CALCULUS

for the Practical Man

by

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Chapter 1

FUNDAMENTAL IDEAS. RATES AND DIFFERENTIALS

1 Rates

The most natural illustration of a rate is that involving motion and time. If an object is moving steadily as time passes, its speed is the distance or space passed over in a specified unit of time, as, for example, 40 miles per hour, 1 mile per minute, 32 feet per second, etc. This speed of motion is the time rate of change of distance, and is found simply by dividing the space passed over by the time required to pass over it, both being expressed in suitable units of measurement. If the motion is such as to increase the distance from a chosen reference point, the rate is taken as positive; if the distance on that same side of the reference point decreases, the rate is said to be negative.

These familiar notions are visualized and put in concise mathematical form by considering a picture or graph representing the motion. Thus in Figure 1 let the motion take place along the straight line OX in the direction from O toward X. Let O be taken as the reference point, and let P represent the position of the moving object or point. The direction of motion is then indicated by the arrow and the distance of P from O at any particular instant is the length OP which is represented by x.

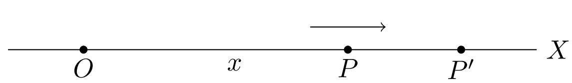

Figure 1: Motion along a straight line.

When the speed is uniform and the whole distance x and the total time t required to reach P are both known, the speed or rate is simply $x \div t$ or $\frac{x}{t}$. If the total distance and time from the starting point are not known, but the rate is still constant, the clock times of passing two points P and P' can be noted and the corresponding distances x and x' measured. Then the rate is

$$\text{Rate} = \frac{x' - x}{t' - t}$$

If the x difference is written dx and the t difference is written dt then the

$$\text{Rate} = \frac{dx}{dt} \tag{1}$$

The symbols dx and dt are not products d times x or d times t as in algebra, but each represents a single quantity, the x or t difference. They are pronounced as one would pronounce his own initials, thus: dx, "dee-ex"; and dt, "dee-tee." These symbols and the quantities which they represent are called differentials. Thus dx is the differential of x and dt the differential of t.

If, in Figure 1, P moves in the direction indicated by the arrow, the rate is taken as positive and the expression (1) is written

$$\text{Rate} = +\frac{dx}{dt}$$

This will apply when P is to the right or left of O, so long as the sense of the motion is toward the right (increasing x) as indicated by the arrow. If it is in the opposite sense, the rate is negative (decreasing x) and is written

$$\text{Rate} = -\frac{dx}{dt}$$

These considerations hold in general and we shall consider always that when the rate of any variable is positive the variable is increasing, when negative it is decreasing.

So far the idea involved is familiar and only the terms used are new. Suppose, however, the object or point P is increasing its speed when we attempt to measure and calculate the rate, or suppose it is slowing down, as when accelerating an automobile or applying the brakes to stop it; what is the speed then, and how shall the rate be measured or expressed in symbols? Or suppose P moves on a circle or other curved path so that its direction is changing, and the arrow in Figure 1 no longer has the significance we have attached to it. How then shall $\frac{dx}{dt}$ be measured or expressed?

These questions bring us to the consideration of variable rates and the heart of the methods of calculus, and we shall find that the scheme given above still applies, the key to the question lying in the differentials dx and dt.

The idea of differentials has here been developed at considerable length because of its extreme importance, and should be mastered thoroughly. The next section will emphasize this statement.

2 Varying Rates

With the method already developed in the preceding section, the present subject can be discussed concisely and more briefly. If the speed of a moving point be not uniform, its numerical measure at any particular instant is the number of units of distance which would be described in a unit of time if the speed were to remain constant from and after that instant. Thus, if a car is speeding up as the engine is accelerated, we would say that it has a velocity of, say, 32 feet per second at any particular instant if it should move for the next second at the same speed it had at that instant and cover a distance of 32 feet. The actual space passed over may be greater if accelerating or less if braking, because of the change in the rate which takes place in that second, but the rate at that instant would be that just stated.

To obtain the measure of this rate at any specified instant, the same principle is used as was used in article 1. Thus, if in Figure 1 dt is any chosen interval of time and $PP' = dx$ is the space which would be covered in that interval, were P to move over the distance PP' with the same speed unchanged which it had at P, then the rate at P is $\frac{dx}{dt}$. The quantity dx is plus or minus according as P moves in the sense of the arrow in Figure 1 or the opposite.

If the point P is moving on a curved path of any kind so that its direction is continually changing, say on a circle, as in Figure 2, then the direction at any instant is that of the tangent to the path at the point P at that instant, as

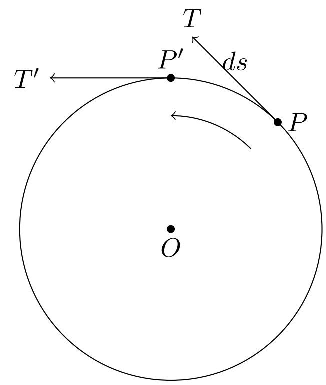

Figure 2: Motion along a circular path.

PT at P and $P'T'$ at P'. The space differential ds is laid off on the direction at P and is taken as the space which P would cover in the time interval dt if the speed and direction were to remain the same during the interval as at P. The rate is then, as usual, $\frac{ds}{dt}$ and is plus or minus according as P moves along the curve in the sense indicated by the curved arrow or the reverse.

3 Differentials

In the preceding discussions the quantities dx or ds and dt have been called the differentials of x, s and t. Now time passes steadily and without ceasing so that dt will always exist. By reference to chosen instants of time the interval dt can be made as great or as small as desired, but it is always formed in the same manner and sense and is always positive, since time never flows backward. The differential of any other variable quantity x may be formed in any way desired if the variation of x is under control and may be great or small, positive or negative, as desired, or if the variable is not under control its differential may be observed or measured and its sense or sign (plus or minus) determined, positive for an increase during the interval dt and negative for a decrease. The rate of x, dx/dt, will then depend on dx and since dt is always positive, dx/dt will be positive or negative according as dx is plus or minus.

From the discussions in articles 1 and 2 it is at once seen that the definition of the differential of a variable quantity is the following:

The differential dx of a variable quantity x at any instant is the change in

x which would occur in the next interval of time dt if x were to continue to change uniformly in the interval dt with the same rate which it has at the beginning of dt.

Using this definition of the differential we then define:

The mathematical rate of x at the specified instant is the quotient of dx by dt, that is, the ratio of the differentials.

The differential of any variable quantity is indicated by writing the letter d before the symbol representing the quantity. Thus the differential of x^2 is written $d(x^2)$, the differential of $\sqrt{x}$ is written $d(\sqrt{x})$. The differential of x^2 or of $\sqrt{x}$ will of course depend on the differential of x itself. Similarly $d(\sin\theta)$ will depend on $d\theta$, $d(\log_b x)$ will depend on dx and also on the base b. When the differentials $d(x^2)$, $d(\sqrt{x})$, $d(\sin\theta)$ are known or expressions for them have been found then the rates of these quantities will be

$$\frac{d(x^2)}{dt}, \frac{d(\sqrt{x})}{dt}, \frac{d(\sin\theta)}{dt}, \text{ etc.},$$

and will depend on the rates $\frac{dx}{dt}$, $\frac{d\theta}{dt}$, etc.

Now, in mathematical problems, such expressions as x^2, $\sqrt{x}$, $\sin\theta$, $\log x$, $x+y$, $x-y$, xy, x/y, etc., are of regular and frequent occurrence. In order to study problems involving changing quantities which contain such expressions as the above, it is necessary to be able to find their differentials and rates.

The finding or calculation of differentials is called differentiation and is one of the most important parts of the subject of calculus, that part of the subject which deals with differentiation and its applications being called the differential calculus.

4 Differential of a Variable with Constant Rate

If a point P moves along a straight line with constant speed k units per unit of time, then at any instant its rate is

$$\frac{dx}{dt} = k \tag{2}$$

At this speed the point P will, in the length of time t, move over a distance equal to kt, the speed multiplied by the time. If at the beginning of this time, the instant of starting, P were already at a certain fixed distance a from the reference point O, then at the end of the time t the total distance is the sum

$$x = a + kt \tag{3}$$

If P starts at the same point and moves in the opposite (negative) direction, then the total distance after the time t is the difference

$$x = a - kt \tag{4}$$

and the rate is

$$\frac{dx}{dt} = -k \tag{5}$$

The several equations (2) to (5) may be combined by saying that if

$$x = a \pm kt, \frac{dx}{dt} = \pm k \tag{6}$$

Considering expressions (2) and (3), since x equals $a + kt$ then of course the rate of x equals the rate of $a + kt$, that is, $\frac{dx}{dt} = \frac{d(a+kt)}{dt}$. But by (2) $\frac{dx}{dt} = k$, therefore

$$\frac{d(a + kt)}{dt} = k \tag{7}$$

In the same manner from (4) and (5) we get

$$\frac{d(a - kt)}{dt} = -k$$

These results will be used in finding the differentials of other simple expressions. It is to be remembered that equation (3) is the expression for the value of any variable x (in this case a distance) at any time t when its rate is constant, and that (7) gives the value of this rate in terms of the right side of (3), which is equal to the variable x.

5 Differential of a Sum or Difference of Variables

We can arrive at an expression for the differential of a sum or difference of two or more variables in an intuitive way by noting that since the sum is made up of the parts which are the several variables, then, if each of the parts changes by a certain amount which is expressed as its differential, the change in the sum, which is its differential, will of course be the sum of the changes in the separate parts, that is, the sum of the several differentials of the parts.

In order to get an exact and logical expression for this differential, however, it is better to base it on the precise results established in the preceding article, which are natural and easily understood, as well as being mathematically correct.

Thus let k denote the rate of any variable quantity x (distance or any other quantity), and k' the rate of another variable y. Then, as in the example of the last article, we can write, as in equation (3),

$$x = a + kt$$

and also

$$y = b + k't$$

the numbers a and b being the constant initial values of x and y. Adding these two equations member by member we get

$$x + y = a + b + kt + k't$$

$$(x + y) = (a + b) + (k + k')t$$

Now, this equation is of the same form as equation (3), $(x + y)$ replacing x and $(a + b)$, $(k + k')$ replacing a, k, respectively. As in (2) and (7), therefore,

$$\frac{d(x + y)}{dt} = (k + k')$$

But k is the rate of x, dx/dt, and k' is the rate of y, dy/dt. Therefore

$$\frac{d(x + y)}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

Multiplying both sides of this equation by dt in order to have differentials instead of rates, there results

$$d(x + y) = dx + dy \tag{8}$$

If instead of adding the two equations above we had subtracted the second from the first, we would have obtained instead of (8) the result

$$d(x - y) = dx - dy$$

This result and (8) may be combined into one by writing

$$d(x \pm y) = dx \pm dy \tag{9}$$

In the same way three or more equations such as (3) above might be written for three or more variables x, y, z, etc., and we would obtain instead of (9) the result

$$d(x \pm y \pm z \pm \cdots) = dx \pm dy \pm dz \pm \cdots \tag{A}$$

the dots meaning "and so on" for as many variables as there may be.

We shall find that formula (A) in which x, y, z, etc., may be any single variables or other algebraic terms is of fundamental importance and very frequent use in the differential calculus.

6 Differential of a Constant and of a Negative Variable

Since a constant is a quantity which does not change, it has no rate or differential, or otherwise expressed, its rate or differential is zero. That is, if c is a constant

$$dc = 0 \tag{B}$$

Then, in an expression like $x + c$, since c does not change, any change in the value of the entire expression must be due simply to the change in the variable x, that is, the differential of $x + c$ is equal simply to that of x and we write

$$d(x + c) = dx \tag{C}$$

This might also have been derived from (8) or (9). Thus,

$$d(x \pm c) = dx \pm dc$$

but by (B) $dc = 0$ and, therefore, $d(x \pm c) = dx$ which is the same as (C), either the plus or minus sign applying in (C).

Consider the expression $y = -x$; then $y+x = 0$ and $d(y+x) = d(0)$, but zero does not change and therefore $d(0) = 0$. Therefore, $d(y + x) = dy + dx = 0$, or $dy = -dx$. But $y = -x$; therefore,

$$d(-x) = -dx \tag{D}$$

7 Differential of the Product of a Constant and a Variable

Let us refer now to formula (A) and suppose all the terms to be the same; then

$$d(x + x + x + \cdots) = dx + dx + dx + \cdots$$

If there are m such terms, with m constant, then the sum of the terms is mx and the sum of the differentials is $m\,dx$. Therefore,

$$d(mx) = mdx \tag{E}$$

Since we might have used either the plus or minus sign in (A) we may write (E) with either $+m$ or $-m$. In general, (E) holds good for any constant m, positive or negative, whole, fractional or mixed, and regardless of the form of the variable which is here represented by x.

Chapter 2

FUNCTIONS AND DERIVATIVES

8 Meaning of a Function

Meaning of a Function.-In the solution of problems in algebra and trigonometry one of the important steps is the expression of one quantity in terms of another. The unknown quantity is found as soon as an equation or formula can be written which contains the unknown quantity on one side of the equation and only known quantities on the other. Even though the equation does not give the unknown quantity explicitly, if any relation can be found connecting the known and unknown quantities it can frequently be solved or transformed in such a way that the unknown can be found if sufficient data are given.

Even though the data may not be given so as to calculate the numerical value of the unknown, if the connecting relation can be found the problem is said to be solved. Thus, consider a right triangle having legs x, y and hypotenuse c and suppose the hypotenuse to retain the same value (c constant) while the legs are allowed to take on different consistent values (x and y variable). Then, to every different value of one of the legs there corresponds a definite value of the other leg. Thus if x is given a particular length consistent with the value of c, y can be determined. This is done as follows: The relation between the three quantities x, y, c is first formulated. For the right triangle, this is,

$$x^2 + y^2 = c^2 \tag{10}$$

Considering this as an algebraic equation, in order to determine y when a value is assigned to x the equation is to be solved for y in terms of x and the constant c. This gives

$$y = \sqrt{c^2 - x^2} \tag{11}$$

In this expression, whenever x is given, y is determined and to every value of x there corresponds a value of y whether it be numerically calculated or not. The variable y is said to be a function of the variable x. The latter is called the independent variable and y is called the dependent variable. We can then in general define a function by saying that,

If when x is given, y is determined, y is a function of x.

Examples of functions occur on every hand in algebra, trigonometry, mechanics, electricity, etc. Thus, in equation (2), x is a function of t; if $y = g(x)$ or $y = x^2$, y is a function of x. If in the right triangle discussed above, $heta$ be the angle opposite the side y, then, from trigonometry, $y = c\sin\theta$ and with c constant y is a function of θ. Also the trigonometric or angle functions sine, cosine, tangent, etc., are functions of their angle; thus $\sin\theta$, $\cos\theta$, $\tan\theta$ are determined as soon as the value of θ is given.

In the mechanics of falling bodies, if a body falls freely from a position of rest then at any time t seconds after it begins to fall it has covered a space $s = 16t^2$ feet and s is a function of t; also when it has fallen through a space s feet it has attained a speed of $v = 8\sqrt{s}$ feet per second, and v is a function of s. If a variable resistance R ohms is inserted in series with a constant electromotive force E volts the electric current I in amperes will vary as R is varied and according to Ohm's Law of the electric circuit is given by the formula $I = E/R$; the current is a function of the resistance.

In general, the study and formulation of relations between quantities which may have any consistent value is a matter of functional relations and when one quantity is expressed by an equation or formula as a function of the other or others the problem is solved. The numerical value of the dependent variable can then by means of the functional expression be calculated as soon as numerical values are known for the independent variable or variables and constants.

In order to state that one quantity y is a function of another quantity x we write $y = f(x)$, $y = f'(x)$, $y = F(x)$, etc., each symbol expressing a different form of function. Thus, in equation (11) above we can say that $y = f(x)$, and similarly in some of the other relations given, $s = F(t)$, $I = \phi(R)$, etc.

If in (11) both x and c are variables, then values of both x and c must be given in order that y may be determined, and y is a function of both x and c. This is expressed by writing $y = f(x, c)$. If in Ohm's Law both E and R are variable, then both must be specified before I can be calculated and I is a function of both, $I = \phi(E, R)$.

In (11) where $y = \sqrt{c^2 - x^2}$ we can also find $x = \sqrt{c^2 - y^2}$ and x is an inverse function of y; similarly if $x = \sin\theta$ then $\theta = \sin^{-1} x$ (read "anti-sine" or "angle whose sine is") is the inverse function. In general if y is a function of x then x is the inverse function of y, and so for any two variables.

9 Classification of Functions

Functions are named or classified according to their form, origin, method of formation, etc. Thus the sine, cosine, tangent, etc., are called the trigonometric or angular (angle) functions. Functions such as x^2, $\sqrt{x}$, $x^2 + \sqrt{x}$, $3\sqrt{x} - 2/x$, formed by using only the fundamental algebraic operations (addition, subtraction, multiplication, division, involution, evolution) are called algebraic functions. A function such as b^x, where b is a constant and x variable, is called an exponential function of x and $\log_b x$ is a logarithmic function of x.

In order to distinguish them from the algebraic functions the trigonometric, exponential and logarithmic functions and certain combinations of these are called transcendental functions. We shall find in a later chapter that transcendental functions are of great importance in both pure and applied mathematics.

Another classification of functions is based on a comparison of equations (10) and (11). In (11) y is given explicitly as a function of x and is said to be an explicit function of x. In (10) if x is taken as independent variable then y can be found but as the equation stands the value of y is not given directly and y is said to be an implicit function of x.

Explicit or implicit functions may be algebraic or transcendental.

10 Differential of a Function of an Independent Variable

If y is a function of x, written $y = f(x)$,

$$y = f(x) \tag{12}$$

then, since a given value of x will determine the corresponding value of y, the rate of y, dy/dt, will depend on both x and the rate dx/dt at any particular instant. Similarly, for the same value of dt, dy will depend on both x and dx.

To differentiate a function is to express its differential in terms of both the independent variable and the differential of the independent variable. Thus, in the case of the function (12) dy will be a function of both x and dx.

If two expressions or quantities are always equal, their rates taken at the same time must evidently be equal and so also their differentials. An equation can, therefore, be differentiated by finding the differentials of its two members and equating them. Thus from the equation

$$(x+c)^2 = x^2 + 2cx + c^2$$

by differentiating both sides and using formula (A) on the right side. Since c, c and 2 are constants, then by formulas (B) and (E) $d(2cx) = 2c\,dx$ and $d(c^2) = 0$. Therefore,

$$d[(x+c)^2] = d(x^2) + 2c\,dx \tag{13}$$

Thus, if the function $(x+c)^2$ is expressed as

$$y = (x+c)^2 \implies dy = d(x^2) + 2c\,dx \tag{13a}$$

and, dividing by dt, the relation between the rates is

$$\frac{dy}{dt} = \frac{d(x^2)}{dt} + 2c\frac{dx}{dt} \tag{13b}$$

From equation (13a) we can express dy in terms of x and dx when we can express $d(x^2)$ in terms of x and dx. This we shall do presently.

11 The Derivative of a Function

If y is any function of x, as in equation (12),

$$y = f(x) \tag{12}$$

then, as seen above, the rate or differential of y will depend on the rate or differential of x and also on x itself. There is, however, another important function of x which can be derived from y which does not depend on dx or dx/dt but only on x. This is true for any ordinary function whatever and will be proven for the general form (12) once for all. The demonstration is somewhat formal, but in view of the definiteness and exactness of the result it is better to give it in mathematical form rather than by means of a descriptive and intuitive form.

In order to determine the value of y in the functional equation (12), let the independent variable x have a particular value a at a particular instant and let dx be purely arbitrary, that is, chosen at will. Then, even though dx is arbitrary, so also is dt, and, therefore, the rate dx/dt can be given any chosen definite, fixed value at the instant when $x = a$. Let this fixed value of the rate be

$$\frac{dx}{dt} = k' \tag{14}$$

The corresponding rate of y will evidently depend on the particular form of the function $f(x)$, as, for example, if the function is $(x + c)^2$ the rate dy/dt is given by equation (13b). Therefore, when dx/dt is definitely fixed, so also is dy/dt. Let this value be represented by

$$\frac{dy}{dt} = k'' \tag{15}$$

Now, since both rates are fixed and definite, so also will be their ratio. Let this ratio be represented by k. Then, from (14) and (15),

$$\frac{dy/dt}{dx/dt} = \frac{k''}{k'} = k$$

Now, $(dy/dt) \div (dx/dt) = dy/dx$ and, therefore,

$$\frac{dy}{dx} = k$$

Since k is definite and fixed, while dx may have any arbitrary value, then k cannot depend on dx. That is, the quantity dy/dx, which is equal to k, cannot depend on dx. It must depend on x alone, that is, dy/dx is a function of x. In general, it is a new function of x different from the original function $f(x)$ from which it was derived. This derived function is denoted by $f'(x)$. We write, then

$$\frac{dy}{dx} = f'(x) \tag{16}$$

This new function is called the derivative of the original function $f(x)$.

Since (16) can also be written as

$$dy = f'(x) \cdot dx \tag{17}$$

in which the differential of the dependent variable is equal to the product of the derivative by the differential of the independent variable, the derivative is also sometimes called the differential coefficient of y regarded as a function of x.

There are thus several ways of viewing the function which we have called the derivative. If we are thinking of a function $f(x)$ as a mathematical expression in any form, then the derivative is thought of as the derived function. If we refer particularly to the dependent variable y as an explicit function of the independent variable x, then we express the derivative as dy/dx (read "dy by dx") and refer to it as the "derivative of y with respect to x."

The derivative was first found, however, as the ratio of the rates of dependent and independent variables, from equations (14) and (15), and this is its proper definition. Using this definition, that is, $(dy/dt) \div (dx/dt) = f'(x)$, we have

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt} \tag{18}$$

and for use in practical problems involving varying quantities this is the most useful way of viewing it. Based on this definition, equation (18) tells us that when we once have an equation expressing one variable as a function of another, the derivative is the function or quantity by which the rate of the independent variable must be multiplied in order to obtain the rate of the dependent variable.

A geometrical interpretation of this important function as applied to graphs will be given later.

In order to find this important function in any particular case equation (16) tells us that we must find the differential of the dependent variable and divide it by the differential of the independent variable. In the next chapter we take up the important matter of finding the differentials and derivatives of some fundamental algebraic functions.

Chapter 3

DIFFERENTIALS OF ALGEBRAIC FUNCTIONS

12 Introduction

In the preceding chapter we saw that in order to find the derivative of a function we must first find its differential, and in Chapter I we saw that in order to find the rate of a varying quantity, we must also first find its differential. We then found the differentials of a few simple but important forms of expressions. These will be useful in deriving formulas for other differentials and are listed here for reference.

$$d(x \pm y \pm z \pm \cdots) = dx \pm dy \pm dz \pm \cdots \tag{A}$$

$$dc = 0 \tag{B}$$

$$d(x + c) = dx \tag{C}$$

$$d(-x) = -dx \tag{D}$$

$$d(mx) = m\,dx \tag{E}$$

In Chapter I we found that when we have given a certain function of an independent variable, the derivative of the function can be obtained by expressing the differential of the function in terms of the independent variable and its differential, and then dividing by the differential of the independent variable. We now proceed to find the differentials of the fundamental algebraic functions, and it is convenient to begin with the square of a variable.

13 Differential of the Square of a Variable

In order to find this differential let us consider the expression mx of formula (E), and let

$$m = x, \text{ then } z = mx = x^2$$

by squaring; z being the dependent variable and m being a constant. Differentiating these two equations by formula (E),

$$dz = m\,dx, \quad d(z^2) = m^2\,d(x^2),$$

and dividing the second of these results by the first, member by member,

$$\frac{d(z^2)}{dz} = m\frac{d(x^2)}{dx}$$

Dividing this result by the original equation $z = m$ to eliminate the constant m there results

$$\frac{1}{z}\frac{d(z^2)}{dz} = \frac{1}{x}\frac{d(x^2)}{dx} \tag{19}$$

Now, $d(x^2)/dx$ is the derivative of x^2, and similarly for z^2. Furthermore, the connecting constant m has been eliminated and has no bearing on the equation (19). This equation therefore tells us that the derivative of the square of a variable divided by the variable itself (multiplied by the reciprocal) is the same for any two variables x and z. It is, therefore, the same for all variables and has a fixed, constant value, say a. Then,

$$\frac{1}{x}\frac{d(x^2)}{dx} = a$$

$$\therefore d(x^2) = ax\,dx \tag{20}$$

In order to know the value of $d(x^2)$, therefore, we must determine the constant a. This is done as follows:

Since equation (20) is true for the square of any variable, it is true for $(x+c)$ where c is a constant. Therefore,

$$d[(x+c)^2] = a(x+c)\,d(x+c)$$

But, by formula (C), $d(x + c) = dx$, hence,

$$d[(x+c)^2] = a(x+c)\,dx = ax\,dx + ac\,dx$$

Also, according to equation (21),

$$d[(x+c)^2] = d(x^2) + 2c\,dx \tag{21}$$

$$= ax\,dx + 2c\,dx \tag{22}$$

by (20). By (21) and (22), therefore,

$$ac\,dx = 2c\,dx \therefore a = 2$$

and this value of a in (20) gives, finally,

$$d(x^2) = 2x\,dx \tag{F}$$

This is the differential of x^2; dividing by dx the derivative of x^2 with respect to x is

$$\frac{d(x^2)}{dx} = 2x \tag{23}$$

These important results can be stated in words by saying that, "the differential of the square of any variable equals twice the variable times its differential," and, "the derivative of the square of any variable with respect to the variable equals twice the variable."

Referring to article 11, formula (F) corresponds to equation (17) and (23) to equation (16) when $f(x) = x^2$, and therefore $f'(x) = 2x$.

14 Differential of the Square Root of a Variable

Let x be the variable and let

$$y = \sqrt{x}, \text{ then } y^2 = x$$

Differentiating the second equation by formula (F),

$$2y\,dy = dx, \text{ or } dy = \frac{dx}{2y}$$

But $y = \sqrt{x}$, therefore,

$$d(\sqrt{x}) = \frac{1}{2}x^{-\frac{1}{2}}\,dx \tag{G}$$

and the derivative is

$$\frac{d(\sqrt{x})}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \tag{24}$$

Formula (G) can be put in a somewhat different form which is sometimes useful. Thus, $\sqrt{x} = x^{1/2}$ and

$$d(x^{1/2}) = \frac{1}{2}x^{-1/2}\,dx \tag{25}$$

15 Differential of the Product of Two Variables

Let x and y be the two variables. We then wish to find $d(xy)$. Since we already have a formula for the differential of a square we first express the product xy in terms of squares. We do this by writing

$$(x+y)^2 = x^2 + 2xy + y^2$$

Transposing and dividing by 2, this gives,

$$xy = \frac{1}{2}(x+y)^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

Differentiating this equation and using formula (A) on the right,

$$d(xy) = d[\frac{1}{2}(x+y)^2] - d(\frac{1}{2}x^2) - d(\frac{1}{2}y^2)$$

Applying formula (F) to each of the squares on the right and handling the constant coefficients by formula (E) we get

$$\begin{aligned} d(xy) &= \frac{1}{2}(x+y)\,d(x+y) - \frac{1}{2}x\,dx - \frac{1}{2}y\,dy \\ &= (x+y)(dx+dy) - x\,dx - y\,dy \\ &= x\,dx + x\,dy + y\,dx + y\,dy - x\,dx - y\,dy \\ \therefore\ d(xy) &= x\,dy + y\,dx \end{aligned} \tag{H}$$

A simple application of this formula gives the differential of the reciprocal of a variable. Let x be the variable and let

$$y = \frac{1}{x}, \text{ then } xy = 1$$

Differentiating the second equation by formula (H), and remembering that by formula (B) $d(1) = 0$, we get,

$$x\,dy + y\,dx = 0, \text{ hence, } dy = -y\,dx$$

But, $y = \frac{1}{x}$, therefore,

$$d(\frac{1}{x}) = -\frac{1}{x^2}\,dx \tag{J}$$

is the differential, and the derivative with respect to x is

$$\frac{d(1/x)}{dx} = -\frac{1}{x^2} \tag{26}$$

Formula (J) can be put into a different form which is often useful. Thus, $1/x = x^{-1}$ and $1/x^2 = x^{-2}$; hence, (J) becomes

$$d(x^{-1}) = -x^{-2}\,dx \tag{27}$$

16 Differential of the Quotient of Two Variables

Let x, y be the variables; we wish to find $d(x/y)$. Now, we can write x/y as

$$(1/y) \cdot x$$

which is in the form of a product. Applying formulas (H) and (J) to this product, we get

$$d(\frac{x}{y}) = \frac{1}{y}\,dx + x\,d(\frac{1}{y}) = \frac{1}{y}\,dx - \frac{x}{y^2}\,dy$$

or, combining these two last terms with a common denominator,

$$d(\frac{x}{y}) = \frac{y\,dx - x\,dy}{y^2} \tag{K}$$

17 Differential of a Power of a Variable

Letting x represent the variable and r represent any constant exponent, we have to find $d(x^r)$. Since a power is the product of repeated multiplication of the same factors, for example, $2^3 = 2 \cdot 2 \cdot 2$, $x^3 = x \cdot x \cdot x$, etc., let us consider formula (H):

$$d(xy) = x\,dy + y\,dx$$

Then

$$\begin{aligned} d(xyz) &= d[(xy)z] = xy\,dz + z\,d(xy) \\ &= xy\,dz + z(y\,dx + x\,dy) = xy\,dz + yz\,dx + xz\,dy \end{aligned}$$

Similarly,

$$d(xyzt) = (xyz)\,dt + (xyt)\,dz + (xzt)\,dy + (yzt)\,dx$$

Extended to the product of any number of factors, this formula says, "To find the differential of the product of any number of factors multiply the differential of each factor by the product of all the other factors and add the results."

Thus,

$$d(x^3) = d(x \cdot x \cdot x) = x \cdot x\,dx + x \cdot x\,dx + x \cdot x\,dx = 3x^2\,dx = 3x^{3-1}\,dx$$

In the same way

$$d(x^4) = d(x \cdot x \cdot x \cdot x) = 4x^3\,dx = 4x^{4-1}\,dx$$

$$d(x^5) = 5x^4\,dx = 5x^{5-1}\,dx$$

and, in general, by extending the same method to any power,

$$d(x^n) = nx^{n-1}\,dx \tag{L}$$

If $y = x^n$ the derivative is

$$\frac{dy}{dx} = \frac{d(x^n)}{dx} = nx^{n-1} \tag{28}$$

Referring now to formulas (F), (25), (27) it is seen that they are simply special cases of the general formula (L) with the exponent $n = 2, \frac{1}{2}, -1$, respectively. Formula (L) holds good for any value of the exponent, positive, negative, whole number, fractional or mixed.

18 Formulas

The formulas derived in this chapter are collected here for reference in connection with the illustrative examples worked out in the next article.

$$d(x^2) = 2x\,dx \tag{F}$$

$$d(\sqrt{x}) = \frac{1}{2\sqrt{x}}\,dx \tag{G}$$

$$d(xy) = x\,dy + y\,dx \tag{H}$$

$$d(\frac{1}{x}) = -\frac{1}{x^2}\,dx \tag{J}$$

$$d(\frac{x}{y}) = \frac{y\,dx - x\,dy}{y^2} \tag{K}$$

$$d(x^n) = nx^{n-1}\,dx \tag{L}$$

19 Illustrative Examples

Example 1

Find the differential of $x^2 - 2x + 3$.

This is the algebraic sum of several terms, therefore by formula (A) we get for the differential of the entire expression

$$d(x^2) - d(2x) + d(3)$$

By formula (F),

$$d(x^2) = 2x\,dx$$

By formula (E),

$$d(2x) = 2\,dx$$

By formula (B),

$$d(3) = 0$$

Therefore

$$d(x^2 - 2x + 3) = 2x\,dx - 2\,dx$$

Example 2

Find $d(2x^3 + 3\sqrt{x} - 3x^2)$

By formula (A) this is equal to $d(2x^3) + d(3\sqrt{x}) - d(3x^2)$

By (E) and (L),

$$d(2x^3) = 2d(x^3) = 2(3x^2\,dx) = 6x^2\,dx$$

By (E) and (G),

$$d(3\sqrt{x}) = 3d(\sqrt{x}) = 3\frac{dx}{2\sqrt{x}} = \frac{3\,dx}{2\sqrt{x}}$$

By (E) and (L)

$$d(3x^2) = 3d(x^2) = 3(2x\,dx) = 6x\,dx$$

Therefore the required differential is

$$6x^2\,dx + \frac{3\,dx}{2\sqrt{x}} - 6x\,dx = (6x^2 + \frac{3}{2\sqrt{x}} - 6x)\,dx$$

Example 3

Differentiate $3xy^2$.

This is the product of x by y^2 with the constant coefficient 3; therefore by formulas (E) and (H) we have

$$d(3xy^2) = 3d(x \cdot y^2)$$

and

$$d(x \cdot y^2) = x \cdot d(y^2) + y^2 \cdot d(x) = x \cdot 2y\,dy + y^2 \cdot dx = 2xy\,dy + y^2\,dx$$

Therefore

$$d(3xy^2) = 3(2xy\,dy + y^2\,dx) = 3y(2x\,dy + y\,dx)$$

Example 4

Differentiate $\sqrt{x^2 - 4}$

By (G),

$$d\sqrt{x^2 - 4} = \frac{d(x^2 - 4)}{2\sqrt{x^2 - 4}}$$

and by (C) and (F),

$$d(x^2 - 4) = d(x^2) = 2x\,dx$$

Therefore,

$$d\sqrt{x^2 - 4} = \frac{2x\,dx}{2\sqrt{x^2 - 4}} = \frac{x\,dx}{\sqrt{x^2 - 4}}$$

Example 5

Differentiate $\frac{u^2}{y}$

This is a quotient; therefore, by (K),

$$d(\frac{u^2}{y}) = \frac{y \cdot d(u^2) - u^2 \cdot d(y)}{y^2}$$

$$= \frac{y \cdot 2u\,du - u^2\,dy}{y^2}$$

Example 6

Differentiate $(x+2)\sqrt{x^2+4x}$

This is the product of the factors $x+2$ and $\sqrt{x^2+4x}$. Therefore, by formula (H),

$$d[(x+2)\sqrt{x^2+4x}] = (x+2)\cdot d(\sqrt{x^2+4x}) + \sqrt{x^2+4x}\cdot d(x+2) \qquad \text{(a)}$$

By formula (G),

$$d(\sqrt{x^2+4x}) = \frac{d(x^2+4x)}{2\sqrt{x^2+4x}}$$

and

$$d(x^2+4x) = d(x^2) + d(4x) = 2x\,dx + 4\,dx = 2(x+2)\,dx$$

Therefore,

$$d(\sqrt{x^2+4x}) = \frac{2(x+2)\,dx}{2\sqrt{x^2+4x}} = \frac{(x+2)\,dx}{\sqrt{x^2+4x}} \qquad \text{(b)}$$

Also

$$\sqrt{x^2+4x}\cdot d(x+2) = \sqrt{x^2+4x}\cdot dx \qquad \text{(c)}$$

Using the results (b) and (c) in expression (a),

$$d[(x+2)\sqrt{x^2+4x}] = (x+2)\cdot\frac{(x+2)\,dx}{\sqrt{x^2+4x}} + \sqrt{x^2+4x}\cdot dx$$

$$= \frac{(x+2)^2\,dx + (x^2+4x)\cdot dx}{\sqrt{x^2+4x}}$$

This can be simplified, if desired, by writing the two expressions in brackets over the common denominator $\sqrt{x^2+4x}$. This gives,

$$\frac{(x+2)^2 + (x^2+4x)\sqrt{x^2+4x}}{\sqrt{x^2+4x}}$$

Simplifying this last expression we get finally

$$(x+2)\sqrt{x^2+4x} = 2(x^2+4x+2)\,dx$$

Example 7

Differentiate $\frac{x+y}{x-y}$

This is the quotient of $(x+y)$ by $(x-y)$; therefore, by (K)

$$d(\frac{x+y}{x-y}) = \frac{(x-y)\cdot d(x+y) - (x+y)\cdot d(x-y)}{(x-y)^2}$$

$$= \frac{(x-y)(dx+dy) - (x+y)(dx-dy)}{(x-y)^2}$$

Now, by multiplication

$$(x-y)(dx+dy) - (x+y)(dx-dy)$$

equals

$$x\,dx + x\,dy - y\,dx - y\,dy - x\,dx + x\,dy - y\,dx + y\,dy$$

$$= x\,dx + x\,dy - y\,dx - y\,dy - x\,dx + x\,dy - y\,dx + y\,dy$$

$$= 2x\,dy - 2y\,dx = 2(x\,dy - y\,dx)$$

Therefore,

$$d(\frac{x+y}{x-y}) = \frac{2(x\,dy - y\,dx)}{(x-y)^2}$$

Example 8

Differentiate $3x^{\frac{1}{2}}$

$$d(3x^{\frac{1}{2}}) = 3d(x^{\frac{1}{2}})$$

and by (E)

$$d(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{1}{2\sqrt{x}}\,dx$$

Therefore,

$$d(3x^{\frac{1}{2}}) = 3\cdot\frac{1}{2\sqrt{x}}\,dx = \frac{3}{2\sqrt{x}}\,dx$$

Example 9

Find the differential of $(x^2+2)^3$

This is a variable (x^2+2) raised to the power 3. Hence by (L)

$$\begin{aligned} d[(x^2+2)^3] &= 3(x^2+2)^2 \cdot d(x^2+2) = 3(x^2+2)^2 \cdot d(x^2) \\ &= 3(x^2+2)^2 \cdot 2x\,dx = 6x(x^2+2)^2\,dx \end{aligned}$$

Example 10

Find $d[(x^2+1)^3(x^2+1)^{\frac{1}{2}}]$

This is a product of two factors and each factor is a power of a variable. Therefore, by formula (H)

$$d[(x^2+1)^3(x^2+1)^{\frac{1}{2}}] = (x^2+1)^3 \cdot d[(x^2+1)^{\frac{1}{2}}] + (x^2+1)^{\frac{1}{2}} \cdot d[(x^2+1)^3] \quad \text{(a)}$$

Also, by formula (L),

$$\begin{aligned} d[(x^2+1)^{\frac{1}{2}}] &= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot d(x^2+1) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot d(x^2) \\ &= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x\,dx = \frac{x\,dx}{\sqrt{x^2+1}} \qquad \text{(b)} \end{aligned}$$

and

$$\begin{aligned} d[(x^2+1)^3] &= 3(x^2+1)^2 \cdot d(x^2+1) = 3(x^2+1)^2 \cdot d(x^2) \\ &= 3(x^2+1)^2 \cdot 2x\,dx = 6x(x^2+1)^2\,dx \qquad \text{(c)} \end{aligned}$$

Using the results (b) and (c) in (a), we get

$$d[(x^2+1)^3(x^2+1)^{\frac{1}{2}}] = (x^2+1)^3 \cdot \frac{x\,dx}{\sqrt{x^2+1}} + (x^2+1)^{\frac{1}{2}} \cdot 6x(x^2+1)^2\,dx$$

By carrying out the indicated multiplications and factoring the resulting expression as follows we get:

$$6x^2(x^2+1)^2(x^2+1)^{\frac{1}{2}}\,dx + 6x(x^2+1)^3(x^2+1)^{\frac{1}{2}}\,dx$$

$$= 6x(x^2+1)^2(x^2+1)^{\frac{1}{2}}[x+(x^2+1)]\,dx$$

Therefore, finally the required differential becomes

$$d[(x^2+1)^3(x^2+1)^{\frac{1}{2}}] = 6x(x^2+1)^2(x^2+1)^{\frac{1}{2}}(x^3+x+1)\,dx$$

Example 11

Differentiate $\frac{2}{x^2+2}$

This is the same as $2 \cdot \frac{1}{x^2+2}$. Therefore by formula (J)

$$d(\frac{2}{x^2+2}) = 2 \cdot d(\frac{1}{x^2+2}) = -2 \cdot \frac{d(x^2+2)}{(x^2+2)^2}$$

$$= -2 \cdot \frac{2x\,dx}{(x^2+2)^2} = -\frac{4x\,dx}{(x^2+2)^2}$$

Example 12

Differentiate $3(y+3)^{-\frac{1}{2}}$

This is the same as

$$3 \cdot \frac{1}{\sqrt{y+3}} = 3(y+3)^{-\frac{1}{2}}$$

Therefore by formula (L)

$$d[3(y+3)^{-\frac{1}{2}}] = 3 \cdot [-\frac{1}{2}(y+3)^{-\frac{3}{2}} \cdot d(y+3)]$$

$$= 3 \cdot [-\frac{1}{2}(y+3)^{-\frac{3}{2}} \cdot dy] = -\frac{3\,dy}{2(y+3)^{\frac{3}{2}}}$$

section*Exercises

Differentiate each of the following expressions:

1. $x^2 - 2x + \sqrt{x}$
2. $x^3 - x^{\frac{3}{5}} + x$
3. $8x^6 + 3x^{\frac{1}{2}} + 10x^{\frac{5}{8}} + 2x + 2$
4. $(x - 3x)^{\frac{1}{2}}$
5. $(4 - 2x)^{\frac{1}{2}}$
6. $\frac{x}{y}$
7. $\frac{1}{x^2} - 2$
8. $\frac{x^2+5}{x}$
9. $\frac{x^4-4\sqrt{1+6x}}{\sqrt{1+x}}$
10. $(x + 3)^3 - 2y + y^2$
11. $(\sqrt{x} + y)^2$
12. $\frac{3-x}{x}$
13. $5x^{\frac{1}{7}} + 6$
14. $4x^{-1} - 7x^{-2}$
15. $6x^{\frac{1}{2}} - 9x^{\frac{1}{3}}$
16. $2\sqrt{x} - 12\sqrt[3]{x}$
17. $\sqrt[3]{3 + 4x}$
18. $\sqrt[4]{4 - 3x}$
19. $\frac{1}{x^n}$, (n is a constant)
20. $\frac{A}{x^n}$, (A and n are constants)

Chapter 4

USE OF RATES AND DIFFERENTIALS IN SOLVING PROBLEMS

20 Introductory Remarks

In the preceding chapter we have worked out a number of examples illustrating the use of the differential formulas. In those illustrations and the corresponding exercises for the reader, the functions or algebraic expressions were given already formed. In applying the principles of rates and differentials to problems which are simply described without being formulated, however, the function must first be formulated mathematically.

In the present chapter we give the detailed solutions of a number of such problems showing the use and application of rates and differentials in cases in which variable quantities are involved. In these solutions the formulation of one variable as a function of another is shown in full and the differentiation is carried out by the appropriate formula in each case. The formula is not designated by letter as in the preceding chapter, however, and the reader is advised to look up the proper formula and follow out its application step by step in each case. The differentiations used involve only the algebraic formulas so far derived.

In each case the results obtained are interpreted when their significance is not immediately obvious.

21 Illustrative Problems

Problem 1

A man walks directly across a street at the rate of five feet per second and his path passes four feet from a lamp post on the opposite side from which he started. The lamp throws his shadow on a wall along the side of the street from which he started. If the lamp is thirty-six feet from the wall, how fast is the shadow moving when the man is sixteen feet from the wall? When he is twenty-six feet away? Where is the man when his shadow moves at the same rate at which he is walking?

Solution

Let AS represent the wall, BL the opposite side of the street, AB the path of the man, L the position of the lamp, P the position of the man and S that of his shadow at any moment, as shown in Fig.3. Then AB = 36 feet, BL = 4 feet, and if we let AP = x represent the distance of the man from the wall at any instant, then the rate at which he is walking is $dx/dt = 5$ ft./sec. If we let AS = y represent the distance of the shadow from the starting point at the same instant, the speed of the shadow is the rate at which y is increasing. That is, we have to find dy/dt.

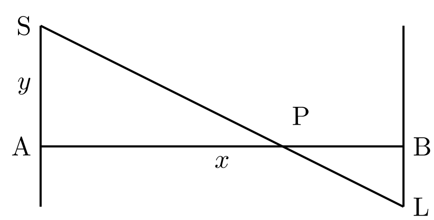

Figure 3: Man walking across street with shadow cast by lamp post.

In order to find dy/dt we must first find the relation between the man's distance x and that of the shadow, y, that is, express y as a function of x.

In order to do this we make use of the relations between the various distances given by the figure.

Since AS and BL are parallel and AB is perpendicular to both, the triangles

APS and BPL are right triangles, and since the vertical angles at P are equal, the two triangles are similar and, therefore, the corresponding sides are proportional. That is,

$$AS : AP :: BL : BP$$

Using the values given above for AS, AP and BL and noticing that BP = AB - AP = 36 - x, this proportion becomes

$$y : x :: 4 : (36 - x)$$

Solving for y,

$$y = \frac{4x}{36 - x}$$

and this is the desired relation between y and x.

We must now find the differential dy, and since y equals the fraction on the right side of the equation the differential of y equals the differential of this fraction. Therefore,

$$dy = \frac{(36 - x) \cdot d(4x) - 4x \cdot d(36 - x)}{(36 - x)^2}$$

and next performing the indicated differentiations in the numerator of the last expression it becomes

$$dy = \frac{(36 - x) \cdot 4\,dx - 4x \cdot (-dx)}{(36 - x)^2} = \frac{144\,dx}{(36 - x)^2}$$

Hence,

$$\frac{dy}{dt} = \frac{144}{(36 - x)^2} \cdot \frac{dx}{dt}$$

But $dx/dt = 5$, therefore,

$$\frac{dy}{dt} = \frac{720}{(36 - x)^2} \tag{a}$$

which gives the rate at which the shadow is moving when the man is at any distance x from the wall.

When $x = 16$, $\frac{dy}{dt} = \frac{720}{(36-16)^2} = 18$ ft./sec.

When $x = 26$, $\frac{dy}{dt} = \frac{720}{(36-26)^2} = 7.2$ ft./sec.

When $\frac{dy}{dt} = \frac{dx}{dt}$, then we have the same factor on both sides in equation (a) above and cancelling this factor gives

$$1 = \frac{144}{(36-x)^2}$$

as the condition, and from this the value of x is to be determined in order to find the position of the man. Taking the square root of both sides of the last expression we get

$$36 - x = \pm 12, \text{ and } x = 36 \pm 12 = 24 \text{ or } 48 \text{ feet}$$

Since the street is only 36 feet wide the 48 is an impossible result and therefore the man is at a distance 24 feet from the wall when his shadow moves at the rate at which he is walking. This is only true for an instant, however; immediately before that time the shadow is moving more slowly and immediately afterwards it is moving more rapidly, as indicated by the rates of the shadow found above when the man is 16 feet and 26 feet from the wall.

The phenomenon considered in this problem has been noticed by everyone, but the problem is not easily solved without the calculus method of rates.

Problem 2

The top of a ladder 20 ft. long is resting against a vertical wall on a level pavement when the ladder begins to slide downward and outward. At the moment when the foot of the ladder is 12 feet from the wall it is sliding away from the wall at the rate of two feet per second. How fast is the top sliding downward at that instant? How far is the foot of the ladder from the wall when it and the top are moving at the same rate?

Solution

In Fig.4 let OA represent the pavement, OB the wall, and AB the ladder; the arrows represent the direction of motion. Let x represent the distance OA of the foot of the ladder from the wall, and y the distance OB of the top from the pavement. We have then from the statement of the problem AB = 20 feet, $dx/dt = 2$ ft./sec., and we are to find dy/dt and also find when $dy/dt = dx/dt$.

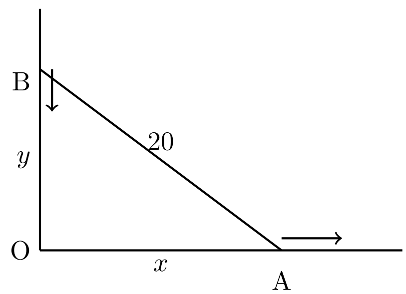

Figure 4: Ladder sliding down wall with motion vectors.

First we must express y in terms of x. This is done from the figure by noting that since OA is horizontal and OB is vertical the triangle AOB is a right triangle. Therefore,

$$OB^2 + OA^2 = AB^2$$

that is,

$$y^2 + x^2 = 20^2 = 400$$

Therefore,

$$y = \sqrt{400 - x^2}$$

is the desired relation between y and x, that is, the expression of y as a function of x. In order to get the rate dy/dt, we must from this equation find dy and then divide by dt.

Differentiating the equation by the square root formula,

$$dy = d(\sqrt{400 - x^2}) = \frac{d(400 - x^2)}{2\sqrt{400 - x^2}} = \frac{-d(x^2)}{2\sqrt{400 - x^2}}$$

$$= \frac{-2x\,dx}{2\sqrt{400 - x^2}} = -\frac{x\,dx}{\sqrt{400 - x^2}}$$

Therefore,

$$\frac{dy}{dt} = -\frac{x}{\sqrt{400 - x^2}} \cdot \frac{dx}{dt} \tag{a}$$

Now we had given $dx/dt = 2$ and are to find dy/dt when the distance $x = 12$. Putting these values in the result (a), we get

$$\frac{dy}{dt} = -\frac{12}{\sqrt{400 - 144}} \cdot 2 = -\frac{24}{\sqrt{256}} = -\frac{24}{16} = -1.5 \text{ ft./sec.}$$

The negative sign of dy/dt indicates that y is decreasing, that is, the top of the ladder is moving downward.

To find when $dy/dt = dx/dt$, put dy/dt for dx/dt in formula (a). Then the factor dy/dt cancels on each side, and we have

$$1 = -\frac{x}{\sqrt{400 - x^2}}$$

Hence,

$$\sqrt{400 - x^2} = -x$$

$$400 - x^2 = x^2$$

$$400 = 2x^2$$

$$x^2 = 200$$

$$x = \sqrt{200} = 14.14 \text{ ft.}$$

That is, at the instant when the foot of the ladder is 14.14 feet from the wall the foot and top are moving at the same rate.

Problem 3

A stone is dropped into a quiet pond and waves move in circles outward from the place where it strikes, at a speed of three inches per second. At the instant when the radius of one of the wave rings is three feet, how fast is its enclosed area increasing?

Solution

Let R be the radius and A the area of one of the circular waves. Then $A = \pi R^2$ and $dA = d(\pi R^2) = 2\pi R\, dR$.

The speed of the wave outward from the center is the rate at which the radius increases, dR/dt. Hence, $dR/dt = 3$ in./sec. $= \frac{1}{4}$ ft./sec., and at the instant when the radius is $R = 3$ feet the area is increasing at the rate

$$\frac{dA}{dt} = 2\pi R \cdot \frac{dR}{dt} = 2\pi \cdot 3 \cdot \frac{1}{4} = \frac{3\pi}{2} \approx 4.71 \text{ sq. ft./sec.}$$

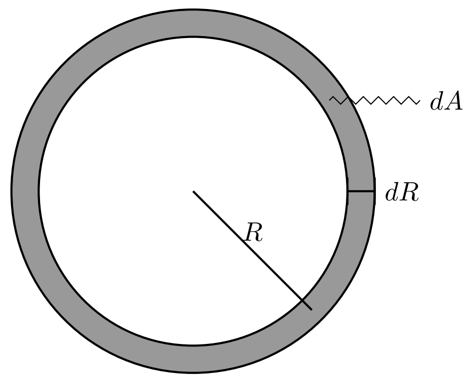

Figure 5: Circular wave expanding in pond with differential area and radius shown.

This problem can be formulated graphically by means of Fig.5. As the circular wave moves from the inner dotted circle to the very near outer one, the differential of the radius is dR and the corresponding differential increase of area is the shaded ring of area dA.

Now the average radius of this ring strip is R and, therefore, its length is $2\pi R$. Its area is the product of length by width:

$$dA = 2\pi R \cdot dR$$

which is the result already obtained by differentiation. From this result, the rate is found as before.

Problem 4

Water runs into a conical paraffine paper cup five inches high and three inches across the top, at the rate of one cubic inch per second. When it is just half filled how rapidly is the surface of the water rising?

Solution

Let Fig.6 represent the shape and dimensions of the cup. Then, if v represents the volume of water already in the cup when h is the height of the surface above the point of the cone, the rate at which water is running in is the rate of increase of the volume, dv/dt, and dh/dt is the rate of increase of the height h, that is, the rate at which the surface is rising.

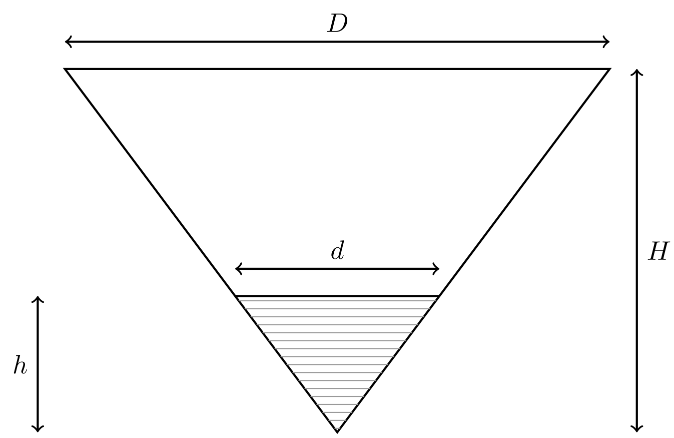

Figure 6: Conical cup filling with water, showing dimensions and water level.

The indicated dimensions d and h (this d has nothing to do with differentiation) are then variables and D, H are constants. We have $dv/dt = 1$ cu. in. per sec. and are to find dh/dt at the instant when h is such that the cup is half filled.

The volume of the cone is one third the base area times the altitude. Hence, the total volume is

$$\frac{\pi D^2 H}{12} = \frac{\pi(3^2)\cdot 5}{12} = 11.7 \text{ cu. in.}$$

with the dimensions given. Therefore, when half filled, the volume of the water is $v = 5.85$ cu. in. We must, therefore, find dh/dt when $v = 5.85$ and $dv/dt = 1$, and to do this we must have a relation between h and v, that is, express h as a function of v.

The volume formula in terms of the altitude h furnishes the relation desired. This formula is, as above,

$$v = \frac{\pi d^2 h}{12}$$

hence,

$$hd^2 = \frac{12v}{\pi} \tag{a}$$

This formula, however, contains not only the desired variables h and v, but also the undesired variable d. This variable must, therefore, be expressed in terms of either h or v. It is simpler to express d in terms of h by means of the proportionality between D, H, which are known, and d, h. In Fig.6 the two inverted triangles of bases (diameters) D, d and heights H, h are similar and therefore $d : h :: D : H$. Therefore,

$$\frac{d}{h} = \frac{D}{H} = \frac{3}{5} = 0.6$$

$$\therefore d = 0.6h, \text{ and } d^2 = 0.36h^2$$

Using this value of d^2 in formula (a) above, it becomes

$$h(0.36h^2) = \frac{12v}{\pi}$$

$$0.36h^3 = \frac{12v}{\pi}$$

hence,

$$h^3 = \frac{12v}{0.36\pi}$$

By taking the cube root of the numerical fraction and expressing the cube root of v as the $\frac{1}{3}$ power, we get finally as the desired functional relation between h and v,

$$h = 2.20v^{\frac{1}{3}} \tag{b}$$

Differentiating by the power formula,

$$dh = 2.2d(v^{\frac{1}{3}}) = 2.2(\frac{1}{3}v^{-\frac{2}{3}}dv) = \frac{0.74}{v^{\frac{2}{3}}}dv$$

Therefore, when $v = 5.85$ (half filled) and $dv/dt = 1$ (rate of inflow),

$$\frac{dh}{dt} = \frac{0.74}{(5.85)^{\frac{2}{3}}} = \frac{0.74}{\sqrt[3]{34.2}} = 0.23 \text{ in./sec.}$$

is the rate at which the surface of the water is rising.

This problem illustrates a condition which is often met in calculus: the algebra and geometry or other calculations which are necessary for the formulation before the calculus can be applied are longer than the direct solution of the calculus problem itself. Thus in this case after h was expressed as a function of v in formula (b) the differentiation and calculation of the rate were simple operations. The tedious part of the solution of the problem consisted not in the application of the calculus, but in deriving the functional relation (b) and in calculating when the cup was half filled.

Problem 5

A ship is sailing due north at the rate of 20 miles per hour. At a certain time another ship crosses its route 40 miles north sailing due east 15 miles per hour. (i) At what rate are the ships approaching or separating after one hour? (ii) After two hours? (iii) After how long are they momentarily neither approaching nor separating? (iv) At that time, how far apart are they?

Solution

We must express the distance between the ships as a function of the time after the second crossed the path of the first. The rate of change of this distance is then their speed of approach or separation. In Fig.7, let P represent the position of the first ship when the second crosses its path at O, 40 miles due north.

After a certain time t hours the ship sailing east will have reached a point A, and the ship sailing north will have reached a point B. The distance between them is then AB.

If we take O as reference point, and let $OA = x$, $OB = y$, $AB = s$, then $OP = 40$ and

$$s^2 = x^2 + (y - 40)^2 \tag{a}$$

Now, the rate of the ship B is $dy/dt = 20$ and of the ship A $dx/dt = 15$ miles per hour. Therefore, after the time t hours has passed, B has covered the distance $PB = 20t$ and the ship A the distance $OA = 15t$. Then,

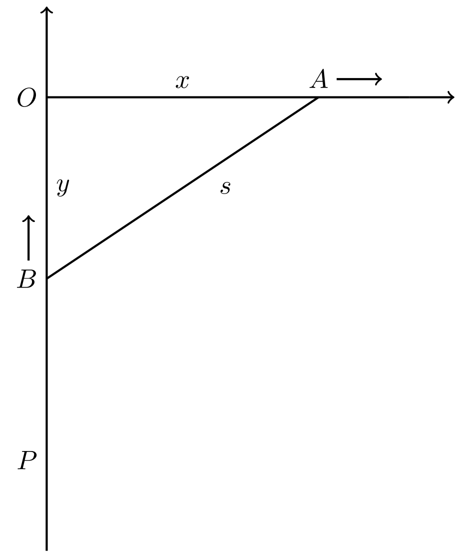

Figure 7: Two ships moving at right angles, showing their relative positions and motion.

$OB = OP + PB = 40 + 20t$. Therefore,

$$x = 15t, \quad y = 40 + 20t \tag{b}$$

Using these values of x, y in equation (a) gives for the distance

$$s^2 = (15t)^2 + (40 + 20t - 40)^2 = 225t^2 + 400t^2 = 625t^2$$

$$\therefore s = 25t \tag{c}$$

This is the desired relation between the distance between the ships, and the time t after the crossing at O. If at any time the rate ds/dt is positive, the distance is increasing, that is, the ships are separating. If at any time it is negative they are approaching. To find the rate ds/dt we must differentiate equation (c). Using the power formula:

$$ds = d(25t) = 25\,dt$$

$$\therefore \frac{ds}{dt} = 25 \tag{d}$$

In order to calculate the required results (i) to (iv) we proceed as follows:

(i) After 1 hour $t = 1$ and $ds/dt = 25$ mi./hr. and the ships are separating.

(ii) After 2 hours $t = 2$ and $ds/dt = 25$ mi./hr. and they are still separating.

(iii) Since ds/dt is constant and positive, the ships are always separating at the same rate. There is no time when they are neither approaching nor separating.

(iv) After 1 hour, equations (b) give

$$x = 15 \text{ mi.}, \quad y = 60 \text{ mi.}$$

The distance between the ships is then according to equation (c)

$$s = 25 \text{ mi.}$$

This completes the solution. This problem illustrates how the derivative can give us information about the relative motion of objects.

Problem 6

An aeroplane flying horizontally in a straight line at a rate of 60 miles an hour and an elevation of 1760 feet crosses at right angles a straight level road just as an automobile passes underneath at 30 miles an hour. How far apart are they and at what rate are they separating one minute later?

Solution

In Fig.8, let C represent the position of the aeroplane at the instant when the automobile is vertically below it at the point O.

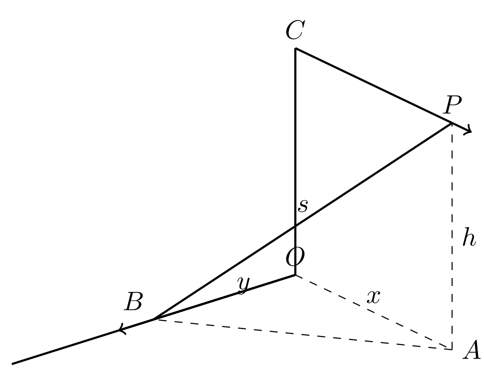

Figure 8: Airplane and automobile moving in perpendicular directions, showing their relative positions and motion.

Then CP is the direction of the aeroplane and OB that of the automobile. If the arrows indicate the motion, then after a time t minutes they will occupy the positions P and B and the straight line distance between them is the inclined diagonal $PB = s$. We have to find an expression for s at any time t and also its rate ds/dt.

Draw OA parallel to CP; draw the vertical line PA to the point vertically under P; draw AB; and denote the distances by x, y, h as shown. Then OAB is a right triangle with legs x and y and hypotenuse AB, and PAB is a right triangle with legs AP and AB and hypotenuse $PB = s$. Therefore,

$$s^2 = AB^2 + h^2, \text{ and } AB^2 = x^2 + y^2$$

$$\therefore s^2 = x^2 + y^2 + h^2 \tag{a}$$

Since the aeroplane is travelling at the rate $dx/dt = 60$ miles per hour $= 1$ mi./min. and the automobile at the rate $dy/dt = 30$ miles per hour $= \frac{1}{2}$ mi./min., then, at the end of t minutes they are at the distances from the crossing point of

$$CP = x = t, \quad OB = y = \frac{1}{2}t, \quad OC = h = \frac{1}{3} \text{ miles}$$

since 1760 feet is one third of a mile. Using these values of x, y, h in (a) we have

$$s^2 = t^2 + \frac{1}{4}t^2 + \frac{1}{9} = \frac{5}{4}t^2 + \frac{1}{9}$$

$$\therefore s = \sqrt{\frac{5}{4}t^2 + \frac{1}{9}} \tag{b}$$

is the distance between the aeroplane and the automobile at any time t minutes after the crossing, and the rate at which they are separating is ds/dt. From (b)

$$ds = d\left(\sqrt{\frac{5}{4}t^2 + \frac{1}{9}}\right) = \frac{d(\frac{5}{4}t^2 + \frac{1}{9})}{2\sqrt{\frac{5}{4}t^2 + \frac{1}{9}}} = \frac{\frac{5}{2}t\,dt}{\sqrt{\frac{5}{4}t^2 + \frac{1}{9}}}$$

$$\therefore \frac{ds}{dt} = \frac{\frac{5}{2}t}{\sqrt{\frac{5}{4}t^2 + \frac{1}{9}}} \tag{c}$$

is the rate at which they are separating at the time t.

Using (b) and (c) we are to calculate the distance and the rate ds/dt at the end of one minute. Thus $t = 1$ and by (b)

$$s = \sqrt{\frac{5}{4} + \frac{1}{9}} = \sqrt{\frac{45}{36} + \frac{4}{36}} = \sqrt{\frac{49}{36}} = \frac{7}{6} \text{ miles}$$

by (c),

$$\frac{ds}{dt} = \frac{\frac{5}{2}}{\sqrt{\frac{49}{36}}} = \frac{\frac{5}{2}}{\frac{7}{6}} = \frac{15}{7} \text{ mi./min.}$$

In the same way formulas (b) and (c) will give the distance and relative velocity of the two at any other time in minutes.

Exercises

1. Air is blown into a spherical rubber balloon at such a rate that the radius is increasing at the rate of one-tenth inch per second. At what rate is the air being blown in when the radius is two inches?

   (Hint: The required rate is dV/dt, where V is the volume.)

2. A metal plate in the shape of an equilateral triangle is being heated in such a way that each of the sides is increasing at the rate of ten inches per hour. How rapidly is the area increasing at the instant when each side is 69.28 inches?

3. Two points move, one on the OX axis and one on OY, in such a manner that in t minutes their distances from O are

   $$x = 2t - 6t, \quad y = 6t - 9$$

   feet.

   (a) At what rate are they approaching or separating after one minute?

   (b) After three minutes?

   (c) When will they be nearest together?

   (Hint: Find s and ds/dt. In (c) put $ds/dt = 0$.)

4. A man whose height is six feet walks directly away from a lamp post at the rate of three miles an hour on a level pavement. If the lamp is ten feet above the pavement, at what rate is the end of his shadow travelling?

   (Suggestion: Draw a figure and denote the variable distance of the man from the post by x, that of the end of the shadow by y, and express y as a function of x by similar triangles.)

5. At what rate does the shadow in Problem 4 increase in length?

6. A man is walking along the straight bank of a river 120 feet wide toward a boat at the bank, at a rate of five feet a second. At the moment when he is still fifty feet from the boat how rapidly is he approaching the point on the opposite bank directly across from the boat?

   (Draw figure, let $x =$ distance to boat, and formulate distance to point opposite.)

7. A man standing on a wharf is hauling in a rope attached to a boat, at the rate of four feet a second. If his hands are nine feet above the point of attachment how fast is the boat approaching the wharf when it is twelve feet away?

8. One end of a wire wound on a reel is fastened to the top of a pole 35 feet high; two men holding the reel on a rod on their shoulders five feet above the level ground walk away from the pole at the rate of five miles an hour, keeping the wire straight. How far are they from the pole when the wire is unwinding at the rate of one mile an hour?

9. A three-mile wind blowing on a level is carrying a kite directly away from a boy. How high is the kite when it is directly over a point 100 feet away and he is paying out the string at the rate of 88 feet a minute?

10. Two automobiles are moving along straight level roads which cross at an angle of sixty degrees, one approaching the crossing at 25 miles an hour and the other leaving it at 30 miles an hour on the same side. How fast are they approaching or separating from each other at the moment when each is ten miles from the crossing?

11. Assuming the volume of a tree to be proportional to the cube of its diameter ($V = k \cdot D^3$ where k is a constant) and that the diameter increases always at the same rate, how much more rapidly is the tree

growing in volume when the diameter is three feet than when it is six inches?

12. In being heated up to the melting point, a brick-shaped ingot of silver expands the thousandth part of each of its three dimensions for each degree temperature increase. At what rate per degree (dV/dT, where T is the temperature) is its volume increasing when the dimensions are $2 \times 3 \times 6$ inches?

13. Sand is being poured from a dumping truck and forms a conical pile with its height equal to one third the base diameter. If the truck is emptying at the rate of 720 cubic feet a minute and the outlet is five feet above the ground, how fast is the pile rising as it reaches the outlet?

14. A block of building stone is to be lifted by a rope 50 ft. long passing over a pulley on a window ledge 25 feet above the level ground. A man takes hold of the loose end of the rope which is held five feet above the ground and walks away from the block at ten feet a second. How rapidly will the block begin to rise?

15. The volume of a sphere is increasing at the rate of 16 cu. in. per second. At the instant when the radius is 6 in. how fast is it increasing?

16. A rope 28 feet long is attached to a block on level ground and runs over a pulley 12 feet above the ground. The rope is stretched taut and the free end is drawn directly away from the block and pulley at the rate of 13 ft. per sec. How fast will the block be moving when it is 5 feet away from the point directly below the pulley?

17. A tank is in the form of a cone with the point downward, and the height and diameter are each 10 feet. How fast is the water pouring in at the moment when it is 5 feet deep and the surface is rising at the rate of 4 feet per minute?

18. The hypotenuse AB of a right triangle ABC remains 5 inches long while the other two sides change, the side AC increasing at the rate of 2 in. per min. At what rate is the area of the triangle changing when AC is just 3 inches?

19. A spherical barrage balloon is being inflated so that the volume increases uniformly at the rate of 40 cu. ft. per min. How fast is the surface area

increasing at the moment when the radius is 8 feet?

20. A fighter plane is flying in a straight line on a level course to cross the course of a bomber which is also flying on a level course in a straight line. The fighter is at a level 500 feet above the bomber, and their courses cross at an angle of 60 degrees. Both planes are headed toward the crossing of their courses and on the same side of it, the bomber flying at 200 miles per hour and the fighter at 300. At the moment when the fighter is 10 miles and the bomber 7 miles from the crossing point, how rapidly are they approaching one another in a straight line joining the two planes?

Chapter 5

DIFFERENTIALS OF TRIGONOMETRIC FUNCTIONS

23 Angle Measure and Angle Functions

If through the center O of the circle in Figure 2 (article 2) we lay off the horizontal line OX of Figure 1 (article 1), join the points O and P, and draw through O the vertical OY, we get Figure 9. Figure 9 is thus a combination of Figure 1 and Figure 2 and there are differentials to be measured horizontally, vertically and tangentially.

Motion measured parallel to OX toward the right is to be taken as positive and toward the left as negative, as in Figure 1; similarly motion parallel to OY and upward is positive, downward is negative. Also as in Figure 2, if the point P moves in the direction opposite to that of the end of the hand of a clock (counter clockwise), the differential of length along the tangent PT is taken as positive, the opposite sense (clockwise) as negative.

As in Figure 1, horizontal distances are denoted by x, similarly vertical distances are denoted by y. Distances measured along the circumference of the circle are denoted by s and differentials of s, taken along the instantaneous tangent at any point P, by ds, as in Figure 2.

With the above system of notation x, y are then called the coordinates of the point P, and if we imagine the point P to move along the tangent line PT with the direction and speed it had at P, covering the differential of distance ds in the time dt as in Figure 2 the coordinate y will change by the differential

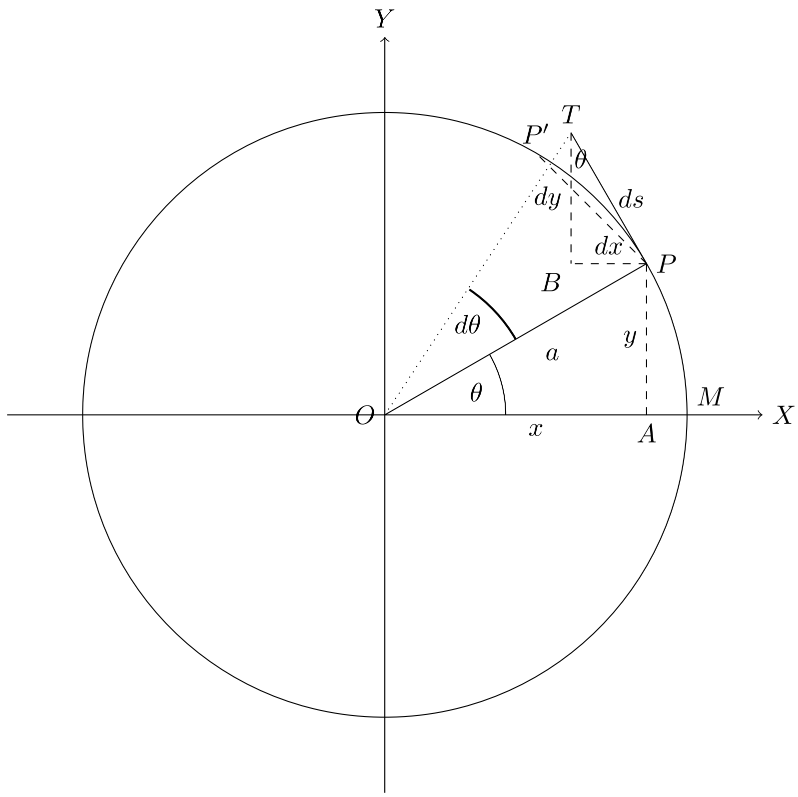

Figure 9: Trigonometric functions and their differentials.

amount dy in the positive sense and x will change by dx in the negative sense. If the angle AOP be represented by the symbol θ (pronounced "theta") then as P moves along PP' the radius $OP = a$ will turn about the center O and θ will increase by the positive differential of angle $d\theta$.

The question we now have to answer is this: When P moves through a space differential ds and the angle θ changes by the corresponding differential $d\theta$, what are the corresponding differentials of the angle functions of θ, that is, what are the values of the differentials $d(\sin\theta)$, $d(\cos\theta)$, etc.?

Since $\sin\theta$, $\cos\theta$, $\tan\theta$, etc., are functions of the independent variable θ, then as we have already seen their differentials will depend on $d\theta$ and also on some other function of θ itself. In the present chapter we shall find the formulas giving these differentials.

In order to find the differentials of the angle functions, or angular functions (also called circular and trigonometric functions, for obvious reasons) we need to understand the method of measurement of θ and $d\theta$.

If the angle is expressed in degree measure, then the unit of angle is such that the arc intercepted on the circumference by a central angle of one degree is equal in length to the 360th part of the circumference. This is the system used in ordinary computation and in the trigonometric tables.

A more convenient system for formulation and analysis is the so-called circular measure, in which the unit of angle is such that the arc intercepted on the circumference by a central angle of one radian is equal in length to the radius of the circle. Then since the entire circumference equals 2π times the radius, it equals 2π times the arc of one radian and there are 2π radians of angle in the circle. Thus, if the radius of the circle is a, one radian intercepts an arc of length a, and an angle of θ radians intercepts an arc of θ times a. If, therefore, the point P in Figure 9 moves a distance along the circumference of $MP = s$ while the line OP turns through an angle θ the length of MP is

$$s = a\theta \tag{29}$$

By the use of the circular measure we have that $360° = 2\pi$ radians, $180° = \pi$ and $90° = \frac{\pi}{2}$, with corresponding conversions for other angles. These values will be found more convenient than the values expressed in degrees, and the formula ((29)) is a much more convenient formula for the length of an arc than the corresponding formula in which the angle is expressed in degrees. Except for numerical computation in which the trigonometric tables have to be used we shall use circular measure throughout this book.[1]

24 Differentials of the Sine and Cosine of an Angle

In Figure 9 we have, by trigonometry, in the right triangle AOP,

$$\sin\theta = \frac{y}{a}, \quad \cos\theta = \frac{x}{a},$$

therefore

$$d(\sin\theta) = \frac{dy}{a}, \quad d(\cos\theta) = \frac{dx}{a} \tag{30}$$

since a is constant. We have also, since PT is a tangent and perpendicular to OP, and BT is perpendicular to OA, the angle PTB is equal to θ. Therefore

[1] A detailed explanation of this system of angle measure is given in the author's "Trigonometry for the Practical Man," published by D. Van Nostrand Company, New York, N.Y.

in the right triangle PTB, by trigonometry,

$$\frac{dy}{ds} = \cos\theta, \quad \frac{-dx}{ds} = \sin\theta,$$

(dx being negative in the figure as already pointed out). Hence

$$dy = (\cos\theta)\cdot ds, \quad dx = -(\sin\theta)\cdot ds. \tag{31}$$

If we now go back to equation (29) and differentiate it by formula (E), a being constant, we get

$$ds = a\,d\theta$$

and this value of ds substituted in the two equations (31) gives

$$dy = (\cos\theta)\cdot a\,d\theta, \quad dx = -(\sin\theta)\cdot a\,d\theta,$$

or

$$\frac{dy}{a} = \cos\theta\cdot d\theta, \quad \frac{dx}{a} = -\sin\theta\cdot d\theta.$$

Substituting these values of dy/a and dx/a in equations (30) we have finally,

$$d(\sin\theta) = \cos\theta\cdot d\theta \tag{M}$$

$$d(\cos\theta) = -\sin\theta\cdot d\theta \tag{N}$$

Thus the answer to our question in the preceding article is that the differential of the sine of an angle is the differential of the angle multiplied by the cosine, and the differential of the cosine is the differential of the angle multiplied by the negative of the sine.

From formulas ((M)) and ((N)), the rates of the sine and cosine are,

$$\frac{d(\sin\theta)}{dt} = \cos\theta\cdot\frac{d\theta}{dt}, \quad \frac{d(\cos\theta)}{dt} = -\sin\theta\cdot\frac{d\theta}{dt} \tag{32}$$

and the derivatives with respect to θ are,

$$\frac{d(\sin\theta)}{d\theta} = \cos\theta, \quad \frac{d(\cos\theta)}{d\theta} = -\sin\theta \tag{33}$$

25 Differentials of the Tangent and Cotangent of an Angle

The differential of the tangent is found by applying the fraction formula (K) to the following expression, which is obtained from trigonometry:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Differentiating this by the formula (K) we get

$$\begin{aligned} d(\tan\theta) = d\left(\frac{\sin\theta}{\cos\theta}\right) &= \frac{\cos\theta\cdot d(\sin\theta) - \sin\theta\cdot d(\cos\theta)}{\cos^2\theta} \\ &= \frac{\cos\theta\cdot(\cos\theta\, d\theta) - \sin\theta\cdot(-\sin\theta\, d\theta)}{\cos^2\theta} \\ &= \frac{(\cos^2\theta + \sin^2\theta)d\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}\, d\theta \end{aligned}$$

since $\cos^2\theta + \sin^2\theta = 1$. Also, $1/\cos\theta = \sec\theta$; therefore,

$$d(\tan\theta) = \sec^2\theta\, d\theta \tag{P}$$

The differential of $\cot\theta$ may be found by using the relation from trigonometry $\cot\theta = \cos\theta/\sin\theta$, and proceeding as in the case of the tangent. The following method is, perhaps, shorter. From trigonometry, $\cot\theta = 1/\tan\theta$; therefore, by (J) and (P),

$$\begin{aligned} d(\cot\theta) = d\left(\frac{1}{\tan\theta}\right) &= -\frac{1}{\tan^2\theta}\, d(\tan\theta) \\ &= -\frac{1}{\tan^2\theta}\cdot\sec^2\theta\, d\theta = -\frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\cos^2\theta}\, d\theta \\ &= -\frac{1}{\sin^2\theta}\, d\theta \end{aligned}$$

since $1/\sin\theta = \csc\theta$,

$$d(\cot\theta) = -\csc^2\theta\, d\theta \tag{Q}$$

26 Differentials of the Secant and Cosecant of an Angle

Since

$$\sec\theta = \frac{1}{\cos\theta}$$

or,

$$d(\sec\theta) = -\frac{1}{\cos^2\theta}\,d(\cos\theta) = \frac{1}{\cos^2\theta}\cdot\sin\theta\,d\theta = \sec\theta\tan\theta\,d\theta \tag{R}$$

Similarly,

$$d(\csc\theta) = d\left(\frac{1}{\sin\theta}\right) = -\frac{1}{\sin^2\theta}\,d(\sin\theta)$$

$$= -\frac{1}{\sin^2\theta}\cdot\cos\theta\,d\theta = -\frac{\cos\theta}{\sin^2\theta}\,d\theta = -\csc\theta\cot\theta\,d\theta \tag{S}$$

This completes the list of differential formulas for the usual trigonometric or angular functions. There are several other angular functions which are of use in certain special branches of applied mathematics, but these are not useful in ordinary work and we will not consider them here. We give next the solution of some examples showing the use and applications of the above formulas.

27 Illustrative Examples Involving the Trigonometric Differentials

In this and the following articles the use of the differential formulas ((M)), ((N)), ((P)), ((Q)), ((R)), ((S)) derived in this chapter will be illustrated by applying them to a few simple examples and problems. Whenever necessary the previous formulas (A) to (I), whose uses have already been illustrated, will be used without referring to them by letter.

Example 1 Find the differential of $3\sin\theta + a\cos\theta$.

Solution. This expression being the sum of two terms, and 3, a being constants, we have

$$\begin{aligned} d(3\sin\theta + a\cos\theta) &= 3\cdot d(\sin\theta) + a\cdot d(\cos\theta) \\ &= 3(\cos\theta\,d\theta) + a(-\sin\theta\,d\theta) \\ &= (3\cos\theta - a\sin\theta)\,d\theta \end{aligned}$$

Example 2 Differentiate $\sin 2\theta$.

Solution. By formula ((M)), 2 being constant,

$$\begin{aligned} d(\sin 2\theta) &= \cos(2\theta)\cdot d(2\theta) \\ &= \cos(2\theta)\cdot 2\,d\theta \\ &= 2\cos 2\theta\,d\theta \end{aligned}$$

Example 3 Differentiate $\sin(x^3)$.

Solution. By formula ((M)), x^3 replacing θ, we get

$$\begin{aligned} d[\sin(x^3)] &= \cos(x^3)\cdot d(x^3) \\ \text{and } d(x^3) &= 3x^2\,dx \\ \therefore d[\sin(x^3)] &= 3x^2\cos(x^3)\,dx \end{aligned}$$

Example 4 Find $d(\cos\sqrt{x})$.

Solution. By formula ((N)), $\sqrt{x}$ replacing θ, we get

$$\begin{aligned} d(\cos\sqrt{x}) &= -\sin\sqrt{x}\cdot d(\sqrt{x}) \\ &= -\sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\,dx \end{aligned}$$

Example 5 Find $d[\tan(2x^2+3)]$.

Solution. By formula ((P)), $(2x^2+3)$ replacing θ, we get

$$\begin{aligned} d[\tan(2x^2+3)] &= \sec^2(2x^2+3)\cdot d(2x^2+3) \\ \text{But } d(2x^2+3) &= d(2x^2)+d(3) = 4x\,dx \\ \therefore d[\tan(2x^2+3)] &= 4x\sec^2(2x^2+3)\,dx \end{aligned}$$

Example 6 Differentiate $\frac{1}{4}\sec(2\theta^3)$.

Solution. By formula ((R)), $2\theta^3$ replacing θ, we have

$$\begin{aligned} d[\frac{1}{4}\sec(2\theta^3)] &= \frac{1}{4}\cdot d[\sec(2\theta^3)] \\ &= \frac{1}{4}\sec(2\theta^3)\tan(2\theta^3)\cdot d(2\theta^3) \\ \text{But } d(2\theta^3) &= 6\theta^2\,d\theta \\ \therefore d[\frac{1}{4}\sec(2\theta^3)] &= \frac{3}{2}\theta^2\sec(2\theta^3)\tan(2\theta^3)\,d\theta \end{aligned}$$

Example 7 Differentiate $(\sin x)^3$.

Solution. This being a power of a variable, we have

$$\begin{aligned} d[(\sin x)^3] &= 3(\sin x)^2 \cdot d(\sin x) \\ &= 3(\sin x)^2 \cdot \cos x\, dx \end{aligned}$$

or since $(\sin x)^3$ is generally written $\sin^3 x$,

$$d(\sin^3 x) = 3\sin^2 x \cos x\, dx$$

Example 8 If $y = \csc^2(4x)$, find $\frac{dy}{dx}$.

Solution. By formula ((S)), $\csc^2$ being a power, we have

$$\begin{aligned} dy &= d[\csc^2(4x)] \\ &= 2\csc(4x) \cdot d[\csc(4x)] \\ &= 2\csc(4x) \cdot [-\csc(4x)\cot(4x) \cdot d(4x)] \\ &= -2\csc^2(4x)\cot(4x) \cdot 4\, dx \\ \therefore \frac{dy}{dx} &= -8\csc^2(4x)\cot(4x) \end{aligned}$$

Example 9 $y = \cot(at)$. Find $\frac{dy}{dt}$.

Solution. By formula ((Q)), a being constant,

$$\begin{aligned} dy &= d[\cot(at)] \\ &= -\csc^2(at) \cdot d(at) \\ &= -a\csc^2(at)\, dt \\ \therefore \frac{dy}{dt} &= -a\csc^2(at) \end{aligned}$$

Example 10 Differentiate $2\sqrt{\tan\theta}$.

Solution. Using formulas (G) and ((P)),

$$\begin{aligned} d(2\sqrt{\tan\theta}) &= 2\cdot d(\sqrt{\tan\theta}) \\ &= 2\cdot\frac{1}{2\sqrt{\tan\theta}}\cdot d(\tan\theta) \\ &= \frac{1}{\sqrt{\tan\theta}}\cdot\sec^2\theta\, d\theta \\ &= \frac{\sec^2\theta}{\sqrt{\tan\theta}}\, d\theta \end{aligned}$$

28 Illustrative Problems

Steam engine crank and rod.

The crank and connecting rod of a steam engine are three and ten feet long respectively, and the crank revolves at a uniform rate of 120 r.p.m. At what rate is the crosshead moving when the crank makes an angle of 45 degrees with the dead center line?

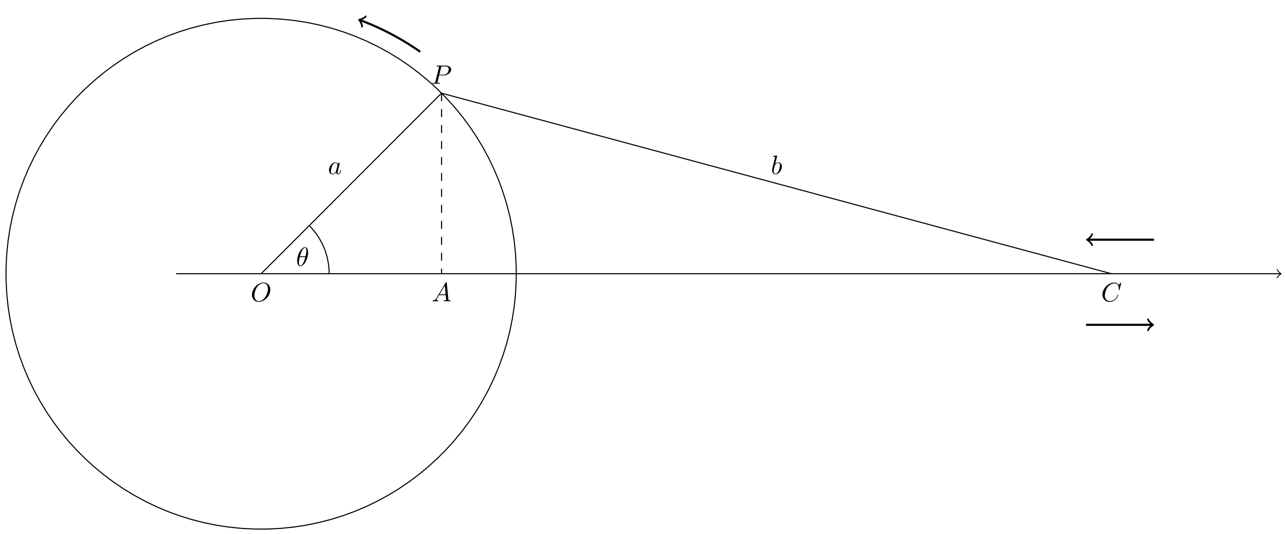

Figure 10: Steam engine crank and connecting rod mechanism.

Solution. In Figure 10 let OC represent the dead-center line and the circle the path of the crank pin P. Then C will represent the crosshead, CP the connecting rod and OP the crank. As P moves steadily round the circle in the direction shown C moves back and forth at different rates along OC. If

$OC = x$ and angle $POC = \theta$, then we are to find dx/dt when $d\theta/dt = 2$ rev. per sec. $= 4\pi$ radians per sec. To do this we must therefore express x as a function of θ.

In the figure let $a =$ crank length $= 3$ feet and $b =$ length of connecting rod $= 10$ feet. Draw PA perpendicular to OC. Then for varying positions of P, A and C will have different positions but always

$$x = OA + AC \tag{a}$$

In the right triangle PAC the hypotenuse formula gives

$$AC = \sqrt{b^2 - AP^2} \tag{b}$$

and in the right triangle POA by trigonometry

$$OA = a\cos\theta, \quad AP = a\sin\theta \tag{c}$$

Substituting this value of AP in (b),

$$AC = \sqrt{b^2 - a^2\sin^2\theta}$$

and this value of AC together with the value of OA in the first of equations (c), when used in equation (a) gives finally

$$x = a\cos\theta + \sqrt{b^2 - a^2\sin^2\theta} \tag{d}$$

which expresses x as a function of the angle θ. In order to find the rate dx/dt this equation must be differentiated to get dx.

Differentiating equation (d) and carrying out the transformations and simplifications, this gives:

$$\begin{aligned}
dx &= d(a\cos\theta) + d(\sqrt{b^2 - a^2\sin^2\theta}) \\
&= a(-\sin\theta\, d\theta) + \frac{1}{2\sqrt{b^2 - a^2\sin^2\theta}} \cdot d(b^2 - a^2\sin^2\theta) \\
&= -a\sin\theta\, d\theta - \frac{a^2\sin\theta\cos\theta\, d\theta}{\sqrt{b^2 - a^2\sin^2\theta}} \\
&= -a\sin\theta \left(1 + \frac{a\cos\theta}{\sqrt{b^2 - a^2\sin^2\theta}}\right) d\theta
\end{aligned}$$

When $\theta = 45°$, $\sin\theta = 0.707$, $\cos\theta = 0.707$, $a = 3$, $b = 10$,

$$\begin{aligned}\frac{dx}{dt} &= -3(0.707)\left(1 + \frac{3(0.707)}{\sqrt{100 - 9(0.707)^2}}\right) \cdot 4\pi \\ &= -32.44 \text{ ft./sec.}\end{aligned}$$

Similarly when $\theta = 270°$, $\sin\theta = -1$, $\cos\theta = 0$,

$$\begin{aligned}\frac{dx}{dt} &= -3(-1)\left(1 + \frac{3(0)}{\sqrt{100 - 9(1)}}\right) \cdot 4\pi \\ &= +37.70 \text{ ft./sec.}\end{aligned}$$

In the same way as here worked out for $\theta = 45°$ and $270°$ the velocity of the crosshead C, dx/dt, could be calculated for any value of θ, that is, any position of the crank OP. In the first case here worked out the negative value of dx/dt means that x is decreasing, that is, C is approaching O, the crosshead is moving toward the left. In the second case the positive sign means that it is moving to the right, x is increasing. In the same way the formula would give a negative value for any position of the crank pin P above the horizontal line OC and a positive value for any position below this line. This serves as a check and test of the differential formula, for it is at once seen by examining the figure that this is correct.

Shadow on a circular path.

A man walks across a diameter, 200 feet, of a circular courtyard at the rate of five feet per second. A lamp on the wall at one end of a diameter perpendicular to his path casts his shadow on the circular wall. How fast is the shadow moving (i) when he is at the center, (ii) when 20 feet from the center, (iii) and at the circumference?

Solution. In Figure 11 let CB be the path of the man, and L the position of the lamp. Let M represent the position of the man at any particular instant and y his distance from the center O. Then P is the position of the shadow on the circular wall and s its distance AP along the curved wall from A, ds being the differential of s in the momentary direction of the tangent, which

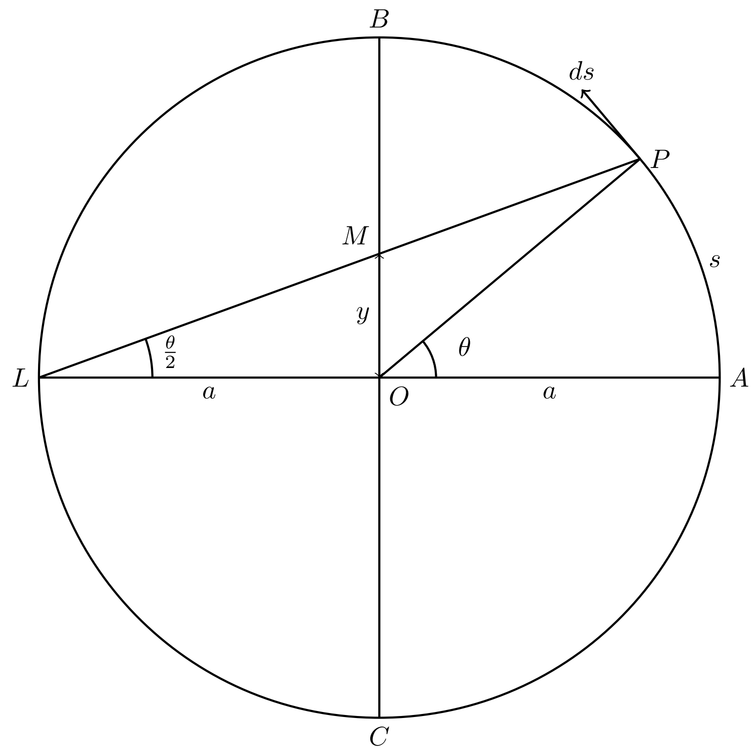

Figure 11: Shadow cast by a moving person in a circular courtyard.

is continually changing so as to follow the curve, as described in article 2. In order to find the rate of the shadow ds/dt, when that of the man dy/dt, is known we must as usual find the relation between y and s.

Draw OP and, as in Figure 11, let angle $AOP = \theta$, arc $AP = s$. Also let the radius $OL = OA = a$; ds is indicated by the arrow. If a line were drawn from the end of this arrow to L, the distance between the point where it crosses OB and the point M would be dy, the change in y corresponding to ds; and if a line were drawn from O to the end of the arrow, the angle between this line and OP would be $d\theta$, the change in θ corresponding to ds.

Using the notation just stated, we have as in Figure 11

$$s = a\theta \tag{a}$$

Also, by geometry, the angle $PLA = \frac{1}{2}$(angle POA), or angle $MLO = \frac{\theta}{2}$ and in the right triangle MOL

$$y = a\tan(\theta/2) \tag{b}$$

Differentiating (b) and (a),

$$dy = a \cdot d[\tan(\theta/2)] = a \sec^2(\theta/2) \cdot d(\theta/2)$$
$$dy = (a/2)\sec^2(\theta/2)\, d\theta; \quad ds = a\, d\theta$$

From the first of these results, $d\theta = \frac{1}{(a/2)\sec^2(\theta/2)}\, dy$, and this value of $d\theta$ in the second gives

$$ds = \frac{2}{\sec^2(\theta/2)}\, dy \tag{c}$$

Since y and a are known at any time, it is convenient to use $\tan\theta$ rather than the secant. In order to make the transformation we use the relation from trigonometry, $\sec^2\theta = 1 + \tan^2\theta$ for any angle. Therefore, equation (c) can be written as

$$ds = \frac{2}{1 + \tan^2(\theta/2)}\, dy \tag{d}$$

This gives the rate of the shadow in terms of the angle MLO and the rate of the man. Now, from the figure $\tan(\theta/2) = y/a$ and we have given $a = 100$. Therefore, $\tan(\theta/2) = y/100$, also $dy/dt = 5$, the rate of the man. Using these values, formula (d) becomes

$$\frac{ds}{dt} = \frac{10}{1 + (y/100)^2}$$

When the position of the man is stated, his distance y from the center is known, and therefore the rate of the shadow on the wall ds/dt, is immediately calculated from this formula. We have three positions of the man given.

(i) When he is at the center $y = 0$, hence $ds/dt = 10$ ft./sec.

(ii) When he is 20 feet from the center $y = 20$, then $y/100 = \frac{1}{5}$, hence

$$\frac{ds}{dt} = \frac{10}{1 + (\frac{1}{5})^2} = 9.6 \text{ ft./sec.}$$

(iii) When he is at the circumference $y = 100$, then $y/100 = 1$ and $ds/dt = 10/2 = 5$ ft./sec.

This is a very interesting application of the trigonometric differentiation formulas, as it also includes the method of circular measure of angles and arcs and a very instructive trigonometric formulation and transformation, beside the differentiation of the tangent of an angle.

An elliptical cam with shaft.

An elliptical cam arranged as in Figure 12 rotates about the focus F as an axis and causes the roller at P to move up and down along the line FP. If the diameters of the cam are six and ten inches and it rotates at the rate of 240 r.p.m., how fast is the roller moving at the moment when the long axis of the cam makes an angle of sixty degrees with the line of motion of the roller?

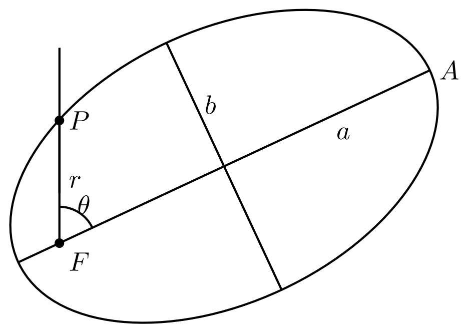

Figure 12: Elliptical cam rotating about focus F with roller at P.

Solution.—Let $FP = r$ and angle $AFP = \theta$, and let the half of the long and short diameters respectively be a and b. We then have given $\frac{d\theta}{dt} = 4$ rev./sec. $= 8\pi$ radians/sec., $a = 5$ in., $b = 3$ in., and are to find $\frac{dr}{dt}$ when $\theta = 60°$. First we must have a relation between r and θ in order to find dr in terms of θ and $d\theta$. For the figure of an ellipse this is known to be

$$r = \frac{b^2}{a(1 - e\cos\theta)} \tag{a}$$

where

$$e = \frac{\sqrt{a^2 - b^2}}{a}. \tag{b}$$

In equation (a) therefore b and e are constants and we have to find dr by differentiating (a). Writing (a) as

$$r = \frac{b^2}{a}\left(\frac{1}{1 - e\cos\theta}\right),$$

$$dr = \frac{b^2}{a} \cdot d\left(\frac{1}{1 - e\cos\theta}\right)$$

and by the formula for a reciprocal,

$$dr = \frac{b^2}{a}\left[-\frac{d(1 - e\cos\theta)}{(1 - e\cos\theta)^2}\right] = \frac{b^2}{a}\left[\frac{d(e\cos\theta)}{(1 - e\cos\theta)^2}\right]$$

$$= \frac{b^2}{a}\left[\frac{-e\sin\theta\, d\theta}{(1 - e\cos\theta)^2}\right]$$

$$\frac{dr}{dt} = -\frac{b^2 e\sin\theta}{a(1 - e\cos\theta)^2}\frac{d\theta}{dt} \tag{c}$$

Since $a = 5$, $b = 3$, equation (b) gives $e = \frac{4}{5}$; also, $b^2 = 9$ and $\frac{d\theta}{dt} = 8\pi$. Using these values, the cam data, (c) becomes

$$\frac{dr}{dt} = \frac{57.6\pi\sin\theta}{5(1 - \frac{4}{5}\cos\theta)^2}$$

This formula gives the instantaneous rate at which the roller is moving for any particular value of the angle between its line of motion and the long axis of the cam, and as the cam rotates about F the angle θ of course varies.

When $\theta = 60°$, $\sin\theta = \frac{1}{2}\sqrt{3}$ and $\cos\theta = \frac{1}{2}$. At this instant

$$\frac{dr}{dt} = \frac{57.6\pi \times \sqrt{3}}{5(1 - \frac{4}{5}\cdot\frac{1}{2})^2} = -16\pi\sqrt{3}$$

$$= -87.1 \text{ in./sec}$$

$$\frac{dr}{dt} = -7.25 \text{ ft./sec}$$

the negative sign indicating that r is decreasing, and therefore the roller is moving downward.

For other values of θ, that is, at other particular instants, the rate may be positive or negative, depending on the signs of the sine and cosine of the angle, for of course the roller will move both upward and downward in turn.

Exercises

Find the differential of each of the following expressions:

1. $\sin 2x + 2\sin x$
2. $\cos^2 x + \sin^3 x$
3. $\cos^3 x + \cos x^3$
4. $\cos \sqrt{1-t}$
5. $\sin x \cdot \cos x$
6. $x \sin x$
7. $\frac{\theta}{2} \cos 2\theta$
8. $\frac{1}{4} \tan 2x \cot 2x$

Find the derivative of each of the following expressions:

9. $\dfrac{\cos x}{x}$
10. $\sin x \cdot \sin 2x$
11. $\sqrt{\sec 2x}$
12. $\frac{1}{4} \tan^3 x - \tan x + x$
13. $\frac{1+\cos x}{1-\cos x}$
14. $\sin(x+a)\cos(x-a)$, a constant

Formulate each of the following problems and solve.

15. A person is approaching a 500-foot tower on a trolley car at the rate of ten miles per hour and looking at the top of the tower. At what rate must he be raising his head (or line of sight) when the car is 500 feet from the tower on level ground?
    (Hint: Draw a figure, call base line x, angle between x and line of sight θ, and find $d\theta/dt$ in radians and degrees per second, decreasing.)

16. A signal observation station sights on a balloon which is rising steadily in a vertical line a mile away. At the moment when the angle of elevation of the telescope is $30°$ and increasing at the rate of 1 radian per minute, how high is the balloon and how fast is it rising?

17. A high-speed motor torpedo boat is moving parallel to a straight shore line at 40 land miles per hour, 1.5 miles from the shore, and is followed by a search-light beam which is trained on the boat from a station half a mile back from the shore. At what rate in radians per minute must the beam turn in order to follow the boat just as the boat passes directly opposite the station, and also when it is half a mile farther along the shore past the station?

18. A fighter plane is travelling at 300 miles per hour in a horizontal straight line and passes an enemy plane travelling in a parallel line at the same level at 250 miles per hour in the same direction. A gunner in the bomber trains his machine gun on the enemy plane as soon as he comes in range and turns the gun to keep his sights on the plane as he passes while firing on it. If the courses of the two planes are 200 yards apart, how rapidly must the machine gun be turned to follow the enemy plane just as they pass, and also half a minute afterwards?

19. If $f = 2\sin\theta - \cos 2\theta$, is f an increasing or decreasing function of θ when $\theta = 45°$ and when $\theta = 150°$? What is the rate of increase in each case, if θ is increasing at the rate of 1 radian per minute?

20. The turning effect of a ship's rudder is shown in the theory of naval engineering to be $T = k\cos\theta\sin^2\theta$, where θ is the angle which the rudder makes with the keel line of the ship. When the rudder is turning at 1 radian per minute, what is the rate at which T is changing, in terms of the constant k, at the moment when $\theta = 30°$?