The Basel Problem

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Here, we present a modern solution to the problem.
We make use of multivariate calculus to find
the value of a certain double integral.

Split the series into even and odd parts . . .

We set

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

then we can split S into even and odd parts

$$S = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{S}{4} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

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which means that

$$S = \frac{4}{3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

now express the series as a double integral . . .

we have shown

$$S = \frac{4}{3} \sum_{n=0}^{\infty} \left(\frac{1}{2n+1} \right) \left(\frac{1}{2n+1} \right)$$

Now we use the fact that any rational number can be expressed as a definite integral of a power function.

$$S = \frac{4}{3} \sum_{n=0}^{\infty} \int_0^1 x^{2n} dx \int_0^1 y^{2n} dy$$

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and collecting terms we have

$$S = \frac{4}{3}\sum_{n=0}^{\infty}\int_0^1\int_0^1 (xy)^{2n}dx\, dy$$

now swap the order ...

swapping the order of summation and integration we have

$$S = \frac{4}{3} \int_0^1 \int_0^1 \left(\sum_{n=0}^{\infty} (xy)^{2n} \right) dx\, dy$$

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summing the inner geometric progression we get

$$S = \frac{4}{3} \int_0^1 \int_0^1 \left(\frac{1}{1-(xy)^2} \right) dx\ dy$$

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so now we need only evaluate this double integral over the unit square which we will do after refreshing our knowledge of change of variable formula for double integrals.

change of variables for double integrals ...

The change of variable formula for a double integral goes like this:

$$\iint_R f(x,y)dx\,dy = \iint_T f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v))} \right| du\,dv$$

where the transformation $(u,v) \to (x(u,v),y(u,v))$ is a *one-to-one* differentiable map from T onto R with Jacobian:

$$J = \left| \frac{\partial(x,y)}{\partial(u,v))} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Back to our problem ...

To evaluate our double integral, the substitution we will attempt is

$$x(u, v) = \frac{\sin u}{\cos v} \quad \text{and} \quad y(u, v) = \frac{\sin v}{\cos u}$$

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$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\cos u}{\cos v} & -\frac{\sin u \sin v}{\cos^2 v} \\ -\frac{\sin v \sin u}{\cos^2 u} & \frac{\cos v}{\cos u} \end{bmatrix}$$

who's determinant evaluates to

$$|J| = 1 - \left(\frac{\sin u \sin v}{\cos v \cos u}\right)^2 = 1 - (xy)^2$$

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Our double integral for the sum of the series transforms as follows

$$\begin{aligned} S &= \frac{4}{3} \iint_{I \times I} \left(\frac{1}{1-(xy)^2}\right) dx\, dy \\ &= \frac{4}{3} \iint_T (1)\, du\, dv = \frac{4}{3} \text{Area}(T) \end{aligned}$$

Finding T . . .

It remains to find a region T in the u v coordinate system that maps to the unit square under the transformation:

$$x(u, v) = \frac{\sin u}{\cos v} \quad \text{and} \quad y(u, v) = \frac{\sin v}{\cos u}$$

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Consider the inequalities that define the unit square:

$$0 \le x \le 1 \quad \text{and} \quad 0 \le y \le 1$$

transforming to u v coordinates and multiplying through by the cosine terms we require:

$$0 \le \sin u \le \cos v \quad \text{and} \quad 0 \le \sin v \le \cos u$$

Sketching T ...

The region T must satisfy:

$$0 \leq \sin u \leq \cos v \quad \text{and} \quad 0 \leq \sin v \leq \cos u$$

The left inequalities are satisfied if both u and v are greater than 0 whilst the right hand inequalities are satisfied if

$$\sin u \leq \sin(\frac{\pi}{2} - v) \quad \text{and} \quad \sin v \leq \sin(\frac{\pi}{2} - u)$$

Both of these are satisfied if $u + v \leq \frac{\pi}{2}$.

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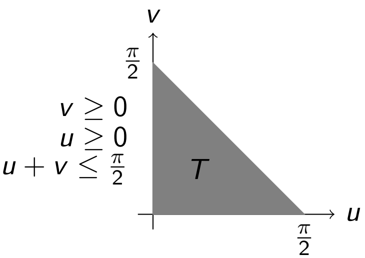

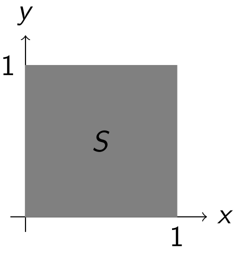

Wrapping it up . . .

We have shown that T is a triangle in the positive quadrant of the $u\ v$ plane with base and height of size $\frac{\pi}{2}$.

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We have shown that T is a triangle in the positive quadrant of the $u\ v$ plane with base and height of size $\frac{\pi}{2}$.
We can now calculate the sum of our series:

$$
\begin{aligned}
S &= \frac{4}{3}\iint_T (1)\,du\,dv \\
&= \frac{4}{3}\text{Area}(T) \\
&= \frac{4}{3}\left(\frac{\pi}{8}\right) \\
&= \frac{\pi}{6}
\end{aligned}
$$

Bibliography

Joe Breen Math, Youtube video, *The Basel problem*, https://www.youtube.com/watch?v=MB2HBH_ykf0