

Strong Spherical Asymptotics for Rotor-Router Aggregation and the Divisible Sandpile

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Abstract

The rotor-router model is a deterministic analogue of random walk. It can be used to define a deterministic growth model analogous to internal DLA. We prove that the asymptotic shape of this model is a Euclidean ball, in a sense which is stronger than our earlier work [12]. For the shape consisting of $n = \omega_d r^d$ sites, where ω_d is the volume of the unit ball in \mathbb{R}^d , we show that the inradius of the set of occupied sites is at least $r - O(\log r)$, while the outradius is at most $r + O(r^\alpha)$ for any $\alpha > 1 - 1/d$. For a related model, the *divisible sandpile*, we show that the domain of occupied sites is a Euclidean ball with error in the radius a constant independent of the total mass. For the classical abelian sandpile model in two dimensions, with $n = \pi r^2$ particles, we show that the inradius is at least $r/\sqrt{3}$, and the outradius is at most $(r + o(r))/\sqrt{2}$. This improves on bounds of Le Borgne and Rossin. Similar bounds apply in higher dimensions.

1 Introduction

Rotor-router walk is a deterministic analogue of random walk, first studied by Priezzhev et al. [16] under the name “Eulerian walkers.” At each site

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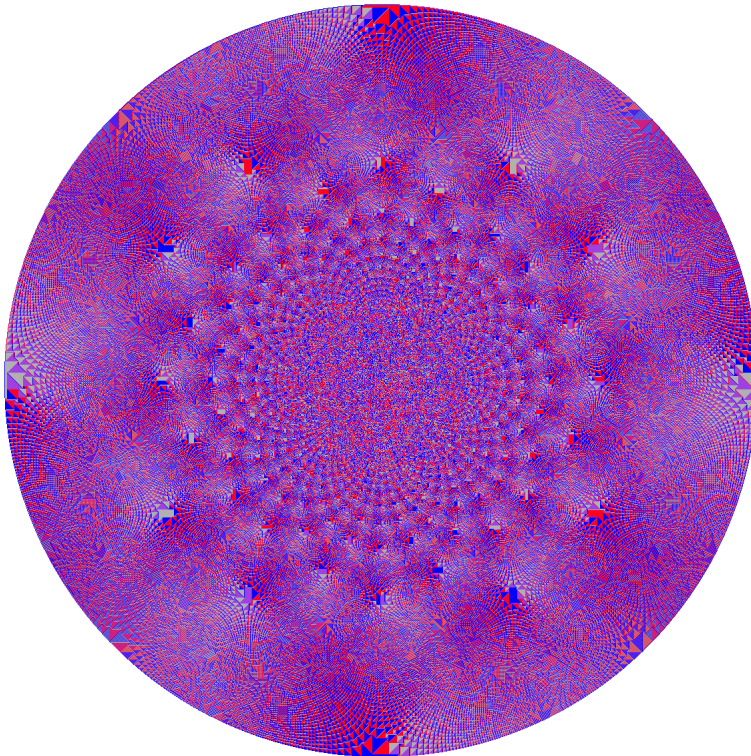


Figure 1: Rotor-router aggregate of one million particles in \mathbb{Z}^2 . Each site is colored according to the direction of its rotor.

in the integer lattice \mathbb{Z}^2 is a *rotor* pointing north, south, east or west. A particle starts at the origin; during each time step, the rotor at the particle's current location is rotated clockwise by 90 degrees, and the particle takes a step in the direction of the newly rotated rotor. In rotor-router aggregation, introduced by Jim Propp, we start with n particles at the origin; each particle in turn performs rotor-router walk until it reaches a site not occupied by any other particles. Let A_n denote the resulting region of n occupied sites. For example, if all rotors initially point north, the sequence will begin $A_1 = \{(0, 0)\}$, $A_2 = \{(0, 0), (1, 0)\}$, $A_3 = \{(0, 0), (1, 0), (0, -1)\}$. The region $A_{1,000,000}$ is pictured in Figure 1. In higher dimensions, the model can be defined analogously by repeatedly cycling the rotors through an ordering of the $2d$ cardinal directions in \mathbb{Z}^d .

Jim Propp observed from simulations in two dimensions that the regions A_n are extraordinarily close to circular, and asked why this was so [5, 17].

Despite the impressive empirical evidence for circularity, the best result known until now [12] says only that if A_n is rescaled to have unit volume, the volume of the symmetric difference of A_n with a ball of unit volume tends to zero as a power of n , as $n \uparrow \infty$. The main outline of the argument is summarized in [13]. Fey and Redig [4] also show that A_n contains a diamond. In particular, these results do not rule out the possibility of “holes” in A_n far from the boundary or of long tendrils extending far beyond the boundary of the ball, provided the volume of these features is negligible compared to n .

Our main result is the following, which rules out the possibility of holes far from the boundary or of long tendrils in the rotor-router shape. For $r \geq 0$ let

$$B_r = \{x \in \mathbb{Z}^d : |x| < r\}.$$

Theorem 1.1. *Let A_n be the region formed by rotor-router aggregation in \mathbb{Z}^d starting from n particles at the origin and any initial rotor state. There exist constants c, c' depending only on d , such that*

$$B_{r-c \log r} \subset A_n \subset B_{r(1+c'r^{-1/d} \log r)}$$

where $r = (n/\omega_d)^{1/d}$, and ω_d is the volume of the unit ball in \mathbb{R}^d .

We remark that the same result holds when the rotors evolve according to stacks of bounded discrepancy; see the remark following Lemma 5.1.

Internal diffusion limited aggregation (“internal DLA”) is an analogous growth model defined using random walks instead of rotor-router walks. Starting with n particles at the origin, each particle in turn performs simple random walk until it reaches an unoccupied site. Lawler, Bramson and Griffeath [8] showed that for internal DLA in \mathbb{Z}^d , the occupied region A_n , rescaled by a factor of $n^{1/d}$, converges with probability one to a Euclidean ball in \mathbb{R}^d as $n \rightarrow \infty$. Lawler [9] estimated the rate of convergence. By way of comparison with Theorem 1.1, if I_n is the internal DLA region formed from n particles started at the origin, the best known bounds [9] are (up to log factors)

$$B_{r-r^{1/3}} \subset I_n \subset B_{r+r^{1/3}}$$

for all sufficiently large n , with probability one.

We also study another model which is slightly more difficult to define, but much easier to analyze. The *divisible sandpile* model is analogous to the abelian sandpile, but uses continuous amounts of mass in place of discrete grains of sand. When it topples, each site keeps mass 1 for itself, and

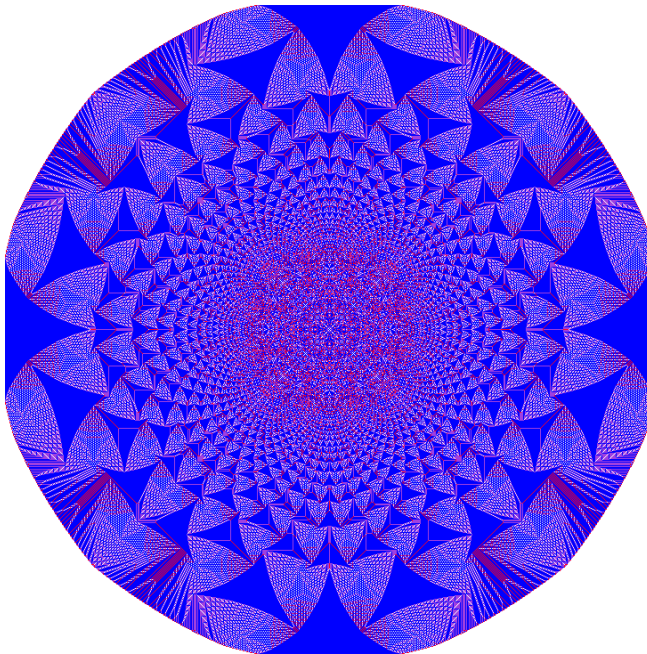


Figure 2: Classical abelian sandpile aggregate of one million particles in \mathbb{Z}^2 . Colors represent the number of grains at each site.

distributes any additional mass equally among its neighbors. As the number of topplings increases, the mass approaches a limiting distribution in which each site has mass ≤ 1 ; this is proved in section 3. Note that individual topplings do not commute; however, the divisible sandpile is “abelian” in the following sense.

Proposition 1.2. *Let $x_1, x_2, \dots \in \mathbb{Z}^d$ be a sequence with the property that for any $x \in \mathbb{Z}^d$ there are infinitely many terms $x_k = x$. Let*

$$\begin{aligned} u_k(x) &= \text{total mass emitted by } x \text{ after toppling } x_1, \dots, x_k; \\ \mu_k(x) &= \text{amount of mass present at } x \text{ after toppling } x_1, \dots, x_k. \end{aligned}$$

Then $u_k \uparrow u$ and $\mu_k \rightarrow \mu$. Moreover, the limits u and μ are independent of the sequence $\{x_k\}$.

This proposition is proved in section 3, along with the following.

Theorem 1.3. *For $m \geq 0$ let $D_m \subset \mathbb{Z}^d$ be the domain of fully occupied sites for the divisible sandpile formed from a pile of size m at the origin. There*

exist constants c, c' depending only on d , such that

$$B_{r-c} \subset D_m \subset B_{r+c'},$$

where $r = (m/\omega_d)^{1/d}$ and ω_d is the volume of the unit ball in \mathbb{R}^d .

The divisible sandpile is similar to the “oil game” studied by Van den Heuvel [19]. In the terminology of [4], it also corresponds to the $h \rightarrow -\infty$ limit of the classical abelian sandpile (defined below), that is, the abelian sandpile started from the initial condition that every site has a very deep “hole.”

In the classical abelian sandpile model [1], each site in \mathbb{Z}^d has an integer number of grains of sand; if a site has at least $2d$ grains, it *topples*, sending one grain to each neighbor. If n grains of sand are started at the origin in \mathbb{Z}^d , write S_n for the set of sites that are visited during the toppling process; in particular, although a site may be empty in the final state, we include it in S_n if it was occupied at any time during the evolution to the final state.

Until now the best known constraints on the shape of S_n were due to Le Borgne and Rossin [10], who proved that in two dimensions

$$\{x \in \mathbb{Z}^2 \mid x_1 + x_2 \leq \sqrt{n/12} - 1\} \subset S_n \subset \{x \in \mathbb{Z}^2 \mid x_1, x_2 \leq \sqrt{n}/2\}.$$

The methods used to prove the near-perfect circularity of the divisible sandpile shape in Theorem 1.3 can be used to give constraints on the shape of the classical abelian sandpile, improving on the bounds of Le Borgne and Rossin.

Theorem 1.4. *Let S_n be the set of sites that are visited by the classical abelian sandpile model in \mathbb{Z}^d , starting from n particles at the origin. Write $n = \omega_d r^d$. Then for any $\epsilon > 0$ we have*

$$B_{c_1 r - c_2} \subset S_n \subset B_{c'_1 r + c'_2}$$

where

$$c_1 = (2d - 1)^{-1/d}, \quad c'_1 = (d - \epsilon)^{-1/d}.$$

The constant c_2 depends only on d , while c'_2 depends only on d and ϵ .

Note that this does not settle the question of the asymptotic shape of S_n , and indeed it is not clear from simulations whether the asymptotic shape in two dimensions is a disc or perhaps a polygon (Figure 2). To our knowledge even the existence of an asymptotic shape is not known.

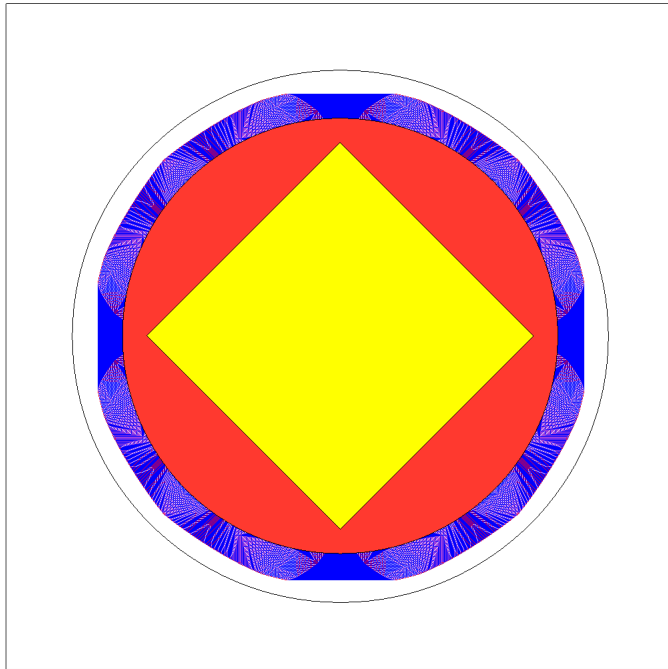


Figure 3: Known bounds on the shape of the classical abelian sandpile in \mathbb{Z}^2 . The inner diamond and outer square are due to Le Borgne and Rossin [10]; the inner and outer circles are those in Theorem 1.4.

The rest of the paper is organized as follows. In section 2, we derive the basic Green's function estimates that are used in the proofs of Theorems 1.1, 1.3 and 1.4. In section 3 we prove Proposition 1.2 and Theorem 1.3 for the divisible sandpile. In section 4 we adapt the methods of the previous section to prove Theorem 1.4 for the classical abelian sandpile model. Section 5 is devoted to the proof of Theorem 1.1.

2 Basic Estimate

Write $(X_k)_{k \geq 0}$ for simple random walk in \mathbb{Z}^d , and denote by

$$g(x) = \mathbb{E}_o \# \{k | X_k = x\}$$

the Green's function in \mathbb{Z}^d for $d \geq 3$. This function has Laplacian $\Delta g(x) = 0$ for $x \neq o$, and $\Delta g(o) = -1$, where Δ is the discrete Laplacian

$$\Delta g(x) = \frac{1}{2d} \sum_{y \sim x} g(y) - g(x).$$

The sum is over the lattice neighbors y of x . In dimension $d = 2$ simple random walk is recurrent, so the expectation defining $g(x)$ is infinite. Here we define the *potential kernel*

$$a(x) = \lim_{n \rightarrow \infty} g_n(o) - g_n(x)$$

where

$$g_n(x) = \mathbb{E}_o \# \{k \leq n | X_k = x\}.$$

The limit defining $a(x)$ is finite [7, 18], and $a(x)$ has Laplacian $\Delta a(x) = 0$ for $x \neq o$, and $\Delta a(o) = 1$.

We will use the following estimates (see [7], Theorems 1.5.4 and 1.6.2 and remarks following).

$$g(x) = a_d |x|^{2-d} + O(|x|^{-d}); \quad (1)$$

$$a(x) = \frac{2}{\pi} \log |x| + \kappa + O(|x|^{-2}). \quad (2)$$

Here $a_d = \frac{2}{(d-2)\omega_d}$, where ω_d is the volume of the unit ball in \mathbb{R}^d , and κ is a constant whose value we will not need to know. For $x \in \mathbb{Z}^d$ we use $|x|$ to denote the Euclidean norm of x . Here and throughout the paper, constants in error terms denoted $O(\cdot)$ depend only on d .

Fix $m > 0$ and consider the function on \mathbb{Z}^d

$$\tilde{\gamma}_d(x) = \begin{cases} |x|^2 - ma(x), & d = 2 \\ |x|^2 + mg(x), & d \geq 3. \end{cases} \quad (3)$$

Let r be such that $m = \omega_d r^d$, and let

$$\gamma_d(x) = \tilde{\gamma}_d(x) - \tilde{\gamma}_d(\lfloor r \rfloor e_1) \quad (4)$$

where e_1 is the first standard basis vector in \mathbb{Z}^d . We will need an estimate for γ_d near the boundary of the ball B_r . We first consider dimension $d = 2$. From (2) we have

$$\tilde{\gamma}_2(x) = \phi(x) - \kappa m + O(m|x|^{-2}), \quad (5)$$

where

$$\phi(x) = |x|^2 - \frac{2m}{\pi} \log |x|.$$

In the Taylor expansion of ϕ about $|x| = r$

$$\phi(x) = \phi(r) - \phi'(r)(r - |x|) + \frac{1}{2}\phi''(t)(r - |x|)^2 \quad (6)$$

the linear term vanishes, leaving

$$\gamma_2(x) = \left(1 + \frac{m}{\pi t^2}\right) (r - |x|)^2 + O(m|x|^{-2}) \quad (7)$$

for some t between $|x|$ and r .

In dimensions $d \geq 3$, from (1) we have

$$\tilde{\gamma}_d(x) = |x|^2 + a_d m |x|^{2-d} + O(m|x|^{-d}).$$

Setting $\phi(x) = |x|^2 + a_d m |x|^{2-d}$, the linear term in the Taylor expansion (6) of ϕ about $|x| = r$ again vanishes, yielding

$$\gamma_d(x) = \left(1 + (d-1)(r/t)^d\right) (r - |x|)^2 + O(m|x|^{-d})$$

for t between $|x|$ and r . Together with (7) this yields the following estimates.

Lemma 2.1. *Let γ_d be given by (4). For all $d \geq 2$ and all $x \in \mathbb{Z}^d$ we have*

$$\gamma_d(x) \geq (r - |x|)^2 + O\left(\frac{r^d}{|x|^d}\right). \quad (8)$$

Lemma 2.2. *Let γ_d be given by (4). Then uniformly in r ,*

$$\gamma_d(x) = O(1), \quad x \in B_{r+1} - B_{r-1}.$$

The following lemma is useful for x near the origin, where the error term in (8) blows up.

Lemma 2.3. *Let γ_d be given by (4). Then for sufficiently large r , we have*

$$\gamma_d(x) \geq \frac{r^2}{4}, \quad \forall x \in B_{r/3}.$$

Proof. Since $\gamma_d(x) - |x|^2$ is superharmonic, it attains its minimum in $B_{r/3}$ at a point z on the boundary. Thus for any $x \in B_{r/3}$

$$\gamma_d(x) - |x|^2 \geq \gamma_d(z) - |z|^2,$$

hence by Lemma 2.1

$$\gamma_d(x) \geq (2r/3)^2 - (r/3)^2 + O(1) > \frac{r^2}{4}. \quad \square$$

Lemmas 2.1 and 2.3 together imply the following.

Lemma 2.4. *Let γ_d be given by (4). There is a constant a depending only on d , such that $\gamma_d \geq -a$ everywhere.*

3 Divisible Sandpile

Let μ_0 be a nonnegative function on \mathbb{Z}^d with finite support. We start with mass $\mu_0(y)$ at each site y . The operation of *toppling* a vertex x yields the mass distribution

$$T_x \mu_0 = \mu_0 + \alpha(x) \Delta \delta_x$$

where $\alpha(x) = \max(\mu_0(x) - 1, 0)$ and Δ is the discrete Laplacian on \mathbb{Z}^d . Thus if $\mu_0(x) \leq 1$ then $T_x \mu_0 = \mu_0$ and no mass topples; if $\mu_0(x) > 1$ then the mass in excess of 1 is distributed equally among the neighbors of x .

Let $x_1, x_2, \dots \in \mathbb{Z}^d$ be a sequence with the property that for any $x \in \mathbb{Z}^d$ there are infinitely many terms $x_k = x$. Let

$$\mu_k(y) = T_{x_k} \dots T_{x_1} \mu_0(y).$$

be the amount of mass present at y after toppling the sites x_1, \dots, x_k in succession. The total mass emitted from y during this process is

$$u_k(y) := \sum_{j \leq k: x_j = y} \mu_{j-1}(y) - \mu_j(y) = \sum_{j \leq k: x_j = y} \alpha_j(y) \quad (9)$$

where $\alpha_j(y) = \max(\mu_j(y) - 1, 0)$.

Lemma 3.1. *As $k \uparrow \infty$ the functions u_k and μ_k tend to limits $u_k \uparrow u$ and $\mu_k \rightarrow \mu$. Moreover, these limits satisfy*

$$\mu = \mu_0 + \Delta u \leq 1.$$

Proof. Write $M = \sum_x \mu_0(x)$ for the total starting mass, and let $B \subset \mathbb{Z}^d$ be a ball centered at the origin containing all points within L^1 -distance M of the support of μ_0 . Note that if $\mu_k(x) > 0$ and $\mu_0(x) = 0$, then x must have received its mass from a neighbor, so $\mu_k(y) \geq 1$ for some $y \sim x$. Since $\sum_x \mu_k(x) = M$, it follows that μ_k is supported on B . Let R be the radius of B , and consider the *quadratic weight*

$$Q_k = \sum_{x \in \mathbb{Z}^d} \mu_k(x) |x|^2 \leq MR^2.$$

Since $\mu_k(x_k) - \mu_{k-1}(x_k) = -\alpha_k(x_k)$ and for $y \sim x_k$ we have $\mu_k(y) - \mu_{k-1}(y) = \frac{1}{2d}\alpha_k(x_k)$, we obtain

$$Q_k - Q_{k-1} = \alpha_k(x_k) \left(\frac{1}{2d} \sum_{y \sim x_k} |y|^2 - |x_k|^2 \right) = \alpha_k(x_k).$$

Summing over k we obtain from (9)

$$Q_k = Q_0 + \sum_{x \in \mathbb{Z}^d} u_k(x).$$

Fixing x , the sequence $u_k(x)$ is thus increasing and bounded above, hence convergent.

Given neighboring vertices $x \sim y$, since y emits an equal amount of mass to each of its $2d$ neighbors, it emits mass $u_k(y)/2d$ to x up to time k . Thus x receives a total mass of $\frac{1}{2d} \sum_{y \sim x} u_k(y)$ from its neighbors up to time k . Comparing the amount of mass present at x before and after toppling, we obtain

$$\mu_k(x) = \mu_0(x) + \Delta u_k(x).$$

Since $u_k \uparrow u$ we infer that $\mu_k \rightarrow \mu := \mu_0 + \Delta u$. Note that if $x_k = x$, then $\mu_k(x) \leq 1$. Since for each $x \in \mathbb{Z}^d$ this holds for infinitely many values of k , the limit satisfies $\mu \leq 1$. \square

A function s on \mathbb{Z}^d is *superharmonic* if $\Delta s \leq 0$. Given a function γ on \mathbb{Z}^d the *least superharmonic majorant* of γ is the function

$$s(x) = \inf\{f(x) \mid f \text{ is superharmonic and } f \geq \gamma\}.$$

The study of the least superharmonic majorant is a classical topic in analysis and PDE; see, for example, [6]. Note that if f is superharmonic and $f \geq \gamma$ then

$$f(x) \geq \frac{1}{2d} \sum_{y \sim x} f(y) \geq \frac{1}{2d} \sum_{y \sim x} s(y).$$

Taking the infimum on the left side we obtain that s is superharmonic.

Lemma 3.2. *The limit u in Lemma 3.1 is given by $u = s + \gamma$, where*

$$\gamma(x) = |x|^2 + \sum_{y \in \mathbb{Z}^d} g(x-y) \mu_0(y)$$

and s is the least superharmonic majorant of $-\gamma$.

Proof. By Lemma 3.1 we have

$$\Delta u = \mu - \mu_0 \leq 1 - \mu_0.$$

Since $\Delta \gamma = 1 - \mu_0$, the difference $u - \gamma$ is superharmonic. As u is nonnegative, it follows that $u - \gamma \geq s$. For the reverse inequality, note that $s + \gamma - u$ is superharmonic on the domain $D = \{x \mid \mu(x) = 1\}$ of fully occupied sites and is nonnegative outside D , hence nonnegative inside D as well. \square

As a corollary of Lemmas 3.1 and 3.2, we obtain the abelian property of the divisible sandpile, Proposition 1.2, which was stated in the introduction.

We now turn to the case of a point source mass m started at the origin: $\mu_0 = m\delta_o$. More general starting distributions are treated in [14], where we identify the scaling limit of the divisible sandpile model and show that it coincides with that of internal DLA and of the rotor-router model. In the case of a point source of mass m , the natural question is to identify the shape of the resulting domain D_m of fully occupied sites, i.e. sites x for which $\mu(x) = 1$. According to Theorem 3.3, D_m is extremely close to a ball of volume m ; in fact, the error in the radius is a constant independent of m . As before, for $r \geq 0$ we write

$$B_r = \{x \in \mathbb{Z}^d : |x| < r\}$$

for the lattice ball of radius r centered at the origin.

Theorem 3.3. *For $m \geq 0$ let $D_m \subset \mathbb{Z}^d$ be the domain of fully occupied sites for the divisible sandpile formed from a pile of size m at the origin. There exist constants c, c' depending only on d , such that*

$$B_{r-c} \subset D_m \subset B_{r+c'},$$

where $r = (m/\omega_d)^{1/d}$ and ω_d is the volume of the unit ball in \mathbb{R}^d .

The idea of the proof is to use Lemma 3.2 along with the basic estimates on γ , Lemmas 2.1 and 2.2, to obtain estimates on the odometer function

$$u(x) = \text{total mass emitted from } x.$$

We will need the following simple observation.

Lemma 3.4. *For every point $x \in D_m - \{o\}$ there is a path $x = x_0 \sim x_1 \sim \dots \sim x_k = o$ in D_m with $u(x_{i+1}) \geq u(x_i) + 1$.*

Proof. If $x_i \in D_m - \{o\}$, let x_{i+1} be a neighbor of x_i maximizing $u(x_{i+1})$. Then $x_{i+1} \in D_m$ and

$$\begin{aligned} u(x_{i+1}) &\geq \frac{1}{2d} \sum_{y \sim x_i} u(y) \\ &= u(x_i) + \Delta u(x_i) \\ &= u(x_i) + 1, \end{aligned}$$

where in the last step we have used the fact that $x_i \in D_m$. \square

Proof of Theorem 3.3. We first treat the inner estimate. Let γ_d be given by (4). By Lemma 3.2 the function $u - \gamma_d$ is superharmonic, so its minimum in the ball B_r is attained on the boundary. Since $u \geq 0$, we have by Lemma 2.2

$$u(x) - \gamma_d(x) \geq O(1), \quad x \in \partial B_r.$$

Hence by Lemma 2.1,

$$u(x) \geq (r - |x|)^2 + O(r^d/|x|^d), \quad x \in B_r. \quad (10)$$

It follows that there is a constant c , depending only on d , such that $u(x) > 0$ whenever $r/3 \leq |x| < r - c$. Thus $B_{r-c} - B_{r/3} \subset D_m$. For $x \in B_{r/3}$, by Lemma 2.3 we have $u(x) > r^2/4 + O(1) > 0$, hence $B_{r/3} \subset D_m$.

For the outer estimate, note that $u - \gamma_d$ is harmonic on D_m . By Lemma 2.4 we have $\gamma_d \geq -a$ everywhere, where a depends only on d . Since u vanishes on ∂D_m it follows that $u - \gamma_d \leq a$ on D_m . Now for any $x \in D_m$ with $r - 1 < |x| \leq r$, we have by Lemma 2.2

$$u(x) \leq \gamma_d(x) + a \leq c'$$

for a constant c' depending only on d . Lemma 3.4 now implies that $D_m \subset B_{r+c'+1}$. \square

4 Classical Sandpile

Consider the odometer function for the classical abelian sandpile

$$u(x) = \text{total number of grains emitted from } x.$$

In the final state, each site retains between 0 and $2d - 1$ grains, hence

$$0 \leq \Delta u + n\delta_o \leq 2d - 1. \quad (11)$$

We can improve the lower bound by averaging over a small box. For $x \in \mathbb{Z}^d$ let

$$Q_k(x) = \{y \in \mathbb{Z}^d : \|x - y\|_\infty \leq k\}$$

be the box of side length $2k + 1$ centered at x , and let

$$u^{(k)}(x) = (2k + 1)^{-d} \sum_{y \in Q_k(x)} u(y).$$

Denote by S_n the set of sites that are visited by the classical abelian sandpile in \mathbb{Z}^d if n grains of sand start at the origin. Write

$$S_n^{(k)} = \{x \mid Q_k(x) \subset S_n\}.$$

Le Borgne and Rossin [10] observe that if S is a set of sites all of which topple, the number of grains remaining in S is at least the number of edges internal to S : indeed, for each internal edge, the endpoint that topples last sends the other a grain which never moves again. Since the box Q_k has $2dk(2k + 1)^{d-1}$ internal edges, we have

$$\Delta u^{(k)}(x) \geq d \frac{2k}{2k + 1}, \quad x \in S_n^{(k)}. \quad (12)$$

The following lemma is analogous to Lemma 3.4.

Lemma 4.1. *For every point $x \in S_n$ adjacent to ∂S_n there is a path $x = x_0 \sim x_1 \sim \dots \sim x_m = o$ in S_n with $u(x_{i+1}) \geq u(x_i) + 1$.*

Proof. By (11) we have

$$\frac{1}{2d} \sum_{y \sim x_i} u(y) \geq u(x_i).$$

Since $u(x_{i-1}) < u(x_i)$, some term $u(y)$ in the sum above must exceed $u(x_i)$. Let $x_{i+1} = y$. □

Proof of Theorem 1.4. Let

$$\tilde{\xi}_d(x) = \begin{cases} (2d - 1)|x|^2 - na(x), & d = 2 \\ (2d - 1)|x|^2 + ng(x), & d \geq 3; \end{cases}$$

and let

$$\xi_d(x) = \tilde{\xi}_d(x) - \tilde{\xi}_d(\lfloor c_1 r \rfloor e_1).$$

Taking $m = n/(2d - 1)$ in Lemma 2.2, we have

$$u(x) - \xi_d(x) \geq -\xi_d(x) = O(1), \quad x \in \partial B_{c_1 r}. \quad (13)$$

By (11), $u - \xi_d$ is superharmonic, so (13) holds in all of $B_{c_1 r}$. Hence by Lemma 2.1

$$u(x) \geq \frac{(r - |x|)^2}{2d - 1} + O(r^d/|x|^d), \quad x \in B_{c_1 r}. \quad (14)$$

It follows that u is positive on $B_{c_1 r - c_2} - B_{c_1 r/3}$ for a suitable constant c_2 . For $x \in B_{c_1 r/3}$, by Lemma 2.3 we have $u(x) > c_1^2 r^2/4 + O(1) > 0$. Thus $B_{c_1 r - c_2} \subset S_n$.

For the outer estimate, let

$$\hat{\psi}_d(x) = \begin{cases} (d - \epsilon)|x|^2 - na(x), & d = 2 \\ (d - \epsilon)|x|^2 + ng(x), & d \geq 3. \end{cases}$$

Choose k large enough so that $d \frac{2k}{2k+1} \geq d - \epsilon$ and define

$$\tilde{\psi}_d(x) = (2k + 1)^{-d} \sum_{y \in Q_k(x)} \hat{\psi}_d(y).$$

Finally, let

$$\psi_d(x) = \tilde{\psi}_d(x) - \tilde{\psi}_d(\lfloor r \rfloor e_1).$$

By (12), $u^{(k)} - \psi_d$ is subharmonic on $S_n^{(k)}$. Taking $m = n/(d - \epsilon)$ in Lemma 2.4, there is a constant a such that $\psi_d \geq -a$ everywhere. Since $u^{(k)} \leq (2d)^{(d+1)k}$ on $\partial S_n^{(k)}$ it follows that $u^{(k)} - \psi_d \leq a + (2d)^{(d+1)k}$ on $S_n^{(k)}$. Now for any $x \in S_n$ with $c'_1 r - 1 < |x| \leq c'_1 r$ we have by Lemma 2.2

$$u^{(k)}(x) \leq \psi_d(x) + a + (2d)^{(d+1)k} \leq \tilde{c}_2$$

for a constant \tilde{c}_2 depending only on d and ϵ . Then $u(x) \leq c'_2 := (2k + 1)^d \tilde{c}_2$. Lemma 4.1 now implies that $S_n \subset B_{c'_1 r + c'_2 + 1}$. \square

5 Rotor-Router Model

Given a function f on \mathbb{Z}^d , for a directed edge (x, y) write

$$\nabla f(x, y) = f(y) - f(x).$$

Given a function s on directed edges in \mathbb{Z}^d , write

$$\operatorname{div} s(x) = \frac{1}{2d} \sum_{y \sim x} s(x, y).$$

The discrete Laplacian of f is then given by

$$\Delta f(x) = \operatorname{div} \nabla f = \frac{1}{2d} \sum_{y \sim x} f(y) - f(x).$$

5.1 Inner Estimate

Fixing $n \geq 1$, consider the odometer function for rotor-router aggregation

$$u(x) = \text{total number exits from } x \text{ by the first } n \text{ particles.}$$

We learned the idea of using the odometer function to study the rotor-router shape from Matt Cook [2].

Lemma 5.1. *For a directed edge (x, y) in \mathbb{Z}^d , denote by $\kappa(x, y)$ the net number of crossings from x to y performed by the first n particles in rotor-router aggregation. Then*

$$\nabla u(x, y) = -2d\kappa(x, y) + R(x, y) \tag{15}$$

for some edge function R which satisfies

$$|R(x, y)| \leq 4d - 2$$

for all edges (x, y) .

Remark. In the more general setting of rotor stacks of bounded discrepancy, the $4d - 2$ will be replaced by a different constant here.

Proof. Writing $N(x, y)$ for the number of particles routed from x to y , we have

$$\frac{u(x) - 2d + 1}{2d} \leq N(x, y) \leq \frac{u(x) + 2d - 1}{2d}$$

hence

$$\begin{aligned} |\nabla u(x, y) + 2d\kappa(x, y)| &= |u(y) - u(x) + 2dN(x, y) - 2dN(y, x)| \\ &\leq 4d - 2. \quad \square \end{aligned}$$

In what follows, C_0, C_1, \dots denote constants depending only on d .

Lemma 5.2. *Let $\Omega \subset \mathbb{Z}^d - \{o\}$ be a finite set. Then*

$$\sum_{y \in \Omega} |y|^{1-d} \leq C_0 \text{Diam}(\Omega).$$

Proof. Let

$$\mathcal{S}_k = \{y \in \mathbb{Z}^d : k \leq |y| < k+1\}.$$

Then

$$\sum_{y \in \mathcal{S}_k} |y|^{1-d} \leq k^{1-d} \#\mathcal{S}_k \leq C_0.$$

Since Ω can intersect at most $\text{Diam}(\Omega) + 1$ distinct sets \mathcal{S}_k , the proof is complete. \square

Lemma 5.3. *Let $G = G_{B_r}$ be the Green's function for random walk stopped on exiting B_r . Let $x \in B_r$ with $|x| > r - \rho$. Then*

$$\sum_{\substack{y \in B_r \\ |x-y| < 3\rho}} \sum_{z \sim y} |G(x, y) - G(x, z)| \leq C_1 \rho. \quad (16)$$

Proof. Let $(X_t)_{t \geq 0}$ denote simple random walk in \mathbb{Z}^d , and let T be the first exit time from B_r . In dimension $d \geq 3$, by counting visits to y by random walk before and after exiting B_r , we obtain the relation $g(x-y) = G(x, y) + \mathbb{E}_x g(X_T - y)$. Similarly, in \mathbb{Z}^2 , the function $A(y) = \mathbb{E}_x a(X_T - y) - a(x-y)$ has Laplacian $\Delta A(y) = -\delta_{xy}$ in B_r and vanishes on ∂B_r , hence $A(y) = G(x, y)$. Thus

$$G(x, y) = \begin{cases} -a(x-y) + \mathbb{E}_x a(X_T - y), & d = 2; \\ g(x-y) - \mathbb{E}_x g(X_T - y), & d \geq 3. \end{cases} \quad (17)$$

Let $y \in B_r$ and $z \sim y$. In dimensions $d \geq 3$, from (1) we have

$$|g(x-y) - g(x-z)| \leq \frac{C_2}{|x-y|^{d-1}}, \quad y, z \neq x.$$

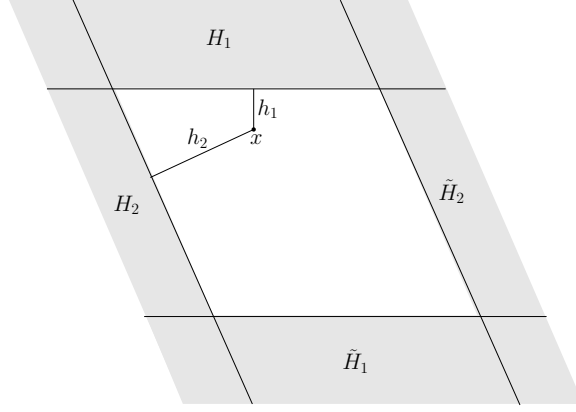


Figure 4: Diagram for the Proof of Lemma 5.4.

Likewise, from (2) we have

$$|a(x-y) - a(x-z)| \leq \frac{C_2}{|x-y|}, \quad y, z \neq x.$$

Using the triangle inequality together with (17), in all dimensions $d \geq 2$ we obtain

$$|G(x, y) - G(x, z)| \leq \sum_{w \in \partial B_r} H_x(w) \frac{C_2}{|w-y|^{d-1}} + \frac{C_2}{|x-y|^{d-1}},$$

where $H_x(w) = \mathbb{P}_x(X_T = w)$. Write $D = \{y \in B_r : |x-y| < 3\rho\}$. Then

$$\sum_{\substack{y \in D \\ y \neq x}} \sum_{\substack{z \sim y \\ z \neq x}} |G(x, y) - G(x, z)| \leq C_2 \sum_{w \in \partial B_r} H_x(w) \sum_{y \in D} |w-y|^{1-d} + C_3 \rho. \quad (18)$$

By Lemma 5.2, the inner sum on the right is at most $C_0 \text{Diam}(D) = 6C_0\rho$, so the sum in (18) is bounded above by $C_1\rho$.

Finally, the terms in which y or z coincides with x make a negligible contribution to the sum in (16), since for $y \sim x \in \mathbb{Z}^2$

$$|G(x, x) - G(x, y)| \leq \mathbb{E}_x |a(X_T - x) - a(X_T - y)| + |a(o) - a(x-y)| \leq C_4.$$

and likewise with a replaced by g in \mathbb{Z}^d for $d \geq 3$. \square

Lemma 5.4. *Let H_1, H_2 be linear half-spaces in \mathbb{Z}^d , not necessarily parallel to the coordinate axes. Let T_i be the first hitting time of H_i . If $x \notin H_1 \cup H_2$, then*

$$\mathbb{P}_x(T_1 > T_2) \leq \frac{5}{2} \frac{h_1 + 1}{h_2} \left(1 + \frac{1}{2h_2}\right)^2$$

where h_i is the distance from x to H_i .

Proof. If one of H_1, H_2 contains the other, the result is vacuous. Otherwise, let \tilde{H}_i be the half-space shifted parallel to H_i^c by distance $2h_2$ in the direction of x , and let \tilde{T}_i be the first hitting time of $H_i \cup \tilde{H}_i$. Let $(X_t)_{t \geq 0}$ denote simple random walk in \mathbb{Z}^d , and write M_t for the (signed) distance from X_t to the hyperplane defining the boundary of H_1 , with $M_0 = h_1$. Then M_t is a martingale with bounded increments. Since $\mathbb{E}_x \tilde{T}_1 < \infty$, we obtain from optional stopping

$$h_1 = \mathbb{E}_x M_{\tilde{T}_1} \geq 2h_2 \mathbb{P}_x(X_{\tilde{T}_1} \in \tilde{H}_1) - \mathbb{P}_x(X_{\tilde{T}_1} \in H_1),$$

hence

$$\mathbb{P}_x(X_{\tilde{T}_1} \in \tilde{H}_1) \leq \frac{h_1 + 1}{2h_2}. \quad (19)$$

Likewise, $dM_t^2 - t$ is a martingale with bounded increments, giving

$$\begin{aligned} \mathbb{E}_x \tilde{T}_1 &\leq d \mathbb{E}_x M_{\tilde{T}_1}^2 \\ &\leq d(2h_2 + 1)^2 \mathbb{P}_x(X_{\tilde{T}_1} \in \tilde{H}_1) \\ &\leq d(h_1 + 1)(2h_2 + 1) \left(1 + \frac{1}{2h_2}\right). \end{aligned} \quad (20)$$

Let $T = \min(\tilde{T}_1, \tilde{T}_2)$. Denoting by D_t the distance from X_t to the hyperplane defining the boundary of H_2 , the quantity

$$N_t = \frac{d}{2}(D_t^2 + (2h_2 - D_t)^2) - t$$

is a martingale. Writing $p = \mathbb{P}_x(T = \tilde{T}_2)$ we have

$$\begin{aligned} dh_2^2 = \mathbb{E} N_0 &= \mathbb{E} N_T \geq p \frac{d}{2}(2h_2)^2 + (1 - p)dh_2^2 - \mathbb{E}_x T \\ &\geq (1 + p)dh_2^2 - \mathbb{E}_x T \end{aligned}$$

hence by (20)

$$p \leq \frac{\mathbb{E}_x T}{dh_2^2} \leq 2 \frac{h_1 + 1}{h_2} \left(1 + \frac{1}{2h_2}\right)^2.$$

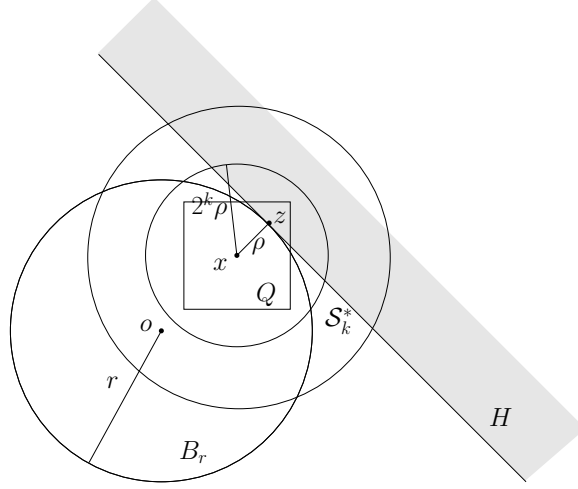


Figure 5: Diagram for the proof of Lemma 5.5.

Finally by (19)

$$\mathbb{P}(T_1 > T_2) \leq p + \mathbb{P}(X_{\tilde{T}_1} \in \tilde{H}_1) \leq \frac{5}{2} \frac{h_1 + 1}{h_2} \left(1 + \frac{1}{2h_2}\right)^2. \quad \square$$

Lemma 5.5. *Let $x \in B_r$ and let $\rho = r + 1 - |x|$. Let*

$$\mathcal{S}_k^* = \{y \in B_r : 2^k \rho < |x - y| \leq 2^{k+1} \rho\}. \quad (21)$$

Let τ_k be the first hitting time of \mathcal{S}_k^ , and T the first exit time from B_r . Then*

$$\mathbb{P}_x(\tau_k < T) \leq C_2 2^{-k}.$$

Proof. Let H be the outer half-space tangent to B_r at the point $z \in \partial B_r$ closest to x . Let Q be the cube of side length $2^k \rho / \sqrt{d}$ centered at x . Then Q is disjoint from \mathcal{S}_k^* , hence

$$\mathbb{P}_x(\tau_k < T) \leq \mathbb{P}_x(T_{\partial Q} < T) \leq \mathbb{P}_x(T_{\partial Q} < T_H)$$

where $T_{\partial Q}$ and T_H are the first hitting times of ∂Q and H . Let H_1, \dots, H_{2d} be the outer half-spaces defining the faces of Q , so that $Q = H_1^c \cap \dots \cap H_{2d}^c$.

By Lemma 5.4 we have

$$\begin{aligned}\mathbb{P}_x(T_{\partial Q} < T_H) &\leq \sum_{i=1}^{2d} \mathbb{P}_x(T_{H_i} < T_H) \\ &\leq \frac{5}{2} \sum_{i=1}^{2d} \frac{\text{dist}(x, H) + 1}{\text{dist}(x, H_i)} \left(1 + \frac{1}{2 \text{dist}(x, H_i)}\right)^2.\end{aligned}$$

Since $\text{dist}(x, H) = |x - z| \leq \rho$ and $\text{dist}(x, H_i) = 2^{k-1}\rho/\sqrt{d}$, and $\rho \geq 1$, taking $C_2 = 20 d^{3/2}(1 + \sqrt{d})^2$ completes the proof. \square

Lemma 5.6. *Let $G = G_{B_r}$ be the Green's function for random walk stopped on exiting B_r . Let $x \in B_r$ and let $\rho = r - |x|$. Then*

$$\sum_{y \in B_r} \sum_{z \sim y} |G(x, y) - G(x, z)| \leq C_3 \rho \log \frac{r}{\rho}.$$

Proof. Let \mathcal{S}_k^* be given by (21), and let

$$W = \{w \in \partial(\mathcal{S}_k^* \cup \partial \mathcal{S}_k^*) : |w - x| < 2^k \rho\}.$$

Let τ_W be the first hitting time of W and T the first exit time from B_r . For $w \in W$ let

$$H_x(w) = \mathbb{P}_x(X_{\tau_W \wedge T} = w).$$

Fixing $y \in \mathcal{S}_k^*$ and $z \sim y$, simple random walk started at x must hit W before hitting either y or z , hence

$$|G(x, y) - G(x, z)| \leq \sum_{w \in W} H_x(w) |G(w, y) - G(w, z)|.$$

For any $w \in W$ we have $|w| > r - (2^k + 1)\rho$ and $|y - w| < 3 \cdot 2^k \rho$. Lemma 5.3 yields

$$\sum_{y \in \mathcal{S}_k^*} \sum_{z \sim y} |G(x, y) - G(x, z)| \leq C_1 (2^k + 1) \rho \sum_{w \in W} H_x(w).$$

By Lemma 5.5 we have $\sum_{w \in W} H_x(w) \leq C_2 2^{-k}$, so the above sum is at most $2C_1 C_2 \rho$. Since the union of shells $\mathcal{S}_0^*, \mathcal{S}_1^*, \dots, \mathcal{S}_{\lceil \log_2(r/\rho) \rceil}^*$ covers all of B_r except for those points within distance ρ of x , and $\sum_{|x-y| \leq \rho} \sum_{z \sim y} |G(x, y) - G(x, z)| \leq C_1 \rho$ by Lemma 5.3, the result follows. \square

Proof of Theorem 1.1, Inner Estimate. Let κ and R be defined as in Lemma 5.1. Since the net number of particles to enter a site $x \neq o$ is at most one, we have $2d \operatorname{div} \kappa(x) \geq -1$. Likewise $2d \operatorname{div} \kappa(o) = n - 1$. Taking the divergence in (15), we obtain

$$\Delta u(x) \leq 1 + \operatorname{div} R(x), \quad x \neq o; \quad (22)$$

$$\Delta u(o) = 1 - n + \operatorname{div} R(o). \quad (23)$$

Let T be the first exit time from B_r , and define

$$f(x) = \mathbb{E}_x u(X_T) - \mathbb{E}_x T + n \mathbb{E}_x \#\{j < T | X_j = 0\}.$$

Then $\Delta f(x) = 1$ for $x \in B_r - \{o\}$ and $\Delta f(o) = 1 - n$. Moreover $f \geq 0$ on ∂B_r . It follows from Lemma 2.2 with $m = n$ that $f \geq \gamma - C_4$ on B_r for a suitable constant C_4 .

We have

$$u(x) - \mathbb{E}_x u(X_T) = \sum_{k \geq 0} \mathbb{E}_x \left(u(X_{k \wedge T}) - u(X_{(k+1) \wedge T}) \right).$$

Each summand on the right side is zero on the event $\{T \leq k\}$, hence

$$\mathbb{E}_x \left(u(X_{k \wedge T}) - u(X_{(k+1) \wedge T}) \mid \mathcal{F}_{k \wedge T} \right) = -\Delta u(X_k) \mathbb{1}_{\{T > k\}}.$$

Taking expectations and using (22) and (23), we obtain

$$\begin{aligned} u(x) - \mathbb{E}_x u(X_T) &\geq \sum_{k \geq 0} \mathbb{E}_x \left[\mathbb{1}_{\{T > k\}} (n \mathbb{1}_{\{X_k = o\}} - 1 - \operatorname{div} R(X_k)) \right] \\ &= n \mathbb{E}_x \#\{k < T | X_k = o\} - \mathbb{E}_x T - \sum_{k \geq 0} \mathbb{E}_x \left[\mathbb{1}_{\{T > k\}} \operatorname{div} R(X_k) \right], \end{aligned}$$

hence

$$u(x) - f(x) \geq -\frac{1}{2d} \sum_{k \geq 0} \mathbb{E}_x \left[\mathbb{1}_{\{T > k\}} \sum_{z \sim X_k} R(X_k, z) \right]. \quad (24)$$

Since random walk exits B_r with probability at least $\frac{1}{2d}$ every time it reaches a site adjacent to the boundary ∂B_r , the expected time spent adjacent to the boundary before time T is at most $2d$. Since $|R| \leq 4d$, the terms in (24) with $z \in \partial B_r$ contribute at most $16d^3$ to the sum. Thus

$$u(x) - f(x) \geq -\frac{1}{2d} \sum_{k \geq 0} \mathbb{E}_x \left[\sum_{\substack{y, z \in B_r \\ y \sim z}} \mathbb{1}_{\{T > k\} \cap \{X_k = y\}} R(y, z) \right] - 8d^2.$$

For $y \in B_r$ we have $\{X_k = y\} \cap \{T > k\} = \{X_{k \wedge T} = y\}$, hence

$$u(x) - f(x) \geq -\frac{1}{2d} \sum_{k \geq 0} \sum_{\substack{y, z \in B_r \\ y \sim z}} \mathbb{P}_x(X_{k \wedge T} = y) R(y, z) - 8d^2. \quad (25)$$

Write $p_k(y) = \mathbb{P}_x(X_{k \wedge T} = y)$. Note that since ∇f and κ are antisymmetric, R is antisymmetric. Thus

$$\begin{aligned} \sum_{\substack{y, z \in B_r \\ y \sim z}} p_k(y) R(y, z) &= - \sum_{\substack{y, z \in B_r \\ y \sim z}} p_k(z) R(y, z) \\ &= \sum_{\substack{y, z \in B_r \\ y \sim z}} \frac{p_k(y) - p_k(z)}{2} R(y, z). \end{aligned}$$

Summing over k and using the fact that $|R| \leq 4d$, we conclude from (25) that

$$u(x) \geq f(x) - \sum_{\substack{y, z \in B_r \\ y \sim z}} |G(x, y) - G(x, z)| - 8d^2,$$

where $G = G_{B_r}$ is the Green's function for simple random walk stopped on exiting B_r . By Lemma 5.6 we obtain

$$u(x) \geq f(x) - C_3(r - |x|) \log \frac{r}{r - |x|} - 8d^2.$$

Using the fact that $f \geq \gamma - C_4$, we obtain from Lemma 2.1

$$u(x) \geq (r - |x|)^2 - C_3(r - |x|) \log \frac{r}{r - |x|} + O\left(\frac{r^d}{|x|^d}\right).$$

The right side is positive provided $r/3 \leq |x| < r - C_5 \log r$. For $x \in B_{r/3}$, by Lemma 2.3 we have $u(x) > r^2/4 - C_3 r \log \frac{3}{2} > 0$, hence $B_{r - C_5 \log r} \subset A_n$. \square

5.2 Outer Estimate

The following result is due to Holroyd and Propp (unpublished); we include a proof for the sake of completeness. Notice that the bound in (26) does not depend on the number of particles.

Proposition 5.7. *Let Γ be a finite connected graph, and let $Y \subset Z$ be subsets of the vertex set of Γ . Let s be a nonnegative integer-valued function on the vertices of Γ . Let $H_w(s, Y)$ be the expected number of particles stopping*

in Y if $s(x)$ particles start at each vertex x and perform independent simple random walks stopped on first hitting Z . Let $H_r(s, Y)$ be the number of particles stopping in Y if $s(x)$ particles start at each vertex x and perform rotor-router walks stopped on first hitting Z . Let $H(x) = H_w(1_x, Y)$. Then

$$|H_r(s, Y) - H_w(s, Y)| \leq \sum_{u \in G} \sum_{v \sim u} |H(u) - H(v)| \quad (26)$$

independent of s and the initial positions of the rotors.

Proof. For each vertex u , arbitrarily choose a neighbor $\eta(u)$. Order the neighbors $\eta(u) = v_1, v_2, \dots, v_d$ of u so that the rotor at u points to v_{i+1} immediately after pointing to v_i (indices mod d). We assign *weight* $w(u, \eta(u)) = 0$ to a rotor pointing from u to $\eta(u)$, and weight $w(u, v_i) = H(u) - H(v_i) + w(u, v_{i-1})$ to a rotor pointing from u to v_i . These assignments are consistent since H is a harmonic function: $\sum_i (H(u) - H(v_i)) = 0$. We also assign weight $H(u)$ to a particle located at u . The sum of rotor and particle weights in any configuration is invariant under the operation of routing a particle and rotating the corresponding rotor. Initially, the sum of all particle weights is $H_w(s, Y)$. After all particles have stopped, the sum of the particle weights is $H_r(s, Y)$. Their difference is thus at most the change in rotor weights, which is bounded above by the sum in (26). \square

For $\rho \in \mathbb{Z}$ let

$$\mathcal{S}_\rho = \{x \in \mathbb{Z}^d : \rho \leq |x| < \rho + 1\}. \quad (27)$$

Then

$$B_\rho = \{x \in \mathbb{Z}^d : |x| < \rho\} = \mathcal{S}_0 \cup \dots \cup \mathcal{S}_{\rho-1}.$$

Note that for simple random walk started in B_ρ , the first exit time of B_ρ and first hitting time of \mathcal{S}_ρ coincide. Our next result is a modification of Lemma 5(b) of [8].

Lemma 5.8. *Fix $\rho \geq 1$ and $y \in \mathcal{S}_\rho$. For $x \in B_\rho$ let $H(x) = \mathbb{P}_x(X_T = y)$, where T is the first hitting time of \mathcal{S}_ρ . Then*

$$H(x) \leq \frac{J}{|x - y|^{d-1}} \quad (28)$$

for a constant J depending only on d .

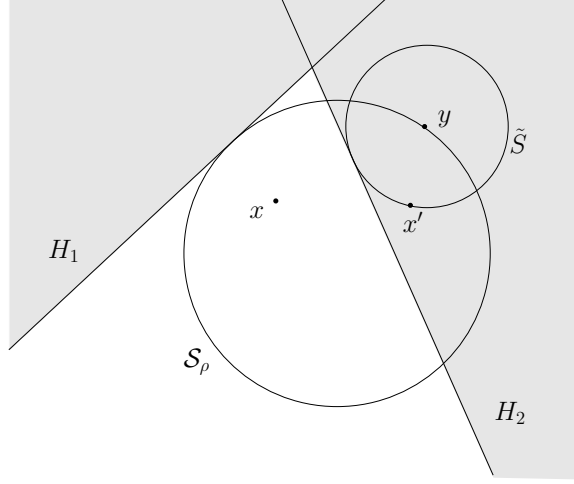


Figure 6: Diagram for the proof of Lemma 5.8.

Proof. We induct on the distance $|x - y|$, assuming the result holds for all x' with $|x' - y| \leq \frac{1}{2}|x - y|$; the base case can be made trivial by choosing J sufficiently large. By Lemma 5(b) of [8], we can choose J large enough so that the result holds provided $|y| - |x| \geq 2^{-d-3}|x - y|$. Otherwise, let H_1 be the outer half-space tangent to \mathcal{S}_ρ at the point of \mathcal{S}_ρ closest to x , and let H_2 be the inner half-space tangent to the ball \tilde{S} of radius $\frac{1}{2}|x - y|$ about y , at the point of \tilde{S} closest to x . By Lemma 5.4 applied to these half-spaces, the probability that random walk started at x reaches \tilde{S} before hitting \mathcal{S}_ρ is at most 2^{1-d} . Writing \tilde{T} for the first hitting time of $\tilde{S} \cup \mathcal{S}_\rho$, we have

$$H(x) \leq \sum_{x' \in \tilde{S}} \mathbb{P}_x(X_{\tilde{T}} = x') H(x') \leq 2^{1-d} J \cdot \left(\frac{|x - y|}{2} \right)^{1-d}$$

where we have used the inductive hypothesis to bound $H(x')$. \square

We will need the following standard result, which can be derived e.g. from the estimates in [15], section II.12; we include a proof for the sake of completeness.

Lemma 5.9. *Given $u \sim v \in \mathbb{Z}^d$, lazy random walks started at u and v can be coupled with probability $1 - C/R$ before either reaches distance R from u , where C depends only on d .*

Proof. First choose one of the d coordinate axes uniformly at random. If u and v agree in this coordinate, let the two walks take the same lazy step so that they still agree in this coordinate. If u and v differ in the chosen coordinate, let the first walk take a step while the second stays in place. With probability $\frac{1}{2}$ the walks will then be coupled. If they are not, they now differ by 2 in this coordinate. From now on, whenever this coordinate is chosen, let the two walks take lazy steps in opposite directions. Let Q be the cube of side length $R/\sqrt{d} + 2$ centered at u . By Lemma 5.4 with $h_1 = 1$ and $h_2 = R/2\sqrt{d}$, the probability that one of the walks exits Q before the walks couple is at most $2d \cdot \frac{5}{2} \frac{h_1+1}{h_2} \left(1 + \frac{1}{2h_2}\right)^2 \leq 20 d^{3/2} (1 + \sqrt{d})^2 / R$. \square

Lemma 5.10. *With H defined as in Lemma 5.8, we have*

$$\sum_{u \in B_\rho} \sum_{v \sim u} |H(u) - H(v)| \leq J' \log \rho$$

for a constant J' depending only on d .

Proof. Given $u \in B_\rho$ and $v \sim u$, by Lemma 5.9, lazy random walks started at u and v can be coupled with probability $1 - 2C/|u - y|$ before either reaches distance $|u - y|/2$ from u . If the walks reach this distance without coupling, by Lemma 5.8 each has still has probability at most $J/|u - y|^{d-1}$ of exiting B_ρ at y . By the strong Markov property it follows that

$$|H(u) - H(v)| \leq \frac{2CJ}{|u - y|^d}.$$

Summing in spherical shells about u , we obtain

$$\sum_{u \in B_\rho} \sum_{v \sim u} |H(u) - H(v)| \leq \sum_{t=1}^{\rho} d\omega_d t^{d-1} \frac{2CJ}{t^d} \leq J' \log \rho. \quad \square$$

Proof of Theorem 1.1, Outer Estimate. Fix integers $\rho \geq r$, $h \geq 1$. In the setting of Proposition 5.7, let G be the lattice ball $B_{\rho+h+1}$, and let $Z = \mathcal{S}_{\rho+h}$. Fix $y \in \mathcal{S}_{\rho+h}$ and let $Y = \{y\}$. For $x \in \mathcal{S}_\rho$ let $s(x)$ be the number of particles stopped at x if all particles in rotor-router aggregation are stopped upon reaching \mathcal{S}_ρ . Write

$$H(x) = \mathbb{P}_x(X_T = y)$$

where T is the first hitting time of $\mathcal{S}_{\rho+h}$. By Lemma 5.8 we have

$$H_w(s, y) = \sum_{x \in \mathcal{S}_\rho} s(x) H(x) \leq \frac{JN_\rho}{h^{d-1}} \quad (29)$$

where

$$N_\rho = \sum_{x \in \mathcal{S}_\rho} s(x)$$

is the number of particles that ever visit the shell \mathcal{S}_ρ in the course of rotor-router aggregation.

By Lemma 5.10 the sum in (26) is at most $J' \log h$, hence from Proposition 5.7 and (29) we have

$$H_r(s, y) \leq \frac{JN_\rho}{h^{d-1}} + J' \log h. \quad (30)$$

Let $\rho(0) = r$, and define $\rho(i)$ inductively by

$$\rho(i+1) = \min \left\{ \rho(i) + N_{\rho(i)}^{2/(2d-1)}, \min \{ \rho > \rho(i) \mid N_\rho \leq N_{\rho(i)}/2 \} \right\}. \quad (31)$$

Fixing $h < \rho(i+1) - \rho(i)$, we have

$$h^{d-1} \log h \leq N_{\rho(i)}^{\frac{2d-2}{2d-1}} \log N_{\rho(i)} \leq N_{\rho(i)};$$

so (30) with $\rho = \rho(i)$ simplifies to

$$H_r(s, y) \leq \frac{CN_{\rho(i)}}{h^{d-1}} \quad (32)$$

where $C = J + J'$.

Since all particles that visit $\mathcal{S}_{\rho(i)+h}$ during rotor-router aggregation must pass through $\mathcal{S}_{\rho(i)}$, we have

$$N_{\rho(i)+h} \leq \sum_{y \in \mathcal{S}_{\rho(i)+h}} H_r(s, y). \quad (33)$$

Let $M_k = \#(A_n \cap \mathcal{S}_k)$. There are at most $M_{\rho(i)+h}$ nonzero terms in the sum on the right side of (33), and each term is bounded above by (32), hence

$$M_{\rho(i)+h} \geq N_{\rho(i)+h} \frac{h^{d-1}}{CN_{\rho(i)}} \geq \frac{h^{d-1}}{2C}$$

where the second inequality follows from $N_{\rho(i)+h} \geq N_{\rho(i)}/2$. Summing over h , we obtain

$$\sum_{\rho=\rho(i)+1}^{\rho(i+1)-1} M_\rho \geq \frac{1}{2dC}(\rho(i+1) - \rho(i) - 1)^d. \quad (34)$$

The left side is at most $N_{\rho(i)}$, hence

$$\rho(i+1) - \rho(i) \leq (2dCN_{\rho(i)})^{1/d} \leq N_{\rho(i)}^{2/(2d-1)}$$

provided $N_{\rho(i)} \geq C' := (2dC)^{2d-1}$. Thus the minimum in (31) is not attained by its first argument. It follows that $N_{\rho(i+1)} \leq N_{\rho(i)}/2$, hence $N_{\rho(a \log r)} < C'$ for a sufficiently large constant a .

By the inner estimate, since the ball $B_{r-c \log r}$ is entirely occupied, we have

$$\begin{aligned} \sum_{\rho \geq r} M_\rho &\leq \omega_d r^d - \omega_d (r - c \log r)^d \\ &\leq cd\omega_d r^{d-1} \log r. \end{aligned}$$

Write $x_i = \rho(i+1) - \rho(i) - 1$; by (34) we have

$$\sum_{i=0}^{a \log r} x_i^d \leq cd\omega_d r^{d-1} \log r,$$

By Jensen's inequality, subject to this constraint, $\sum x_i$ is maximized when all x_i are equal, in which case $x_i \leq C'' r^{1-1/d}$ and

$$\rho(a \log r) = r + \sum x_i \leq r + C'' r^{1-1/d} \log r. \quad (35)$$

Since $N_{\rho(a \log r)} < C'$ we have $N_{\rho(a \log r)+C'} = 0$; that is, no particles reach the shell $\mathcal{S}_{\rho(a \log r)+C'}$. Taking $c' = C' + C''$, we obtain from (35)

$$A_n \subset B_{r(1+c'r^{-1/d} \log r)}. \quad \square$$

6 Concluding Remarks

A number of intriguing questions remain unanswered. Although we have shown that the asymptotic shape of the rotor-router model is a ball, the

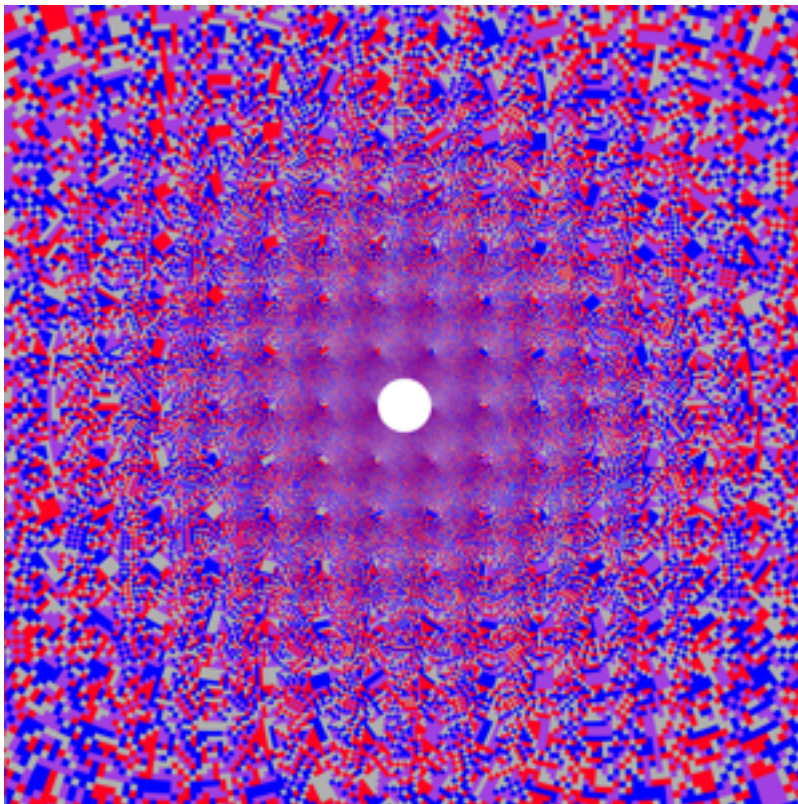


Figure 7: Image of the rotor-router aggregate of one million particles under the map $z \mapsto 1/z^2$. The colors represent the rotor directions. The white disc in the center is the image of the complement of the occupied region.

near perfect circularity found in Figure 1 remains a mystery. In particular, we do not know whether an analogue of Theorem 1.3 holds for the rotor-router model, with constant error in the radius as the number of particles grows.

Equally mysterious are the patterns in the rotor directions evident in Figure 1. The rotor directions can be viewed as values of the odometer function mod $2d$, but our control of the odometer is not fine enough to provide useful information about the patterns. If the rescaled occupied region $\sqrt{\pi/n} A_n$ is viewed as a subset of the complex plane, it appears that the monochromatic regions visible in Figure 1, in which all rotors point in the same direction, occur at points of the form $(1+2z)^{-1/2}$, where $z = a+bi$ is a Gaussian integer (i.e. $a, b \in \mathbb{Z}$). We do not even have a heuristic explanation

for this phenomenon. Figure 7 shows the image of $A_{1,000,000}$ under the map $z \mapsto 1/z^2$.

László Lovász (personal communication) has asked whether the occupied region A_n is simply connected, i.e. whether its complement is connected. While Theorem 1.1 shows that A_n cannot have any holes far from the boundary, we cannot answer his question at present.

A final question is whether our methods could be adapted to internal DLA to show that if $n = \omega_d r^d$, then with high probability $B_{r-c \log r} \subset I_n$, where I_n is the internal DLA cluster of n particles. The current best bound is due to Lawler [9], who proves that with high probability $B_{r-r^{1/3}(\log r)^2} \subset I_n$.

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