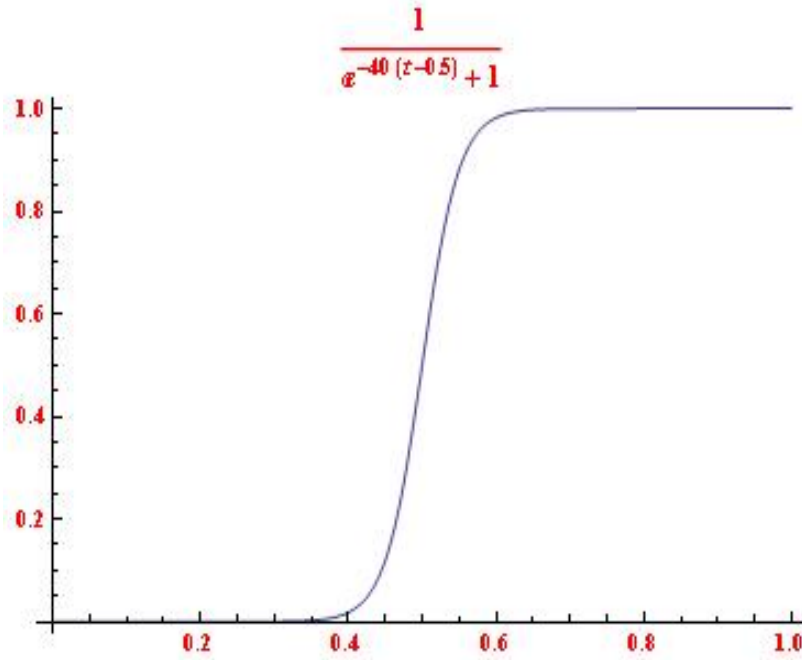
Figure 1.8: A logistic function shifted with $r = 0.5$.

If a particular reactant is involved in several reactions, then the rate of change of this reactant is made by adding up all positive rates and subtracting all negative ones [12].

### 1.3.1  A simple autocatalytic reaction

Consider the following autocatalytic reversible reaction mechanism:

$$X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} X + X, \tag{1.13}$$

which can be also written as $X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$.

Applying the mass action law we obtain the Verhulst model:

$$x' = kx - k_{-1}x^2 = kx\left(1 - \frac{k_{-1}}{k}x\right). \tag{1.14}$$

The stationary point is $x^* = \dfrac{k}{k_{-1}}$.

Another kinetic mechanism inducing Verhulst model that seems theoretically possible and better practically justified follows.
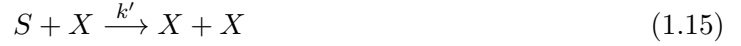
### 1.3.2   Autocatalytic reaction involving nutrient substrate

Consider the following autocatalytic reaction equation:

$$S + X \xrightarrow{k'} X + X \tag{1.15}$$

(or S+X$\xrightarrow{k'}$ $2X$ ), where $S$ is a nutrient substance, $X$ is a population and $k'$ is the specific growth rate of the particular population.

The biological (biochemical) interpretation of reaction equation (1.15) is that the substrate $S$ is utilized by the population $X$ leading to the reproduction of the population (simple binary fusion reproduction in the case of bacterial cells population).

Denoting the biomass (or density) of $X$ by $x$ and the mass (concentration) of $S$ by $s$ and applying the mass action law, one obtains the following dynamical system for functions $s(t), x(t)$:

$$\begin{aligned}
\frac{ds}{dt} &= -k'xs, \\
\frac{dx}{dt} &= k'xs, \ \ s(0) = s_0, \ \ x(0) = x_0.
\end{aligned} \tag{1.16}$$

The solutions $s, x$ of (1.16) for reaction rate $k = 40$ and initial conditions $s_0 = 1$, $x_0 = 1. \times 10^{-9}$ are iluustrated on Fig. 1.9, see Appendix 2.

Noticing that $\frac{ds}{dt} + \frac{dx}{dt} = 0$, hence $s + x = \text{const} = x_0 + s_0 = a$, we can substitute $s = a - x$ in the differential equation for $x$ to obtain the differential equation $\frac{dx}{dt} = k'sx = k'x(a - x)$ also known as Verhulst model [15]–[17]

$$\frac{dx}{dt} = k'x(a - x). \tag{1.17}$$

Clearly, the solution $x$ of the initial problem (1.16) coincides with the solution $x$ of problem (1.17) with initial condition $x(0) = x_0$:

$$\frac{dx}{dt} = k'x(a - x). \ \ x(0) = x_0. \tag{1.18}$$

Conversely, the solution of (1.18) coincides with the solution $x$ of the inital problem (1.16) whenever $s_0 = a - x_0$.

The above can be summarized in the following:
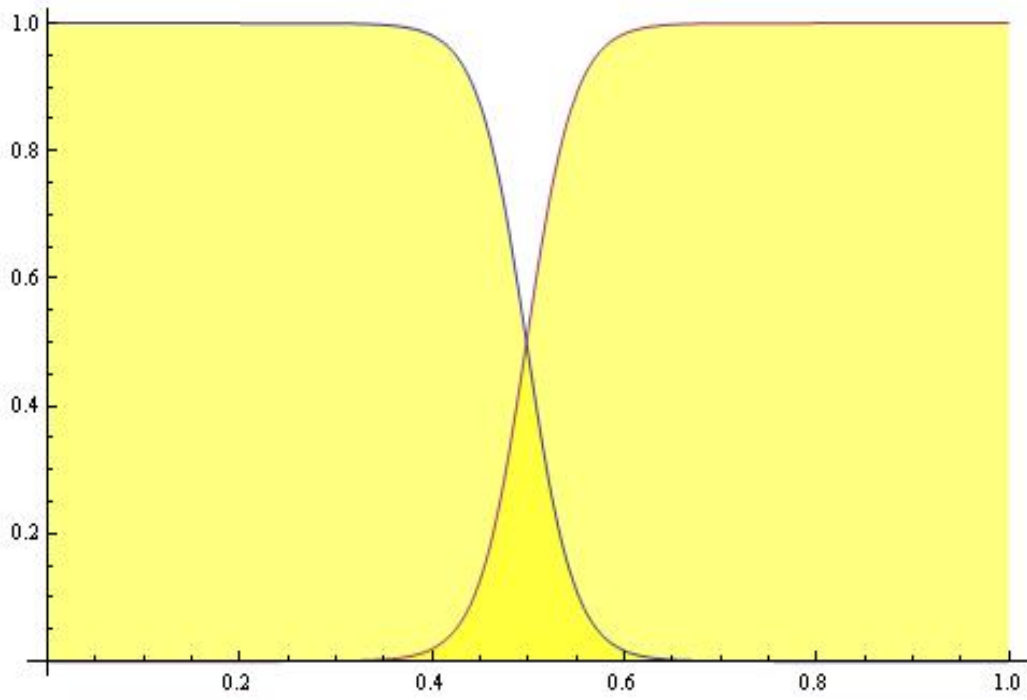
Figure 1.9: Reaction rate $k = 40$; $s_0 = 1$, $x_0 = 1. \times 10^{-9}$

**Theorem 1.3** *The autocatalytic reaction (1.15) via mass action kinetic induces the dynamic model (1.16).*

*Models (1.16) and (1.18) are equivalent in the sense that their solutions $x$ coincide (for $x_0 + s_0 = a$).*

We see that the underlying mechanism in Verhulst model (1.18) is a biochemical reproduction reaction (1.15) based on the utilization of a nutrient substrate.

This explains the important versatile applications of Verhulst model.

The Verhulst model is usually written with a normalized rate constant $k = k'/a$ as

$$\frac{dx}{dt} = \frac{k}{a}x(a - x) = kx\left(1 - \frac{x}{a}\right). \tag{1.19}$$

The solution $x$ to equation (1.20) with initial condition $x_0 = a/2$, is the (basic) logistic sigmoidal function:

$$s_0(t) = \frac{a}{1 + be^{-kt}}; \quad b = \frac{a - x_0}{x_0} = 1,$$

that is (1.6).
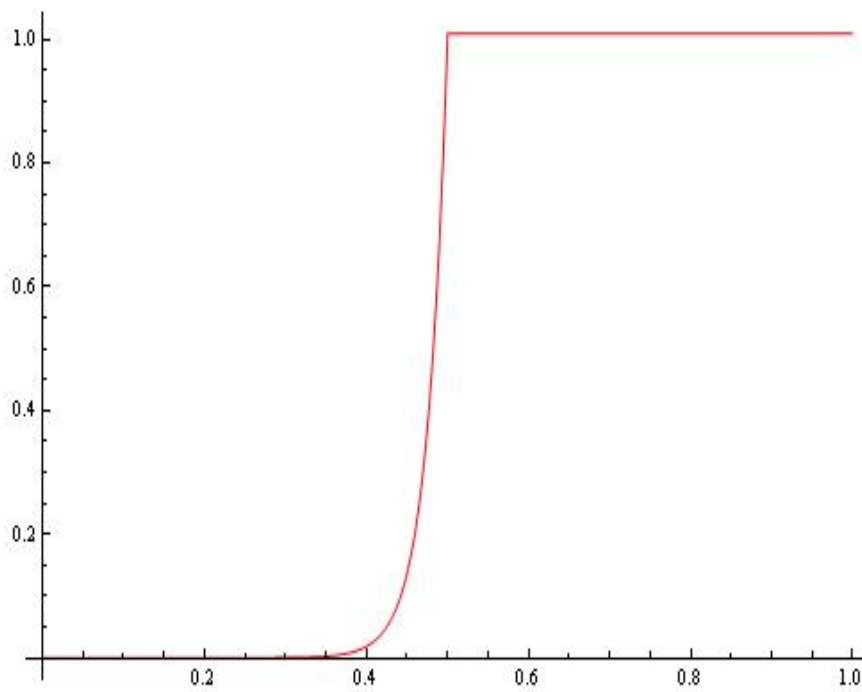
Figure 1.10: Reaction rate $k = 40$, $r = 0.5$, $x_0 = 1.85 \times 10^{-9}$

### 1.3.3   Nutrient supply as input function

The Verhulst model can be considered as a prototype of models used in bioreactor modelling.

There, especially in the case of continuous bioreactore, the nutrient supply is considered as an input function $s(t)$ as follows:

$$\frac{dx}{dt} = kx(t)s(t), \tag{1.20}$$

where $s$ is additionally specified.

The solution $x$ for the input nutrient function $s(t) = \dfrac{Sign(r - t) + 1}{2}$ obtained by a *Mathematica* module is given in Appendix 4, see Fig. 1.10.

### 1.3.4   Computational issues

The shifted logistic function (1.12) can be computed as a solution of the Verhulst model (1.17); in that case we need to know a suitable initial condition for equation (1.17).

The shifted (by $r$) logistic function (1.12) can be considered as solution of the initial problem:

$$\frac{dx}{dt} = \frac{k}{a}x(a - x), \; x(0) = x_0 = a/(1 + e^{kr}). \tag{1.21}$$
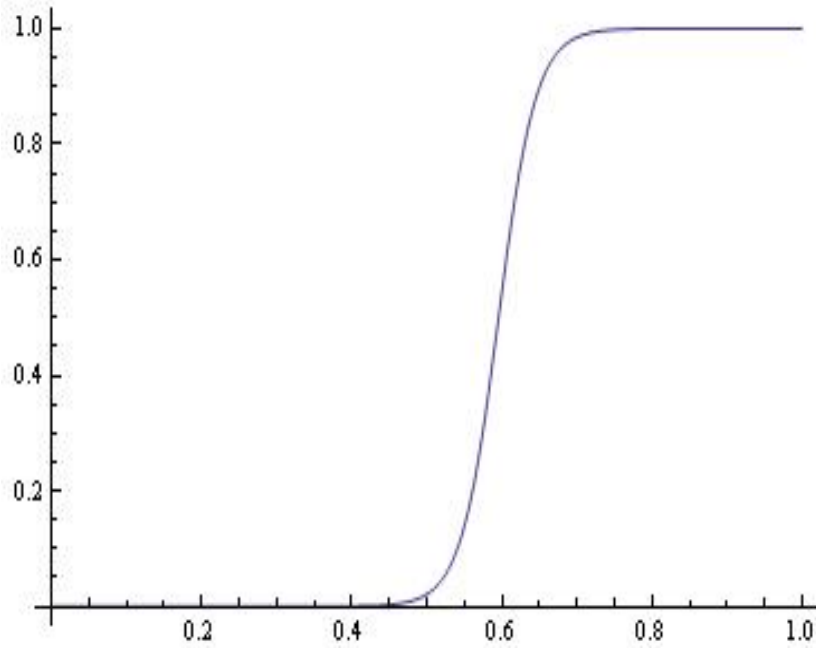
Figure 1.11: Reaction rate $k = 40$; $r = 0.6$

As $r$ increases the computational time for solving problem (1.21) increases rapidly if large values of $k$ have been used (in order to achieve good approximation of the step function).

As an example, on CAS *Mathematica* for $kr > 30$ the computations fails, whenever using formula (1.21) for the computation of the shifted logistic function, see Fig. 1.11 corresponding to a case $kr \le 30$ and Fig. 1.12 for the case $kr > 30$, cf. Appendix 3.

We note that within such an approach very small values (for the distance $d$) and very large values (for the rate $k$) are simultaneously involved in the computation.

In order to reduce the computational problems in CAS *Mathematica* the user should take care with specifying facilities such as AccuracyGoal, PrecisionGoal, and WorkingPrecision.

## 1.4 Conclusions

H-continuous step functions and smooth sigmoid functions are used to model biological dynamic processes, in particular certain enzyme kinetic and population growth processes which develop almost step-wise [9].
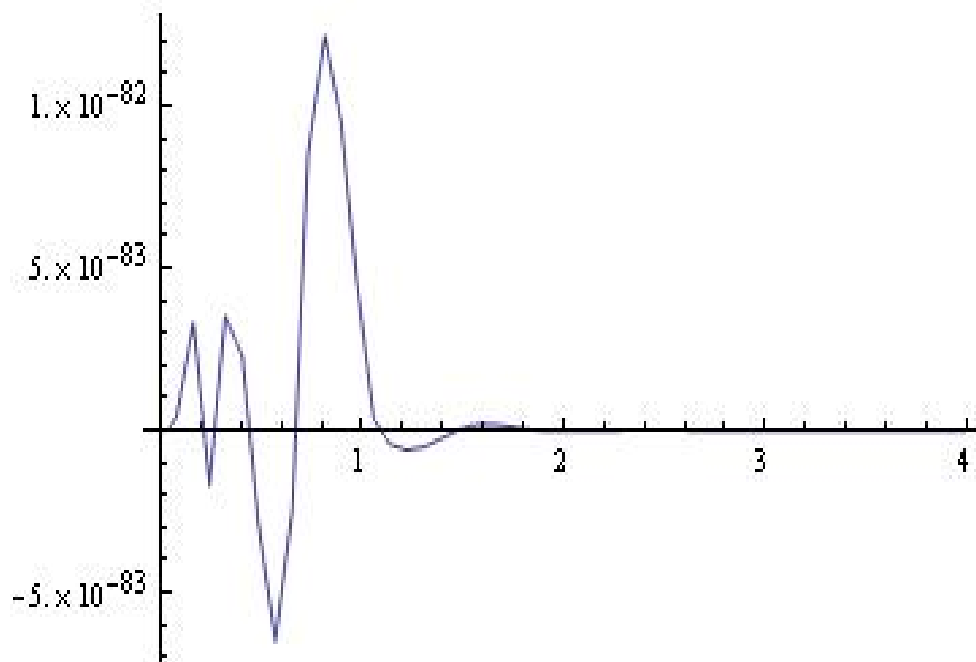
Figure 1.12: Solution of (1.21) fails for $kr > 30$

Such processes are usually described or approximated by smooth sigmoid functions (especially in the theory of neural networks), however, H-continuous step-wise functions can be also conveniently used.

To substitute a sigmoid function by a step function (or conversely) we need to know the approximation error as given in Theorem 1.2.

Biological processes are often very sensitive and can be effectively studied within the frames of interval analysis [8].

Verhulst model is an important classical example involving a simple logistic sigmoid function as solution.

We demonstrate that this model is induced by simple autocatalytic reactions that describe certain reproduction biochemical mechanisms.

On a number of computational examples we demonstrate the applicability of the logistic function to approximate the Heaviside step function and consequently to be emploit in fitting time course experimental data related to population dynamics.

## 1.5 Supplements

### 1.5.1 Supplement 1. More precise bounds for $d(k)$.

In this section we prove more precise boubds for the H–distance $d$ between the Heaviside step function $h_0$ and the sigmoid Verhulst function $s_0$.

The following theorem gives upper and lower bounds for $d(k)$.

**Theorem 1.4** *[27]. For the H–distance $d = \rho(h_0, s_0)$ between the Heaviside step function $h_0$ and the sigmoid Verhulst function $s_0$ the following inequalities hold for $k \geq 2$:*

$$\tilde{d}_l = \frac{\ln(k+1)}{k+1} - \frac{\ln\ln(k+1)}{(k+1)\left(1 + \frac{1}{\ln(k+1)}\right)} < d < \frac{\ln(k+1)}{k+1} + \frac{\ln\ln(k+1)}{(k+1)\left(\frac{\ln\ln(k+1)}{1-\ln(k+1)} - 1\right)} = \tilde{d}_r. \quad (1.22)$$

**Proof.** Evidently, the second derivative of $g$ (see, Theotem 1.2)

$$g''(d) = -\frac{1}{d^2} < 0$$

has a constant sign on $\left[\frac{1}{k+1}, \frac{\ln(k+1)}{k+1}\right]$.

The straight line, defined by the points $\left(\frac{1}{k+1}, g\left(\frac{1}{k+1}\right)\right)$ and $\left(\frac{\ln(k+1)}{k+1}, g\left(\frac{\ln(k+1)}{k+1}\right)\right)$, and the tangent to $g(d)$ at the point $\left(\frac{\ln(k+1)}{k+1}, g\left(\frac{\ln(k+1)}{k+1}\right)\right)$ cross the abscissa at the points

$$\frac{\ln(k+1)}{k+1} + \frac{\ln\ln(k+1)}{(k+1)\left(\frac{\ln\ln(k+1)}{1-\ln(k+1)} - 1\right)},$$

$$\frac{\ln(k+1)}{k+1} - \frac{\ln\ln(k+1)}{(k+1)\left(1 + \frac{1}{\ln(k+1)}\right)},$$

respectively.

This completes the proof of the Theorem 1.4.

We note that the improved bounds (1.22) are more precise than (1.10).
Some computational examples using relations (1.22) are presented in Table 1.3.

We have the following Theorem

**Theorem 1.5** *[27]. Using the same notations and conditions as in Theorem 1.4, we have*

$$\breve{d}_l(k) = \frac{1}{\frac{k}{2} + 2} < d(k) < \frac{\ln(k+1)}{k+1}. \quad (1.23)$$

| $k$ | $\tilde{d}_l(k)$ | $root$ | $\tilde{d}_r(k)$ |
|------|------------------|-----------|------------------|
| 10 | 0.1413863 | 0.1633510 | 0.169081814 |
| 40 | 0.0587838 | 0.0661748 | 0.06900431 |
| 100 | 0.03058471 | 0.0335928 | 0.03505368 |
| 300 | 0.013177444 | 0.0141467 | 0.014736887 |
| 1000 | 0.004971273 | 0.00524519 | 0.00544691 |
| 2000 | 0.0027852088 | 0.00291712 | 0.0030233927 |
| 3000 | 0.0019748445 | 0.00206086 | 0.0021335086 |

Table 1.3: Bounds for $d(k)$ computed by (1.22) for various rates $k$

**Proof.** We define the function

$$G(d) = \frac{k}{2} \ln \frac{1+d}{1-d} - \ln \frac{1}{d} - \ln (1-d). \tag{1.24}$$

Then

$$G(d) - f(d) = \frac{k}{2} \ln \frac{1+d}{1-d} - kd.$$

We examine $G(d) - f(d) \approx 0$, i.e.

$$\frac{k}{2} \ln \frac{1+d}{1-d} = kd,$$

$$\ln \left( \frac{1+d}{1-d} \right)^{\frac{k}{2}} = kd,$$

$$\left( \frac{1+d}{1-d} \right)^{\frac{k}{2}} = e^{kd}.$$

According to [18]

$$\left( \frac{1+d}{1-d} \right)^{\frac{k}{2}} - e^{kd} = O(d^3)$$

we see that the function $G(d)$ approximates $f(d)$ with $d \to 0$ as $O(d^3)$ (cf. Fig.1.13). From

$$G(d) = \frac{k}{2} \ln \frac{1+d}{1-d} + \ln \frac{d}{1-d} = 0$$

we get

$$\frac{k}{2} \ln \frac{1+d}{1-d} = -\ln \frac{d}{1-d} = \ln \frac{1-d}{d},$$

$$\ln \left( \frac{1+d}{1-d} \right)^{\frac{k}{2}} = \ln \frac{1-d}{d},$$

$$\left(\frac{1+d}{1-d}\right)^{\frac{k}{2}} = \frac{1-d}{d},$$

$$d = \frac{1}{1 + \left(\dfrac{1+d}{1-d}\right)^{\frac{k}{2}}}.$$

From Taylor expansion

$$\frac{1}{1 + \left(\dfrac{1+d}{1-d}\right)^{\frac{k}{2}}} = \frac{1}{2} - \frac{kd}{4} + O(d^3)$$

we obtain

$$\breve{d}_l = \frac{1}{\dfrac{k}{2} + 2}.$$

In addition $G'(d) > 0$, $G(\frac{1}{\frac{k}{2}+2}) < 0$ and $G(\frac{\ln(k+1)}{k+1}) > 0$.

This completes the proof of the inequalities (1.23).

Evidently $\breve{d}_l(k) = \dfrac{1}{\frac{k}{2}+2} > \dfrac{1}{k+1} = \dot{d}_l(k)$ for $k > 2$ (see, (1.9)).

We can obtain improved upper and lower bounds for $d(k)$, corresponding to (1.23). The proof follows the ideas given in Theorem 1.4, and will be omitted.

To obtain precise estimates of $d(k)$ one can use the well-known inequality (see [19]):

$$\frac{(d+1)^{\frac{k}{2}}}{d} > \frac{k}{2}\left(1 + \frac{1}{\frac{k}{2}-1}\right)^{\frac{k}{2}-1},$$

which is fulfilled for $kd > 1$, as well as the limit

$$\lim_{k\to\infty}\left(1 + \frac{1}{k}\right)^k = e.$$

We will explicitly note that such improved estimates are labor-intensive from computational point of view, which severely limit the possibility of their practical use.
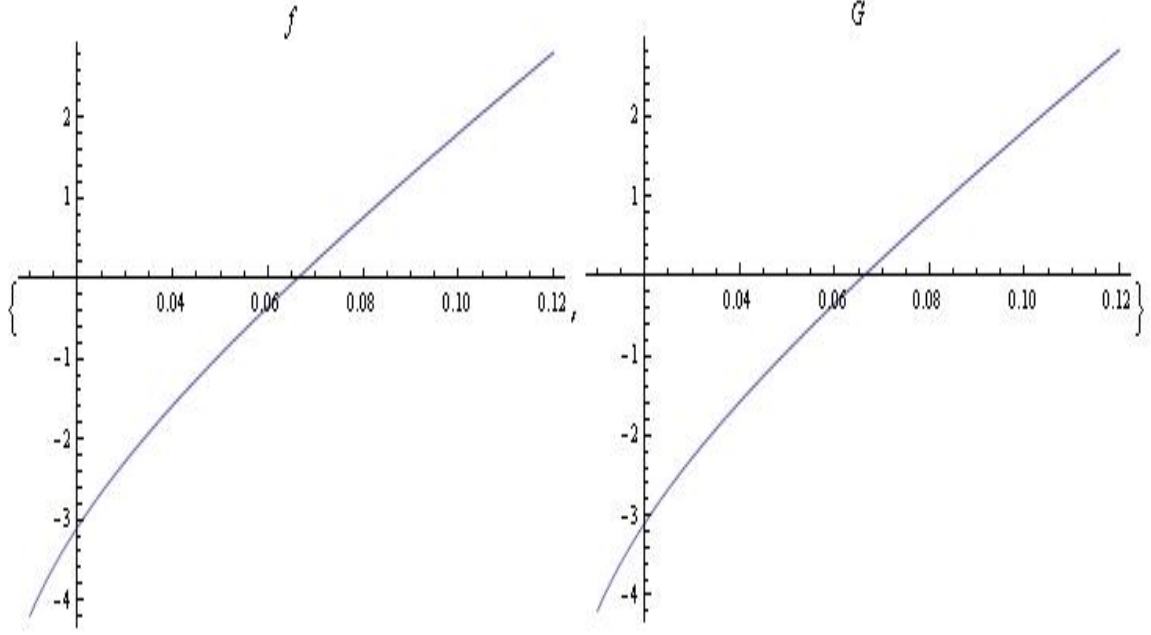
Figure 1.13: The functions $f(d)$ and $G(d)$ for $k = 40$

## 1.5.2   Supplement 2. Approximation of continuous functions by multilayer perceptrons.

The approximation of continuous functions by multilayer perceptrons is considered in [6].

Definition 1. A linear perceptron is a perceptron which given inputs $x_i$, $i \in \{1, \ldots, n\}$ has output

$$Q = \sum_{i \in I} \omega_i x_i + \theta,$$

where $\theta \in R$ is the bias and the $\omega_i \in R$ are the weights, ie, the output is a linear combination of the inputs [24].

Definition 2. A sigmoid perceptron is a perceptron which given inputs $x_i$, $i \in \{1, \ldots, n\}$ has output

$$Q = \frac{1}{1 + e^{-\sum\limits_{i \in I} \omega_i x_i + \theta}},$$

where $\theta \in R$ is the bias and the $\omega_i \in R$ are the weights [25].

The sigmoid perceptron with a simple input $x$ has output function:

$$Q_{(\alpha, \beta)} = \frac{1}{1 + e^{-\alpha(x + \beta)}},$$

Figure 1.14: The perceptron of Rosenblatt [24].
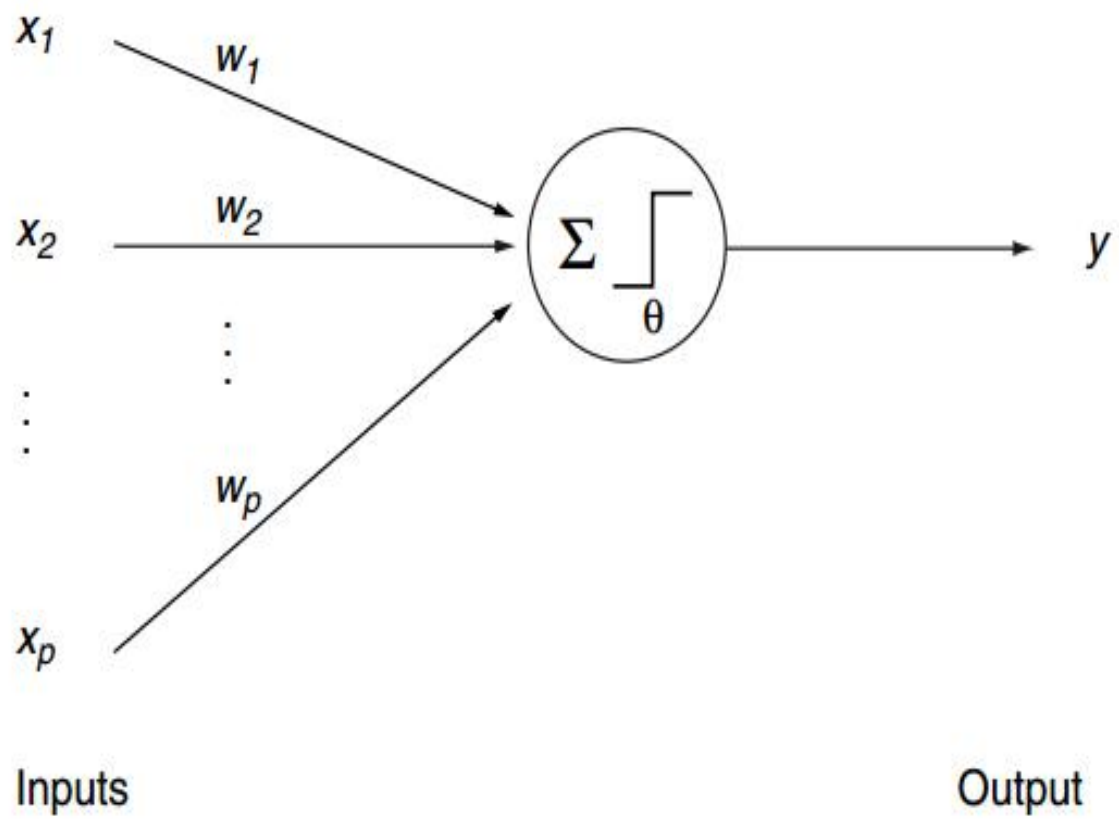
$$y(\mathbf{x}) = \textit{heaviside}(\sum_{j=1}^{p} w_j x_j - \theta)$$

$$= \textit{heaviside}(\sum_{j=0}^{p} w_j x_j) \quad \text{with } w_0 = -\theta, \ x_0 = 1$$
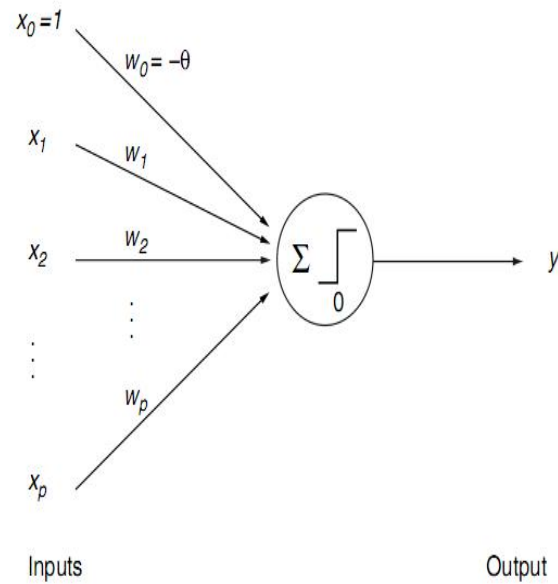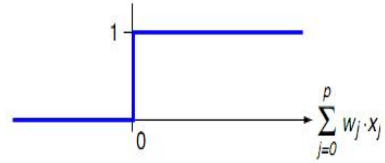


Figure 1.15: Computation in the perceptron (continued) [25].

Now consider the Heaviside step function $H(x)$ given by

$$H(x) = \begin{cases} 0, & \text{if} \quad x < 0, \\ [0,1], & \text{if} \quad x = 0, \\ 1, & \text{if} \quad x > 0. \end{cases}$$

In [6] the following Theorem is proved

**Theorem 1.6** *[6] Given $\epsilon > 0$ we can find $\alpha(\epsilon)$ such that*

$$||H - Q_{(\alpha,0)}||_{[a,b]} < (b-a)\epsilon, \tag{1.25}$$

*where*

$$||f||_{[a,b]} = \int_{[a,b]} |f(x)|dx.$$

Consider the following conditions:

$$Q_{(\alpha,0)}(x) < \epsilon \text{ for } x < -k, \ k \in R^+ \tag{1.26}$$

$$\frac{1+\epsilon}{\epsilon} \quad < \quad e^{-\alpha x} \text{ for } x < -k, \ k \in R^+$$
$$< \quad e^{-\alpha k} \tag{1.27}$$

$$Q_{(\alpha,0)}(x) > 1 - \epsilon \text{ for } x > k. \tag{1.28}$$

Now

$$\alpha k \le \ln \epsilon - \ln(1 + \epsilon). \tag{1.29}$$

This can be achieved by setting

$$\alpha = \frac{1}{k} \left( \ln \epsilon - \ln(1 + \epsilon) \right). \tag{1.30}$$

Here we give an estimate similar to (1.25) in integral metric.

The parameter $\alpha$ can be expressed in terms of the Hausdorff distance $\epsilon$ as follows

$$\alpha = \frac{1}{\epsilon} \left( \ln(1 - \epsilon) - \ln \epsilon \right).$$

From this we obtain the following theorem similar to Theorem 1.6

**Theorem 1.7** *[27]*

$$||H - Q_{(\alpha,0)}||_{[a,b]} < (b-a).\frac{\ln(\alpha+1)}{\alpha+1} \left( 1 + O \left( \frac{\ln \ln(\alpha+1)}{\ln(\alpha+1)} \right) \right). \tag{1.31}$$

About approximation of the cut function by sigmoid and squashing functions see, [26].

**Appendix 1**. Calculation of the value of the Hausdorff distance d between the Heaviside step function $h$ and the sigmoid Verhulst function $s$ in terms of the reaction rate k

```
Print["Calculation of the value of the Hausdorff distance d between the Heaviside step function h
and the sigmoidal Verhulst function s in terms of the reaction rate k "];
k = Input["  k"];(*100 *)
Print["The reaction rate -  k = ", k];
Print["The following nonlinear equation is used to determination of the Hausdorff distance d: "];


m = 1 / (1 + Exp[-d * k]) - 1 + d;


Print[m, " = 0"];
Print["The roots of this equation are: "];
NSolve[m == 0, ε];
Print[TableForm[%]];
Print["The unique positive root of the equation is the searched value of d: "];


positiveReals = Solve[Reduce[{m == 0, d > 0}, d], d];
If[Length[positiveReals] > 0,
  If[Length[positiveReals] == 1,
   Print["There exists an unique ", Length[positiveReals], " positive root: "]
  ];
  Print[TableForm[N[positiveReals]]];
 ];
```

```
Calculation of the value of the Hausdorff distance d between the Heaviside step function h
and the sigmoidal Verhulst function s in terms of the reaction rate k

The reaction rate -  k = 1000

The following nonlinear equation is used to determination of the Hausdorff distance d:
```

$$-1 + d + \frac{1}{1 + e^{-1000\,d}} = 0$$

```
The roots of this equation are:

{}

The unique positive root of the equation is the searched value of d:

There exists an unique 1 positive root:
```

$$d \to 0.00524519$$

**Appendix 2**. A kinetic mechanism yielding Verhulst model

```mathematica
Print["The reaction equation:"];

Print["S + X  →ᵏ   X + X "];

Print["The Law of Mass Action applied to the above reaction leads to the following system
 of differential equations: "];

Print["s'[t] ==-k*s[t]*x[t]"];
Print["x'[t] ==k*s[t]*x[t]"];

k = Input["Input Rate constant -  k"]; (* 40 *)
Print["Rate constant -  k = ", k];

s0 = Input["Input initial condition - s[0]"]; (* 0.999999999 *)
Print["Initial condition s0 = ", s0];

x0 = Input["Input initial condition - x[0]"]; (* 0.000000001 *)
Print["Initial condition x0 = ", x0];

t0 = Input["Input t0"];
Print["t0 = ", t0];

t1 = Input["Input t1"];
Print["t1 = ", t1];

Print["The solutions of the above reaction equation are visualized on the next figure"];

NDSolve[{s'[t] == -k*s[t]*x[t], x'[t] == k*s[t]*x[t],
    s[0] == s0, x[0] == x0}, {s, x}, {t, t0, t1}];

Plot[Evaluate[{s[t], x[t]} /. First[%]], {t, t0, t1}, Filling → Axis,
 FillingStyle -> Directive[Opacity[0.5], Yellow]]
```

**Appendix 3**. The general case

```
Print["x'[t] ==k*x*(1-x)"];
k = Input["Input -  k"]; (* 40 *)
Print[" k = ", k];
r = Input["Input -  r"]; (* 0.6 *)
Print[" r = ", r];
t0 = Input["Input t0"];
Print["t0 = ", t0];
t1 = Input["Input t1"];
Print["t1 = ", t1];


s = NDSolve[{x'[t] == k*x[t]*(1-x[t]), x[0] == 1/(1+Exp[k*r])}, x, {t, t0, t1}];
Plot[Evaluate[x[t] /. s], {t, t0, t1}, PlotRange->All]
```

**Appendix 4**. Nutrient supply as input function

```
Print["x'[t] ==k*x[t]*S[t]"];
Print["S[t] ==(Sign[r-t]+1)/2"];
k = Input["Input -  k"]; (* 40 *)
Print[" k = ", k];
r = Input["Input -  r"]; (* 0.5 *)
Print[" r = ", r];
x0 = Input["Input initial condition - x[0]"]; (* 0.00000000185 *)
Print["Initial condition x0 = ", x0];
t0 = Input["Input t0"];
Print["t0 = ", t0];
t1 = Input["Input t1"];
Print["t1 = ", t1];


sol1 = NDSolve[{x'[t] == k*x[t]*(Sign[r-t]+1)/2, x[0] == x0}, x, {t, t0, t1}];
Plot[Evaluate[x[t] /. sol1], {t, t0, t1}, PlotRange->All, PlotStyle->{Red}]
```

# Chapter 2

# Approximation of the cut function

We study the uniform approximation of the sigmoid cut function by smooth functions such as the logistic and the squashing functions.

The limiting case of the interval-valued step function is also discussed which imposes the use of Hausdorff metric.

Numerical examples are presented using *CAS MATHEMATICA*.

## 2.1 Introduction

In this Chapter we discuss some computational, modelling and approximation issues related to several classes of sigmoid functions.

Sigmoid functions find numerous applications in various fields related to life sciences, chemistrty, physics, artificial inteligence, etc.

In fields such as signal processes, patern recognition, machine learning, neural networks, sigmoid functions of many variables are also known as "activation functions".

A practically important class of sigmoid functions is the class of cut functions including the step function as a limiting case.

Cut functions are continuous but they are not smooth (differentiable) at the two endpoints of the interval where they increase.

In some applications smooth sigmoid functions are preferred; some authors even require smoothness in the definition of sigmoid functions.

Two familiar classes of smooth sigmoid functions are the logistic and squashing functions.

There are situations when one needs to pass from nonsmooth sigmoid functions (e. g. cut functions) to smooth sigmoid functions, and vice versa.

Such a necessity rises in a natural way the issue of approximating nonsmooth sigmoid functions by smooth sigmoid functions.

Section 2.2 contains preliminary definitions and motivations.

In Section 2.3 we study the uniform and Hausdorff approximation of the cut functions by logistic functions.

We find an expression for the error of the best uniform approximation.

Curiously, the uniform distance between a cut function and the logistic function of best uniform approximation is an absolute constant (not depending on the slope $k$).

By contrast, it turns out that the Hausdorff distance (H-distance) depends on the slope $k$ and tends to zero with $k \to \infty$.

Showing that the family of logistic functions cannot approximate the cut function arbitrary well, we then consider the limiting case when the cut function tends to the Heaviside step function (in Hausdorff sense).

In this way we obtain an extension of a previous result on the Hausdorff approximation of the step function by logistic functions [2].

In Section 4 we discuss the approximation of the cut function by the family of squashing functions.

Reminding first a known result on the uniform approximation [26], we then propose a new estimate for the H-distance between a cut function and its best approximating squashing function.

Our estimate is then extended to cover the limiting case of the Heaviside step function.

Numerical examples are presented using the computer algebra system *MATHEMATICA*.

## 2.2   Preliminaries

### 2.2.1   Sigmoid functions

Sigmoid functions can be defined as bounded monotone non-decreasing functions on the real line.

One usually makes use of normalized sigmoid functions defined as monotone non-decreasing functions $s(t), t \in$, such that $\lim s(t)_{t \to -\infty} = 0$ and $\lim s(t)_{t \to \infty} = 1$.

In the fields of neural networks and machine learning sigmoid-like functions of many variables are used, familiar under the name *activation functions*.

### 2.2.2   Piece-wise linear sigmoid functions: cut function

Let $\Delta = [\gamma - \delta, \gamma + \delta]$ be an interval on the real line with centre $\gamma \in R$ and radius $\delta \in R^+$.

A cut function (on $\Delta$) is defined as follows:

The cut function $c_\Delta$ on $\Delta$ is defined for $t \in R$ by

$$c_\Delta(t) = c_{\gamma,\delta}(t) = \begin{cases} 0, & \text{if } t < \Delta, \\ \dfrac{t - \gamma + \delta}{2\delta}, & \text{if } t \in \Delta, \\ 1, & \text{if } \Delta < t. \end{cases} \tag{2.1}$$

Note that the slope of function $c_\Delta(t)$ on the interval $\Delta$ is $1/(2\delta)$ (the slope is constant in the whole interval $\Delta$).

Two special cases and a limiting case are of interest for our discussion in the sequel.

**Special case 1**.
For $\gamma = 0$ we obtain the special cut function on the interval $\Delta = [-\delta, \delta]$:

$$c_{0,\delta}(t) = \begin{cases} 0, & \text{if } t < -\delta, \\ \dfrac{t + \delta}{2\delta}, & \text{if } -\delta \le t \le \delta, \\ 1, & \text{if } \delta < t. \end{cases} \tag{2.2}$$

**Special case 2**.
For $\gamma = \delta$ we obtain the special cut function on the interval $\Delta = [0, 2\delta]$:

$$c_{\delta,\delta}(t) = \begin{cases} 0, & \text{if } t < 0, \\ \dfrac{t}{2\delta}, & \text{if } 0 \le t \le 2\delta, \\ 1, & \text{if } 2\delta < t. \end{cases} \tag{2.3}$$

**A limiting case**.
If $\delta \to 0$, then $c_{\delta,\delta}$ tends (in Hausdorff sense) to the Heaviside step function

$$c_0 = c_{0,0}(t) = \begin{cases} 0, & \text{if } t < 0, \\ [0, 1], & \text{if } t = 0, \\ 1, & \text{if } t > 0, \end{cases} \tag{2.4}$$

which is an interval-valued function [2], [23], [47], [50].

To prove that (2.3) tends to (2.4) let $h$ be the H-distance using a square (box) unit ball between the step function (2.4) and the cut function (2.3).

By the definition of H-distance, $h$ is the side of the unit square, hence we have $1 - c_{\delta,\delta}(h) = h$, that is $1 - h/(2\delta) = h$, implying

$$h = \frac{2\delta}{1 + 2\delta} = 2\delta + O(\delta^2). \tag{2.5}$$

For the sake of simplicity throughout the paper we shall work with the special cut function (2.3) instead of the more general (arbitrary shifted) cut function (2.1); this special choice will not lead to any loss of generality concerning the results obtained.

### 2.2.3  Smooth sigmoid functions: logistic function

Smooth sigmoid functions are prized because their derivatives are easy to calculate.

The most popular smooth sigmoid functions are the Gompertz, the logistic and the squashing functions.

The Gompertz functions are introduced by Benjamin Gompertz for the study of demographic phenomena [43].

The sigmoid Gompertz function $\sigma_{\alpha,\beta}(t)$ is defined for $\alpha$, $\beta > 0$ by: $\sigma_{\alpha,\beta}(t) = a \exp(-\alpha \exp(-\beta t))$, where $a$ is the upper asymptote when $t \to \infty$.

Gompertz functions find applications in biology and medecine, e.g., in modeling tumor growth [33], [37] [51].

Various approximation results for sigmoid functions, including Gompertz functions, are obtained in [4], [38].

Below we shall focus on the logistic functions intrduced by P.-F. Verhulst [15]–[17].

In what follows we shall be interested in the approximation of cut function by smooth sigmoid functions, more specifically the logistic and squashing functions.

## 2.3    Approximation of the cut function by logistic functions

Define the logistic (Verhulst) function $v$ on $R$ as [15]

$$v_{\gamma,k}(t) = \frac{1}{1 + e^{-4k(t-\gamma)}}. \tag{2.6}$$

 Note that the logistic function (2.6) has an inflection at its "centre" $\gamma$ and its slope at $\gamma$ is equal to $k$.

**Theorem 2.1** *The function $v_{\gamma,k}(t)$ defined by (2.6) with $k = \dfrac{1}{2\delta}$: i) is the logistic function of best uniform one-sided approximation to function $c_{\gamma,\delta}(t)$ in the interval $[\gamma, \infty)$ (as well as in the interval $(-\infty, \gamma]$); ii) approximates the cut function $c_{\gamma,\delta}(t)$ in uniform metric with an error*

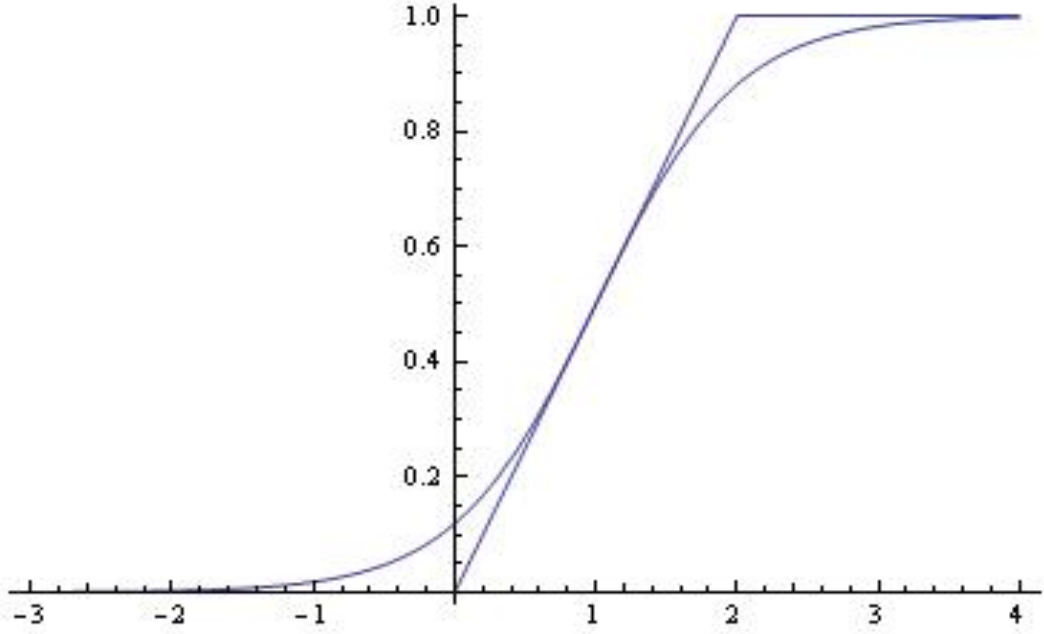$$\rho = \rho(c, v) = \frac{1}{1 + e^2} = 0.11920292.... \tag{2.7}$$

Figure 2.1: The cut and logistic functions for $\gamma = \delta = 1$, $k = 1/2$.

**Proof.** Consider functions (2.1) and (2.6) with same centres $\gamma = \delta$, that is functions $c_{\delta,\delta}$ and $v_{\delta,k}$.

In addition chose $c$ and $v$ to have same slopes at their coinciding centres, that is assume $k = \dfrac{1}{2\delta}$, cf. Figure 2.1.

Then, noticing that the largest uniform distance between the cut and logistic functions is achieved at the endpoints of the underlying interval $[0, 2\delta]$, we have:

$$\rho = v_{\delta,k}(0) - c_{\delta,\delta}(0) = \frac{1}{1 + e^{4k\delta}} = \frac{1}{1 + e^2}. \tag{2.8}$$

This completes the proof of the theorem.

We note that the uniform distance (2.8) is an absolute constant that does not depend on the width of the underlying interval $\Delta$, resp. on the slope $k$.

The next theorem shows that this is not the case whenever H-distance is used.

**Theorem 2.2** *The function $v_{\gamma,k}(t)$ with $k = \frac{1}{2\delta}$ is the logistic function of best Hausdorff one-sided approximation to function $c_{\gamma,\delta}(t)$ in the interval $[\gamma, \infty]$ (resp. in the interval $[-\infty, \gamma]$).*

*The function $v_{\gamma,k}(t)$, $k = \frac{1}{2\delta}$, approximates function $c_{\gamma,\delta}(t)$ in H-distance with an error $h = h(c, v)$ that satisfies the relation:*

$$\ln \frac{1 - h}{h} = 2 + 4kh. \tag{2.9}$$

**Proof.** Using $\delta = \frac{1}{2k}$ we can write $\delta + h = \frac{1+2hk}{2k}$, resp.:

$$v(-\delta - h) = \frac{1}{1 + e^{2(1+2hk)}}. \tag{2.10}$$

The H-distance $h$ using square unit ball (with a side $h$) satisfies the relation $v(-\delta - h) = h$, which implies (2.9).

This completes the proof of the theorem.

Relation (2.9) shows that the H-distance $h$ depends on the slope $k$, $h = h(k)$.

The next result gives additional information about this dependence.

**Theorem 2.3** *For the H-distance $h(k)$ the following holds for $k > 5$:*

$$\frac{1}{4k+1} < h(k) < \frac{\ln(4k+1)}{4k+1}. \tag{2.11}$$

**Proof.** We need to express $h$ in terms of $k$, using (2.9).

Let us examine the function

$$f(h) = 2 + 4hk - \ln(1-h) - \ln\frac{1}{h}. \tag{2.12}$$

From

$$f'(h) = 4k + \frac{1}{1-h} + \frac{1}{h} > 0 \tag{2.13}$$

we conclude that function $f$ is strictly monotone increasing.

Consider function

$$g(h) = 2 + h(1 + 4k) - \ln\frac{1}{h}. \tag{2.14}$$

Then $g(h) - f(h) = h + \ln(1-h) = O(h^2)$ using the Taylor expansion $\ln(1-h) = -h + O(h^2)$. Hence $g(h)$ approximates $f(h)$ with $h \to 0$ as $O(h^2)$.

In addition $g'(h) = 1 + 4k + 1/h > 0$, hence function $g$ is monotone increasing.

Further, for $k \geq 5$

$$g\left(\frac{1}{1+4k}\right) = 3 - \ln(1+4k) < 0,$$

$$g\left(\frac{\ln(4k+1)}{4k+1}\right) = 2 + \ln\ln(1+4k) > 0.$$

This completes the proof of the theorem.

Relation (2.11) implies that when the slope $k$ of functions $c$ and $v$ tends to infinity, the h-distance $h(c,v)$ between the two functions tends to zero (differently to the uniform distance $\rho(c,v)$ which remains constant).

The following proposition gives more precise upper and lower bounds for $h(k)$.

For brevity denote $K = 4k + 1$.

**Theorem 2.4** *For the H-distance h the following inequalities hold for $k \geq 5$:*

$$\frac{\ln K}{K} - \frac{2 + \ln \ln K}{K \left(1 + \frac{1}{\ln K}\right)} < h(k) < \frac{\ln K}{K} + \frac{2 + \ln \ln K}{K \left(\frac{\ln \ln K}{1 - \ln K} - 1\right)}, \tag{2.15}$$

*where $K = 4k + 1$.*

**Proof.** Evidently, the second derivative of $g(h) = 2 + h(1 + 4k) - \ln \frac{1}{h}$, namely $g''(h) = -\frac{1}{h^2} < 0$, has a constant sign on $\left[\frac{1}{K}, \frac{\ln K}{K}\right]$.

The straight line, defined by the points $\left(\frac{1}{K}, g\left(\frac{1}{K}\right)\right)$ and $\left(\frac{\ln K}{K}, g\left(\frac{\ln K}{K}\right)\right)$, and the tangent to $g$ at the point $\left(\frac{\ln K}{K}, g\left(\frac{\ln K}{K}\right)\right)$ cross the abscissa at the points

$$\frac{\ln K}{K} + \frac{2 + \ln \ln K}{K \left(\frac{\ln \ln K}{1 - \ln K} - 1\right)}, \quad \frac{\ln K}{K} - \frac{2 + \ln \ln K}{K \left(1 + \frac{1}{\ln K}\right)},$$

respectively.

This completes the proof of the theorem.

Theorems (2.2), (2.3) and (2.4) extend similar results from [2] stating that the Heaviside interval-valued step function is approximated arbitrary well by logistic functions in Hausdorff metric.

The Hausdorff approximation of the Heaviside step function by sigmoid functions is discussed from various computational and modelling aspects in [20].

## 2.4 Approximation of the cut function by squashing functions

The results obtained in Section 2.3 state that the cut function cannot be approximated arbitrary well by the family of logistic functions.

This result justifies the study of other families of smooth sigmoid functions having better approximating properties.

Such are the squashing functions to be considered below.

The squashing function $s_\Delta$ on the interval $\Delta = [\gamma - \delta, \gamma + \delta]$ is defined by

$$s_\Delta^{(\beta)}(t) = s_{\gamma,\delta}^{(\beta)}(t) = \frac{1}{2\delta} \ln \left(\frac{1 + e^{\beta(t - \gamma + \delta)}}{1 + e^{\beta(t - \gamma - \delta)}}\right)^{\frac{1}{\beta}}. \tag{2.16}$$

Note that the squashing function (2.16) has an inflection at its "centre" $\gamma$ and its slope at $\gamma$ is equal to $(2\delta)^{-1}$.

The squashing function (2.16) with centre $\gamma = \delta$:

$$s_{\delta,\delta}^{(\beta)}(t) = \frac{1}{2\delta} \ln \left( \frac{1 + e^{\beta t}}{1 + e^{\beta(t-2\delta)}} \right)^{\frac{1}{\beta}}, \tag{2.17}$$

approximates the cut function (2.3).

Indeed, functions $c_{\delta,\delta}$ and $s_{\gamma,\delta}^{(\beta)}$ have same centre $\gamma = \delta$ and equal slopes $\frac{1}{2\delta}$ at their coinciding centres.

As in the case with the logistic function, one observes that the uniform distance $\rho = \rho(c, s)$ between the cut and squashing function is achieved at the endpoints of the interval $\Delta$, more specifically at the origin.

Denoting the width of the interval $\Delta$ by $w = 2\delta$ we obtain

$$\rho = s_{\delta,\delta}^{(\beta)}(0) = \frac{1}{w} \ln \left( \frac{2}{1 + e^{\beta(-w)}} \right)^{\frac{1}{\beta}} < \frac{\ln 2}{w} \cdot \frac{1}{\beta} = \text{const } \frac{1}{\beta}. \tag{2.18}$$

The estimate (2.18) has been found by Dombi and Gera [26].

This result shows that any cut function $c_\Delta$ can be approximated arbitrary well by squashing functions $s_\Delta^{(\beta)}$ from the class (2.16).

The approximation becomes better with the increase of the value of the parameter $\beta$.

Thus $\beta$ affects the quality of the approximation; as we shall see below the practically interesting values of $\beta$ are integers greater than 4.

In what follows we aim at an analogous result using Hausdorff distance.

Let us fix again the centres of the cut and squashing functions to be $\gamma = \delta$ so that the form of the cut function is $c_{\delta,\delta}$, namely (2.3), whereas the form of the squashing function is $s_{\delta,\delta}^{(\beta)}$ as given by (2.17).

Both functions $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ have equal slopes $\frac{1}{w}$, $w = 2\delta$, at their centres $\delta$.

Denoting the square-based H-distance between $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ by $d = d(w; \beta)$, we have the relation

$$s_{\delta,\delta}^{(\beta)}(w + d) = \frac{1}{w} \ln \left( \frac{1 + e^{\beta(w+d)}}{1 + e^{\beta d}} \right)^{\frac{1}{\beta}} = 1 - d$$
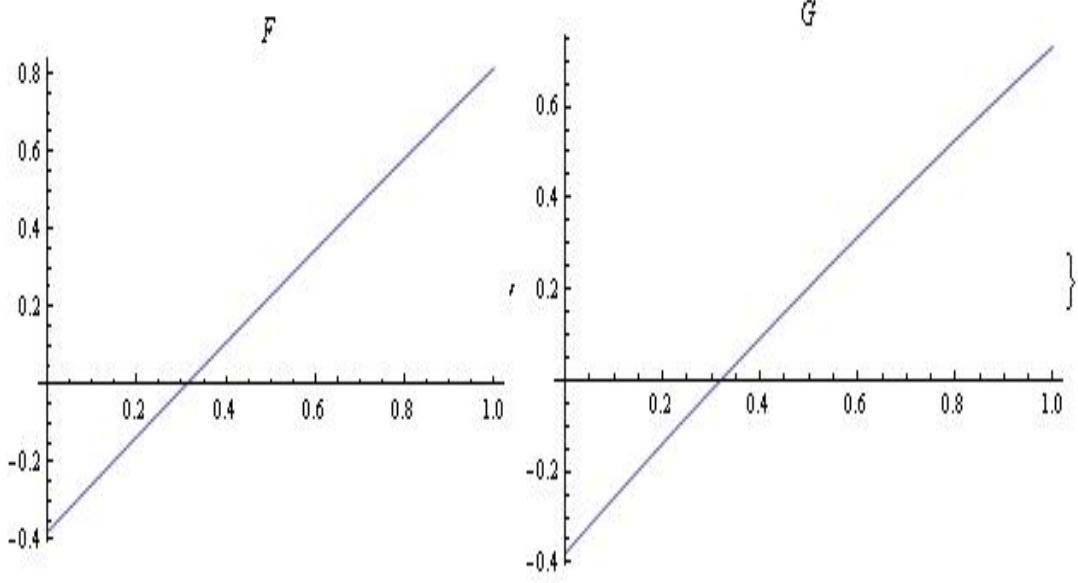
or

$$\ln \frac{1 + e^{\beta(w+d)}}{1 + e^{\beta d}} = \beta w (1 - d). \tag{2.19}$$

The following theorem gives an upper bound for $d = d(w; \beta)$ as implicitly defined by (2.19):

**Theorem 2.5** *For the distance $d$ the following holds for $\beta \geq 5$:*

$$d < \ln 2 \, \frac{\ln(4\beta w + 1)}{4w\beta + 1}. \tag{2.20}$$

Figure 2.2: The functions $F(d)$ and $G(d)$.

**Proof.** We examine the function:

$$F(d) = -\beta w(1-d) + \ln(1 + e^{\beta(w+d)}) + \ln \frac{1}{1 + e^{\beta d}}. \tag{2.21}$$

From $F'(d) > 0$ we conclude that function $F(d)$ is strictly monotone increasing. We define the function

$$G(d) = -\beta w + \ln(1 + e^{\beta w}) + d\beta \left( w + \frac{e^{\beta w}}{1 + e^{\beta w}} \right) + \ln \frac{1}{1 + e^{\beta d}}. \tag{2.22}$$

We examine $G(d) - F(d)$:

$$G(d) - F(d) = \ln(1 + e^{\beta w}) + \frac{e^{\beta w} \beta d}{1 + e^{\beta w}} - \ln(1 + e^{\beta(w+d)}).$$

From Taylor expansion

$$\ln(1 + e^{\beta(w+d)}) = \ln(1 + e^{\beta w}) + \frac{e^{\beta w} \beta d}{1 + e^{\beta w}} + O(d^2)$$

we see that function $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (cf. Fig. 2.2).

In addition $G(0) < 0$ and $G\left(\ln 2. \frac{\ln(4\beta w + 1)}{4w\beta + 1}\right) > 0$ for $\beta \geq 5$.

This completes the proof of the theorem.

Some computational examples using relation (2.20) for various $\beta$ and $w$ are presented in Table 2.1.

| $w$ | $\beta$ | $d(w;\beta)$ from   (2.19) | $d(w;\beta)$ from   (2.20) |
|---|---|---|---|
| 1 | 30 | 0.016040 | 0.027472 |
| 5 | 10 | 0.012639 | 0.018288 |
| 6 | 100 | 0.001068 | 0.002247 |
| 14 | 5 | 0.009564 | 0.013908 |
| 50 | 100 | 0.000137 | 0.000343 |
| 500 | 1000 | $1.38 \times 10^{-6}$ | $5.02 \times 10^{-6}$ |
| 1000 | 5000 | $1.3 \times 10^{-7}$ | $5.8 \times 10^{-7}$ |

Table 2.1: Bounds for $d(w;\beta)$ computed by (14) and (15) respectively
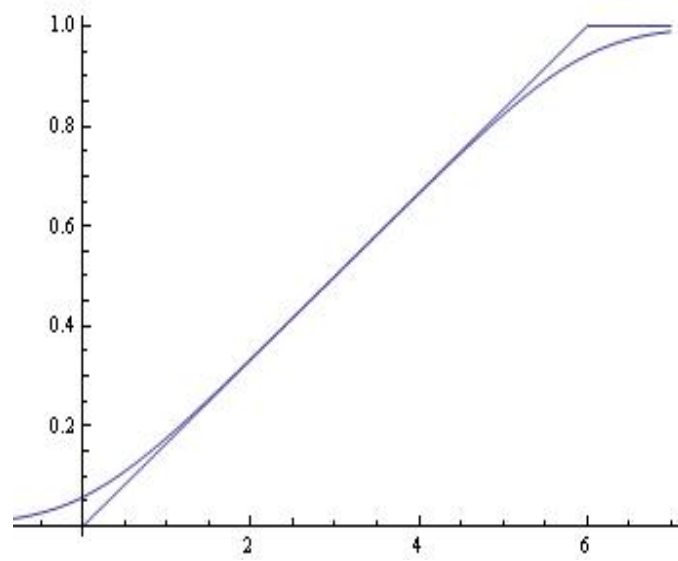


Figure 2.3: Functions $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ for $\beta = 2$, $w = 6$; $d = 0.0534283$.

The numerical results are plotted in Fig. 2.3 (for the case $\beta = 5$, $w = 3$; $d = 0.0398921$), Fig.2.4 (for the case $\beta = 2$, $w = 6$; $d = 0.0534283$) and Fig.2.5 (for the case $\beta = 10$, $w = 4$; $d = 0.0154697$).

**Conclusions**. In this chapter we discuss several computational, modelling and approximation issues related to two familiar classes of sigmoid functions—these are the families of logistic and squashing functions.

Both classes find numerous applications in various fields of life sciences [20], [9], [13].

We study the uniform and Hausdorff approximation of the cut functions by logistic functions.

We demonstrate that the best uniform approximation between a cut function and the respective logistic function is an absolute constant not depending on the slope $k$.
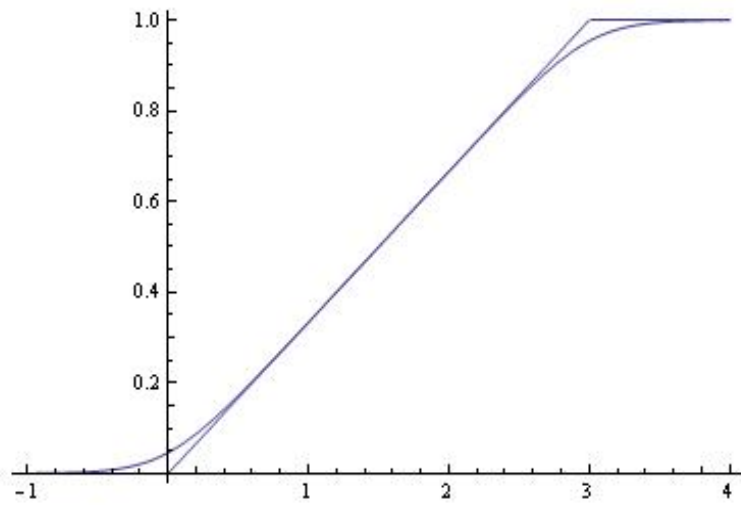
Figure 2.4: Functions $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ for $\beta = 5$, $w = 3$; $d = 0.0398921$.
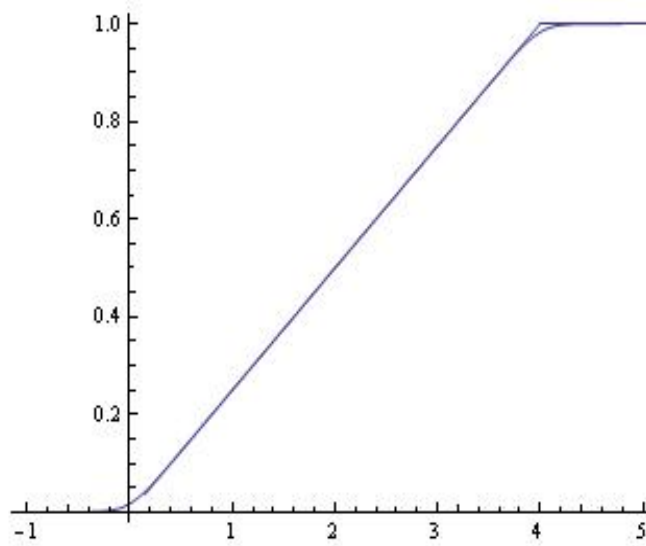


Figure 2.5: Functions $c_{\delta,\delta}$ and $s_{\delta,\delta}^{(\beta)}$ for $\beta = 10$, $w = 4$; $d = 0.0154697$.

On the other side we show that the Hausdorff distance (H-distance) depends on the slope $k$ and tends to zero whenever $k \to \infty$.

We also discuss the limiting case when the cut function tends to the step function in Hausdorff sense.

The approximation of the cut function by the family of squashing functions is also discussed.

We propose a new estimate for the H-distance between a cut function and its best approximating squashing function.

Our estimate extends an well-known result stating that the cut function can be approximated arbitrary well by squashing functions [26].

Our estimate is also extended to cover the limiting case of the step function.

For basic results on H-continuous functions and their application to problems in abstract areas such as Real Analysis, Approximation Theory, Set-valued Analysis and Fuzzy Sets and Systems we recommend [23], [39]– [42].

**Appendix**.  The Module "Computation of the distance $d$ and visualization of the cut function $c_\Delta$ and squashing function $s_\Delta^{(\beta)}$" in CAS MATHEMATICA

```
Print["Calculation of the value of the distance d (see Eq. (14)) and graphical visualization
of the generalized cut function [x]_{0,r} and the interval squashing function S_{0,r}^k"];


k = Input["  k"]; (*5 *)
Print["The parameter -  k = ", k];


r = Input["  r"]; (*5 *)
Print["The parameter -  r = ", r];


Print["The following nonlinear equation is used to determination of the distance d: "];


m = Log[(1 + Exp[k * (r + d)]) / (1 + Exp[d * k])] - k * r * (1 - d);


Print[m, "  = 0"];
Print["The unique positive root of the equation is the searched value of d:"];
FindRoot[m == 0, {d, 0}];
Print[TableForm[%]];


Print["Graphical visualization of the generalized cut function [x]_{0,r}:"];
pw = Piecewise[{{0, x ≤ 0}, {x / r, 0 < x < r}, {1, r ≤ x}}]
g1 = Plot[pw, {x, -1, 7}]


Print["Graphical visualization of the the interval squashing function S_{0,r}^k:"];
g2 = Plot[1 / r * Log[((1 + Exp[k * t]) / (1 + Exp[k * (t - r)]))^{1 / k}], {t, -1, 7}]


Print["Comparing of both graphical visualizations:"];
Show[g1, g2]
```

Figure 2.6: Module in programming environment MATHEMATICA.

Calculation of the value of the distance d (see Eq. (14)) and graphical visualization
of the generalized cut function [x]_{0,r} and the interval squashing function S_{0,r}^k

The parameter -  k = 5

The parameter -  r = 5

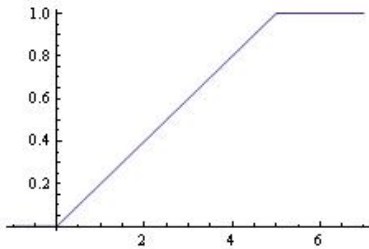The following nonlinear equation is used to determination of the distance d:

$$-25\,(1-d) + \mathrm{Log}\left[\frac{1 + e^{5\,(5+d)}}{1 + e^{5\,d}}\right] \;=\; 0$$

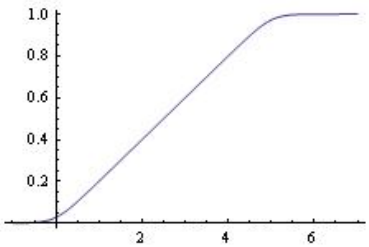The unique positive root of the equation is the searched value of d:

$d \rightarrow 0.0252779$

Graphical visualization of the generalized cut function [x]_{0,r}:

$$\begin{cases} 0 & x \leq 0 \\ \frac{x}{5} & 0 < x < 5 \\ 1 & 5 \leq x \\ 0 & \text{True} \end{cases}$$



Graphical visualization of the the interval squashing function S_{0,r}^k:
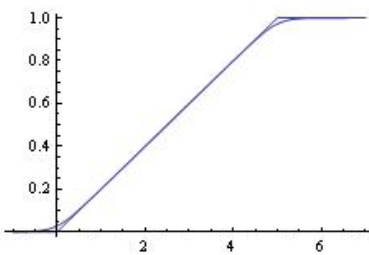


Comparing of both graphical visualizations:



Figure 2.7: The test provided on our control example.

# Chapter 3

# Log–logistic and transmuted log–logistic functions

In this Chapter the Hausdorff approximation of the step function by sigmoid log–logistic and transmuted log–logistic functions arising from biological applications is considered and precise upper and lower bounds for the Hausdorff distance are obtained. Numerical examples, that illustrate our results are given, too.

## 3.1 Introduction

The log–logistic distribution is used to model in fields such as biostatistics, population dynamic, medical research [29] and economics.

The log–logistic distribution is also known as the Fisk distribution.

Definition. The cumulative distribution function $f(t; a, b, c)$ is defined for $b, c > 0$ by:

$$f(t; a, b, c) = \frac{1}{1 + \left(\dfrac{t - a}{b}\right)^{-c}}.$$  (3.1)

**Special case**.
For $b = -a$ we obtain the special log–logistic function:

$$f(t; a, b = -a, c) = \frac{1}{1 + \left(\dfrac{t - a}{-a}\right)^{-c}},$$  (3.2)

for which

$$f(0; a, -a, c) = \frac{1}{2}.$$

The Fisk function, or log–logistic function have is introduced by P. Fisk [28].

Shaw et al. [30], Gupta et. al. [31] study a new model which generalizes the log–logistic function [77].

Definition.  The cumulative distribution function $H(t; a, b, c, \lambda)$ for the generalized log–logistic model is defined for $b, c > 0$, $-1 < \lambda < 1$ by:

$$H(t; a, b, c, \lambda) = \frac{1 + \lambda}{1 + \left(\frac{t - a}{b}\right)^{-c}} - \frac{\lambda}{\left(1 + \left(\frac{t - a}{b}\right)^{-c}\right)^2}. \tag{3.3}$$

We examine the special case

$$H(0; a, b, c, \lambda) = \frac{1}{2} = \frac{1 + \lambda}{1 + \left(\frac{-a}{b}\right)^{-c}} - \frac{\lambda}{\left(1 + \left(\frac{-a}{b}\right)^{-c}\right)^2}, \quad 0 < \lambda < 1. \tag{3.4}$$

Let

$$u = \frac{1}{1 + \left(\frac{-a}{b}\right)^{-c}}. \tag{3.5}$$

From (3.4) we have

$$\lambda u^2 - (1 + \lambda)u + \frac{1}{2} = 0, \quad u_{1,2} = \frac{1 + \lambda \pm \sqrt{1 + \lambda^2}}{2\lambda}.$$

We are interested in the solution $u = \dfrac{1 + \lambda - \sqrt{1 + \lambda^2}}{2\lambda}$.

**Special case**.

The special transmuted log–logistic function for $b, c > 0$, $0 < \lambda < 1$ is defined by:

$$H(t; a, b, c, \lambda) = \frac{1 + \lambda}{1 + \left(\frac{t - a}{b}\right)^{-c}} - \frac{\lambda}{\left(1 + \left(\frac{t - a}{b}\right)^{-c}\right)^2},$$

$$\frac{1}{1 + \left(\frac{-a}{b}\right)^{-c}} = \frac{1 + \lambda - \sqrt{1 + \lambda^2}}{2\lambda}, \tag{3.6}$$
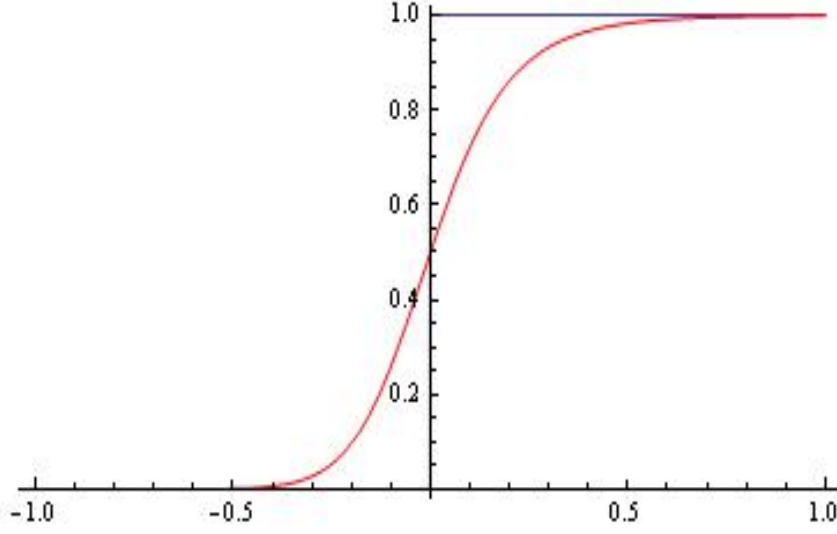
$$H(0; a, b, c, \lambda) = \frac{1}{2}.$$

Figure 3.1: The log–logistic function (3.2) with $a = -1$, $c = 10$; H–distance $d = 0.1709984$.

## 3.2 Approximation of the step function by log-logistic function

We study the Hausdorff approximation [14] of the Heaviside step function $h_0(t)$ by sigmoid log–logistic functions of the forms (3.2) and find an expression for the error of the best approximation.

The Hausdorff distance $d = d(a, c)$ between the Heaviside step function $h_0(t)$ and the sigmoid log–logistic function $f(t; a, -a, c)$ satisfies the relation

$$f(d; a, -a, c) = \frac{1}{1 + \left(\dfrac{d - a}{-a}\right)^{-c}} = 1 - d, \tag{3.7}$$

or

$$\ln \frac{1 - d}{d} - c \ln \left(1 - \frac{d}{a}\right) = 0. \tag{3.8}$$

The following Theorem gives upper and lower bounds for $d(a, c)$

**Theorem 3.1** *The Hausdorff distance $d = d(a, c)$ between the Heaviside step function $h_0$ and the sigmoid log–logistic function (3.2) can be expressed in terms of the parameters $a < 0$ and $c > 0$ for any real $-\frac{c}{a} \geq 2$ as follows:*

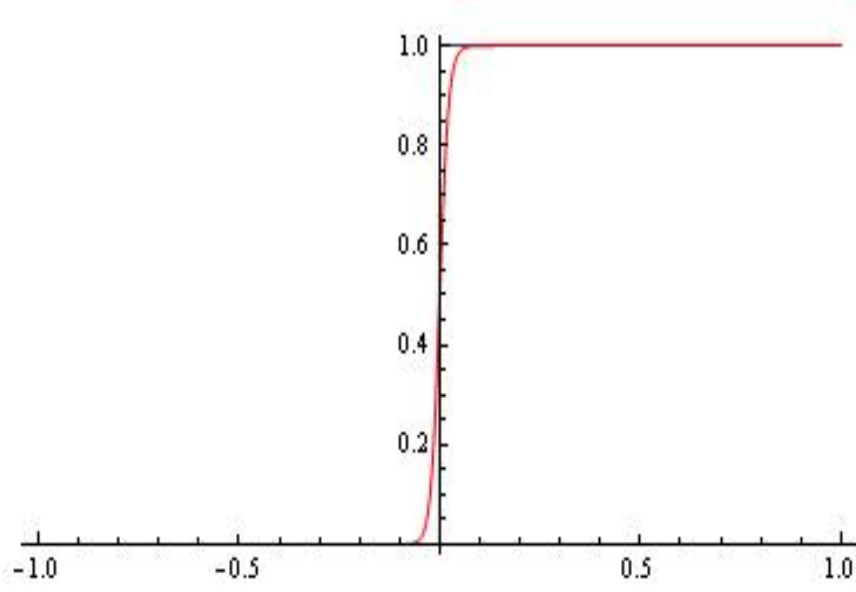$$\frac{1}{1 - \dfrac{c}{a}} < d < \frac{\ln \left(1 - \dfrac{c}{a}\right)}{1 - \dfrac{c}{a}}. \tag{3.9}$$

Figure 3.2: The log–logistic function (3.2) with $a = -1$, $c = 90$; H–distance $d = 0.0369065$.

**Proof.** Let us examine the function (see, (3.8)

$$F(d) = c \ln \left( 1 - \frac{d}{a} \right) - \ln \frac{1-d}{d} = c \ln \left( 1 - \frac{d}{a} \right) - \ln(1-d) - \ln \frac{1}{d}. \qquad (3.10)$$

From

$$F'(d) = -\frac{c}{a} \cdot \frac{1}{1 - \frac{d}{a}} + \frac{1}{1-d} + \frac{1}{d} > 0 \qquad (3.11)$$

we conclude that the function $F$ is strictly monotonically increasing.

Consider function

$$G(d) = \left( 1 - \frac{c}{a} \right) d - \ln \frac{1}{d}. \qquad (3.12)$$

From Taylor expansion

$$\left( 1 - \frac{c}{a} \right) d - c \ln \left( 1 - \frac{d}{a} \right) + \ln(1-d) = O(d^2)$$

we obtain

$$G(d) - F(d) = \left( 1 - \frac{c}{a} \right) d - c \ln \left( 1 - \frac{d}{a} \right) + \ln(1-d) = O(d^2).$$

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$.

In addition

$$G'(d) = 1 - \frac{c}{a} + \frac{1}{d} > 0.$$

Further, for $-\frac{c}{a} \geq 2$ we have

$$G\left(\frac{1}{1-\frac{c}{a}}\right) = 1 - \ln\left(1 - \frac{c}{a}\right) < 0,$$

$$G\left(\frac{\ln\left(1-\frac{c}{a}\right)}{1-\frac{c}{a}}\right) = \ln\ln\left(1 - \frac{c}{a}\right) > 0.$$

This completes the proof of the theorem.

Some computational examples using relation (3.8) are presented in Table 3.1 for various $a$ and $c$.

| $a$ | $c$ | $d(a,c)$ |
|---|---|---|
| $-1$ | 1 | 0.1709984 |
| $-1$ | 90 | 0.0369065 |
| $-2$ | 10 | 0.241953 |
| $-2$ | 100 | 0.0569386 |
| $-3$ | 100 | 0.0759188 |
| $-3$ | 500 | 0.0226684 |

Table 3.1: Bounds for $d$ computed by (3.8) for various $a$ and $c$.

We prove next more precise bounds. Using the same notations as in Theorem 3.1, we state the following

**Theorem 3.2** . *For $a, b, s, d \in R$*

$$\frac{\ln(1-\frac{c}{a})}{1-\frac{c}{a}} - \frac{\ln\ln(1-\frac{c}{a})}{\left(1-\frac{c}{a}\right)\left(1+\frac{1}{\ln(1-\frac{c}{a})}\right)} < d < \frac{\ln(1-\frac{c}{a})}{1-\frac{c}{a}} + \frac{\ln\ln(1-\frac{c}{a})}{\left(1-\frac{c}{a}\right)\left(\frac{\ln\ln(1-\frac{c}{a})}{1-\ln(1-\frac{c}{a})}-1\right)} \qquad (3.13)$$

**Proof.** Evidently, the second derivative of $G(d)$

$$G''(d) = -\frac{1}{d^2} < 0$$

has a constant sign on $\left[\frac{1}{1-\frac{c}{a}}, \frac{\ln(1-\frac{c}{a})}{1-\frac{c}{a}}\right].$

The straight line, defined by the points $\left(\frac{1}{1-\frac{c}{a}}, G(\frac{1}{1-\frac{c}{a}})\right)$ and $\left(\frac{\ln(1-\frac{c}{a})}{1-\frac{c}{a}}, G(\frac{\ln(1-\frac{c}{a})}{1-\frac{c}{a}})\right),$
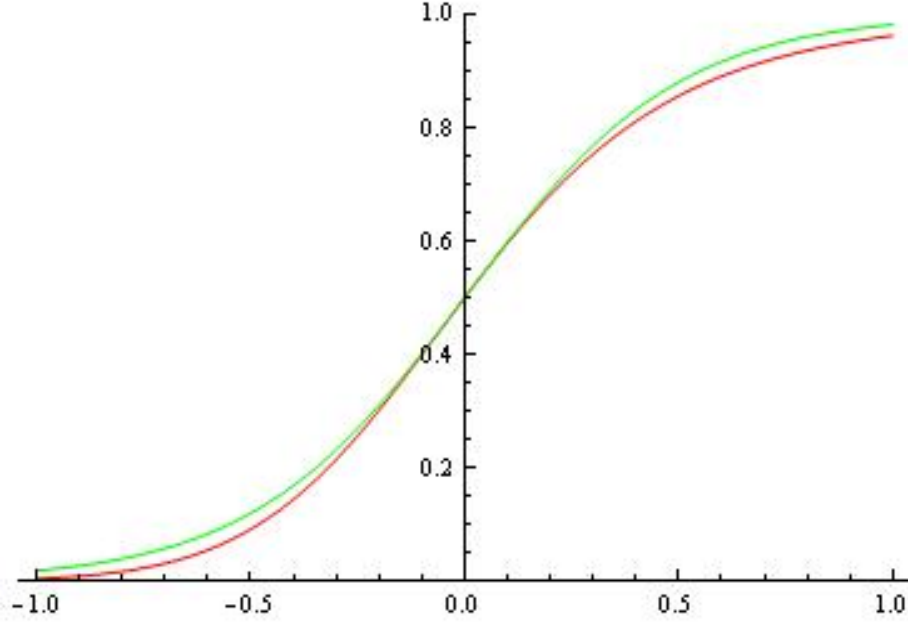
Figure 3.3: Comparison of the log–logistic function (3.2) with $a = -2$, $b = -a$, $c = 8$ and Verhulst logistic finction for $k = -\frac{c}{a} = 4$.

and the tangent to $G(d)$ at the point $\left( \dfrac{\ln(1 - \frac{c}{a})}{1 - \frac{c}{a}}, G(\dfrac{\ln(1 - \frac{c}{a})}{1 - \frac{c}{a}}) \right)$ cross the abscissa at the points

$$\frac{\ln(1 - \frac{c}{a})}{1 - \frac{c}{a}} + \frac{\ln \ln(1 - \frac{c}{a})}{(1 - \frac{c}{a}) \left( \frac{\ln \ln(1 - \frac{c}{a})}{1 - \ln(1 - \frac{c}{a})} - 1 \right)},$$

$$\frac{\ln(1 - \frac{c}{a})}{1 - \frac{c}{a}} - \frac{\ln \ln(1 - \frac{c}{a})}{(1 - \frac{c}{a}) \left( 1 + \frac{1}{\ln(1 - \frac{c}{a})} \right)},$$

respectively.

This completes the proof of the theorem.

We note that the improved bounds (3.13) are more precise than (3.9).

**Remark.** We note that for large $k = -\frac{c}{a}$ the bounds (3.13) and (3.14) are consistent.

Some comparisons of the log–logistic function (3.2) and Verhulst logistic finction for various $a$, $b$ $c$ and $k$ are plotted in Fig.3.3 and Fig.3.4.
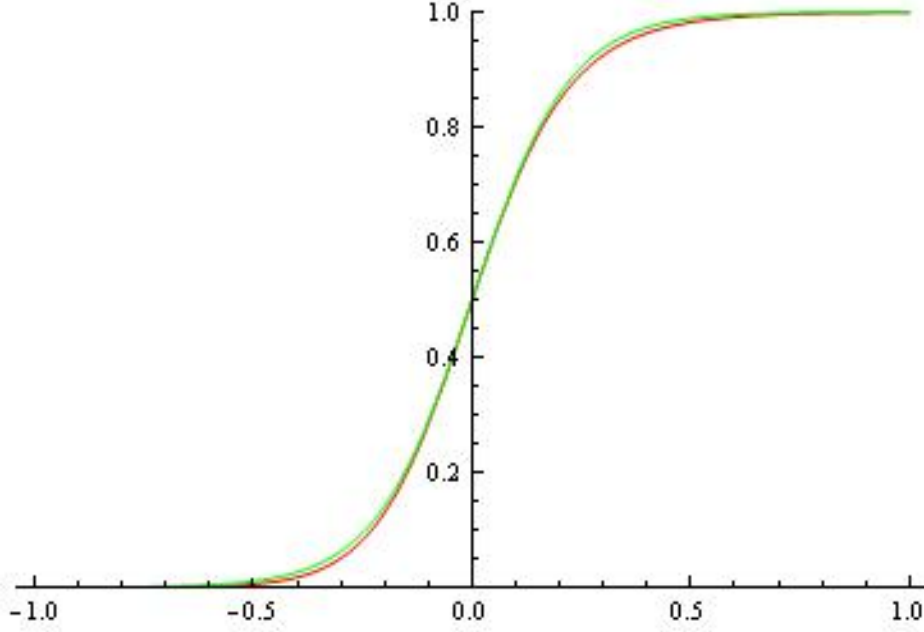
Figure 3.4: Comparison of the log–logistic function (3.2) with $a = -2$, $b = -a$, $c = 18$ and Verhulst logistic finction for $k = -\frac{c}{a} = 9$.

## 3.3 Approximation of $h_0(t)$ by transmuted log-logistic function

We study the Hausdorff distance between the Heaviside step function $h_0(t)$ and the transmuted sigmoid log–logistic functions of the forms (3.6) and find an expression for the error of the best approximation.

$$H(d; a, b, c, \lambda) = 1 - d = \frac{1 + \lambda}{1 + \left(\frac{d-a}{b}\right)^{-c}} - \frac{\lambda}{\left(1 + \left(\frac{d-a}{b}\right)^{-c}\right)^2} \tag{3.14}$$

or

$$\frac{1 + \lambda}{1 + \left(\frac{d-a}{b}\right)^{-c}} - \frac{\lambda}{\left(1 + \left(\frac{d-a}{b}\right)^{-c}\right)^2} - 1 + d = 0. \tag{3.15}$$

We note that from (3.6) we find

$$c = \frac{\ln \frac{1 + \lambda - \sqrt{1 + \lambda^2}}{\lambda - 1 + \sqrt{1 + \lambda^2}}}{\ln \left(-\frac{a}{b}\right)} \tag{3.16}$$
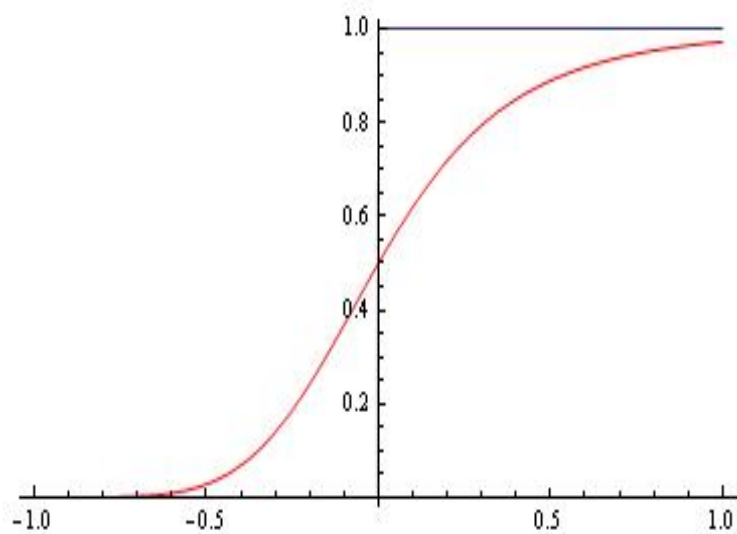
Figure 3.5: The transmuted log–logistic function (3.6) with $a = -1$, $h = 1.05$, $b = -ah = 1.05$, $\lambda = 0.25$, $c = 5.07206$; H–distance $d = 0.244635$.
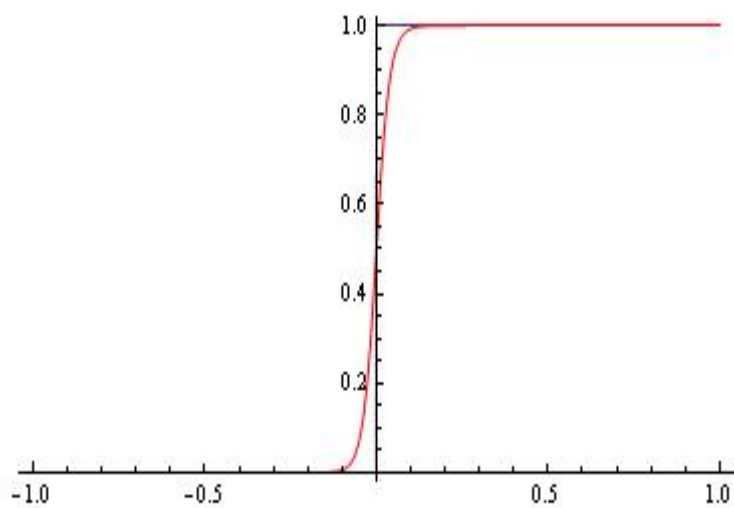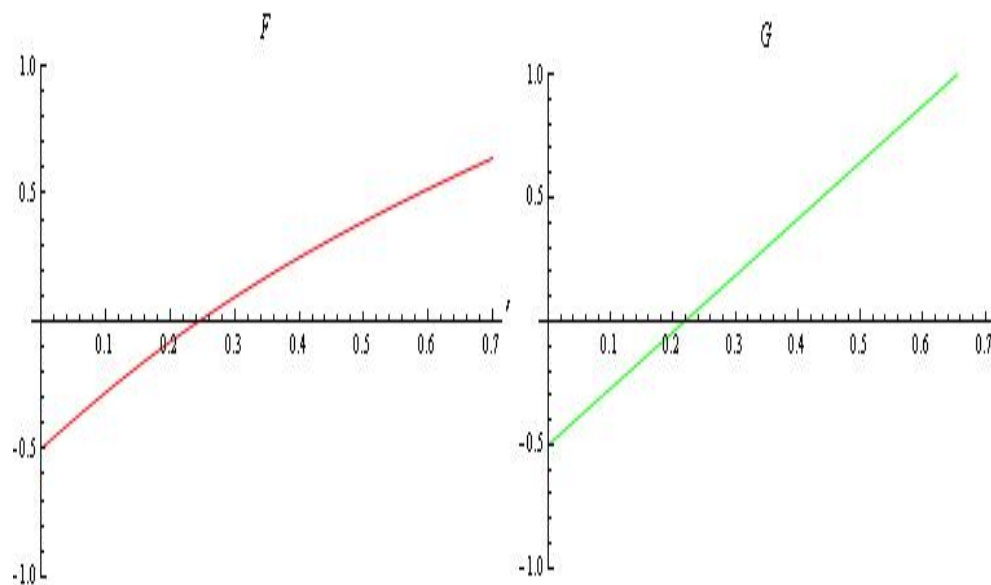


Figure 3.6: The transmuted log–logistic function (3.6) with $a = -1$, $h = 1.01$, $b = -ah = 1.01$, $\lambda = 0.5$, $c = 48.3614$; H–distance $d = 0.0563904$.

Figure 3.7: The functions $F(d, a, b, c, \lambda)$ and $G(d, a, b, c, \lambda)$.

Some computational examples using relation (3.15) are presented in Table 3.2 for various $a$, $h$, $b = -ah$, $\lambda$ and $c$.

| $a$ | $k$ | $b = ah$ | $\lambda$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $-1$ | 1.01 | 1.01 | 0.25 | 24.8702 | 0.0940255 |
| $-1$ | 1.01 | 1.01 | 0.5 | 48.3614 | 0.0563904 |
| $-1$ | 1.05 | 1.05 | 0.25 | 5.07206 | 0.244635 |
| $-0.8$ | 1.05 | 0.84 | 0.75 | 14.2067 | 0.108187 |
| $-2$ | 1.001 | 2.002 | 0.5 | 481.452 | 0.0162858 |
| $-3$ | 1.0001 | 3.0003 | 0.75 | 6931.82 | 0.00233178 |
| $-4$ | 1.00001 | 4.00004 | 0.9 | 80887.1 | 0.000324614 |

Table 3.2: Bounds for $d$ computed by (3.15) for various $a$, $k$, $b = -ak$, $\lambda$ and $c$.

**Theorem 3.3** *For the Hausdorff distance $d$ between the Heaviside step function $h_0(t)$ and the transmuted log–logistic function (3.6) the following lower bound holds*

$$\frac{t_1}{t_2} < d \tag{3.17}$$

*where*

$$t_1 = \frac{1 + \left(-\frac{a}{b}\right)^c (1 - \lambda)}{\left(1 + \left(-\frac{a}{b}\right)^c\right)^2}$$

$$t_2 = 1 - \frac{2\left(-\frac{a}{b}\right)^{-1-c} c\lambda}{\left(1 + \left(-\frac{a}{b}\right)^{-c}\right)^3 b} + \frac{\left(-\frac{a}{b}\right)^{-1-c} c(1 + \lambda)}{\left(1 + \left(-\frac{a}{b}\right)^{-c}\right)^2 b}$$

**Proof.** Let

$$F(d, a, b, c, \lambda) = \frac{1 + \lambda}{1 + \left(\frac{d - a}{b}\right)^{-c}} - \frac{\lambda}{\left(1 + \left(\frac{d - a}{b}\right)^{-c}\right)^2} - 1 + d$$

and

$$G(d, a, b, c, \lambda) = \frac{-1 + \left(-\frac{a}{b}\right)^c (\lambda - 1)}{(1 + \left(-\frac{a}{b}\right)^c)^2} + \left(1 - \frac{2\left(-\frac{a}{b}\right)^{-1-c} c\lambda}{\left(1 + \left(-\frac{a}{b}\right)^{-c}\right)^3 b} + \frac{\left(-\frac{a}{b}\right)^{-1-c} c(1 + \lambda)}{\left(1 + \left(-\frac{a}{b}\right)^{-c}\right)^2 b}\right) d$$

We note that from Taylor expansion

$$G(d) - F(d) = O(d^2)$$

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see, Fig.3.7).

From this we obtain (3.17).

# Chapter 4

# The Gompertz sigmoid function

In this Chapter we study the Hausdorff approximation of the Heaviside step function by the Gompertz function and obtain precise upper and lower bounds for the Hausdorff distance. Numerical examples, illustrating our results are given.

## 4.1    Introduction

Definition 1. The Gompertz function $\sigma_{\alpha,\beta}(t)$ is defined for $\alpha$, $\beta > 0$ by [43]:

$$\sigma_{\alpha,\beta}(t) = ae^{-\alpha e^{-\beta t}}, \tag{4.1}$$

where $a$ is the upper asymptote when time approaches $+\infty$.

**Special case**. For $\alpha^* = 0.69314718...$ and $a = 1$ we obtain the special Gompertz function:

$$\sigma_{\alpha^*,\beta}(t) = e^{-\alpha^* e^{-\beta t}}, \tag{4.2}$$

such that

$$\sigma_{\alpha^*,\beta}(0) = \frac{1}{2}.$$

Definition 2. The Heaviside step function $h_0(t)$ is defined by

$$h_0(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ [0,1], & \text{if} \quad t = 0 \\ 1, & \text{if} \quad t > 0 \end{cases} \tag{4.3}$$

Gompertz functions, are introduced by Benjamin Gompertz for the study of his demographic model, which represents a refinement of the Malthus model.

Thse functions find applications in modeling tumor growth and in population aging description [33], [37], [52], [54–59], [65–76].

The Hausdorff approximation of the Heaviside interval step function by sigmoid functions is discussed from various computational and biological applications in [2], [20] and [32].
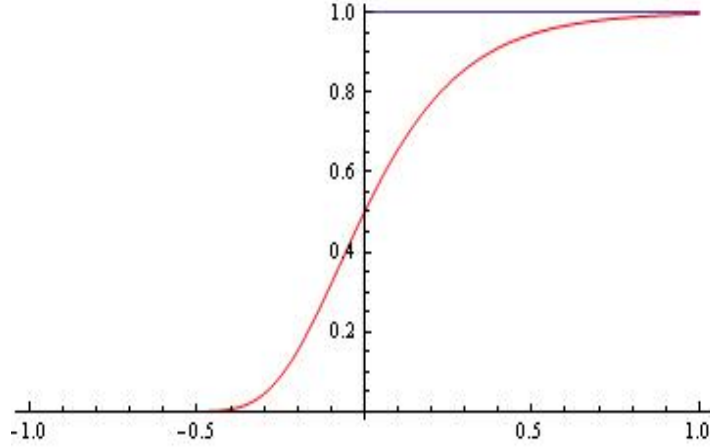
Figure 4.1: The Gompertz function with $\alpha = 0.69314718...$ and $\beta = 5$; H–distance $d = 0.212765$.

For other results, see [21], [22] and [23].

We study the Hausdorff approximation [14] of the Heaviside step function $h_0(t)$ by sigmoid Gompertz functions of the form (4.2) and find an expression for the error of the best approximation.

## 4.2   Approximation of the step function by Gompertz function

The Hausdorff distance $d = d(\alpha, \beta)$ between the Heaviside step function $h_0(t)$ and the sigmoid Gompertz function $\sigma_{\alpha,\beta}(t)$ satisfies the relation

$$\sigma_{\alpha,\beta}(d) = e^{-\alpha e^{-\beta d}} = 1 - d, \tag{4.4}$$

or

$$\ln(1 - d) + \alpha e^{-\beta d} = 0. \tag{4.5}$$

The following theorem gives upper and lower bounds for $d(\alpha, \beta)$

**Theorem 4.1** *The Hausdorff distance $d = d(\alpha, \beta)$ between the step function $h_0$ and the sigmoid Gompertz function $\sigma_{\alpha,\beta}(t)$ ($\alpha = 0.69314718...$) can be expressed in terms of the parameter $\beta$ for any real $\beta \geq 2$ as follows:*

$$\frac{2\alpha - 1}{1 + \alpha\beta} < d < \frac{\ln(1 + \alpha\beta)}{1 + \alpha\beta}. \tag{4.6}$$

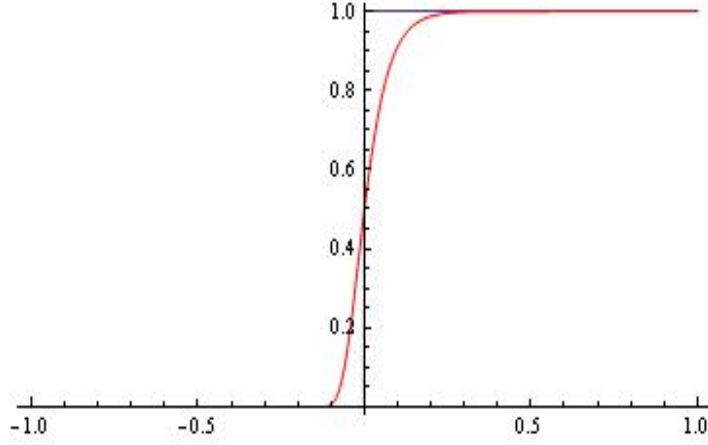**Proof.** We need to express $d$ in terms of $\alpha$ and $\beta$, using (4.5).

Figure 4.2: The Gompertz function with $\alpha = 0.69314718...$ and $\beta = 20$; H–distance $d = 0.0962215$.

Let us examine the function

$$F(d) = \ln(1 - d) + \alpha e^{-\beta d}. \tag{4.7}$$

From

$$F'(d) = -\frac{1}{1 - d} - \alpha \beta e^{-\beta d} < 0 \tag{4.8}$$

we conclude that the function $F$ is strictly monotone decreasing.

Consider function

$$G(d) = \alpha - (1 + \alpha\beta)d. \tag{4.9}$$

From Taylor expansion

$$\alpha - (1 + \alpha\beta)d - \ln(1 - d) - \alpha e^{-\beta d} = O(d^2)$$

we obtain $G(d) - F(d) = \alpha - (1 + \alpha\beta)d - \ln(1 - d) - \alpha e^{-\beta d} = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$.

In addition $G'(d) = -(1 + \alpha\beta) < 0$.

Further, for $\beta \geq 2$

$$G\left(\frac{2\alpha - 1}{1 + \alpha\beta}\right) = 1 - \alpha > 0,$$

$$G\left(\frac{\ln(1 + \alpha\beta)}{1 + \alpha\beta}\right) = \alpha - \ln(1 + \alpha\beta) < 0.$$

This completes the proof of the theorem.

Some computational examples using relation (4.5) are presented in Table 4.1.

| $\beta$ | $d(\alpha, \beta)$ |
|---|---|
| 2 | 0.310825 |
| 3 | 0.267231 |
| 5 | 0.212765 |
| 10 | 0.147136 |
| 20 | 0.0962215 |
| 50 | 0.0514763 |
| 100 | 0.0309364 |
| 500 | 0.00873829 |
| 1000 | 0.00494117 |

Table 4.1: Bounds for $d(0.69314718..., \beta)$ computed by (4.5) for various $\beta$

The calculation of the value of the Hausdorff distance between the Gompertz sigmoid function and the Heaviside function is given in Appendix 1.

The numerical results are plotted in Fig.4.1 (for the case $\alpha^* = 0.69314718...$, $\beta = 5$, H–distance $d = 0.212765$) and Fig. 4.2. (for the case $\alpha^* = 0.69314718...$, $\beta = 20$, H–distance $d = 0.0962215$).

## 4.3   Remarks

**1.** Focusing on the shifted logistic function

$$s_r(t) = s_0(t - r) = \frac{1}{1 + e^{-k(t-r)}} \tag{4.10}$$

and the shifted Gompertz function

$$\sigma_{\alpha,\beta,r}(t) = e^{-\alpha e^{-\beta(t-r)}}. \tag{4.11}$$

For $r \in R$ denote by $h_r(t)$ the (interval) Heaviside step function given by

$$h_r(t) = \begin{cases} 0, & \text{if} \quad t < r, \\ [0, 1], & \text{if} \quad t = r, \\ 1, & \text{if} \quad t > r, \end{cases} \tag{4.12}$$

We note that for the Hausdorff distanse between the shifted Heaviside function and shifted Gompertz function the same bounds (4.6) are valid.

**2.** In some cases the approximation of Heaviside function by Gompertz function is better in comparison to its approximation by logistic function (see, Figure 4.4)

$$s_0(t) = \frac{1}{1 + e^{-kt}}. \tag{4.13}$$

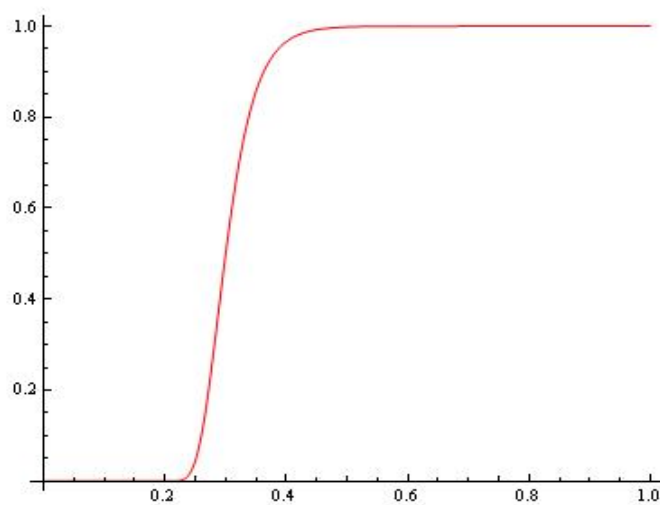For some comparisons of the Gompertz and logistic equation see [60].

Figure 4.3: The approximation of the shifted Heaviside function (4.11) by shifted Gompertz function (4.10) with $\alpha^* = 0.69314718...$, $\beta = 30$ and $r = 0.3$.
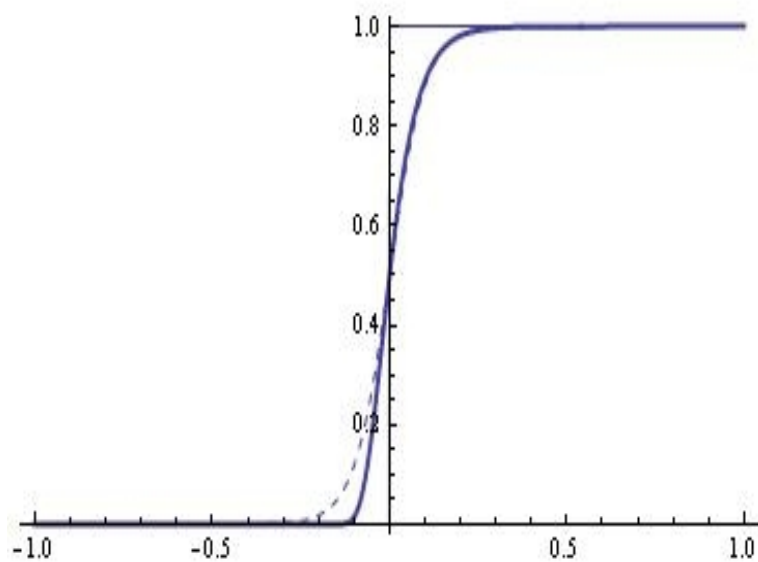


Figure 4.4: The Gompertz function with $\alpha^* = 0.69314718...$ and $\beta = 18$ (thick) and the logistic function with $k = 20$ (dashed).

**3.** In biology, the Gompertz curve or Gompertz function is commonly used to model growth process where the period of increasing growth is shorter than the period in which growth decreases.

The model is given by [61]:

$$Y_{ij} = e^{-\alpha e^{-\beta t_i}} + \varepsilon_{ij}, \tag{4.14}$$

where $Y_{ij}$ is the $i$-th observation at time $t_i$, $\varepsilon_{ij}$ are normal measurement errors with 0 means, independent of the random effects and the parameters $\alpha$, $\beta$ are constrained to be positive.

A number of models have been used to represent the growth curve [64]:

$$\ln \frac{N}{N_0} = ae^{-e^{(b-ct)}}, \tag{4.15}$$

where $\frac{N}{N_0}$ is the ratio of number of bacteria $N$, to the initial population $N_0$, and $a, b, c$ are constants, which are obtained by fitting experimental data.

We note that

$$\ln \frac{N}{N_0} = ae^{-\alpha e^{-\beta}}$$

with $\alpha = e^b$, $\beta = c$, i.e. (4.15) is the "Log–Gompertz model".

**4.** Consider the differential equation

$$y' = ke^{-\beta t}y, \quad \frac{k}{\beta} = \alpha. \tag{4.16}$$

For $\beta \neq 0$ we have

$$\frac{dy}{dt} = ke^{-\beta t}y; \quad \frac{dy}{y} = ke^{-\beta t}dt$$

$$\ln y = -\frac{k}{\beta}e^{-\beta t} = -\alpha e^{-\beta t}; \quad y = e^{-\alpha e^{-\beta t}}.$$

We see that the solution of differential equation (4.16) is the Gompertz function $\sigma_{\alpha,\beta}(t)$ (4.1) (or the "Gompertz growth law" see, [62]).

Savageau [63] recast equation (4.16) into a system of autonomous differential equations from the class of "synergistic and saturable systems":

$$y' = \beta xy; \quad x' = -\beta x, \tag{4.17}$$

where $x = \frac{kP}{\beta} = \alpha P$.

This interpretation is inaccurate because $x$ is only proportional, and not equal to the growth fraction $P$ (see, [62] for more details).

| $t$ | $Biomass$ | $Substrate$ |
|-----|-----------|-------------|
| 0 | 0.083 | 1. |
| 24 | 0.4 | 0.5 |
| 48 | 0.733 | 0.24 |
| 72 | 0.88 | 0.18 |
| 90 | 0.81 | 0.11 |

Table 4.2: The experimental data (biomass and substrate) for Xantobacter autotrophycum without electric field
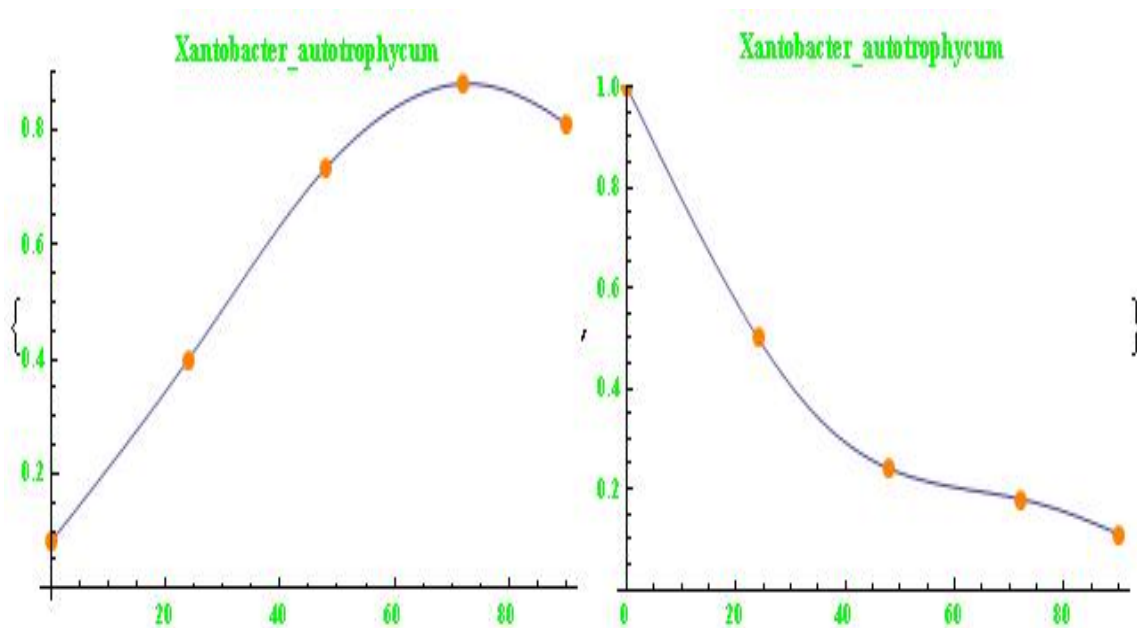


Figure 4.5: Cubic interpolation of the experimental data.

## 4.4 Some applications

We examine the following experimental data for biomass and substrare for Xantobacter autotrophycum without electric field (see, Table 4.2).

A cubic interpolation of the experimental data is plotted in Figure 4.5.

This gives us a general idea of the dynamics of the biomass and the substrate.

We next choose to fit the Gompertz sigmoid model (4.1) to the available time course measurment data.

The appropriate least–square fitting of the experimental data (for the biomass) by the Gompertz model yields for $\alpha = 2.66$ and $\beta = 0.044$ and is visualized on Figure 4.6.
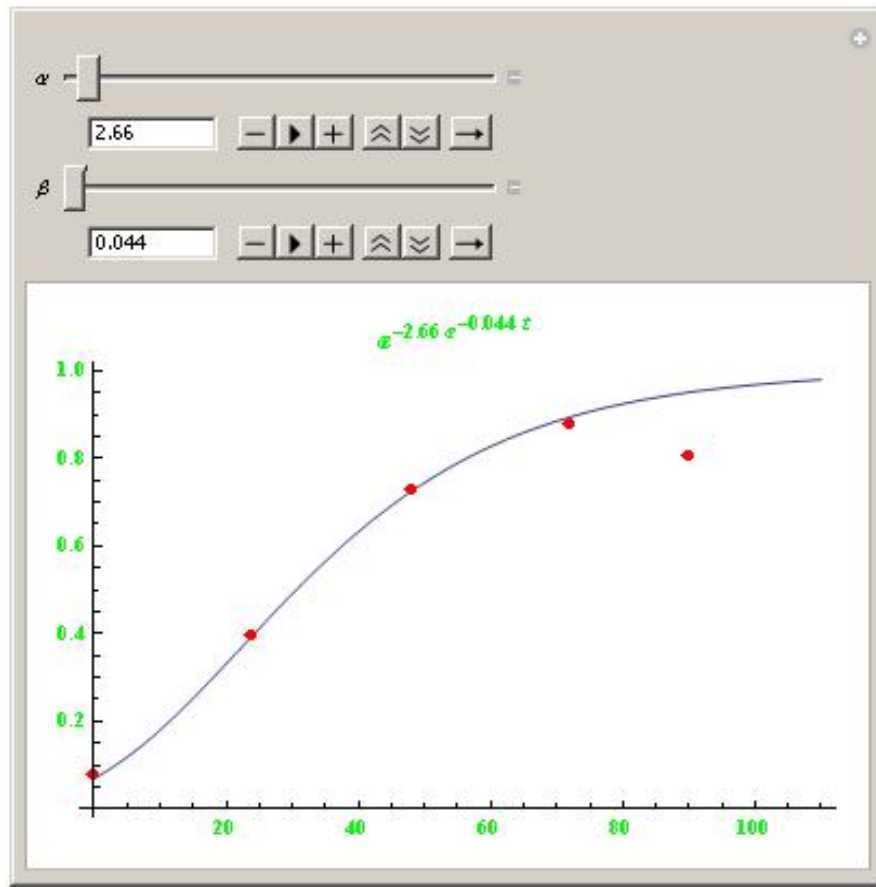
Figure 4.6: The least–square fitting of the experimental data (for the biomass) by the Gompertz model (4.1) yields $\alpha = 2.66$, $\beta = 0.044$.

We have

$$X(t) = e^{-2.66e^{-0.044t}}. \tag{4.18}$$

A possible way to model the substrate - $S(t)$ is by means of the following differential equation

$$dS/dt = -\beta S(t)(x_0 + s_0 - X(t)), \tag{4.19}$$

where $x_0 = x(0) = 0.083$, $s_0 = s(0) = 1$.

Figure 4.7 visualizes the functions $X(t)$ and $S(t)$.

The numerical results demonstrate the advantages of the approximate model in comparison to the "classical" model.

**Appendix 1.** Calculation of the value of the H–distance $d$ between the Gompertz sigmoid function and the Heaviside step function

```
Print["s'[t] ==-β*s[t]*(x0+s0-x[t])"];
s0 = Input["Input initial condition - s[0]"]; (* 1 *)
Print["Initial condition s0 = ", s0];
x0 = Input["Input initial condition - x[0]"]; (* 0.083 *)
Print["Initial condition x0 = ", x0];
α = Input["Input α"]; (* 2.66 *)
Print["α = ", α];
β = Input["Input β"]; (* 0.044 *)
Print["β = ", β];
sol = NDSolve[{s'[t] == -β*s[t]*(x0 + s0 - Exp[-α*Exp[-β*t]]),
    s[0] == s0}, s, {t, 0, 100}];
Show[{Plot[Evaluate[s[t] /. sol], {t, 0, 100}, PlotRange → {0, 20}, PlotStyle → {Red}],
  ListPlot[{{0, 1}, {24, 0.5}, {48, 0.24},
    {72, 0.18}, {90, 0.11}}, PlotStyle → Red, PlotMarkers → {Automatic, Small}],
  Plot[Exp[-α*Exp[-β*t]], {t, 0, 100}, PlotRange → {0, 20}, PlotStyle → {Green}],
  ListPlot[{{0, 0.083}, {24, 0.4}, {48, 0.733}, {72, 0.88}, {90, 0.81}},
    PlotStyle → Green, PlotMarkers → {Automatic, Small}]}, PlotRange → All]
s'[t] ==-β*s[t]*(x0+s0-x[t])
Initial condition s0 = 1
Initial condition x0 = 0.083
α = 2.66
β = 0.044
```
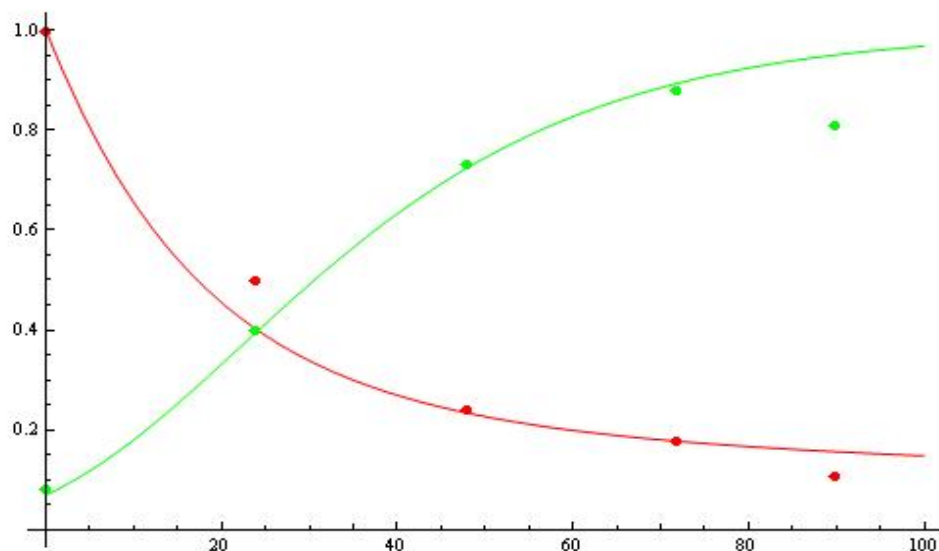


Figure 4.7: The approximate solutions $X(t)$ (green) and $S(t)$ (red).

```
Print["Calculation of the value of the Hausdorff distance d between the Gompertz
sigmoidal function and the Heaviside function"];

α = Input["   α"]; (*0.69314718 *)
Print["   α = ", α];
β = Input["   β"]; (*1000 *)
Print["   β = ", β];

Print["The following nonlinear equation is used to determination of the Hausdorff
distance d: "];

m = Log[1 - d] + α * Exp[-β * d];

Print[m, "  = 0"];
Print["The roots of this equation are: "];

NSolve[m == 0, ε];
Print[TableForm[%]];

Print["The unique positive root of the equation is the searched value of d: "];

positiveReals = Solve[Reduce[{m == 0, d > 0}, d], d];
If[Length[positiveReals] > 0,
  If[Length[positiveReals] == 1,
    Print["There exists an unique ", Length[positiveReals], " positive root: "]
  ];

  Print[TableForm[N[positiveReals]]];
];
```

```
Calculation of the value of the Hausdorff distance d between the Gompertz
sigmoidal function and the Heaviside function
  α = 0.693147
  β = 1000
The following nonlinear equation is used to determination of the Hausdorff
distance d:
0.693147 e^{-1000 d} + Log[1 - d] = 0
The roots of this equation are:
{}
The unique positive root of the equation is the searched value of d:
There exists an unique 1 positive root:
d → 0.00494117
```

Figure 4.8: Modul in CAS MATHEMATICA for calculation of the value of the Hausdorff distance $d$ between the Gompertz sigmoid function and the Heaviside step function.
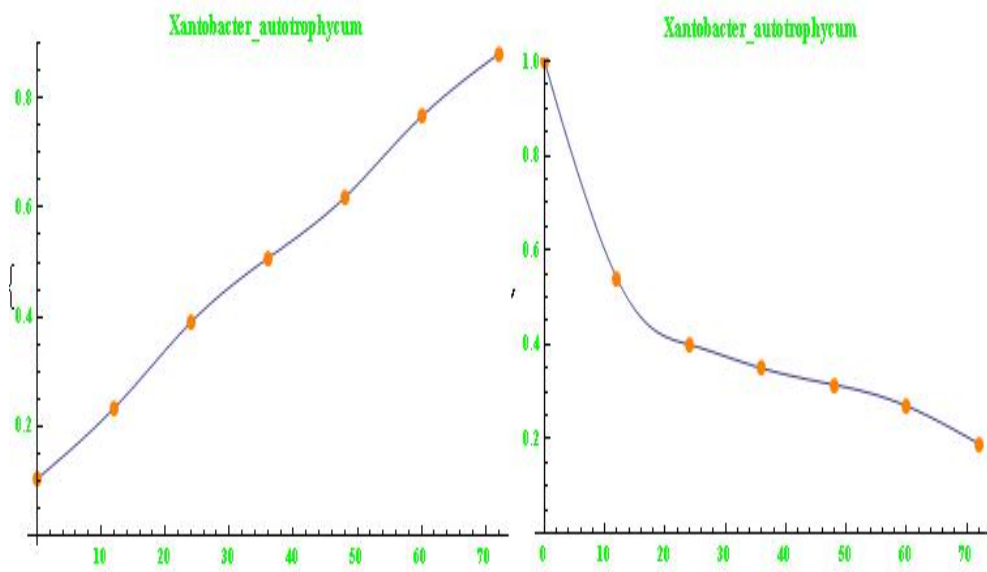
Figure 4.9: The cubic interpolation of the experimental data.

Case study 2. We examine the following experimental data for biomass and substrare for Xantobacter autotrophycum with electric field (see, Table 4.3).

| $t$ | $Biomass$ | $Substrate$ |
|---|---|---|
| 0 | 0.104 | 1. |
| 12 | 0.233 | 0.54 |
| 24 | 0.39 | 0.4 |
| 36 | 0.507 | 0.35 |
| 48 | 0.618 | 0.315 |
| 60 | 0.766 | 0.27 |
| 72 | 0.88 | 0.19 |

Table 4.3: The experimental data (biomass and substrate) for Xantobacter autotrophycum with electric field

The cubic interpolation of the experimental data is plotted in Figure 4.9.

The appropriate fitting of the experimental data (for the biomass) by the classical Gompertz model for $\alpha = 2.25$ and $\beta = 0.035$ is visualized on Figure 4.10.

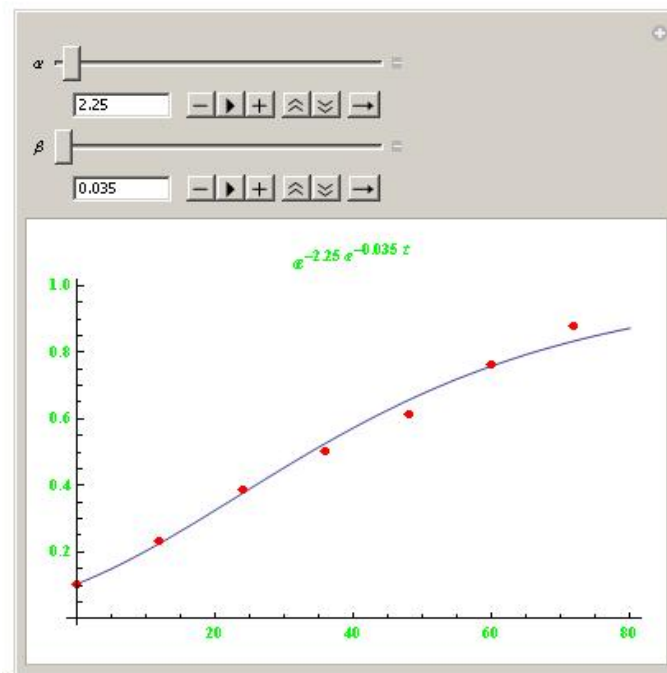Figure 4.11 visualizes approximate solutions $X(t)$ and $S(t)$.

Figure 4.10: The appropriate fitting of the experimental data (for the biomass) by the classical Gompertz model (4.1); $\alpha = 2.25$, $\beta = 0.035$.
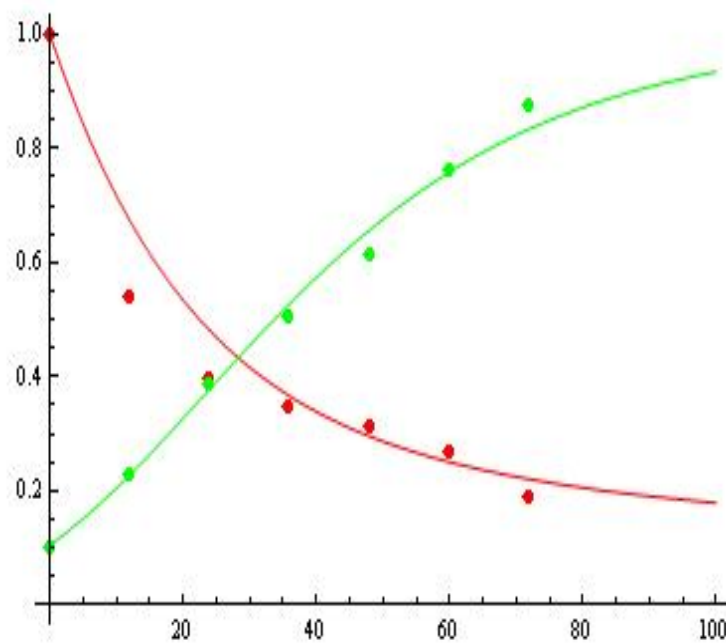


Figure 4.11: The approximate solutions $X(t)$ (green) and $S(t)$ (red).

## 4.5 Approximation of the $h_0(t)$ by transmuted Gompertz function

**Definition.** The cumulative distribution function $\sigma_{\alpha,\beta,\lambda}(t)$ for the generalized Gompertz model is defined for $\alpha, \beta > 0$, $-1 < \lambda < 1$ by:

$$\sigma_{\alpha,\beta,\lambda}(t) = (1+\lambda)e^{-\alpha e^{-\beta t}} - \lambda\left(e^{-\alpha e^{-\beta t}}\right)^2. \tag{4.20}$$

We examine the special case

$$\sigma_{\alpha,\beta,\lambda}(0) = \frac{1}{2} = (1+\lambda)e^{-\alpha} - \lambda e^{-2\alpha}, \quad 0 < \lambda < 1. \tag{4.21}$$

Let

$$u = e^{-\alpha}. \tag{4.22}$$

From (4.21) we have

$$\lambda u^2 - (1+\lambda)u + \frac{1}{2} = 0, \quad u_{1,2} = \frac{1+\lambda \pm \sqrt{1+\lambda^2}}{2\lambda}.$$

We are interested in the solution $u = \dfrac{1+\lambda - \sqrt{1+\lambda^2}}{2\lambda}$.

**Special case.** The special transmuted Gompertz function for $\alpha, \beta > 0$, $0 < \lambda < 1$ is defined by:

$$\sigma_{\alpha,\beta,\lambda}(t) = (1+\lambda)e^{-\alpha e^{-\beta t}} - \lambda e^{-2\alpha e^{-\beta t}},$$

$$e^{-\alpha} = \frac{1+\lambda - \sqrt{1+\lambda^2}}{2\lambda}, \tag{4.23}$$

$$\sigma_{\alpha,\beta,\lambda}(0) = \tfrac{1}{2}.$$

We study the Hausdorff approximation [14] of the Heaviside step function $h_0(t)$ by transmuted sigmoid Gompertz functions of the forms (4.23) and find an expression for the error of the best approximation.

The H–distance $d$ between the step function $h_0(t)$ and the transmuted Gompertz function (4.23) satisfies the relation

$$\sigma_{\alpha,\beta,\lambda}(d) = (1+\lambda)e^{-\alpha e^{-\beta d}} - \lambda e^{-2\alpha e^{-\beta d}} = 1 - d \tag{4.24}$$

or

$$(1+\lambda)e^{-\alpha e^{-\beta d}} - \lambda e^{-2\alpha e^{-\beta d}} - 1 + d = 0. \tag{4.25}$$

We note that from (4.23) we find

$$\alpha = -\ln\frac{1+\lambda - \sqrt{1+\lambda^2}}{2\lambda} \tag{4.26}$$

Some computational examples using relation (4.25) are presented in Table 4.4 for various $\beta$, $\lambda$ and $\alpha$ (computed by (4.26)).
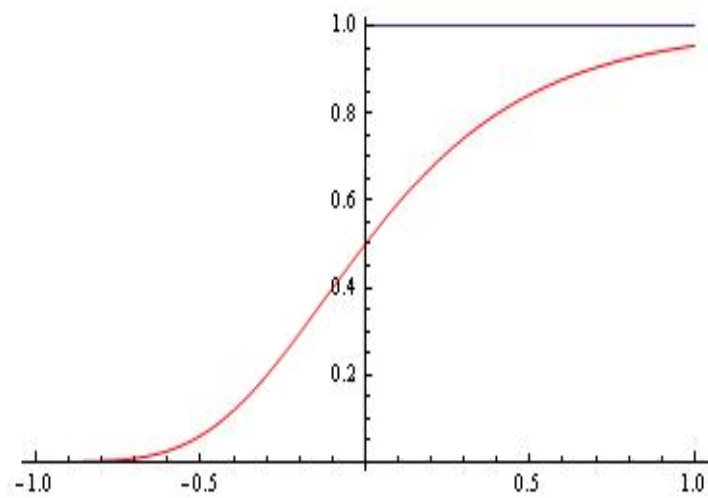
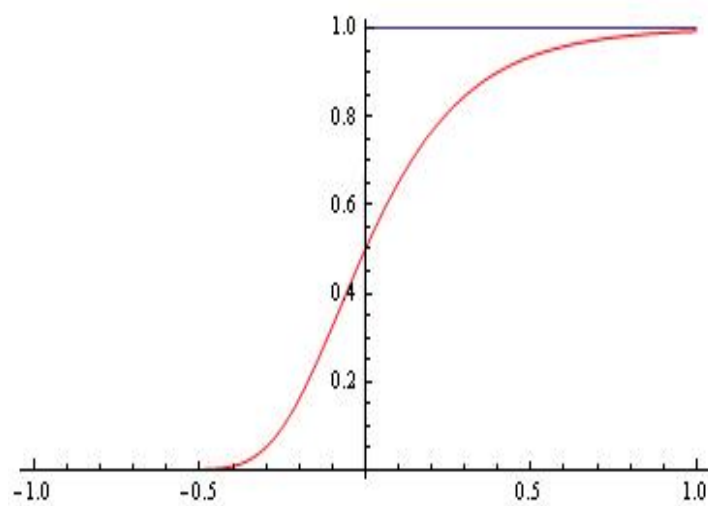Figure 4.12: The transmuted Gompertz function with $\beta = 2.4$, $\lambda = 0.5$; H–distance $d = 0.273635$.



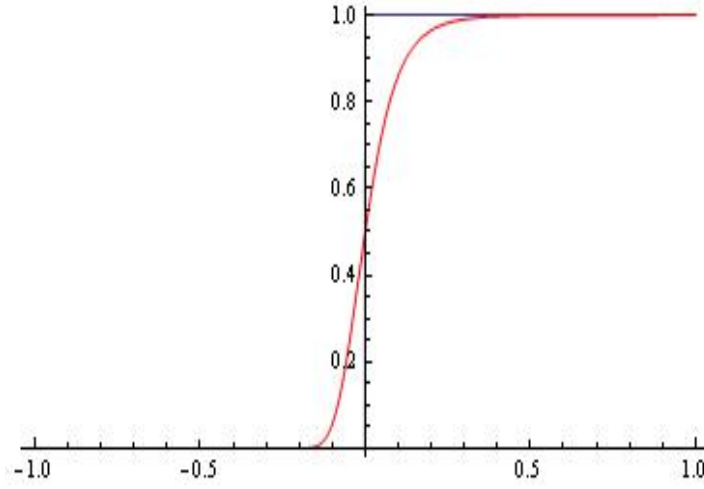Figure 4.13: The transmuted Gompertz function with $\beta = 4.5$, $\lambda = 0.25$; H–distance $d = 0.217538$.

Figure 4.14: The transmuted Gompertz function with $\beta = 11.4$, $\lambda = 0.75$; H–distance $d = 0.114365$.

| $\beta$ | $\lambda$ | $\alpha$ | $d$ |
|------|------|----------|------------|
| 2.4 | 0.5 | 0.962424 | 0.273635 |
| 4.5 | 0.25 | 0.824516 | 0.217538 |
| 11.4 | 0.75 | 1.09861 | 0.114365 |
| 15 | 0.95 | 1.20277 | 0.0871638 |
| 50 | 0.95 | 1.20277 | 0.0369552 |
| 100 | 0.95 | 1.20277 | 0.0217796 |
| 1000 | 0.95 | 1.20277 | 0.00340655 |

Table 4.4: Bounds for $d$ computed by (4.25) for various $\beta$ and $\lambda$.

**Theorem 4.2** *For the H–distance $d$ between the Heaviside step function $h_0(t)$ and the transmuted Gompertz function the following lower bound hold*

$$\frac{e^{-2\alpha}(e^\alpha - 1)(e^\alpha - \lambda)}{1 + \alpha\beta e^{-\alpha}(1 - 2e^{-\alpha}\lambda + \lambda)} < d. \tag{4.27}$$

**Proof.** Let

$$F(d) = (1 + \lambda)e^{-\alpha e^{-\beta d}} - \lambda e^{-2\alpha e^{-\beta d}} - 1 + d$$

and

$$G(d) = -e^{-2\alpha}(e^\alpha - 1)(e^\alpha - \lambda) + \left(1 + \alpha\beta e^{-\alpha}(1 - 2e^{-\alpha}\lambda + \lambda)\right)d.$$
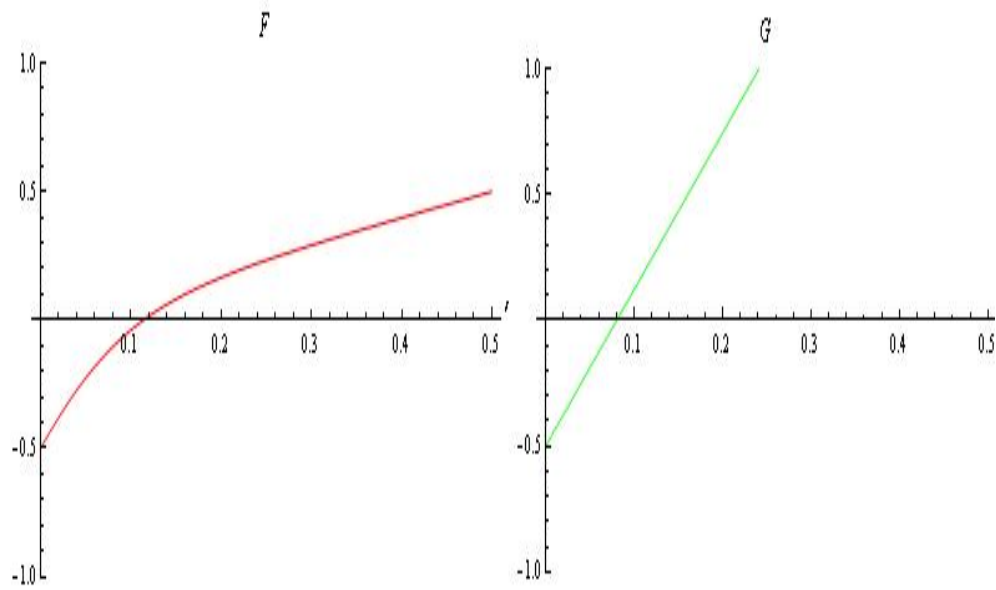
We note that from Taylor expansion

$$G(d) - F(d) = O(d^2)$$

Figure 4.15: The functions $F(d)$ and $G(d)$.

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see, Fig.4.15).

From this we obtain (4.27).

# Chapter 5

# The Rayleigh sigmoid function

In this Chapter the Hausdorff approximation of the shifted Heaviside step function by transmuted sigmoid Rayleigh function arising from biological, medical, finansial and engineering applications is considered and lower bound for the Hausdorff distance has been obtained.

Numerical examples, that illustrate our results are given, too.

## 5.1 Introduction

Definition 1. A random variable $T$ is said to have a transmuted distribution if its cumulative distribution function (cdf) is given by

$$G(t) = (1 + \lambda)F(t) - \lambda F^2(t), \quad |\lambda| \leq 1, \tag{5.1}$$

where $F(t)$ is the cdf of the base distribution.

Definition 2. For $r \in R$ denote by $h_r \in H(R)$ the shifted (interval) Heaviside step function given by

$$h_r(t) = \begin{cases} 0, & \text{if} \quad t < r, \\ [0,1], & \text{if} \quad t = r, \\ 1, & \text{if} \quad t > r. \end{cases} \tag{5.2}$$

Definition 3. [78] The transmuted sigmoid Rayleigh function is:

$$G(t, \sigma, \lambda) = \left(1 - e^{-\frac{t^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{t^2}{2\sigma^2}}\right), \tag{5.3}$$

where $t \geq 0$.

We examine the special case

$$G(r, \sigma, \lambda) = \frac{1}{2} = \left(1 - e^{-\frac{r^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{r^2}{2\sigma^2}}\right). \tag{5.4}$$

Let

$$u = e^{-\frac{r^2}{2\sigma^2}}. \tag{5.5}$$

From (5.4) we have

$$\lambda u^2 + (1 - \lambda)u - \frac{1}{2} = 0, \quad u_{1,2} = \frac{\lambda - 1 \pm \sqrt{1 + \lambda^2}}{2\lambda}.$$

We are interested in the solution $u = \dfrac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\lambda}$.

**Special case**. The special transmuted Rayleigh function for $\sigma > 0$, $0 < \lambda < 1$ is defined by:

$$G(t, \sigma, \lambda) = \left(1 - e^{-\frac{t^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{t^2}{2\sigma^2}}\right),$$

$$e^{-\frac{r^2}{2\sigma^2}} = \frac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\lambda}, \tag{5.6}$$

$$G(r, \sigma, \lambda) = \tfrac{1}{2}.$$

We study the Hausdorff approximation [14] of the shifted Heaviside step function $h_r(t)$ by transmuted sigmoid Rayleigh function of the form (5.6) and find an expression for the error of the best approximation.

For related results see, [79–82].

## 5.2  Approximation of the $h_r(t)$ by transmuted Rayleigh function

The H–distance $d$ between the step function $h_r(t)$ and the transmuted Rayleigh function (5.6) satisfies the relation

$$G(r + d, \sigma, \lambda) = \left(1 - e^{-\frac{(r+d)^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{(r+d)^2}{2\sigma^2}}\right) = 1 - d \tag{5.7}$$

or

$$\left(1 - e^{-\frac{(r+d)^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{(r+d)^2}{2\sigma^2}}\right) - 1 + d = 0. \tag{5.8}$$

From (5.6) we obtain

$$\sigma = r\sqrt{-0.5\frac{1}{\ln\frac{\lambda - 1 + \sqrt{1 + \lambda^2}}{2\lambda}}}. \tag{5.9}$$

Some computational examples using relation (5.8) are presented in Table 5.1 for various $r$, $\lambda$ and $\sigma$ (computed by (5.9)).

Figures 5.1–5.3 illustrates some of the possible shapes of the cdf of a transmuted Rayleigh function (5.6) for selected values of the parameters.
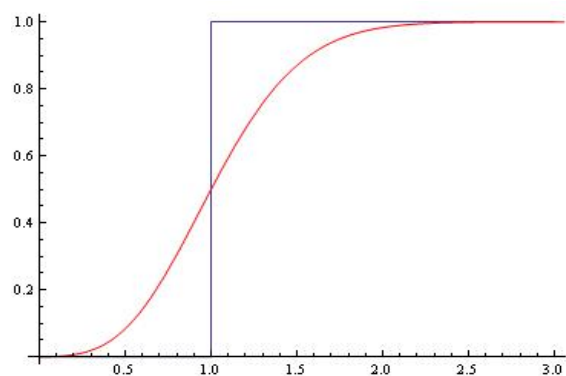
Figure 5.1: The transmuted Rayleigh function with $r = 1$, $\lambda = -0.9$; H–distance $d = 0.266739$.



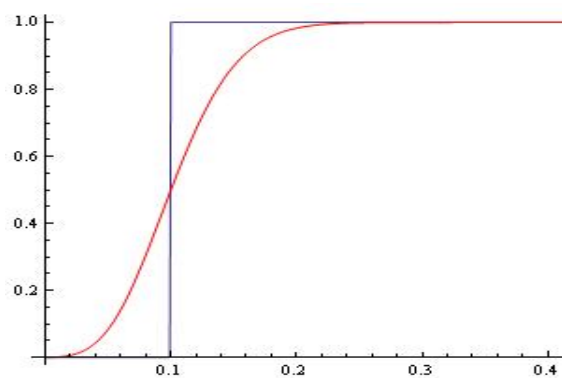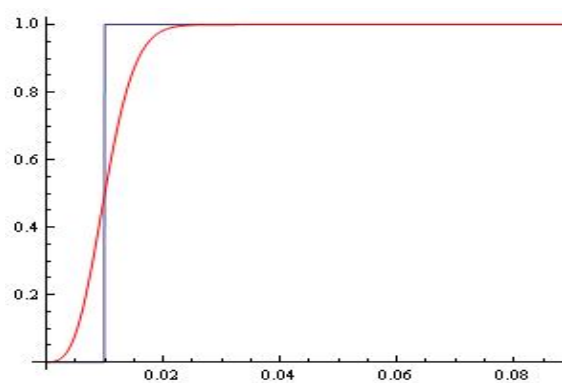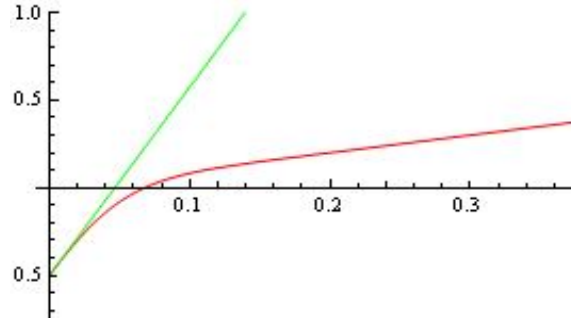Figure 5.2: The transmuted Rayleigh function with $r = 0.1$, $\lambda = -0.9$; H–distance $d = 0.0678183$.



Figure 5.3: The transmuted Rayleigh function with $r = 0.01$, $\lambda = -0.9$; H–distance $d = 0.0109272$.

| $r$ | $\lambda$ | $\sigma$ | $d$ |
|---|---|---|---|
| 0.05 | $-0.9$ | 0.0325855 | 0.0403306 |
| 0.1 | $-0.9$ | 0.0651711 | 0.0678183 |
| 0.005 | $-0.9$ | 0.00325855 | 0.00604863 |
| 0.01 | $-0.9$ | 0.00651711 | 0.0109272 |
| 0.001 | $-0.9$ | 0.000651711 | 0.0014672 |
| 0.0001 | $-0.9$ | 0.000065171 | 0.000180488 |
| 0.2 | 0.5 | 0.203867 | 0.146201 |
| 0.1 | 0.8 | 0.112872 | 0.0917928 |
| 0.05 | 0.9 | 0.0582531 | 0.0549035 |

Table 5.1: Bounds for $d$ computed by (5.8) for various $r$ and $\lambda$.



Figure 5.4: The functions $F(d)$ and $G(d)$.

**Theorem 5.1** *For the H–distance $d$ between the Heaviside step function $h_r(t)$ and the transmuted Rayleigh sigmoid function (5.6) the following lower bound hold*

$$\frac{e^{-\frac{r^2}{2\sigma^2}}\left(1 - \lambda + \lambda e^{-\frac{r^2}{2\sigma^2}}\right)}{1 + \frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\left(1 - \lambda + 2\lambda e^{-\frac{r^2}{2\sigma^2}}\right)} < d. \tag{5.10}$$

**Proof.** Let

$$F(d) = \left(1 - e^{-\frac{d^2}{2\sigma^2}}\right)\left(1 + \lambda e^{-\frac{d^2}{2\sigma^2}}\right) - 1 + d$$

and

$$G(d) = -e^{-\frac{r^2}{2\sigma^2}}\left(1 - \lambda + \lambda e^{-\frac{r^2}{2\sigma^2}}\right) + \left(1 + \frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}\left(1 - \lambda + 2\lambda e^{-\frac{r^2}{2\sigma^2}}\right)\right)d.$$

We note that from Taylor expansion

$$G(d) - F(d) = O(d^2)$$

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see, Fig.5.4).

From this we obtain (5.10).

## 5.3 Approximation of the $h_r(t)$ by the generalized Rayleigh function

**Definition 4.** [78] The transmuted generalized sigmoid Rayleigh function is:

$$G(t, \alpha, \beta, \lambda) = \left(1 - e^{-(\beta t)^2}\right)^{\alpha} \left(1 + \lambda - \lambda \left(1 - e^{-(\beta t)^2}\right)^{\alpha}\right), \qquad (5.11)$$

where $\alpha$, $\beta > 0$, $|\lambda| \leq 1$.

We examine the special case

$$G(r, \alpha, \beta, \lambda) = \frac{1}{2} = \left(1 - e^{-(\beta r)^2}\right)^{\alpha} \left(1 + \lambda - \lambda \left(1 - e^{-(\beta r)^2}\right)^{\alpha}\right). \qquad (5.12)$$

Let

$$u = e^{-(\beta r)^2}, \quad (1 - u)^{\alpha} = t. \qquad (5.13)$$

From (5.12) and (5.13) we have

$$\lambda t^2 - (1 + \lambda)t + \frac{1}{2} = 0, \quad t_{1,2} = \frac{1 + \lambda \pm \sqrt{1 + \lambda^2}}{2\lambda}.$$

We interested from the solution $t = \dfrac{1 + \lambda - \sqrt{1 + \lambda^2}}{2\lambda}$.

Further

$$\beta(r, \alpha, \lambda) = \sqrt{\ln\left(1 - \left(\frac{1 + \lambda - \sqrt{1 + \lambda^2}}{2\lambda}\right)^{\frac{1}{\alpha}}\right)^{-\frac{1}{r^2}}}. \qquad (5.14)$$

**Special case.** The special transmuted generalized Rayleigh function for $\alpha, \beta > 0$, $0 < \lambda < 1$ is defined by:

$$G(t, \alpha, \beta, \lambda) = \left(1 - e^{-(\beta t)^2}\right)^{\alpha} \left(1 + \lambda - \lambda \left(1 - e^{-(\beta t)^2}\right)^{\alpha}\right),$$

$$\beta(r, \alpha, \lambda) = \sqrt{\ln\left(1 - \left(\frac{1 + \lambda - \sqrt{1 + \lambda^2}}{2\lambda}\right)^{\frac{1}{\alpha}}\right)^{-\frac{1}{r^2}}}, \qquad (5.15)$$

$$G(r, \alpha, \beta, \lambda) = \tfrac{1}{2}.$$

We study the Hausdorff approximation [14] of the shifted Heaviside step function $h_r(t)$ by transmuted generalized sigmoid Rayleigh function of the form (5.15) and find an expression for the error of the best approximation.

For related results see, [83].

The H–distance $d$ between the step function $h_r(t)$ and the transmuted generalized Rayleigh function (5.15) satisfies the relation
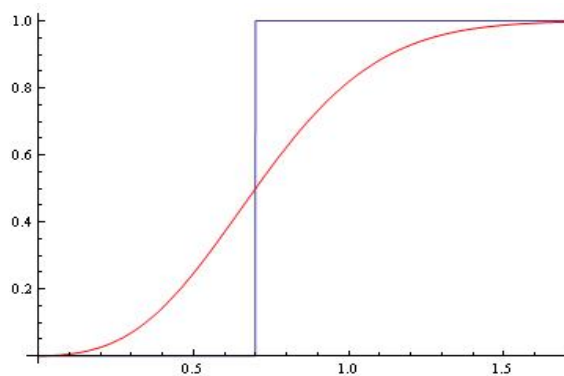
Figure 5.5: The transmuted generalized Rayleigh function with $r = 0.7$, $\alpha = 0.9$, $\lambda = -0.85$, $\beta = 1.47475$ (computed by (5.14)); H–distance $d = 0.235033$.
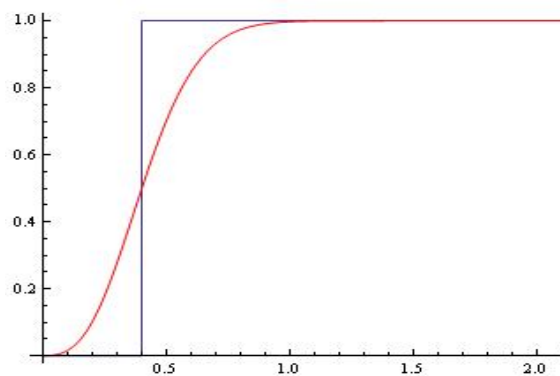


Figure 5.6: The transmuted generalized Rayleigh function with $r = 0.4$, $\alpha = 0.8$, $\lambda = -0.95$, $\beta = 2.52647$ (computed by (5.14)); H–distance $d = 0.178209$.

$$G(r + d, \alpha, \beta, \lambda) = \left(1 - e^{-(\beta(r+d))^2}\right)^{\alpha} \left(1 + \lambda - \lambda \left(1 - e^{-(\beta(r+d))^2}\right)^{\alpha}\right) = 1 - d \qquad (5.16)$$

or

$$\left(1 - e^{-(\beta(r+d))^2}\right)^{\alpha} \left(1 + \lambda - \lambda \left(1 - e^{-(\beta(r+d))^2}\right)^{\alpha}\right) - 1 + d = 0. \qquad (5.17)$$

Some computational examples using relation (5.15) are presented in Table 5.2 for various $r$, $\alpha$, $\lambda$ and $\beta$ (computed by (5.14)).

Figures 5.5–5.7 visualize some of the possible shapes of the cdf of a transmuted Rayleigh function (5.15) for selected values of the parameters.

Let

$$F(d) = \left(1 - e^{-(\beta(r+d))^2}\right)^{\alpha} \left(1 + \lambda - \lambda \left(1 - e^{-(\beta(r+d))^2}\right)^{\alpha}\right) - 1 + d$$
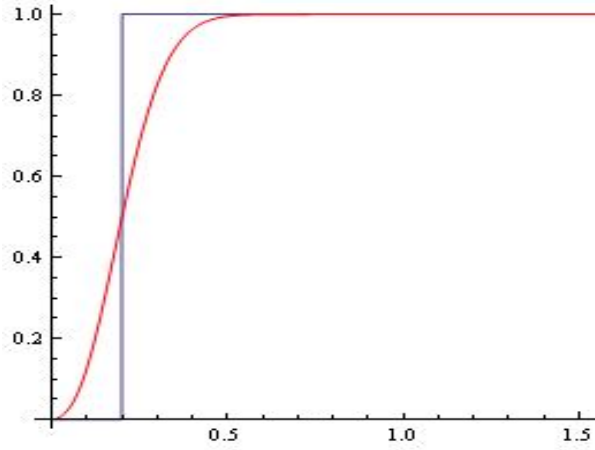
Figure 5.7: The transmuted generalized Rayleigh function with $r = 0.2$, $\alpha = 0.7$, $\lambda = -0.9$, $\beta = 4.72658$ (computed by (5.14)); H–distance $d = 0.124428$.

| $r$ | $\alpha$ | $\lambda$ | $\beta$ | $d$ |
|---|---|---|---|---|
| 0.7 | 0.9 | $-0.85$ | 1.47475 | 0.235033 |
| 0.4 | 0.8 | $-0.95$ | 2.52647 | 0.178209 |
| 0.2 | 0.7 | $-0.90$ | 4.72658 | 0.124428 |
| 0.1 | 0.85 | $-0.99$ | 10.4367 | 0.0704303 |
| 0.05 | 0.95 | $-1$ | 21.7737 | 0.040099 |
| 0.01 | 0.99 | $-0.99$ | 110.20652 | 0.0107146 |
| 0.001 | 1 | $-1$ | 1108.12778 | 0.00142879 |

Table 5.2: Bounds for $d$ computed by (5.16) for various $r$, $\alpha$, $\lambda$, $\beta$ (computed by (5.14)).

and

$$G(d) = -a + bd,$$

where

$$a = \left(-1 + \left(1 - e^{-r^2\beta^2}\right)^{\alpha}\right)\left(-1 + \left(1 - e^{-r^2\beta^2}\right)^{\alpha}\lambda\right),$$

$$b = \frac{-1 + e^{r^2\beta^2} + 2\left(1 - e^{-r^2\beta^2}\right)^{\alpha} r\alpha\beta^2 + 2\left(1 - e^{-r^2\beta^2}\right)^{\alpha} r\alpha\beta^2\lambda - 4\left(1 - e^{-r^2\beta^2}\right)^{2\alpha} r\alpha\beta^2\lambda}{-1 + e^{r^2\beta^2}}.$$

**Theorem 5.2** *For the H–distance $d$ between the Heaviside step function $h_r(t)$ and the transmuted generalized Rayleigh sigmoid function (5.15) the following lower bound hold*

$$\frac{a}{b} < d. \tag{5.18}$$

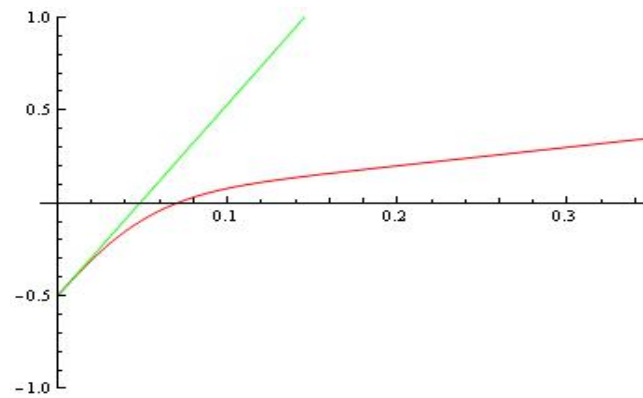**Proof.** The proof follows the ideas given in this Chapter and will be omitted.

Figure 5.8: The functions $F(d)$ and $G(d)$.

We note that from Taylor expansion we have

$$G(d) - F(d) = O(d^2)$$

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see, Fig.5.8).
From this we obtain (5.18).

For related results, see [84–92].

# Chapter 6

# The Lindley sigmoid function

In this Chapter the Hausdorff approximation of the shifted Heaviside step function by the sigmoid Lindley function arising from biomedical, finansial, engineering, economic and demographic applications is considered and lower bound for the Hausdorff distance has been obtained.

Numerical examples, which illustrate our results are given, too.

## 6.1   Introduction

Definition 1. [86] The transmuted sigmoid Lindley function is:

$$F_1(t,\theta,\lambda) = \left(1 - \frac{\theta+1+\theta t}{\theta+1}e^{-\theta t}\right)\left(1 + \lambda\frac{\theta+1+\theta t}{\theta+1}e^{-\theta t}\right), \tag{6.1}$$

where $t \geq 0$, $\theta > 0$ and $|\lambda| \leq 1$.

We examine the special case

$$F_1(r,\theta,\lambda) = \frac{1}{2} = \left(1 - \frac{\theta+1+\theta r}{\theta+1}e^{-\theta r}\right)\left(1 + \lambda\frac{\theta+1+\theta r}{\theta+1}e^{-\theta r}\right). \tag{6.2}$$

Let

$$u = e^{-\theta r}, \quad A = \frac{\theta+1+\theta r}{\theta+1}. \tag{6.3}$$

From (6.2) we have

$$\lambda A^2 u^2 + A(1-\lambda)u - \frac{1}{2} = 0, \quad u_{1,2} = \frac{\lambda - 1 \pm \sqrt{1+\lambda^2}}{2A\lambda}.$$

We are interested in the solution $u = \dfrac{\lambda - 1 + \sqrt{1+\lambda^2}}{2A\lambda}$.

For given $r$ and $\lambda$ the parameter $\theta$ is the solution of the nonlinear equation:

$$\ln\frac{(\lambda - 1 + \sqrt{1+\lambda^2})(\theta+1)}{2\lambda(\theta+1+\theta r)} + \theta r = 0. \tag{6.4}$$

The equation (6.4) has unique positive solution.

**Special case**. The special transmuted Lindley function for $t \geq 0$, $\theta > 0$ and $|\lambda| \leq 1$ is defined by:

$$F_1(t, \theta, \lambda) = \left(1 - \frac{\theta+1+\theta t}{\theta+1}e^{-\theta t}\right)\left(1 + \lambda\frac{\theta+1+\theta t}{\theta+1}e^{-\theta t}\right),$$

$$\ln \frac{(\lambda-1+\sqrt{1+\lambda^2})(\theta+1)}{2\lambda(\theta+1+\theta r)} + \theta r = 0, \qquad (6.5)$$

$$F_1(r, \theta, \lambda) = \tfrac{1}{2}.$$

We study the Hausdorff approximation [14] of the shifted step function $h_r(t)$ by transmuted sigmoid Lindley function of the form (6.5) and find an expression for the error of the best approximation.

For other results see, [84], [85], [86] and [87].

## 6.2   Approximation of the $h_r(t)$ by transmuted Lindley function

The H–distance $d$ between the function $h_r(t)$ and the transmuted Lindley function (6.5) satisfies the relation

$$F_1(r+d, \theta, \lambda) = \left(1 - \frac{\theta+1+\theta(r+d)}{\theta+1}e^{-\theta(r+d)}\right)\left(1 + \lambda\frac{\theta+1+\theta(r+d)}{\theta+1}e^{-\theta(r+d)}\right) = 1 - d$$

$$(6.6)$$

or

$$\left(1 - \frac{\theta+1+\theta(r+d)}{\theta+1}e^{-\theta(r+d)}\right)\left(1 + \lambda\frac{\theta+1+\theta(r+d)}{\theta+1}e^{-\theta(r+d)}\right) - 1 + d = 0. \qquad (6.7)$$

Some computational examples using relation (6.7) are presented in Table 6.1 for various $r$, $\lambda$ and $\theta$ (computed by (6.4)).

Figures 6.1–6.3 illustrate some of the possible shapes of the cdf of a transmuted Lindley function (6.5) for selected values of the parameters.

Let

$$F(d) = \left(1 - \frac{\theta+1+\theta(r+d)}{\theta+1}e^{-\theta(r+d)}\right)\left(1 + \lambda\frac{\theta+1+\theta(r+d)}{\theta+1}e^{-\theta(r+d)}\right) - 1 + d$$

and

$$G(d) = -a + bd,$$

where

$$a = 1 - \left(1 - \frac{(1+\theta+r\theta)e^{-r\theta}}{1+\theta}\right)\left(1 + \frac{(1+\theta+r\theta)\lambda e^{-r\theta}}{1+\theta}\right),$$

Figure 6.1: The transmuted Lindley function with $r = 0.1$, $\lambda = -0.05$, $\theta = 8.03648$ (computed by (6.4)); H–distance $d = 0.159454$.



Figure 6.2: The transmuted Lindley function with $r = 0.1$, $\lambda = -1$, $\theta = 13.1682$ (computed by (6.4)); H–distance $d = 0.123399$.
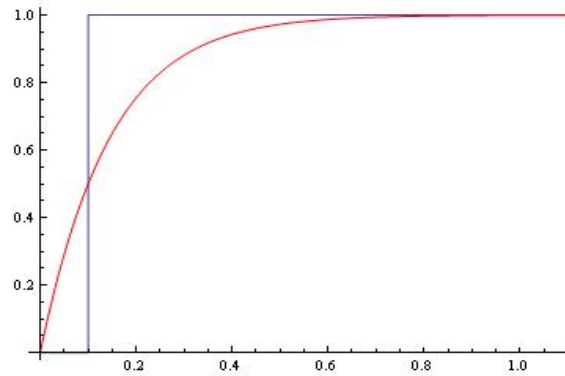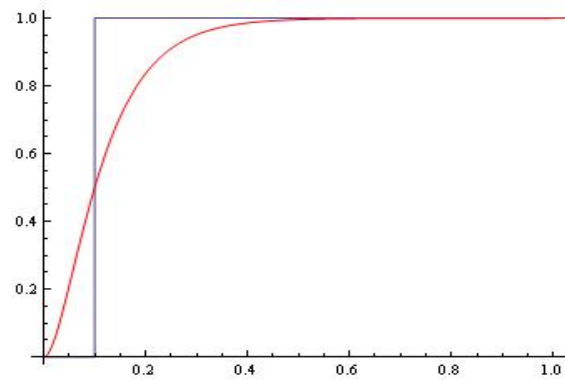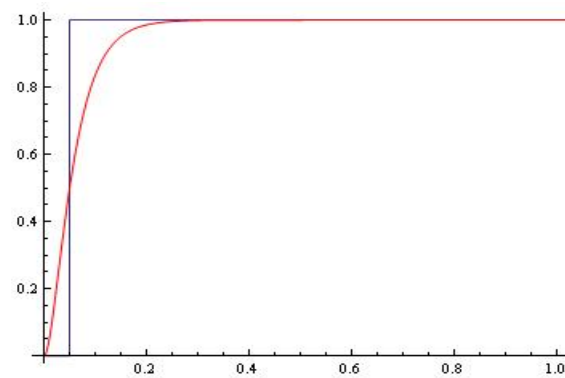


Figure 6.3: The transmuted Lindley function with $r = 0.05$, $\lambda = -1$, $\theta = 25.4988$ (computed by (6.4)); H–distance $d = 0.080034$.

| $r$ | $\lambda$ | $\theta$ | $d$ |
|------|--------|----------|-----------|
| 0.1 | $-0.05$ | 8.03648 | 0.159454 |
| 0.1 | $-1$ | 13.1682 | 0.123399 |
| 0.05 | $-1$ | 25.4988 | 0.080034 |
| 0.01 | $-1$ | 123.7828 | 0.0254762 |
| 0.001 | $-0.99$ | 1223.94 | 0.00406381 |

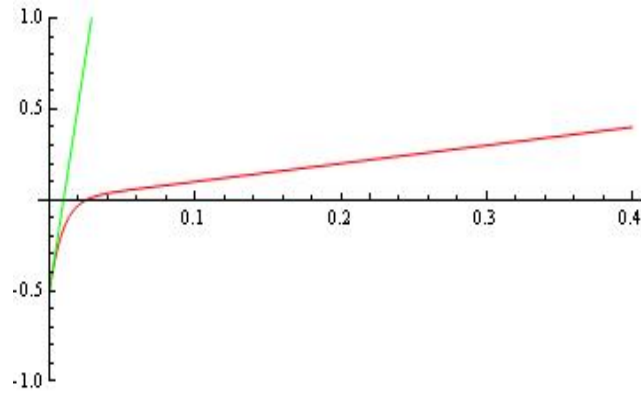Table 6.1: Bounds for $d$ computed by (6.7) for various $r$, $\lambda$ and $\theta$ (computed by (6.4)).



Figure 6.4: The functions $F(d)$ and $G(d)$ for $r = 0.01$, $\lambda = -1$, $\theta = 123.782$.

$$
\begin{aligned}
b \;=\; & 1 + \frac{\left(e^{-r\theta}\theta - e^{-r\theta}\theta(1+\theta+r\theta)\right)\left(1 - \frac{e^{-r\theta}(1+\theta+r\theta)}{1+\theta}\right)\lambda}{1+\theta} \\[2ex]
& + \; \frac{\left(e^{-r\theta}\theta + e^{-r\theta}\theta(1+\theta+r\theta)\right)\left(1 + \frac{e^{-r\theta}(1+\theta+r\theta)\lambda}{1+\theta}\right)}{1+\theta}
\end{aligned}
$$

**Theorem 6.1** *For the Hausdorff distance $d$ between the Heaviside step function $h_r(t)$ and the transmuted Lindley sigmoid function the following lower bound hold*

$$\frac{a}{b} < d. \tag{6.8}$$

**Proof.** The proof follows the ideas given in this Chapter and will be omitted.
   We note that from Taylor expansion we have

$$G(d) - F(d) = O(d^2)$$

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see, Fig.6.4).
   From this we obtain (6.8).

## 6.3   Approximation of the $h_r(t)$ by generalized Lindley function

**Definition 2.** [87] The generalized sigmoidal Lindley function is:

$$F_2(t, \theta, \lambda, \alpha, \delta) = (1 + \lambda)\left(1 - \frac{\theta + 1 + \theta t}{\theta + 1}e^{-\theta t}\right)^{\delta} - \lambda\left(1 - \frac{\theta + 1 + \theta t}{\theta + 1}e^{-\theta t}\right)^{\alpha}, \tag{6.9}$$

where $\theta > 0$, $\alpha > 0$ and $\delta > 0$ for $-1 < \lambda < 0$ and $\theta > 0$, $\alpha > 0$, $\frac{\alpha}{2} \leq \delta \leq \alpha + \frac{\alpha}{4}$ for $0 < \lambda < 1$.

### 6.3.1   Special cases of the generalized Lindley function

**I.** For $\lambda = 0$ and $\delta = 1$ we have the Lindley function [84], [85]:

$$F_2(t, \theta, 0, 1) = 1 - \frac{\theta + 1 + \theta t}{\theta + 1}e^{-\theta t}. \tag{6.10}$$

We examine the special case

$$F_2(r, \theta, 0, 1) = \frac{1}{2} = 1 - \frac{\theta + 1 + \theta r}{\theta + 1}e^{-\theta r}. \tag{6.11}$$

Let

$$u = e^{-\theta r}, \quad A = \frac{\theta + 1 + \theta r}{\theta + 1}. \tag{6.12}$$

From (6.11) we have

$$Au = \frac{1}{2}, \quad u = \frac{\theta + 1}{2(\theta + 1 + \theta r)}.$$

For given $r$ the parameter $\theta$ is the solution of the following nonlinear equation:

$$\ln\frac{\theta + 1}{2(\theta + 1 + \theta r)} + \theta r = 0. \tag{6.13}$$

The equation (6.13) has unique positive solution.

**Special case.** The special Lindley function for $t > 0$, $\theta > 0$ is defined by:

$$F_2(t, \theta, 0, 1) = 1 - \frac{\theta + 1 + \theta t}{\theta + 1}e^{-\theta t}$$

$$\ln\frac{\theta + 1}{2(\theta + 1 + \theta r)} + \theta r = 0 \tag{6.14}$$

$$F_2(r, \theta, 0, 1) = \frac{1}{2}.$$

We study the Hausdorff approximation [14] of the shifted Heaviside step function $h_r(t)$ by Lindley function of the form (6.14) and find an expression for the error of the best approximation.

The H–distance $d$ between the function $h_r(t)$ and the Lindley function (6.14) satisfies the relation

$$F_2(r + d, \theta, 0, 1) = 1 - \frac{\theta + 1 + \theta(r + d)}{\theta + 1}e^{-\theta(r + d)} = 1 - d \tag{6.15}$$

or

$$-\frac{\theta + 1 + \theta(r + d)}{\theta + 1}e^{-\theta(r+d)} + d = 0. \tag{6.16}$$

Let

$$F(d) = -\frac{\theta + 1 + \theta(r + d)}{\theta + 1}e^{-\theta(r+d)} + d$$

and

$$G(d) = -a + bd,$$

where

$$a = \frac{(1 + \theta + r\theta)e^{-r\theta}}{1 + \theta},$$

$$b = 1 + \frac{e^{-r\theta}\theta^2(1 + r)}{1 + \theta}.$$

We note that from Taylor expansion we have

$$G(d) - F(d) = O(d^2)$$

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$.

The above can be summarized in the following

**Theorem 6.2** *For the H–distance $d$ between the Heaviside step function $h_r(t)$ and the Lindley sigmoid function (6.14) the following lower bound holds*

$$\frac{a}{b} < d. \tag{6.17}$$

**II.** For $\lambda = 0$ we have the exponentiated Lindley function [87]:

$$F_3(t, \theta, \delta) = \left(1 - \frac{\theta + 1 + \theta t}{\theta + 1}e^{-\theta t}\right)^\delta. \tag{6.18}$$

We examine the special case

$$F_3(r, \theta, \delta) = \frac{1}{2} = \left(1 - \frac{\theta + 1 + \theta r}{\theta + 1}e^{-\theta r}\right)^\delta. \tag{6.19}$$

Let

$$u = e^{-\theta r}, \quad A = \frac{\theta + 1 + \theta r}{\theta + 1}. \tag{6.20}$$

From (6.19) we have

$$Au = 1 - \left(\frac{1}{2}\right)^{\frac{1}{\delta}}, \quad u = \left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\delta}}\right)\frac{\theta + 1}{\theta + 1 + \theta r}.$$

For given $r$ and $\delta$ the parameter $\theta$ is the solution of the nonlinear equation:

$$\ln\left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\delta}}\right)\frac{\theta+1}{\theta+1+\theta r} + \theta r = 0. \tag{6.21}$$

The equation (6.21) has unique positive solution.

**Special case**. The special exponentiated Lindley function for $t > 0$, $\theta > 0$, $\delta > 0$ is defined by:

$$F_3(t,\theta,\delta) = \left(1 - \frac{\theta+1+\theta t}{\theta+1}e^{-\theta t}\right)^\delta$$

$$\ln\left(1 - \left(\frac{1}{2}\right)^{\frac{1}{\delta}}\right)\frac{\theta+1}{\theta+1+\theta r} + \theta r = 0 \tag{6.22}$$

$$F_3(r,\theta,\delta) = \frac{1}{2}.$$

We study the Hausdorff approximation [14] of the shifted Heaviside step function $h_r(t)$ by exponentiated Lindley function of the form (6.22) and find an expression for the error of the best approximation.

The Hausdorff distance $d$ between the function $h_r(t)$ and the exponentiated Lindley function (6.22) satisfies the relation

$$F_3(r+d,\theta,\delta) = \left(1 - \frac{\theta+1+\theta(r+d)}{\theta+1}e^{-\theta(r+d)}\right)^\delta = 1 - d. \tag{6.23}$$

Let

$$F(d) = \left(1 - \frac{\theta+1+\theta(r+d)}{\theta+1}e^{-\theta(r+d)}\right)^\delta - 1 + d$$

and

$$G(d) = -a + bd,$$

where

$$a = 1 - \left(1 - \frac{(1+\theta+r\theta)e^{-r\theta}}{1+\theta}\right)^\delta,$$

$$b = 1 + \frac{e^{-r\theta}\delta\theta^2\left(1 - \frac{e^{-r\theta}(1+\theta+\theta r)}{1+\theta}\right)^{-1+\delta}}{1+\theta}$$

$$+ \frac{e^{-r\theta}r\delta\theta^2\left(1 - \frac{e^{-r\theta}(1+\theta+\theta r)}{1+\theta}\right)^{-1+\delta}}{1+\theta}.$$

We note that from Taylor expansion we have

$$G(d) - F(d) = O(d^2)$$

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$.

The above can be summarized in the following

**Theorem 6.3** *For the Hausdorff distance $d$ between the Heaviside step function $h_r(t)$ and the exponentiated Lindley function (6.22) the following lower bound holds*

$$\frac{a}{b} < d. \tag{6.24}$$

## 6.4   Approximation of the $h_r(t)$ by exponentiated Lindley function

Definition 3. [93] The exponentiated power Lindley function with 2 parameters is defined by:

$$F(t, \theta, \beta) = 1 - \left(1 + \frac{\theta t^\beta}{\theta + 1}\right) e^{-\theta t^\beta}, \tag{6.25}$$

where $t > 0$, $\theta > 0$ and $\beta > 0$.

We examine the special case

$$F(r, \theta, \beta) = \frac{1}{2} = 1 - \left(1 + \frac{\theta r^\beta}{\theta + 1}\right) e^{-\theta r^\beta}. \tag{6.26}$$

For given $r$ and $\beta$ the parameter $\theta$ is the solution of the following nonlinear equation:

$$\left(1 + \frac{\theta r^\beta}{\theta + 1}\right) e^{-\theta r^\beta} - \frac{1}{2} = 0. \tag{6.27}$$

The equation (6.27) has unique positive solution.

**Special case**. The special exponentiated power Lindley function for $t > 0$, $\theta > 0$ and $\beta > 0$ is defined by:

$$F(t, \theta, \beta) = 1 - \left(1 + \frac{\theta t^\beta}{\theta+1}\right) e^{-\theta t^\beta}$$

$$\left(1 + \frac{\theta r^\beta}{\theta+1}\right) e^{-\theta r^\beta} - \frac{1}{2} = 0 \tag{6.28}$$

$$F(r, \theta, \beta) = \frac{1}{2}.$$

We study the Hausdorff approximation [14] of the shifted Heaviside step function $h_r(t)$ by special exponentiated power Lindley function of the form (6.28) and find an expression for the error of the best approximation.

The Hausdorff distance $d$ between the function $h_r(t)$ and the function (6.28) satisfies the relation

$$F(r + d, \theta, \beta) = 1 - \left(1 + \frac{\theta(r+d)^\beta}{\theta + 1}\right) e^{-\theta(r+d)^\beta} = 1 - d \tag{6.29}$$
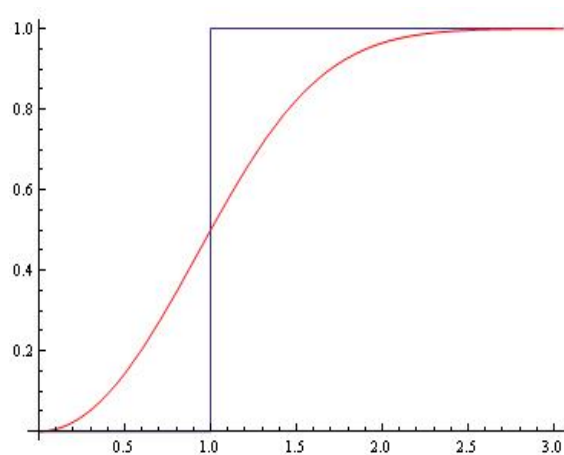
Figure 6.5: The exponentiated power Lindley function with $r = 1$, $\beta = 2$, $\theta = 1.11684$ (computed by (6.29)); H–distance $d = 0.291704$.
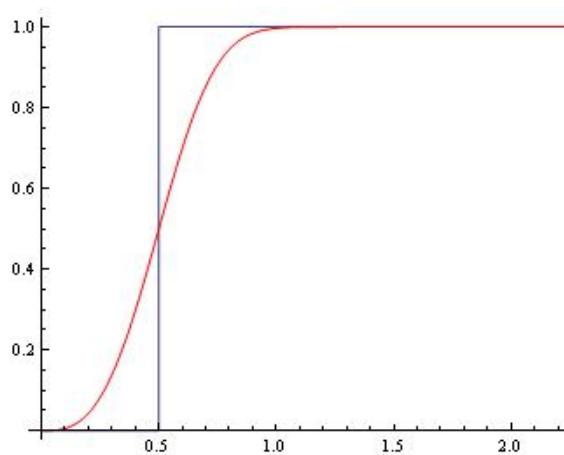


Figure 6.6: The exponentiated power Lindley function with $r = 0.5$, $\beta = 3$, $\theta = 6.36585$ (computed by (6.29)); H–distance $d = 0.176549$.
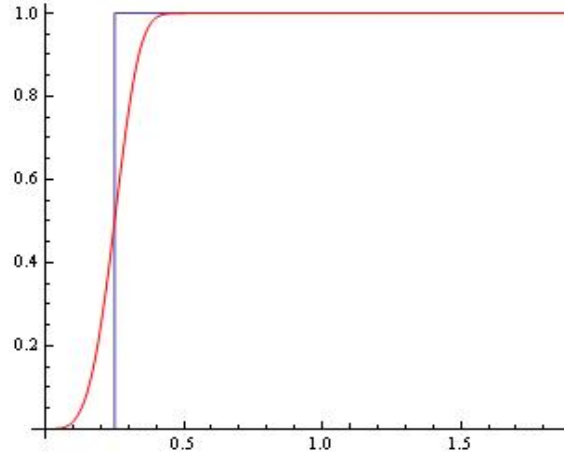
Figure 6.7: The exponentiated power Lindley function with $r = 0.25$, $\beta = 4$, $\theta = 178.438$ (computed by (6.29)); H–distance $d = 0.0909372$.

or

$$-\left(1 + \frac{\theta(r+d)^{\beta}}{\theta+1}\right) e^{-\theta(r+d)^{\beta}} + d = 0. \tag{6.30}$$

Let

$$F(d) = -\left(1 + \frac{\theta(r+d)^{\beta}}{\theta+1}\right) e^{-\theta(r+d)^{\beta}} + d$$

and

$$G(d) = -a + bd,$$

where

$$a = e^{-r^{\beta}\theta}\frac{1+\theta+r^{\beta}\theta}{1+\theta},$$

$$b = \frac{e^{-r^{\beta}\theta}\left(e^{r^{\beta}\theta}r + e^{r^{\beta}\theta}r\theta + r^{\beta}\beta\theta^2 + r^{2\beta}\beta\theta^2\right)}{r(1+\theta)}$$

We note that from Taylor expansion we have

$$G(d) - F(d) = O(d^2)$$

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)(see, Fig.6.8)$.

The above can be summarized in the following

**Theorem 6.4** *For the H–distance $d$ between the Heaviside step function $h_r(t)$ and the exponentiated power Lindley function (6.28) the following lower bound holds*
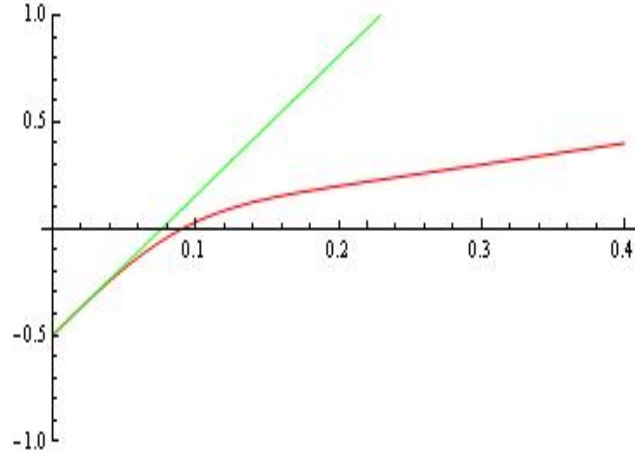
$$\frac{a}{b} < d. \tag{6.31}$$

Figure 6.8: The functions $F(d)$ and $G(d)$ for $r = 0.25$, $\beta = 4$, $\theta = 178.438$.

For related results, see [95–108].

Definition 4. [93] The exponentiated power Lindley function with three parameters is defined by:

$$F(t, \theta, \beta, \alpha) = \left(1 - \left(1 + \frac{\theta t^\beta}{\theta + 1}\right) e^{-\theta t^\beta}\right)^\alpha,$$ (6.32)

where $t > 0$, $\theta > 0$, $\alpha$ and $\beta > 0$.

We examine the special case

$$F(r, \theta, \beta, \alpha) = \frac{1}{2} = \left(1 - \left(1 + \frac{\theta r^\beta}{\theta + 1}\right) e^{-\theta r^\beta}\right)^\alpha.$$ (6.33)

For given $r$, $\alpha$ and $\beta$ the parameter $\theta$ is the solution of the following nonlinear equation:

$$\ln\left(1 - \left(1 + \frac{\theta r^\beta}{\theta + 1}\right) e^{-\theta r^\beta}\right) - \frac{1}{\alpha} \ln 2 = 0.$$ (6.34)

The equation (6.34) has unique positive solution.

**Special case**. The special exponentiated power Lindley function with 3 parameters for $t > 0$, $\theta > 0$ and $\beta > 0$ is defined by:

$$F(t, \theta, \beta, \alpha) = \left(1 - \left(1 + \frac{\theta t^\beta}{\theta+1}\right) e^{-\theta t^\beta}\right)^\alpha$$

$$\ln\left(1 - \left(1 + \frac{\theta r^\beta}{\theta+1}\right) e^{-\theta r^\beta}\right) - \frac{1}{\alpha} \ln 2 = 0$$ (6.35)

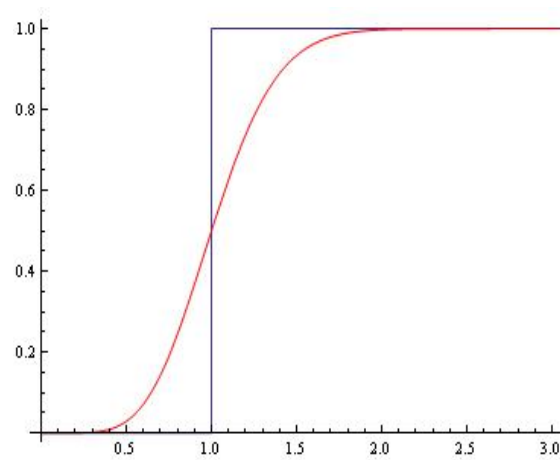$$F(r, \theta, \beta, \alpha) = \frac{1}{2}.$$

Figure 6.9: The exponentiated power Lindley function with 3 parameters for $r = 1$, $\beta = 2$, $\alpha = 3$, $\theta = 2.0954$ (computed by (6.34)); H–distance $d = 0.232062$.
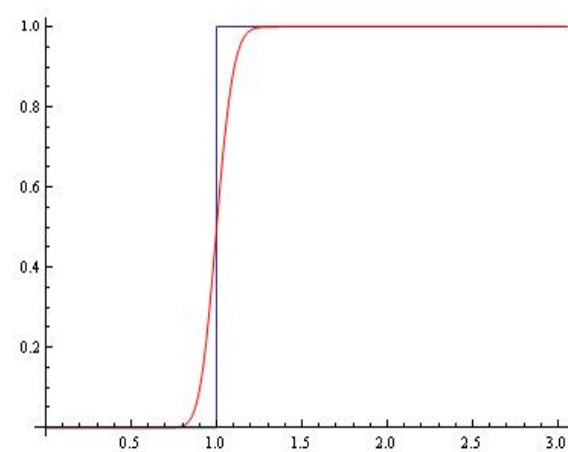


Figure 6.10: The exponentiated power Lindley function with 3 parameters for $r = 1$, $\beta = 5$, $\alpha = 10$, $\theta = 3.27223$ (computed by (6.34)); H–distance $d = 0.102878$.
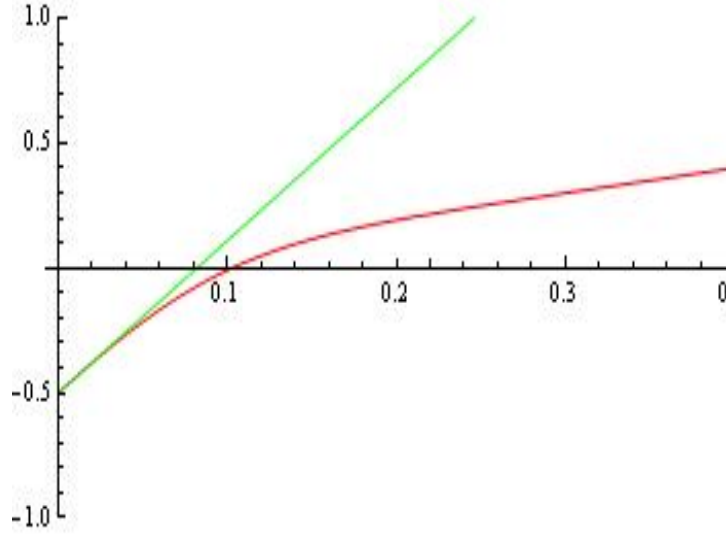
Figure 6.11: The functions $F(d)$ and $G(d)$ for $r = 1$, $\beta = 5$, $\alpha = 10$, $\theta = 3.27223$.

We study the Hausdorff approximation [14] of the shifted Heaviside step function $h_r(t)$ by special exponentiated power Lindley function of the form (6.35) and find an expression for the error of the best approximation.

The Hausdorff distance $d$ between the function $h_r(t)$ and the function (6.35) satisfies the relation

$$F(r+d, \theta, \beta, \alpha) = \left(1 - \left(1 + \frac{\theta(r+d)}{\beta}\theta + 1\right)e^{-\theta(r+d\beta)}\right)^\alpha = 1 - d. \tag{6.36}$$

or

$$\left(1 - \left(1 + \frac{\theta(r+d)}{\beta}\theta + 1\right)e^{-\theta(r+d\beta)}\right)^\alpha - 1 + d = 0. \tag{6.37}$$

Figures 6.9–6.10 illustrates some of the possible shapes of the cdf of a exponentiated power Lindley function (6.35) for selected values of the parameters.

Let

$$F(d) = \left(1 - \left(1 + \frac{\theta(r+d)}{\beta}\theta + 1\right)e^{-\theta(r+d\beta)}\right)^\alpha - 1 + d$$

and

$$G(d) = -a + bd,$$

where

$$a = 1 - \left(1 - e^{-r^\beta\theta} - \frac{e^{-r^\beta\theta}r^\beta\theta}{1+\theta}\right)^\alpha,$$

$$b = 1 + \alpha\left(1 - e^{-r^\beta\theta} - \frac{e^{-r^\beta\theta}r^\beta\theta}{1+\theta}\right)^{-1+\alpha}\left(e^{-r^\beta\theta}r^{-1+\beta}\beta\theta - \frac{e^{-r^\beta\theta}r^{-1+\beta}\beta\theta}{1+\theta} + \frac{e^{-r^\beta\theta}r^{-1+2\beta}\beta\theta^2}{1+\theta}\right)$$

We note that from Taylor expansion we have

$$G(d) - F(d) = O(d^2)$$

and $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)(see, Fig.6.11)$.

The above can be summarized in the following

**Theorem 6.5** *For the H–distance $d$ between the Heaviside step function $h_r(t)$ and the exponentiated power Lindley function (6.35) the following lower bound holds*

$$\frac{a}{b} < d. \tag{6.38}$$

# Chapter 7

# The Burr sigmoid function

In this Chapter the Hausdorff approximation of the shifted Heaviside step function $h_r(t)$ by sigmoid Burr function is considered and lower bound for the Hausdorff distance has been obtained.

Numerical examples, which illustrate our results are given, too.

## 7.1   Introduction

Definition 1. [109] The cumulative Burr function is:

$$F(t, c, k) = 1 - \frac{1}{(1 + t^c)^k}, \tag{7.1}$$

where $t > 0$, $c > 0$ and $k > 0$.

We examine the special case

$$F(r, c, k) = \frac{1}{2} = 1 - \frac{1}{(1 + r^c)^k}. \tag{7.2}$$

From (7.2) we have

$$k = \frac{\ln 2}{\ln(1 + r^c)}.$$

**Special case**. The special sigmoid Burr function is defined by:

$$F(t, c, k) = 1 - \frac{1}{(1+t^c)^k},$$

$$k = \frac{\ln 2}{\ln(1+r^c)}, \tag{7.3}$$

$$F(r, c, k) = \frac{1}{2}.$$

We study the Hausdorff approximation [14] of the shifted step function $h_r(t)$ by sigmoid Burr function of the form (7.3) and find an expression for the error of the best approximation.

## 7.2   Approximation of the $h_r(t)$ by sigmoid Burr function

The Hausdorff distance $d$ between the function $h_r(t)$ and the sigmoid Burr function (7.3) satisfies the relation

$$F(r+d,c,k) = 1 - \frac{1}{(1+(r+d)^c)^k} = 1 - d, \qquad (7.4)$$

or

$$\frac{1}{(1+(r+d)^c)^k} - d = 0. \qquad (7.5)$$

The equation (7.5) has unique positive solution.

Let

$$f(d) = \frac{1}{(1+(r+d)^c)^k} - d$$

and

$$g(d) = -a + bd,$$

where

$$a = (1+r^c)^{-k},$$

$$b = 1 + ckr^{-1+c}(1+r^c)^{-1-k}.$$

**Theorem 7.1** *For the H–distance $d$ between the Heaviside step function $h_r(t)$ and the sigmoid Burr function the following lower bound holds*

$$\frac{a}{b} < d. \qquad (7.6)$$

The proof follows the ideas given in this book and will be omitted (see, Fig. 7.4).

Figures 7.1–7.3 illustrates some of the possible shapes of a sigmoidal Burr function (7.3) for selected values of the parameters.

When $c = 1$, the Burr distribution becomes the Pareto distribution.

When $k = 1$, the Burr distribution is a special case of the Champernowne distribution, often referred to as the Fisk distribution.

For related results, see [110–125].
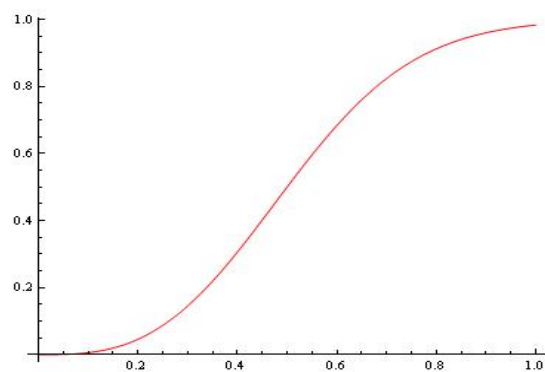
Figure 7.1: The sigmoid Burr function with $r = 0.5$, $c = 3$, $k = 5.88495$ (computed by (7.3)); H–distance $d = 0.189057$.
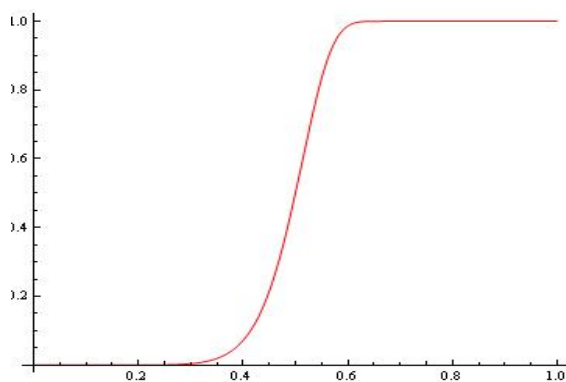


Figure 7.2: The sigmoid Burr function with $r = 0.5$, $c = 10$, $k = 710.129$ (computed by (7.3)); H–distance $d = 0.0715597$.
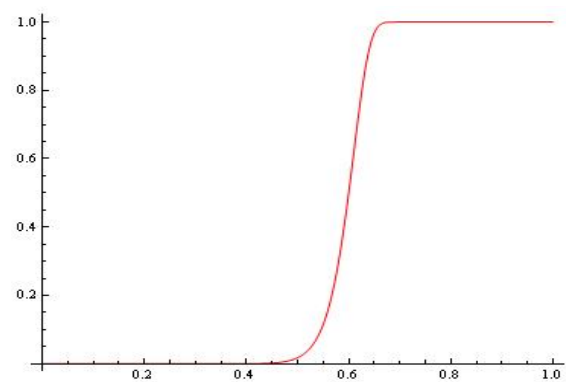


Figure 7.3: The sigmoid Burr function with $r = 0.6$, $c = 20$, $k = 18958.7$ (computed by (7.3)); H–distance $d = 0.0463638$.
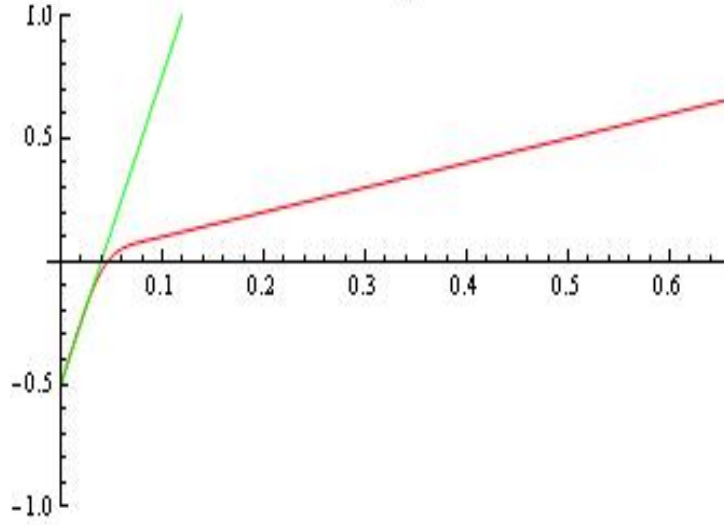
Figure 7.4: The functions $f(d)$ and $g(d)$ for $r = 0.6$, $c = 20$, $k = 18958.7$, $d = 0.0463638$.

## 7.3   Approximation of the $h_r(t)$ by generalized Burr function

Definition 2. The cumulative three–parameters Burr function is:

$$F_1(t, \alpha, c, k) = 1 - \frac{1}{\left(1 + \left(\frac{t}{\alpha}\right)^c\right)^k}, \tag{7.7}$$

where $t > 0$, $\alpha > 0$, $c > 0$ and $k > 0$.

We examine the special case

$$F_1(r, \alpha, c, k) = \frac{1}{2} = 1 - \frac{1}{\left(1 + \left(\frac{r}{\alpha}\right)^c\right)^k}. \tag{7.8}$$

From (7.8) we have

$$k = \frac{\ln 2}{\ln(1 + \left(\frac{r}{\alpha}\right)^c)}.$$

**Special case**. The special three–parameters Burr function is defined by:

$$F_1(t, \alpha, c, k) = 1 - \frac{1}{\left(1 + \left(\frac{t}{\alpha}\right)^c\right)^k},$$

$$k = \frac{\ln 2}{\ln(1 + \left(\frac{r}{\alpha}\right)^c)}, \tag{7.9}$$

$$F_1(r, \alpha, c, k) = \frac{1}{2}.$$
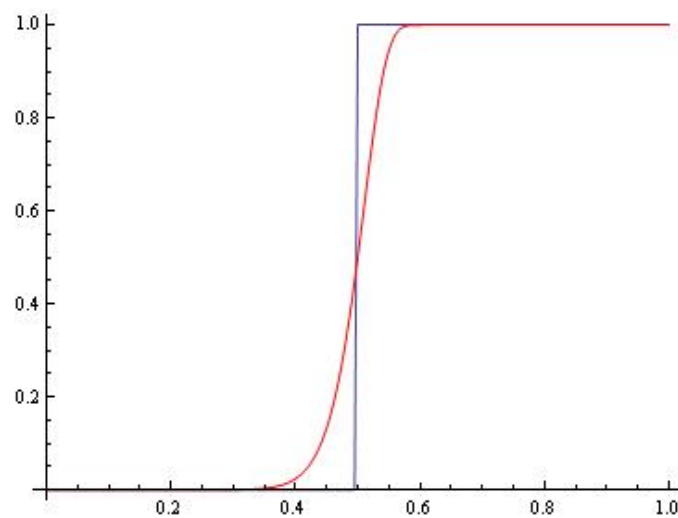
Figure 7.5: The sigmoid three–parameters Burr function with $r = 0.5$, $c = 15$, $\alpha = 0.9$, $k = 4676.76$ (computed by $(7.9)$); H–distance $d = 0.0510122$.
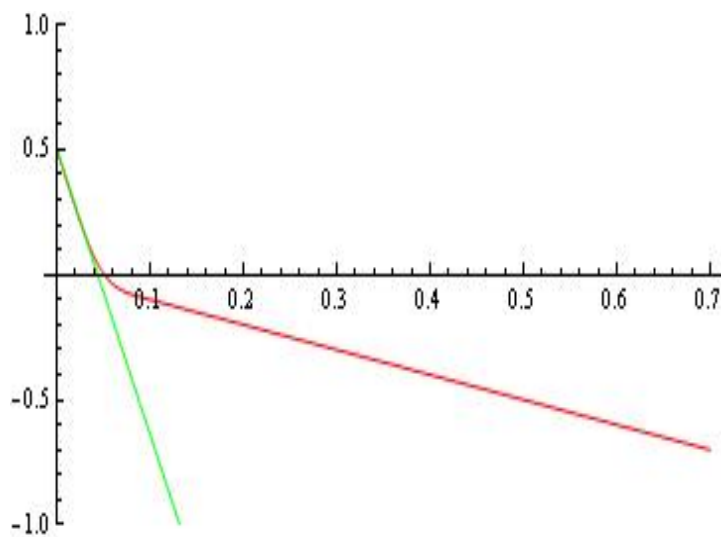


Figure 7.6: The functions $f(d)$ and $g(d)$ for $r = 0.5$, $c = 15$, $\alpha = 0.9$, $k = 4676.76$.

The Hausdorff distance $d$ between the function $h_r(t)$ and the three–parameters Burr function (7.9) satisfies the relation

$$F_1(r+d, \alpha, c, k) = 1 - \frac{1}{\left(1 + \left(\frac{r+d}{\alpha}\right)^c\right)^k} = 1 - d, \qquad (7.10)$$

or

$$\frac{1}{(1 + (\frac{r+d}{\alpha})^c)^k} - d = 0. \qquad (7.11)$$

The equation (7.11) has unique positive solution.

Let

$$f(d) = \frac{1}{(1 + (r+d)^c)^k} - d$$

and

$$g(d) = a - bd,$$

where

$$a = \left(1 + \left(\frac{r}{\alpha}\right)^c\right)^{-k},$$

$$b = 1 + \frac{ck\left(1 + \left(\frac{r}{\alpha}\right)^c\right)^{-1-k}\left(\frac{r}{\alpha}\right)^{-1+c}}{\alpha}.$$

**Theorem 7.2** *For the H–distance $d$ between the Heaviside step function $h_r(t)$ and the sigmoid three–parameters Burr function the following lower bound holds*

$$\frac{a}{b} < d. \qquad (7.12)$$

The proof follows the ideas given in this book and will be omitted (see, Fig. 7.6).

**Remark.** In the present book we do not consider sigmoid functions generated as cumulative functions of other probabilistic distributions, such as the Lomax, Pareto, Gumbel (also known as Fisher-Tippett), Log-Gumbel, and Weibull distributions, see [126–142].

Based on the methodology proposed in the present book, the reader may formulate the corresponding approximation problems on his/her own.

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