Mathematical Surveys and Monographs

Number 7
Part 1

The Algebraic Theory of Semigroups

A. H. Clifford
G. B. Preston

American Mathematical Society

http://dx.doi.org/10.1090/surv/007.1

The Algebraic Theory of Semigroups

This page intentionally left blank

Mathematical
Surveys
and
Monographs

Number 7
Part I

The Algebraic Theory of Semigroups

A. H. Clifford
G. B. Preston

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 20-XX.

International Standard Serial Number 0076-5376

International Standard Book Number 0-8218-0271-2

Library of Congress Catalog Card Number: 61-15686

Second Edition, 1964
Reprinted with corrections, 1977.

♾ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at URL: `http://www.ams.org/`

10 9 8 7 6 5 06 05 04 03 02 01

TABLE OF CONTENTS

This page intentionally left blank

PREFACE

So far as we know, the term "semigroup" first appeared in mathematical literature on page 8 of J.-A. de Séguier's book, *Élements de la Théorie des Groupes Abstraits* (Paris, 1904), and the first paper about semigroups was a brief one by L. E. Dickson in 1905. But the theory really began in 1928 with the publication of a paper of fundamental importance by A. K. Suschkewitsch. In current terminology, he showed that every finite semigroup contains a "kernel" (a simple ideal), and he completely determined the structure of finite simple semigroups. A brief account of this paper is given in Appendix A.

Unfortunately, this result of Suschkewitsch is not in a readily usable form. This defect was removed by D. Rees in 1940 with the introduction of the notion of a matrix over a group with zero, and, moreover, the domain of validity was extended to infinite simple semigroups containing primitive idempotents. The Rees Theorem is seen to be the analogue of Wedderburn's Theorem on simple algebras, and it has had a dominating influence on the later development of the theory of semigroups. Since 1940, the number of papers appearing each year has grown fairly steadily to a little more than thirty on the average.

It is in response to this developing interest that this book has been written. Only one book has so far been published which deals predominantly with the algebraic theory of semigroups, namely one by Suschkewitsch, *The Theory of Generalized Groups* (Kharkow, 1937); this is in Russian, and is now out of print. A chapter of R. H. Bruck's *A Survey of Binary Systems* (Ergebnisse der Math., Berlin, 1958) is devoted to semigroups. There is, of course, E. Hille's book, *Functional Analysis and Semi-groups* (Amer. Math. Soc. Colloq. Publ., 1948), and the 1957 revision thereof by Hille and R. S. Phillips; but this deals with the analytic theory of semigroups and its application to analysis. The time seems ripe for a systematic exposition of the algebraic theory. (Since the above words were written, there has appeared such an exposition, in Russian: *Semigroups*, by E. S. Lyapin, Moscow, 1960.)

The chief difficulty with such an exposition is that the literature is scattered over extremely diverse topics. We have met this situation by confining ourselves to a portion of the existing theory which has proved to be capable of a well-knit and coherent development. All of Volume 1 and the first half of Volume 2 center around the structure of semigroups of certain types (such as simple semigroups, inverse semigroups, unions of groups, semigroups with minimal conditions, etc.) and their representation by mappings or by matrices. The second half of Volume 2 treats the theory of congruences and the embedding of semigroups in groups, including a modest account of the active French school of semigroups (which they call "demi-groupes") founded in 1941 by P. Dubreil.

In order to keep our book within reasonable bounds, moreover, we have construed the term "algebraic" in a somewhat narrow sense: the semigroups under consideration are not endowed with any further structure. This has the effect of excluding not only topological semigroups, but ordered semigroups as well. Fortunately, a good account of lattice-ordered semigroups and groups is to be found in G. Birkhoff's *Lattice Theory* (Amer. Math. Soc. Colloq. Publ., 1940; revised 1948). It also excludes P. Lorenzen's generalization of multiplicative ideal theory (see, for example, §5 of W. Krull's *Idealtheorie*, Ergebnisse der Math., Berlin, 1935) to any commutative semigroup S with cancellation, in which S (or its quotient group) is endowed with a family of subsets called r-ideals, satisfying certain conditions analogous to those for closed sets in topology.

The book aims at being largely self-contained, but it is assumed that the reader has some familiarity with sets, mappings, groups, and lattices. The material on these topics in an introductory text such as Birkhoff and MacLane, *A Survey of Modern Algebra* (New York, Revised Edition, 1953) should suffice. Only in Chapter 5 will more preliminary knowledge be required, and even there the classical definitions and theorems on the matrix representations of algebras and groups are summarized.

We have included a number of exercises at the end of each section. These are intended to illuminate and supplement the text, and to call attention to papers not cited in the text. They can all be solved by applying the methods and results of the text, and often more simply than in the paper cited.

Each volume has a separate bibliography listing those papers referred to in that volume. No attempt has been made to list those papers on semigroups to which no reference has been made in the text or exercises. The combined bibliography contains about half of the papers which have appeared in the (strictly) algebraic theory of semigroups. (The bibliography in Lyapin's book appears to be complete.) Whenever possible, the reference to the review of each paper in the Mathematical Reviews has been given, (MR x, y) denoting page y of volume x. English translations of Russian titles are those given in the Mathematical Reviews.

The material in Volume 1 (more or less) was presented in a second-year graduate course at Tulane University during the academic year 1958–1959, and this volume has benefited greatly from the students' criticisms. The authors would also like to express their gratitude to Professors A. D. Wallace, D. D. Miller, and P. F. Conrad for many useful suggestions; and, above all, to Dr. W. D. Munn for his very valuable criticisms, especially of Chapter 5, and for his permission to draw on unpublished material from his dissertation (Cambridge University, 1955) for Sections 3.4 and 4.5. We deeply appreciate the thoughtful kindness of Professor Š. Schwarz and the Central Library of the Slovakian Academy of Sciences for sending us (unsolicited) a photostat print of Suschkewitsch's book. Our thanks go also to Mrs. Anna L. McGinity for typing all of Volume 1. Finally, the authors gratefully acknowledge partial support for this work from the National Science Foundation (U.S.A.).

A. H. C.
G. B. P.

July 28, 1960

THE TULANE UNIVERSITY OF LOUISIANA

THE ROYAL MILITARY COLLEGE OF SCIENCE

This page intentionally left blank

NOTATION USED IN VOLUME ONE

Square brackets are used for alternative readings and for reference to the bibliography.

Let A and B be sets.

$A \subset B$ (or $B \supset A$) means A is properly contained in B.

$A \subseteq B$ (or $B \supseteq A$) means $A \subset B$ or $A = B$.

$A \setminus B$ means the set of elements of A which are not in B.

$A \times B$ means the set of all ordered pairs (a, b) with a in A, b in B.

The signs $\cup$ and $\cap$ are reserved for union and intersection, respectively, of sets and relations. The signs $\vee$ and $\wedge$ will be used for join and meet in [semi]lattices.

$|A|$ means the cardinal number of the set A.

The sign $\circ$ is used for composition of relations (§1.4), but is usually omitted for composition of mappings.

$\square$ denotes the empty set, mapping, or relation.

ι [ι_A] denotes the identity mapping or relation [on the set A].

If ϕ is a mapping whose domain includes A, then $\phi|A$ means ϕ restricted to A.

$\{a_1, \cdots, a_n\}$ means the set whose members are $a_1, \cdots, a_n$. Braces are sometimes omitted on single elements, for example $A \cup b$ instead of $A \cup \{b\}$.

If $P(x)$ is a proposition for each element x of a set X, then the set of all x in X for which $P(x)$ is true is denoted by either $\{x \in X : P(x)\}$ or $\{x : P(x), x \in X\}$.

If $M(x)$ is a set for each x in a set X, then the union of all the sets $M(x)$ with x in X is denoted by either $\bigcup_{x \in X} M(x)$ or $\bigcup\{M(x) : x \in X\}$.

If $F(x)$ is a member of a set C for each x in a set X, then the subset of C consisting of all $F(x)$ with x in X is denoted by $\{F(x) : x \in X\}$. If $X = A \times B$, we may write $\{F(a, b) : a \in A, b \in B\}$ instead of $\{F(a, b) : (a, b) \in A \times B\}$.

If A is a subset of a semigroup S, then $\langle A \rangle$ denotes the subsemigroup of S generated by A. If S is a group, then the subgroup of S generated by A is $\langle A \cup A^{-1} \rangle$, where $A^{-1} = \{a^{-1} : a \in A\}$.

If A and B are subsets of a semigroup S, then AB means $\{ab : a \in A, b \in B\}$.

S^1 [S^0] means the semigroup $S \cup 1$ [$S \cup 0$] arising from a semigroup S by the adjunction of an identity element 1 [a zero element 0], unless S already has an identity [has a zero, and $|S| > 1$], in which case $S^1 = S$ [$S^0 = S$]. (§1.1)

$a|b$ means "a divides b", that is, $b \in aS^1$, where a and b are elements of a commutative semigroup S. (§4.3)

ρ_a [λ_a] denotes the inner right [left] translation $x \to xa$ [$x \to ax$] of a semigroup S, where a is a fixed element of S. (§1.3)

If ρ is an equivalence relation on a set X, and if $(a, b) \in \rho$, then we write $a \rho b$ and say that a and b are *ρ-equivalent*, and that they belong to the same *ρ-class*.

If ρ is a congruence relation on a semigroup S, then S/ρ denotes the factor semigroup of S modulo ρ, and $\rho^\natural$ denotes the natural mapping of S upon S/ρ. (§1.5) S/J denotes the Rees factor semigroup of S modulo an ideal J.

Let S be a semigroup, and let $a \in S$. (Following from §2.1)

$L(a)$ denotes the principal left ideal S^1a.

$R(a)$ denotes the principal right ideal aS^1.

$J(a)$ denotes the principal two-sided ideal S^1aS^1.

$\mathscr{L}$ means $\{(a, b) \in S \times S : L(a) = L(b)\}$.

$\mathscr{R}$ means $\{(a, b) \in S \times S : R(a) = R(b)\}$.

$\mathscr{J}$ means $\{(a, b) \in S \times S : J(a) = J(b)\}$.

$\mathscr{H}$ means $\mathscr{L} \cap \mathscr{R}$.

$\mathscr{D}$ means $\mathscr{L} \circ \mathscr{R}$ $(= \mathscr{R} \circ \mathscr{L})$.

L_a, R_a, J_a, H_a, D_a mean respectively the $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$, $\mathscr{H}$, $\mathscr{D}$-class containing a.

$I(a)$ means $J(a) \backslash J_a$. (It is empty or an ideal of S.)

$J(a)/I(a)$ is the principal factor of S corresponding to a. (§2.6)

$\mathscr{T}_X$ means the semigroup of all transformations of a set X. (§1.1)

$\mathscr{G}_X$ means the group of all permutations of a set X. (§1.1)

$\mathscr{I}_X$ means the symmetric inverse semigroup on a set X. (§1.9)

$\mathscr{B}_X$ means the semigroup of all binary relations on X. (§1.4)

$\mathscr{F}_X$ means the free semigroup on X. (§1.12)

$\mathscr{F}\mathscr{G}_X$ means the free group on X. (§1.12)

$\mathscr{C}$ means the bicyclic semigroup. (§1.12)

$\mathscr{M}^0(G; I, \Lambda; P)$ means the Rees $I \times \Lambda$ matrix semigroup over the group with zero G^0, with $\Lambda \times I$ sandwich matrix P.

$\mathscr{M}(G; I, \Lambda; P)$ means the Rees $I \times \Lambda$ matrix semigroup without zero over the group G, with $\Lambda \times I$ sandwich matrix P. (§3.1)

$\mathscr{L}\mathscr{T}(V)$ means the algebra of all linear transformations of a vector space V, or the multiplicative semigroup thereof. (§§2.2, 5.1)

$(\mathfrak{A})_n$ means the algebra of all $n \times n$ matrices over an algebra $\mathfrak{A}$. (§5.1)

$\Phi[S]$ means the algebra of a semigroup S over a field Φ. (§5.2)

$\cong$ means "isomorphic". (§1.3)

$\sim$ means "homomorphic", and sometimes "equivalent". (§1.3)

$\oplus$ is used for the direct sum of algebras, vector spaces, and representations. (§5.1)

The $n \times n$ identity matrix is denoted by:

I_n when it is over a field (§§5.2, 5.3),

U_n when it is over an algebra with identity element u (§5.1),

Δ_n when it is over a group with zero (§3.1).

Γ^m denotes the representation of $(\mathfrak{A})_m$ *associated* with the representation Γ of $\mathfrak{A}$. (§5.1)

M_L, M_R, M_J denote the minimal conditions on the set of principal left, right, two-sided ideals, respectively, of a semigroup. (A partially ordered set P is said *to satisfy the minimal condition* if each non-empty subset A of P contains at least one *minimal element*, i.e., an element x of A such that $y < x$ $(y \in P)$ implies $y \notin A$.) (§§5.3, 5.4)

This page intentionally left blank

http://dx.doi.org/10.1090/surv/007.1/01

CHAPTER 1

ELEMENTARY CONCEPTS

In this introductory chapter, we present a number of elementary concepts and propositions on semigroups, most of which will be indispensable for the remainder of the book. This chapter has also been written with the aim of giving the casual reader a broad survey of the subject which is also well-rounded and not too superficial. This explains why certain topics are treated here, which could well have been postponed to later chapters (especially in the later sections of this chapter).

1.1 Basic definitions

A *binary operation* on a set S is a mapping of $S \times S$ into S, where $S \times S$ is the set of all ordered pairs of elements of S. If the mapping is denoted by a dot $(\cdot)$, the image in S of the element (a, b) of $S \times S$ (a and b in S) will be denoted by $a \cdot b$. Frequently we shall omit the dot, writing ab for $a \cdot b$. Other symbols which we may use to denote binary operations are $+$, $\circ$, and $*$.

A *groupoid* is a system $S(\cdot)$ consisting of a non-empty set S together with a binary operation $(\cdot)$ on S. We shall usually write S instead of $S(\cdot)$ when there is no danger of ambiguity.

A *partial binary operation* on a set S is a mapping of a non-empty subset of $S \times S$ into S. By a *partial groupoid* we shall mean a system $S(\cdot)$ consisting of a non-empty set S together with a partial binary operation $(\cdot)$ on S.

A binary operation $(\cdot)$ on a set S is called *associative* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all a, b, c in S. A *semigroup* is a groupoid $S(\cdot)$ such that the operation $(\cdot)$ is associative. We frequently use the expression, "S is a semigroup with respect to $(\cdot)$," to mean that $(\cdot)$ is an associative binary operation on S. Frequently this is further abbreviated to "S is a semigroup".

The object of investigation of this book is semigroups and not groupoids or partial groupoids. The latter more general systems are, however, occasionally useful in the theory of semigroups, and so must be taken into account.

One exception we shall make to the above terminology is the concept of *Brandt groupoid* (§3.3), which is really a partial groupoid satisfying several rather stringent conditions.

By a *transformation* of a set X we shall mean a single-valued mapping of X into itself. Except in Chapter 5, we shall denote the image of an element x of X under a transformation or mapping α by $x\alpha$ (rather than αx or $\alpha(x)$). By the *product* (or *iterate* or *composition*) of two transformations α and β of X we mean the transformation $\alpha\beta$ defined by $x(\alpha\beta) = (x\alpha)\beta$ for all x in X (that is, α followed by β). The associative law $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds since, for every x in X,

$$x((\alpha\beta)\gamma) = (x(\alpha\beta))\gamma = ((x\alpha)\beta)\gamma = (x\alpha)(\beta\gamma) = x(\alpha(\beta\gamma)).$$

Hence the set $\mathscr{T}_X$ of all transformations of X is a semigroup with respect to iteration. We call $\mathscr{T}_X$ the *full transformation semigroup on* X.

A mapping α of a set X into a set Y will be said to be a mapping of X *onto* Y (or *upon* Y) if every element of Y is the image under α of at least one element of X. A mapping α of X into Y is said to be (1, 1) or *one-to-one* if distinct elements of X are mapped by α into distinct elements of Y. A one-to-one mapping of a set X upon itself will be called a *permutation* of X, even when X is infinite. The *symmetric group* $\mathscr{G}_X$ *on* X consists of all permutations of X under the operation of iteration.

If X is a finite set $\{x_1, \cdots, x_n\}$, and if $y_1, \cdots, y_n$ are elements of X, not necessarily distinct, then we shall use the classical notation

$$\alpha = \begin{pmatrix} x_1 x_2 \cdots x_n \\ y_1 y_2 \cdots y_n \end{pmatrix}$$

to mean that α is the transformation of X defined by

$$x_i\alpha = y_i \quad (i = 1, 2, \cdots, n).$$

If $a_1, a_2, \cdots, a_n$ *are elements of a semigroup* S, *and we set*

$$a_1a_2\cdots a_n = a_1(a_2(a_3\cdots(a_{n-1}a_n)\cdots)),$$

then every other possible meaningful expression A *obtained by inserting parentheses in the finite sequence* $a_1, a_2, \cdots, a_n$ *is equal to* $a_1a_2\cdots a_n$.

This is trivial for $n = 2$. Assume by way of induction that it is true for all expressions of length less than n. Now A, to be meaningful, must be a product BC of a meaningful expression B in $a_1, a_2, \cdots, a_r$ and a meaningful expression C in $a_{r+1}, \cdots, a_n$, for some r such that $1 \leqq r < n$. By hypothesis for induction, $B = a_1a_2\cdots a_r = a_1(a_2\cdots a_r)$. Hence, by associativity,

$$A = BC = (a_1(a_2\cdots a_r))C = a_1((a_2\cdots a_r)C).$$

But $(a_2\cdots a_r)C$ is a meaningful expression of length $n-1$ in $a_2, \cdots, a_n$, and so, by hypothesis for induction again, must equal $a_2a_3\cdots a_n$, yielding the desired conclusion.

For any positive integer n, the nth *power* a^n of an element a of a semigroup S is defined to be $a_1a_2\cdots a_n$ with $a_1 = a_2 = \cdots = a_n = a$. The first two "laws of exponents",

$$a^{m+n} = a^m a^n, \qquad (a^m)^n = a^{mn},$$

evidently hold for any a in S and any positive integers m and n.

A non-empty subset T of a groupoid S is called a *subgroupoid* of S if $a \in T$ and $b \in T$ imply $ab \in T$. The intersection of any set of subgroupoids of S is evidently either empty or a subgroupoid of S. If A is any non-empty subset of S, the intersection of all subgroupoids of S containing A (S itself being one such) is a subgroupoid $\langle A \rangle$ of S containing A and contained in every other subgroupoid of S containing A. We call $\langle A \rangle$ the *subgroupoid of S generated by A*. The subgroupoid $\langle A \rangle$ can also be described as the set of all elements of S expressible as finite products of elements of A. If $\langle A \rangle = S$, then A will be called a *set of generators* of S. If S is a semigroup, then any subgroupoid of S is also a semigroup, and we shall use the term *subsemigroup* rather than subgroupoid.

If S is a groupoid, the cardinal number $|S|$ of the set S is called the *order* of S. If $|S|$ is finite, we can exhibit the binary operation in S by means of its *Cayley multiplication table* as for finite groups, and this is often a useful picture even for infinite S. The Cayley table is a square matrix of elements of S, the rows and columns of which are labelled by the elements of S, such that the element in the a-row and b-column (a, b in S) is the product ab.

An element a of a groupoid S is said to be *left* [*right*] *cancellable* if, for any x and y in S, $ax = ay$ [$xa = ya$] implies $x = y$. A groupoid S is called *left* [*right*] *cancellative* if every element of S is left [right] cancellable. We say that S is *cancellative* (or is a *cancellation groupoid*) if it is both left and right cancellative.

Two elements a and b of a semigroup S are said to *commute* with each other if $ab = ba$. If this is the case, the third "law of exponents", $(ab)^n = a^n b^n$, holds. A semigroup S is called *commutative* if all of its elements commute with each other. An element of a semigroup S which commutes with every element of S is called a *central element* of S. The set of all central elements of S is either empty or a subsemigroup of S, and in the latter case is called the *center* of S. *If $a_1, a_2, \cdots, a_n$ are elements of a commutative semigroup S, and ϕ is any permutation of the set $\{1, 2, \cdots, n\}$, then*

$$a_{1\phi} a_{2\phi} \cdots a_{n\phi} = a_1 a_2 \cdots a_n.$$

This is easily proved by induction on n.

An element e of a groupoid S is called a *left* [*right*] *identity element of S* if $ea = a$ [$ae = a$] for all a in S. An element e of S called a *two-sided identity* (or simply *identity*) *element* of S if it is both a left and a right identity element of S. We note that if S contains a left identity e and a right identity f, then $e = f$; for $ef = f$ since e is a left identity, and $ef = e$ since f is a right identity. As a consequence of this, we see that *exactly one of the following statements must hold for a groupoid S*:

(1) *S has no left and no right identity element*;

(2) *S has one or more left identity elements, but no right identity element*;

(3) *S has one or more right identity elements, but no left identity element*;

(4) *S has a unique two-sided identity element, and no other right or left identity element.*

An element z of a groupoid S is called a *left* [*right*] *zero element* if $za = z$ [$az = z$] for every a in S. An element z of S is called a *zero element* of S if it is both a left and a right zero element of S. Any left zero of S must coincide with any right zero of S, and hence *the foregoing tetrachotomy holds if we replace the word "identity" by "zero"*.

Let X be any set, and define a binary operation $(\circ)$ in X by $x \circ y = y$ for every x, y in X. Associativity is quickly verified. We call $X(\circ)$ the *right zero semigroup on* X. Every element of $X(\circ)$ is both a right zero and a left identity. The *left zero semigroup* $X(*)$ on X is defined by $x * y = x$. In spite of their triviality, these semigroups arise naturally in a number of investigations, for example in Theorem 1.27 below.

A semigroup S with a zero element 0 will be called a *zero* or *null semigroup* if $ab = 0$ for all a, b in S.

Let S be any semigroup, and let 1 be a symbol not representing any element of S. Extend the given binary operation in S to one in $S \cup 1$ by defining $11 = 1$ and $1a = a1 = a$ for every a in S. It is quickly verified that $S \cup 1$ is a semigroup with identity element 1. We speak of the passage from S to $S \cup 1$ as "the adjunction of an identity element to S". Similarly one may adjoin a zero element 0 to S by defining $00 = 0a = a0 = 0$ for all a in S. *Throughout the book we shall adhere to the following notation*:

$$S^1 = \begin{cases} S & \textit{if } S \textit{ has an identity element,} \\ S \cup 1 & \textit{otherwise}; \end{cases}$$

$$S^0 = \begin{cases} S & \textit{if } S \textit{ has a zero element, and } |S| > 1, \\ S \cup 0 & \textit{otherwise.} \end{cases}$$

An element e of a groupoid S is called *idempotent* if $ee = e$. One-sided identity and zero elements are idempotent. The converse is in general false, but note Exercise 1 below, and Lemma 1.26. If every element of a semigroup S is idempotent, we shall say that S itself is *idempotent*, or that S is a *band*. Bands were introduced by Klein-Barmen [1940], who used the term "Schief".

H. Weber (*Lehrbuch der Algebra*, vol. 2 (1896), pp. 3–4) effectively defined a *group* as a semigroup G such that, for any given elements a and b of G, there exist unique elements x and y of G such that $ax = b$ and $ya = b$. E. V. Huntington (*Simplified definition of a group*, Bull. Amer. Math. Soc., 8 (1901–1902), 296–300) showed that it is not necessary to postulate the uniqueness of x and y, that this followed as a consequence.

An equivalent definition of *group* was given by L. E. Dickson (*Definitions of a group and a field by independent postulates*, Trans. Amer. Math. Soc., 6 (1905), 198–204), namely that a group is a semigroup G containing a left identity element e such that, for any a in G there exists y in G such that $ya = e$. Such an element y is called a *left inverse of* a *with respect to* e. Dickson showed that e is also a right identity of G (and so is the unique identity of G), and that every left inverse of a is also a right inverse, and is unique. The inverse of a will, as usual, be denoted by a^{-1}. The unique solutions of $ax = b$ and $ya = b$ are then $x = a^{-1}b$ and $y = ba^{-1}$.

The first published system of group axioms of this nature is that of J. Pierpont (*Galois theory of algebraic equations*. II, Annals of Math. 2 (1900–1901), 22–56; see p. 47); he postulated a two-sided identity element e and a two-sided inverse a' for each element a of the set: $aa' = a'a = e$.

By a *subgroup* of a semigroup S we mean a subsemigroup T of S which is also a group with respect to its binary operation. This is equivalent to saying that T is a subsemigroup of S such that if $a, b \in T$ then there exist x, y in T such that $ax = ya = b$. From this it is evident that *a subset T of a semigroup S is a subgroup of S if and only if $aT = Ta = T$ for every a in T.* (Example: if X is a set, $\mathscr{G}_X$ is a subgroup of $\mathscr{T}_X$.)

The identity element e of a subgroup T of a semigroup S is an idempotent element of S; it need not be the identity element of S.

If G is a group, then, by the convention given above, G^0 means $G \cup 0$, i.e., G with a zero element adjoined. By a *group with zero* we mean G^0 where G is a group. For example, let $R(\cdot, +)$ be a ring. Then $R(\cdot)$ is a semigroup which we call the multiplicative semigroup of $R(\cdot, +)$. Evidently $R(\cdot, +)$ is a division ring if and only if $R(\cdot)$ is a group with zero.

By the *dual* of a proposition or concept we mean the proposition or concept obtained by replacing every product ab in the statement thereof by ba. Thus "left identity" and "right identity" are dual concepts. The dual of Dickson's definition of a group is that a group is a semigroup G containing a right identity e such that every element of G possesses a right inverse with respect to e. The Weber-Huntington definition is self-dual.

Let $d(A)$ denote the dual of a proposition A. If a proposition has the form "A implies B" then its dual has the form "$d(A)$ implies $d(B)$". Clearly, if one is true then so is the other. *Many non-self-dual theorems will be established in the book, and their duals will be taken for granted without comment.*

If A and B are subsets of a groupoid S, then by the *set product* AB of A and B we shall mean the set of all elements ab of S with a in A and b in B. If $A = \{a\}$ then we also write AB as aB, and similarly if $B = \{b\}$. Thus

$$AB = \bigcup\{Ab : b \in B\} = \bigcup\{aB : a \in A\}.$$

By a *left* [*right*] *ideal* of a groupoid S we mean a non-empty subset A of S such that $SA \subseteq A$ [$AS \subseteq A$]. By *two-sided ideal*, or simply *ideal*, we mean a subset of S which is both a left and a right ideal of S. A groupoid S is called *left* [*right*] *simple* if S itself is the only left [right] ideal of S. Likewise S is called *simple* if it contains no proper (two-sided) ideal.

If A is any non-empty subset of a groupoid S, the intersection of all left ideals of S containing A (S itself being one such) is a left ideal of S containing A and contained in every other such left ideal of S. We call it the *left ideal* of S *generated by* A. If S is a semigroup, the left ideal of S generated by A is simply $A \cup SA = S^1A$. With the analogous definitions, we see that the right ideal of S generated by A is $A \cup AS = AS^1$, and that the (two-sided) ideal of S generated by A is $A \cup SA \cup AS \cup SAS = S^1AS^1$. If, in particular,

A consists of a single element a, then we call $L(a) = S^1a$, $R(a) = aS^1$, and $J(a) = S^1aS^1$ the *principal* left, right, and two-sided ideal of S, respectively, generated by a.

A semigroup S is right simple if and only if $aS = S$ for every a in S. For if $aS \neq S$, then aS is a proper right ideal of S; and if R is a proper right ideal of S, and $a \in R$, then $aS \subseteq R \subset S$, so $aS \neq S$. But to say that $aS = S$ for every a in S is equivalent to saying that, for every a and b in S, there exists x in S such that $ax = b$. Combining this with the dual proposition, and recalling the Weber-Huntington axioms for a group, we see that *a semigroup is a group if and only if it is both left and right simple.*

EXERCISES FOR §1.1

1. (a) If e is an idempotent element of a left cancellative semigroup S, then e is a left identity element of S.

(b) A cancellative semigroup can contain at most one idempotent element, namely an identity element.

2. (a) If S is a cancellative semigroup, so is S^1.

(b) Let S be a left zero semigroup with $|S| > 1$. Then S is right cancellative, but S^1 is not.

3. Let a be an element of a semigroup S, and let $A = \{x : axa = a, x \in S\}$. If $A \neq \square$, then Aa $[aA]$ is a left [right] zero subsemigroup of S. (Bruck [1958], pp. 25–26.)

4. A left zero semigroup S is left simple, and each element of S forms by itself a right ideal of S.

5. Let S be a semigroup such that if $ab = cd$ (a, b, c, d in S) then either $a = c$ or $b = d$. Then S is either a left zero semigroup or a right zero semigroup. (Thierrin [1952*b*].)

6. If S is a semigroup having a right zero element, then the set K of right zero elements of S is a right zero subsemigroup of S, and is a two-sided ideal of S contained in every two-sided ideal of S.

7. The right zero elements of $\mathscr{T}_X$ are just the "constant" transformations, mapping every element of X onto a single fixed element of X. There are no left zeros in $\mathscr{T}_X$ if $|X| > 1$.

8. Let K be the set of right zero elements of a semigroup S, and assume $K \neq \square$. Then $S \cong \mathscr{T}_K$ if and only if (i) $xa = xb$ (a, b in S) for all x in K implies $a = b$, and (ii) if α is any transformation of K, there exists a in S such that $xa = x\alpha$ for all x in K. (Malcev [1952].)

9. An element α of $\mathscr{T}_X$ is idempotent if and only if it is the identical mapping when restricted to $X\alpha$.

10. Let X be a finite set of cardinal n. Then $\mathscr{T}_X$ contains the symmetric group $\mathscr{G}_X$ of degree n. If $\alpha \in \mathscr{T}_X$, define the *rank* r of α to be $|X\alpha|$, and the *defect* of α to be $n - r$.

(a) If β is an element of $\mathscr{T}_X$ of rank $r < n$, there exist elements γ and δ

of $\mathscr{T}_X$ such that γ has rank $r + 1$, δ has rank $n - 1$, and $\beta = \gamma\delta$. (We can choose δ to be idempotent, and γ so as to differ from β at only one point of X.) By induction, every element of $\mathscr{T}_X$ of defect k ($1 \leqq k \leqq n - 1$) is expressible as the product of an element of $\mathscr{G}_X$ and k (idempotent) elements of defect 1.

(b) If α is an element of $\mathscr{T}_X$ of defect 1, then every other element of $\mathscr{T}_X$ of defect 1 can be expressed in the form $\lambda\alpha\mu$ with λ and μ in $\mathscr{G}_X$.

(c) If α is an element of $\mathscr{T}_X$ of defect 1, then $\langle \mathscr{G}_X, \alpha \rangle = \mathscr{T}_X$. (Vorobev [1953*b*].)

1.2 Light's associativity test

To test a finite groupoid $S(\cdot)$ for associativity, when the operation $(\cdot)$ is given by a Cayley multiplication table, is usually quite a tedious business. The following procedure was suggested to one of the authors by Dr. F. W. Light in 1949.

This procedure is to be carried out for each element a of the groupoid S. However, we shall show below that it suffices to carry it out for each element a of a set of generators of S.

Consider the two binary operations $(*)$ and $(\circ)$ defined in S as follows:

$$x * y = x \cdot (a \cdot y), \quad x \circ y = (x \cdot a) \cdot y.$$

Associativity holds in $S(\cdot)$ if and only if, for each fixed element a of S, these two binary operations coincide. The idea is essentially to construct the Cayley tables for $(*)$ and $(\circ)$ and then see if they are the same.

The $(*)$-table is obtained from the original $(\cdot)$-table by replacing, for each y in S, the y column by the $a \cdot y$ column. Similarly, to make up the $(\circ)$-table we simply copy down in the x row the $x \cdot a$ row of the $(\cdot)$-table. However, we do not need to write out the $(\circ)$-table, since we can check directly whether the x row of the $(*)$-table coincides with the $x \cdot a$ row of the $(\cdot)$-table.

For convenience in performing the test, we replace the top index line of the $(*)$-table by the a row of the $(\cdot)$-table, and the left-hand index column by the a column of the $(\cdot)$-table. For each entry $a \cdot y$ in the a row of the $(\cdot)$-table tells us what column of the $(\cdot)$-table to copy down as the y column of the $(*)$-table, and each entry $x \cdot a$ in the a column of the $(\cdot)$-table tells us which row of the $(\cdot)$-table should be compared with x row of the $(*)$-table. For example, let $S(\cdot)$ be defined by the table:

	a	b	c	d	e
a	a	a	a	d	d
b	a	b	c	d	d
c	a	c	b	d	d
d	d	d	d	a	a
e	d	e	e	a	a

The set $\{c, e\}$ generates S, since $a = e \cdot e$, $b = c \cdot c$, and $d = c \cdot e$. The $(*)$-tables (with index rows and columns modified as described above) for the elements c and e are as follows:

c	a	c	b	d	d
a	a	a	a	d	d
c	a	c	b	d	d
b	a	b	c	d	d
d	d	d	d	a	a
e	d	e	e	a	a

e	d	e	e	a	a
d	d	d	d	a	a
d	d	d	d	a	a
d	d	d	d	a	a
a	a	a	a	d	d
a	a	a	a	d	d

Thus, to form the c-table, copy the c row $(a\ c\ b\ d\ d)$ from the $(\cdot)$-table into the upper index line, and similarly the c column into the left-hand index column. Now copy the columns of the $(\cdot)$-table in the order specified by the upper index line, i.e., the a column, the c column, etc. We now verify that the rows of the c-table thus formed are just those of the $(\cdot)$-table labelled by the left-hand index column. One may prefer to copy the rows of the $(\cdot)$-table as specified by the left-hand index column, and then check that the columns are correctly labelled. Since this is found to check for both the c-table and the e-table, we conclude that $S(\cdot)$ is associative.

That it suffices to carry out Light's procedure only for a set of generators of S is an immediate consequence of the fact that *the set of all elements a of a groupoid S that associate with all elements of S, in the sense that $x(ay) = (xa)y$ for all x, y in S, is a subsemigroup of S.* For if we assume that a and b are elements of S such that $x(ay) = (xa)y$ and $x(by) = (xb)y$ for all x, y in S, then

$$\begin{aligned} x((ab)y) &= x(a(by)) = (xa)(by) \\ &= ((xa)b)y = (x(ab))y. \end{aligned}$$

Thus if a and b associate with all elements of S, so does ab.

EXERCISES FOR §1.2

1. Check for associativity:

	e	f	g	a
e	e	e	e	e
f	f	f	f	f
g	g	g	g	g
a	e	e	f	e

2. Check for associativity:

	e	f	g	a	0
e	e	a	e	a	0
f	0	f	g	0	0
g	g	f	g	f	0
a	0	a	e	0	0
0	0	0	0	0	0

3. The set of all elements a of a groupoid S such that $a(xy) = (ax)y$ for all x, y in S is a subsemigroup of S.

1.3 Translations and the regular representation

Let S and S' be groupoids. A mapping ϕ of S into S' is called a *homomorphism* if $(ab)\phi = (a\phi)(b\phi)$ for all a, b in S. The range $S\phi$ of ϕ, i.e., the set of all elements $a\phi$ of S' with a in S, is a subgroupoid of S'. We say that S is homomorphic with $S\phi$, and that $S\phi$ is a homomorphic image of S; we write $S \sim S\phi$. *If S is a semigroup, so is $S\phi$.* A one-to-one homomorphism ϕ of S into S' is called an *isomorphism* of S into S'. The groupoids S and $S\phi$ are then said to be isomorphic, and we write $S \cong S\phi$. A homomorphism of S into itself is called an *endomorphism*, and an isomorphism of S upon itself is called an *automorphism.*

A mapping ϕ of a groupoid S into a groupoid S' is called an *anti-homomorphism* if $(ab)\phi = (b\phi)(a\phi)$ for all a, b in S. The terms *anti-isomorphism, anti-endomorphism,* and *anti-automorphism* are defined analogously. A transformation $x \to x^*$ of a groupoid S is called an *involutorial anti-automorphism* if $(x^*)^* = x$ and $(xy)^* = y^*x^*$.

Let S be a groupoid, X any set, and $\mathscr{T}_X$ the full transformation semigroup on X. An [anti-] homomorphism ϕ of S into $\mathscr{T}_X$ is called an *[anti-] representation* of S by transformations of X. If T is a subgroupoid of S, then $\phi|T$ (ϕ restricted to T) is evidently an [anti-] representation of T, said to be that *induced* in T by ϕ. An [anti-] representation ϕ of S is called *faithful* or *true* if it is one-to-one.

With each element a of a groupoid S we associate a transformation $\rho_a[\lambda_a]$ of S defined by $x\rho_a = xa$ $[x\lambda_a = ax]$ for all x in S. We call $\rho_a[\lambda_a]$ the *inner right [left] translation* of S corresponding to the element a of S. The transformations ρ_a and λ_a are, of course, elements of $\mathscr{T}_S$.

From $x\rho_{ab} = x(ab)$ and $x\rho_a\rho_b = (xa)b$, we see that S is associative if and only if $\rho_{ab} = \rho_a\rho_b$, i.e., if and only if the mapping $a \to \rho_a$ is a representation of S by transformations of S. Similarly, S is associative if and only if $\lambda_{ab} = \lambda_b\lambda_a$, i.e., if and only if the mapping $a \to \lambda_a$ is an anti-representation of S. If S is a semigroup, the mapping $a \to \rho_a$ $[a \to \lambda_a]$ will be called the *regular [anti-] representation* of S. By the *extended regular [anti-] representation* of S we shall mean the representation induced in S by the regular [anti-] representation of S^1. The extended regular [anti-] representation of S is always faithful.

A semigroup S is called *left [right] reductive* if $xa = xb$ $[ax = bx]$ for all x in S implies $a = b$ (a, b in S). The regular [anti-] representation of S is faithful if and only if S is left [right] reductive. In particular, it is faithful if S has a left [right] identity element, or if S is left [right] cancellative. We note also that the regular [anti-] representation of S is by one-to-one transformations of S if and only if S is right [left] cancellative.

If a semigroup S has no idempotent element except possibly an identity element, we shall say briefly that "S has no idempotent $\neq 1$".

LEMMA 1.0. *A semigroup S can be faithfully represented as a semigroup of one-to-one mappings of a set into itself if and only if it is right cancellative and has no idempotent $\neq 1$. If this is the case, then:*

(i) *if a and b are elements of S such that $ab = b$, then $a = 1$ (and $S = S^1$);*

(ii) *S^1 is a right cancellative semigroup having no idempotent $\neq 1$;*

(iii) *the extended regular representation of S is a faithful representation of S as a semigroup of one-to-one mappings of S^1 into itself.*

PROOF. Let S be a semigroup of one-to-one mappings of a set X into itself. Let α, β, γ be elements of S such that $\alpha\gamma = \beta\gamma$. Then $x\alpha\gamma = x\beta\gamma$ for every x in X. Since γ is one-to-one, we infer that $x\alpha = x\beta$ for every x in X, whence $\alpha = \beta$. Thus S is right cancellative.

If ϵ is an idempotent element of S, then $x\epsilon\epsilon = x\epsilon$ for every x in X. Since ϵ is one-to-one, $x\epsilon = x$ for every x in X. In other words, ϵ is the identical mapping of X, and so is the identity element of S.

Conversely, assume that S is a right cancellative semigroup having no idempotent $\neq 1$. We shall prove (i), (ii), and (iii); the latter then implies the "if" part of the first assertion of the lemma.

To show (i), let a and b be elements of S such that $ab = b$. Then $a^2b = ab$, and $a^2 = a$ by right cancellation. Since S has no idempotent $\neq 1$, it follows that a must be the identity element of S, and hence that $S = S^1$.

(ii) is trivial if $S = S^1$, and so we may assume $S \neq S^1$. Assume, by way of contradiction, that a, b, c are elements of S^1 such that $ac = bc$, but $a \neq b$. Then $c \neq 1$, so that $c \in S$. Since S is right cancellative, a and b cannot both be in S. Hence we can assume $a \in S$ and $b = 1$. But then $ac = c$ with $a \neq 1$, contrary to (i). Hence S^1 is right cancellative, and clearly it cannot contain an idempotent $\neq 1$.

To show (iii), let ϕ be the extended regular representation $a \rightarrow \rho_a$ of S ($a \in S$), where ρ_a is the inner right translation $x \rightarrow x\rho_a = xa$ of S^1 ($x \in S^1$). Then, as noted above, ϕ is faithful. If x and y are elements of S^1 such that $x\rho_a = y\rho_a$, that is, $xa = ya$, then $x = y$ by (ii). Thus each element ρ_a of $S\phi$ is a one-to-one mapping of S^1 into itself.

The general theory of representations of a semigroup by transformations of a set will be taken up in Chapter 11. We go no further into this theory at present, but devote the remainder of this section to translations of a semigroup. This will be used in the extension theory (§4.4).

A transformation ρ of a semigroup S is called a *right translation* of S if $x(y\rho) = (xy)\rho$ for all x, y in S. A transformation λ of S is called a *left translation* of S if $(x\lambda)y = (xy)\lambda$ for all x, y in S. A left translation λ and a right translation ρ are said to be *linked* if $x(y\lambda) = (x\rho)y$ for all x, y in S. For example, if $a \in S$, then the inner translations λ_a and ρ_a are linked.

The set of right [left] translations of a semigroup S is a subsemigroup $\mathrm{P}[\Lambda]$ of $\mathscr{T}_S$. For if $\lambda_1, \lambda_2 \in \Lambda$ and $x, y \in S$, then

$$(x(\lambda_1\lambda_2))y = ((x\lambda_1)\lambda_2)y = ((x\lambda_1)y)\lambda_2 = ((xy)\lambda_1)\lambda_2 = (xy)(\lambda_1\lambda_2),$$

whence $\lambda_1\lambda_2 \in \Lambda$. The proof that $\rho_1, \rho_2 \in \mathrm{P}$ implies $\rho_1\rho_2 \in \mathrm{P}$ is similar. The set of inner right [left] translations of S is a subsemigroup P_0 of P [Λ_0 of Λ]. The mapping $a \to \rho_a$ [$a \to \lambda_a$] is an [anti-] homomorphism of S upon P_0 [Λ_0], in fact just the regular [anti-] representation of S.

Lemma 1.1. *Let λ and ρ be left and right translations, respectively, of a semigroup S. Let $a \in S$. Then*

$$\lambda_a\lambda = \lambda_{a\lambda}, \qquad \rho_a\rho = \rho_{a\rho}.$$

If λ and ρ are linked, then

$$\lambda\lambda_a = \lambda_{a\rho}, \qquad \rho\rho_a = \rho_{a\lambda}.$$

Proof. For any x in S, we have

$$x(\lambda_a\lambda) = (x\lambda_a)\lambda = (ax)\lambda = (a\lambda)x = x\lambda_{a\lambda},$$

$$x(\rho_a\rho) = (x\rho_a)\rho = (xa)\rho = x(a\rho) = x\rho_{a\rho}.$$

Assume now that λ and ρ are linked. Then, for any x in S, we have

$$x(\lambda\lambda_a) = (x\lambda)\lambda_a = a(x\lambda) = (a\rho)x = x\lambda_{a\rho},$$

$$x(\rho\rho_a) = (x\rho)\rho_a = (x\rho)a = x(a\lambda) = x\rho_{a\lambda}.$$

We define the *translational hull* $\bar{S}$ of a semigroup S to be the set of all pairs (λ, ρ) of linked left and right translations λ and ρ of S. If (λ_1, ρ_1) and (λ_2, ρ_2) are in $\bar{S}$, then so is $(\lambda_2\lambda_1, \rho_1\rho_2)$, since, for any x and y in S, we have

$$\begin{aligned} x(y(\lambda_2\lambda_1)) &= x((y\lambda_2)\lambda_1) = (x\rho_1)(y\lambda_2) \\ &= ((x\rho_1)\rho_2)y = (x(\rho_1\rho_2))y. \end{aligned}$$

We may therefore define a binary operation in $\bar{S}$ by

$$(\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_2\lambda_1, \rho_1\rho_2).$$

Associativity is evident, and so $\bar{S}$ is a semigroup.

Let $\bar{S}_0$ be the subset of $\bar{S}$ consisting of all pairs (λ_a, ρ_a) with a in S. We see that $\bar{S}_0 \subseteq \bar{S}$ since λ_a and ρ_a are linked. For any a and b in S, we have

$$(\lambda_a, \rho_a)(\lambda_b, \rho_b) = (\lambda_b\lambda_a, \rho_a\rho_b) = (\lambda_{ab}, \rho_{ab}).$$

Hence $\bar{S}_0$ is a subsemigroup of $\bar{S}$, and the mapping $a \to (\lambda_a, \rho_a)$ is a homomorphism of S upon $\bar{S}_0$. It is an isomorphism if and only if $\lambda_a = \lambda_b$ and $\rho_a = \rho_b$ together imply $a = b$, in other words $ax = bx$ and $xa = xb$ for all x in S imply $a = b$. A semigroup S having this property will be called *weakly reductive*.

Lemma 1.2. *Let S be a weakly reductive semigroup, and let us identify S*

with the inner part $\bar{S}_0$ of the translational hull $\bar{S}$ of S. Then S is an ideal of $\bar{S}$, and, if $a \in S$ and $(\lambda, \rho) \in \bar{S}$, we have

$$(\lambda, \rho)a = a\lambda, \qquad a(\lambda, \rho) = a\rho.$$

PROOF. By Lemma 1.1,

$$(\lambda, \rho)(\lambda_a, \rho_a) = (\lambda_a \lambda, \rho\rho_a) = (\lambda_{a\lambda}, \rho_{a\lambda}),$$

$$(\lambda_a, \rho_a)(\lambda, \rho) = (\lambda\lambda_a, \rho_a\rho) = (\lambda_{a\rho}, \rho_{a\rho}).$$

If we now identify the element x of S with the element (λ_x, ρ_x) of $\bar{S}_0$, which is permissible since S is weakly reductive and hence $x \to (\lambda_x, \rho_x)$ is an isomorphism of S upon $\bar{S}_0$, we obtain the desired conclusion.

The following considerations show that the translational hull of a semigroup bears some analogy to the holomorph of a group.

If S is an ideal of a semigroup T, then every inner right [left] translation of T induces a right [left] translation of S. For if $t \in T$ and $x \in S$, then $x\rho_t = xt \in S$ since S is an ideal of T, and evidently

$$(xy)\rho_t = (xy)t = x(yt) = x(y\rho_t).$$

Similarly, $x\lambda_t \in S$ and $\lambda_t|S$ is a left translation of S. *What are necessary and sufficient conditions on a semigroup S in order that we can embed S in a semigroup T such that* (1) *S is an ideal of T, and* (2) *every left and every right translation of S is induced by some inner translation of T?* The following theorem answers this question for weakly reductive semigroups; the general case was solved by T. Tamura and N. Graham [1964] (see Vol. II, p. 339).

THEOREM 1.3. *A weakly reductive semigroup S can be embedded in a semigroup T with the foregoing properties* (1) *and* (2) *if and only if* (3) *each left translation of S is linked with some right translation of S, and vice versa.*

PROOF. Let S be a semigroup which can be embedded in a semigroup T with properties (1) and (2). Let λ be any left translation of S. By (2) there exists t in T such that $\lambda = \lambda_t|S$. Then $\rho_t|S$ is a right translation of S linked with λ. Similarly, every right translation of S is linked with some left translation of S.

Conversely, let S be a weakly reductive semigroup satisfying (3). Let T be the translational hull $\bar{S}$ of S. Then S is an ideal of T by Lemma 1.2. Let λ be any left translation of S. By (3), there exists a right translation ρ of S linked with λ. Then $t = (\lambda, \rho) \in T$, and $\lambda_t|S = \lambda$ by Lemma 1.2. The proof of the dual statement in (2) is similar.

EXERCISES FOR §1.3

1. Let ϕ be a homomorphism of a groupoid S into a groupoid T. If I is a left [right] ideal of S, then $I\phi$ is a left [right] ideal of $S\phi$. Conversely, if J is a left [right] ideal of T, then $J\phi^{-1}$ if not empty is a left [right] ideal of S.

2. (a) A semigroup S is right cancellative and has no idempotent $\neq 1$ if and only if S^1 is right cancellative.

(b) A right cancellative, left reductive semigroup has no idempotent $\neq 1$.

3. A groupoid S is a semigroup if and only if every inner right translation of S is a right translation of S.

4. If a semigroup S contains a right identity element, then every right translation of S is inner.

5. A transformation of a groupoid S is a left translation of S if and only if it commutes with every inner right translation of S.

6. If S is a semigroup such that $S^2 = S$, then every right translation of S commutes with every left translation of S. (Clifford [1950].)

7. A semigroup S is a right zero semigroup if and only if it has either of the following properties:

(a) every transformation of S is a right translation of S;

(b) the only left translation of S is the identity mapping. (Posey [1949], (a); Tamura [1955*b*], (a) and (b).)

8. The translational hull $\bar{S}$ of a right zero semigroup S is isomorphic with the full transformation semigroup $\mathscr{T}_S$ on S. Identifying $\bar{S}$ with $\mathscr{T}_S$ and $\bar{S}_0$ with S, the set of right zeros of $\bar{S}$ is S.

9. Let S be the semigroup $\{e, f, g, a\}$ defined by the Cayley table in Exercise 1 of §1.2. It is weakly (in fact, right) reductive. The transformation

$$\begin{pmatrix} e & f & g & a \\ g & g & e & g \end{pmatrix}$$

is a left translation of S which is not linked with any right translation of S.

1.4 The semigroup of relations on a set

By a (*binary*) *relation* on a set X we mean a subset ρ of the direct product $X \times X$ of X with itself. If $(a, b) \in \rho$, where a and b are elements of X, we may also write $a \, \rho \, b$, and say that "a bears the relation ρ to b".

If ρ and σ are relations on X, their *composition* $\rho \circ \sigma$ is defined as follows: $(a, b) \in \rho \circ \sigma$ if there exists x in X such that $(a, x) \in \rho$ and $(x, b) \in \sigma$. The binary operation $(\circ)$ is associative. For if ρ, σ, and τ are relations on X, the two assertions $(a, b) \in (\rho \circ \sigma) \circ \tau$ and $(a, b) \in \rho \circ (\sigma \circ \tau)$ are each equivalent to asserting the existence of x and y in X such that $(a, x) \in \rho$, $(x, y) \in \sigma$, and $(y, b) \in \tau$. Hence the set $\mathscr{B}_X$ of all binary relations on X is a semigroup with respect to $(\circ)$.

We shall denote by ι the *equality relation* (or "diagonal" of $X \times X$), defined by $(a, b) \in \iota$ if and only if $a = b$. Clearly ι is the identity element of $\mathscr{B}_X$. We shall denote by ω the *universal relation*, defined by $(a, b) \in \omega$ for all a, b in X, that is, $\omega = X \times X$. The *empty relation* $\square$ is the zero element of $\mathscr{B}_X$.

The *converse* ρ^{-1} of a relation ρ is defined by $(a, b) \in \rho^{-1}$ if and only if $(b, a) \in \rho$. We note that

$$(\rho^{-1})^{-1} = \rho, \qquad (\rho \circ \sigma)^{-1} = \sigma^{-1} \circ \rho^{-1}.$$

In other words, the mapping $\rho \to \rho^{-1}$ is an involutorial anti-automorphism of the semigroup $\mathscr{B}_X$.

The relation $\rho \subseteq \sigma$ means that ρ is a subset of σ. It is equivalent to the implication: $a \rho b$ implies $a \sigma b$. Since $\mathscr{B}_X$ consists of all subsets of the set $X \times X$, we can perform the Boolean operations of union, intersection, and complement in $\mathscr{B}_X$. Some of the many formulae holding among these and product and converse will be given in the exercises.

A relation ρ is said to be (1) *reflexive* if $\iota \subseteq \rho$, (2) *symmetric* if $\rho \subseteq \rho^{-1}$ (and hence $\rho = \rho^{-1}$), and (3) *transitive* if $\rho \circ \rho \subseteq \rho$. A relation ρ on a set X is called an *equivalence relation* on X if it is reflexive, symmetric, and transitive. An equivalence relation on X is an idempotent element of $\mathscr{B}_X$.

If ρ is any relation on a set X, and $a \in X$, we set $\rho a = \{x \in X : x \rho a\}$ and $a\rho = \{y \in X : a \rho y\}$. If ρ is an equivalence relation, then (1) $a \in a\rho$ for every a in X, and (2) if $a\rho \cap b\rho \neq \square$, then $a\rho = b\rho$. Thus the family of sets $a\rho$ with a in X is a *partition* of X, i.e., they are mutually disjoint and their union is X; we denote this family by X/ρ. We call $a\rho$ *the equivalence class of* X *modulo* ρ *containing* a. Conversely, any partition $\mathscr{P}$ of X determines an equivalence relation ρ such that $\mathscr{P} = X/\rho$, namely $a \rho b$ if and only if a and b belong to the same member of $\mathscr{P}$. We call the mapping $a \to a\rho$ the *natural* or *canonical* mapping of X upon X/ρ, and denote it by $\rho^\natural$. We observe that $a\rho = a\rho^\natural$ for every a in X, but it would be confusing to use the same symbol ρ for the equivalence relation on X and for the natural mapping of X upon X/ρ.

If ρ is any relation on X, we define the *transitive closure* ρ^t of ρ by

$$\rho^t = \bigcup_{n=1}^{\infty} \rho^n = \rho \cup (\rho \circ \rho) \cup (\rho \circ \rho \circ \rho) \cup \cdots.$$

Clearly ρ^t is transitive, and is contained in every transitive relation on X containing ρ.

If ρ_0 is any relation on X, the relation $\rho_1 = \rho_0 \cup \rho_0^{-1} \cup \iota$ is the smallest reflexive and symmetric relation on X containing ρ_0. The transitive closure $\rho = \rho_1^t$ of ρ_1 is an equivalence relation on X, and is contained in every equivalence relation on X containing ρ_0. We call ρ *the equivalence relation on* X *generated by* ρ_0.

The intersection of any set of equivalence relations is an equivalence relation, but this is not true for the union of even two equivalence relations. By the *join* $\rho \vee \sigma$ of two equivalence relations ρ and σ we mean the equivalence relation generated by $\rho \cup \sigma$. In general, all we can say is that $\rho \vee \sigma$ is the transitive closure of $\rho \cup \sigma$.

Lemma 1.4. *If ρ and σ are equivalence relations on a set X, and if $\rho \circ \sigma = \sigma \circ \rho$, then $\rho \circ \sigma$ is an equivalence relation on X, and is the join $\rho \vee \sigma$ of ρ and σ.*

Proof. Since $\rho \circ \sigma$ is clearly contained in $\rho \vee \sigma$, all we need to show is that $\rho \circ \sigma$ is an equivalence relation. From $\iota \subseteq \rho \subseteq \rho \circ \sigma$, we see that $\rho \circ \sigma$ is reflexive. From

$$(\rho \circ \sigma)^{-1} = \sigma^{-1} \circ \rho^{-1} = \sigma \circ \rho = \rho \circ \sigma,$$

it follows that $\rho \circ \sigma$ is symmetric. Finally, for the transitivity, we have

$$(\rho \circ \sigma) \circ (\rho \circ \sigma) = \rho \circ \sigma \circ \rho \circ \sigma = \rho \circ \rho \circ \sigma \circ \sigma = \rho \circ \sigma.$$

If ρ is a relation on X such that, for each x in X, $|x\rho| = 1$, then we may identify the one-element set $x\rho$ with its member, and regard ρ as the transformation $x \rightarrow x\rho$ of X. If σ is another such relation on X, then so is $\rho \circ \sigma$, and $\rho \circ \sigma$ is the same as the iterate of ρ and σ regarded as transformations. Dually, if $|\rho x| = 1$, then we may regard $x \rightarrow \rho x$ as a transformation of X. In this case $\rho \circ \sigma$ is the iterate of σ and ρ in that order. Thus $\mathscr{B}_X$ contains $\mathscr{T}_X$ as a subsemigroup, and also a subsemigroup $\mathscr{T}_X{}^*$ anti-isomorphic with $\mathscr{T}_X$.

Let ϕ be a mapping of one set X into another set X'. Then ϕ may be regarded as a relation on $X \cup X'$. For each x' in X', $x'\phi^{-1} = \{x \in X : x\phi = x'\}$. The relation $\phi \circ \phi^{-1}$ is contained in $X \times X$, hence can be regarded as a relation on X, and we see that $(x, y) \in \phi \circ \phi^{-1}$ (x, y in X) if and only if $x\phi = y\phi$. From this it is clear that $\phi \circ \phi^{-1}$ is an equivalence relation, and ϕ induces in an obvious way a one-to-one mapping of $X/\phi \circ \phi^{-1}$ upon $X\phi$. We call $\phi \circ \phi^{-1}$ *the equivalence on X naturally induced by ϕ.*

Exercises for §1.4

1. Let $\mathscr{B}_X$ be the semigroup of relations on a set X. Let Ω be an index class with typical element α. Let ρ, ρ_α, σ, τ denote arbitrary elements of $\mathscr{B}_X$. Then the following relations hold in $\mathscr{B}_X$.

(a) $\rho \subseteq \sigma$ implies $\rho \circ \tau \subseteq \sigma \circ \tau$ and $\tau \circ \rho \subseteq \tau \circ \sigma$.

(b) $\sigma \circ (\bigcup_{\alpha \in \Omega} \rho_\alpha) = \bigcup_{\alpha \in \Omega} \sigma \circ \rho_\alpha$.

(c) $\sigma \circ (\bigcap_{\alpha \in \Omega} \rho_\alpha) \subseteq \bigcap_{\alpha \in \Omega} \sigma \circ \rho_\alpha$.

(d) $\rho \subseteq \sigma$ implies $\rho^{-1} \subseteq \sigma^{-1}$.

(e) $(\bigcup_{\alpha \in \Omega} \rho_\alpha)^{-1} = \bigcup_{\alpha \in \Omega} \rho_\alpha^{-1}$.

(f) $(\bigcap_{\alpha \in \Omega} \rho_\alpha)^{-1} = \bigcap_{\alpha \in \Omega} \rho_\alpha^{-1}$.

2. To see that equality does not hold in general in (1c), let ι' be the inequality relation $x \neq y$ on X (complement of ι). If $|X| > 1$, then

$$\omega \circ (\iota \cap \iota') = \square \quad \text{but} \quad (\omega \circ \iota) \cap (\omega \circ \iota') = \omega.$$

3. If ρ and σ are symmetric relations such that $\rho \circ \sigma \subseteq \sigma \circ \rho$, then $\rho \circ \sigma = \sigma \circ \rho$.

1.5 CONGRUENCES, FACTOR GROUPOIDS AND HOMOMORPHISMS

A relation ρ on a groupoid S is said to be *right* [*left*] *compatible* (or *regular* or *homogeneous*) if $a \, \rho \, b$ (a, b in S) implies $ac \, \rho \, bc$ [$ca \, \rho \, cb$] for every c in S. A right [left] compatible equivalence relation on S will be called a *right* [*left*] *congruence* on S. By a *congruence* on S we mean an equivalence relation on S which is both right and left compatible.

Let ρ be a congruence on a groupoid S. Let A and B be members of S/ρ, i.e., equivalence classes of S mod ρ. Let $a_1, a_2 \in A$ and let $b_1, b_2 \in B$. From $a_1 \, \rho \, a_2$ we have $a_1b_1 \, \rho \, a_2b_1$, since ρ is right compatible. From $b_1 \, \rho \, b_2$ we have $a_2b_1 \, \rho \, a_2b_2$, since ρ is left compatible. From the transitivity of ρ, we conclude that $a_1b_1 \, \rho \, a_2b_2$. Hence the set product AB of A and B is contained in some equivalence class C. If we define $A \circ B = C$, then S/ρ becomes a groupoid which we call the *factor* (or *quotient*) *groupoid* of S modulo ρ.

As in §1.4, we denote by $a\rho$ (a in S) the equivalence class mod ρ to which a belongs. The above definition of ($\circ$) is simply $a\rho \circ b\rho = (ab)\rho$, for all a, b in S. Denoting by $\rho^\natural$ the natural mapping of S upon S/ρ, we have $a\rho = a\rho^\natural$ for all a in S, and so $a\rho^\natural \circ b\rho^\natural = (ab)\rho^\natural$. Thus $\rho^\natural$ is a homomorphism, and we call it the *natural* (or *canonical*) *homomorphism* of S upon S/ρ. Since $\rho^\natural$ is a homomorphism, *S/ρ is a semigroup if S is a semigroup.*

The foregoing shows that every factor groupoid of a groupoid S is a homomorphic image of S. The following theorem shows conversely that every homomorphic image of S is isomorphic with a factor groupoid of S. Thus, if we do not distinguish between isomorphic groupoids, the extrinsic problem of finding all homomorphic images of a given groupoid S can be replaced by the intrinsic problem of finding all congruences on S.

THEOREM 1.5 (Main Homomorphism Theorem). *Let θ be a homomorphism of a groupoid S upon a groupoid S'. Let $\rho = \theta \circ \theta^{-1}$, i.e., $a \, \rho \, b$ (a, b in S) if and only if $a\theta = b\theta$. Then ρ is a congruence on S, and there exists an isomorphism ψ of S/ρ upon S' such that $\rho^\natural \psi = \theta$ where $\rho^\natural$ is the natural homomorphism of S upon S/ρ.*

PROOF. If $a \, \rho \, b$ and $c \in S$ then

$$(ac)\theta = (a\theta)(c\theta) = (b\theta)(c\theta) = (bc)\theta,$$

so that $ac \, \rho \, bc$. Similarly, $ca \, \rho \, cb$. Since ρ is evidently an equivalence relation on S, it is a congruence.

For each element A of S/ρ, define $A\psi = a_1\theta$ where $a_1 \in A$. To see that ψ is a single-valued mapping (of S/ρ into S'), we note that if $a_2 \in A$ then $a_1 \,\rho\, a_2$, and hence $a_1\theta = a_2\theta$. Since θ maps S upon S' we see that ψ maps S/ρ upon S'. To show that ψ is a homomorphism, let $A, B \in S/\rho$, and let $a \in A$, $b \in B$. Then $ab \in A \circ B$, and so

$$(A \circ B)\psi = (ab)\theta = (a\theta)(b\theta) = (A\psi)(B\psi).$$

To show that ψ is one-to-one, let $A\psi = B\psi$ (A, B in S/ρ), and let $a \in A$, $b \in B$. Then $a\theta = A\psi = B\psi = b\theta$, whence $a \,\rho\, b$, and so $A = B$. Hence ψ is an isomorphism of S/ρ upon S'.

If $a \in A \in S/\rho$, then $a\rho^\natural = A$. Hence $a\theta = A\psi = (a\rho^\natural)\psi = a(\rho^\natural\psi)$. Since this holds for any element a of S, we conclude that $\theta = \rho^\natural\psi$.

In general, congruences on a semigroup S are not determined by any single congruence class (or "kernel"), as they are for groups (Exercise 1 below), but certain classes of congruences on S may be so determined. For example, every congruence ρ such that S/ρ is a group (or a group with zero) is determined by the congruence class which is the identity element of the group (or group with zero) S/ρ (Chapter 9). Another example which will be of constant application throughout the book is the following.

Let I be an ideal of a semigroup S. Define $a \,\rho\, b$ (a, b in S) to mean that either $a = b$ or else both a and b belong to I. We call ρ the *Rees congruence modulo I*. The equivalence classes of S mod ρ are I itself and every one-element set $\{a\}$ with a in $S \backslash I$. We shall write S/I instead of S/ρ, and we call S/I the *Rees factor semigroup of S modulo I*. We may describe S/I as the result of collapsing I into a single (zero) element, while the elements of S outside of I retain their identity. This concept was introduced by Rees in [1940].

THEOREM 1.6 (Induced Homomorphism Theorem). *Let ϕ_1 and ϕ_2 be homomorphisms of a groupoid S upon groupoids S_1 and S_2, respectively, such that $\phi_1 \circ \phi_1^{-1} \subseteq \phi_2 \circ \phi_2^{-1}$. Then there exists a unique homomorphism θ of S_1 upon S_2 such that $\phi_1\theta = \phi_2$.*

PROOF. Let $a_1 \in S_1$, and let a be an element of S such that $a\phi_1 = a_1$. Define $a_1\theta = a\phi_2$. This is single-valued, for if $b\phi_1 = a_1$ (b in S), we have $(a, b) \in \phi_1 \circ \phi_1^{-1} \subseteq \phi_2 \circ \phi_2^{-1}$, so that $a\phi_2 = b\phi_2$. It is clear that $\phi_1\theta = \phi_2$, and the assertion that θ is a homomorphism follows from this:

$$\begin{aligned}[(a\phi_1)(b\phi_1)]\theta &= [(ab)\phi_1]\theta = (ab)\phi_2 = (a\phi_2)(b\phi_2) \\ &= [(a\phi_1)\theta][(b\phi_1)\theta].\end{aligned}$$

The uniqueness of θ is evident; for, if θ is to satisfy $\phi_1\theta = \phi_2$ we are compelled to define θ as above.

COROLLARY 1.6a. *If ρ_1 and ρ_2 are congruences on a groupoid S such that $\rho_1 \subseteq \rho_2$, then $S/\rho_1 \sim S/\rho_2$.*

PROOF. Let $\phi_1 = \rho_1^\natural$, $\phi_2 = \rho_2^\natural$, $S_1 = S/\rho_1$, and $S_2 = S/\rho_2$. Since $\rho_1 = \phi_1 \circ \phi_1^{-1}$ and $\rho_2 = \phi_2 \circ \phi_2^{-1}$, the hypotheses of Theorem 1.6 are satisfied, and we conclude that there is a homomorphism θ of S_1 upon S_2.

It is easily verified that the intersection of any set of congruences on a groupoid S is also a congruence on S. The following principle is due to Tamura and Kimura [1954, 1955]. (See also Vol. II, p. 275.)

PROPOSITION 1.7 (Principle of the Maximal Homomorphic Image of Given Type). *Let $\mathscr{C}$ be a given type of groupoid such that if one of two isomorphic groupoids has type $\mathscr{C}$, then so also has the other. We say that a congruence σ on a groupoid S has type $\mathscr{C}$ if S/σ has type $\mathscr{C}$. Suppose that the intersection ρ of all congruences σ on S having type $\mathscr{C}$ also has type $\mathscr{C}$. Then S/ρ is the maximal homomorphic image of S of type $\mathscr{C}$ in the sense that it has type $\mathscr{C}$, and every other homomorphic image of S of type $\mathscr{C}$ is a homomorphic image of S/ρ.*

PROOF. If T is any homomorphic image of S of type $\mathscr{C}$, then, by the Main Homomorphism Theorem, $T \cong S/\sigma$ for some congruence σ on S. By hypothesis on $\mathscr{C}$, S/σ has type $\mathscr{C}$. Hence σ has type $\mathscr{C}$, and so $\rho \subseteq \sigma$ by definition of ρ. By Corollary 1.6a, $S/\rho \sim S/\sigma$, and hence $S/\rho \sim T$.

As examples of the application of this principle, we offer the following. (1) Every groupoid has a maximal homomorphic semigroup image. (2) Every semigroup has a maximal homomorphic commutative image.

We may replace "commutative" in (2) by "cancellative" or "idempotent", or by any combination of these. The most fruitful so far has proved to be "commutative and idempotent" (= "semilattice", §1.8), which was the first type considered by Tamura and Kimura [1954], and will play an important part in Chapter 4. On the other hand, a semigroup does not in general possess a maximal homomorphic group image (Exercise 6 of §1.6). For an abstract discussion, see Kimura [1958*d*].

If ρ_0 is any relation on a groupoid S, there exists at least one congruence on S containing ρ_0, namely the universal relation $\omega = S \times S$. Hence the intersection ρ of all congruences on S containing ρ_0 exists. We call ρ the *congruence on S generated by ρ_0*.

If S is a semigroup, we may describe ρ more usefully as follows. Let $\rho_1 = \rho_0 \cup \rho_0^{-1} \cup \iota$. For a, b in S, define $a\,\rho_2\,b$ to mean that

$$a = xcy, \qquad b = xdy, \qquad \text{and } c\,\rho_1\,d,$$

for some c, d in S and x, y in S^1. We describe the passage from a to b, or vice-versa, as an *elementary ρ_0-transition*. Evidently ρ_2 is reflexive, symmetric, and compatible, and $\rho_0 \subseteq \rho_1 \subseteq \rho_2 \subseteq \rho$. Finally, the transitive closure ρ_2^t of ρ_2 is a congruence on S contained in ρ, and hence equal to ρ. Thus $a\,\rho\,b$ if and only if there exist $c_1, c_2, \cdots, c_n$ in S such that $a\,\rho_2\,c_1, \cdots, c_i\,\rho_2\,c_{i+1}, \cdots, c_n\,\rho_2\,b$. We formulate the foregoing in a theorem.

THEOREM 1.8. *Let ρ_0 be a relation on a semigroup S, and let ρ be the congruence on S generated by ρ_0. Then $a \rho b$ (a, b in S) if and only if b can be obtained from a by a finite sequence of elementary ρ_0-transitions.*

EXERCISES FOR §1.5

1. If H is a subgroup of a group G, then the relation ρ on G defined by $a \rho b$ (a, b in G) if and only if $ab^{-1} \in H$ is a right congruence on G, and every right congruence on G is obtained in this way. The equivalence classes of ρ are the right cosets Ha of H in G. Moreover, ρ is a congruence if and only if H is normal in G. (Dubreil [1941].)
2. The following consequence of Theorem 1.6 is sometimes called the Induced Homomorphism Theorem for Groups. Let ϕ [ϕ'] be a homomorphism of a group G [G'] upon a group H [H'], and let θ be a homomorphism of G into G' which maps the kernel of ϕ into the kernel of ϕ'. Then there exists a unique homomorphism θ' of H into H' such that $\phi\theta' = \theta\phi'$.

1.6 CYCLIC SEMIGROUPS

If a is any element of a semigroup S, then the subsemigroup $\langle a \rangle$ of S generated by a consists of all the positive integral powers of a:

$$\langle a \rangle = \{a, a^2, a^3, \cdots\}.$$

If $\langle a \rangle = S$, then S is called a *cyclic semigroup*. In the general case, we call $\langle a \rangle$ the *cyclic subsemigroup of S generated by a*. The *order* of a is defined to be the order of $\langle a \rangle$. There are just two possibilities:

(1) No two powers of a are equal. Then evidently a has (countably) infinite order.

(2) There exist positive integers r and s with $r < s$ such that $a^r = a^s$. Then, as we proceed to show, a has finite order.

Let s be chosen as small as possible, that is, a^s is the first power of a which is equal to some lower power. Then $a^s = a^r$ for some $r < s$. Since $a, a^2, \cdots, a^{s-1}$ must be distinct, r is the only positive integer such that $r < s$ and $a^r = a^s$.

Let $m = s - r$. By multiplying $a^r = a^{m+r}$ successively by a^m, we obtain $a^{r+km} = a^r$ for every non-negative integer k. If n is any positive integer, we can write $n = km + i$ with k and i integers such that $k \geqq 0$ and $0 \leqq i < m$. From

$$a^{r+n} = a^{r+km+i} = a^{r+i}$$

we see that every power of a beyond a^r is equal to one of the set

$$K_a = \{a^r, a^{r+1}, \cdots, a^{r+m-1}\}.$$

It is now clear that the order of a is finite, namely $r + m - 1$. We call r the *index* and m the *period* of a and of $\langle a \rangle$. Observe the relation

$$\text{index} + \text{period} = \text{order} + 1.$$

The set K_a is clearly a subsemigroup of S. If to each member a^n of K_a ($r \leqq n \leqq r + m - 1$) we let correspond the residue class $(m) + n$ of integers mod m containing n, then this mapping $a^n \to (m) + n$ is evidently an isomorphism of K_a upon the additive group $I/(m)$ of all residue classes mod m. Hence K_a is a cyclic group of order m.

The results just described were first found by Frobenius (*Über endliche Gruppen*, Sitzungsber. Preuss. Akad. Wiss. Berlin, 1895, pp. 163–194), not for single elements of a semigroup, but for subsets (complexes) of a group (see Exercise 2 below). They were also found by: Morgan Ward in 1933 (unpublished); Suschkewitsch [1937], Chapter 2, §19; Poole [1937]; Rees [1940]; and Climescu [1946]. We formulate them in the following theorem.

THEOREM 1.9. *Let a be an element of a semigroup S, and let $\langle a \rangle$ be the cyclic subsemigroup of S generated by a. If $\langle a \rangle$ is infinite, all the powers of a are distinct. If $\langle a \rangle$ is finite, there exist two positive integers, the index r and the period m of a, such that $a^{m+r} = a^r$ and*

$$\langle a \rangle = \{a, a^2 \cdots, a^{m+r-1}\},$$

the order of $\langle a \rangle$ being $m + r - 1$. The set

$$K_a = \{a^r, a^{r+1}, \cdots, a^{m+r-1}\}$$

is a cyclic subgroup of S of order m.

There exists a cyclic semigroup of any preassigned index r and period m, for example that generated by the transformation

$$\begin{pmatrix} 0 & 1 & 2 \cdots r-1 & r & \cdots r+m-2 & r+m-1 \\ 1 & 2 & 3 \cdots \quad r & r+1 & \cdots r+m-1 & r \end{pmatrix}$$

of the set $\{0, 1, 2, \cdots, r + m - 1\}$. Clearly two finite cyclic semigroups are isomorphic if and only if they have the same index and period.

A semigroup S is called *periodic* if every element of S has finite order; in particular, every finite semigroup is periodic. If a is an element of finite order, then $\langle a \rangle$ contains exactly one idempotent, namely the identity element of K_a. The specification of this element given in Exercise 1 below is taken directly from the above cited paper of Frobenius. That *some power of every element of a finite semigroup is idempotent* was also shown by E. H. Moore (*A definition of abstract groups*, Trans. Amer. Math. Soc. 3 (1902), 485–492).

EXERCISES FOR §1.6

1. Let a be an element of finite order of a semigroup S, and let r be the index and m the period of a. Then the identity element of the subgroup K_a of $\langle a \rangle$ is a^n, where n is divisible by m and $r \leqq n < r + m$. (Frobenius, 1895, paper cited in text.)

2. Let G be a group such that the subsemigroup of G generated by any finite subset of G is finite. Then any subsemigroup of G is actually a subgroup of G, so that our hypothesis is equivalent to saying that G is locally finite. If $A \subseteq G$, A finite, and $1 \in A$, then in the sequence of powers $A, A^2, \cdots$ only a finite number of them are distinct; exactly one of these is a subgroup of G, and is the subgroup of G generated by A. (Frobenius, 1895, paper cited in text.)

3. If S is a right cancellative semigroup, every element of finite order has index 1.

4. Let S be a periodic commutative semigroup, and let E be the set of idempotents of S. For each e in E, let S_e be the set of all x in S such that $x^n = e$ for some positive integer n. Then S_e and S_f are disjoint if $e \neq f$ in E, and S is the union of all the S_e. Each S_e is a subsemigroup of S containing e but no other idempotent; and $S_eS_f \subseteq S_{ef}$ for all e, f in E. We call the S_e the *maximal one-idempotent* (or *unipotent*) *subsemigroups of* S. (Schwarz [1953a, 1954a].)

5. A commutative semigroup every element of which has index 1 is a union of disjoint periodic groups. (Poole [1937].)

6. Every homomorphic group image of an infinite cyclic semigroup C is a finite cyclic group, and every finite cyclic group is a homomorphic image of C. Hence C has no maximal homomorphic group image. (See remarks after Proposition 1.7. Exercise 16 of §2.7 gives a class of semigroups which do have maximal homomorphic group images.)

1.7 Units and maximal subgroups

Let S be a semigroup with identity element 1. If p and q are elements of S such that $pq = 1$, then p will be called a *left inverse* of q and q a *right inverse* of p. (The phrase "with respect to 1" will be omitted.) A *right* [*left*] *unit* in S is defined to be an element of S having a right [left] inverse in S. Thus, if $pq = 1$, p is a right and q a left unit. By a *unit* in S we mean an element of S having both a right and a left inverse in S.

Theorem 1.10. *Let S be a semigroup with identity element* 1.

(i) *The set P [Q] of all right [left] units of S is a right [left] cancellative subsemigroup of S containing* 1.

(ii) *The set U of all units of S is a subgroup of S, and $U = P \cap Q$. Each unit has a unique two-sided inverse in U, and has no other left or right inverse in S.*

(iii) *Every subgroup of S containing* 1 *is contained in U.*

Remark. We call P [Q] the *right* [*left*] *unit subsemigroup of* S, and U the *group of units of* S.

Proof. (i) If $pq = p'q' = 1$ then $(pp')(q'q) = 1$, which shows that P and Q are subsemigroups of S. They evidently contain 1. If $ap = bp$ with a, b

in S and p in P, then p has a right inverse q, and $a = a1 = apq = bpq = b1 = b$. Similarly, Q is left cancellative.

(ii) It is clear that $U = P \cap Q$, and so U is a subsemigroup of S. If $u \in U$, then there exist x and y in S such that $xu = uy = 1$. Let x and y be any such elements of S. Then $x = x1 = xuy = 1y = y$. Hence, every left inverse of u is equal to every right inverse, and so u has a unique two-sided inverse u', and no other one-sided inverse. From $uu' = u'u = 1$ it follows that $u' \in U$, and hence U is a group.

(iii) Let G be any subgroup of S containing 1, and let $a \in G$. Let a^{-1} be the inverse of a in G. From $aa^{-1} = a^{-1}a = 1$ it follows that $a \in U$, and so $G \subseteq U$.

A semigroup need not contain a subgroup. For example, the infinite cyclic semigroup does not. It is clear that a semigroup S will contain a subgroup if and only if it contains an idempotent. If e is an idempotent element of a semigroup S, then eS consists of all elements a of S for which e is a left identity, that is, $ea = a$. For if $a = ex$ for some x in S, then $ea = e^2x = ex = a$, and the converse is evident. Similarly, Se consists of all elements of S for which e is a right identity, and eSe is the set of all elements of S for which e is a two-sided identity. We evidently have

$$eSe = eS \cap Se.$$

Now eS [Se] is the principal right [left] ideal of S generated by e. In particular, eS and Se are subsemigroups of S, and hence so is their intersection eSe. Moreover, eSe has the identity element e, and so we may speak of the group of units of eSe, which we shall denote by H_e.

THEOREM 1.11. *Let e be any idempotent element of a semigroup S, and let H_e be the group of units of eSe. Then H_e contains every subgroup G of S that meets H_e.*

PROOF. Let f be the identity element of G. We show first that $f = e$. By hypothesis, $G \cap H_e$ is not empty; let a be an element thereof. Let b and c be the inverses of a in the groups G and H_e, respectively. Then

$$e = ca = caf = ef = eab = ab = f.$$

Since e is a two-sided identity of G, it follows that $G \subseteq eSe$. From Theorem 1.10 (iii) we conclude that $G \subseteq H_e$.

A subgroup G of a semigroup S is called a *maximal subgroup* of S if it is not properly contained in any other subgroup of S. If e is the identity element of a maximal subgroup G of S, then G meets H_e in e at least, and so $G \subseteq H_e$ by Theorem 1.11. From the maximality of G, we have $G = H_e$. Conversely, if e is any idempotent of S, it is evident from Theorem 1.11 that H_e is a maximal subgroup of S. We conclude that *the groups H_e of Theorem* 1.11 *are just the maximal subgroups of S.*

It is also evident from Theorem 1.11 that if e and f are distinct idempotent elements of S, then H_e and H_f are disjoint. We may visualize the maximal subgroups of S as islands in a sea.

The existence of maximal subgroups in a semigroup S was noted first by Schwarz [1943] for periodic S, and by Wallace [1953] and Kimura [1954] for arbitrary S.

Exercises for §1.7

1. Let P [Q] be the right [left] unit subsemigroup of a semigroup S with identity element 1, and let U be the group of units of S.

(a) The following three conditions on S are equivalent: (i) $ab = 1$ (a, b in S) implies $ba = 1$; (ii) $P = U$; (iii) $Q = U$.

(b) The conditions in (a) hold if S is periodic, or if S is right cancellative.

(c) The conditions in (a) hold for the semigroups P and Q.

2. Let $\mathscr{T}_X$ be the full transformation semigroup on a set X (§1.1).

(a) The right unit subsemigroup of $\mathscr{T}_X$ consists of all one-to-one transformations of X into X.

(b) The left unit subsemigroup of $\mathscr{T}_X$ consists of all transformations of X upon X.

(c) The group of units of $\mathscr{T}_X$ is the symmetric group $\mathscr{G}_X$ on X. (Suschkewitsch [1937], Chapter 1, §7; [1940a, b].)

3. The maximal subgroup H_e of a semigroup S containing an idempotent element e of S can be characterized as the set of all elements a of S such that (i) $ea = ae = a$ and (ii) there exist x, y in S such that $xa = ay = e$.

4. A finite cyclic semigroup $\langle a \rangle$ contains a single maximal subgroup, namely K_a (in the notation of §1.6).

5. If a semigroup is a union of groups, it is a union of disjoint groups. (Clifford [1941].)

6. (a) A periodic semigroup S is a union of groups if and only if each element of S has index 1 (§1.6).

(b) Every right cancellative, periodic semigroup is a union of groups (note Exercise 3 of §1.6).

(c) Every cancellative periodic semigroup is a group.

1.8 Bands and semilattices; bands of semigroups

We recall that a relation $\leqq$ on a set X is called a *partial ordering* of X if (1) $a \leqq a$, (2) $a \leqq b$ and $b \leqq a$ imply $a = b$, and (3) $a \leqq b$ and $b \leqq c$ imply $a \leqq c$ (a, b, c in X). In other words, a partial ordering is a reflexive, antisymmetric, and transitive relation. We write $a < b$ if $a \leqq b$ and $a \neq b$. The converse or dual of the relation $\leqq$ [$<$] is denoted as usual by $\geqq$ [$>$].

The following example is of great importance to us. Let E be the set of idempotent elements of a semigroup S. Define $e \leqq f$ (e, f in E) to mean $ef = fe = e$. If $e \leqq f$ we say that e is *under* f and that f is *over* e. To see that $\leqq$ is a partial ordering of E, let $e, f, g \in E$. (1) $e^2 = e$, and hence $e \leqq e$. (2) If $e \leqq f$ and $f \leqq e$ then $ef = fe = e$ and $fe = ef = f$, whence $e = f$. (3) If $e \leqq f$ and $f \leqq g$ then $ef = fe = e$ and $fg = gf = f$, whence

$$eg = (ef)g = e(fg) = ef = e,$$
$$ge = g(fe) = (gf)e = fe = e.$$

Hence $e \leqq g$. We shall call $\leqq$ the *natural partial ordering of E*.

An element b of a partially ordered set X is called an *upper bound* (UB) of a subset Y of X if $y \leqq b$ for every y in Y. An upper bound b of Y is called a *least upper bound* (LUB) or *join* of Y if $b \leqq c$ for every upper bound c of Y. If Y has a join in X, it is clearly unique. *Lower bound* (LB) and *greatest lower bound* (GLB) or *meet* are defined dually. A partially ordered set X is called an *upper* [*lower*] *semilattice* if every two-element subset $\{a, b\}$ of X has a join [meet] in X; it follows that every finite subset of X has a join [meet]. The join [meet] of $\{a, b\}$ will be denoted by $a \vee b$ [$a \wedge b$]. A *lattice* is a partially ordered set which is both an upper and a lower semilattice. A lattice X is said to be *complete* if every subset of X has a join and a meet.

For example, let X be the set of all subgroupoids of a groupoid S, together with the empty set; X is partially ordered by set-theoretical inclusion. Since the intersection of any set of subgroupoids of S is either empty or a subgroupoid, X is a complete lattice. The meet of a subset Y of X is the set-theoretical intersection of the members of Y, while the join of Y is the subgroupoid of S generated by the set-theoretical union of the members of Y. The foregoing holds verbatim if we replace "subgroupoid or empty subset of S" by "congruence on S".

On the other hand, the set of all left [right, two-sided] ideals of a groupoid S, together with the empty set, is closed under set-theoretical union as well as intersection, and so is a complete sublattice of the Boolean algebra of all subsets of S.

We recall (§1.1) that a *band* is a semigroup S every element of which is idempotent. Thus $S = E$ if S is a band, and so the *natural partial ordering* ($a \leqq b$ if and only if $ab = ba = a$) applies to all of S.

THEOREM 1.12. *A commutative band S is a lower semilattice with respect to the natural partial ordering of S. The meet $a \wedge b$ of two elements a and b of S is just their product ab. Conversely, a lower semilattice is a commutative band with respect to the meet operation.*

REMARK. We could of course make S an upper semilattice by defining $a \leqq b$ to mean $ab = b$, but for the sake of uniformity we shall adhere to the definition given above. We shall use the term *semilattice* as synonymous with *commutative band*. Consequently we agree that the term *semilattice* will mean *lower semilattice* unless the contrary is specified.

PROOF. That $\leqq$ is a partial ordering of S ($= E$) was shown above. We must show that the product ab ($= ba$) of two elements a and b of S is effective as the meet of $\{a, b\}$. From $(ba)a = ba^2 = ba$ and $(ab)b = ab^2 = ab$, we see that $ab \leqq a$ and $ab \leqq b$. Suppose $c \leqq a$ and $c \leqq b$. Then $(ab)c = a(bc) = ac = c$, and similarly $c(ab) = c$, whence $c \leqq ab$.

The converse is evident.

As an example of a non-commutative band, let X and Y be any two sets, and define a binary operation in $S = X \times Y$ as follows:

$$(x_1, y_1)(x_2, y_2) = (x_1, y_2) \quad (x_1, x_2 \text{ in } X;\ y_1, y_2 \text{ in } Y).$$

Associativity and idempotence are immediate. We shall call S the *rectangular band* on $X \times Y$. The reason for the term is the following. Think of $X \times Y$ as a rectangular array of points, the point (x, y) lying in the x-row and y-column of the array. Then $a_1 = (x_1, y_1)$ and $a_2 = (x_2, y_2)$ are opposite vertices of a rectangle, the other two vertices of which are $a_1a_2 = (x_1, y_2)$ and $a_2a_1 = (x_2, y_1)$. The rectangular bands on $X \times Y$ and $X' \times Y'$ are isomorphic if and only if $|X| = |X'|$ and $|Y| = |Y'|$.

If $|X| = 1$ [$|Y| = 1$] then the rectangular band on $X \times Y$ is isomorphic with the right [left] zero semigroup on Y [X] (§1.1).

The theory of semigroups does not include the vast theory of lattices and semilattices, any more than it includes the theory of groups. If a type of semigroup can be described explicitly in terms of groups and semilattices, we regard the further determination of the structure of such semigroups as lying outside the theory of semigroups. On the other hand, we consider that the study of bands does belong to the theory of semigroups. At the present time, we are a long way from giving a description of the structure of bands which is complete modulo semilattices.

By a *decomposition* of a semigroup S we mean a partition of S into the union of disjoint subsemigroups S_α ($\alpha \in \Omega$). For this to have any value, the S_α should be semigroups of some more restricted type than S, for example simple semigroups or groups.

Suppose that $S = \bigcup\{S_\alpha : \alpha \in \Omega\}$ is a decomposition of S such that, for every pair of elements α, β of the index set Ω there is an element γ of Ω such that $S_\alpha S_\beta \subseteq S_\gamma$. If we define a product in Ω by $\alpha\beta = \gamma$ if $S_\alpha S_\beta \subseteq S_\gamma$, then Ω becomes thereby a band. We say that *S is the union of the band Ω of semigroups S_α* ($\alpha \in \Omega$). The mapping ϕ defined by $a\phi = \alpha$ if $a \in S_\alpha$ is a homomorphism of S upon Ω and the S_α are the congruence classes of the congruence $\phi \circ \phi^{-1}$ (§§1.4 and 1.5). Conversely, if ϕ is a homomorphism of a semigroup S upon a band Ω, then the inverse image $S_\alpha = \alpha\phi^{-1}$ of each element α of Ω is a subsemigroup of S, and S is the union of the band Ω of semigroups S_α ($\alpha \in \Omega$). If Ω is commutative, we say that *S is the union of the semilattice Ω of semigroups S_α* ($\alpha \in \Omega$). For example, the conclusion of Exercise 4 of §1.6 asserts that S is the union of the semilattice E of semigroups S_e ($e \in E$).

If Ω and each S_α ($\alpha \in \Omega$) is of known structure, then we know what might be called the "gross structure" of S. The "fine structure" of S, just how the products are formed between different S_α, is a more difficult problem. These matters will be discussed in Chapter 4.

We shall also use the abbreviated expression, *S is a band [semilattice] of semigroups of type* $\mathscr{C}$, to mean that S is the union of a band [semilattice] Ω of semigroups S_α $(\alpha \in \Omega)$, where each S_α is of type $\mathscr{C}$. For example, the result of Exercise 4 of §1.6, due to Schwarz, may be expressed in part as follows: *every periodic commutative semigroup is a semilattice of one-idempotent semigroups.*

This result was extended by Numakura [1954], who showed that any commutative semigroup is a semilattice of semigroups containing at most one idempotent. Tamura and Kimura [1954] showed that, for any commutative semigroup S, there is a finest congruence relation η on S such that S/η is a semilattice, and so S/η is the maximal semilattice homomorphic image of S (Proposition 1.7). They gave an explicit description of η which will be given in Theorem 4.12 below. They give an example of a semigroup on which the congruence used by Numakura differs from η. Yamada [1955*b*] gave an explicit description of the finest congruence ϕ on any semigroup S such that S/ϕ is a semilattice; a somewhat simpler description is given in the review of his paper [MR 17, 584].

EXERCISES FOR §1.8

1. A semigroup S is said to be *nowhere commutative* if $ab = ba$ (a, b in S) implies $a = b$. A rectangular band is nowhere commutative.

2. An idempotent element e of a semigroup S is said to be *primitive* if the only idempotents of S under e are e itself and 0 (if S has a zero), and $e \neq 0$. A semigroup S is nowhere commutative if and only if it is a band without zero in which every element is primitive, or $|S| = 1$.

3. A semigroup S is called by Thierrin [1954*b*] *strongly reversible* if, for every a, b in S there exist positive integers r, s, t such that $(ab)^r = a^s b^t = b^t a^s$. A periodic semigroup is a semilattice of one-idempotent semigroups with commuting idempotents if and only if it is strongly reversible. (Iséki [1956*a*]; this generalizes Exercise 4 of §1.6.)

4. A semigroup S has the property that every transformation of S is an endomorphism of S if and only if S is either a left zero semigroup or a right zero semigroup. (Posey [1949].)

1.9 REGULAR ELEMENTS AND INVERSES; INVERSE SEMIGROUPS

An element a of a semigroup S is called *regular* if $a \in aSa$, that is, if $axa = a$ for some x in S. A semigroup S is called regular if every element of S is regular.

We note that if $axa = a$ then $e = ax$ is an idempotent element of S such that $ea = a$. For $e^2 = (ax)(ax) = (axa)x = ax = e$, and $ea = axa = a$. Similarly, $f = xa$ is an idempotent such that $af = a$. We also note that if a is a regular element of S, then the principal right ideal $aS^1 = a \cup aS$ generated by a is just aS, for $a = af$ implies $a \in aS$. Similarly $S^1a = Sa$. These two remarks will be used in the sequel without comment.

The notion of regularity was introduced by J. von Neumann for rings (*On regular rings*, Proc. Nat. Acad. Sci. U.S.A. 22 (1936), 707–713), and the following lemma is the straightforward analogue of Lemma 6 in the paper cited.

LEMMA 1.13. *An element a of a semigroup S is regular if and only if the principal right [left] ideal of S generated by a has an idempotent generator e; that is, $aS^1 = eS^1$ $[S^1a = S^1e]$.*

PROOF. If a is regular, then $axa = a$ for some x in S, and $e = ax$ is an idempotent element of S such that $ea = a$. Clearly $aS^1 = eS^1$. Assume conversely that $aS^1 = eS^1$ with $e^2 = e$. Then $a = ex$ with x in S^1, so $ea = e^2x = ex = a$; and $e = ay$ with y in S^1, so that $a = ea = aya$. If $y = 1$, then $a = a^2$, and $a = aaa$. Hence, in any case, $a \in aSa$, and a is regular.

Two elements a and b of a semigroup S are said to be *inverses* of each other if

$$aba = a \qquad \text{and} \qquad bab = b.$$

This notion was introduced by Vagner [1952*b*] under the name *generalized inverses*; by Thierrin [1952*a*] who called a and b *reciprocal*; and by Preston [1954*a*]. If a and b are elements of the same maximal subgroup H of S, in particular if S is itself a group, then a and b are (generalized) inverses of each other if and only if they are group-inverses in the usual sense in H. Hence the term "inverse" in the new sense should not cause confusion. We remark that Bruck uses the term "relative inverse" for this concept in his *Survey of Binary Systems*. The same term was used by Clifford [1941] meaning inverses within the same maximal subgroup of S. It will not be used in this book.

If an element a of a semigroup S has an inverse in S, then a is evidently regular. The converse (Lemma 1.14) was noted by Thierrin [1952*a*]. Thus a regular semigroup is one in which every element has at least one inverse.

LEMMA 1.14. *If a is a regular element of a semigroup S, say $axa = a$ with x in S, then a has at least one inverse in S, in particular xax.*

PROOF. Let $b = xax$. Then

$$\begin{aligned} aba &= a(xax)a = ax(axa) = axa = a, \\ bab &= (xax)a(xax) = x(axa)(xax) = xa(xax) \\ &= x(axa)x = xax = b. \end{aligned}$$

Hence b is an inverse of a.

LEMMA 1.15. *Two elements of a semigroup S are group-inverses of each* *other within some subgroup of S if and only if they are inverses of, and commute with, each other.*

PROOF. The "only if" is evident. To prove the "if", let a and b be commuting inverse elements of S. Let $e = ab$ $(= ba)$. Then e is an idempotent such that $ea = ae = a$ and $eb = be = b$. Hence a and b are units in eSe, and so belong to the maximal subgroup H_e of S containing e (Theorem 1.11). Since $ab = ba = e$, a and b are group-inverses in H_e.

Inverses need not be unique. As an extreme example, any two elements of a rectangular band (§1.8) are inverses of each other. By an *inverse semigroup* we mean a semigroup in which every element has a unique inverse. Vagner [1952*b*] used the term *generalized group*. Inverse semigroups constitute probably the most promising class of semigroups for study at the present time, since they are not too far away from groups. Chapter 7 is devoted to them, and in §4.2 we give a complete description of inverse semigroups which are unions of groups. We shall devote the rest of this section to giving some of their more basic properties.

In Theorem 1.17 below, we give two other useful characterizations of inverse semigroups. That (i) implies (iii) was first shown by Vagner [1952*b*] and independently by Preston [1954*a*]. That (iii) implies (i) was shown by Liber [1954]; this and also the equivalence with (ii) was shown by Munn and Penrose [1955]. First we need a lemma.

LEMMA 1.16. *If e, f, ef, and fe are all idempotent elements of a semigroup, then ef and fe are inverses of each other.*

PROOF. $(ef)(fe)(ef) = ef^2e^2f = efef = (ef)^2 = ef$, and, by symmetry, $(fe)(ef)(fe) = fe$.

THEOREM 1.17. *The following three conditions on a semigroup S are equivalent:*

(i) *S is regular, and any two idempotent elements of S commute with each other;*

(ii) *every principal right ideal and every principal left ideal of S has a unique idempotent generator;*

(iii) *S is an inverse semigroup (i.e., every element of S has a unique inverse in S).*

PROOF. (i) implies (ii). By Lemma 1.13, every principal right ideal of S has at least one idempotent generator. Suppose e and f are idempotents generating the same principal right ideal, so that $eS = fS$. Then $ef = f$ and $fe = e$. But, by (i), $ef = fe$, and hence $e = f$.

(ii) implies (iii). By Lemma 1.13, S is regular and so we need only show that inverses are unique. Let b and c be inverses of a, so that

$$aba = a, \qquad bab = b,$$
$$aca = a, \qquad cac = c.$$

Then $abS = aS = acS$ and $Sba = Sa = Sca$, so that $ab = ac$ and $ba = ca$ by (ii). Hence
$$b = bab = bac = cac = c.$$

(iii) implies (i). An inverse semigroup is clearly regular, so we need only show that any two idempotent elements of S commute. First we show that the product ef of two idempotent elements e and f of S is idempotent. Let a be the (unique) inverse of ef, so that
$$(ef)a(ef) = ef, \qquad a(ef)a = a.$$
Let $b = ae$. Then
$$(ef)b(ef) = efae^2f = efaef = ef,$$
$$b(ef)b = ae^2fae = aefae = ae = b.$$

Hence b is also an inverse of ef. By (iii), $ae = b = a$. Similarly we can show that $fa = a$. Hence
$$a^2 = (ae)(fa) = a(ef)a = a.$$

But an idempotent is an inverse of itself, and, again using (iii), we conclude that $a = ef$. Hence ef is idempotent.

Now let e and f be any two idempotents of S. By the foregoing, ef and fe are also idempotent. By Lemma 1.16, they are inverses of each other. Thus ef and fe are both inverses of ef, and so $ef = fe$.

By a *one-to-one partial transformation* of a set X we mean a one-to-one mapping α of a subset Y of X upon a subset $Y' = Y\alpha$ of X. By the *inverse* α^{-1} of α we mean the mapping of $Y\alpha$ upon Y which is inverse to α in the usual sense of mappings, i.e., $y'\alpha^{-1} = y$ ($y \in Y$, $y' \in Y\alpha$) if and only if $y' = y\alpha$. Let $\mathscr{I}_X$ denote the set of all one-to-one partial transformations of X, including that of the empty subset $\square$ of X upon itself; this "empty transformation" will be denoted by 0. The product $\alpha\beta$ of two elements α and β of $\mathscr{I}_X$ is defined as follows. Let Y be the domain of α and Z that of β. If $Y\alpha \cap Z = \square$, we define $\alpha\beta = 0$. Otherwise, let $W = (Y\alpha \cap Z)\alpha^{-1}$. Then we define $\alpha\beta$ to be the iterate of $\alpha | W$ and $\beta | W\alpha$ in the usual sense. Clearly $\alpha\beta$ is a one-to-one transformation of W upon $W\alpha\beta$, and so belongs to $\mathscr{I}_X$. Associativity is easily verified. Hence $\mathscr{I}_X$ is a semigroup, which we call *the symmetric inverse semigroup on the set* X. This concept was introduced by Vagner [1952*a*].

We must show that $\mathscr{I}_X$ is an inverse semigroup. Since α^{-1} is evidently inverse to α in $\mathscr{I}_X$, that is, $\alpha\alpha^{-1}\alpha = \alpha$ and $\alpha^{-1}\alpha\alpha^{-1} = \alpha^{-1}$, it is clear that $\mathscr{I}_X$ is regular. An element of $\mathscr{I}_X$ is idempotent if and only if it is the identity mapping on some subset of X, from which it is clear that any two idempotents commute. That $\mathscr{I}_X$ is an inverse semigroup thus follows from Theorem 1.17.

We remark that the definition of product in $\mathscr{I}_X$ is consistent with that of relations (§1.4) on X. In fact $\mathscr{I}_X$ may be regarded as the subsemigroup of $\mathscr{B}_X$ consisting of all one-to-one relations.

Vagner [1952b] and Preston [1954c] have shown that every inverse semigroup can be embedded in some symmetric inverse semigroup. This is analogous to Cayley's Theorem for groups, and to the regular representation of semigroups (§1.3), but, as we shall see, is much harder to prove. For another approach, see Preston [1957].

In what follows, S denotes an inverse semigroup. The (unique) inverse of an element a of S will be denoted by a^{-1}. We thus have

$$aa^{-1}a = a \quad \textit{and} \quad a^{-1}aa^{-1} = a^{-1}.$$

The idempotent $e = aa^{-1}$ [$f = a^{-1}a$] will be called the *left* [*right*] *unit of* a; it is characterized as the unique idempotent generator of aS [Sa]. These remarks, and the commutativity of idempotents (Theorem 1.17), will be used without comment. The set of idempotents of S will be denoted by E; it is a subsemigroup, and in fact a subsemilattice (§1.8), of S. We say that a subsemigroup T of S is an *inverse subsemigroup* of S if $a \in T$ implies $a^{-1} \in T$.

LEMMA 1.18. *For any elements a, b of an inverse semigroup S, we have*

$$(a^{-1})^{-1} = a \quad \textit{and} \quad (ab)^{-1} = b^{-1}a^{-1}.$$

PROOF. The first is obvious. For the second,

$$(ab)(b^{-1}a^{-1})(ab) = a(bb^{-1})(a^{-1}a)b = a(a^{-1}a)(bb^{-1})b = ab,$$

$$(b^{-1}a^{-1})(ab)(b^{-1}a^{-1}) = b^{-1}(a^{-1}a)(bb^{-1})a^{-1} = b^{-1}(bb^{-1})(a^{-1}a)a^{-1} = b^{-1}a^{-1}.$$

Hence $b^{-1}a^{-1}$ is the inverse of ab.

LEMMA 1.19. *If e and f are idempotent elements of an inverse semigroup S, then*

$$Se \cap Sf = Sef \; (= Sfe).$$

PROOF. If $a \in Se \cap Sf$, then $ae = af = a$, so that $aef = af = a$, and $a \in Sef$. Conversely, if $a \in Sef$ $(= Sfe)$, then $aef = afe = a$, and so $ae = af = a$, that is, $a \in Se \cap Sf$.

THEOREM 1.20. *An inverse semigroup S is isomorphic with an inverse subsemigroup of the symmetric inverse semigroup $\mathscr{I}_S$ of all one-to-one partial transformations of S.*

PROOF. For each a in S we define ρ_a to be the mapping $x \to x\rho_a = xa$ of Sa^{-1} $(= Saa^{-1})$ into $Sa^{-1}a$ $(= Sa)$. Note that we are restricting the domain of ρ_a to Sa^{-1}. Evidently $\rho_{a^{-1}}$ maps Sa $(= Sa^{-1}a)$ into Saa^{-1} $(= Sa^{-1})$. If $x \in Saa^{-1}$ and $y \in Sa^{-1}a$, then

$$x\rho_a\rho_{a^{-1}} = xaa^{-1} = x,$$
$$y\rho_{a^{-1}}\rho_a = ya^{-1}a = y,$$

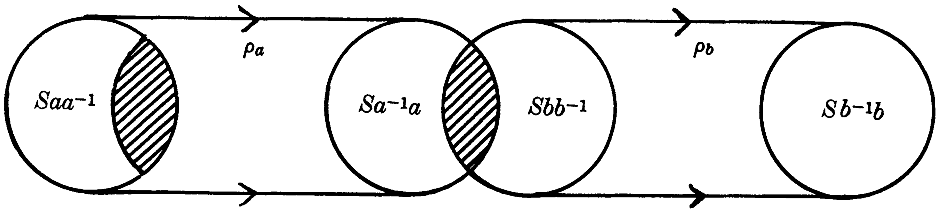
since any idempotent e is a right identity element of Se. Hence ρ_a and $\rho_{a^{-1}}$ are mutually inverse, one-to-one mappings of Saa^{-1} and $Sa^{-1}a$ upon each other. Thus $\rho_a \in \mathscr{I}_S$ and $\rho_{a^{-1}} = \rho_a^{-1}$. We shall show that $a \rightarrow \rho_a$ is an isomorphism of S into $\mathscr{I}_S$.

Suppose first that $\rho_a = \rho_b$ (a, b in S). Then $Saa^{-1} = Sbb^{-1}$, so that $aa^{-1} = bb^{-1}$ by Theorem 1.17 (ii), and $x \in Saa^{-1}$ implies $xa = x\rho_a = x\rho_b = xb$. Since $a^{-1} \in Saa^{-1}$, we conclude that $a^{-1}a = a^{-1}b$. Hence

$$a = aa^{-1}a = aa^{-1}b = bb^{-1}b = b.$$

Hence the mapping $a \rightarrow \rho_a$ is one-to-one.

Finally, we must show that $\rho_a\rho_b = \rho_{ab}$ (a, b in S). Since $(xa)b = x(ab)$ for every x in S, all we need to show is that $\rho_a\rho_b$ and ρ_{ab} have the same domain.



The domain of ρ_{ab} is $S(ab)(ab)^{-1}$. That of $\rho_a\rho_b$ is $(Sa^{-1}a \cap Sbb^{-1})\rho_{a^{-1}}$, indicated by the left-hand shaded area in the diagram. By Lemma 1.19,

$$Sa^{-1}a \cap Sbb^{-1} = Sa^{-1}abb^{-1} = Sabb^{-1}.$$

Hence, by Lemma 1.18,

$$(Sa^{-1}a \cap Sbb^{-1})\rho_{a^{-1}} = Sabb^{-1}a^{-1} = S(ab)(ab)^{-1}.$$

Let A be a subset $\neq \square$ of an inverse semigroup S. The intersection A^* of all inverse subsemigroups of S containing A is an inverse subsemigroup of S, contained in all other such. We call A^* the *inverse subsemigroup of S generated by A*. Let A^{-1} be the set of all inverses of elements of A. By virtue of Lemma 1.18, the subsemigroup of S generated by $A \cup A^{-1}$ is an inverse subsemigroup of S containing A and evidently contained in all other such. Hence $A^* = \langle A \cup A^{-1} \rangle$. In other words, A^* consists of all finite products of elements of A and inverses of elements of A.

We conclude this section with a notion that will serve several purposes, and has considerable intrinsic interest. Let S be a right cancellative semigroup not having any idempotent $\neq 1$. Let ϕ be the extended regular representation $a \rightarrow \rho_a$ of S, where ρ_a is the inner right translation $x \rightarrow x\rho_a = xa$ of S^1 ($a \in S$, $x \in S^1$). By Lemma 1.0, ϕ is a faithful representation of S, and $S\phi$

consists of one-to-one mappings of S^1 into itself. Hence $S\phi$ is contained in the symmetric inverse semigroup $\mathscr{I}_{S^1}$ on S^1. Since ρ_a maps S^1 in one-to-one fashion upon S^1a, its inverse ρ_a^{-1} in $\mathscr{I}_{S^1}$ maps S^1a one-to-one upon S^1. By the *inverse hull of* S we shall mean the inverse subsemigroup Σ of $\mathscr{I}_{S^1}$ generated by $S\phi$. This concept (but not the name) was introduced by Rees [1948*b*] for the purpose of embedding a cancellative semigroup in a group (see §1.10). Part (ii) of Lemma 1.21 generalizes Lemma 2.11 of his paper.

An element α of $\mathscr{I}_{S^1}$ is called a *one-to-one partial right translation* of S^1 if the following holds: if s is an element of S^1 such that $s\alpha$ is defined, then $(rs)\alpha$ is defined for every r in S^1, and $(rs)\alpha = r(s\alpha)$.

LEMMA 1.21. *Let S be a right cancellative semigroup having no idempotent $\neq 1$, and let Σ be its inverse hull.*

(i) *The set of one-to-one partial right translations of S^1 is an inverse subsemigroup of $\mathscr{I}_{S^1}$ containing Σ.*

(ii) *The domain $U(\alpha)$ and the range $V(\alpha)$ of a non-empty one-to-one partial right translation α of S^1 (in particular of an element α of Σ) are each left ideals of S^1.*

PROOF. Let α and β be one-to-one partial right translations (p.r.t.'s) of S^1. Let $r, s \in S^1$, and assume that $s(\alpha\beta)$ is defined. By definition of product in $\mathscr{I}_{S^1}$, this means that both $s\alpha$ and $(s\alpha)\beta$ are defined, and $s(\alpha\beta) = (s\alpha)\beta$. Since α is a p.r.t. of S^1, and $s\alpha$ is defined, we conclude that $(rs)\alpha$ is defined and equals $r(s\alpha)$. Since β is a p.r.t. of S^1, and $(s\alpha)\beta$ is defined, we conclude that $[r(s\alpha)]\beta$ is defined and equals $r[(s\alpha)\beta]$. Now

$$[r(s\alpha)]\beta = [(rs)\alpha]\beta = (rs)(\alpha\beta)$$

and

$$r[(s\alpha)\beta] = r[s(\alpha\beta)].$$

Hence $(rs)(\alpha\beta)$ is defined and equals $r[s(\alpha\beta)]$, which proves that $\alpha\beta$ is a p.r.t. of S^1.

Now let α be a p.r.t. of S^1. Let $r, s \in S^1$, and assume that $s\alpha^{-1}$ is defined. Let $q = s\alpha^{-1}$. Then $q\alpha$ is defined and equals s. Since α is a p.r.t. of S^1, $(rq)\alpha$ is defined and equals $r(q\alpha) = rs$. But this implies that $(rs)\alpha^{-1}$ is defined and equals $rq = r(s\alpha^{-1})$. Hence α^{-1} is a p.r.t. of S^1.

Hence the set of p.r.t.'s of S^1 is an inverse subsemigroup of $\mathscr{I}_{S^1}$. It obviously contains $S\phi = \{\rho_a : a \in S\}$, and hence contains the inverse subsemigroup Σ of $\mathscr{I}_{S^1}$ generated by $S\phi$. This concludes the proof of (i).

Turning to (ii), let α be a p.r.t. of S^1. If $s \in U(\alpha)$ and $r \in S^1$, then $rs \in U(\alpha)$ by definition of p.r.t. Let $q \in V(\alpha)$. Then $q = s\alpha$ for some s in S^1. Let $r \in S^1$. Since $s\alpha$ is defined, and α is a p.r.t. of S^1, $(rs)\alpha$ is defined and equals $r(s\alpha) = rq$. But this implies that $rq \in V(\alpha)$. Hence $U(\alpha)$ and $V(\alpha)$ are left ideals of S^1.

In Theorem 1.10, we saw that the right unit subsemigroup of a semigroup with identity is a right cancellative semigroup with identity. The converse is immediate from the following theorem.

THEOREM 1.22. *Let S be a right cancellative semigroup not containing any idempotent $\neq 1$. Let ϕ be the extended regular representation of S. Let Σ be the inverse hull of S, and let ι be the identical mapping of S^1. Then $\iota \cup S\phi$ is the right unit subsemigroup of Σ, and the set of all inverses in Σ of elements of $\iota \cup S\phi$ is the left unit subsemigroup of Σ.*

REMARK. If S does not contain an identity element, then ι is the only two-sided unit of Σ, and $S\phi$ consists of all the right units of Σ except ι. No element of $S\phi$ can be a left unit, for a left unit must have range S^1, whereas every element of $S\phi$ maps S^1 into S.

PROOF. If $a \in S$, then $\rho_a\rho_a^{-1} = \iota$. Hence every element ρ_a of $S\phi$ is a right unit of Σ. Suppose conversely that α is a right unit of Σ, so that $\alpha\beta = \iota$ for some β in Σ. Then $\alpha = \iota\alpha = \alpha\beta\alpha$ and $\beta = \beta\iota = \beta\alpha\beta$, so that $\beta = \alpha^{-1}$.

From $\alpha\alpha^{-1} = \iota$ it is clear that the domain $U(\alpha)$ of α must be all of S^1. In particular, $1 \in U(\alpha)$. By Lemma 1.21 (i), α is a partial right translation of S^1. Hence, for every x in S^1, $x\alpha = (x1)\alpha = x(1\alpha)$, and so $\alpha = \rho_{1\alpha} \in \iota \cup S\phi$. Hence $\iota \cup S\phi$ is the right unit subsemigroup of Σ.

Since $\rho_a\rho_a^{-1} = \iota$ for every a in S, ρ_a^{-1} is a left unit of Σ. Conversely, if α is a left unit of Σ, then we can show as above that $\alpha^{-1}\alpha = \iota$. Then $\alpha^{-1}(\alpha^{-1})^{-1} = \iota$ so that $\alpha^{-1} \in \iota \cup S\phi$; in other words, α is the inverse of an element of $\iota \cup S\phi$.

EXERCISES FOR §1.9

1. The full transformation semigroup $\mathscr{T}_X$ on a set X is regular. (Doss [1955].)
2. Let S be a left zero semigroup ($xy = x$ for all x, y in S) such that $|S| > 1$. Then every principal right ideal of S has a unique idempotent generator, but S is not an inverse semigroup. (Note Theorem 1.17 (ii).)
3. A semigroup S has the property that any two elements of S are inverses of each other if and only if S is nowhere commutative (Exercise 1 of §1.8).
4. A regular semigroup containing exactly one idempotent is a group.
5. A regular cancellative semigroup is a group. (Thierrin [1951].)
6. A regular commutative semigroup is a union of groups.
7. Let a be an element of a semigroup S. Let $A = \{x : axa = a, x \in S\}$. Then AaA is the set of inverses of a. (Bruck [1958], pp. 25–26; cf. Exercise 3 of §1.1.)
8. The symmetric inverse semigroup $\mathscr{I}_X$ on a set X contains the symmetric group $\mathscr{G}_X$ on X, and its semilattice of idempotents is isomorphic with the Boolean algebra of all subsets of X.
9. Applied to a semilattice S, the proof of Theorem 1.20 reduces to the usual method of embedding S in the Boolean algebra of all subsets of S.

10. If every element of an inverse semigroup S commutes with its inverse, then S is a union of groups.

11. A semigroup S is regular if and only if $A \cap B = AB$ for every right ideal A and every left ideal B of S. (Iséki [1956*d*].)

12. Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be classes of semigroups such that $\mathscr{B} \subseteq \mathscr{A} \subseteq \mathscr{C}$. Then $\mathscr{B}$ is called a *basis class* for $\mathscr{A}$ within $\mathscr{C}$ if (i) every semigroup in $\mathscr{A}$ is a union of subsemigroups each of which belongs to $\mathscr{B}$, (ii) no proper subclass $\mathscr{B}'$ of $\mathscr{B}$ exists having property (i), and (iii) every semigroup in $\mathscr{C}$ which is a union of subsemigroups in $\mathscr{B}$ belongs to $\mathscr{A}$. (When $\mathscr{C}$ is the class of all semigroups, this notion was introduced by Lyapin [1954].)

By an *elementary inverse semigroup* we mean an inverse semigroup generated by two mutually inverse elements. The class of all elementary inverse semigroups is a basis class for that of all inverse semigroups within the class of all semigroups whose idempotents commute, but not within the class of all semigroups. (Gluskin [1957].)

1.10 EMBEDDING SEMIGROUPS IN GROUPS

A commutative semigroup can be embedded in a group if and only if it is cancellative. The usual procedure for doing this, by means of ordered pairs, is just like that of embedding an integral domain in a field (see, for example, van der Waerden, *Modern Algebra*, §13). In fact, it is easier, since there is only one instead of two binary operations to consider.

For non-commutative semigroups, cancellation is an evidently necessary condition for embeddability in a group, but it is far from sufficient. A very useful sufficient condition is due to Ore (Ann. of Math. 32 (1931), 463–477). Following Dubreil [1941], we call a semigroup S *right reversible* if any two principal left ideals of S intersect: $Sa \cap Sb \neq \square$ for all a, b in S. Ore showed that any right reversible ring without divisors of zero can be embedded in a division ring. We can extract immediately from his proof the theorem that any right reversible, cancellative semigroup can be embedded in a group, and we shall refer to this as Ore's Theorem.

This can be done, as Ore did it, by means of ordered pairs, and we shall do just this in Chapter 10 for a more general embedding theorem. The main objective of the present section is to give an elegant proof of Ore's Theorem due to Rees [1948*b*]. Since a commutative semigroup is evidently right reversible, the embedding of a cancellative, commutative semigroup in a group is a consequence of Ore's Theorem.

Right reversibility is sufficient, but not necessary for embeddability in general. (Note, however, Dubreil's Theorem 1.24 below.) The first necessary and sufficient conditions for the embeddability of a semigroup in a group were given by Malcev [1939]. An account of this, and related work, will be given in Chapter 10.

THEOREM 1.23 (Ore). *Any right reversible, cancellative semigroup can be embedded in a group.*

PROOF (Rees). Let S be a right reversible, cancellative semigroup. Let $\mathscr{I}_S$ be the symmetric inverse semigroup of all one-to-one partial transformation of S. Let ϕ be the regular representation of S, and let Σ be the inverse hull of S (§1.9), i.e., the inverse subsemigroup of $\mathscr{I}_S$ generated by $S\phi$.

For each α in Σ, we denote by $U(\alpha)$ and $V(\alpha)$ the domain and range, respectively, of α. If $\alpha, \beta \in \Sigma$, we define $\alpha \subseteq \beta$ to mean that $U(\alpha) \subseteq U(\beta)$ and $x\alpha = x\beta$ for every x in $U(\alpha)$. This is consistent with the relation of inclusion in the semigroup $\mathscr{B}_S$ of all binary relations on S, of which $\mathscr{I}_S$ is a subsemigroup.

We define a relation $\sim$ on Σ as follows: $\alpha \sim \beta$ (α, β in Σ) if there exists γ in Σ such that $\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$. We proceed to show that $\sim$ is a congruence relation on Σ.

The reflexive and symmetric properties are immediate from the definition of $\sim$. To show transitivity, assume $\alpha \sim \beta$ and $\beta \sim \gamma$ (α, β, γ in Σ). Then there exist elements δ_1, δ_2 of Σ such that $\delta_1 \subseteq \alpha$, $\delta_1 \subseteq \beta$, $\delta_2 \subseteq \beta$, and $\delta_2 \subseteq \gamma$. Let $\delta = \delta_2\delta_2^{-1}\delta_1$. Then $U(\delta) \subseteq U(\delta_2)$. Hence, if $x \in U(\delta)$, then

$$x\delta = ((x\delta_2)\delta_2^{-1})\delta_1 = x\delta_1 = x\beta = x\delta_2.$$

Hence $\delta \subseteq \delta_1 \subseteq \alpha$ and $\delta \subseteq \delta_2 \subseteq \gamma$, so that $\alpha \sim \gamma$.

Now suppose that $\alpha \sim \alpha'$ and $\beta \sim \beta'$ ($\alpha, \alpha', \beta, \beta'$ in Σ). Then there exist γ and δ in Σ such that $\gamma \subseteq \alpha$, $\gamma \subseteq \alpha'$, $\delta \subseteq \beta$, and $\delta \subseteq \beta'$. It follows that $\gamma\delta \subseteq \alpha\beta$ and $\gamma\delta \subseteq \alpha'\beta'$, and so $\alpha\beta \sim \alpha'\beta'$. Hence $\sim$ is a congruence relation on Σ.

We show next that the factor semigroup (§1.5) $G = \Sigma/\sim$ of Σ modulo $\sim$ is a group. For this it suffices to show that, for given α, β in Σ, there exist ξ and η in Σ such that $\alpha\xi \sim \eta\alpha \sim \beta$. It suffices to take $\xi = \alpha^{-1}\beta$ and $\eta = \beta\alpha^{-1}$, for $\alpha\alpha^{-1}\beta \subseteq \beta$ and $\beta\alpha^{-1}\alpha \subseteq \beta$.

So far we have made no use of right reversibility. Thanks to this condition, Σ *does not contain the empty mapping*. (If the contrary, G would reduce to a single element!) To show this, let α and β be any two non-empty elements of Σ. By Lemma 1.21 (ii), $V(\alpha)$ and $U(\beta)$ are left ideals of S. But right reversibility is equivalent to the assertion that any two left ideals of S intersect. Hence $V(\alpha) \cap U(\beta) \neq \square$, and we conclude that $\alpha\beta$ is not empty. Since Σ is generated by the non-empty elements ρ_a, ρ_a^{-1} (a in S) of $\mathscr{I}_S$, no element of Σ can be empty.

Finally, we show that $(S\phi/\sim) \cong S$ by showing that $\rho_a \sim \rho_b$ implies $a = b$. If $\rho_a \sim \rho_b$ then there exists γ in Σ such that $\gamma \subseteq \rho_a$ and $\gamma \subseteq \rho_b$. Since Σ does not contain the empty mapping, $U(\gamma) \neq \square$. Let $x \in U(\gamma)$. Then $x\rho_a = x\rho_b$, that is, $xa = xb$, which implies $a = b$ since S is cancellative. The mapping sending each element a of S into the congruence class of Σ mod $\sim$ containing ρ_a is thus an isomorphism of S into the group G.

Although Ore's Theorem is phrased as a sufficient condition for embeddability in a group, it was noted by Dubreil ([1943] and his book, *Algèbre* (1954), p. 269) that right reversibility is nonetheless a necessary and sufficient condition that the manner of embedding be of the following simple type. We say that G is a *group of left quotients* of S if G is a group containing the semigroup S such that every element of G is expressible in the form $a^{-1}b$ with a and b in S.

THEOREM 1.24. *A cancellative semigroup S can be embedded in a group of left quotients of S if and only if it is right reversible.*

PROOF. Let G be a group of left quotients of S, and let $a, b \in S$. Then the element ab^{-1} of G must be expressible in the form $ab^{-1} = x^{-1}y$ for some x and y in S. Then $xa = yb \in Sa \cap Sb$, and S is right reversible.

Conversely, let S be right reversible. By Theorem 1.23, S can be embedded in a group G. Let G_1 be the set of all elements of G of the form $a^{-1}b$, with a and b in S. We proceed to show that G_1 is a subgroup of G, and it is evidently then a group of left quotients of S. If $a^{-1}b \in G_1$, then $(a^{-1}b)^{-1} = b^{-1}a \in G_1$, so that G_1 is closed under inverses. Let $a^{-1}b$ and $c^{-1}d$ be arbitrary elements of G_1 (a, b, c, d in S). By hypothesis, there exist x and y in S such that $xb = yc$. Then $bc^{-1} = x^{-1}y$ in G, and so

$$a^{-1}b \cdot c^{-1}d = a^{-1}x^{-1}yd = (xa)^{-1}(yd).$$

This lies in G_1, since xa and yd lie in S. Hence G_1 is a subgroup of G.

Our last theorem shows that the group of left quotients of an Ore semigroup S is uniquely determined by S.

THEOREM 1.25. *Let S be a right reversible cancellative semigroup. Let G $(\cdot)$ and G' $(\circ)$ be two groups of left quotients of S. Then there exists an isomorphism of G upon G' leaving the elements of S fixed.*

PROOF. In G, $a^{-1}b = c^{-1}d$ (a, b, c, d in S) if and only if the following holds: $xa = yc$ (x, y in S) if and only if $xb = yd$. Moreover, $(a^{-1}b) \cdot (c^{-1}d) = (xa)^{-1} \cdot (yd)$ where x and y are any elements of S such that $xb = yc$. But these same conditions for equality and product hold in G', so that $a^{-1}b \to a^{-1} \circ b$ is an isomorphism of G upon G' leaving the elements of S fixed.

If S is a cancellative commutative semigroup, then $Sa \cap Sb$ is never empty (a, b in S), since it contains ab $(= ba)$. The group of (left) quotients of S is easily seen to be commutative. We readily find that $a^{-1}b = c^{-1}d$ (a, b, c, d in S) if and only if $ad = bc$, and that $(a^{-1}b)(c^{-1}d) = (ac)^{-1}(bd)$.

EXERCISES FOR §1.10

1. Let S be the set of ordered pairs (i, j) of non-negative integers i, j, with product defined as follows:

$$(i, j)(k, l) = (i + k, 2^k j + l).$$

(We note that S is the semigroup generated by two elements p and q subject to the relation $qp = pq^2$; cf. §1.12.) Then S is left reversible but not right reversible. (For further examples, see Tamari [1948].)

2. Any commutative semigroup S can be embedded in a semigroup S^* with identity element such that (i) each cancellable element of S is a unit in S^*, and (ii) each element of S^* can be expressed in the form ab^{-1} with a and b in S^1, and b cancellable. Such a semigroup S^* must be commutative, and is unique to within isomorphism. (Vandiver [1940].)

3. Let S be a right reversible, cancellative semigroup, and let G be a group containing S as a subsemigroup, and generated by S. Then G is a group of left quotients of S. (P. F. Conrad, verbal remark.)

1.11 Right groups

We recall (§1.1) that a semigroup S is called *right* [*left*] *simple* if it contains no proper right [left] ideal, and that a group is just a semigroup that is both left and right simple. In this section we consider certain right simple semigroups, leaving the general theory to Chapter 8.

A semigroup S is called a *right group* if it is right simple and left cancellative. This is equivalent to saying that, for any elements a and b of S, there exists one and only one element x of S such that $ax = b$. *Left group* is defined dually.

If S and T are semigroups, then by the *direct product* of S and T we mean (as in group theory) the set $S \times T$ of all ordered pairs (s, t) of elements s of S and t of T, with product defined by

$$(s, t)(s', t') = (ss', tt')$$

for all s, s' in S and t, t' in T. Clearly the direct product of two right simple semigroups is right simple; for to solve $(a, b)(x, y) = (c, d)$ we need only solve $ax = c$ and $by = d$ separately. It is also clear that the direct product of two left cancellative semigroups is left cancellative, and hence the direct product of two right groups is a right group.

We recall (§1.1) that a semigroup E is called a *right zero semigroup* if every element of E is a right zero of E, i.e., $xy = y$ for all x, y in E. Evidently E is a right group.

Lemma 1.26. *Every idempotent element of a right simple semigroup S is a left identity element of S.*

Proof. Let e be an idempotent element of S, and let a be any element of S. Since S is right simple, there exists x in S such that $ex = a$. Then $ea = e^2x = ex = a$.

The following theorem completely determines the structure of all right groups, modulo that of all groups. It was found by Suschkewitsch [1928] for finite semigroups, and, as he remarks in his book [1937], Chapter 3, §43, the proof carries over to the general case. It was stated without proof by

Clifford [1933] for arbitrary semigroups; alternative characterizations of right groups from this paper are given in Exercise 1 below.

This theorem, or a variant thereof (such as in Exercise 1), has been rediscovered by Schwarz [1943], Mann [1944], Prachar [1947], Ballieu [1950], Skolem [1951], and Stolt [1956]. Other characterizations of a right group have been given by Tamura [1950], Hashimoto [1954], Thierrin [1954*a*], and Szép [1956].

THEOREM 1.27. *The following assertions concerning a semigroup S are equivalent:*

(i) *S is a right group.*

(ii) *S is right simple, and contains an idempotent.*

(iii) *S is the direct product $G \times E$ of a group G and a right zero semigroup E.*

PROOF. (i) *implies* (ii). A right group S is right simple by definition. Let $a \in S$. By right simplicity, there exists e in S such that $ae = a$. Hence $ae^2 = ae$, and, by left cancellation, $e^2 = e$.

(ii) *implies* (iii). Let E be the set of idempotents of S. By (ii), $E \neq \square$. By Lemma 1.26, *every element of E is a left identity element of S.* In particular, $ef = f$ for every e, f in E, and so E is a right zero subsemigroup of S.

We show next that *S is left cancellative*, thereby incidentally proving that (ii) implies (i). Let $ca = cb$ $(a, b, c$ in $S)$. Let $f \in E$, and solve $cx = f$ for x in S. Let $e = xc$. Then $e^2 = xcxc = xfc = xc = e$. Hence $a = ea = xca = xcb = eb = b$.

If $e \in E$, then Se is a subsemigroup of S with right (as well as left) identity element e. If $a \in Se$, we can solve $ax = e$ for x in S. But then $a(xe) = e^2 = e$, and so a has the right inverse xe in Se. Hence *Se is a subgroup of S.*

Let g be a fixed element of E, and let G be the group Sg. Form the direct product $G \times E$, and define the mapping ϕ of $G \times E$ into S by

$$(a, e)\phi = ae \quad (a \text{ in } G,\ e \text{ in } E).$$

Then, with a, b in G and e, f in E, we have

$$[(a, e)(b, f)]\phi = (ab, ef)\phi = (ab)(ef) = abf,$$

$$[(a, e)\phi][(b, f)\phi] = (ae)(bf) = a(eb)f = abf.$$

Hence ϕ is a homomorphism.

To see that ϕ is one-to-one, suppose $(a, e)\phi = (b, f)\phi$, that is, $ae = bf$ $(a, b$ in G; e, f in $E)$. Since g is the identity element of G, $a = ag = aeg = bfg = bg = b$. Hence $ae = af$. Since S is left cancellative, $e = f$.

Finally, to show that ϕ maps $G \times E$ upon S, let $a \in S$. Solve $ae = a$ for e in S. Then $ae^2 = ae$, and $e^2 = e$ by left cancellation; hence $e \in E$. Then $ag \in Sg = G$, and $(ag, e)\phi = age = ae = a$. Hence ϕ is an isomorphism of $G \times E$ upon S, and (iii) is established.

(iii) *implies* (i). Since the direct product of two right groups is a right group, and E and G are both right groups, it follows that $G \times E$ is a right group.

Considerable work has been done on axiomatics of groups. If A is a system of axioms including closure and associativity, then the proposition $P(A)$, asserting that any semigroup satisfying A is a group, or the negation of $P(A)$, logically belongs to the theory of semigroups. If $P(A)$ is false, the system A may possibly define an interesting class of semigroups. Questions of independence of axioms lead to assertions about the inclusion pattern of various classes of semigroups.

In practice, such axiomatic studies have seldom produced interesting material for the theory of semigroups. An exception, however, is provided by the work of Baer and Levi [1932], who investigated the logical relations among the following possible axioms on a semigroup S:

(RS) S is right simple.
(LS) S is left simple.
(RC) S is right cancellative.
(LC) S is left cancellative.

It is easy to see that the conjunction of any three of these implies that S is a group. We can classify their conjunctions in pairs as follows.

I. (RS) and (LS): S is a group.
II. (RC) and (LC): S is a cancellative semigroup.
III. (RS) and (LC): S is a right group.
(LS) and (RC): S is a left group.
IV. (RS) and (RC): ?
(LS) and (LC): ?

An example of a semigroup in Class IV was given by Baer and Levi in the paper cited. Semigroups of this type will be discussed in Chapter 8.

Exercises for §1.11

1. Consider the following possible conditions on a semigroup S.

I L [R] There exists a left identity element e of S such that $e \in Sa$ [$e \in aS$] for every element a of S.

II L [R] For each element a of S, Sa [aS] contains a left identity element of S.

I L is equivalent to the statement that S is a group, while I R, II L, and II R are each separately equivalent to the statement that S is a right group. (Clifford [1933].)

2. A semigroup S is a right group if and only if it is a union of disjoint groups such that the set of identity elements of the groups is a right zero subsemigroup of S.

3. A right group is the union of a set of isomorphic disjoint groups. If

e and f are distinct idempotents of a right group S, then $x \to xf$ (x in Se) is an isomorphism of the group Se upon the group Sf.

4. A semigroup S is a right group if and only if it is regular and left cancellative. (Munn, unpublished.)

5. A periodic semigroup S is a right group if and only if every idempotent element of S is a left identity element of S. (Suschkewitsch [1937], Chapter 3, §24.)

6. Let S be the set of non-zero complex numbers. Define a product ($\circ$) in S by $a \circ b = |a|b$ (a, b in S). Then S ($\circ$) is a right group. The idempotents of S are the complex numbers of modulus 1. Each maximal subgroup of S ($\circ$) is isomorphic with the multiplicative group of the positive real numbers. (E. T. Bell; see Clifford [1933], p. 871.)

Exercises 7–9 are taken from Helen Bradley Grimble [1950]. Let e be a fixed left identity element of a semigroup S. Let U be the set of *left divisors of* e ($a \in U$ if $e \in aS$). Let V be the set of *right divisors of* e ($a \in V$ if $e \in Sa$). Let W be the set of *interior divisors of* e ($a \in W$ if $e \in SaS$). An element a of S is called a *universal left* [*interior*] *divisor of* S if $aS = S$ [$SaS = S$]. A left [two-sided] ideal A of S is called *universally maximal* if $A \neq S$ and A contains every proper left [two-sided] ideal of S. An ideal A (of any type) is called *prime* if $S \setminus A$ is a subsemigroup of S.

7. (a) U consists of all universal left divisors of S. It is a subsemigroup of S containing all the left identity elements of S, and no further idempotents.

(b) If $U = S$, then S is a right group. If $U \neq S$, then $S \setminus U$ is a universally maximal right ideal of S, and is prime.

8. (a) V is a left-cancellative subsemigroup of S containing e and no further idempotents.

(b) $U \cap V = H_e$ (the maximal subgroup of S containing e).

(c) If $V = S$, then S is a group. If $V \neq S$, then $S \setminus V$ is a prime left ideal of S.

9. (a) W consists of all universal interior divisors of S.

(b) If $W = S$, then S is simple. If $W \neq S$, then $S \setminus W$ is a universally maximal two-sided ideal of S.

(c) $W = U$ if and only if $V \subseteq U$ if and only if $V = H_e$. This is the case, for example, if every element of V has finite order.

(d) $W = V$ if and only if $U \subseteq V$ if and only if $U = V$ if and only if $U = H_e$. In this case, e is the only left identity element of S.

1.12 FREE SEMIGROUPS AND GENERATING RELATIONS; THE BICYCLIC SEMIGROUP

Let X be any set, and let $\mathscr{F}_X$ consist of all finite sequences of elements of X. If $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_n)$ are elements of $\mathscr{F}_X$ ($x_i \in X$, $i = 1, \cdots, m$; $y_j \in X$, $j = 1, \cdots, n$) then we define their product by simple juxtaposition:

$$(x_1, \cdots, x_m)(y_1, \cdots, y_n) = (x_1, \cdots, x_m, y_1, \cdots, y_n).$$

$\mathscr{F}_X$ becomes thereby a semigroup; we call it the *free semigroup on* X. The elements of $\mathscr{F}_X$ will be called *words*. If we identify the element x of X with the sequence (x) of length 1, then, by the above definition of product,

$$(x_1, x_2, \cdots, x_m) = (x_1)(x_2)\cdots(x_m) = x_1x_2\cdots x_m.$$

Thus X is a set of generators of $\mathscr{F}_X$, and is clearly the only such not containing superfluous elements.

It is often convenient to work with $\mathscr{F}_X^1$ rather than $\mathscr{F}_X$. The adjoined identity element 1 may be regarded as the "empty word".

Suppose now that we wish to impose some "generating relations" among the elements of X, for example

$$x_1x_2 = x_3x_4^2, \qquad x_1^3 = x_4\, x_1x_2.$$

Let us say that we wish to impose the generating relations $u_\lambda = v_\lambda$ $(\lambda \in \Lambda)$, where, for each element λ of the index set Λ, u_λ and v_λ are elements of $\mathscr{F}_X$. Let $\rho_0 = \{(u_\lambda, v_\lambda) : \lambda \in \Lambda\}$, and let ρ be the congruence relation on $\mathscr{F}_X$ generated by ρ_0 (§1.5). Let $\rho^\natural$ be the natural homomorphism of $\mathscr{F}_X$ upon $\mathscr{F}_X/\rho$. Then the set $\{x\rho^\natural : x \in X\}$ generates $\mathscr{F}_X/\rho$, and $u_\lambda\rho^\natural = v_\lambda\rho^\natural$ for all λ in Λ, that is, the generators of $\mathscr{F}_X/\rho$ actually satisfy the generating equations. We call $\mathscr{F}_X/\rho$ *the semigroup generated by* X *subject to the generating relations* $u_\lambda = v_\lambda$ $(\lambda \in \Lambda)$, although it is really generated by the set $X\rho^\natural$.

LEMMA 1.28. *Let $\mathscr{F}_X$ be the free semigroup on a set X. Let S be any semigroup, and let ϕ_0 be any mapping of X into S. Then ϕ_0 can be extended in one and only one way to a homomorphism ϕ of $\mathscr{F}_X$ into S.*

PROOF. If ϕ is any homomorphism of $\mathscr{F}_X$ into S coinciding with ϕ_0 on X, then, for arbitrary $x_1, x_2, \cdots, x_n$ in X,

$$(x_1\, x_2 \cdots x_n)\phi = (x_1\phi_0)(x_2\phi_0)\cdots(x_n\phi_0).$$

Hence there can be at most one such ϕ. But the foregoing may be taken as the definition of a mapping ϕ of $\mathscr{F}_X$ into S which is clearly a homomorphism and extends ϕ_0.

THEOREM 1.29. *Let $\mathscr{F}_X$ be the free semigroup on a set X. Let ρ_0 be any relation on $\mathscr{F}_X$, and let ρ be the congruence relation on $\mathscr{F}_X$ generated by ρ_0. Let $\rho^\natural$ be the natural homomorphism of $\mathscr{F}_X$ upon $\mathscr{F}_X/\rho$. Let S be any semigroup, and let ϕ be a homomorphism of $\mathscr{F}_X$ into S such that $u\phi = v\phi$ for every (u, v) in ρ_0. Then there exists a homomorphism θ of $\mathscr{F}_X/\rho$ into S such that $\rho^\natural\theta = \phi$.*

PROOF. We show first that if w and w' are elements of $\mathscr{F}_X$ such that $w\,\rho\, w'$, then $w\phi = w'\phi$.

By hypothesis, $(u,v) \in \rho_0$ implies $u\phi = v\phi$, and hence $\rho_0 \subseteq \phi \circ \phi^{-1}$. Since

ρ is the smallest congruence on S containing ρ_0, and $\phi \circ \phi^{-1}$ is a congruence, we conclude that $\rho \subseteq \phi \circ \phi^{-1}$. Hence $(w, w') \in \rho$ implies $w\phi = w'\phi$.

We now define a mapping θ of $\mathscr{F}_X/\rho$ into S by $(w\rho^\natural)\theta = w\phi$, for every w in $\mathscr{F}_X$. That θ is single-valued follows from what we have just shown, that $w\rho^\natural = w'\rho^\natural$ (w, w' in $\mathscr{F}_X$), that is, $w \,\rho\, w'$, implies $w\phi = w'\phi$. The domain of θ is all of $\mathscr{F}_X/\rho$ since every element of $\mathscr{F}_X/\rho$ has the form $w\rho^\natural$ for some w in $\mathscr{F}_X$.

Since $\rho^\natural\theta = \phi$ is now evident, all that remains to be shown is that θ is a homomorphism. Let w and w' be any two elements of $\mathscr{F}_X$. Then

$$\begin{aligned}[(w\rho^\natural)(w'\rho^\natural)]\theta &= [(ww')\rho^\natural]\theta = (ww')\phi \\ &= (w\phi)(w'\phi) = [(w\rho^\natural)\theta][(w'\rho^\natural)\theta].\end{aligned}$$

Hence θ is a homomorphism.

Let us reformulate the foregoing theorem in more vivid, if less accurate, fashion.

Let $X = \{x_\tau : \tau \in \Omega\}$ and $\rho_0 = \{(u_\lambda, v_\lambda) : \lambda \in \Lambda\}$. Let $x_\tau\phi = a_\tau$. If $w = w(x)$ is any word in $\mathscr{F}_X$, then $w\phi = w(a)$ is obtained by "substituting the value a_τ in S for the variable x_τ" wherever it appears in the word w, and we do this for each x_τ that appears in w. We customarily write the generating relations in ρ_0 as equalities:

$$u_\lambda(x) = v_\lambda(x) \quad (\lambda \in \Lambda). \tag{1}$$

We think of these as "equations in the variables x_τ". To say that $u\phi = v\phi$ for every (u, v) in ρ_0 is to say that the system of values $\{a_\tau : \tau \in \Omega\}$ "satisfies these equations"; in other words, $u_\lambda(a) = v_\lambda(a)$, for every λ in Λ, is an actual equality in S.

Let $\Phi = \mathscr{F}_X/\rho$, and let $\xi_\tau = x_\tau\rho^\natural$. Then $\{\xi_\tau : \tau \in \Omega\}$ is a set of generators of Φ and this system satisfies (1).

If, in Theorem 1.29, ϕ maps $\mathscr{F}_X$ upon S, then θ maps Φ upon S. This is the case if and only if $\{a_\tau : \tau \in \Omega\}$ generates S. Let us then say that "S has a system of generators $\{a_\tau : \tau \in \Omega\}$ satisfying (1)". Theorem 1.29 makes precise the sense in which Φ is the "most general" semigroup having a system of generators satisfying (1).

COROLLARY 1.29a. *If S is any semigroup having a system of generators $\{a_\tau : \tau \in \Omega\}$ satisfying* (1), *then the mapping $\xi_\tau \to a_\tau$ (every τ in Ω) can be extended to a homomorphism θ of Φ upon S.*

It follows at once that no equality can hold among the ξ_τ in Φ that could not be deduced from (1) in any semigroup. For if $w(\xi) = w'(\xi)$, then we obtain $w(a) = w'(a)$ in S by applying θ. This holds just as well when $\{a_\tau : \tau \in \Omega\}$ does not generate S. On reversing this proposition, we infer the following corollary of Theorem 1.29, which we shall use in Example 2 below.

COROLLARY 1.30. *If we can exhibit a semigroup S with a system of elements $\{a_\tau : \tau \in \Omega\}$ satisfying* (1), *and for which $w(a) \neq w'(a)$, where $w, w' \in \mathscr{F}_X$, then we can deduce that $w(\xi) \neq w'(\xi)$ in Φ.*

It may be that we would like a certain word $w(x)$ to map into the identity element of $\mathscr{F}_X/\rho$. For this we include in ρ_0 the generating relations $(x_\tau w, x_\tau)$ and (wx_τ, x_τ), all τ in Ω. To avoid this nuisance, we may work with $\mathscr{F}^1_X$ instead of $\mathscr{F}_X$, and include the single relation $(w, 1)$ instead of the foregoing. This causes no essential difference in Lemma 1.28 or Theorem 1.29.

EXAMPLE 1. *The free group $\mathscr{F}\mathscr{G}_X$ on a set X.*

Let X' be a set disjoint from X and such that $|X'| = |X|$. Let $x \to x'$ be a fixed one-to-one mapping of X upon X'. Let $\mathscr{F}^1$ be the free semigroup with identity on $X \cup X'$, and let

$$\rho_0 = \{(xx', 1) : x \in X\} \cup \{(x'x, 1) : x \in X\}.$$

Let ρ be the congruence on $\mathscr{F}^1$ generated by ρ_0. Then we define $\mathscr{F}\mathscr{G}_X = \mathscr{F}^1/\rho$.

It is clear that $\mathscr{F}\mathscr{G}_X$ is a group, and that X generates $\mathscr{F}\mathscr{G}_X$ in the sense of group theory. Let G be any group, and let ϕ_1 be any mapping of X into G. Let ϕ_0 be the extension of ϕ_1 to $X \cup X'$ defined by $x\phi_1 = x\phi_0$ and $x'\phi_0 = (x\phi_1)^{-1}$ for all x in X. By Lemma 1.28 (modified to include the empty word), ϕ_0 can be extended to a homomorphism ϕ of $\mathscr{F}^1 = \mathscr{F}^1_{X \cup X'}$ into G. Then, for each x in X, $(xx')\phi = (x\phi)(x'\phi) = (x\phi)(x\phi)^{-1} = 1 = 1\phi$, and similarly $(x'x)\phi = 1\phi$. Hence, $u\phi = v\phi$ for every (u, v) in ρ_0. Applying Theorem 1.29 (similarly modified), we conclude that there exists a homomorphism θ of $\mathscr{F}^1/\rho = \mathscr{F}\mathscr{G}_X$ into G such that $\rho^\natural\theta = \phi$. We have thus shown that any mapping ϕ_1 of X into G can be extended to a homomorphism θ of $\mathscr{F}\mathscr{G}_X$ into G. This property justifies our calling $\mathscr{F}\mathscr{G}_X$ the free group on X.

EXAMPLE 2. *The bicyclic semigroup $\mathscr{C}$.*

This is a very useful semigroup in the theory of simple semigroups. We shall give more of its properties and applications in §2.7. It is the simplest member of an extensive class of semigroups which will be studied in Chapter 8 (bisimple inverse semigroups with identity element); it is also an "elementary inverse semigroup" (Exercise 12 of §1.9). It first appeared in print in Lyapin [1953*b*]. It was found independently by Rees and one of us prior to 1943 (but not published) as an example of a simple semigroup containing a non-primitive idempotent. In his unpublished thesis [1952], Olaf Andersen showed that every such semigroup contains a bicyclic subsemigroup (Theorem 2.54 below).

We define the *bicyclic semigroup* $\mathscr{C}$ to be the semigroup with identity element generated by a two-element set $X = \{x_1, x_2\}$ subject to the single generating relation $x_1x_2 = 1$. Here ρ_0 consists of the single pair $(x_1x_2, 1)$, and if, as above, we let ρ be the congruence on $\mathscr{F}^1_X$ generated by ρ_0, then $\mathscr{C} = \mathscr{F}^1_X/\rho$. $\mathscr{C}$ is of course actually generated by the congruence classes $p = x_1\rho^\natural$ and $q = x_2\rho^\natural$, which satisfy $pq = 1$, and we shall write $\mathscr{C} = \mathscr{C}(p, q)$.

We show first that $qp \neq 1$. By Corollary 1.30, it suffices to give an example of a semigroup S with identity element 1 containing two elements a and b such that $ab = 1$ but $ba \neq 1$.

Let S be the full transformation semigroup $\mathscr{T}_N$ on the set N of non-negative integers. Let the transformations α and β be defined as follows, where n denotes an arbitrary element of N:

$$n\alpha = n + 1; \tag{2}$$

$$n\beta = \begin{cases} 0 & \text{if } n = 0, \\ n - 1 & \text{if } n > 0. \end{cases} \tag{3}$$

Then $\alpha\beta = \iota$ where ι is the identity element of $\mathscr{T}_N$, but

$$n\beta\alpha = \begin{cases} 1 \text{ if } n = 0, \\ n \text{ if } n > 0. \end{cases}$$

Hence $\beta\alpha \neq \iota$ and we can conclude that $qp \neq 1$ in $\mathscr{C}$.

Every element of $\mathscr{C}$ is, of course, expressible as a product of p's and q's. By use of $pq = 1$, we can reduce any such expression to the form $q^m p^n$, where m and n are non-negative integers; we agree that $p^0 = q^0 = 1$. We wish to show that this expression is unique, that is, if $q^i p^j = q^m p^n$ for some non-negative integers i, j, m, n, then $i = m$ and $j = n$. We could do this by remarking that $\beta^i \alpha^j = \beta^m \alpha^n$ implies $i = m, j = n$, where α and β are the transformations of N defined by (2) and (3) above; see Exercise 1 below. It is, however, also a consequence of the following lemma, together with the result of $qp \neq 1$ established above. We shall need the added generality of this lemma later (§2.7).

LEMMA 1.31. *Let e, a, b be elements of a semigroup S such that $ea = ae = a$, $eb = be = b$, $ab = e$, $ba \neq e$. Then every element of the subsemigroup $\langle a, b\rangle$ of S generated by a and b is uniquely expressible in the form $b^m a^n$, with m and n non-negative integers (and $a^0 = b^0 = e$), and hence $\langle a, b\rangle$ is isomorphic with the bicyclic semigroup $\mathscr{C}$.*

PROOF. Clearly $e \in \langle a, b\rangle$, and e is the identity element of $\langle a, b\rangle$. By means of $ab = e$, every element of $\langle a, b\rangle$ is expressible in the form $b^m a^n$. All that needs proof is the uniqueness of m and n. We first prove three preliminary propositions.

(i) *The elements a and b must have infinite order.* Suppose, by way of contradiction, that $a^{h+k} = a^h$ for some positive integers h and k. Multiplying on the right by b^h, we obtain $a^k = e$. Then $b = eb = a^k b = a^{k-1} e = a^{k-1}$ and $ba = a^k = e$, contrary to the hypothesis $ba \neq e$. The proof that b has infinite order is similar.

(ii) *If $a^h = b^k$ for some non-negative integers h and k, then $h = k = 0$.* For then $a^{h+k} = a^k b^k = e$, whence $h + k = 0$ by (i).

(iii) *If* $b^h a^k = e$ *for some non-negative integers* h *and* k, *then* $h = k = 0$. If $k = 0$, then $h = 0$ by (i). We show that $k > 0$ is impossible. For if $k > 0$, then $b = eb = b^h a^k b = b^h a^{k-1}$, and $ba = b^h a^k = e$, contrary to hypothesis.

Now assume, by way of contradiction, that $b^m a^n = b^i a^j$, where m, n, i, j are non-negative integers such that either $i \neq m$ or $j \neq n$. We shall treat the case $i \neq m$, the other being similar. Without loss of generality, we can assume $i < m$. Multiplying on the left by a^i, we get $b^{m-i} a^n = a^j$. If $j \geqq n$, multiplication on the right by b^n gives $b^{m-i} = a^{j-n}$, with $m - i > 0$, contrary to (ii). If $j \leqq n$, multiplication on the right by b^j gives $b^{m-i} a^{n-j} = e$, with $m - i > 0$, contrary to (iii).

By Corollary 1.29a, the mapping $p \rightarrow a$, $q \rightarrow b$ induces a homomorphism θ of $\mathscr{C}(p, q)$ into S, and hence upon $\langle a, b \rangle$, namely $(q^m p^n)\theta = b^m a^n$. From the uniqueness just proved, it follows that θ is an isomorphism.

Corollary 1.32. *If* ϕ *is a homomorphism of a bicyclic semigroup* $\mathscr{C}$ *into a semigroup* S, *then either* ϕ *is an isomorphism of* $\mathscr{C}$ *into* S, *or else* $\mathscr{C}\phi$ *is a cyclic group.*

Proof. Let $\mathscr{C} = \mathscr{C}(p, q)$ with $pq = 1$, and let $a = p\phi$, $b = q\phi$, and $e = 1\phi$. Then e, a, b satisfy the hypotheses of Lemma 1.31, except possibly $ba \neq e$. If $ba \neq e$, then ϕ is an isomorphism; for if $(q^m p^n)\phi = (q^i p^j)\phi$, then $b^m a^n = b^i a^j$, whence $m = i$ and $n = j$ by Lemma 1.31. If $ba = e$, then $\mathscr{C}\phi$ is the cyclic group generated by a.

It is also a consequence of Lemma 1.31 that the subsemigroup $\langle \alpha, \beta \rangle$ of $\mathscr{T}_N$, where α and β are defined by (2) and (3) above, is isomorphic with $\mathscr{C}$. We could have used $\langle \alpha, \beta \rangle$ as our definition of $\mathscr{C}$.

By Lemma 1.31 and $qp \neq 1$, the mapping $(m, n) \rightarrow q^m p^n$ is one-to-one from $N \times N$ upon $\mathscr{C}(p, q)$. Noting Exercise 2 below, we can make $N \times N$ a semigroup by defining a product in $N \times N$ by

$$(k, l)(m, n) = (k + m - \min(l, m), l + n - \min(l, m)).$$

We could have used this to define $\mathscr{C}$; we must then, of course, check associativity.

The foregoing exemplifies three useful methods of constructing semigroups: (1) by generating relations imposed on a free semigroup; (2) by transformations of a set; (3) by ordered pairs.

Exercises for §1.12

1. If α and β are the transformations of the set N of non-negative integers defined by equations (2) and (3) of the text, then

$$k\beta^m\alpha^n = \begin{cases} n & \text{if } k \leqq m, \\ k - m + n & \text{if } k \geqq m, \end{cases}$$

for all k, m, n in N. From this we conclude that $\beta^i\alpha^j = \beta^m\alpha^n$ (i, j, m, n in N)

implies $i = m$ and $j = n$. The same conclusion then holds for the elements $q^m p^n$ of the bicyclic semigroup $\mathscr{C}(p, q)$ by Corollary 1.30.

2. The product of the elements $q^k p^l$ and $q^m p^n$ of $\mathscr{C}(p, q)$, (k, l, m, n in N), is $q^i p^j$, where $i = k + m - \min(l, m)$ and $j = l + n - \min(l, m)$.

3. The finite cyclic semigroup of index r and period m may be described as the semigroup generated by a symbol x subject to the one relation $x^{m+r} = x^r$.

4. Let $P = \langle p \rangle$ be an infinite cyclic semigroup. Then $\mathscr{C}(p, q)$ is isomorphic with the inverse hull (§1.9) of P^1.

5. An element a of a semigroup S is called a *left increasing element* of S if there exists a proper subset N of S such that $aN = S$.

(a) Let S be a semigroup with identity element. The left increasing elements of S are precisely the right units of S which are not left units.

(b) Let S be a semigroup with identity element, and let a be a left increasing element of S. Then there exists b in S such that $\langle a, b \rangle$ is a bicyclic subsemigroup of S. (Lyapin [1953c].)

http://dx.doi.org/10.1090/surv/007.1/02

CHAPTER 2

IDEALS AND RELATED CONCEPTS

Two elements of a semigroup S are said to be $\mathscr{L}$-equivalent if they generate the same principal left ideal of S. $\mathscr{R}$-equivalence is defined dually. The join of the equivalence relations $\mathscr{L}$ and $\mathscr{R}$ is denoted by $\mathscr{D}$ and their intersection by $\mathscr{H}$. These fundamental equivalence relations, definable in any semigroup, were first introduced and studied by Green [1951]. They have shed a great deal of light on the structure of semigroups in general.

In particular, they enable us to give a proof of the Rees Theorem (3.5) which is more enlightening than the original proof given by Rees [1940]. We shall show that a 0-simple semigroup is completely 0-simple if and only if it contains at least one 0-minimal left ideal and at least one 0-minimal right ideal (Theorem 2.48). Two non-zero elements of a completely 0-simple semigroup S are $\mathscr{L}$ [$\mathscr{R}$]-equivalent if and only if they belong to the same 0-minimal left [right] ideal of S. We develop the structure of completely 0-simple semigroups in §2.7 from this point of view.

2.1 Green's relations

All the results of this section except Theorem 2.4 are due to J. A. Green [1951]. An elementary discussion of relations in general was given in §1.4.

We define $a\mathscr{L}b$ (a, b in S) to mean that a and b generate the same principal left ideal of S. In other words, $\mathscr{L}$ is the subset of $S \times S$ consisting of all pairs (a, b) such that $a \cup Sa = b \cup Sb$. The latter is equivalent to $S^1a = S^1b$, where (as in §1.1 and throughout the book) we define S^1 to be S if S has an identity element, and otherwise to be S with an identity element 1 adjoined. Clearly $\mathscr{L}$ is an equivalence relation such that if $a\mathscr{L}b$ then $ac\mathscr{L}bc$ for all c in S, that is, $\mathscr{L}$ is a right congruence (§1.5). If $a\mathscr{L}b$, we say that a and b are *$\mathscr{L}$-equivalent*. By L_a we mean the set of all elements of S which are $\mathscr{L}$-equivalent to a (a in S), that is, L_a is the equivalence class mod $\mathscr{L}$ containing a; we call L_a the *$\mathscr{L}$-class* containing a.

Dually we define $a\mathscr{R}b$ to mean $aS^1 = bS^1$ (a, b in S), and note that $\mathscr{R}$ is a left congruence on S. By R_a we mean the equivalence class of S mod $\mathscr{R}$, or the *$\mathscr{R}$-class*, containing a.

Lemma 2.1. *The relations $\mathscr{L}$ and $\mathscr{R}$ commute, and so the relation $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ is the smallest equivalence relation $\mathscr{L} \vee \mathscr{R}$ containing both $\mathscr{L}$ and $\mathscr{R}$.*

PROOF. Once we show that $\mathscr{L}$ and $\mathscr{R}$ commute, the rest of the lemma is immediate from Lemma 1.4. For this it suffices to show that $\mathscr{L} \circ \mathscr{R} \subseteq \mathscr{R} \circ \mathscr{L}$ (Exercise 3 of §1.4). Let a and b be elements of a semigroup S such that $a(\mathscr{L} \circ \mathscr{R})b$. By definition of the product of relations (§1.4), this means that there exists c in S such that $a\mathscr{L}c$ and $c\mathscr{R}b$. By definition of $\mathscr{L}$ and $\mathscr{R}$, these imply that there exist u and v in S^1 such that $a = uc$ and $b = cv$. Let $d = av = ucv = ub$. Since $\mathscr{L}$ is a right congruence, $a\mathscr{L}c$ implies $av\mathscr{L}cv$, that is, $d\mathscr{L}b$. Since $\mathscr{R}$ is a left congruence, $c\mathscr{R}b$ implies $uc\mathscr{R}ub$, that is, $a\mathscr{R}d$. From $a\mathscr{R}d$ and $d\mathscr{L}b$, and the evident fact that $d \in S$, we infer that $a(\mathscr{R} \circ \mathscr{L})b$. Hence, $\mathscr{L} \circ \mathscr{R} \subseteq \mathscr{R} \circ \mathscr{L}$, which concludes the proof of the lemma.

The $\mathscr{D}$-class containing an element a of S will be denoted by D_a.

We define $a\mathscr{J}b$ (a, b in S) to mean $S^1aS^1 = S^1bS^1$, that is, a and b are *$\mathscr{J}$-equivalent* if and only if they generate the same two-sided principal ideal. It is evident that $\mathscr{J}$ is an equivalence relation. Green denoted this relation by $\mathscr{F}$, but we have changed it to $\mathscr{J}$ because a number of papers on topological semigroups have used the notation $J(a) = S^1aS^1$ and J_a for the set of all generators of S^1aS^1, that is, for the $\mathscr{J}$-class containing a. We shall do likewise. Since $\mathscr{L} \subseteq \mathscr{J}$ and $\mathscr{R} \subseteq \mathscr{J}$, we have $\mathscr{D} \subseteq \mathscr{J}$; in general, $\mathscr{D} \neq \mathscr{J}$.

Finally, we define $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$. Naturally, $\mathscr{H}$ is an equivalence relation. We denote the $\mathscr{H}$-class containing a by H_a. Clearly $H_a = R_a \cap L_a$.

We remark that *if R is an $\mathscr{R}$-class of a semigroup S, and L is an $\mathscr{L}$-class of S, then R meets L if and only if they are both contained in the same $\mathscr{D}$-class of S.* For, let $a \in R$ and $b \in L$. Then $a\mathscr{D}b$ if and only if there exists c in S such that $a\mathscr{R}c$ and $c\mathscr{L}b$. But this condition on c is equivalent to $c \in R$ and $c \in L$, that is, to $c \in R \cap L$. Hence $a\mathscr{D}b$ if and only if $R \cap L \neq \square$. On the other hand, it is clear that $a\mathscr{D}b$ if and only if the $\mathscr{D}$-classes containing R and L coincide.

It is a great help to visualize a $\mathscr{D}$-class D of a semigroup S in the following way, which we call the *egg-box picture*. Imagine the elements of D arranged in a rectangular pattern, like an egg-box, the rows corresponding to the $\mathscr{R}$-classes and the columns to the $\mathscr{L}$-classes contained in D. Each cell of the egg-box corresponds to an $\mathscr{H}$-class contained in D, and the foregoing remark shows that no cell is empty. We do not arrange the elements in the $\mathscr{H}$-classes in any particular way. As we shall see presently, the $\mathscr{H}$-classes contained in D all have the same cardinal number; thus the cells of the egg-box are, so to speak, equally full of elements of S.

The egg-boxes ($\mathscr{D}$-classes) themselves may be thought of as strung together by opposite vertices, as in the diagrams given in the next section.

If a and b are elements of a semigroup S, we shall write $J_a \leqq J_b$ if $S^1aS^1 \subseteq S^1bS^1$, that is, if $a \in J(b)$. The relation $\leqq$ is a partial ordering of the set of $\mathscr{J}$-classes of S.

We note that a semigroup is left [right] simple if and only if it consists of a single $\mathscr{L}$ [$\mathscr{R}$] class, and that S is simple if and only if it consists of a single

$\mathscr{J}$-class. We say that a semigroup S is *$\mathscr{D}$-simple* or *bisimple* if it consists of a single $\mathscr{D}$-class. Since $\mathscr{D} \subseteq \mathscr{J}$, every bisimple semigroup is also simple. Exercise 10 below shows that not every simple semigroup is bisimple, and this of course shows that $\mathscr{D} \neq \mathscr{J}$ in general. Since $\mathscr{L} \subseteq \mathscr{D}$ and $\mathscr{R} \subseteq \mathscr{D}$, every left simple and every right simple semigroup is also bisimple.

LEMMA 2.2 (Green). *Let a and b be $\mathscr{R}$-equivalent elements of a semigroup S, and let s and s' be elements of S^1 such that $as = b$ and $bs' = a$. (Such elements s and s' must exist.) Then the mappings $x \to xs$ ($x \in L_a$) and $y \to ys'$ ($y \in L_b$) are mutually inverse, $\mathscr{R}$-class preserving, one-to-one mappings of L_a upon L_b, and of L_b upon L_a, respectively.*

PROOF. Denote the two mappings by σ and σ'. We note that σ [σ'] is the inner right translation ρ_s [$\rho_{s'}$] restricted to L_a [L_b].

Let $x \in L_a$. Since $\mathscr{L}$ is a right congruence, $x\mathscr{L}a$ implies $xs\mathscr{L}as = b$, so that $xs \in L_b$. Thus σ maps L_a into L_b, and similarly σ' maps L_b into L_a.

Again let $x \in L_a$. Then there exists t in S^1 such that $x = ta$, and so

$$x\sigma\sigma' = xss' = tass' = tbs' = ta = x.$$

Thus $\sigma\sigma'$ is the identity transformation on L_a. Similarly, $\sigma'\sigma$ is the identity transformation on L_b, and so σ and σ' are mutually inverse, one-to-one mappings of L_a and L_b upon each other.

To see that σ is $\mathscr{R}$-class preserving, we note that if $x \in L_a$ and $y = x\sigma = xs$, then $ys' = x$, so that $y\mathscr{R}x$. Similarly, σ' is also $\mathscr{R}$-class preserving.

THEOREM 2.3. *Let a and c be $\mathscr{D}$-equivalent elements of a semigroup S. Then there exists b in S such that $a\mathscr{R}b$ and $b\mathscr{L}c$, and hence $as = b$, $bs' = a$, $tb = c$, $t'c = b$, for some s, s', t, t' in S^1. The mappings $x \to txs$ ($x \in H_a$) and $z \to t'zs'$ ($z \in H_c$) are mutually inverse, one-to-one mappings of H_a and H_c upon each other. It follows that any two $\mathscr{H}$-classes contained in the same $\mathscr{D}$-class have the same cardinal number.*

PROOF. By the dual of Green's Lemma, the mappings $\tau : y \to ty$ ($y \in R_b$) and $\tau' : z \to t'z$ ($z \in R_c$) are mutually inverse, $\mathscr{L}$-class preserving, one-to-one mappings of R_b and R_c upon each other.

Let σ and σ' be as in Green's Lemma, but restricted to H_a and H_b, respectively. (Since the unrestricted σ and σ' are $\mathscr{R}$-class preserving, they map H_a and H_b upon each other in one-to-one fashion.) Similarly, let τ and τ' be restricted to H_b and H_c, respectively. Then $\sigma\tau$ and $\tau'\sigma'$ are mutually inverse, one-to-one mappings of H_a and H_c upon each other. But these are just the mappings defined in the statement of the theorem.

All the results of this section so far are due to Green [1951]. We conclude with a result due to Miller and Clifford [1956], which we shall need later.

THEOREM 2.4. *The set product LR of any $\mathscr{L}$-class L and any $\mathscr{R}$-class R of a semigroup S is always contained in a single $\mathscr{D}$-class of S.*

PROOF. The statement of the theorem is equivalent to asserting that if a, a', b, b' are elements of a semigroup S such that $a\mathscr{L}a'$ and $b\mathscr{R}b'$, then $ab\mathscr{D}a'b'$. Since $\mathscr{L}$ is a right congruence, $a\mathscr{L}a'$ implies $ab\mathscr{L}a'b$. Since $\mathscr{R}$ is a left congruence, $b\mathscr{R}b'$ implies $a'b\mathscr{R}a'b'$. But $ab\mathscr{L}a'b$ and $a'b\mathscr{R}a'b'$ imply $ab\mathscr{D}a'b'$.

EXERCISES FOR §2.1

1. Any two elements of a subgroup of a semigroup are $\mathscr{H}$-equivalent.
2. Each $\mathscr{L}$-class of a right group S (§1.11) is also an $\mathscr{H}$-class, and is a subgroup of S.
3. If S is a right cancellative semigroup without idempotents, then every $\mathscr{L}$-class of S consists of a single element.
4. Let S be a right simple semigroup. If S is left cancellative, then every $\mathscr{H}$-class of S is a group. If S is right cancellative, then either (i) S is a group, or (ii) S has no idempotent, and every $\mathscr{H}$-class of S consists of a single element.
5. A rectangular band (§1.8) is bisimple. Its rows are its $\mathscr{R}$-classes, and its columns are its $\mathscr{L}$-classes. Each $\mathscr{H}$-class consists of a single element.
6. The bicyclic semigroup $\mathscr{C}(p, q)$ (§1.12) is bisimple. If we write the elements of $\mathscr{C}$ in the array

$$\begin{array}{cccc} 1 & p & p^2 & \cdots \\ q & qp & qp^2 & \cdots \\ q^2 & q^2p & q^2p^2 & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

then the $\mathscr{R}$- [$\mathscr{L}$-] classes of $\mathscr{C}$ are the rows [columns] of the array. Each $\mathscr{H}$-class consists of a single element.

7. Let S be a semigroup with identity element 1. The right [left] unit subsemigroup P [Q] of S (§1.7) consists of all elements $\mathscr{R}$- [$\mathscr{L}$-] equivalent to 1. The semigroup S is bisimple if and only if $S = QP$. (Clifford [1953].)

Exercises 8–10 are taken from Andersen [1952].

8. Let G be a set in which two binary operations $(+)$ and $(\cdot)$ are defined such that the following axioms hold:

I. $G(+)$ is a semigroup, and $G(\cdot)$ is a group.
II. $(a + b)c = ac + bc$.

Then: (a) $G + G = G$.

(b) If $G(+)$ contains an idempotent, every element of $G(+)$ is idempotent.

Example 1. Let G be the positive real numbers with the usual $(+)$ and $(\cdot)$ (or more generally, let G be the positive part of any ordered field).

Example 2. Let G be the positive rationals, with $(\cdot)$ meaning ordinary multiplication, and $(+)$ meaning least common multiple (or, more generally, let G be any lattice-ordered group).

9. Let $A = G \times G$, and define a product in A as follows:

$$(a, b)(c, d) = (ac, bc + d), \quad (a, b, c, d \text{ in } G).$$

Then:

(a) A is a simple semigroup.

(b) The set of all elements of A of the form $(1, b)$, b in G, where 1 is the identity element of $G(\cdot)$, is a subsemigroup B of A isomorphic with $G(+)$.

(c) If $G(+)$ has no idempotents, neither does A. If $G(+)$ has idempotents, then B is the set of idempotents of A, and A is regular.

(d) $(a, b)\mathscr{L}(c, d)$ in A implies $b\mathscr{L}d$ in $G(+)$; $(a, b)\mathscr{R}(c, d)$ in A implies $ba^{-1}\mathscr{R}dc^{-1}$ in $G(+)$. The converse of the first [second] implication holds if $b \neq d[ba^{-1} \neq dc^{-1}]$.

10. (a) Let $G(+)$ be a cancellative semigroup without identity element (e.g., as in Example 1). Then A is a cancellative (simple) semigroup without identity element.

(b) Under the assumptions in (a), every $\mathscr{D}$-class of A consists of a single element. (Thus A is a simple semigroup which is as far from being bisimple as any semigroup could be.)

(If G is as in Example 1, A is a simple semigroup which can be embedded in a group. Does there exist a cancellative simple semigroup which can not be embedded in a group?)

11. Any cancellative simple semigroup containing an idempotent is a group.

12. Lemma 2.1 is an immediate corollary of Lemma 2.2.

2.2 $\mathscr{D}$-STRUCTURE OF THE FULL TRANSFORMATION SEMIGROUP $\mathscr{T}_X$ ON A SET X

The purpose of the present section is to illustrate the concepts introduced in the previous section, using the semigroup $\mathscr{T}_X$. The results of this section culminating in Theorem 2.9 are due to D. D. Miller and C. G. Doss [1955]. Lemmas 2.5 and 2.6 are also given by Suschkewitsch in [1937], Chapter 3, §31. A similar example is provided by the multiplicative semigroup $\mathscr{L}\mathscr{T}(V)$ of all linear transformations of a vector space V; this is presented in Exercise 6 below.

With each element α of $\mathscr{T}_X$ we associate two things: (1) the *range* $X\alpha$ of α, and (2) the *partition* $\pi_\alpha = \alpha \circ \alpha^{-1}$ of X corresponding to α, i.e., the equivalence relation on X defined by $x\pi_\alpha y$ (x, y in X) if $x\alpha = y\alpha$ (§1.4).

Let $\pi_\alpha^\natural$ be the natural mapping of X upon the set X/π_α of equivalence classes of X mod π_α. Then $x\pi_\alpha^\natural \rightarrow x\alpha$ is a one-to-one mapping of X/π_α upon

$X\alpha$. It follows that $|X/\pi_\alpha| = |X\alpha|$, and this cardinal number is called the *rank* of α.

If $y \in X$ and $\alpha \in \mathscr{T}_X$, we define $y\alpha^{-1}$ to be the set of all x in X such that $x\alpha = y$.

LEMMA 2.5. *Given α and β in $\mathscr{T}_X$, there exists ξ in $\mathscr{T}_X$ such that $\xi\alpha = \beta$ if and only if $X\alpha \supseteq X\beta$. Hence $\alpha\mathscr{L}\beta$ if and only if $X\alpha = X\beta$.*

PROOF. If $\xi\alpha = \beta$, then $X\beta = (X\xi)\alpha \subseteq X\alpha$. Conversely, assume $X\beta \subseteq X\alpha$. Define a transformation ξ of X as follows: for each y in $X\beta$, let ξ map all the elements of the set $y\beta^{-1}$ upon a single element in $y\alpha^{-1}$. Then $\xi\alpha = \beta$.

LEMMA 2.6. *Given α and β in $\mathscr{T}_X$, there exists ξ in $\mathscr{T}_X$ such that $\alpha\xi = \beta$ if and only if $\pi_\alpha \subseteq \pi_\beta$. Hence $\alpha\mathscr{R}\beta$ if and only if $\pi_\alpha = \pi_\beta$.*

PROOF. If $\alpha\xi = \beta$ and $x\pi_\alpha y$, then $x\beta = x\alpha\xi = y\alpha\xi = y\beta$, and so $x\pi_\beta y$; thus $\alpha\xi = \beta$ implies $\pi_\alpha \subseteq \pi_\beta$. Conversely, assume that $\pi_\alpha \subseteq \pi_\beta$. Define ξ on $X\alpha$ by $x\alpha\xi = x\beta$ (x in X). To see that ξ is single-valued on X, suppose $x\alpha = y\alpha$. Then $x\beta = y\beta$ by hypothesis. Define ξ to be the identity on $X \setminus X\alpha$. Then $\alpha\xi = \beta$.

LEMMA 2.7. *Let π be a partition of X, and let Y be a subset of X such that $|X/\pi| = |Y|$. Then there exists α in $\mathscr{T}_X$ such that $\pi_\alpha = \pi$ and $X\alpha = Y$.*

PROOF. Since $|X/\pi| = |Y|$, there exists a one-to-one mapping ϕ of X/π upon Y. Then $\alpha = \pi^\natural\phi$ has the desired properties.

LEMMA 2.8. *Two elements of $\mathscr{T}_X$ are $\mathscr{D}$-equivalent if and only if they have the same rank.*

PROOF. Let $\alpha, \beta \in \mathscr{T}_X$. If $\alpha\mathscr{D}\beta$, then $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$ for some γ in $\mathscr{T}_X$. By Lemma 2.5, α and γ have the same range, and hence the same rank. By Lemma 2.6, γ and β have the same partition, and hence the same rank.

Conversely, assume that α and β have the same rank. Then $|X\alpha| = |X/\pi_\beta|$. By Lemma 2.7, there exists γ in $\mathscr{T}_X$ such that $X\gamma = X\alpha$ and $\pi_\gamma = \pi_\beta$. By Lemmas 2.5 and 2.6, $\alpha\mathscr{L}\gamma$ and $\gamma\mathscr{R}\beta$, whence $\alpha\mathscr{D}\beta$.

THEOREM 2.9. *Let $\mathscr{T}_X$ be the full transformation semigroup on a set X.*

(i) *In the semigroup $\mathscr{T}_X$, we have $\mathscr{D} = \mathscr{J}$.*

(ii) *There is a one-to-one correspondence between the set of all principal ideals of $\mathscr{T}_X$ and the set of all cardinal numbers $r \leqq |X|$ such that the principal ideal corresponding to r consists of all elements of $\mathscr{T}_X$ of rank $\leqq r$.*

(iii) *There is a one-to-one correspondence between the set of all $\mathscr{D}$-classes of $\mathscr{T}_X$ and the set of all cardinal numbers $r \leqq |X|$ such that the $\mathscr{D}$-class D_r corresponding to r consists of all elements of $\mathscr{T}_X$ of rank r.*

(iv) *Let r be a cardinal number $\leqq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{L}$-classes in D_r and the set of all subsets Y of X*

of cardinal r such that the $\mathscr{L}$-class corresponding to Y consists of all elements of $\mathscr{T}_X$ having range Y.

(v) *Let r be a cardinal number $\leqq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{R}$-classes contained in D_r and the set of all partitions π of X for which $|X/\pi| = r$ such that the $\mathscr{R}$-class corresponding to π consists of all elements of $\mathscr{T}_X$ having partition π.*

(vi) *Let r be a cardinal number $\leqq |X|$. There is a one-to-one correspondence between the set of all $\mathscr{H}$-classes in D_r and the set of all pairs (π, Y), where π is a partition of X and Y is a subset of X such that $|X/\pi| = |Y| = r$, such that the $\mathscr{H}$-class corresponding to (π, Y) consists of all elements of $\mathscr{T}_X$ having partition π and range Y.*

PROOF. Let $\alpha, \beta \in \mathscr{T}_X$. We show first that $\beta \in J(\alpha)$ if and only if rank of $\beta \leqq$ rank of α. If $\beta \in J(\alpha)$, then $\beta = \xi\alpha\eta$ for some ξ and η in $\mathscr{T}_X$, and so $|X\beta| = |X\xi\alpha\eta| \leqq |X\alpha\eta| \leqq |X\alpha|$. Conversely, assume rank of $\beta \leqq$ rank of α. Let Y be any subset of X of cardinal $|X\alpha|$ containing $X\beta$, and let γ be any element of $\mathscr{T}_X$ having range Y. Since $|X\alpha| = |Y| = |X\gamma|$, it follows from Lemma 2.8 that $\gamma\mathscr{D}\alpha$. Since $\mathscr{D} \subseteq \mathscr{J}$, we have $J(\gamma) = J(\alpha)$. From $X\gamma \supseteq X\beta$ and Lemma 2.5, there exists ξ in $\mathscr{T}_X$ such that $\xi\gamma = \beta$. Hence, $\beta \in J(\gamma) = J(\alpha)$.

It follows that $J(\alpha) = J(\beta)$, that is, $\alpha\mathscr{J}\beta$, if and only if α and β have the same rank. By Lemma 2.8, $\alpha\mathscr{J}\beta$ if and only if $\alpha\mathscr{D}\beta$, which establishes (i).

From the foregoing, and the fact that $\mathscr{T}_X$ contains elements of arbitrary rank $\leqq |X|$, it follows that the mapping $J(\alpha) \to$ (rank of α) is a one-to-one mapping of the set of principal ideals of $\mathscr{T}_X$ upon the set of all cardinals $\leqq |X|$ with the property asserted in (ii).

Similarly, it follows from Lemma 2.8 that the mapping $D_\alpha \to$ (rank of α), α in $\mathscr{T}_X$, is a one-to-one mapping of the set of $\mathscr{D}$-classes of $\mathscr{T}_X$ upon the set of all cardinals $\leqq |X|$ with the property asserted in (iii).

Now let r be a cardinal $\leqq |X|$, and D_r the $\mathscr{D}$-class of $\mathscr{T}_X$ consisting of all elements of $\mathscr{T}_X$ of rank r. If Y is a subset of X such that $|Y| = r$, then $\mathscr{T}_X$ evidently contains an element α such that $X\alpha = Y$, and clearly $\alpha \in D_r$. Similarly, if π is a partition of X such that $|X/\pi| = r$, then D_r contains an element β such that $\pi_\beta = \pi$. Parts (iv), (v), and (vi) of the theorem are immediate consequences of these remarks, together with Lemmas 2.5, 2.6, and 2.7.

We now proceed to locate the idempotent elements of $\mathscr{T}_X$. Part (ii) of the next theorem shows that an $\mathscr{H}$-class of $\mathscr{T}_X$ is a subgroup of $\mathscr{T}_X$ if (and evidently only if) it contains an idempotent. We shall see that this is true for any semigroup (Green's Theorem 2.16). Hence there can be at most one idempotent in an $\mathscr{H}$-class. Part (i) of the next theorem shows just which $\mathscr{H}$-classes of $\mathscr{T}_X$ contain idempotents.

THEOREM 2.10. *Let Y be a subset of the set X, and let π be a partition of X such that $|Y| = |X/\pi|$. Let H be the $\mathscr{H}$-class of $\mathscr{T}_X$ determined by the pair (π, Y) as in Theorem* 2.9 (vi).

(i) *H contains an idempotent if and only if Y meets each equivalence class of X* mod π *in exactly one element (i.e., Y is a "cross-section" of π).*

(ii) *If H contains an idempotent, then H induces and is isomorphic with the symmetric group $\mathscr{G}_Y$ on Y.*

PROOF. (i) Let ϵ be an idempotent element of H. Thus $Y = X\epsilon$, $\pi = \pi_\epsilon$, and $\epsilon^2 = \epsilon$. The mapping ϵ leaves every element of Y fixed, and moves every element of $X \setminus Y$. Let $x \in X$. Since $x\epsilon = (x\epsilon)\epsilon$ it follows from $\pi = \pi_\epsilon$ that $x\pi x\epsilon$. On the other hand, if y and y' are elements of Y such that $y\pi y'$, then $y = y\epsilon = y'\epsilon = y'$. Hence each equivalence class of X mod π contains exactly one element of Y, and ϵ maps every element of $y\pi^\natural$ (y in Y) upon y.

Conversely, assume that Y is a cross-section of π. Then the element of $\mathscr{T}_X$ which maps each element x of X upon the element y of Y such that $x\pi y$ is clearly an idempotent element of H.

(ii) Assume that H contains an idempotent ϵ. Let $\alpha \in H$. For each x in X, $x\alpha \in X\alpha = Y = X\epsilon$, and so $x\alpha\epsilon = x\alpha$; this implies that $\alpha\epsilon = \alpha$. For each x in X, $x\pi x\epsilon$ (shown above), and since $\pi_\alpha = \pi = \pi_\epsilon$, we have $x\alpha = (x\epsilon)\alpha$; this implies that $\epsilon\alpha = \alpha$.

We now show that α induces a permutation of Y. If $y\alpha = y'\alpha$ (y, y' in Y) then $y\pi y'$, and so $y = y'$. Given y in $Y = X\alpha$, there exists x in X such that $x\alpha = y$. Then $(x\epsilon) \in Y$ and $(x\epsilon)\alpha = x\alpha = y$. Hence, $(\alpha|Y) \in \mathscr{G}_Y$.

Every element ϕ of $\mathscr{G}_Y$ is induced by some element α of H, namely that defined by $x\alpha = (x\epsilon)\phi$. Moreover, α is uniquely determined by ϕ. For if $y\alpha = y\beta$ for all y in Y, with α and β in H, then $x\epsilon\alpha = x\epsilon\beta$ for all x in X, so that $\alpha = \epsilon\alpha = \epsilon\beta = \beta$. Hence the mapping $\alpha \to \phi = \alpha|Y$ is a one-to-one mapping of H upon $\mathscr{G}_Y$, evidently an isomorphism. Hence H is a subgroup of $\mathscr{T}_X$ isomorphic with $\mathscr{G}_Y$.

As an example, we write out all the $\mathscr{D}$-classes of $\mathscr{T}_4$ ($\mathscr{T}_X$ with $|X| = 4$). Let $X = \{1, 2, 3, 4\}$. We shall write $(i\,j\,k\,l)$ for the mapping $1 \to i$, $2 \to j$, $3 \to k$, $4 \to l$. There are four $\mathscr{D}$-classes D_r ($r = 1, 2, 3, 4$), where D_r is the set of all elements of rank r. The headings for the rows are partitions of $\{1, 2, 3, 4\}$; those for the columns are subsets of $\{1, 2, 3, 4\}$. We omit D_4, which consists of a single $\mathscr{H}$-class; it is just the symmetric group of degree 4 (order 24) on $\{1, 2, 3, 4\}$. Starred elements are idempotent; these show which cells are groups. Table 4 gives the whole $\mathscr{D}$-picture of $\mathscr{T}_4$, the numbers 1, 2, 6, 24 giving merely the number of elements in each $\mathscr{H}$-class.

TABLE 1

D_1	{1}	{2}	{3}	{4}
{1234}	(1111)*	(2222)*	(3333)*	(4444)*

TABLE 2

D_2	{12}	{13}	{14}	{23}	{24}	{34}
{1} {234}	(1222)* (2111)	(1333)* (3111)	(1444)* (4111)	(2333) (3222)	(2444) (4222)	(3444) (4333)
{2} {134}	(1211)* (2122)	(1311) (3133)	(1411) (4144)	(2322) (3233)*	(2422) (4244)*	(3433) (4344)
{3} {124}	(1121) (2212)	(1131)* (3313)	(1141) (4414)	(2232)* (3323)	(2242) (4424)	(3343) (4434)*
{4} {123}	(1112) (2221)	(1113) (3331)	(1114)* (4441)	(2223) (3332)	(2224)* (4442)	(3334)* (4443)
{12} {34}	(1122) (2211)	(1133)* (3311)	(1144)* (4411)	(2233)* (3322)	(2244)* (4422)	(3344) (4433)
{13} {24}	(1212)* (2121)	(1313) (3131)	(1414)* (4141)	(2323) (3232)*	(2424) (4242)	(3434)* (4343)
{23} {14}	(1221)* (2112)	(1331)* (3113)	(1441) (4114)	(2332) (3223)	(2442) (4224)*	(3443) (4334)*

TABLE 3

D_3	{123}	{124}	{134}	{234}
{1} {2} {34}	(1233)* (2133) (2311) (3211) (3122) (1322)	(1244)* (2144) (2411) (4211) (4122) (1422)	(1344) (3144) (3411) (4311) (4133) (1433)	(2344) (3244) (3422) (4322) (4233) (2433)
{1} {3} {24}	(1232)* (2131) (2313) (3212) (3121) (1323)	(1242) (2141) (2414) (4212) (4121) (1424)	(1343) (3141) (3414) (4313) (4131) (1434)*	(2343) (3242) (3424) (4323) (4232) (2434)
{1} {4} {23}	(1223) (2113) (2331) (3221) (3112) (1332)	(1224)* (2114) (2441) (4221) (4112) (1442)	(1334)* (3114) (3441) (4331) (4113) (1443)	(2334) (3224) (3442) (4332) (4223) (2443)
{2} {3} {14}	(1231)* (2132) (2312) (3213) (3123) (1321)	(1241) (2142) (2412) (4214) (4124) (1421)	(1341) (3143) (3413) (4314) (4134) (1431)	(2342) (3243) (3423) (4324) (4234)* (2432)
{2} {4} {13}	(1213) (2123) (2321) (3231) (3132) (1312)	(1214)* (2124) (2421) (4241) (4142) (1412)	(1314) (3134) (3431) (4341) (4143) (1413)	(2324) (3234)* (3432) (4342) (4243) (2423)
{3} {4} {12}	(1123) (2213) (2231) (3321) (3312) (1132)	(1124) (2214) (2241) (4421) (4412) (1142)	(1134)* (3314) (3341) (4431) (4413) (1143)	(2234)* (3324) (3342) (4432) (4423) (2243)

TABLE 4

"EGG-BOX PICTURE" OF $\mathscr{T}_4$ — NUMBER OF ELEMENTS

D_4	24															$1 \cdot 1 \cdot 24 = 24$
		6	6	6	6											
		6	6	6	6											
		6	6	6	6											$6 \cdot 4 \cdot 6 = 144$
D_3		6	6	6	6											
		6	6	6	6											
		6	6	6	6											
						2	2	2	2	2	2					
						2	2	2	2	2	2					
						2	2	2	2	2	2					$7 \cdot 6 \cdot 2 = 84$
D_2						2	2	2	2	2	2					
						2	2	2	2	2	2					
						2	2	2	2	2	2					
						2	2	2	2	2	2					
D_1												1	1	1	1	$1 \cdot 4 \cdot 1 = 4$
															Total	$4^4 = 256$

EXERCISES FOR §2.2

1. (a) Let Y be a non-empty subset of a set X. There exists in $\mathscr{T}_X$ at least one projection of X upon Y, that is, a mapping ϵ of X upon Y leaving each element of Y fixed. The idempotent elements in the $\mathscr{L}$-class of $\mathscr{T}_X$ corresponding to Y by Theorem 2.9 (iv) are just the projections of X upon Y.

(b) Let π be any partition of X. There exists (by the Axiom of Choice) at least one cross-section Y of π. For each x in X, let $x\epsilon$ be the element y of Y such that $x\pi y$. We call ϵ a representative mapping of π. The idempotents in the $\mathscr{R}$-class of $\mathscr{T}_X$ corresponding to π by Theorem 2.9 (v) are just the representative mappings of π.

(c) By Lemma 1.13, $\mathscr{T}_X$ is regular (cf. Exercise 1 of §1.9).

2. (a) Each $\mathscr{L}$-class of rank r in $\mathscr{T}_n$ $(= \mathscr{T}_X$ with $|X| = n)$ contains r^{n-r} idempotents.

(b) Each $\mathscr{R}$-class of $\mathscr{T}_n$ corresponding to a partition $n = n_1 + n_2 + \cdots + n_r$ of n contains $n_1 n_2 \cdots n_r$ idempotents.

3. (a) Every ideal of $\mathscr{T}_n$ (n finite) is principal.

(b) The only non-principal ideal of $\mathscr{T}_X$ with countable X is the set of all transformations of X of finite rank.

4. (a) The right unit subsemigroup P of $\mathscr{T}_X$ is the $\mathscr{R}$-class corresponding to the identity partition of X, and so consists of all one-to-one mappings of X into X (cf. §1.7, Exercise 2).

(b) The left unit subsemigroup Q of $\mathscr{T}_X$ is the $\mathscr{L}$-class consisting of all elements of $\mathscr{T}_X$ with range X.

(c) The $\mathscr{H}$-class $P \cap Q$ is the symmetric group $\mathscr{G}_X$.

5. The numbers of elements in the five $\mathscr{D}$-classes of $\mathscr{T}_5$ are: 5, 300, 1500, 1200, 120.

6. Let Φ be a field, and V a vector-space over Φ. By the *dimension* $\dim V$ of V we mean the cardinal number of a basis of V over Φ. Let $\mathscr{LT}(V)$ be the multiplicative semigroup (that is, under iteration) of all linear transformations of V. With each element A of $\mathscr{LT}(V)$ we associate two subspaces of V: (1) the *range* VA of A, consisting of all xA with x in V, and (2) the *null-space* N_A of A, consisting of all y in V such that $yA = 0$.

(a) Let $A \in \mathscr{LT}(V)$. Let W be a subspace of V complementary to N_A, so that $V = N_A \oplus W$. Then A induces a non-singular linear transformation of W upon VA. Hence $\dim (V/N_A) = \dim W = \dim VA$; this cardinal number we call the *rank* of A. (Here we have denoted by V/N_A the difference or quotient space of V modulo N_A, frequently denoted by $V - N_A$. If $\dim V$ is finite, this notion of rank is the usual one for matrices A, since VA is the row-space of A. Also, N_A is the orthogonal complement of the column-space of A.)

(b) Two elements of $\mathscr{LT}(V)$ are $\mathscr{L}$-[$\mathscr{R}$-] equivalent if and only if they have the same range [null-space].

(c) If N and W are subspaces of V such that $\dim (V/N) = \dim W$, then there exists at least one element A of $\mathscr{LT}(V)$ such that $N = N_A$ and $W = VA$.

(d) Two elements of $\mathscr{LT}(V)$ are $\mathscr{D}$-equivalent if and only if they have the same rank.

(e) Theorem 2.9 holds for $\mathscr{LT}(V)$ instead of $\mathscr{T}_X$ if we replace "subset Y of X" by "subspace W of V", $|Y|$ by $\dim W$, "partition π of X" by "subspace N of V", and $|X/\pi|$ by $\dim (V/N)$.

(f) Let N and W be subspaces of V such that $\dim (V/N) = \dim W$. Let H be the $\mathscr{H}$-class consisting of all elements of $\mathscr{LT}(V)$ with null-space N and range W. Then H contains an idempotent if and only if N and W are complementary in V, and this idempotent is the projection of V upon W which annuls N. If this is the case, H induces and is isomorphic with the full linear group $\mathscr{GL}(W)$ on W, consisting of all non-singular linear transformations of W.

(g) $\mathscr{LT}(V)$ is regular.

7. Let X be a countably infinite set. Let π and ρ be elements of $\mathscr{T}_X$ such that $\pi\rho = \iota$, and let G be a subgroup of the symmetric group $\mathscr{G}_X$ on X. Then $\rho G \pi$ is a subgroup of $\mathscr{T}_X$, isomorphic with G, consisting of elements of $\mathscr{T}_X$ of infinite rank. Conversely, any subgroup of infinite rank of $\mathscr{T}_X$ is obtained in this way. (Suschkewitsch [1940*a*].)

8. Let S be a subsemigroup of the semigroup $\mathscr{T}_X$ of all transformations of a finite set X. S is a left group if and only if all its members have the same range; S is a right group if and only if all its members have the same partition of X. (See Exercise 5 of §1.11.) (Suschkewitsch [1937], Chapter 3, §31.)

2.3 REGULAR $\mathscr{D}$-CLASSES

We recall (§1.9) that an element a of a semigroup S is called *regular* if $a = axa$ for some x in S. A $\mathscr{D}$-class D (or in fact any subset) of S is called regular if every element of D is regular. The next theorem shows that if D is not regular, then no element of D is regular; in this case we call D irregular. In the present section we consider the theory of regular $\mathscr{D}$-classes in an arbitrary semigroup.

THEOREM 2.11. (i) *If a $\mathscr{D}$-class D of a semigroup S contains a regular element, then every element of D is regular.*

(ii) *If D is regular, then every $\mathscr{L}$-class and every $\mathscr{R}$-class contained in D contains an idempotent.*

PROOF. We may rephrase Lemma 1.13 as follows: *an element a of a semigroup S is regular if and only if R_a [L_a] contains an idempotent.* It follows that if an $\mathscr{R}$-class R [$\mathscr{L}$-class L] contains a regular element, then it contains an idempotent, and every element of R [L] is regular. Since every $\mathscr{R}$-class of S contained in D meets every $\mathscr{L}$-class of S contained in D, (i) is evident. But then (ii) is also immediate from Lemma 1.13.

We recall (§1.9) that two elements a and a' of a semigroup S are said to be *inverse* elements if $aa'a = a$ and $a'aa' = a'$. The next two lemmas are evident.

LEMMA 2.12. *If a and a' are inverse element of a vanigroup S, then $e = aa'$ and $f = a'a$ are idempotents such that $ea = af = a$ and $a'e = fa' = a'$. Hence $e \in R_a \cap L_{a'}$ and $f \in R_{a'} \cap L_a$. The elements a, a', e, f all belong to the same $\mathscr{D}$-class of S.*

LEMMA 2.13. (i) *If a is a regular element of a semigroup S, then $aS^1 = aS$ and $S^1a = Sa$.*

(ii) *If a and b are regular elements of S, then $a\mathscr{L}b$ [$a\mathscr{R}b$] if and only if $Sa = Sb$ [$aS = bS$].*

LEMMA 2.14. *Any idempotent element e of a semigroup S is a right identity element of L_e, a left identity element of R_e, and a two-sided identity element of H_e.*

PROOF. If $a \in L_e$, then $a \in Se$, and hence $ae = a$. If $a \in R_e$, then $a \in eS$, and $ea = a$. If $a \in H_e = R_e \cap L_e$, then $ea = ae = a$.

LEMMA 2.15. *No $\mathscr{H}$-class can contain more than one idempotent.*

PROOF. If e and f are idempotents such that $H_e = H_f$, then, by Lemma 2.14, each is a two-sided identity of the other, and hence $e = f$.

The following important theorem is due to Green [1951].

THEOREM 2.16 (Green). *If a, b and ab all belong to the same $\mathscr{H}$-class H of a semigroup S, then H is a subgroup of S. In particular, any $\mathscr{H}$-class containing an idempotent is a subgroup of S.*

PROOF. We first remark that if h and hs $[sh]$ both belong to the same $\mathscr{H}$-class H of S, then $Hs = H$ $[sH = H]$. For then $h\mathscr{R}hs$, and it follows from Green's Lemma (2.2) that the mapping $x \to xs$ is a one-to-one mapping of H_h upon H_{hs}, that is, of H upon itself. The bracketed statement follows from the dual of Green's Lemma.

Now let a, b and ab all belong to the $\mathscr{H}$-class H. By the remark, $Hb = H$. Let c and d be arbitrary elements of H. Then $cb \in Hb = H$. Since b and cb both belong to H, it follows from the bracketed remark that $cH = H$. Then $cd \in H$, and we use the remark again to see that $Hd = H$. From $cH = Hd = H$ for arbitrary c and d in H, it follows that H is a subgroup of S.

The next theorem is due to Miller and Clifford [1956].

THEOREM 2.17. *If a and b are elements of a semigroup S, then $ab \in R_a \cap L_b$ if and only if $R_b \cap L_a$ contains an idempotent. If this is the case, then*

$$aH_b = H_ab = H_aH_b = H_{ab} = R_a \cap L_b.$$

PROOF. Assume first that $ab \in R_a \cap L_b$. From $ab \in R_a$ we infer the existence of b' in S such that $(ab)b' = a$. By Green's Lemma (2.2), the mappings $\sigma : x \to xb (x \in L_a)$ and $\sigma' : y \to yb' (y \in L_{ab})$ are mutually inverse, $\mathscr{R}$-class preserving, one-to-one mappings of L_a upon L_{ab} and of L_{ab} upon L_a, respectively. But $ab \in L_b$, and so $L_{ab} = L_b$. Thus σ' maps the element b of L_b upon the element bb' of L_a, and moreover $bb' \in R_b$ since σ' is $\mathscr{R}$-class preserving. Hence $bb' \in R_b \cap L_a$. If $x \in L_a$, then $xbb' = x\sigma\sigma' = x$; putting $x = bb'$, we conclude that bb' is idempotent.

Conversely, assume that $R_b \cap L_a$ contains an idempotent e. Then $eb = b$ by Lemma 2.14. Since $e\mathscr{R}b$, it follows from Green's Lemma that $\sigma : x \to xb (x \in L_e)$ is an $\mathscr{R}$-class preserving, one-to-one mapping of L_e upon L_b. Since $a \in L_e$, $ab \in L_b$; moreover, $ab \in R_a$ since σ is $\mathscr{R}$-class preserving. Hence $ab \in R_a \cap L_b$.

Continuing with the hypothesis that $R_b \cap L_a$ contains an idempotent e, let $x \in H_a$ and $y \in H_b$. Then $e \in R_y \cap L_x$, and we conclude from what has been shown that $xy \in R_x \cap L_y = R_a \cap L_b$. Hence $H_aH_b \subseteq R_a \cap L_b$. Since

$L_e = L_a$ and $L_b = L_{ab}$, $\sigma : x \to xb$ maps L_a upon L_{ab}. Since σ is $\mathscr{R}$-preserving, it maps H_a upon H_{ab}, and so $H_a b = H_{ab}$. Hence,

$$H_a b \subseteq H_a H_b \subseteq R_a \cap L_b = H_{ab} = H_a b,$$

and equalities hold all down the line. Dually, $aH_b = H_{ab}$.

The next theorem, due to Miller and Clifford [1956], locates all the inverses of a regular element a of a semigroup S. (Of course an irregular element has none.) It shows that there is a one-to-one correspondence between the set of all inverses a' of a and the set of all pairs (e, f) of idempotent elements with e in R_a and f in L_a. The a' corresponding to (e, f) lies in $R_f \cap L_e$. (The egg-box picture helps one to visualize the situation.)

THEOREM 2.18. *Let a be a regular element of a semigroup S.*

(i) *Every inverse of a lies in D_a.*

(ii) *An $\mathscr{H}$-class H_b contains an inverse of a if and only if both of the $\mathscr{H}$-classes $R_a \cap L_b$ and $R_b \cap L_a$ contain idempotents.*

(iii) *No $\mathscr{H}$-class contains more than one inverse of a.*

PROOF. (i) is immediate from Lemma 2.12.

To show (ii), assume first that H_b contains an inverse a' of a. By Lemma 2.12, the $\mathscr{H}$-classes $R_a \cap L_b$ $(= R_a \cap L_{a'})$ and $R_b \cap L_a$ $(= R_{a'} \cap L_a)$ contain the idempotents aa' and $a'a$, respectively.

Conversely, assume that e is an idempotent in $R_a \cap L_b$, and that f is an idempotent in $R_b \cap L_a$. From $a\mathscr{R}e$ and $a\mathscr{L}f$ we have $ea = a = af$, by Lemma 2.14, and $e = ax$, $f = ya$, for some x, y in S, by Lemma 2.13. Let $a' = fxe$. Then

$$fa' = a'e = a',$$

$$aa' = afxe = axe = e^2 = e,$$

$$a'a = fa'a = yaa'a = yea = ya = f.$$

Since $aa'a = ea = a$ and $a'aa' = a'e = a'$, a and a' are mutually inverse. From $fa' = a'$ and $a'a = f$, we have $a'\mathscr{R}f$. From $a'e = a'$ and $aa' = e$, we have $a'\mathscr{L}e$. Hence $a' \in R_f \cap L_e = R_b \cap L_b = H_b$.

To show (iii), let b and c be $\mathscr{H}$-equivalent inverse elements of a. By Lemma 2.12, ab is an idempotent element in $R_a \cap L_b$, and ac is an idempotent in $R_a \cap L_c$. But $L_b = L_c$, and so $ab = ac$ by Lemma 2.15. Similarly, from $R_b = R_c$, we conclude that $ba = ca$. Hence $b = bab = cab = cac = c$.

COROLLARY 2.19. (i) *A semigroup S is an inverse semigroup if and only if each $\mathscr{L}$-class and each $\mathscr{R}$-class of S contains exactly one idempotent.*

(ii) *If D is a $\mathscr{D}$-class of an inverse semigroup S, then there is a one-to-one correspondence between the set of $\mathscr{L}$-classes contained in D and the set of $\mathscr{R}$-classes contained in D, whereby an $\mathscr{L}$-class L and an $\mathscr{R}$-class R correspond if and only if $R \cap L$ contains an idempotent.*

PROOF. By Theorem 2.18, (i) is precisely the condition that each element of S have exactly one inverse, and then (ii) is immediate.

Part (i) of the corollary forms part of Theorem 1.17, namely the equivalence of parts (ii) and (iii) of the latter. The present theory serves to illuminate this equivalence. It does not assist in the implication of (i) by (iii) since this entails idempotents lying in different $\mathscr{D}$-classes.

The significance of Corollary 2.19 (ii) for the egg-box picture is that we may imagine the $\mathscr{L}$-classes and $\mathscr{R}$-classes contained in a $\mathscr{D}$-class D of the inverse semigroup S arranged in such an order that the $\mathscr{H}$-classes containing idempotents come on the main diagonal. Then Theorem 2.18 shows that the inverse a^{-1} of an element a of D lies in the $\mathscr{H}$-class situated symmetrically to H_a with respect to the main diagonal. An illustration of this is afforded by the bicyclic semigroup (Exercise 6 of §2.1).

We conclude this section with a theorem due to Green [1951], which shows that if two $\mathscr{H}$-classes in the same $\mathscr{D}$-class are groups, then they are isomorphic.

THEOREM 2.20. *Let e and f be $\mathscr{D}$-equivalent idempotents of a semigroup S. Let a be an arbitrary but fixed element of $R_e \cap L_f$, and let a' be the inverse of a in $R_f \cap L_e$ (cf. Theorem 2.18). Then the mappings $x \to a'xa$ and $y \to aya'$ are mutually inverse isomorphisms of H_e upon H_f, and of H_f upon H_e, respectively.*

PROOF. Let $x \in H_e$. By two applications of Theorem 2.17, we see that $xa \in R_e \cap L_a$ and $a'xa \in R_{a'} \cap L_{xa} = R_{a'} \cap L_a = H_f$. Similarly, $y \in H_f$ implies $aya' \in H_e$. If $x \in H_e$, then $a(a'xa)a' = exe = x$; and if $y \in H_f$, then $a'(aya')a = fyf = y$. Hence the mappings $x \to a'xa$ and $y \to aya'$ are mutually inverse, one-to-one mappings of H_e and H_f upon each other. To show that $x \to a'xa$ is an isomorphism, let $x_1, x_2 \in H_e$. Then

$$(a'x_1a)(a'x_2a) = a'x_1ex_2a = a'(x_1x_2)a.$$

EXERCISES FOR §2.3

1. The maximal subgroups of a semigroup S are precisely the $\mathscr{H}$-classes of S containing idempotents.

2. Let R be an $\mathscr{R}$-class and L an $\mathscr{L}$-class of a semigroup S such that $R \cap L$ contains an idempotent. Let D be the $\mathscr{D}$-class containing R and L. Then $LR = D$. (Exercise 7 of §2.1 is a consequence of this. The condition that $R \cap L$ contain an idempotent is not necessary for $LR = D$; necessary and sufficient conditions are unknown.)

3. (a) An inverse of an idempotent element need not be idempotent; for example, the elements g and a of the semigroup of Exercise 2 of §1.2 are inverses of each other.

(b) Any inverse g' of an idempotent element g of a semigroup S is the product of two idempotents, namely $g' = fe$, where $e = gg'$ and $f = g'g$. (Miller and Clifford [1956].)

4. Let e and f be $\mathscr{D}$-equivalent idempotents of a semigroup S. For each x in $R_e \cap L_f$, let x' be the inverse of x in $R_f \cap L_e$ (Theorem 2.18).

(a) Let $x, y \in R_e \cap L_f$. Then in the group H_e the inverse of xy' is yx', and in the group H_f the inverse of $x'y$ is $y'x$.

(b) Let a be a fixed element of $R_e \cap L_f$. For x and y in $A = R_e \cap L_f$, define $x \circ y = xa'y$. For u and v in $A' = R_f \cap L_e$, define $u * v = uav$. Then $A(\circ)$ and $A'(*)$ are groups, and $x \to x'$ is an anti-isomorphism of A upon A'.

(c) With the notation of part (b), the mappings $\mu_a : x \to ax$ and $\sigma_a : y \to ya$ are isomorphisms of the group A' upon the groups H_e and H_f. If, instead of a, we select a different element b of $R_e \cap L_f$, then $\mu_a \neq \mu_b$ and $\sigma_a \neq \sigma_b$. (Miller and Clifford [1956].)

5. Let S be the semigroup generated by p and q subject to the generating relations (§1.12)

$$pqp = p, \quad qpq = q.$$

Every $\mathscr{D}$-class of S has four elements, every $\mathscr{L}$-class and every $\mathscr{R}$-class has two elements, and every $\mathscr{H}$-class has one element. The only regular $\mathscr{D}$-class is

$$\begin{Bmatrix} pq & p \\ q & qp \end{Bmatrix}.$$

6. If a regular $\mathscr{D}$-class D of a semigroup S is a subsemigroup of S, then D is bisimple. (It is not known if this holds without the assumption of regularity.)

7. (a) A regular semigroup S is an inverse semigroup if and only if it possesses an involutorial anti-automorphism ("i.a.a."—see §1.3) which leaves every idempotent element of S fixed. (Munn, unpublished.)

(b) Let $a \to a^*$ be an i.a.a. of a semigroup S. Then there is at most one mapping $a \to a^\dagger$ of S into itself such that

$$aa^\dagger a = a, \qquad a^\dagger a a^\dagger = a^\dagger,$$

$$(aa^\dagger)^* = aa^\dagger, \qquad (a^\dagger a)^* = a^\dagger a,$$

for all a in S.

8. Let V be a linear space of finite dimension n over the field C of complex numbers. In the notation of Exercise 6 of §2.2, let $\mathscr{L}\mathscr{T}(V)$ be the multiplicative semigroup of all linear transformations of V. By fixing a basis in V, we may regard the elements A of $\mathscr{L}\mathscr{T}(V)$ as $n \times n$ matrices over C. Let A^* be the transpose of the complex conjugate of A. If W is a subspace of V, denote by $W_\perp$ the (unitary) orthogonal complement of W, consisting of all vectors v of V such that $vw^* = v_1\bar{w}_1 + \cdots + v_n\bar{w}_n = 0$ for every w in W.

(a) If an element A of $\mathscr{L}\mathscr{T}(V)$ has range W and null-space N, then A^* has range $N_\perp$ and null-space $W_\perp$. (Jacobson, *Lectures in abstract algebra*, vol. 2, *Linear algebra*, Van Nostrand, New York (1953); Theorem 11 on p. 59.)

(b) An idempotent element E of $\mathscr{L}\mathscr{T}(V)$ is hermitian ($E^* = E$) if and only if its range and null space are orthogonal. There exists exactly one hermitian idempotent in each $\mathscr{L}$-class and in each $\mathscr{R}$-class of $\mathscr{L}\mathscr{T}(V)$.

(c) For A in $\mathscr{L}\mathscr{T}(V)$, let $E\,[F]$ be the hermitian idempotent which is $\mathscr{R}$- [$\mathscr{L}$-] equivalent to A, and (in accordance with Theorem 2.18) let $A^\dagger$ be the inverse of A which is $\mathscr{R}$- [$\mathscr{L}$-] equivalent to $F\,[E]$. Then the mapping $A \to A^\dagger$ of $\mathscr{L}\mathscr{T}(V)$ into itself has the properties stated in Exercise 7 (b) above. The matrix $A^\dagger$ can also be described as the inverse of A in $\mathscr{L}\mathscr{T}(V)$ which is $\mathscr{H}$-equivalent to A^*.

NOTE. The mapping $A \to A^\dagger$ was discovered by E. H. Moore, who called $A^\dagger$ the *general reciprocal* of A. It reduces to the ordinary inverse of A if A is non-singular. It was rediscovered by R. Penrose.

E. H. Moore, *On the reciprocal of the general algebraic matrix* (abstract), Bull. Amer. Math. Soc. 26 (1920), 394–5.

E. H. Moore, *General Analysis*, vol. I, Memoirs of the Amer. Phil. Soc., vol. 1, Philadelphia (1935); cf. p. 8 and Chapter 3, Section 29.

R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.

T. N. E. Greville, *The pseudoinverse of a rectangular or singular matrix and its application to the solution of systems of linear equations*, SIAM Newsletter 5 (1957) no. 2, pp. 3–6 (MR 19, 243).

2.4 The Schützenberger group of an $\mathscr{H}$-class

In the previous section, we saw that any $\mathscr{H}$-class containing an idempotent is a group (Green's Theorem 2.16), and that two such $\mathscr{H}$-classes in the same $\mathscr{D}$-class are isomorphic (Theorem 2.20). In the present section, we give an account of the discovery by M. P. Schützenberger [1957*a*] that there is a group $\Gamma(H)$ associated with any $\mathscr{H}$-class H whatever, even in an irregular $\mathscr{D}$-class. If H and H' are two $\mathscr{H}$-classes in the same $\mathscr{D}$-class D, then $\Gamma(H)$ and $\Gamma(H')$ are isomorphic, so that Γ really depends on D. If H is itself a group, then $\Gamma(H) \cong H$.

Let A be any subset of a semigroup S. Let $T = T(A)$ be the set of all elements t of S^1 such that $At \subseteq A$. Evidently T is a subsemigroup of S^1. To say that $t \in T$ is to say that A is invariant (as a set) under the inner right translation ρ_t of S^1. Thus ρ_t induces a transformation $\gamma_t = \rho_t|A$ of A. Let $\Gamma = \Gamma(A)$ be the set of all γ_t as t ranges over $T(A)$. Evidently Γ is a semigroup, and $t \to \gamma_t$ is a homomorphism of T upon Γ. We call $\Gamma(A)$ the semigroup of transformations of A induced by inner right translations of S^1. We are interested in the case $A = H$, an $\mathscr{H}$-class of S.

The following lemma is an immediate consequence of Green's Lemma (2.2.).

LEMMA 2.21. *Let H be an $\mathscr{H}$-class of a semigroup S. Let $h_0 \in H$, and let t be an element of S^1 such that $h_1 = h_0 t \in H$. Then $h_0 = h_1 t'$ for some t' in S^1, and the mappings $\gamma_t : x \to xt$ and $\gamma_{t'} : x \to xt'$ are mutually inverse permutations of H. Thus t and t' belong to $T(H)$, and $\gamma_t \gamma_{t'} = \gamma_{t'} \gamma_t = \gamma_1$.*

If L is the $\mathscr{L}$-class containing H, then the mappings $x \rightarrow xt$ and $x \rightarrow xt'$ are mutually inverse, one-to-one, $\mathscr{R}$-class preserving mappings of L upon itself. Thus $T(H') = T(H)$ if H' is any $\mathscr{H}$-class of S contained in L.

We recall that if Σ is a set of transformations of a set X, then Σ is said to be *transitive* [*simply transitive*] if, given any two elements x and y of X, there is at least one [exactly one] element of Σ mapping x upon y.

The following theorem, due to Schützenberger [1957*a*], shows that $\Gamma(H)$ is a group. We call $\Gamma(H)$ *the Schützenberger group of H*.

THEOREM 2.22. *Let H be an $\mathscr{H}$-class of a semigroup S. Then the semigroup $\Gamma(H)$ of transformations of H induced by the inner right translations of S^1 is a simply transitive group of permutations of H. It follows that $|\Gamma(H)| = |H|$. If H is itself a subgroup of S, then $\Gamma(H) \cong H$; in fact, $\Gamma(H)$ is then the image of the regular representation of H.*

PROOF. Let $\gamma_t \in \Gamma(H)$ with t in $T(H)$. If $h_0 \in H$, then $h_1 = h_0 t \in H$, and, by Lemma 2.21, γ_t has a group-inverse $\gamma_{t'}$ in $\Gamma(H)$. Hence $\Gamma(H)$ is a group.

To show that $\Gamma(H)$ is simply transitive, let h_0 and h_1 be any two elements of H. From $h_0 \mathscr{R} h_1$ we have $h_0 t = h_1$ for some t in S^1; by Lemma 2.21, $t \in T(H)$ and $h_0 \gamma_t = h_1$. To show that γ_t is the only element of $\Gamma(H)$ mapping h_0 upon h_1, suppose $h_0 \gamma_s = h_1$ (that is, $h_0 s = h_1$) for some s in $T(H)$. Let x be an arbitrary element of H. From $x \mathscr{L} h_0$ we have $x = y h_0$ for some y in S^1, and so

$$x\gamma_t = xt = yh_0 t = yh_1 = yh_0 s = xs = x\gamma_s.$$

Hence $\gamma_s = \gamma_t$.

Now suppose that H is a group. Let e be the identity element of H, and let h be an arbitrary element of H. From the foregoing, there is exactly one element of $\Gamma(H)$ mapping e upon h. But γ_h maps e upon h. Thus $\Gamma(H) = \{\gamma_h : h \in H\}$, and the mapping $h \rightarrow \gamma_h$ is just the regular representation of H.

We consider now the dual of Theorem 2.22. Let $T'(H)$ be the set of all elements u of S^1 such that $uH \subseteq H$. Let λ_u be the inner left translation $x \rightarrow ux$ of S^1 corresponding to u, and let $\gamma'_u = \lambda_u | H$. Let $\Gamma'(H)$ be the set of all γ'_u with u in $T'(H)$. For all u, v in $T'(H)$ we have $\gamma'_{uv} = \gamma'_v \gamma'_u$, so that $u \rightarrow \gamma'_u$ is an anti-homomorphism of $T'(H)$ upon $\Gamma'(H)$. By the dual of Theorem 2.22, $\Gamma'(H)$ is also a simply transitive group of permutations of H; we call it the *dual Schützenberger group of H*.

Since every inner left translation of S commutes with every inner right translation of S, it is clear that *every element of $\Gamma'(H)$ commutes with every element of $\Gamma(H)$*. It will then follow from the next lemma that *$\Gamma'(H)$ and $\Gamma(H)$ are anti-isomorphic.*

LEMMA 2.23. *If Γ and Γ' are two simply transitive groups of permutations of a set H such that every element of Γ commutes with every element of Γ', then Γ and Γ' are anti-isomorphic.*

PROOF. Let h_0 be a fixed element of H. For each h in H there is a unique γ in Γ such that $h_0\gamma = h$ and a unique γ' in Γ' such that $h_0\gamma' = h$. The mapping $\phi : \gamma \rightarrow \gamma'$ is clearly a one-to-one mapping of Γ upon Γ'. We show that it is an anti-isomorphism.

Let $\gamma_1, \gamma_2 \in \Gamma$. Then, using the fact that $\gamma_1\phi$ and γ_2 commute (by hypothesis), and the fact that $h_0(\gamma\phi) = h_0\gamma' = h_0\gamma$ (by definition of ϕ), we have

$$\begin{aligned} h_0[(\gamma_1\gamma_2)\phi] &= h_0(\gamma_1\gamma_2) = (h_0\gamma_1)\gamma_2 = [h_0(\gamma_1\phi)]\gamma_2 \\ &= h_0[(\gamma_1\phi)\gamma_2] = h_0[\gamma_2(\gamma_1\phi)] \\ &= (h_0\gamma_2)(\gamma_1\phi) = [h_0(\gamma_2\phi)](\gamma_1\phi) \\ &= h_0[(\gamma_2\phi)(\gamma_1\phi)]. \end{aligned}$$

Hence,

$$(\gamma_1\gamma_2)\phi = (\gamma_2\phi)(\gamma_1\phi).$$

Let us recapitulate what has been shown above.

THEOREM 2.24. *Let H be an $\mathscr{H}$-class of a semigroup S. Let $\Gamma(H)$ be the Schützenberger group of H, and $\Gamma'(H)$ the dual Schützenberger group. Then $\Gamma(H)$ and $\Gamma'(H)$ are simply transitive groups of permutations of H such that every element of $\Gamma(H)$ commutes with every element of $\Gamma'(H)$. The groups $\Gamma(H)$ and $\Gamma'(H)$ are anti-isomorphic.*

THEOREM 2.25. *Let H and H' be $\mathscr{H}$-classes of a semigroup S both contained in the same $\mathscr{D}$-class of S. Then $\Gamma(H) \cong \Gamma(H')$.*

PROOF. Let $a \in H$ and $b \in H'$. Since $a\mathscr{D}b$, there exists c in S such that $a\mathscr{L}c$ and $c\mathscr{R}b$. By Lemma 2.21, $T(H_a) = T(H_c)$, and for each t in $T(H_a)$, the mapping $\rho_t|L_a$ is an $\mathscr{R}$-class preserving permutation of L_a ($= L_c$). For any s and t in $T(H_a)$, $\rho_s|L_a = \rho_t|L_a$ if and only if $\rho_s|H_a = \rho_t|H_a$. Hence the mapping $\rho_t|H_a \rightarrow \rho_t|H_c$ is an isomorphism of $\Gamma(H_a)$ upon $\Gamma(H_c)$. Dually, $\Gamma'(H_c) \cong \Gamma'(H_b)$. Using Theorem 2.24 twice, we conclude that $\Gamma(H_a) \cong \Gamma(H_b)$.

EXERCISES FOR §2.4

1. Let H be a set, and let Γ and Γ' be simply transitive groups of permutations of H such that every element of Γ commutes with every element of Γ'. Then we can define a binary operation $(\circ)$ in H such that $H(\circ)$ is a group, and such that Γ is the image of the regular representation of H, and Γ' that of the regular anti-representation of H. Any element of H can be chosen to be the identity element of $H(\circ)$; once this choice is made, $(\circ)$ is uniquely determined. Conversely, if H is a group, the images Γ and Γ' of the regular representation and anti-representation, respectively, of H, have the above properties.

2. Let H be a set, and let Γ and Γ' be simply transitive groups of permutations of H such that every element of Γ commutes with every element

of Γ'. Let T and T' be semigroups disjoint from H and from each other. Let ϕ be a homomorphism of T upon Γ and let ϕ' be an anti-homomorphism of T' upon Γ'. For any h in H, t in T, and u in T', define

$$ht = h(t\phi), \qquad uh = h(u\phi').$$

Let 0 be a symbol not representing any element of H, T, or T'. Let $S = H \cup T \cup T' \cup \{0\}$. Define product in S by means of the table, where $h_1, h_2 \in H$; $t_1, t_2 \in T$; $u_1, u_2 \in T'$. Then S is a semigroup in which H is an $\mathscr{H}$-class having Schützenberger group Γ. Moreover, H coincides with the $\mathscr{D}$-class D of S containing it, and D is irregular; in fact $D^2 = 0$.

	h_2	t_2	u_2	0
h_1	0	h_1t_2	0	0
t_1	0	t_1t_2	0	0
u_1	u_1h_2	0	u_1u_2	0
0	0	0	0	0

We may modify the foregoing, taking $T = T'$, without altering the conclusion.

3. An alternative proof of Theorem 2.25 is based on the following. Let $\{R_i : i \in I\}$ and $\{L_\lambda : \lambda \in \Lambda\}$ be the $\mathscr{R}$-classes and $\mathscr{L}$-classes, respectively, of S contained in the $\mathscr{D}$-class D. Let $H = H_{11}$ and $H' = H_{i\lambda}$. By Green's Lemma (2.2) there exist elements q_λ and q'_λ of S^1 such that $x \to xq_\lambda$ and $y \to yq'_\lambda$ are mutually inverse, one-to-one, $\mathscr{R}$-class preserving mappings of L_1 upon L_λ and vice-versa. Write $\gamma(t)$ instead of γ_t. For w in $T(H_{i\lambda})$, let $\delta(w) = \rho_w | H_{i\lambda}$. Then $\gamma(t) \to \delta(q'_\lambda t q_\lambda)$, with t ranging over $T(H)$, is an isomorphism of $\Gamma(H)$ upon $\Gamma(H_{i\lambda})$.

4. Let D be a regular $\mathscr{D}$-class of a semigroup S, and let H be an $\mathscr{H}$-class of S contained in D. Let e be any idempotent in the same $\mathscr{L}$-class as H. Then $H_e \subseteq T(H) = T$. Furthermore, $Te = T \cap L_e$, and Te is the union of those $\mathscr{H}$-classes in L_e which are groups. In particular, therefore, the union of the set of subgroups in an arbitrary $\mathscr{L}$-class of a semigroup S is either empty or a subsemigroup of S; in the latter case, it is a left group. Moreover, Te is a two-sided ideal of the semigroup T. Being simple, in fact left simple, Te properly contains no ideal of T (and so is the "kernel" of T; see §2.5). (R. J. Koch, unpublished.)

2.5 0-MINIMAL IDEALS AND 0-SIMPLE SEMIGROUPS

As in §1.1, we call a semigroup S *simple* [*left simple*, *right simple*] if it does not properly contain any two-sided [left, right] ideal. We saw in §1.1 that S is both left simple and right simple if and only if it is a group. Right simple semigroups which are not groups were studied in §1.11. In the exercises for §2.1 we met a number of simple semigroups which are not right or left simple. Many others will occur in the course of this and the next chapter, and Chapter 8 will be devoted entirely to the theory of simple semigroups.

A two-sided [left, right] ideal M of a semigroup S is called *minimal* if it does not properly contain any two-sided [left, right] ideal of S. If A is any other ideal of S of the same type as M, either $M \subseteq A$ or $M \cap A = \square$. In particular, two distinct minimal ideals of the same type are disjoint. For example, the rows of a rectangular band are minimal right ideals, and they are evidently mutually disjoint.

Since two two-sided ideals A and B of a semigroup S always contain their set product AB, it follows that there can be at most one minimal two-sided ideal of S. If S has a minimal two-sided ideal K, then K is called the *kernel* of S. Since K is contained in every two-sided ideal of S, it may be characterized as the intersection of all the two-sided ideals of S. If this intersection is empty, then S does not have a kernel; this is the case, for example, for an infinite cyclic semigroup. Every finite semigroup evidently has a kernel. In fact, the algebraic theory of semigroups originated with the determination by Suschkewitsch [1928] of the structure of the kernel (which he called the "Kerngruppe") of any finite semigroup (see Appendix A). We shall see (Corollary 2.30) that the kernel of a semigroup, if it exists, is always itself a simple semigroup.

The theory of minimal ideals in a semigroup S with a zero element 0 becomes trivial. For this reason, and also to accord with the theory of minimal ideals in rings, we introduce the notion of 0-minimality. A two-sided [left, right] ideal M of S is called *0-minimal* if (i) $M \neq 0$, and (ii) 0 is the only two-sided [left, right] ideal of S properly contained in M.

If M is a 0-minimal two-sided [left, right] ideal of a semigroup S with zero 0, then M^2 is an ideal of the same type as M contained in M, and so we must have either $M^2 = M$ or $M^2 = 0$. As in §1.1, we call a semigroup with zero 0 a *zero* or *null semigroup* if the product of any two of its elements is 0. Hence either $M^2 = M$ or M is a null subsemigroup of S.

It is clear that any two distinct 0-minimal ideals of S of the same type are *0-disjoint* in the sense that their intersection is 0.

We also have the corresponding notions of 0-simplicity. A semigroup S with zero element 0 is called *0-simple* [*left 0-simple, right 0-simple*] if (i) $S^2 \neq 0$, and (ii) 0 is the only proper two-sided [left, right] ideal of S.

LEMMA 2.26. *Let S be a semigroup with 0 such that 0 is the only proper two-sided ideal of S. Then either S is 0-simple or S is the null semigroup of order 2.*

PROOF. Evidently $S^2 = S$ or $S^2 = 0$. In the former case, S is 0-simple; for $S^2 = 0$ would imply $S = 0$, in which case 0 would not be a proper ideal of S. In the latter case, if a is any element $\neq 0$ of S, then $\{0, a\}$ is an ideal $\neq 0$ of S, and so $\{0, a\} = S$.

The following theorem shows that there is no essential difference between "right simple" and "right 0-simple", in the sense that every right 0-simple semigroup arises from a right simple semigroup by the adjunction of a zero element. On the other hand, there is a big difference between "0-simple semigroup" and "simple semigroup with zero adjoined".

THEOREM 2.27. *If S is a right [left] 0-simple semigroup, then $S\setminus 0$ is a right [left] simple subsemigroup of S.*

PROOF. We show first that $S\setminus 0$ is a subsemigroup of S, that is, that S contains no *proper divisors of zero*. Suppose, by way of contradiction, that $a, b \in S\setminus 0$ but $ab = 0$. The set of all x in S such that $ax = 0$ is a right ideal of S containing $\{0, b\} \neq 0$, and hence coincides with S. But then $\{0, a\}$ is a right ideal $\neq 0$ of S, and so $\{0, a\} = S$. But then $S^2 = 0$, contrary to the definition of right 0-simplicity.

To show that $S\setminus 0$ is right simple, let R be any right ideal of $S\setminus 0$. Then clearly $R \cup 0$ is a right ideal of S. Since $R \neq \square$, $R \cup 0 \neq 0$. Hence $R \cup 0 = S$, and we conclude that $R = S\setminus 0$.

Let S be a semigroup without zero, and let $S^0 = S \cup 0$ be the semigroup arising from S by the adjunction of a zero element (§1.1). Then $A \to A \cup 0$ is a one-to-one mapping of the set of all two-sided [left, right] ideals A of S upon the set of all non-zero two-sided [left, right] ideals of S^0. This mapping preserves inclusion, and, in particular, A is minimal if and only if $A \cup 0$ is 0-minimal. Consequently, *any theorem concerning 0-minimal ideals implies an evident corollary concerning minimal ideals in a semigroup without zero.* Similarly, *any theorem concerning 0-simple semigroups implies an evident corollary concerning simple semigroups.* These evident corollaries will be explicitly stated only if emphasis is desired, as in the case of Corollary 2.30 below.

LEMMA 2.28. *Let S be a semigroup with zero 0, and such that $S \neq 0$. Then S is 0-simple if and only if $SaS = S$ for every element $a \neq 0$ of S.*

REMARK. This condition is, of course, equivalent to: given a and b in S with $a \neq 0$, we can always solve $xay = b$ for x and y in S.

PROOF. Assume that S is 0-simple. Let B be the set of all elements b of S such that $SbS = 0$. Clearly B is an ideal of S, and hence either $B = S$ or $B = 0$. The former would imply $S^3 = 0$; this is impossible, since $S^2 = S$ and hence $S^3 = S^2 = S$. Hence $B = 0$, and we conclude that $SaS \neq 0$ for every $a \neq 0$ in S. But SaS is an ideal $\neq 0$ of S, and so $SaS = S$.

Conversely, assume $SaS = S$ for every element $a \neq 0$ of S. Let A be an ideal $\neq 0$ of S, and let a be a non-zero element of A. Then $S = SaS \subseteq SAS \subseteq A$, so that $A = S$. Since $S \neq 0$ by hypothesis, S contains an element $a \neq 0$. From $S = SaS \subseteq S^2$, we see that $S^2 \neq 0$, and hence S is 0-simple.

The remaining results of this section are taken from Clifford [1949].

THEOREM 2.29. *Let M be a 0-minimal (two-sided) ideal of a semigroup S with zero 0. Then either $M^2 = 0$ or M is a 0-simple subsemigroup of S.*

PROOF (Munn). Assume $M^2 \neq 0$. Then, as noted above, $M^2 = M$. Let $a \in M$, $a \neq 0$. Since S^1aS^1 is a non-zero ideal of S contained in M, it follows that $S^1aS^1 = M$. Hence $M = M^3 = MS^1aS^1M \subseteq MaM \subseteq M$. Consequently, $MaM = M$, and M is 0-simple by Lemma 2.28.

COROLLARY 2.30. *If a semigroup S contains a kernel K, then K is a simple subsemigroup of S.*

LEMMA 2.31. *If L is a 0-minimal left ideal of a semigroup S with 0 such that $L^2 \neq 0$, then $L = Sa$ for any element $a \neq 0$ of L.*

PROOF. Clearly Sa is a left ideal of S contained in L. Were $Sa = 0$, $\{0, a\}$ would be a left ideal $\neq 0$ of S contained in L, and hence $\{0, a\} = L$. But then $L^2 = 0$, contrary to hypothesis. Hence $Sa \neq 0$, and so $Sa = L$.

REMARK. In contrast to the situation for 0-minimal two-sided ideals, a 0-minimal left ideal L such that $L^2 \neq 0$ need not be a left 0-simple, or even a 0-simple, semigroup. We may take, for example, the semigroup of Exercise 2 of §1.2, and let $L = \{0, f, a\}$.

LEMMA 2.32. *Let L be a 0-minimal left ideal of a semigroup S with 0, and let $c \in S$. Then Lc is either 0 or a 0-minimal left ideal of S.*

PROOF. Assume $Lc \neq 0$. Evidently Lc is a left ideal of S. To show that it is 0-minimal, let A be a left ideal of S contained in Lc. Let B be the set of all elements b of L such that $bc \in A$. Then $Bc \subseteq A$. Since every element of A has the form xc for some x in L, and any such x is in B, it follows that $Bc = A$. If $b \in B$ and $s \in S$, then $sbc \in sA \subseteq A$, and moreover $sb \in sL \subseteq L$. Hence $sb \in B$, which shows that B is a left ideal of S. From the 0-minimality of L, either $B = 0$ or $B = L$, and we have correspondingly $A = 0$ or $A = Lc$.

THEOREM 2.33. *Let S be a semigroup with 0. Let M be a 0-minimal ideal of S containing at least one 0-minimal left ideal of S. Then M is the union of all the 0-minimal left ideals of S contained in M.*

PROOF. Let A be the union of all the 0-minimal left ideals of S contained in M. We are to show that $A = M$. Clearly A is a left ideal of S. We proceed to show that A is also a right ideal. Let $a \in A$ and $c \in S$. By definition of A, $a \in L$ for some 0-minimal left ideal L of S contained in M. By Lemma 2.32, $Lc = 0$ or Lc is a 0-minimal left ideal of S. Moreover, $Lc \subseteq Mc \subseteq M$, and hence $Lc \subseteq A$. Consequently $ac \in A$. Now $A \neq 0$ since it contains at least one 0-minimal left ideal of S. Hence A is a non-zero two-sided ideal of S contained in M, whence $A = M$ by the 0-minimality of M.

LEMMA 2.34. *If M is a 0-minimal ideal of a semigroup S with 0 such that $M^2 \neq 0$, and if L is a non-zero left ideal of S contained in M, then $L^2 \neq 0$.*

PROOF. Since LS is an ideal of S contained in M, we must have either $LS = M$ or $LS = 0$. If $LS = 0$, then L is an ideal of S, whence $L = M$, and so $M^2 = LM \subseteq LS = 0$, contrary to hypothesis. Hence $LS = M$, and from $M = M^2 = LSLS \subseteq L^2S$, we conclude that $L^2 \neq 0$.

THEOREM 2.35. *Let M be a 0-minimal ideal of a semigroup S with 0 such that $M^2 \neq 0$, and assume that M contains at least one 0-minimal left ideal of S. Then every left ideal of M is also a left ideal of S.*

PROOF. Let L be a non-zero left ideal of M, and let $a \in L \backslash 0$. Then $Ma \neq 0$, for M is 0-simple by Theorem 2.29, and hence $MaM = M$ by Lemma 2.28.

By Theorem 2.33, there is a 0-minimal left ideal L_0 of S such that $a \in L_0 \subseteq M$. Since Ma is a non-zero left ideal of S contained in L_0, we conclude that $Ma = L_0$, and in particular $a \in Ma$. Hence, $L = \bigcup\{Ma : a \in L\}$. But the union of a set of left ideals of S is a left ideal of S.

REMARK. It was shown by W. E. Clark [1965] (see Vol. II, p. 334) that Theorem 2.35 would be false without the hypothesis that M contains at least one 0-minimal left ideal of S. Clearly it would suffice to assume that $a \in Ma$ for every a in M.

EXERCISES FOR §2.5

1. Let S be a semigroup with zero element 0. Then S is both left 0-simple and right 0-simple if and only if it is a group with zero.
2. A minimal left ideal of a semigroup S is a left simple subsemigroup of S.
3. Let S be a simple semigroup (without zero) containing an idempotent e and containing at least one minimal left ideal.
   (a) S is the union of its minimal left ideals.
   (b) The minimal left ideal L of S containing e is a left group, and eL is a group (cf. Theorem 1.27).
   (c) eS is a minimal right ideal of S. (Schwarz [1951].)

Exercises 4–7 are taken from Clifford and Miller [1948]. An element u of a semigroup S is called a *left* [*right*] *zeroid* of S if, for every element a of S, there exists x in S such that $xa = u$ [$ax = u$], that is, $u \in Sa$ [$u \in aS$]. An element of S is called a *zeroid* if it is both a left and a right zeroid. A left ideal L of S is called *universally minimal* in S if it is contained in every left ideal of S.

4. A semigroup S contains a left zeroid if and only if it contains a universally minimal left ideal L, and then L consists of all the left zeroids of S. By Lemma 2.32, L is also a right ideal, and is the kernel of S.
5. If a semigroup S contains a subgroup G which is also an ideal of S, then G is the kernel of S, and consists of all the zeroids of S.
6. If a semigroup S contains a zeroid, then every left zeroid is also a right zeroid, and vice versa, and the set K of all the zeroids of S is the kernel of S. Moreover, K is both left and right simple, and hence is a subgroup of S.
7. Let S be a semigroup with zeroid group K. Then the identity element e of K commutes with every element of S, and the mapping $x \to ex$ of S upon

K is a homomorphism of S upon K leaving the elements of K fixed. (If S is a topological semigroup, K is a homomorphic retract of S.) (Also found by Suschkewitsch: [1937], Chapter 3, §28, Theorem 5.)

8. (a) Let S be a semigroup containing exactly one idempotent e. Then e is a left zeroid of S if and only if it is a right zeroid of S, and then H_e is the group of zeroids of S. (Tamura [1954*b*].)

(b) Let S be a finite semigroup containing exactly one idempotent e. Then H_e is the group of zeroids of S, and $S^n = H_e$ for some positive integer n. (Tamura [1954*a*], where the structure of S is given completely when $n = 2$.)

9. (a) If a 0-simple semigroup contains no non-zero nilpotent elements, then it contains no proper divisors of zero.

(b) Let M be a maximal proper two-sided ideal of a semigroup S. Then M is prime (i.e., $S \setminus M$ is a subsemigroup of S) if and only if M is semiprime (i.e., $x^2 \in M$ implies $x \in M$—see §4.1). (Helen B. Grimble [1950].)

2.6 Principal factors of a semigroup

In this section we give a general definition of the principal factors of any semigroup S, and show (Theorem 2.40) that these are isomorphic with the factors of any principal series of S, if such exists. This definition and theorem are due to Green [1951].

An analogue for semigroups of the Jordan-Hölder-Schreier Theorem was stated and proved by Rees [1940]. We shall state but not prove it, since we shall have no occasion to use it in this book.

We begin with analogues of two of the isomorphism theorems for groups.

Theorem 2.36. *Let J be an ideal and T a subsemigroup of a semigroup S such that $J \cap T \neq \square$. Then $J \cap T$ is an ideal of T, $J \cup T$ is a subsemigroup of S, and*

$$(J \cup T)/J \cong T/(J \cap T).$$

Proof. Since

$$(J \cup T)^2 = J^2 \cup JT \cup TJ \cup T^2 \subseteq J \cup T,$$

$J \cup T$ is a subsemigroup of S. It is evident that $J \cap T$ is an ideal of T, and that J is an ideal of $J \cup T$. Hence the Rees quotients $(J \cup T)/J$ and $T/(J \cap T)$ are defined. Let their respective zero elements be denoted by 0 and 0′. Then

$$(J \cup T)/J = [(J \cup T)\setminus J] \cup 0 = (T \setminus J) \cup 0,$$

$$T/(J \cap T) = [T \setminus (J \cap T)] \cup 0' = (T \setminus J) \cup 0'.$$

Hence each of the two Rees quotients in question consists of the set of all elements of T not in J, together with a zero element. Not only are they isomorphic; they are even identical, if we identify their zero elements.

Theorem 2.37. *Let J be an ideal of a semigroup S, and let θ be the natural* *homomorphism of S upon the Rees factor S/J. Then θ induces a one-to-one, inclusion-preserving mapping $A \to A\theta = A/J$ of the set of all ideals A of S containing J upon the set of all ideals of S/J, and*

$$(S/J)\,/\,(A/J) \cong S/A.$$

PROOF. Let $S/J = (S\backslash J) \cup 0$. Then $A\theta = A/J = (A\backslash J) \cup 0$. Since θ is a homomorphism, $A\theta$ is an ideal of S/J. If Q is any ideal of S/J, then $A = Q\theta^{-1}$ is an ideal of S containing J $(= 0\theta^{-1})$, and clearly $A\theta = Q$. If $J \subseteq A \subset B$, where A and B are ideals of S, then $A\backslash J \subset B\backslash J$, whence (adding the element 0 to both sides) $A/J \subset B/J$. Finally if A and B are ideals of S containing J, and $A\theta = B\theta$, then $A\backslash J = B\backslash J$, and hence $A = B$.

To prove the last assertion in the theorem, let A be an ideal of S containing J. Let $0'$ and $0''$ be the zero elements of the Rees factor semigroups in question, so that

$$(S/J)\,/\,(A/J) = [(S/J)\backslash(A/J)] \cup 0',$$

$$S/A = (S\backslash A) \cup 0''.$$

Since $(S/J)\backslash(A/J) = S\backslash A$, the two factor semigroups are not only isomorphic, but (as in Theorem 2.36) essentially identical.

Part (i) of the following corollary is immediate from Theorem 2.37 and Theorem 2.29; part (ii) is likewise immediate from Theorem 2.37 and Lemma 2.26.

COROLLARY 2.38. (i) *If J and J' are ideals of a semigroup S with $J \subset J'$, then J is maximal in J' (in the sense that there is no ideal of S lying properly between them) if and only if J'/J is a 0-minimal ideal of S/J. If this is the case, then J'/J is either a 0-simple semigroup or a null semigroup.*

(ii) *An ideal J of a semigroup S is a maximal (proper) ideal of S if and only if S/J has no proper non-zero ideal, hence if and only if S/J is either 0-simple or the null semigroup of order two.*

We remark that in the case $J' \neq S$, J'/J may be a null semigroup of order greater than two. (See Exercise 3 below.)

Let a be an element of a semigroup S. As in §2.1, we denote by $J(a)$ the principal ideal S^1aS^1 of S generated by a, and by J_a the $\mathcal{J}$-class containing a, that is, the set of generators of $J(a)$. Let $I(a)$ consist of all those elements of $J(a)$ which do not generate $J(a)$, that is, $I(a) = J(a)\backslash J_a$. If $I(a)$ is not empty, it is an ideal of S. For suppose that $b \in I(a)$ and $c \in S$. Then $bc \in J(a)$ since $b \in J(a)$ and $J(a)$ is an ideal. Since $J(bc) \subseteq J(b) \subset J(a)$, we conclude that $bc \in I(a)$. Similarly, $cb \in I(a)$.

Since $I(a)$ is an ideal of S, it is in particular an ideal of $J(a)$. Each Rees factor semigroup $J(a)/I(a)$, with a in S, is called a *principal factor* of S. We make the convention that *if T is any semigroup then $T/\square$ means just T itself.*

LEMMA 2.39. *Each principal factor of any semigroup S is 0-simple, simple, or null. Only if S has a kernel is there a simple principal factor, and in this case the kernel is the only simple principal factor.*

PROOF. Let S be a semigroup, and let $a \in S$. We show first that the ideal $I(a)$ of S is maximal in $J(a)$. For suppose B is an ideal of S such that $I(a) \subset B \subseteq J(a)$. Let $b \in B \setminus I(a)$. Then $b \in J(a) \setminus I(a) = J_a$, so that $J(b) = J(a)$. But $J(b) \subseteq B$, and hence $B = J(a)$.

If $I(a)$ is empty, then the foregoing shows that $J(a)$ is minimal in S, and hence must be the kernel of S; it is simple by Corollary 2.30. If $I(a) \neq \Box$ then $J(a)/I(a)$ is 0-simple or null by Corollary 2.38 (i).

By a *principal series* of a semigroup S we mean a chain

$$S = S_1 \supset S_2 \supset \cdots \supset S_m \supset S_{m+1} = \Box \tag{1}$$

of ideals S_i $(i = 1, \cdots, m)$ of S, beginning with S and ending with the empty set, and such that there is no ideal of S strictly between S_i and S_{i+1} $(i = 1, \cdots, m)$. By the *factors* of the principal series (1) we mean the Rees factor semigroups S_i/S_{i+1} $(i = 1, \cdots, m)$. By Corollary 2.38 (i), S_i/S_{i+1} is either 0-simple (simple if $i = m$) or null.

THEOREM 2.40. *Let S be a semigroup admitting a principal series* (1). *Then the factors of* (1) *are isomorphic in some order to the principal factors of S. In particular, any two principal series of S have isomorphic factors. The last term in any principal series of S is the kernel of S.*

PROOF. Consider one of the factors S_i/S_{i+1} of the principal series (1). Let $a \in S_i \setminus S_{i+1}$. Clearly $J(a) \cup S_{i+1}$ is an ideal of S between S_{i+1} and S_i, and properly containing S_{i+1}, since $a \notin S_{i+1}$. Hence $J(a) \cup S_{i+1} = S_i$.

Let $b \in I(a)$. Then $b \in S_{i+1}$, for otherwise we would conclude (as for a) that $J(b) \cup S_{i+1} = S_i$, and hence that $a \in J(b)$, contrary to $b \in I(a)$. Hence $I(a) \subseteq S_{i+1}$.

On the other hand, if $c \in J(a) \cap S_{i+1}$, then $J(c) \subseteq S_{i+1}$; hence $J(c) \neq J(a)$, so that $c \in I(a)$. We conclude that $I(a) = J(a) \cap S_{i+1}$.

By Theorem 2.36,

$$J(a)/(J(a) \cap S_{i+1}) \cong (J(a) \cup S_{i+1})/S_{i+1}.$$

But the left-hand factor is just $J(a)/I(a)$, and the right-hand factor is S_i/S_{i+1}. We conclude that S_i/S_{i+1} is isomorphic with the principal factor $J(a)/I(a)$.

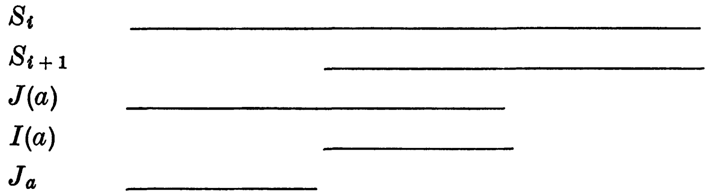
Moreover,

$$J_a = J(a) \setminus I(a) = (J(a) \cup S_{i+1}) \setminus (I(a) \cup S_{i+1}) = S_i \setminus S_{i+1}.$$

Hence if $a' \in S_i \setminus S_{i+1}$, then $J(a') = J(a)$, so that the principal factor $J(a)/I(a)$ corresponding to S_i/S_{i+1} is independent of the choice of the element a in $S_i \setminus S_{i+1}$. On the other hand, if a is any element of S, there must exist

$i (1 \leqq i \leqq m)$ such that $a \in S_i$ and $a \notin S_{i+1}$; for $a \in S_1$ and $a \notin S_{m+1}$. Consequently the mapping $S_i/S_{i+1} \to J(a)/I(a)$ is one-to-one from the set of factors of (1) upon the set of principal factors of S. The last assertion of the theorem follows from the last assertion of Lemma 2.39.

This concludes the proof of Theorem 2.40, but it is instructive to indicate schematically the inclusion pattern of the sets S_i, S_{i+1}, $J(a)$, $I(a)$, and J_a when $a \in S_i \backslash S_{i+1}$:



A chain (1) of subsets S_i of S is called a *relative ideal series* of S if each S_{i+1} is an ideal of S_i $(i = 1, \cdots, m-1)$; and the Rees factors S_i/S_{i+1} $(i = 1, \cdots, m)$ are called the *factors* of the series. Two relative ideal series are called *isomorphic* if their factors are isomorphic in some order. One relative ideal series is said to be *refinement* of a second relative ideal series if each term of the second series is also a term of the first series. A relative ideal series is called a *composition series* if it does not possess a proper refinement. From Corollary 2.38 (ii), each factor of a composition series (1) is either 0-simple or the null semigroup of order two, except for the last factor $S_m/\square = S_m$, which is always simple (and S_m is the kernel of S).

The analogue for semigroups of the Jordan-Hölder-Schreier Theorem asserts that *any two relative ideal series of a semigroup S have isomorphic refinements; in particular, any two composition series of S are isomorphic.* This is proved by Rees [1940] using the Zassenhaus method (H. Zassenhaus, *The Theory of Groups*, translated by S. Kravetz, Chelsea, New York, 1949; Chapter II, §5). Once we have the two isomorphism theorems (2.36 and 2.37), the proof (which we omit) closely parallels that for groups.

We call a semigroup S *semisimple* if every principal factor of S is 0-simple or simple. This amounts to excluding null factors (Lemma 2.39).

THEOREM 2.41. *Every ideal of an ideal of a semisimple semigroup S is an ideal of S.*

PROOF. Let S be a semisimple semigroup. Let A be an ideal of S, and let B be an ideal of the semigroup A. Now ABA is an ideal of S contained in B. If $ABA = B$, then B is an ideal of S, as claimed by the theorem. We proceed to show that $ABA \subset B$ leads to a contradiction.

Let $b \in B \backslash ABA$. Since, by hypothesis, S is semisimple, the principal factor $J(b)/I(b)$ is 0-simple or simple, and so

$$(J(b)/I(b))^3 = J(b)/I(b).$$

Hence,

$$J(b)^3 \cup I(b) = J(b).$$

Now

$$S^1bS^1S^1bS^1S^1bS^1 \subseteq S^1bS^1bS^1bS^1,$$

in other words,

$$J(b)^3 \subseteq J(b)bJ(b).$$

Moreover, $J(b) \subseteq A$ and $b \in B$. Hence

$$J(b)^3 \subseteq ABA.$$

Consequently,

$$J(b) = J(b)^3 \cup I(b) \subseteq ABA \cup I(b).$$

But this is absurd, since $b \in J(b)$, yet b belongs to neither ABA nor $I(b)$.

The concluding corollary, due to Munn [1955*b*], is immediate from Theorem 2.41.

COROLLARY 2.42. *The terms of any relative ideal series of a semisimple semigroup S are ideals of S. In particular, there is no distinction between principal series and composition series of a semisimple semigroup.*

EXERCISES FOR §2.6

1. Let T be the full transformation semigroup $\mathscr{T}_X$ on a finite set X of cardinal n.
   (a) Every ideal of T is principal.
   (b) The semigroup T has one and only one principal series,

$$T = T_n \supset T_{n-1} \supset \cdots \supset T_1 \supset T_0 = \square,$$

where T_r consists of all elements of T of rank $\leqq r$ $(r = 1, \cdots, n)$; note Theorem 2.9.
   (c) The semigroup T is semisimple.

2. If a semigroup S has a composition series it also has a principal series. (Munn [1955*a*].)

3. The following is an example of a semigroup S having a principal series, but not having any composition series. Let $S = A \cup B \cup \{0\}$ where A is the infinite cyclic group generated by an element a; where

$$B = \{\cdots, b_{-2}, b_{-1}, b_0, b_1, b_2, \cdots\};$$

and products in S are defined as follows (i and j any integers):

$$a^ib_j = b_{i+j},\ b_ja^i = b_ib_j = 0,$$

and 0 is the zero of S. (Munn [1955*a*].)

4. (a) If A is an ideal of a semigroup S, and if B is an ideal of A such that $B^2 = B$, then B is an ideal of S. (Carman [1949].)

(b) If S_i is a term of a relative ideal series of a semigroup S, and $S_i^2 = S_i$ then S_i is an ideal of S. (Munn [1955*b*].)

5. Let S be a semigroup having a composition series. Then every simple or 0-simple principal factor of S is also a composition factor, and conversely. More specifically, let $\cdots \supset S_i \supset S_{i+1} \supset \cdots$ be part of a principal [composition] series of S such that S_i/S_{i+1} is simple or 0-simple. Then $S_i \backslash S_{i+1}$ is a $\mathscr{J}$-class of S, and there exists a composition [principal] series of S with a part $\cdots \supset T_j \supset T_{j+1} \supset \cdots$ such that $T_j \backslash T_{j+1} = S_i \backslash S_{i+1}$.

6. If a semigroup S admits a principal series, and if A is any ideal of S, then there exists a principal series of S having A as a term.

7. (a) A semigroup S is semisimple if and only if $A^2 = A$ for every ideal A of S.

(b) Let A be an ideal of a semigroup S. Then S is semisimple if and only if both A and S/A are semisimple. (Munn [1955*b*].)

2.7 COMPLETELY 0-SIMPLE SEMIGROUPS

Let E be the set of idempotents of a semigroup S. If $e, f \in E$, we define $e \leqq f$ to mean $ef = fe = e$. It was shown in §1.8 that $\leqq$ is a partial ordering of E. If S contains a zero element 0, then $0 \leqq e$ for every e in E. An idempotent element f of S is called *primitive* if $f \neq 0$ and if $e \leqq f$ implies $e = 0$ or $e = f$. (This is the usual definition if S is a ring.)

By a *completely* [0-] *simple semigroup* we mean a [0-] simple semigroup containing a primitive idempotent.

For example, *any finite* [0-] *simple semigroup is completely* [0-] *simple.* For S must contain an idempotent (§1.6), so that $E \neq \square$. Furthermore $E \neq 0$, since $E = 0$ would imply that every element of S, and hence S itself, is nilpotent, contrary to $S^2 = S$. It is then clear that the finite partially ordered set $E \backslash 0$ must contain a minimal element, that is, a primitive idempotent.

We shall show in this section (Theorem 2.48) that a 0-simple semigroup S is completely 0-simple if and only if S contains at least one 0-minimal left ideal and at least one 0-minimal right ideal. We shall also show (Theorem 2.51) that a completely 0-simple semigroup S is 0-*bisimple*, that is, that $S \backslash 0$ is a $\mathscr{D}$-class of S. The results of this section center around the 0-minimal one-sided ideals of a completely 0-simple semigroup S, and the $\mathscr{D}$-structure of S. The complete determination of all possible completely 0-simple semigroups, which is given by the Rees Theorem (3.5), will be one of the main objectives of the next chapter.

We recall once more the remark made in §2.5 that the results of this section apply to a completely simple semigroup S; one simply adjoins a zero element to S.

LEMMA 2.43. *If L is a 0-minimal left ideal of a semigroup S with zero 0, then $L \backslash 0$ is an $\mathscr{L}$-class of S.*

Proof. Let $a \in L \backslash 0$. Then either $Sa = L$ or $Sa = 0$. If $Sa = L$ for every a in $L \backslash 0$, then $S^1a = S^1b$ for every a and b in $L \backslash 0$, so that $L \backslash 0 \subseteq L_a$. If $c \in L_a$ then $c \in S^1a = L$, so that $L_a \subseteq L \backslash 0$. Hence $L \backslash 0$ is the $\mathscr{L}$-class L_a.

Suppose $Sa = 0$ for some a in $L \backslash 0$. Then $\{0, a\}$ is a non-zero left ideal of S contained in L, whence $L = \{0, a\}$. Then $S^1a = L$, and $S^1x = S^1a$ implies $x = a$. Hence in this case also, $L \backslash 0 = \{a\} = L_a$.

Lemma 2.44. *Let S be a 0-simple semigroup containing a 0-minimal left ideal and a 0-minimal right ideal. Then to each 0-minimal left ideal L of S corresponds at least one 0-minimal right ideal R of S such that $LR \neq 0$.*

Proof. We note that LS is a (two-sided) ideal of S, and hence $LS = S$ or $LS = 0$. If $LS = 0$ then $L^2 = 0$, contrary to Lemma 2.34. Hence $LS = S$. In particular, $Lc \neq 0$ for some c in S. By the dual of Theorem 2.33, S is the union of 0-minimal right ideals. Hence $c \in R$ for some 0-minimal right ideal R of S, and evidently $LR \neq 0$.

Lemma 2.45. *Let L be a 0-minimal left ideal of a 0-simple semigroup S, and let $a \in L \backslash 0$. Then $Sa = L$.*

Proof. Since Sa is a left ideal of S contained in L, it follows that $Sa = 0$ or $Sa = L$. The case $Sa = 0$ is ruled out by Lemma 2.28.

Lemma 2.46. *Let S be a 0-simple semigroup. Let L and R be 0-minimal left and right ideals of S, respectively, such that $LR \neq 0$. Then:* (i) *$LR = S$;* (ii) *RL is a group with zero;* (iii) *$RL = R \cap L$. Let e be the identity element of the group $RL \backslash 0$. Then:* (iv) *$R = eS$, $L = Se$, and $RL = eSe$;* (v) *e is a primitive idempotent of S.*

Proof. (i) Since LR is a non-zero two-sided ideal of the 0-simple semigroup S, we must have $LR = S$.

(ii) From $S = S^2 = LRLR$, we conclude that $RL \neq 0$. We prove that RL is a group with zero by showing that $RLa = aRL = RL$ for any element $a \neq 0$ of RL. Let $a \in RL \backslash 0$. Then $a \in R \backslash 0$, and so $aS = R$ by the dual of Lemma 2.45. Since $S = LR = LaS$, it follows that $La \neq 0$. Thus La is a non-zero left ideal of S contained in L (since $a \in L$), and hence $La = L$. Consequently $RLa = RL$. The proof that $aRL = RL$ is dual.

(iii) Let e be the identity element of the group $RL \backslash 0$. By Lemma 2.43, $(R \backslash 0) \cap (L \backslash 0)$ is an $\mathscr{H}$-class of S. It contains the idempotent e, and is therefore a group by Green's Theorem (2.16). Hence $R \cap L$ is a group with zero. If $a \in R \cap L$ then $a = ae \in RL$ since $a \in R$ and $e \in L$. Thus $R \cap L \subseteq RL$. Since we always have $RL \subseteq R \cap L$, equality follows.

(iv) Since $e \in L \backslash 0$, it follows from Lemma 2.45 that $Se = L$. Dually, $eS = R$. Evidently, $RL = eSSe = eSe$.

(v) Suppose f is an idempotent of S such that $f \leqq e$. Then $f \in eSe$. But $eSe = RL$ by (iv), and RL is a group with zero by (ii). Since the only idempotents in a group with zero are the identity and the zero, it follows that $f = e$ or $f = 0$, and hence e is primitive.

LEMMA 2.47. *Let S be a completely 0-simple semigroup, and let e be a primitive idempotent of S. Then $L = Se$ and $R = eS$ are 0-minimal left and right ideals of S, respectively, such that RL $(= eSe = R \cap L)$ is a group with zero having e as identity element.*

PROOF. To show that $R = eS$ is 0-minimal, we first note that $R \neq 0$ since $e \in R$. Let A be a non-zero right ideal of S contained in R, and let $a \in A \setminus 0$. Since $a \in eS$, we have $ea = a$. Since S is 0-simple, and $a \neq 0$, we have $SaS = S$ (Lemma 2.28), and so there exist x' and y' in S such that $x'ay' = e$. Setting $x = ex'e$ and $y = y'e$, we have

$$xay = e, \qquad ex = xe = x, \qquad ye = y.$$

Setting $f = ayx$, we have

$$\begin{aligned} f^2 &= ay(xay)x = ayex = ayx = f, \\ ef &= (ea)yx = ayx = f, \\ fe &= ay(xe) = ayx = f. \end{aligned}$$

Furthermore,

$$e = e^2 = x(ayx)ay = xfay,$$

and so $f \neq 0$. Thus f is a non-zero idempotent under e. By hypothesis, e is primitive, and hence $f = e$. It follows that $e = ayx \in aS$, and so $R = eS \subseteq aS^2 \subseteq A$. Hence $A = R$, and R is 0-minimal.

Dually, we can prove that L is 0-minimal. Since $LR = SeS = S \neq 0$, it follows from Lemma 2.46 that RL is a group with zero. Since $e \in eSe = eS^2e = RL$, and $e \neq 0$, it is clear that e is the identity element of RL.

The next theorem is due to Clifford [1949].

THEOREM 2.48. *Let S be a 0-simple semigroup. Then S is completely 0-simple if and only if it contains at least one 0-minimal left ideal and at least one 0-minimal right ideal.*

PROOF. If S is completely 0-simple, then it contains a primitive idempotent e. By Lemma 2.47, $L = Se$ and $R = eS$ are 0-minimal left and right ideals of S, respectively.

Conversely, assume that S contains at least one 0-minimal left ideal and at least one 0-minimal right ideal. Let L be a 0-minimal left ideal of S. By Lemma 2.44, there exists a 0-minimal right ideal R of S such that $LR \neq 0$. It then follows from Lemma 2.46 (v) that S contains a primitive idempotent, and so is completely 0-simple.

COROLLARY 2.49. *A completely 0-simple semigroup is the union of its 0-minimal left [right] ideals.*

PROOF. This is immediate from Theorem 2.48 and Theorem 2.33.

The following is due to Rich [1949].

COROLLARY 2.50. *Let M be a 0-minimal ideal of a semigroup S such that $M^2 \neq 0$. Assume, moreover, that M contains at least one 0-minimal left ideal of S and at least one 0-minimal right ideal of S. Then M is a completely 0-simple subsemigroup of S.*

PROOF. By Theorem 2.29, M is a 0-simple subsemigroup of S. By Theorem 2.35, the 0-minimal left [right] ideals of S contained in M are also 0-minimal as left [right] ideals of M. By Theorem 2.48, M is completely 0-simple.

Rich (loc. cit.) also established a converse of Corollary 2.50. This is given as Exercise 6 below.

The following theorem is due to Green [1951].

THEOREM 2.51. *A completely 0-simple semigroup is 0-bisimple and regular.*

PROOF. Let S be a completely 0-simple semigroup. Let a and b be non-zero elements of S; we are to show that $a\mathscr{D}b$. By Corollary 2.49, a belongs to some 0-minimal left ideal L of S, and b belongs to some 0-minimal right ideal R of S. By Lemma 2.45, $L = Sa$ and $R = bS$. By Lemma 2.43 and its dual, $L_a = L\setminus 0$ and $R_b = R\setminus 0$. Since $a \in L$ and $b \in R$, $bSa \subseteq R \cap L$. Since S is 0-simple, and $a \neq 0$, $b \neq 0$, we have $SaS = S$ and $SbS = S$. Hence,

$$S = S^2 = SbSSaS \subseteq S(bSa)S,$$

so that $bSa \neq 0$. Since $R_b \cap L_a$ contains the non-empty set $bSa\setminus 0$, we conclude that $a\mathscr{D}b$.

By definition of complete 0-simplicity, the $\mathscr{D}$-class $S\setminus 0$ contains a (primitive) idempotent. By Theorem 2.11 (i), every element of $S\setminus 0$ is regular. Since 0 is regular, we conclude that S is regular.

THEOREM 2.52. *Let S be a completely 0-simple semigroup.*

(i) *If $a \in S$ and $a^2 \neq 0$, then $a^2 \in H_a$, and H_a is a group.*

(ii) *If $a, b \in S$ and $ab \neq 0$, then $ab \in R_a \cap L_b$.*

(iii) *If $a, b \in S$, then $H_aH_b = 0$ or $H_aH_b = R_a \cap L_b$; in either case, $H_aH_b = H_{ab}$.*

PROOF. By Corollary 2.49, a belongs to some 0-minimal left ideal L of S. Then $a^2 \in L$ also. By Lemma 2.43, $L\setminus 0$ is an $\mathscr{L}$-class of S. Since $a^2 \neq 0$ by hypothesis, and hence also $a \neq 0$, both a and a^2 belong to $L\setminus 0$, so that $a\mathscr{L}a^2$. By the dual argument, $a\mathscr{R}a^2$. Hence $a\mathscr{H}a^2$, and by Green's Theorem (2.16) it follows that H_a is a group.

(ii) From $ab \neq 0$ we have $a \neq 0$ and $b \neq 0$. By Theorem 2.51, $a\mathscr{D}b$, and hence $R_b \cap L_a \neq \square$. Let $c \in R_b \cap L_a$. Then $c^2 \in L_aR_b$. By Theorem 2.4, either $L_aR_b = 0$ or $L_aR_b \subseteq S\setminus 0$. The former is excluded by $ab \neq 0$. Hence $c^2 \neq 0$. By (i), H_c $(= R_b \cap L_a)$ is a group. By Theorem 2.17, $ab \in R_a \cap L_b$.

(iii) If $ab = 0$, then $H_aH_b \subseteq L_aR_b = 0$, by Theorem 2.4. In this case, $H_aH_b = 0 = H_0 = H_{ab}$. If $ab \neq 0$, then $ab \in R_a \cap L_b$ by (ii), and hence $H_aH_b = R_a \cap L_b = H_{ab}$ by Theorem 2.17.

Let S be a completely 0-simple semigroup. Let $\{R_i^0 : i \in I\}$ and $\{L_\lambda^0 : \lambda \in \Lambda\}$ be the 0-minimal right ideals and the 0-minimal left ideals of S, respectively. Here I and Λ are index sets, non-empty by Theorem 2.48. For each i in I and λ in Λ, let

$$R_i = R_i^0 \setminus 0, \qquad L_\lambda = L_\lambda^0 \setminus 0, \qquad H_{i\lambda} = R_i \cap L_\lambda.$$

By Lemma 2.43 and its dual, each L_λ $[R_i]$ is an $\mathscr{L}$ $[\mathscr{R}]$-class of S, and $H_{i\lambda}$ is an $\mathscr{H}$-class. By Theorem 2.51, $S \setminus 0$ is a $\mathscr{D}$-class of S, and, by Corollary 2.49, it is the union of all the R_i $(i \in I)$, of all the L_λ $(\lambda \in \Lambda)$, and of all the $H_{i\lambda}$ $(i \in I,\ \lambda \in \Lambda)$. The following is just a restatement of Theorem 2.52.

COROLLARY 2.52a. *Let S be a completely 0-simple semigroup. Then, with the notation just introduced, the following assertions are true.*

(i) *For every i in I and λ in Λ, either $H_{i\lambda}$ is a (maximal) subgroup of S, or else $H_{i\lambda}^2 = 0$.*

(ii) *For every i, j in I and λ, μ in Λ, the product $H_{i\lambda}H_{j\mu}$ is equal either to $H_{i\mu}$ or to 0.*

The following specialization to the case when S has no zero was obtained by Suschkewitsch for finite simple semigroups in his 1928 paper; see Appendix A.

COROLLARY 2.52b. *Let S be a completely simple semigroup, and let $\{R_i : i \in I\}$ and $\{L_\lambda : \lambda \in \Lambda\}$ be its minimal right ideals and its minimal left ideals, respectively. Then, for every i in I and λ in Λ, the intersection $H_{i\lambda} = R_i \cap L_\lambda$ is a (maximal) subgroup of S. Moreover, for every i, j in I and λ, μ in Λ, we have $H_{i\lambda}H_{j\mu} = H_{i\mu}$.*

REMARK. If we define a product $\circ$ on the set $I \times \Lambda$ by

$$(i, \lambda) \circ (j, \mu) = (i, \mu), \quad (i, j \text{ in } I \text{ and } \lambda, \mu \text{ in } \Lambda),$$

then $I \times \Lambda$ becomes what we called a rectangular band in §1.8. In terms of the concept of a band of semigroups, also introduced in §1.8, Corollary 2.52b asserts that *every completely simple semigroup is a rectangular band of groups.* The converse is easily shown (Exercise 4 below).

In §1.12 we defined the *bicyclic semigroup* to be the semigroup $\mathscr{C}(p, q)$ with identity element generated by two symbols p and q subject to the single generating relation $pq = 1$.

THEOREM 2.53. *The bicyclic semigroup $\mathscr{C} = \mathscr{C}(p, q)$ is a bisimple inverse semigroup with identity element. The idempotent elements of $\mathscr{C}$ are $e_n = q^np^n$ $(n = 0, 1, 2, \cdots)$. These satisfy $1 = e_0 > e_1 > e_2 > \cdots > e_n > e_{n+1} > \cdots$, and so $\mathscr{C}$ contains no primitive idempotent.*

PROOF. It is a matter of straightforward computation to show (cf. Exercise 2 of §1.12) that two elements $q^k p^l$ and $q^m p^n$ of $\mathscr{C}$ (k, l, m, n non-negative integers) multiply as follows:

$$(q^k p^l)(q^m p^n) = q^i p^j,$$

where

$$i = k + m - \min(l, m),$$

$$j = l + n - \min(l, m).$$

We note that $i \geqq k$. Hence $i \geqq k$ if $q^i p^j$ belongs to the principal right ideal $R(q^k p^l)$ generated by $q^k p^l$. Conversely, if $i \geqq k$, then $q^i p^j \in R(q^k p^l)$; we need only take $m = l + i - k$ and $n = j$.

If we write out the elements of $\mathscr{C}$ in the array

$$\begin{array}{cccc} 1 & p & p^2 & \cdots \\ q & qp & qp^2 & \cdots \\ q^2 & q^2p & q^2p^2 & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

we can describe $R(q^k p^l)$ as the union of all the rows of the array from the $(k + 1)$th on down. It follows that the $\mathscr{R}$-class containing $q^k p^l$ is just the $(k + 1)$th row. Similarly, the $\mathscr{L}$-class containing $q^k p^l$ is just the $(l + 1)$th column. Thus the $\mathscr{R}$-classes of $\mathscr{C}$ are the rows, and the $\mathscr{L}$-classes of $\mathscr{C}$ are the columns, of the array. The $\mathscr{H}$-classes of $\mathscr{C}$ thus reduce to single elements. Since each $\mathscr{R}$-class meets each $\mathscr{L}$-class, the only $\mathscr{D}$-class of $\mathscr{C}$ is $\mathscr{C}$ itself, and so $\mathscr{C}$ is bisimple.

Suppose now that $q^m p^n$ is idempotent. Then $q^m p^n = q^i p^j$ with $i = 2m - \min(m, n)$ and $j = 2n - \min(m, n)$. By Lemma 1.31, we must have $m = i$ and $n = j$. This implies $m = \min(m, n)$ and $n = \min(m, n)$, hence, $m = n$. Conversely, it is immediately seen that $e_n = q^n p^n$ is idempotent. If $m < n$, then direct calculation shows that $e_m e_n = e_n e_m = e_n$, so that $e_m \geqq e_n$. By Lemma 1.31, $e_m \neq e_n$ if $m \neq n$. Hence, $m < n$ implies $e_m > e_n$, and it is then clear that $\mathscr{C}$ contains no primitive idempotent.

Since $\mathscr{C}$ contains an idempotent, and consists of a single $\mathscr{D}$-class, it is regular by Theorem 2.11. Since $e_m e_n = e_n = e_n e_m$ if $m < n$, the idempotents of $\mathscr{C}$ commute, and $\mathscr{C}$ is an inverse semigroup by Theorem 1.17.

The next theorem, due to Olaf Andersen [1952], shows that a non-completely 0-simple semigroup containing an idempotent is a kind of web of bicyclic semigroups. It is also useful in proving certain criteria for complete 0-simplicity, such as Theorem 2.55 and Corollary 2.56 below.

THEOREM 2.54. *If e is any non-zero idempotent of a 0-simple semigroup S which is not completely 0-simple, then S contains a bicyclic subsemigroup having e as identity element.*

PROOF. The idempotent e is not primitive, since otherwise S would be completely 0-simple. Hence, there exists a non-zero idempotent f in S such that $e > f$, that is, $ef = fe = f$ and $e \neq f$. Since $f \neq 0$ and S is 0-simple, $SfS = S$, and so there exist x' and y' in S such that $x'fy' = e$. Setting $x = ex'f$ and $y = fy'e$, we have

$$ex = xf = x, \qquad fy = ye = y, \qquad \text{and} \qquad xy = e.$$

Let $g = yx$. Then

$$g^2 = yxyx = yex = yx = g,$$
$$fg = fyx = yx = g,$$
$$gf = yxf = yx = g.$$

Thus g is an idempotent under f, and hence under e. In fact $g < e$ since $g \leqq f$ and $f < e$.

Noting that e is a two-sided identity for x and y, that $xy = e$, and that $yx \neq e$, it follows from Lemma 1.31 that $\langle x, y \rangle$ is a bicyclic subsemigroup of S with identity element e.

The next theorem is due to Munn [1961].

THEOREM 2.55. *A 0-simple semigroup S is completely 0-simple if and only if some power of each element of S lies in a subgroup of S.*

PROOF. If S is completely 0-simple, then the square of each element of S lies in a subgroup of S, by Theorem 2.52 (i).

Assume conversely that some power of each element of S lies in a subgroup of S. We show first that not every element of S is nilpotent. Let $a \neq 0$ in S. Then $a \in SaS$ by Lemma 2.28, and so $a = xay$ for some x and y in S. By repeated multiplication by x on the left and y on the right, we have $a = x^n a y^n$ for every positive integer n. Since $a \neq 0$ we conclude that $x^n \neq 0$ for every n, that is, x is not nilpotent.

It follows that S must contain a non-zero idempotent. For if x is a non-nilpotent element of S, then (by hypothesis) x^n belongs to a subgroup G of S for some positive integer n, and clearly the identity element of G is not 0.

Let e be a non-zero idempotent of S. If S were not completely 0-simple, then, by Theorem 2.54, S would contain a bicyclic subsemigroup $\langle p, q \rangle$ with identity element e, and with $pq = e$ and $qp \neq e$. We proceed to show that this is impossible.

By hypothesis, p^n belongs to some subgroup G of S for some positive integer n. Let f be the identity element of G, and let r be the inverse of p^n in G. From $p^n q^n = e$ we have

$$fe = fp^n q^n = p^n q^n = e.$$

From $rp^n = f$ we have

$$fe = rp^n e = rp^n = f.$$

Hence, $e = f$, and so

$$q^n = eq^n = fq^n = rp^nq^n = re = rf = r.$$

But then

$$q^np^n = rp^n = f = e,$$

whereas, $q^np^n < e$ in the bicyclic semigroup $\langle p, q\rangle$.

Corollary 2.56. *Any periodic (in particular, any finite) 0-simple semigroup is completely 0-simple.*

This was shown by Rees [1940]. A quick proof for the finite case was given in the third paragraph of the present section.

Exercises for §2.7

1. A completely 0-simple semigroup contains an identity element if and only if it is a group with zero. (Rees [1940].)
2. A non-zero right ideal of a completely 0-simple semigroup is a principal right ideal if and only if it is 0-minimal.
3. Every right ideal of the bicyclic semigroup $\mathscr{C}$ is principal. There is a one-to-one mapping $n \to R_n$ of the set N of non-negative integers n upon the set of right ideals R_n of $\mathscr{C}$ such that

$$\mathscr{C} = R_0 \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset R_{n+1} \supset \cdots.$$

It is evident from this that $\mathscr{C}$ contains no minimal right ideal.

4. A rectangular band of groups (§1.8) is completely simple. (Note the remark after Corollary 2.52b.)
5. Every non-zero idempotent element of a completely 0-simple semigroup is primitive. (Rees [1941].)
6. If M is an ideal of a semigroup S with zero, and if M is a completely 0-simple subsemigroup of S, then M is a 0-minimal ideal of S containing at least one 0-minimal right ideal of S and at least one 0-minimal left ideal of S. (Rich [1949]; this is the converse of Corollary 2.50.)
7. Let I be a set. Let $S = (I \times I) \cup \{0\}$. For i, j, k, l in I, define

$$(i,j) \cdot (k,l) = \begin{cases} (i,l) & \text{if } j = k, \\ 0 & \text{if } j \neq k; \end{cases}$$

$$0 \cdot (i,j) = (i,j) \cdot 0 = 0 \cdot 0 = 0.$$

Then S is a completely 0-simple semigroup; we call it the *semigroup of $I \times I$ matrix units.*

8. A direct product of semigroups is [completely] simple if and only if each direct factor is [completely] simple. (Ivan [1953].)
9. Let S be a completely simple semigroup. Then $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{H}$ are congruences on S, and $S/\mathscr{H}$ is a rectangular band isomorphic to the rectangular band on $S/\mathscr{R} \times S/\mathscr{L}$.

10. Let e be a primitive idempotent of a regular semigroup S with zero. Then eS is a 0-minimal right ideal of S.

11. A semigroup S with zero is completely 0-simple if and only if it satisfies the following three conditions:

(i) S is regular;

(ii) every non-zero idempotent of S is primitive;

(iii) if e and f are non-zero idempotents of S, then $eSf \neq 0$.

12. A 0-simple semigroup is completely 0-simple if and only if it contains a 0-minimal left ideal and a non-zero idempotent element. (Schwarz [1951].)

13. (a) If L is a minimal left ideal of a semigroup S, and A is any ideal of S, then $L \subseteq A$. (Consider the non-empty set AL.)

(b) A semigroup S containing a minimal left ideal has a kernel K, and K is the union of all the minimal left ideals of S. If S also contains a minimal right ideal, then K is completely simple.

(c) If L and R are minimal left and right ideals, respectively, of a semigroup S, then $L \cap R$ is a (maximal) subgroup of S. (Suschkewitsch [1928] for finite S (see Appendix A); Clifford [1948].)

14. Let e be an idempotent element of a semigroup S without zero. Then the following are mutually equivalent:

(i) Se is a minimal left ideal of S;

(ii) SeS is the kernel K of S, and K is completely simple;

(iii) eSe is the maximal subgroup H_e of S containing e. (Koch [1953]; stated as Theorem 4.1 in Wallace [1955].)

15. Call a subsemigroup B of a semigroup S a *bi-ideal* of S if $BSB \subseteq B$.

(a) If C is any non-empty subset of S, then $C \cup C^2 \cup CSC$ is the smallest bi-ideal of S containing C.

(b) S is a group if and only if it contains no proper bi-ideal.

(c) Let B be a subgroup and a bi-ideal of S, and let e be the identity element of B. Then $eSe = B = H_e$, so that (by Exercise 14) SeS is the kernel of S, and is completely simple.

(d) Let B be a bi-ideal of S, and let e be an idempotent element in the kernel of the semigroup B. Then e belongs to the kernel of S.

(e) Let $S = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_n$ be a finite sequence of subsemigroups of S such that B_i is a bi-ideal of B_{i-1} $(i = 1, 2, \cdots, n)$, and such that B_n has no proper bi-ideal. Then B_n is a maximal subgroup of S contained in the kernel K of S, and K is completely simple. (Good and Hughes [1952].)

16. Let S be a semigroup having a completely simple kernel K, and let e be an idempotent element of K. Then any homomorphism of S upon a group G induces a homomorphism of H_e upon G. Hence if H_e is itself a homomorphic image of S, then it is a maximal group homomorphic image. By Exercise 7 of §2.5, this is the case if S is a semigroup having zeroid elements. (Good and Hughes [1952].)

17. Call a subset $A \neq \square$ of a semigroup S a *quasi-ideal* of S if $AS \cap SA \subseteq A$.

(a) If C is any subset $\neq \square$ of S, then $(C \cup SC) \cap (C \cup CS)$ is the smallest quasi-ideal of S containing C.

(b) A subset of S is a quasi-ideal of S if and only if it is the intersection of a right ideal of S and a left ideal of S.

(c) S is a group if and only if it contains no proper quasi-ideal.

(d) A semigroup S contains a minimal quasi-ideal if and only if it contains a completely simple kernel K. If this is so, then the minimal quasi-ideals of S are just the maximal subgroups of K.

(e) If Q is a 0-minimal quasi-ideal of a semigroup S with zero, then either Q is a group with zero, or else Q is a null semigroup. (Steinfeld [1956, 1957].)

18. (a) Every quasi-ideal of S is also a bi-ideal of S. (See Exercises 15 and 17 for definitions.)

(b) Let B be a bi-ideal of S, let $R = B \cup BS$, and let $L = B \cup SB$. Then $RL \subseteq B \subseteq R \cap L$.

(c) If R is a right ideal and L a left ideal of S, then any subset B of S satisfying $RL \subseteq B \subseteq R \cap L$ is a bi-ideal of S.

(d) S has the property that every bi-ideal is also a quasi-ideal if S is regular (cf. Exercise 11 of §1.9). (Lajos [1961].[1])

19. (a) Every subsemigroup T of a finite simple semigroup S is also simple. In fact, if S has the structure described in Corollary 2.52b, then there are subsets I' of I and Λ' of Λ such that $H'_{i\lambda} = T \cap H_{i\lambda} \neq \square$ if and only if $i \in I'$ and $\lambda \in \Lambda'$, and T is the union of the groups $H'_{i\lambda}$ (i in I', λ in Λ').

(b) If K is the kernel of a finite semigroup S, and T is a subsemigroup of S such that $T \cap K \neq \square$, then $T \cap K$ is the kernel of T.

(c) If S is a finite semigroup, and $a \in S$, then the kernel of the semigroup Sa is the union of some or all of the minimal left ideals of S.

(d) If S is a finite semigroup, and if $a, b \in S$, then the kernel of the semigroup aSb is the union of some or all of the maximal subgroups of S contained in the kernel of S. (Suschkewitsch [1937], Chapter 3, §27.)

[1] S. Lajos, *Generalized ideals in semigroups*, Acta Sci. Math. Szeged 22 (1961), 217-222.

http://dx.doi.org/10.1090/surv/007.1/03

CHAPTER 3

REPRESENTATION BY MATRICES OVER A GROUP WITH ZERO

Hitherto, especially in Chapter 1, we have represented semigroups by means of (possibly partial) transformations of a set. In Chapter 5 we shall consider representations of semigroups by matrices over a field, exactly as for groups and algebras in the classical representation theory. In the present chapter, we consider representations by means of matrices the elements of which belong to some group with zero G^0. We can multiply such matrices by the usual row-by-column rule provided we never have to add two elements of G. This imposes limitations on the occurrence of non-zero entries. For example, in the Rees representation (§3.2), the representing matrices have at most one non-vanishing element. In the Schützenberger representations (§3.5), the representing matrices are either row-monomial, that is, have at most one non-vanishing entry in each row, or column-monomial.

The Rees Theorem (3.5) faithfully represents any completely 0-simple semigroup as the semigroup of all matrices, over a certain group with zero, of a certain width and depth having at most one non-vanishing element, and multiplying by means of a certain "sandwich" matrix P, the product $A \circ B$ of two Rees matrices A and B being the matrix product APB. In this way, the structure of all completely 0-simple semigroups is made clear. The Rees Theorem plays the same rôle for completely 0-simple semigroups that the Second Wedderburn Theorem (stated in §5.1) does for linear associative algebras of finite order. This theorem is applied in §3.3 to elucidate the structure of Brandt groupoids.

The following considerations link the Wedderburn and Rees Theorems. Let A be a simple linear associative algebra of finite order over a field Φ. Let the identity element e of A be expressed as a sum $e = e_1 + e_2 + \cdots + e_n$ of pair-wise orthogonal, primitive idempotents e_i of A. Let $A_{ij} = e_iAe_j$. Then the algebra A is the direct sum of the algebras A_{ij} $(i, j = 1, 2, \cdots, n)$. The various A_{ij} all contain the same cardinal number of elements, and the A_{ii} are isomorphic division algebras over Φ. (See, for example, A. A. Albert, *Structure of Algebras*, Amer. Math. Soc. Colloquium Publications, vol. 24, 1939; Chapter III, Theorem 9.) Let Δ be a division algebra over Φ isomorphic with the A_{ii}. The union S of the sets A_{ij} is readily seen to be closed under the multiplication defined in A, and, in fact, we easily see that, relative to this operation, S forms a completely 0-simple semigroup. The sets $A_{ij}\backslash 0$ are the $\mathscr{H}$-classes of S in the $\mathscr{D}$-class $S\backslash 0$. The Wedderburn representation of A as the full $n \times n$ matrix algebra over Δ induces the Rees representation of S, regarding Δ as a group with zero under multiplication. Here the sandwich matrix P is the $n \times n$ identity matrix, S being in fact a Brandt semigroup (§3.3).

In §3.5 we discuss the two Schützenberger representations of any semigroup S associated with any $\mathscr{D}$-class of S, and show their connection with the Rees representation when S is completely 0-simple. Finally, in §3.6, we show that the direct sum of all the Schützenberger representations of a regular semigroup is faithful, a result due to Preston [1958].

In view of the fact that any semigroup can be embedded in a regular semigroup (in fact in a regular bisimple semigroup with identity, as we shall show in Chapter 8), it follows that any semigroup can be represented faithfully by means of the Schützenberger representations of some containing semigroup. No real use has as yet been made of the Schützenberger representation in semigroup theory. In the process of generalizing from the Rees representation of a completely 0-simple semigroup to the Schützenberger representations of wider classes of semigroups, an essential feature of the former has been lost. The Rees representation enables completely 0-simple semigroups to be characterized as full semigroups of matrices—the image of a Rees representation consists of *all* matrices of a certain kind. No such characterization of a wider class of semigroups has yet been obtained by means of the Schützenberger representation.

3.1 Matrix semigroups over a group with zero

Let G be a group, and let $G^0 = G \cup 0$ be the group with zero arising from G by the adjunction of a zero element 0 (§1.1). (For the basic definitions of this section, G could be any semigroup.)

Let X be any set, and let $i \rightarrow a_i$ be a mapping of X into G^0. If $a_i = 0$ for every i in X, we define $\sum_{i \in X} a_i = 0$. If $a_j \neq 0$ for a certain element j of X, and $a_i = 0$ if $i \neq j$, we define $\sum_{i \in X} a_i = a_j$. If $a_j \neq 0$ and $a_k \neq 0$ for $j \neq k$ in X, then $\sum_{i \in X} a_i$ is not defined.

Let X and Y be any sets. By an *$X \times Y$ matrix over G^0* we mean a mapping A of $X \times Y$ into G^0. If (i, j) is a typical element of $X \times Y$ $(i \in X, j \in Y)$, and $a_{ij} = (i, j)A$, then we may write $A = (a_{ij})$, and speak of a_{ij} as the element, or entry, of A lying in the ith row and jth column of A.

Let X, Y, and Z be sets. Let $A = (a_{ij})$ be an $X \times Y$ matrix over G^0, and let $B = (b_{jk})$ be a $Y \times Z$ matrix over G^0. If, for every pair (i, k) in $X \times Z$, the sum $c_{ik} = \sum_{j \in Y} a_{ij}b_{jk}$ is defined, then we define the *matrix product* $C = AB$ of A and B to be the $X \times Z$ matrix $C = (c_{ik})$ over G^0.

Let S be a set of $X \times X$ matrices over G^0 such that if A and B belong to S, then AB exists and belongs to S. Then S is a semigroup; associativity is proved as for matrices over a ring. (See also Exercise 1 below.) An example of this which is of importance for us in this chapter is the following.

A matrix A over G^0 is called *row-monomial* if each row of A contains at most one non-zero element of G^0. The set of all row-monomial $X \times X$ matrices over G^0 is a semigroup.

We proceed now to describe another kind of semigroup of matrices over G^0 which is of the utmost importance in the algebraic theory of semigroups. Let I and Λ be arbitrary sets. The elements of I will be denoted by $i, j, k, \cdots$; those of Λ by $\lambda, \mu, \nu, \cdots$. By a *Rees* $I \times \Lambda$ *matrix over* G^0 we mean an $I \times \Lambda$ matrix over G^0 having at most one non-zero element. If $a \in G$, $i \in I$, and $\lambda \in \Lambda$, then $(a)_{i\lambda}$ will denote the Rees $I \times \Lambda$ matrix over G^0 having a in the ith row and λth column, its remaining entries being 0. For any i in I and λ in Λ, the expression $(0)_{i\lambda}$ will mean the $I \times \Lambda$ zero matrix, which will also be denoted by 0.

Now let $P = (p_{\lambda i})$ be an arbitrary but fixed $\Lambda \times I$ matrix over G^0. We use P to define a binary operation $(\circ)$ in the set of Rees $I \times \Lambda$ matrices over G^0 as follows:

$$A \circ B = APB.$$

If A and B are Rees $I \times \Lambda$ matrices over G^0, then so is $A \circ B$. In fact, if $A = (a)_{i\lambda}$ and $B = (b)_{j\mu}$, then we easily find that

$$(a)_{i\lambda} \circ (b)_{j\mu} = (ap_{\lambda j}b)_{i\mu} \quad (a, b \in G;\ i, j \in I;\ \lambda, \mu \in \Lambda). \tag{1}$$

Moreover, the operation $(\circ)$ is associative; for

$$A \circ (B \circ C) = AP(BPC) = (APB)PC = (A \circ B) \circ C.$$

Hence the set of all Rees $I \times \Lambda$ matrices over G^0 is a semigroup with respect to the binary operation $(\circ)$; we call it the *Rees* $I \times \Lambda$ *matrix semigroup over the group with zero* G^0 *with sandwich matrix* P, and denote it by $\mathscr{M}^0(G; I, \Lambda; P)$. We call G the *structure group* of $\mathscr{M}^0$.

Another approach to Rees matrix semigroups is to begin with the set $G^0 \times I \times \Lambda$ consisting of all triples $(a; i, \lambda)$ with a in G^0, i in I, and λ in Λ, and to define $(\circ)$ by the analogue of (1), namely,

$$(a; i, \lambda) \circ (b; j, \mu) = (ap_{\lambda j}b; i, \mu). \tag{1'}$$

Associativity is easily verified. Next we observe that the set $0 \times I \times \Lambda$ of all triples $(0; i, \lambda)$ is an ideal, and we take the Rees factor semigroup modulo this ideal. The result is $\mathscr{M}^0(G; I, \Lambda; P)$.

If P contains no zero entry, then there are no proper divisors of zero in $\mathscr{M}^0(G; I, \Lambda; P)$. The semigroup $\mathscr{M}^0 \setminus 0$ we call the *Rees* $I \times \Lambda$ *matrix semigroup without zero over the group* G *with sandwich matrix* P, and denote by $\mathscr{M}(G; I, \Lambda; P)$. This may be regarded as the semigroup $G \times I \times \Lambda$ of triples $(a; i, \lambda)$ with product defined by (1′). In this formulation, it is not necessary to adjoin a zero to G. But if we wish to think of the triple $(a; i, \lambda)$ as the Rees matrix $(a)_{i\lambda}$, then it is of course necessary to adjoin a zero to G.

In the sequel, we shall use whichever notation $(a)_{i\lambda}$ or $(a; i, \lambda)$ is most convenient, and we shall not distinguish between "Rees matrix" and "triple".

Accordingly we shall identify all triples of the form $(0; i, \lambda)$ just as we do all the notations $(0)_{i\lambda}$ for the $I \times \Lambda$ zero matrix.

We remark also that *from every theorem on Rees matrix semigroups over G^0 we can read off in an obvious way a theorem on Rees matrix semigroups without zero over G*, and the latter will not usually be mentioned explicitly.

In the present section we shall deal only with Rees matrix semigroups. All the results are due to D. Rees [1940]. In §§3.5 and 3.6 we shall be dealing mostly with the row-monomial (or column-monomial) matrices defined above.

LEMMA 3.1. *The Rees $I \times \Lambda$ matrix semigroup $\mathscr{M}^0(G; I, \Lambda; P)$ over a group with zero G^0, and with sandwich matrix P, is regular if and only if each row and each column of P contains a non-zero entry.*

PROOF. Let $P = (p_{\lambda i})$. Let $a, b \in G$; $i, j \in I$; $\lambda, \mu \in \Lambda$. Then

$$(a)_{i\lambda} \circ (b)_{j\mu} \circ (a)_{i\lambda} = (ap_{\lambda j}bp_{\mu i}a)_{i\lambda}.$$

This is equal to $(a)_{i\lambda}$ if and only if $p_{\lambda j}bp_{\mu i} = a^{-1}$. With $(a)_{i\lambda}$ given, there exists such an element $(b)_{j\mu}$ in $\mathscr{M}^0$ if and only if $p_{\lambda j} \neq 0$ and $p_{\mu i} \neq 0$ for some j in I and μ in Λ, that is, if and only if the λth row and ith column of P each contain a non-zero element of G^0.

Because of Lemma 3.1, we shall say that a matrix P over a group with zero is *regular* if and only if each row and each column of P contains a non-zero entry.

In the next lemma, and later in this chapter, we shall use the following notation, applying to a Rees $I \times \Lambda$ matrix semigroup $\mathscr{M}^0(G; I, \Lambda; P)$ over a group with zero G^0 and with sandwich matrix $P = (p_{\lambda i})$. Denote the elements of $\mathscr{M}^0$ by $(a)_{i\lambda}$ with a in G^0, i in I, and λ in Λ. Let

$$R_i = \{(a)_{i\lambda} : a \in G, \lambda \in \Lambda\} \text{ and } R_i^0 = R_i \cup 0;$$

$$L_\lambda = \{(a)_{i\lambda} : a \in G, i \in I\} \text{ and } L_\lambda^0 = L_\lambda \cup 0;$$

$$H_{i\lambda} = R_i \cap L_\lambda = \{(a)_{i\lambda} : a \in G\}.$$

LEMMA 3.2.

(i) *For each i in I, R_i^0 is a right ideal of $\mathscr{M}^0$; any two $\mathscr{R}$-equivalent elements of $\mathscr{M}^0 \setminus 0$ must belong to the same R_i, for some i in I.*

(ii) *If P is regular, then, for each i in I, R_i^0 is a 0-minimal right ideal of $\mathscr{M}^0$, and R_i is an $\mathscr{R}$-class.*

(iii) *If, for some i in I, $p_{\lambda i} = 0$ for every λ in Λ, then R_i^0 is a two-sided ideal of $\mathscr{M}^0$ such that $\mathscr{M}^0 \circ R_i^0 = 0$; in particular, $(R_i^0)^2 = 0$.*

(iv) *The set $H_{i\lambda}$ (i in I, λ in Λ) contains an idempotent element if and only if $p_{\lambda i} \neq 0$. If $p_{\lambda i} \neq 0$, then $H_{i\lambda}$ is an $\mathscr{H}$-class of $\mathscr{M}^0$, and is a subgroup of $\mathscr{M}^0$ with identity element $e_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda}$. The mapping $a \rightarrow (ap_{\lambda i}^{-1})_{i\lambda}$ is an isomorphism of G upon $H_{i\lambda}$.*

(v) *For every i, j in I and λ, μ in Λ, we have*

$$H_{i\lambda} \circ H_{j\mu} = \begin{cases} H_{i\mu} & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0. \end{cases}$$

PROOF. (i) That R_i^0 is a right ideal of $\mathscr{M}^0$ is evident from (1). Let $(a)_{i\lambda}$ and $(b)_{j\mu}$ be non-zero, $\mathscr{R}$-equivalent elements of $\mathscr{M}^0$. Then there exists $(c)_{k\nu}$ in $\mathscr{M}^0$ such that $(a)_{i\lambda} \circ (c)_{k\nu} = (b)_{j\mu}$, hence, $(ap_{\lambda k}c)_{i\nu} = (b)_{j\mu}$. Since $b \neq 0$, this requires $j = i$.

(ii) Assume that P is regular, and let $(a)_{i\lambda}$ and $(b)_{i\mu}$ be (non-zero) elements of R_i. Since P is regular, there exists k in I such that $p_{\lambda k} \neq 0$. Then $(a)_{i\lambda} \circ (c)_{k\mu} = (b)_{i\mu}$ if $c = p_{\lambda k}^{-1}a^{-1}b$. This shows that R_i^0 is a 0-minimal right ideal of $\mathscr{M}^0$, and that any two elements of R_i are $\mathscr{R}$-equivalent. That R_i is an $\mathscr{R}$-class then follows from (i) or Lemma 2.43.

(iii) Assume $p_{\lambda i} = 0$ for every λ in Λ. If $(a)_{i\lambda} \in R_i^0$ and $(b)_{j\mu} \in \mathscr{M}^0$, then $(b)_{j\mu} \circ (a)_{i\lambda} = (bp_{\mu i}a)_{j\lambda} = 0$. Hence $\mathscr{M}^0 \circ R_i^0 = 0$. From this and (i) it follows that R_i^0 is a two-sided ideal.

(iv) Let $(a)_{i\lambda} \in H_{i\lambda}$. From $(a)_{i\lambda} \circ (a)_{i\lambda} = (ap_{\lambda i}a)_{i\lambda}$ we see that $(a)_{i\lambda}$ is idempotent if and only if $p_{\lambda i} \neq 0$ and $a = p_{\lambda i}^{-1}$. Assume $p_{\lambda i} \neq 0$. Then $H_{i\lambda}$ contains the idempotent $e_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda}$. For a in G, let $a\phi = (ap_{\lambda i}^{-1})_{i\lambda}$. Then, for a and b in G, $(a\phi) \circ (b\phi) = (ap_{\lambda i}^{-1}p_{\lambda i}bp_{\lambda i}^{-1})_{i\lambda} = (ab)\phi$, and, since ϕ is evidently a one-to-one mapping of G upon $H_{i\lambda}$, it follows that ϕ is an isomorphism of G upon $H_{i\lambda}$. Hence $H_{i\lambda}$ is a subgroup of $\mathscr{M}^0$ with identity element $e_{i\lambda}$.

Let H be the $\mathscr{H}$-class of $\mathscr{M}^0$ containing $e_{i\lambda}$. Evidently $H_{i\lambda} \subseteq H$ since $H_{i\lambda}$ is a group. But by (i) and its dual, $H \subseteq R_i \cap L_\lambda = H_{i\lambda}$, and hence $H = H_{i\lambda}$.

(v) Let $(a)_{i\lambda} \in H_{i\lambda}$ and $(b)_{j\mu} \in H_{j\mu}$. We have

$$(a)_{i\lambda} \circ (b)_{j\mu} = (ap_{\lambda j}b)_{i\mu}.$$

If $p_{\lambda j} = 0$, this is 0. If $p_{\lambda j} \neq 0$, it belongs to $H_{i\mu}$. In the latter event, we can obtain any element $(c)_{i\mu}$ of $H_{i\mu}$ as such a product by taking $a = p_{\lambda j}^{-1}$ and $b = c$.

THEOREM 3.3. *A Rees matrix semigroup is 0-simple if and only if it is regular, and if so it is completely 0-simple.*

PROOF. Let $\mathscr{M}^0(G; I, \Lambda; P)$ be a Rees matrix semigroup. Suppose first that $\mathscr{M}^0$ is not regular. By Lemma 3.1, there is a row or column of P which consists of zeros, say the ith column: $p_{\lambda i} = 0$ for all λ in Λ. By Lemma 3.2 (iii), R_i^0 is a non-zero nilpotent ideal of $\mathscr{M}^0$, and so $\mathscr{M}^0$ can not be 0-simple.

Assume conversely that $\mathscr{M}^0$ is regular. Let $(a)_{i\lambda}$ and $(b)_{j\mu}$ be any two elements of $\mathscr{M}^0$ with $a \neq 0$. By Lemma 3.1, there exist ν in Λ and k in I such that $p_{\nu i} \neq 0$ and $p_{\lambda k} \neq 0$. Let $c = b(p_{\nu i}ap_{\lambda k})^{-1}$, and let e be the identity element of G. Then $(c)_{j\nu} \circ (a)_{i\lambda} \circ (e)_{k\mu} = (b)_{j\mu}$, and it follows from Lemma 2.28 that $\mathscr{M}^0$ is 0-simple.

By Lemma 3.2 (iv), the non-zero idempotents of $\mathscr{M}^0$ are the elements

$e_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda}$. There is one such for each pair i, λ (i in I, λ in Λ) such that $p_{\lambda i} \neq 0$. If $e_{i\lambda} \circ e_{j\mu} = e_{j\mu} \circ e_{i\lambda} = e_{j\mu}$, then $i = j$ and $\lambda = \mu$, so that $e_{i\lambda} = e_{j\mu}$. Thus every non-zero idempotent of $\mathscr{M}^0$ is primitive, and so $\mathscr{M}^0$ is completely 0-simple.

When $\mathscr{M}^0$ is completely 0-simple, and thus P is regular, Lemma 3.2 (ii) and its dual assert that each R_i is an $\mathscr{R}$-class and each L_λ is an $\mathscr{L}$-class. Hence, the notation and results of Lemma 3.2 in the regular case are in accord with those of Corollary 2.52a.

Exercises for §3.1

1. (a) Let A, B, C be respectively a $W \times X$, $X \times Y$, $Y \times Z$ matrix over a group with zero G^0, where W, X, Y, Z are any sets. If AB and BC are defined, then $(AB)C$ is defined if and only if $A(BC)$ is defined, and if so $(AB)C = A(BC)$.

(b) There exist 2×2 matrices A, B, C over G^0 such that AB and $(AB)C$ are defined, but BC is not defined.

2. Let I be any set. The semigroup of $I \times I$ matrix units (Exercise 7 of §2.7) is isomorphic with the Rees $I \times I$ matrix semigroup $\mathscr{M}^0(G; I, I; \Delta)$, where G is a one-element group $\{e\}$, and Δ is the $I \times I$ *identity matrix* (δ_{ij}) over G^0, that is,

$$\delta_{ij} = \begin{cases} e & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

3. A rectangular band (§1.8) is isomorphic with a Rees matrix semigroup without zero over a one-element group.

4. A Rees matrix semigroup without zero is a rectangular band of groups (§1.8).

5. Let A and B be elements of a Rees matrix semigroup over a group with zero. If $A \circ B \circ A = A \neq 0$, then B is an inverse of A (§1.9).

6. Let $\mathscr{M}^0(G; I, \Lambda; P)$ be a Rees matrix semigroup. In the notation of Lemma 3.2, an element in $H_{i\lambda}$ has an inverse in $H_{j\mu}$ if and only if both $p_{\lambda j} \neq 0$ and $p_{\mu i} \neq 0$. (This illustrates Theorem 2.18.)

3.2 The Rees Theorem

According to Theorem 3.3, a Rees matrix semigroup $\mathscr{M}^0(G; I, \Lambda; P)$ over a group with zero G^0 is completely 0-simple if the sandwich matrix P is regular, i.e., if no row or column of P consists wholly of zeros. This is the "easy half" of the important theorem due to Rees [1940], that a semigroup is completely 0-simple if and only if it is isomorphic with a regular Rees matrix semigroup over a group with zero (Theorem 3.5). We shall derive it from a somewhat more general theorem (Theorem 3.4), due to Miller and Clifford [1956], which applies to any regular $\mathscr{D}$-class of any semigroup.

Let D be a regular $\mathscr{D}$-class of a semigroup S. Let $\{R_i : i \in I\}$ and $\{L_\lambda : \lambda \in \Lambda\}$ be the sets of $\mathscr{R}$-classes and $\mathscr{L}$-classes, respectively, of S contained in D. Then the set of $\mathscr{H}$-classes of S contained in D is $\{H_{i\lambda}: i \in I, \lambda \in \Lambda\}$, where $H_{i\lambda} = R_i \cap L_\lambda$.

Select an $\mathscr{H}$-class of D containing an idempotent e; such exists by Theorem 2.11. We shall denote R_e and L_e by R_1 and L_1, respectively, and consequently H_e by H_{11}. We thereby assume that I and Λ have the element 1 in common; this causes no confusion or loss of generality. By Green's Theorem (2.16), H_{11} is a subgroup of S.

For each i in I and λ in Λ, select and fix an element r_i of H_{i1} and an element q_λ of $H_{1\lambda}$. Define the $\Lambda \times I$ matrix $P = (p_{\lambda i})$ over H_{11}^0 as follows:

$$p_{\lambda i} = \begin{cases} q_\lambda r_i & \text{if } q_\lambda r_i \in H_{11}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $\mathscr{M}^0(H_{11}; I, \Lambda; P)$ be the Rees $I \times \Lambda$ matrix semigroup over the group with zero H_{11}^0 with sandwich matrix P. The binary operation in $\mathscr{M}^0$ will be denoted by $(\circ)$. By Theorem 2.17, $q_\lambda r_i \in H_{11}$ if and only if $H_{i\lambda}$ contains an idempotent. By Theorem 2.11, it follows that P has a non-zero entry in each row and in each column. By Lemma 3.1, $\mathscr{M}^0$ is regular; and by Theorem 3.3, $\mathscr{M}^0$ is completely 0-simple.

If we made a (possibly) different selection of H_{11} and of the elements r_i and q_λ, we would obtain a (possibly) different Rees matrix semigroup $\mathscr{M}_1^0$. One can establish by explicit calculation an isomorphism between $\mathscr{M}^0$ and $\mathscr{M}_1^0$. We shall avoid this labor in the proof of the next theorem by means of the notion of the *trace* T of a $\mathscr{D}$-class D. Let 0 be a symbol not representing any element of D, and let $T = D \cup 0$. Define a product $(*)$ in T as follows, where a and b denote arbitrary elements of D:

$$a * b = \begin{cases} ab & \text{if } ab \in R_a \cap L_b, \\ 0 & \text{otherwise;} \end{cases} \tag{2}$$
$$a * 0 = 0 * a = 0 * 0 = 0.$$

The set T becomes thereby a groupoid $T(*)$. It is easy to show, on the basis of results in §2.3, that $(*)$ is associative. This will, however, be a by-product of the next theorem. For we shall show that there is an isomorphism of $\mathscr{M}^0$ upon $T(*)$, and clearly an isomorphic image of a semigroup is a semigroup (thanks to E. J. Tully for this remark).

The notation established so far in this section is taken for granted in the next theorem.

THEOREM 3.4. *Every element of D is uniquely representable in the form $r_i a q_\lambda$ with a in H_{11}, i in I, and λ in Λ. The one-to-one mapping ϕ of $\mathscr{M}^0$ upon T defined by*

$$(a)_{i\lambda}\phi = \begin{cases} r_i a q_\lambda & \textit{if } a \neq 0, \\ 0 & \textit{if } a = 0, \end{cases}$$

is an isomorphism. When restricted to $\mathscr{M}^0 \setminus 0$, ϕ is a partial isomorphism of $\mathscr{M}^0 \setminus 0$ upon D.

REMARK. By a *partial homomorphism* of a partial groupoid S into a partial groupoid S' we mean a mapping ϕ of S into S' such that if a and b are elements of S such that ab is defined in S, then the product $(a\phi)(b\phi)$ is defined in S' and is equal to $(ab)\phi$. By a *partial isomorphism* of S upon S' we mean a one-to-one partial homomorphism ϕ of S upon S'; we do not require that ϕ^{-1} be a partial isomorphism of S' upon S.

Thus the last assertion of the theorem is that if $(a)_{i\lambda}$ and $(b)_{j\mu}$ are elements of $\mathscr{M}^0 \backslash 0$ such that $(a)_{i\lambda} \circ (b)_{j\mu} \neq 0$, then $[(a)_{i\lambda}\phi][(b)_{j\mu}\phi]$ lies in D and is equal to $[(a)_{i\lambda} \circ (b)_{j\mu}]\phi$. On the other hand, it may happen that $[(a)_{i\lambda}\phi][(b)_{j\mu}\phi]$ lies in D, but in the "wrong" $\mathscr{H}$-class (the "right" one being $H_{i\mu}$), in which case $p_{\lambda j} = 0$ and $(a)_{i\lambda} \circ (b)_{j\mu}$ does not lie in $\mathscr{M}^0 \backslash 0$.

PROOF. For each λ in Λ, let e_λ be an idempotent in L_λ; such exists by Theorem 2.11. By Theorem 2.18, q_λ has a unique inverse q'_λ in $R_{e_\lambda} \cap L_1$. Then $eq_\lambda = q_\lambda$ and $q_\lambda q'_\lambda = e$, where e is the idempotent in H_{11}. By Green's Lemma (2.2), the mappings

$$x \to xq_\lambda \ (x \in L_1) \quad \text{and} \quad y \to yq'_\lambda \ (y \in L_\lambda)$$

are mutually inverse, $\mathscr{R}$-class preserving, one-to-one mappings of L_1 upon L_λ and vice-versa.

Dually, for each i in I, there exists an inverse r'_i of r_i in R_1, and the mappings

$$x \to r_i x \ (x \in R_1) \quad \text{and} \quad y \to r'_i y \ (y \in R_i)$$

are mutually inverse, $\mathscr{L}$-class preserving, one-to-one mappings of R_1 upon R_i and vice-versa.

Combining these two, as in Theorem 2.3, the mappings

$$x \to r_i x q_\lambda \ (x \in H_{11}) \quad \text{and} \quad y \to r'_i y q'_\lambda \ (y \in H_{i\lambda})$$

are (for each i in I and λ in Λ) mutually inverse, one-to-one mappings of H_{11} upon $H_{i\lambda}$ and vice-versa.

Since every element of D belongs to exactly one $H_{i\lambda}$, it follows that the mapping ϕ defined in the theorem is a one-to-one mapping of $\mathscr{M}^0$ upon $D \cup 0 = T$. We show next that ϕ is an isomorphism of $\mathscr{M}^0$ upon T, that is,

$$[(a)_{i\lambda} \circ (b)_{j\mu}]\phi = [(a)_{i\lambda}\phi] * [(b)_{j\mu}\phi], \tag{3}$$

for arbitrary elements $(a)_{i\lambda}$ and $(b)_{j\mu}$ of $\mathscr{M}^0$. Clearly we may assume $a \neq 0$ and $b \neq 0$. The following "eggbox" diagram of a portion of D may help us to visualize the situation.

	L_1	L_λ	L_μ	
R_1	e, a, b $q_\lambda r_j$?	q_λ	q_μ	
R_i	r_i	$r_i a q_\lambda$	$(r_i a q_\lambda)(r_j b q_\mu)$?	
R_j	r_j	an idempotent ?	$r_j b q_\mu$	

All the unqueried elements definitely belong in the $\mathscr{H}$-cells in which they appear in the diagram. As for the queried elements, two applications of Theorem 2.17 show that $q_\lambda r_j \in H_{11}$ if and only if $H_{j\lambda}$ contains an idempotent, and that this in turn is so if and only if $(r_i a q_\lambda)(r_j b q_\mu) \in H_{i\mu}$. Hence, the answers to the three queries are either all "yes" or all "no". Moreover, by the definition (1) of P, the answer is "yes" if and only if $p_{\lambda j} \neq 0$.

First, let us assume that the answer is "yes". Since $r_i a q_\lambda \in R_i$, $r_j b q_\mu \in L_\mu$, and their product belongs to $R_i \cap L_\mu$, it follows from the definition (2) of $(*)$ in T that $(r_i a q_\lambda) * (r_j b q_\mu)$ is just their product in D. Hence, using (1) also,

$$\begin{aligned}[(a)_{i\lambda}\phi] * [(b)_{j\mu}\phi] &= (r_i a q_\lambda) * (r_j b q_\mu) = r_i a q_\lambda r_j b q_\mu \\ &= r_i a p_{\lambda j} b q_\mu = (a p_{\lambda j} b)_{i\mu}\phi = [(a)_{i\lambda} \circ (b)_{j\mu}]\phi.\end{aligned}$$

Now let us assume that the answer is "no". Then $p_{\lambda j} = 0$, $H_{j\lambda}$ does not contain an idempotent, and $(r_i a q_\lambda)(r_j b q_\mu) \notin H_{i\mu}$. By (1),

$$(r_i a q_\lambda) * (r_j b q_\mu) = 0 \text{ in } T.$$

Hence (3) reduces to $0\phi = 0$, which is true by definition of ϕ.

It is now immediate that the restriction of ϕ to $\mathscr{M}^0 \backslash 0$ is a partial isomorphism. For if $(a)_{i\lambda} \circ (b)_{j\mu} \neq 0$, that is, if $p_{\lambda j} \neq 0$, then we are in the "yes" case considered above, and

$$[(a)_{i\lambda}\phi][(b)_{j\mu}\phi] = (r_i a q_\lambda)(r_j b q_\mu) \in H_{i\mu} \subseteq D.$$

THEOREM 3.5 (Rees). *A semigroup is completely* 0*-simple if and only if it is isomorphic with a regular Rees matrix semigroup over a group with zero.*

PROOF. If a semigroup is isomorphic with a regular Rees matrix semigroup over a group with zero, then it is completely 0-simple by Theorem 3.3.

Conversely, let S be a completely 0-simple semigroup. By Theorem 2.51, S is 0-bisimple, so that $D = S\backslash 0$ is a $\mathscr{D}$-class of S. Construct $\mathscr{M}^0(H_{11}; I, \Lambda; P)$ for D as in Theorem 3.4. Now Theorem 2.52 (ii) shows that S is isomorphic with the trace $T = D \cup 0$ of D, product $(*)$ therein being defined by (2). By Theorem 3.4, T and $\mathscr{M}^0$ are isomorphic. Hence, S and $\mathscr{M}^0$ are isomorphic.

It is possible for two regular Rees $I \times \Lambda$ matrix semigroups

$$S = \mathscr{M}^0(G; I, \Lambda; P) \qquad \text{and} \qquad S' = \mathscr{M}^0(G; I, \Lambda; P')$$

over the same group with zero G^0 to be isomorphic, without the sandwich matrices P and P' being the same. The theory of homomorphisms of regular Rees matrix semigroups will be treated fully in §3.4. Here we give a sufficient condition for S and S' to be isomorphic which enables us to make a useful normalization of P. (Regularity is not needed.)

LEMMA 3.6. *Two Rees $I \times \Lambda$ matrix semigroups $S = \mathscr{M}^0(G; I, \Lambda; P)$ and $S' = \mathscr{M}^0(G; I, \Lambda; P')$ over the same group with zero G^0 are isomorphic if there* *exist a mapping $i \to u_i$ of I into G and a mapping $\lambda \to v_\lambda$ of Λ into G such that $p'_{\lambda i} = v_\lambda p_{\lambda i} u_i$ for all i in I and λ in Λ, where $P = (p_{\lambda i})$ and $P' = (p'_{\lambda i})$.*

Remark. We may express the stated relation between P and P' in matrix form: $P' = VPU$, where V is the "diagonal" $\Lambda \times \Lambda$ matrix having v_λ in the (λ, λ) position, and zeros off the main diagonal, and U is the diagonal $I \times I$ matrix having u_i in the (i, i) position.

Proof. Denote the elements of S by $(a)_{i\lambda}$ and those of S' by $[a]_{i\lambda}$. Let

$$[a]_{i\lambda}\phi = (u_i a v_\lambda)_{i\lambda}.$$

Then ϕ is evidently a one-to-one mapping of S' upon S. From

$$\begin{aligned}([a]_{i\lambda} \circ [b]_{j\mu})\phi &= [ap'_{\lambda j}b]_{i\mu}\phi \\ &= (u_i a p'_{\lambda j} b v_\mu)_{i\mu} = (u_i a v_\lambda p_{\lambda j} u_j b v_\mu)_{i\mu} \\ &= (u_i a v_\lambda)_{i\lambda} \circ (u_j b v_\mu)_{j\mu} \\ &= [a]_{i\lambda}\phi \circ [b]_{j\mu}\phi\end{aligned}$$

we see that ϕ is an isomorphism of S' upon S.

In representing a given completely 0-simple semigroup as a regular Rees matrix semigroup, Lemma 3.6 enables us to replace P by $P' = VPU$ with "invertible" (see §3.4) diagonal matrices V and U. For example, we may thereby "normalize" P so that all the elements in a given row and in a given column are either 0 or the identity element of the structure group G. As an application, we note that Theorem 1.27 is an immediate consequence of the Rees Theorem and Lemma 3.6.

We conclude this section with an example. Let T be the full transformation semigroup $\mathscr{T}_X$ on a finite set X of cardinal n, and let T_r be the set of all elements of T of rank $\leqq r$ (§2.2), $1 \leqq r \leqq n$. Then (Exercise 1 of §2.6) T_r is an ideal of T, and T has a unique principal series

$$T = T_n \supset T_{n-1} \supset \cdots \supset T_1 \supset T^0 = \square.$$

Moreover, T_r/T_{r-1} is completely 0-simple. In fact the set $D_r = T_r \setminus T_{r-1}$ of all elements of T of rank r is a $\mathscr{D}$-class of T (Theorem 2.9), so that T_r/T_{r-1} is 0-simple; since it is finite, Corollary 2.56 applies. We proceed to set up a Rees matrix representation of the principal factor T_r/T_{r-1} of T, as suggested by Hewitt and Zuckerman [1957].

Let $X = N_n = \{1, 2, \cdots, n\}$. By Theorem 2.9, we specify an $\mathscr{H}$-class of T in D_r by selecting (i) a subset A of N_n of cardinal r, and (ii) a partition (or equivalence relation) ξ of N_n such that $|N_n/\xi| = r$; then $\mathscr{H}(\xi, A)$ consists of all transformations of N_n (of rank r) having range A and partition ξ. If $X_1, \cdots, X_r$ are the equivalence classes of N_n mod ξ, if $A = \{a_1, \cdots, a_r\}$, and if $\alpha \in \mathscr{H}(\xi, A)$, then for each i $(1 \leqq i \leqq r)$, α maps every element of X_i into the same element a_j of A, and every element of N_n mapped into a_j by α belongs to X_i. Clearly the mapping $i \to j$ set up in this way by α is a permutation ϕ of $\{1, \cdots, r\}$; we write $j = i\phi$.

In order to express the foregoing without ambiguity in the notation of permutations, we ascribe the natural order to the elements of A, and so assume $a_1 < a_2 < \cdots < a_r$; and we order the equivalence classes mod ξ by $X_1^* < X_2^* < \cdots < X_r^*$, where, for any subset Y of N_n, Y^* denotes the least element of N_n contained in Y. We then write

$$\alpha = \begin{pmatrix} X_1 X_2 \cdots X_r \\ a_{1\phi} a_{2\phi} \cdots a_{r\phi} \end{pmatrix} = (\phi\,;\, \xi,\, A).$$

Two elements of $\mathscr{H}(\xi, A)$ differ only in the permutation ϕ.

Now let β be any other element of T of rank r, and let $\beta \in \mathscr{H}(\eta, B)$, where η is a partition and B a subset of N_n of cardinal r. Then

$$\beta = \begin{pmatrix} Y_1 Y_2 \cdots Y_r \\ b_{1\psi} b_{2\psi} \cdots b_{r\psi} \end{pmatrix} = (\psi\,;\, \eta,\, B),$$

where $Y_1, \cdots, Y_r$ are the r equivalence classes of N_n mod η ordered such that $Y_1^* < Y_2^* < \cdots < Y_r^*$; $B = \{b_1, \cdots, b_r\}$ with $b_1 < b_2 < \cdots < b_r$; and ψ is the permutation of $\{1, \cdots, r\}$ defined by the assertion that β maps all of Y_i upon the element $b_{i\psi}$ of B $(1 \leqq i \leqq r)$.

We note that the product $\alpha\beta$ of α and β will have rank r if and only if no two elements of A lie in the same equivalence class mod η. If this is the case, we can define a permutation $\pi = \pi_{A,\eta}$ of $\{1, \cdots, r\}$ by $a_i \in Y_{i\pi}$ $(i = 1, \cdots, r)$. Then $a_i\beta = b_{i\pi\psi}$, and so $X_i\alpha\beta = a_{i\phi}\beta = b_{i\phi\pi\psi}$. Hence,

$$\alpha\beta = (\phi\,;\, \xi,\, A) \circ (\psi\,;\, \eta,\, B) = (\phi\pi_{A,\eta}\psi\,;\, \xi,\, B). \tag{4}$$

In the contrary case, define $\pi_{A,\eta} = 0$, where 0 is a zero element adjoined to the symmetric group $\mathscr{G}_r = \mathscr{G}_{N_r}$ on the set $\{1, \cdots, r\}$. In this case $\alpha\beta \in T_{r-1}$, and so (4) still holds if we construe the product $\alpha\beta$ as an element of T_r/T_{r-1}. Hence, (4) gives the desired Rees representation:

$$T_r/T_{r-1} \cong \mathscr{M}^0(\mathscr{G}_r\,;\, I,\, \Lambda\,;\, \Pi),$$

where $\mathscr{G}_r$ is the symmetric group on N_r, I is the set of all partitions of N_n into r subsets, Λ is the set of all subsets of N_n of cardinal r, and Π is the $\Lambda \times I$ matrix $(\pi_{A,\eta})$ defined above.

EXERCISES FOR §3.2

1. Taking the case $n = 4$, $r = 3$ of the example at the end of the section, and adopting the same order for the subsets and partitions of N_4 as that used in the table of the $\mathscr{D}$-class D_3 given in §2.2, the sandwich matrix $\Pi = (\pi_{A,\eta})$ is as follows:

$$\begin{pmatrix} (1) & (1) & 0 & (1) & 0 & 0 \\ (1) & 0 & (1) & 0 & (1) & 0 \\ 0 & (23) & (1) & 0 & 0 & (1) \\ 0 & 0 & 0 & (123) & (12) & (1) \end{pmatrix}.$$

Here (1) means the identity element of $\mathscr{G}_3$, while the other entries in $\mathscr{G}_3$ are in the usual cycle notation.

2. (a) A nowhere commutative semigroup (Exercise 1 of §1.8 and Exercise 3 of §1.9) is completely simple, and hence must be a rectangular band.

(b) If the set E of idempotents of a completely simple semigroup S is a subsemigroup of S, then E is a rectangular band, and $S \cong E \times G$, with G a group.

(c) Let S be a finite simple semigroup such that the product of two idempotents of S is idempotent. Then S is indecomposable as a direct product if and only if either (i) S is an indecomposable group, or (ii) the order of S is prime. (Ivan [1954].)

3. The trace of an irregular $\mathscr{D}$-class is a null semigroup.

4. The trace of the bicyclic semigroup (§1.12) is isomorphic to the semigroup of $N \times N$ matrix units (Exercise 7 of §2.7 and Exercise 2 of §3.1), where N is the set of natural numbers.

5. Let Ω be an index set. For each ω in Ω, let S_ω be a Rees $I_\omega \times \Lambda_\omega$ matrix semigroup $\mathscr{M}(G_\omega; I_\omega, \Lambda_\omega; P_\omega)$ without zero over a group G_ω and with $\Lambda_\omega \times I_\omega$ sandwich matrix P_ω. Then the (unrestricted) direct product S of all the S_ω is isomorphic with $\mathscr{M}(G; I, \Lambda; P)$, where $I[\Lambda]$ is the Cartesian product of the sets I_ω $[\Lambda_\omega]$, where G is the direct product of the groups G_ω, and where, if we regard the groups G_ω as subgroups of G, the matrix P is the direct product of the matrices P_ω. (Compare with Exercise 8 of §2.7. For definition of direct product of matrices, see for example J. H. M. Wedderburn, *Lectures on matrices*, Amer. Math. Soc. Colloquium Publications, vol. 17, 1934, p. 74.)

6. (a) Let S be a semigroup, and let Z be the Cartesian product $S \times S \times S$. Define product in Z by $(x, y, z)(x', y', z') = (x, yzx'y', z')$. Then Z is a semigroup. Define a mapping μ of Z into S by $(x, y, z)\mu = xyz$. Then μ is a homomorphism of Z into S, and is upon S if and only if $S^2 = S$.

(b) Let E be the set of idempotents of S, and let $e \in E$. Then $Z_e = (L_e \cap E) \times H_e \times (R_e \cap E) \subseteq Z$, and $K_e = (L_e \cap E)H_e(R_e \cap E) \subseteq S$. We have $eK_e e = H_e$. The mapping $\mu | Z_e$ is a partial isomorphism of Z_e upon K_e, and its inverse is a partial isomorphism of K_e upon Z_e. The set K_e is a union of $\mathscr{H}$-classes of S contained in D_e. The set Z_e is a subsemigroup of Z if and only if every $\mathscr{H}$-class contained in K_e is a group. If this is the case, then Z_e is a Rees $(L_e \cap E) \times (R_e \cap E)$ matrix semigroup without zero over H_e, and K_e is a completely simple subsemigroup of S isomorphic with Z_e. Consequently, if D_e is a union of groups then it is a completely simple subsemigroup of S. In particular, a bisimple semigroup which is a union of groups is isomorphic to a Rees matrix semigroup without zero over a group. (Wallace [1957].)

7. Let a, b, x, y be elements of a semigroup S. The four elements

$$\begin{matrix} ax & ay \\ bx & by \end{matrix}$$

are at the vertices of a rectangle in the Cayley multiplication table of S. We call S *rectangular* if, whenever three of these elements are equal, all four are equal.

(a) If e is an idempotent element of a rectangular semigroup S, then $aeb = ab$ for all a, b in S.

(b) The set E of idempotent elements of a rectangular semigroup is (if not empty) a rectangular band. (Note Exercise 3 of §1.9, and Exercise 2 (a) above.) In particular, if a band is rectangular, it is a rectangular band. (Thierrin [1955*a*].)

8. A semigroup S is called *E-inversive* if, for each a in S, there exists x in S such that ax is idempotent.

(a) If a is an element of an E-inversive semigroup S, there exists y in S such that ay and ya are both idempotent. (Croisot, verbal communication to Thierrin.)

(b) Let S be an E-inversive, rectangular semigroup. Then S^2 is the kernel of S, and $S^2 \cong E \times G$, where E is a rectangular band and G is a group. (Thierrin [1955*a*].)

9. A semigroup S is called *stationary on the right* if $ab = ac$ (a, b, c in S) implies $xb = xc$ for all x in S.

(a) A semigroup which is stationary on the right is rectangular.

(b) An E-inversive, rectangular semigroup is stationary on the right (and on the left). (Thierrin [1955*a*]. See also Thierrin [1955*b*].)

10. Let T be a semigroup. With each element α of T, associate a set X_α containing α such that the sets X_α ($\alpha \in T$) are mutually disjoint. Let $S = \bigcup_{\alpha \in T} X_\alpha$, and let the product in T be extended to a product in S by defining $ab = \alpha\beta$ if $a \in X_\alpha$ and $b \in X_\beta$ (α, β in T). Then S is a semigroup which we call an *inflation* of T.

(a) With the above notation, T is a subsemigroup of S such that $S^2 \subseteq T$. If we define $a\theta = \alpha$ if $a \in X_\alpha$, then (i) θ maps S upon T, (ii) $\theta^2 = \theta$, and (iii) $(a\theta)(b\theta) = ab$ for all a, b in S.

(b) Let T be a subsemigroup of a semigroup S such that $S^2 \subseteq T$, and let θ be a transformation of S having properties (i), (ii), and (iii) of part (a). Then S is an inflation of T.

11. A semigroup S is E-inversive and rectangular if and only if it is an inflation of a direct product of a group and a rectangular band. (The mapping θ of Exercise 10 is given by $a\theta = ae$ ($a \in S$), where e is the identity element of the maximal subgroup of S to which a^2 belongs.) (Yamada [1955*a*].)

12. An element u of a semigroup S is called a *middle unit* of S if $aub = ab$ for all a, b in S. (We have $u^3 = u^2$, and so u^2 is idempotent, but u itself need not be idempotent.) The semigroup S is called *M-inversive* if, for each a in S, there exist x and y in S such that ax and ya are both middle units. A semigroup S is M-inversive if and only if it is E-inversive and rectangular. (Yamada [1955*a*].)

13. Let $\mathscr{T}_X$ be the full transformation semigroup on a set X. Let G be a subgroup of the symmetric group $\mathscr{G}_X$ on X. Let π_λ $(\lambda \in \Lambda)$ and ρ_i $(i \in I)$ be elements of $\mathscr{T}_X$ such that (i) $\pi_\lambda \in G\pi_\mu$ (λ, μ in Λ) implies $\lambda = \mu$; (ii) $\rho_i \in \rho_j G$ (i, j in I) implies $i = j$; and (iii) $\pi_\lambda \rho_i \in G$ for every λ in Λ, i in I. Let $S = \bigcup\{\rho_i G \pi_\lambda : \lambda \in \Lambda, i \in I\}$. Then $S \cong \mathscr{M}(G; I, \Lambda; P)$, with $P = (\pi_\lambda \rho_i)$. (Suschkewitsch [1940a].)

3.3 Brandt groupoids

In 1927, H. Brandt [1927] introduced and determined the structure of a binary system in which products are not always defined, but satisfying some fairly restrictive axioms. This system, known as a Brandt groupoid, is an abstraction of the system of normal ideals in a semisimple linear algebra under "proper" multiplication. A good account of this is given in Chapter 6 (pp. 67–78) of M. Deuring's *Algebren* (Ergebnisse der Math., vol. 4 (1935), Springer, Berlin). No knowledge thereof is required for the present section.

Also in 1927, A. Loewy [1927] introduced a partial groupoid which he called a *mixed group*, and which turned out to be equivalent to a Brandt groupoid. A full account of this is given by Suschkewitsch in his book [1937], Chapter 5, §§54–61.

As we shall show, if we adjoin a new symbol 0 to a Brandt groupoid B, and define $ab = 0$ if a and b are elements of B such that the product ab is undefined in B, then $B^0 = B \cup 0$ is a completely 0-simple semigroup of especially simple structure. We shall call B^0 a Brandt semigroup. In the Rees representation (Theorem 3.5), and in the notation introduced in §3.1 above, $B^0 \cong \mathscr{M}^0(G; I, I; \Delta)$, where G is any group, I is any set, and the sandwich matrix is the identity matrix Δ over G^0. By the latter, we mean that $\Delta = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} e & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and e is the identity element of G. If $(a)_{ij}$ and $(b)_{kl}$ are any two elements of $\mathscr{M}^0$, with a, b in G^0 and i, j, k, l in I, then

$$(a)_{ij} \circ (b)_{kl} = (a\delta_{jk}b)_{il} = \begin{cases} (ab)_{il} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

In terms of the original Brandt groupoid B, we can express this result as follows. The elements of B can be uniquely represented as triples $(a)_{ij}$ with a in G and i, j in I, every such triple representing an element of B, and with the product of two such triples $(a)_{ij}$ and $(b)_{kl}$ defined to be $(ab)_{il}$ if $j = k$, and undefined if $j \neq k$. Brandt obtained these results in [1927]. We shall obtain them as an application of the Rees Theorem (3.5).

Brandt defines a *Gruppoid*, which we shall call a *Brandt groupoid*, to be a partial groupoid B (see §1.1) satisfying the following axioms.

(B1) If $ab = c$ (a, b, c in B), then each of the three elements a, b, c is uniquely determined by the other two.

(B2) Let a, b, c be elements of B.

(i) If ab and bc are defined, so are $(ab)c$ and $a(bc)$, and these are equal.

(ii) If ab and $(ab)c$ are defined, so are bc and $a(bc)$, and $a(bc) = (ab)c$.

(iii) If bc and $a(bc)$ are defined, so are ab and $(ab)c$, and $(ab)c = a(bc)$.

(B3) To each element a of B there correspond unique elements e, f and a' of B such that $ea = af = a$ and $a'a = f$. (These are called the left identity of a, the right identity of a, and the inverse of a, respectively.)

(B4) If $e^2 = e$ and $f^2 = f$ (e, f in B), there exists an element a in B such that $ea = af = a$.

Now let 0 be a symbol not representing any element of B, and let $B^0 = B \cup 0$. Define a product $(\circ)$ in B^0 as follows (wherein $a, b \in B$):

$$(1) \qquad a \circ b = \begin{cases} ab & \text{if } ab \text{ is defined in } B, \\ 0 & \text{otherwise;} \end{cases}$$

$$a \circ 0 = 0 \circ a = 0 \circ 0 = 0.$$

(Brandt suggested this possibility in [1927], but saw no advantage therein.) The following lemma characterizes partial groupoids that arise from semigroups with zero when we discard the zero. It was found by Conrad [1957].

LEMMA 3.7. *Let B be a partial groupoid, and let $B^0 = B \cup 0$. Define a binary operation $(\circ)$ in B^0 by* (1). *Then B^0 is a semigroup with respect to $(\circ)$ if and only if B satisfies Brandt's axioms* B2 (ii) *and* B2 (iii).

PROOF. Assume B2 (ii) and B2 (iii). Let $a, b, c \in B^0$. If any one of a, b, c is 0, then $a \circ (b \circ c)$ and $(a \circ b) \circ c$ are both equal to 0 by (1), and so we may assume that a, b, c all belong to B.

If $a \circ (b \circ c) \neq 0$, then $b \circ c \neq 0$; it then follows from (1) that bc and $a(bc)$ are defined and equal to $b \circ c$ and $a \circ (b \circ c)$, respectively. By B2 (iii), ab and $(ab)c$ are defined; by (1), $ab = a \circ b$ and $(ab)c = (a \circ b) \circ c$. Moreover, by B2 (iii), $a(bc) = (ab)c$, and hence $a \circ (b \circ c) = (a \circ b) \circ c$ in this case. Similarly, by B2 (ii), this is true if $(a \circ b) \circ c \neq 0$. But the only remaining case is when both $a \circ (b \circ c)$ and $(a \circ b) \circ c$ are equal to 0, and hence they are equal in all cases.

Conversely, assume that the binary operation defined by (1) is associative. Let a, b, c be elements of B such that bc and $a(bc)$ are defined. Then, by (1), $b \circ c = bc$ and $a \circ (b \circ c) = a(bc) \neq 0$. But $(a \circ b) \circ c = a \circ (b \circ c)$ by assumption. Since this implies that $(a \circ b) \circ c \neq 0$, and hence $a \circ b \neq 0$, we conclude from (1) that ab and $(ab)c$ are defined in B and are equal respectively to $a \circ b$ and $(a \circ b) \circ c$. From $(a \circ b) \circ c = a \circ (b \circ c)$ we conclude $(ab)c = a(bc)$. This proves B2 (iii), and the proof of B2 (ii) is similar.

By a *Brandt semigroup* we mean the semigroup $B^0 = B \cup 0$ arising from a Brandt groupoid B by the adjunction of a zero element, the product $(\circ)$

in B^0 being defined by (1). That B^0 is a semigroup follows from Lemma 3.7. We shall now drop the symbol ($\circ$) and rephrase the axioms B1–4 in the language of semigroups as follows:

A *Brandt semigroup* is a semigroup S with zero satisfying the following axioms:

(A1) If a, b, c are elements of S such that $ac = bc \neq 0$ or $ca = cb \neq 0$, then $a = b$.

(A2) If a, b, c are elements of S such that $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$.

(A3) To each element $a \neq 0$ of S there correspond a unique element e of S such that $ea = a$, a unique element f of S such that $af = a$, and a unique element a' of S such that $a'a = f$.

(A4) If e and f are non-zero idempotents of S, then $eSf \neq 0$.

We hasten to show that A1 and A2 are consequences of A3. Brandt was aware that his axioms were not independent, and he gave several independent systems of axioms in a later paper [1940].

Lemma 3.8. *Axioms* A1 *and* A2 *are consequences of* A3. *Hence a semigroup S with zero is a Brandt semigroup if and only if it satisfies* A3 *and* A4.

Proof. Assume that S satisfies A3. We observe first that the elements e, f, and a' in A3 satisfy the relations

$$e^2 = e, \qquad f^2 = f, \qquad fa' = a'e = a', \qquad aa' = e.$$

The first follows from $e^2a = ea = a$ and the uniqueness of e postulated in A3, and the proof of $f^2 = f$ is similar. From $a = af = aa'a$ and the uniqueness of e, we have $aa' = e$. From $a'ea = a'a = f$ and the uniqueness of a', we have $a'e = a'$; the proof of $fa' = a'$ is similar. The foregoing relations show that $(a')' = a$.

We observe next that $e^2 = e$, $f^2 = f$, and $ef \neq 0$ imply $e = f$. Let a' be the inverse of $a = ef$. From $e = aa' = efa' = ea'$ we conclude $a' = e$. Hence $a = (a')' = e' = e$. From $ee = e = a = ef$ we conclude $e = f$. It follows immediately that a product $ab \neq 0$ if and only if the right identity of a is the same as the left identity of b. (For the "if" part, note that $af = a$, $fb = b$, $a'a = f$, and $bb' = f$ imply $f = f^2 = a'(ab)b'$.)

We are now ready to prove A1. Suppose $ac = bc \neq 0$. Then the left identity e of c is also the right identity of a and b. If c' is the inverse of c, we have $cc' = e$, and so $a = ae = acc' = bcc' = be = b$. Similarly, $ca = cb \neq 0$ implies $a = b$.

To prove A2, let $ab \neq 0$ and $bc \neq 0$. Then the right identity of a is the same as the left identity of b, and the latter is clearly also the left identity of bc. Hence $a(bc) \neq 0$.

In the next theorem, the equivalence of (i) and (iii) was found by Clifford [1942]; that of (ii) and (iii) was found by Munn [1957*a*].

THEOREM 3.9. *The following three conditions on a semigroup S with zero are equivalent.*

(i) *S is a Brandt semigroup.*

(ii) *S is a completely 0-simple inverse semigroup.*

(iii) *S is isomorphic with a (regular) Rees $I \times I$ matrix semigroup $\mathscr{M}^0(G; I, I; \Delta)$ over a group with zero G^0 and with the $I \times I$ identity matrix Δ as sandwich matrix.*

PROOF. (i) *implies* (ii). We show first that S is 0-simple. By Lemma 2.28, it suffices to show that if a and b are non-zero elements of S, there exist x and y in S such that $yax = b$. Let e be the left identity of a, and f the right identity of b. By A4, there exists $c \neq 0$ in eSf. Let a' be the inverse of a, and c' that of c. Then $aa' = e$ and $c'c = f$. Let $x = a'c$ and $y = bc'$. Then $yax = bc'aa'c = bc'ec = bc'c = bf = b$.

From A3 it is clear that S contains idempotents. Let e and f be idempotents of S such that $0 < f \leqq e$. From $ef = ff = f \neq 0$ and the uniqueness in A3, we conclude that $e = f$. Hence e is a primitive idempotent of S, and S is completely 0-simple.

It is clear from A3 that S is regular ($aa'a = af = a$). Let R be a non-zero $\mathscr{R}$-class of S, and let e and f be idempotents of S contained in R. By Lemma 2.14, $ef = f$. From this, $ff = f$, and A3, it follows that $e = f$. Similarly, each $\mathscr{L}$-class of S contains precisely one idempotent. By Corollary 2.19, S is an inverse semigroup.

(ii) *implies* (iii). Assume (ii). By the Rees Theorem (3.5), S is isomorphic to a regular Rees $I \times \Lambda$ matrix semigroup $\mathscr{M}^0(G; I, \Lambda; P)$ over a group with zero G^0 with sandwich matrix P. Let us adopt the notation introduced just before Lemma 3.2. By part (iv) of this lemma, $H_{i\lambda}$ contains an idempotent if and only if $p_{\lambda i} \neq 0$. By hypothesis, S is an inverse semigroup, and, by Corollary 2.19, each $\mathscr{R}$-class and each $\mathscr{L}$-class of S contains exactly one idempotent. Hence, each row and each column of P contains exactly one non-zero entry. Hence I and Λ have the same cardinal, and we may assume them ordered so that the non-zero entries of P occur on the main diagonal. Since I and Λ are merely index classes, we may take $\Lambda = I$. Then $p_{ii} \neq 0$ for each i in I, and $p_{ji} = 0$ if $i \neq j$; in other words, P is a diagonal matrix. By Lemma 3.6, we can replace $P = (p_{ji})$ by $P' = (p'_{ji}) = (v_j p_{ji} u_i)$ with arbitrary u_i and v_j in G. Taking $u_i = p_{ii}^{-1}$ and $v_j = e$, we get $P' = \Delta$.

(iii) *implies* (i). Taking $S = \mathscr{M}^0(G; I, I; \Delta)$, it suffices to prove A3 and A4, by Lemma 3.8. To prove A3, let $(a)_{ij}$ be a given non-zero element of S. Then $(x)_{kl}(a)_{ij} = (x\delta_{li}a)_{kj} = (a)_{ij}$ if and only if $k = i$, $l = i$, and $xa = a$, hence if and only if $(x)_{kl} = (e)_{ii}$. Similarly, $(e)_{jj}$ is the unique right unit, and $(a^{-1})_{ji}$ the unique inverse, of $(a)_{ij}$. As for A4, $(e)_{ii}S(e)_{jj}$ contains the non-zero element $(e)_{ij}$.

Exercises for §3.3

1. Axiom A3 is equivalent to the following: to each element $a \neq 0$ of S corresponds a unique element a' such that $aa'a = a$.

2. A semigroup S with zero satisfies A3 if and only if S is an inverse semigroup in which every non-zero idempotent is primitive.

It follows that Brandt semigroups can be characterized as inverse semigroups satisfying A4, and in which every non-zero idempotent is primitive. If we replace "inverse" by "regular" in this, we obtain a characterization of completely 0-simple semigroups (cf. Exercise 11 of §2.7).

3. Each principal factor of a finite inverse semigroup is a Brandt semigroup or a group. (Munn [1955*b*].)

4. Define a *partial group* to be a partial groupoid satisfying Brandt's axioms B1, 2, 3. (By Exercise 2 and Lemma 3.8, a partial group is $S\backslash 0$ for some inverse semigroup S in which every non-zero idempotent is primitive.)

(a) Let ρ be an equivalence relation on a set X, and define a product on ρ by $(x, y)(w, z) = (x, z)$ provided $y = w$ and $(x, z) \in \rho$, and undefined otherwise (x, y, w, z in X). Then ρ is a partial group.

(b) The above-defined partial group is a Brandt groupoid if and only if ρ is the universal relation ω on X. (Croisot [1948*a*]. See also Croisot [1948*b*].)

3.4 Homomorphisms of a regular Rees matrix semigroup

By a *non-trivial* homomorphic image of a semigroup we mean one of order greater than one. Let S be a regular Rees matrix semigroup. From Lemma 3.10 below and the Rees Theorem (3.5), every non-trivial homomorphic image of S is isomorphic with a regular Rees matrix semigroup.

The objective of this section is to determine all homomorphisms of S into a Rees matrix semigroup S^* which need not be regular. The basic result, Theorem 3.11, is due to Munn [1955*a*]. This generalizes an earlier theorem due to Rees [1940], which we derive as Corollary 3.12.

Lemma 3.10. *Let θ be a non-trivial homomorphism of a completely 0-simple semigroup S upon a semigroup S'. Then θ maps non-zero elements of S onto non-zero elements of S', and S' is also completely 0-simple.*

Proof. Evidently $0' = 0\theta$ is the zero element of S', and $0'\theta^{-1}$ is an ideal of S. Were $0'\theta^{-1} = S$, we would have $S' = S\theta = 0'$, and θ would be trivial. Hence, $0'\theta^{-1} = 0$, and so $a \neq 0$ in S implies $a\theta \neq 0'$ in S'.

Let $a', b' \in S'$ with $a' \neq 0'$. Since $S' = S\theta$, there exist a, b in S with $a\theta = a'$ and $b\theta = b'$. Clearly $a \neq 0$, and hence, by Lemma 2.28, there exist x and y in S such that $xay = b$. Then $b' = x'a'y'$ with $x' = x\theta$, $y' = y\theta$, and, again by Lemma 2.28, S' is 0-simple.

By Theorem 2.48, S possesses a 0-minimal right ideal R. By the foregoing, $R\theta \neq 0'$, and clearly $R\theta$ is a 0-minimal right ideal of S'. Similarly, S'

possesses a 0-minimal left ideal. Again by Theorem 2.48, S' is completely 0-simple.

THEOREM 3.11. *Let $S = \mathscr{M}^0(G; I, \Lambda; P)$ be a Rees $I \times \Lambda$ matrix semigroup over a group with zero G^0 with sandwich matrix $P = (p_{\lambda i})$. Let*

$$S^* = \mathscr{M}^0(G^*; I^*, \Lambda^*; P^*)$$

be a Rees $I^* \times \Lambda^*$ matrix semigroup over a group with zero $(G^*)^0$ with sandwich matrix $P^* = (p^*_{\lambda^* i^*})$.

Let $i \to u_i$ and $\lambda \to v_\lambda$ be mappings of I and Λ, respectively, into G^*. Let ϕ and ψ be mappings of I into I^* and Λ into Λ^*, respectively. Let ω be a non-trivial homomorphism of G^0 into $(G^*)^0$ such that

$$p_{\lambda i}\omega = v_\lambda p^*_{\lambda\psi,\, i\phi} u_i \tag{1}$$

for all λ in Λ and i in I. For each element $(a; i, \lambda)$ of S, define

$$(a; i, \lambda)\theta = [u_i(a\omega)v_\lambda; i\phi, \lambda\psi], \tag{2}$$

the square bracket indicating an element of S^*. Then θ is a non-trivial homomorphism of S into S^*. Conversely, if S is regular, then every non-trivial homomorphism of S into S^* is obtained in this fashion.

PROOF. Let $(a; i, \lambda)$ and $(b; j, \mu)$ be elements of S. Then, from (1) and (2),

$$\begin{aligned}((a; i, \lambda) \circ (b; j, \mu))\theta &= (ap_{\lambda j}b; i, \mu)\theta \\ &= [u_i(ap_{\lambda j}b)\omega v_\mu; i\phi, \mu\psi] \\ &= [u_i(a\omega)(p_{\lambda j}\omega)(b\omega)v_\mu; i\phi, \mu\psi] \\ &= [u_i(a\omega)v_\lambda p^*_{\lambda\psi,\, j\phi} u_j(b\omega)v_\mu; i\phi, \mu\psi] \\ &= [u_i(a\omega)v_\lambda; i\phi, \lambda\psi] \circ [u_j(b\omega)v_\mu; j\phi, \mu\psi] \\ &= (a; i, \lambda)\theta \circ (b; j, \mu)\theta.\end{aligned}$$

Since ω is non-trivial by hypothesis, it maps the zero element of G^0 into that of $(G^*)^0$, and maps G into G^*. It follows from (2) that θ is non-trivial.

Conversely, assume that S is regular, and let θ be a non-trivial homomorphism of S into S^*. If 0 denotes the zero of S, then 0θ is an idempotent of S^*. If 0θ is not the zero of S^*, then, by Lemma 3.2 (iv), $0\theta = [p^{*-1}_{\lambda^* i^*}; i^*, \lambda^*]$ for some i^* in I^*, λ^* in Λ^*. Now 0θ is clearly the zero of $S\theta$. Hence, for any $[x; j^*, \mu^*]$ in $S\theta$,

$$[x; j^*, \mu^*] \circ 0\theta = 0\theta = 0\theta \circ [x; j^*, \mu^*].$$

From this it follows easily that $j^* = i^*$, $\mu^* = \lambda^*$, and $x = p^{*-1}_{\lambda^* i^*}$. Thus, $S\theta$ consists of a single element, contrary to assumption. Hence 0θ must be the zero of S^*. Furthermore, it follows from Lemma 3.10 that θ maps non-zero elements of S into non-zero elements of S^*.

It is evident that any homomorphism of one semigroup into another maps

$\mathscr{R}$ [$\mathscr{L}$]-equivalent elements into $\mathscr{R}$ [$\mathscr{L}$]-equivalent elements. We shall use the notation R_i (i in I) and L_λ (λ in Λ) introduced prior to Lemma 3.2, and correspondingly $R^*_{i^*}$ (i^* in I^*) and $L^*_{\lambda^*}$ (λ^* in Λ^*) for S^*. By Lemma 3.2 (ii) and the assumption that S is regular, each R_i is an $\mathscr{R}$-class of S, and so $R_i\theta$ is contained in some $\mathscr{R}$-class of S^*. By Lemma 3.2 (i) applied to S^*, it follows that $R_i\theta \subseteq R^*_{i^*}$ for some i^* in I^*. Since $R_i\theta \neq 0$, $i \to i^*$ is a mapping ϕ of I into I^*. Similarly, $L_\lambda\theta \subseteq L^*_{\lambda^*}$ for some λ^* in Λ^*, and $\lambda \to \lambda^*$ is a mapping ψ of Λ into Λ^*. From $H_{i\lambda} = R_i \cap L_\lambda$ follows $H_{i\lambda}\theta \subseteq R^*_{i\phi} \cap L^*_{\lambda\psi}$.

Now select a particular $\mathscr{H}$-class H_{11} of S containing an idempotent e_{11}; then H_{11} is a subgroup of S isomorphic with G, and $p_{11} \neq 0$. Evidently $e_{11}\theta$ is a non-zero idempotent element of S^* in $R^*_{1\phi} \cap L^*_{1\psi}$, and, by Lemma 3.2 (iv), $p^*_{1\psi,\,1\phi} \neq 0$ and $H^*_{1\phi,\,1\psi} = R^*_{1\phi} \cap L^*_{1\psi}$ is an $\mathscr{H}$-class of S^*, and a subgroup of S^* isomorphic with G^*. The equation

$$(3) \qquad (p_{11}^{-1}x;\, 1,\, 1)\theta = [p^{*-1}_{1\psi,1\phi}(x\omega);\, 1\phi,\, 1\psi]$$

defines a mapping ω of G into G^*. If x and y are elements of G,

$$\begin{aligned}
[p^{*-1}_{1\psi,1\phi}(xy)\omega;\, 1\phi,\, 1\psi] &= (p_{11}^{-1}xy;\, 1,\, 1)\theta \\
&= ((p_{11}^{-1}x;\, 1,\, 1)\circ(p_{11}^{-1}y;\, 1,\, 1))\theta \\
&= (p_{11}^{-1}x;\, 1,\, 1)\theta\circ(p_{11}^{-1}y;\, 1,\, 1)\theta \\
&= [p^{*-1}_{1\psi,1\phi}(x\omega);\, 1\phi,\, 1\psi]\circ[p^{*-1}_{1\psi,1\phi}(y\omega);\, 1\phi,\, 1\psi] \\
&= [p^{*-1}_{1\psi,1\phi}(x\omega)(y\omega);\, 1\phi,\, 1\psi],
\end{aligned}$$

whence $(xy)\omega = (x\omega)(y\omega)$, that is, ω is a homomorphism of G into G^*.

We extend ω to G^0 by defining 0ω to be the zero element of $(G^*)^0$.

We now define the elements u_i (i in I) and v_λ (λ in Λ) of G^* by

$$(4) \qquad (e;\, i,\, 1)\theta = [u_i;\, i\phi,\, 1\psi],$$

$$(5) \qquad (p_{11}^{-1};\, 1,\, \lambda)\theta = [p^{*-1}_{1\psi,1\phi}v_\lambda;\, 1\phi,\, \lambda\psi].$$

Now, for any element $(a;\, i,\, \lambda)$ of S, we have

$$(a;\, i,\, \lambda) = (e;\, i,\, 1)\circ(p_{11}^{-1}a;\, 1,\, 1)\circ(p_{11}^{-1};\, 1,\, \lambda).$$

Applying the homomorphism θ, and using (3), (4), and (5), we obtain

$$(a;\, i,\, \lambda)\theta = [u_i;\, i\phi,\, 1\psi]\circ[p^{*-1}_{1\psi,1\phi}(a\omega);\, 1\phi,\, 1\psi]\circ[p^{*-1}_{1\psi,1\phi}v_\lambda;\, 1\phi,\, \lambda\psi],$$

which evidently reduces to (2).

From (2) we deduce

$$\begin{aligned}
((e;\, i,\, \lambda)\circ(e;\, i,\, \lambda))\theta &= (p_{\lambda i};\, i,\, \lambda)\theta \\
&= [u_i(p_{\lambda i}\omega)v_\lambda;\, i\phi,\, \lambda\psi]
\end{aligned}$$

and

$$\begin{aligned}
(e;\, i,\, \lambda)\theta\circ(e;\, i,\, \lambda)\theta &= [u_iv_\lambda;\, i\phi,\, \lambda\psi]\circ[u_iv_\lambda;\, i\phi,\, \lambda\psi] \\
&= [u_iv_\lambda p^*_{\lambda\psi,\,i\phi}u_iv_\lambda;\, i\phi,\, \lambda\psi].
\end{aligned}$$

Since θ is a homomorphism of S into S^*, we conclude that

$$u_i(p_{\lambda i}\omega)v_\lambda = u_i v_\lambda p^*_{\lambda\psi, i\phi} u_i v_\lambda,$$

which reduces to (1).

We now proceed to derive a corollary and two supplementary theorems. The corollary is the original theorem of Rees [1940], p. 397. To express the latter, we define an $I \times I^*$ matrix U over a group with zero G^0 to be *invertible* if each row and each column of U contains exactly one non-zero element of G^0. This clearly implies that $|I| = |I^*|$. Also, if ω is a homomorphism of G^0 into a group with zero $(G^*)^0$, and $P = (p_{\lambda i})$ is any $\Lambda \times I$ matrix over G^0, then by $P\omega$ we mean the $\Lambda \times I$ matrix $(p_{\lambda i}\omega)$.

COROLLARY 3.12. *Two regular Rees matrix semigroups* $\mathscr{M}^0(G; I, \Lambda; P)$ *and* $\mathscr{M}^0(G^*; I^*, \Lambda^*; P^*)$ *are isomorphic if and only if there exist an isomorphism* ω *of* G^0 *upon* $(G^*)^0$, *an invertible* $I^* \times I$ *matrix* U, *and an invertible* $\Lambda \times \Lambda^*$ *matrix* V, *such that* $P\omega = VP^*U$.

PROOF. We note that if θ in Theorem 3.11 is an isomorphism onto S^*, then the mappings ϕ, ψ, and ω are one-to-one and upon. For θ clearly induces a one-to-one mapping of the $\mathscr{R}$ [$\mathscr{L}$]-classes of S upon those of S^*. We define the $I^* \times I$ matrix $U = (u_{i^*, j})$ by

$$(6) \qquad u_{i^*, j} = \begin{cases} u_j & \text{if } i^* = j\phi, \\ 0 & \text{otherwise}; \end{cases}$$

and we define the $\Lambda \times \Lambda^*$ matrix $V = (v_{\lambda, \mu^*})$ by

$$(7) \qquad v_{\lambda, \mu^*} = \begin{cases} v_\lambda & \text{if } \mu^* = \lambda\psi, \\ 0 & \text{otherwise.} \end{cases}$$

Then $P\omega = VP^*U$.

Conversely, let there exist an isomorphism ω of G^0 upon $(G^*)^0$ and invertible matrices U and V such that $P\omega = VP^*U$. Let u_j be the non-zero element of G^0 in the jth column of U, and let u_j be in the i^*th row. Then the mapping $j \to i^*$ is a one-to-one mapping ϕ of I upon I^*, and (6) holds. Similarly, (7) holds for suitable v_λ and ψ, and the equation $P\omega = VP^*U$ becomes (1). Since ϕ, ψ, and ω are one-to-one and upon, (2) defines a one-to-one mapping θ of S upon S^*, which is an isomorphism by Theorem 3.11.

It follows from Corollary 3.12 that, if $S = \mathscr{M}^0(G; I, \Lambda; P)$, we can replace P by $P^* = V^{-1}(P\omega)U^{-1}$ without changing S; we merely obtain a different representation of S as a Rees matrix semigroup. Conversely, Corollary 3.12 assures us that this is the full extent of the liberties we can take with P.

The direct half of Corollary 3.12 is true for an irregular S. Regard S as "coördinatized" by the triples $(a; i, \lambda)$ with a ranging over G^0, i over I, and λ over Λ, and let us introduce new coördinates $(a; i, \lambda)'$ as follows: Let U and V be invertible $I \times I$ and $\Lambda \times \Lambda$ matrices, respectively, over G^0, say

$$U = (u_{i\phi, i}), \; V = (v_{\lambda, \lambda\psi}),$$

where $\phi[\psi]$ is a permutation of I $[\Lambda]$, and let ω be an automorphism of G^0. We then let

$$(a; i, \lambda)' = (u_{i\phi, i}(a\omega)v_{\lambda, \lambda\psi}; i\phi, \lambda\psi). \tag{8}$$

By direct calculation we find that

$$(a; i, \lambda)' \circ (b; j, \mu)' = (aq_{\lambda j}\, b; i, \mu)', \tag{9}$$

where

$$q_{\lambda j} = (v_{\lambda, \lambda\psi}\, p_{\lambda\psi, j\phi}\, u_{j\phi, j})\omega^{-1}. \tag{10}$$

Consequently the new triples $(a; i, \lambda)'$ multiply, under the given binary operation $\circ$ in S, exactly like Rees triples with sandwich matrix

$$Q = (VPU)\omega^{-1}, \tag{10'}$$

and we therefore feel justified in writing $S = \mathscr{M}^0(G; I, \Lambda; Q)$. We emphasize that there has been no change in the definition of product in S, only a change in coördinates which results in exhibiting S as a Rees matrix semigroup with sandwich matrix Q instead of P. We shall call the triples $(a; i, \lambda)'$ *coördinates of the elements of S relative to Q.*

The derivation of (9) from (8) is simplified if we write these equations in matrix form. If, for every $I \times \Lambda$ Rees matrix A, we define

$$A' = U(A\omega)V, \tag{8'}$$

then

$$A'PB' = (AQB)', \tag{9'}$$

where Q is given by (10′). In his dissertation [1960], Tully gives an interpretation of the change from A to A' (and the "contragredient" change from P to Q) similar to the change undergone by the matrix representing a linear transformation of one vector space into another when we change to new bases in the two spaces.

The following theorem, however, does not depend on our giving any interpretation to the change of coördinates (8). Moreover, we shall need only the case where U and V are diagonal, and ω is the identity automorphism. This special case was first considered in Lemma 3.6, and has already been useful for "normalizing" the sandwich matrix. It should be remarked that the normalization of P in Exercise 1 below succeeds in giving a simple expression of all homomorphisms of a given completely simple semigroup S into a given group G^*, whereas the normalization of P and P^* in Theorem 3.13 below, that eliminates the u_i and v_λ in the expression of the homomorphism θ, depends on θ itself. Whether such a normalization can be found which does this for all θ, we do not know, but it does not seem plausible.

THEOREM 3.13. *Let S and S^* be as in Theorem 3.11, and let θ be a non-trivial homomorphism of S into S^*. Let the objects ϕ, ψ, ω, u_i, and v_λ be* *those determined by* θ *in accordance with Theorem* 3.11, *so that* (1) *and* (2) *hold. Then we can choose new sandwich matrices* Q *for* S *and* Q^* *for* S^* *such that*

(1′) $$q_{\lambda i}\omega = q^*_{\lambda\psi, i\phi} \text{ (all } \lambda \text{ in } \Lambda,\ i \text{ in } I)$$

and

(2′) $$(a;\, i,\, \lambda)'\theta = [a\omega;\, i\phi,\, \lambda\psi]',$$

where the primes indicate the coördinates of elements of S *and* S^* *relative to* Q *and* Q^*, *respectively.*

It follows that $S\theta$ *is the regular Rees matrix semigroup* $\mathscr{M}^0(G';\, I',\, \Lambda';\, Q')$, *where* $G' = G\omega$, $I' = I\phi$, $\Lambda' = \Lambda\psi$, *and* Q' *is the* $\Lambda' \times I'$ *submatrix of* Q^*.

PROOF. For each i^* in $I\phi$, let an element i_0 be chosen in I such that $i_0\phi = i^*$. Let α denote the mapping $i^* \rightarrow i_0$, so that α is a mapping of $I\phi$ into I such that $i^*\alpha\phi = i^*$ for every i^* in $I\phi$. Similarly, let β be a mapping of $\Lambda\psi$ into Λ such that $\beta\psi$ is the identity on $\Lambda\psi$. Let

$$u_{i^*} = u_{i^*\alpha}, \quad v_{\lambda^*} = v_{\lambda^*\beta}.$$

For i^* in $I^*\backslash I\phi$, let $u_{i^*} = e^*$ (the identity element of G^*), and for λ^* in $\Lambda^*\backslash\Lambda\psi$, let $v_{\lambda^*} = e^*$. Finally, let

$$q^*_{\lambda^*, i^*} = v_{\lambda^*} p^*_{\lambda^*, i^*} u_{i^*} \text{ (all } \lambda^* \text{ in } \Lambda^*,\ i^* \text{ in } I^*).$$

As remarked above, $S^* = \mathscr{M}^0(G^*;\, I^*,\, \Lambda^*;\, Q^*)$ with $Q^* = (q^*_{\lambda^*, i^*})$.

Let i be any element of I, and let $i_0 = i\phi\alpha$; then $i\phi = i_0\phi$. Since S is regular, by hypothesis, there exists λ in Λ such that $p_{\lambda i_0} \neq 0$. By (1),

$$p_{\lambda i}\omega = v_\lambda p^*_{\lambda\psi, i\phi} u_i$$

and

$$p_{\lambda i_0}\omega = v_\lambda p^*_{\lambda\psi, i_0\phi} u_{i_0}.$$

From $i\phi = i_0\phi$, we infer that

$$(p_{\lambda i}\omega)u_i^{-1} = (p_{\lambda i_0}\omega)u_{i_0}^{-1};$$

hence $p_{\lambda i} \neq 0$ and

$$u_{i_0}^{-1} u_i = (p_{\lambda i_0}\omega)^{-1}(p_{\lambda i}\omega) = (p_{\lambda i_0}^{-1} p_{\lambda i})\omega.$$

Thus, $u_{i_0}^{-1} u_i \in G\omega$. Select, for each i in I, an element x_i of G such that $u_{i_0}^{-1} u_i = x_i\omega$. Since $u_{i_0} = u_{i\phi\alpha} = u_{i\phi}$, we have

$$u_i = u_{i\phi}(x_i\omega).$$

Similarly, for each λ in Λ, there exists y_λ in G such that

$$v_\lambda = (y_\lambda\omega)v_{\lambda\psi}.$$

Let

$$q_{\lambda i} = y_\lambda^{-1} p_{\lambda i} x_i^{-1}.$$

Then, $S = \mathscr{M}^0(G; I, \Lambda; Q)$, with $Q = (q_{\lambda i})$, and

$$\begin{aligned} q_{\lambda i}\omega &= (y_\lambda\omega)^{-1}(p_{\lambda i}\omega)(x_i\omega)^{-1} \\ &= v_{\lambda\psi}\, v_\lambda^{-1}\,(p_{\lambda i}\omega)\, u_i^{-1}\, u_{i\phi} \\ &= v_{\lambda\psi}\, p^*_{\lambda\psi,\, i\phi}\, u_{i\phi} = q^*_{\lambda\psi,\, i\phi}, \end{aligned}$$

which establishes (1′).

According to equation (8), applied to the present situation, the coördinates of the elements of S and S^* relative to Q and Q^*, respectively, are given by

$$(a, i, \lambda)' = (x_i^{-1} a y_\lambda^{-1}; i, \lambda)$$

and

$$[a^*; i^*, \lambda^*]' = [u_{i^*}\, a^* v_{\lambda^*}; i^*, \lambda^*].$$

Thus, from (2),

$$\begin{aligned} (a; i, \lambda)'\theta &= [u_i(x_i^{-1}\, a y_\lambda^{-1})\omega v_\lambda; i\phi, \lambda\psi] \\ &= [u_i(x_i\omega)^{-1}(a\omega)(y_\lambda\omega)^{-1}v_\lambda; i\phi, \lambda\psi] \\ &= [u_{i\phi}(a\omega)v_{\lambda\psi}; i\phi, \lambda\psi] \\ &= [a\omega; i\phi, \lambda\psi]'. \end{aligned}$$

The final assertion of the theorem is now evident; we remark only that (1′) shows that the entries of Q' lie in G'^0, and that Q' is regular (thereby corroborating Lemma 3.10).

The following is a modification of Theorem 3.11 which will be used in the theory of extensions (§4.5).

THEOREM 3.14. *If we modify Theorem* 3.11 *in that we assume that* (1) *holds if* $p_{\lambda i} \neq 0$, *but not necessarily if* $p_{\lambda i} = 0$, *then* θ *defined by* (2) *is a partial homomorphism of* $S \setminus 0$ *into* $S^* \setminus 0$. *Conversely, if* S *is regular, then every partial homomorphism of* $S \setminus 0$ *into* $S^* \setminus 0$ *is obtained in this fashion.*

PROOF. Let $(a; i, \lambda)$ and $(b; j, \mu)$ be elements of $S \setminus 0$. If

$$(a; i, \lambda) \circ (b; j, \mu) \in S \setminus 0,$$

then $p_{\lambda j} \neq 0$. The condition (1) therefore holds, and the proof of the direct half of Theorem 3.11 holds in the present case too.

Conversely, assume that S is regular and that θ is a partial homomorphism of $S \setminus 0$ into $S^* \setminus 0$. Then, trivially, θ maps non-zero elements of S into non-zero elements of S^*. If A and B are distinct elements of $S \setminus 0$ such that $A \mathscr{R} B$, then $AX = B$ and $BY = A$ for some X and Y in $S \setminus 0$, and $(A\theta)(X\theta) = B\theta$ and $(B\theta)(Y\theta) = A\theta$, so that $(A\theta)\mathscr{R}(B\theta)$ in $S^* \setminus 0$. The proof of Theorem 3.11 is then seen to hold up through equation (5) and the proof of equation (2). Thereafter, we assume that i and λ are elements of I and Λ such that $(e; i, \lambda) \circ (e; i, \lambda) \neq 0$, that is, $p_{\lambda i} \neq 0$, and the proof of (1) restricted to $p_{\lambda i} \neq 0$ is the same as before.

EXERCISE FOR §3.4

1. (a) Let S be a Rees $I \times \Lambda$ matrix semigroup $\mathscr{M}(G; I, \Lambda; P)$ without zero over a group G, with sandwich matrix $P = (p_{\lambda i})$, and let G^* be a group. Assume, moreover, that P has been normalized so that $p_{\lambda 1} = p_{1i} = e$, the identity element of G, for all λ in Λ and i in I, this being possible by Lemma 3.6. Let ω be a homomorphism of G into G^* such that $p_{\lambda i}\omega = e^*$, the identity element of G^*, for all i in I and λ in Λ. Then $(a; i, \lambda)\theta = a\omega$ defines a homomorphism θ of S into G^*, and every such is obtained in this way.

(b) Continuing with the situation in (a), let N be the normal subgroup of G generated by the entries of P (in normal form). Then $S\theta$ is a homomorphic image of G/N, which shows that G/N is the maximal group homomorphic image (§1.5) of S. (Stoll [1951].)

3.5 THE SCHÜTZENBERGER REPRESENTATIONS

Let S be a semigroup, G a group, and I a set. By a *representation* M of S by $I \times I$ matrices over G^0 we mean a mapping $s \to M(s)$ of S into the set of all $I \times I$ matrices over G^0 such that if s and t are any elements of S then the product $M(s)M(t)$ of the matrices $M(s)$ and $M(t)$ is defined and equal to $M(st)$. If $M(s)M(t) = M(ts)$ instead of $M(st)$, then M is called an *anti-representation.*

We do not treat the general subject of such representations, a development of which has been begun by Tully [1960], but we give in this section some particular representations of this type discovered by Schützenberger [1957*a*]. His original version contained an unnecessary restriction later removed by Preston [1958]. Schützenberger gives a generalization in [1958].

There is one Schützenberger representation M_D of S and one anti-representation M'_D for each $\mathscr{D}$-class D of S. The latter can be converted into a direct representation M^*_D of S, which we call the dual Schützenberger representation of S corresponding to D.

LEMMA 3.15. *Let H be an $\mathscr{H}$-class of a semigroup S, and let R and L be the $\mathscr{R}$- and $\mathscr{L}$-classes of S containing H.*

(i) *For every s in S, either $Hs \cap R = \square$ or else Hs is an $\mathscr{H}$-class contained in R, and Ls is the $\mathscr{L}$-class of S containing Hs.*

(ii) *If $Hs \cap R = \square$, then $Hst \cap R = \square$ for every t in S.*

PROOF. (i) Suppose $Hs \cap R \neq \square$. Let $b \in Hs \cap R$. Then $b = as$ for some a in H. Since $a\mathscr{R}b$, $a = bs'$ for some s' in S. By Green's Lemma (2.2), $x \to xs$ is an $\mathscr{R}$-class preserving, one-to-one mapping of L_a upon L_b. Hence $Hs = H_b \subseteq R$, and $Ls = L_b \supseteq H_b$.

(ii) Were $Hst \cap R \neq \square$, and b an element of $Hst \cap R$, then $b = ast$ for some a in H, and $b\mathscr{R}a$, so that $a = bt'$ for some t' in S. But the equations

$b = (as)t$ and $as = b(t's)$ imply that $b\mathscr{R}as$, whence $as \in Hs \cap R$, contrary to the hypothesis that $Hs \cap R = \square$.

Let D be a $\mathscr{D}$-class of a semigroup S. Let $\{R_i : i \in I\}$ and $\{L_\lambda : \lambda \in \Lambda\}$ be the $\mathscr{R}$-classes and $\mathscr{L}$-classes, respectively, of S contained in D. Then the $\mathscr{H}$-classes of S contained in D are the sets $H_{i\lambda} = R_i \cap L_\lambda$ (i in I, λ in Λ). We assume that I and Λ have an element 1 in common. This entails no loss of generality. We are simply selecting a particular $\mathscr{H}$-class $H = H_{11}$ of S contained in D which will play a special part in the following.

For each λ in Λ, pick an element h_λ in $H_{1\lambda}$. Since $h_\lambda \mathscr{R} h_1$, there exist elements q_λ and q'_λ in S^1 such that $h_\lambda = h_1 q_\lambda$ and $h_1 = h_\lambda q'_\lambda$. By Green's Lemma (2.2), the mappings $x \to xq_\lambda$ and $y \to yq'_\lambda$ are mutually inverse, one-to-one, $\mathscr{R}$-class preserving mappings of L_1 and L_λ upon each other. For each λ in Λ, make and fix a selection of such elements q_λ and q'_λ of S^1. Dually, for each i in I there exist elements r_i and r'_i of S^1 such that the mappings $x \to r_i x$ and $y \to r'_i y$ are mutually inverse, one-to-one, $\mathscr{L}$-class preserving mappings of R_1 and R_i upon each other. For each i in I, make and fix a selection of such elements r_i and r'_i. (Unlike the situation in §3.2, we cannot assume that q_λ, q'_λ, r_i, and r'_i lie in D.)

As in §2.4, we let $T(H)$ be the set of all elements t of S^1 such that $Ht \subseteq H$, and we let $\gamma_t = \rho_t | H$. The set $\Gamma(H)$ of all γ_t as t ranges over $T(H)$ is the Schützenberger group of H (Theorem 2.22). Dually, we let $T'(H)$ be the set of all elements u of S^1 such that $uH \subseteq H$, and we let $\gamma'_u = \lambda_u | H$. The set $\Gamma'(H)$ of all γ'_u as u ranges over $T'(H)$ is the dual Schützenberger group of H, and is anti-isomorphic with $\Gamma(H)$ by Theorem 2.24. In the sequel it will be convenient to write $\gamma(t)$ instead of γ_t, and $\gamma'(u)$ instead of γ'_u.

Corresponding to each element s of S we define a $\Lambda \times \Lambda$ matrix $M_D(s) = (m_{\lambda\mu}(s))$ over $\Gamma(H)^0$, and an $I \times I$ matrix $M'_D(s) = (m'_{ij}(s))$ over $\Gamma'(H)^0$, as follows (λ, μ in Λ; i, j in I):

$$m_{\lambda\mu}(s) = \begin{cases} \gamma(q_\lambda s q'_\mu) & \text{if } H_{1\lambda}s = H_{1\mu}, \\ 0 & \text{otherwise;} \end{cases} \tag{1}$$

$$m'_{ij}(s) = \begin{cases} \gamma'(r'_j s r_i) & \text{if } sH_{i1} = H_{j1}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We must show, of course, that the expressions on the right are defined. Suppose $H_{1\lambda}s = H_{1\mu}$. Then $hq_\lambda s \in H_{1\mu}$ for any h in H, since $hq_\lambda \in H_{1\lambda}$. Since $H_{1\mu}q'_\mu = H_{11} = H$, it follows that $hq_\lambda s q'_\mu \in H$. Thus $q_\lambda s q'_\mu \in T(H)$, and so $\gamma(q_\lambda s q'_\mu)$ is defined. Dually, we can show that $\gamma'(r'_j s r_i)$ is defined if $sH_{i1} = H_{j1}$.

THEOREM 3.16. *The mapping $s \to M_D(s)$ is a representation of S by row-monomial $\Lambda \times \Lambda$ matrices over $\Gamma(H)^0$. For given λ and μ in Λ and $\gamma(t)$ in $\Gamma(H)$, (t in $T(H)$), there exists an element s of S^1 such that $m_{\lambda\mu}(s) = \gamma(t)$.*

The mapping $s \to M'_D(s)$ is an anti-representation of S by row-monomial $I \times I$ matrices over $\Gamma'(H)^0$. For given i and j in I and $\gamma'(u)$ in $\Gamma'(H)$ (u in $T'(H)$), there exists an element s of S^1 such that $m'_{ij}(s) = \gamma'(u)$.

REMARKS. We call M_D $[M'_D]$ the *Schützenberger* [*anti-*] *representation* of S corresponding to the $\mathscr{D}$-class D. If we take a different $\mathscr{H}$-class in D to begin with, or make a different choice of the elements q_λ and q'_λ [r_i and r'_i], then the form (1) [(2)] of the Schützenberger [anti-] representation changes in an unessential manner (Exercise 1 below).

Exercise 2 below shows that the semigroup $M_D(S)$ of representing matrices does not in general have more properties than the two stated: (i) every matrix in $M_D(S)$ is row-monomial; (ii) $m_{\lambda\mu}(S) \supseteq \Gamma(H)$ for each (λ, μ) in $\Lambda \times \Lambda$.

PROOF. We show first that each $M_D(s)$ is row-monomial. Let $s \in S$, and let $\lambda \in \Lambda$. Now $H_{1\lambda} \subseteq R_1$; and if we apply Lemma 3.15 (i) to $H_{1\lambda}$ (instead of H) we see that either $H_{1\lambda}s \cap R_1 = \square$ or else $H_{1\lambda}s = H_{1\mu}$ for some μ in Λ. In the first case, the λ-row of $M_D(s)$ consists of zeros. In the second case, it has $\gamma(q_\lambda s q'_\mu)$ in the μ-column and zeros elsewhere.

To show the other property stated in the theorem, we need only take $s = q'_\lambda t q_\mu$. For then

$$H_{1\lambda}s = H_{1\lambda}q'_\lambda t q_\mu = Htq_\mu = Hq_\mu = H_{1\mu},$$

and so $m_{\lambda\mu}(s) = \gamma(q_\lambda s q'_\mu)$. If h is any element of H, then

$$hq_\lambda s q'_\mu = hq_\lambda q'_\lambda t q_\mu q'_\mu = ht$$

because $xq_\lambda q'_\lambda = x$ and $xq_\mu q'_\mu = x$ for every x in H. Hence $\gamma(q_\lambda s q'_\mu) = \gamma(t)$.

We are to show that $M_D(s)M_D(t) = M_D(st)$ for every s and t in S; in other words, that

$$\sum_{\mu \in \Lambda} m_{\lambda\mu}(s)m_{\mu\nu}(t) = m_{\lambda\nu}(st) \tag{3}$$

for every λ, ν in Λ.

Suppose first that $m_{\lambda\nu}(st) \neq 0$. Then $H_{1\lambda}st = H_{1\nu}$, and

$$m_{\lambda\nu}(st) = \gamma(q_\lambda st q'_\nu). \tag{4}$$

By Lemma 3.15 (ii), $H_{1\lambda}s \cap R_1 \neq \square$, since otherwise $H_{1\nu} = H_{1\lambda}st \cap R_1 = \square$. Hence, by Lemma 3.15 (i), $H_{1\lambda}s = H_{1\kappa}$ for some κ in Λ, and so

$$m_{\lambda\kappa}(s) = \gamma(q_\lambda s q'_\kappa),$$

$$m_{\lambda\mu}(s) = 0 \quad \text{for } \mu \neq \kappa.$$

Since $H_{1\nu} = H_{1\lambda}st = H_{1\kappa}t$, it follows that

$$m_{\kappa\nu}(t) = \gamma(q_\kappa t q'_\nu).$$

Hence the left-hand side of (3) is

$$\gamma(q_\lambda s q'_\kappa)\gamma(q_\kappa t q'_\nu) = \gamma(q_\lambda s q'_\kappa q_\kappa t q'_\nu). \tag{5}$$

To show that the expressions (4) and (5) are equal, it suffices to show that

$$hq_\lambda s q'_\kappa q_\kappa t q'_\nu = hq_\lambda st q'_\nu$$

for every h in H. But this is so because $hq_\lambda s \in H_{1\lambda}s = H_{1\kappa}$, and $x \rightarrow xq'_\kappa q_\kappa$ is the identity mapping of $H_{1\kappa}$ upon itself.

Now suppose that $m_{\lambda\nu}(st) = 0$, so that $H_{1\lambda}st \neq H_{1\nu}$. We are to show that the left-hand side of (3) is also 0. If $H_{1\lambda}s \cap R_1 = \square$, then $m_{\lambda\mu}(s) = 0$ for every μ in Λ, and so the left-hand side of (3) is 0 in this case. If $H_{1\lambda}s \cap R_1 \neq \square$, then, by Lemma 3.15 (i), $H_{1\lambda}s = H_{1\kappa}$ for some κ in Λ. Since $m_{\lambda\mu}(s) = 0$ for every $\mu \neq \kappa$, it suffices to show that $m_{\kappa\nu}(t) = 0$. But this is so because $H_{1\kappa}t = H_{1\lambda}st \neq H_{1\nu}$.

The proof of the second part of Theorem 3.16 is similar, so we shall give only a portion thereof. We are to show that $M'_D(s)M'_D(t) = M'_D(ts)$, that is,

$$\sum_{j \in I} m'_{ij}(s)m'_{jl}(t) = m'_{il}(ts). \tag{6}$$

Assume $m'_{il}(ts) \neq 0$. Then $tsH_{i1} = H_{l1}$, and

$$m'_{il}(ts) = \gamma'(r'_l tsr_i). \tag{7}$$

By the dual of Lemma 3.15 (ii), $sH_{i1} \cap L_1 \neq \square$, since otherwise $tsH_{i1} \cap L_1 = \square$. Hence, by the dual of Lemma 3.15 (i), $sH_{i1} = H_{k1}$ for some k in I, and so

$$m'_{ik}(s) = \gamma'(r'_k sr_i),$$

$$m'_{ij}(s) = 0 \quad \text{for } j \neq k.$$

Since $H_{l1} = tsH_{i1} = tH_{k1}$, it follows that

$$m'_{kl}(t) = \gamma'(r'_l tr_k).$$

Hence the left-hand side of (6) is

$$\gamma'(r'_k sr_i)\gamma'(r'_l tr_k) = \gamma'(r'_l tr_k r'_k sr_i), \tag{8}$$

since $\gamma'(u)\gamma'(v) = \gamma'(vu)$ for all u, v in $T'(H)$. For any h in H we have

$$r'_l tr_k r'_k sr_i h = r'_l tsr_i h$$

since $sr_i h \in H_{k1}$, and $x \rightarrow r_k r'_k x$ is the identity mapping of H_{k1} upon itself. Hence the expressions (7) and (8) are equal. The case $m'_{il}(ts) = 0$ is handled similarly to the dual case, and likewise for the row-monomiality and the property $m_{ij}(S) \supseteq \Gamma'(H)$.

We now wish to modify (2) so as to obtain a direct representation of S. By the *dual Schützenberger representation* of S corresponding to D we shall mean the mapping $s \rightarrow M^*_D(s)$, where $M^*_D(s)$ is obtained from $M'_D(s)$ by transposing the matrix and taking the dual of $\Gamma'(H)$, that is, essentially taking $\Gamma(H)$. Thus M^*_D is a (direct) representation of S by column-monomial $I \times I$ matrices over $\Gamma(H)^0$.

We can make this more explicit in the following way. Let h_1 be a fixed element of H. Then there exists a mapping θ of $T'(H)$ into $T(H)$ such that

$uh_1 = h_1(u\theta)$ for every u in $T'(H)$. For each u in $T'(H)$, let $\gamma^*(u) = \gamma(u\theta)$. For any u, v in $T'(H)$ we have

$$h_1((uv)\theta) = uvh_1 = uh_1(v\theta) = h_1(u\theta)(v\theta),$$

whence

$$\gamma((uv)\theta) = \gamma((u\theta)(v\theta)) = \gamma(u\theta)\gamma(v\theta),$$

and so

$$\gamma^*(uv) = \gamma^*(u)\gamma^*(v).$$

We may take $M_D^*(s) = (m_{ij}^*(s))$ with

$$m_{ij}^*(s) = \begin{cases} \gamma^*(r_i'sr_j) & \text{if } sH_{j1} = H_{i1}, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

We cannot simply take $\gamma(r_i'sr_j)$ on the right, since $r_i'sr_j$ need not lie in $T(H)$. However, the elements of the matrix $M_D^*(s)$ do belong to $\Gamma(H)^0$.

THEOREM 3.17. *Let S be a regular Rees matrix semigroup $\mathscr{M}^0(G; I, \Lambda; P)$. Then the Schützenberger representations of S corresponding to the $\mathscr{D}$-class $D = S \setminus 0$ can be taken to be*

$$M_D(s) = Ps, \qquad M_D^*(s) = sP,$$

where s denotes an arbitrary element of S.

PROOF. We shall prove the second of these. The proof of the first is similar, only easier since it lacks the dualizing complication.

We assume that I and Λ have an element 1 in common, and that $p_{11} \neq 0$. Moreover we can assume that p_{11} is the identity element e of G; this normalization of P is possible by Lemma 3.6. Then $a \to (a)_{11}$ is an isomorphism of G upon $H_{11} = H$.

Let $r_i = (e)_{i1}$. Since P is regular, there exists, for each i in I, an element $\kappa = \kappa_i$ of Λ such that $p_{\kappa i} \neq 0$. Let $r_i' = (p_{\kappa i}^{-1})_{1\kappa}$. Then $x \to r_i x$ and $y \to r_i' y$ are mutually inverse one-to-one mappings of H upon H_{i1} and vice versa.

In the present case, $H \subseteq T(H) \cap T'(H)$, and $\Gamma(H) \cong H$ by Theorem 2.22. We shall replace each element $\gamma(h)$ of $\Gamma(H)$ by the corresponding element h of H in the dual Schützenberger representation of S, and this in turn by the corresponding element of G.

In the paragraph preceding the statement of the theorem, take h_1 to be the identity element of H. Then $\theta | H$ is the identity mapping, and

$$\gamma^*(h) = \gamma(h\theta) = \gamma(h).$$

Hence we may replace each element $\gamma^*(h)$ in (9) by h. If $sH_{j1} = H_{i1}$ then $sr_j \in H_{i1}$ since $r_j \in H_{j1}$ in the present case; hence $r_i'sr_j \in H_{11}$. Consequently, (9) reduces to

$$m_{ij}^*(s) = \begin{cases} r_i'sr_j & \text{if } sH_{j1} = H_{i1}, \\ 0 & \text{otherwise.} \end{cases}$$

Take $s = (a)_{k\lambda}$ with a in G, k in I, and λ in Λ. Then $sH_{j1} \subseteq H_{k1} \cup 0$, and so $m^*_{ij}(s) = 0$ if $i \neq k$. If $p_{\lambda j} \neq 0$, then $sH_{j1} = H_{k1}$, and

$$m^*_{kj}(s) = r'_k s r_j = (p^{-1}_{\kappa k})_{1\kappa}(a)_{k\lambda}(e)_{j1} = (ap_{\lambda j})_{11}.$$

If $p_{\lambda j} = 0$ then $sH_{j1} = 0$, so that $m^*_{kj}(s) = 0$; the foregoing thus holds in this case also. We conclude that, for $s = (a)_{k\lambda}$,

$$m^*_{ij}(s) = \begin{cases} (ap_{\lambda j})_{11} & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Thus $M^*_D(s)$ is a matrix whose only non-vanishing row is the kth, and the element thereof in the jth column is $(ap_{\lambda j})_{11}$. But $sP = (a)_{k\lambda}P$ is an $I \times I$ matrix over G^0 whose only non-vanishing row is the kth, and the element thereof in the jth column is $ap_{\lambda j}$. If we identify the isomorphic groups H_{11} and G, then $M^*_D(s) = sP$.

Exercises for §3.5

1. The form (1) of the Schützenberger representation $s \to M_D(s)$ of a semigroup S corresponding to a $\mathscr{D}$-class D of S depends upon the selection of (i) an $\mathscr{H}$-class $H = H_{11}$ of S in D, and (ii) elements q_λ and q'_λ of S^1 such that $x \to xq_\lambda$ and $y \to yq'_\lambda$ are mutually inverse one-to-one mappings of H_{11} upon $H_{1\lambda}$ and vice versa.

(a) Suppose we select new elements $\bar{q}_\lambda$ and $\bar{q}'_\lambda$ in (ii). Then $t_\lambda = \bar{q}_\lambda q'_\lambda$ and $t'_\lambda = q_\lambda \bar{q}'_\lambda$ are elements of $T(H_{11})$, and $\gamma(t_\lambda)$ and $\gamma(t'_\lambda)$ are mutually inverse elements of $\Gamma(H_{11})$. Let V be the diagonal $\Lambda \times \Lambda$ matrix having $\gamma(t_\lambda)$ in the (λ, λ)-position. Then the new form of the Schützenberger representation is $s \to VM_D(s)V^{-1}$.

(b) Suppose we select another $\mathscr{H}$-class $H_{i\kappa}$ in D instead of H_{11}. By Exercise 3 of §2.4, the mapping $\gamma(t) \to \delta(q'_\kappa t q_\kappa)$ is an isomorphism θ of $\Gamma(H_{11})$ upon $\Gamma(H_{i\kappa})$. If we choose $\bar{q}_\lambda = q'_\kappa q_\lambda$ and $\bar{q}'_\lambda = q'_\lambda q_\kappa$, then $x \to x\bar{q}_\lambda$ and $y \to y\bar{q}'_\lambda$ are mutually inverse one-to-one mappings of $H_{i\kappa}$ upon $H_{i\lambda}$ and vice versa. The new Schützenberger representation $s \to N_D(s)$ differs from the old one only in that each entry $\gamma(t)$ of $M_D(s)$ is replaced by $\delta(q'_\kappa t q_\kappa)$; in other words, $N_D(s) = (n_{\lambda\mu}(s))$ with

$$n_{\lambda\mu}(s) = \begin{cases} \delta(q'_\kappa q_\lambda s q'_\mu q_\kappa) & \text{if } H_{i\lambda}s = H_{i\mu}, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that $H_{1\lambda}s = H_{1\mu}$ if and only if $H_{i\lambda}s = H_{i\mu}$.) Thus $N_D(s) = M_D(s)\theta$, and we see that the representation M_D essentially depends only on D, which justifies the notation M_D.

2. Let G be a group, and let I and Λ be sets. Let X be a semigroup possessing a representation $x \to N(x) = (n_{\lambda\mu}(x))$ by $\Lambda \times \Lambda$ row-monomial matrices over G^0 such that, for given λ and μ in Λ and a in G, there exists x in X such that $n_{\lambda\mu}(x) = a$. Let Y be a semigroup possessing a representation $y \to N^*(y) = (n^*_{ij}(y))$ by $I \times I$ column-monomial matrices over G^0 such that, for given i and j in I and a in G, there exists y in Y such that $n^*_{ij}(y) = a$.

Let D be the set of all non-zero Rees $I \times \Lambda$ matrices over G^0, and let Z be the $I \times \Lambda$ zero matrix over G^0. Assume furthermore that the sets X, Y, D, and $\{Z\}$ are mutually disjoint, and let $S = X \cup Y \cup D \cup Z$. Define product in S as follows. For x in X, y in Y, and A in D, define

$$Ax = AN(x) \quad \text{and} \quad yA = N^*(y)A,$$

where on the right we mean matrix product. Products within X and Y are defined as originally given. All other products are defined to be Z. Then S is a semigroup, and D is a $\mathscr{D}$-class of S.

For s in $S \setminus X$, define $N(s)$ to be the $\Lambda \times \Lambda$ zero matrix over G^0. For s in $S \setminus Y$, define $N^*(s)$ to be the $I \times I$ zero matrix over G^0. Then

$$s \to N(s)\, [s \to N^*(s)]$$

is the [dual] Schützenberger representation of S determined by the $\mathscr{D}$-class D.

3. Let S be a regular Rees $I \times \Lambda$ matrix semigroup over a group with zero G^0. The semigroup P of all right translations of S (§1.3) is isomorphic with the semigroup of all row-monomial $\Lambda \times \Lambda$ matrices over G^0. (In fact, if ρ is any right translation of S, and $\lambda \in \Lambda$, then $(e; 1, \lambda)\rho = (c_\lambda; 1, \lambda')$ for some c_λ in G^0 and some λ' in Λ. Let C be the row-monomial $\Lambda \times \Lambda$ matrix having c_λ in the (λ, λ') position and zeros elsewhere. Then $A\rho = AC$ for every A in S.) The mapping $A \to \rho_A$ of S upon the semigroup P_0 of inner right translations of S is equivalent to the Schützenberger representation of S. Dual remarks hold for the left translations of S, column-monomial $I \times I$ matrices over G^0, and the dual Schützenberger representation of S. Since S is weakly reductive, the mapping $A \to (\lambda_A, \rho_A)$ is one-to-one.

4. (a) Let $\mathscr{T}_X$ be the semigroup of all transformations of a set X. Let E be the one-element group $\{e\}$. Let $\mathscr{V}_X$ be the semigroup of all strictly row-monomial $X \times X$ matrices over E^0, that is, such that each row contains exactly one non-vanishing element. For each α in $\mathscr{T}_X$, let $V(\alpha)$ be the matrix $(v_{xy}(\alpha))$ in $\mathscr{V}_X$ defined as follows:

$$v_{xy}(\alpha) = \begin{cases} e & \text{if } x\alpha = y, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\alpha \to V(\alpha)$ is an isomorphism of $\mathscr{T}_X$ upon $\mathscr{V}_X$. We call it the *natural* isomorphism.

(b) Let s be an element of a semigroup S. Let $N(s) = (n_{ab}(s))$ be the $S \times S$ matrix over E^0 defined by

$$n_{ab}(s) = \begin{cases} e & \text{if } as = b, \\ 0 & \text{otherwise.} \end{cases}$$

Then $s \to N(s)$ is a representation of S by strictly row-monomial $S \times S$ matrices over E^0. It is essentially the regular representation of S, in the sense that $N(s) = V(\rho_s)$, with V defined in (a) and ρ_s defined as usual by $x\rho_s = xs$ for all x, s in S.

(c) Let S be a right simple, right cancellative semigroup without an idempotent element. (Semigroups of this type will be treated in Chapter 8 below.) Then S is an $\mathscr{R}$-class (hence also a $\mathscr{D}$-class) of itself, and each $\mathscr{L}$-class (hence also each $\mathscr{H}$-class) of S consists of a single element of S. Then the Schützenberger representation of S is the representation $s \to N(s)$ defined in (b); it is faithful if S is left reductive.

3.6 A FAITHFUL REPRESENTATION OF A REGULAR SEMIGROUP

Let S be a semigroup. Let Ω be an index set, and for each ω in Ω let $s \to M_\omega(s)$ be a representation of S by matrices over a group with zero G_ω^0. Let $M_\omega(S)$ be the image of S under the homomorphism M_ω, and let T be the direct product of the semigroups $M_\omega(S)$. For each s in S, let $M(s)$ be the element of T whose ω-component is $M_\omega(s)$. Clearly $s \to M(s)$ is a homomorphism M of S into T; we call M the *direct sum* of the representations M_ω of S. We can interpret M as a representation of S by matrices over a group with zero if we wish (Exercise 1 below).

Now let Ω be the set of all $\mathscr{D}$-classes of a semigroup S, and form the direct sum M [M^*] of all the [dual] Schützenberger representations M_D [M_D^*] of S. These were discussed in §3.5. We may also form the direct sum $M \oplus M^*$ of M and M^*. In Theorem 3.19 we give necessary and sufficient conditions on S for M, M^*, and $M \oplus M^*$ to be faithful representations of S. As consequences of this theorem, we show that M and M^* are faithful if S is an inverse semigroup, and that $M \oplus M^*$ is faithful if S is regular. These results are due to Preston [1958].

LEMMA 3.18. *Let D be a $\mathscr{D}$-class of a semigroup S. Let $s \to M_D(s)$ be the Schützenberger representation of S corresponding to D. Let s and t be elements of S. Then $M_D(s) = M_D(t)$ if and only if s and t have the following property: if d is an element of D such that either $ds\mathscr{R}d$ or $dt\mathscr{R}d$, then $ds = dt$.*

PROOF. We shall take $M_D(s)$ as defined by equation (1) of §3.5, and shall adopt the notation of §3.5 introduced prior to (1).

Assume first that $M_D(s) = M_D(t)$, and that d is an element of D such that either $ds\mathscr{R}d$ or $dt\mathscr{R}d$. We are to prove $ds = dt$. By symmetry, we may assume that $ds\mathscr{R}d$.

Since $d \in D$, it follows that $d \in H_{i\lambda}$ for some i in I and λ in Λ. Since $ds\mathscr{R}d$, it follows that $ds \in H_{i\mu}$ for some μ in Λ. By Green's Lemma (2.2), $x \to xs$ is a one-to-one, $\mathscr{R}$-class preserving mapping of L_λ upon L_μ. Restricted to $H_{1\lambda}$, it is a one-to-one mapping of $H_{1\lambda}$ upon $H_{1\mu}$, and so $H_{1\lambda}s = H_{1\mu}$. From this and equation (1) of §3.5, we have $m_{\lambda\mu}(s) = \gamma(q_\lambda s q'_\mu)$. By hypothesis, $m_{\lambda\mu}(s) = m_{\lambda\mu}(t)$, and from this we infer that $H_{1\lambda}t = H_{1\mu}$ and $\gamma(q_\lambda s q'_\mu) = \gamma(q_\lambda t q'_\mu)$. By Green's Lemma, $x \to xt$ is a one-to-one mapping of $H_{i\lambda}$ upon $H_{i\mu}$, and so

$dt \in H_{i\mu}$. Now $hq_\lambda sq'_\mu = hq_\lambda tq'_\mu$ for every h in H. Since $hq_\lambda s$ and $hq_\lambda t$ both belong to $H_{1\mu}$, and q'_μ maps $H_{1\mu}$ one-to-one upon H_{11}, we infer that $hq_\lambda s = hq_\lambda t$ for all h in H. Now $d = r_i hq_\lambda$ for some h in H, and so

$$ds = r_i hq_\lambda s = r_i hq_\lambda t = dt.$$

Conversely, assume that s and t have the property stated in the lemma. Let λ and μ be elements of Λ. We are to show that $m_{\lambda\mu}(s) = m_{\lambda\mu}(t)$.

First assume that $H_{1\lambda}s = H_{1\mu}$. Let $d \in H_{1\lambda}$. Then $ds \in H_{1\mu}$, and so $ds\mathscr{R}d$. By hypothesis, this implies $ds = dt$. If $x \in H_{1\lambda}$, then $x = yd$ for some y in S^1, and so $xs = yds = ydt = xt$. Hence $x \to xt$ is the same mapping of $H_{1\lambda}$ upon $H_{1\mu}$ as $x \to xs$. This implies that $H_{1\lambda}t = H_{1\mu}$ and that $(hq_\lambda)t = (hq_\lambda)s$ for every h in H. Hence $hq_\lambda tq'_\mu = hq_\lambda sq'_\mu$, and so $\gamma(q_\lambda tq'_\mu) = \gamma(q_\lambda sq'_\mu)$. From (1) we conclude that $m_{\lambda\mu}(s) = m_{\lambda\mu}(t)$ in this case.

Now assume that $H_{1\lambda}s \neq H_{1\mu}$. Then $H_{1\lambda}t \neq H_{1\mu}$, since otherwise we could reverse s and t in the preceding paragraph, and infer that $H_{1\lambda}s = H_{1\mu}$. Hence, $m_{\lambda\mu}(s)$ and $m_{\lambda\mu}(t)$ are both 0 in this case.

We now define two binary relations α and β on a semigroup S as follows, wherein s and t denote elements of S:

$$\alpha = \{(s, t)\colon \text{if either } xs\mathscr{R}x \text{ or } xt\mathscr{R}x, \text{ then } xs = xt\},$$

$$\beta = \{(s, t)\colon \text{if either } sx\mathscr{L}x \text{ or } tx\mathscr{L}x, \text{ then } sx = tx\}.$$

The next theorem shows that $\alpha = M \circ M^{-1}$ and $\beta = M^* \circ M^{*-1}$, and hence that α and β are congruences on S. (It is easy to verify the latter directly.)

THEOREM 3.19. *Let S be a semigroup and let M [M^*] be the direct sum of all the [dual] Schützenberger representations M_D [M_D^*] of S. Let s and t be elements of S. Then $M(s) = M(t)$ if and only if $s\alpha t$, and $M^*(s) = M^*(t)$ if and only if $s\beta t$. The representation M, M^*, or $M \oplus M^*$ of S is faithful if and only if, respectively, α, β or $\alpha \cap \beta$ is equal to the identical congruence on S.*

PROOF. If, for each $\mathscr{D}$-class D of S, we define $s\alpha_D t$ to mean that s and t have the property stated in Lemma 3.18, then α is the intersection of all the α_D as D ranges over the set of $\mathscr{D}$-classes of S. Hence, by Lemma 3.18, $s\alpha t$ if and only if $M_D(s) = M_D(t)$ for every $\mathscr{D}$-class D of S, hence if and only if $M(s) = M(t)$. That $M^*(s) = M^*(t)$ if and only if $s\beta t$ follows similarly from the dual of Lemma 3.18, and the last assertion is immediate from these.

LEMMA 3.20. *Let S be a regular semigroup. Let s and t be elements of S such that $s\alpha t$. Then s and t have the same inverses in S, and $s\mathscr{L}t$.*

PROOF. Let x be an inverse of s, and let y be an inverse of t. We proceed to show that y is also an inverse of s. (It will then follow by symmetry that x is also an inverse of t.)

Since $xsx = x$, we have $xs\mathscr{R}x$; since $s\alpha t$, we conclude that $xs = xt$. Similarly, $yty = y$ implies $yt = ys$. Hence

$$\begin{aligned} s &= s(xs) = s(xt) = (sx)t = sx(tyt) \\ &= s(xt)(yt) = s(xs)(ys) = sys, \\ y &= (yt)y = ysy, \end{aligned}$$

and so y is an inverse of s.

From $s = (sx)t$ shown above, and, by symmetry, $t = (ty)s$, it follows that $s\mathscr{L}t$.

Exercise 3 below shows that the converse of Lemma 3.20 is false.

THEOREM 3.21. *Let S be a regular semigroup, and let M [M^*] be the direct sum of all the [dual] Schützenberger representations of S. Then $M \oplus M^*$ is a faithful representation of S. If S is an inverse semigroup, then M and M^* are separately faithful representations of S.*

PROOF. The last assertion is immediate from Theorem 3.19 and Lemma 3.20, and its dual; for if two elements of an inverse semigroup have the same inverse, they coincide.

Let S be a regular semigroup. By Theorem 3.19, to show that $M \oplus M^*$ is faithful, it suffices to show that if $(s, t) \in \alpha \cap \beta$ then $s = t$. Assume that $(s, t) \in \alpha \cap \beta$. From $(s, t) \in \alpha$ and Lemma 3.20, we have $s\mathscr{L}t$. From $(s, t) \in \beta$ and the dual of Lemma 3.20, we have $s\mathscr{R}t$. Hence $s\mathscr{H}t$. Furthermore, by Lemma 3.20, s and t have the same inverses in S. Let x be a common inverse of s and t. Then s and t are inverses of x lying in the same $\mathscr{H}$-class of S, and, by Theorem 2.18, this requires $s = t$.

On the basis of Exercise 2 below, and its dual, it is easy to construct a regular semigroup which is not an inverse semigroup, yet for which M and M^* are separately faithful. Likewise, from the same exercise, it is easy to construct a regular semigroup for which neither M nor M^* is faithful. Finally, Exercise 4 (c) of §3.5 provides an example of an irregular semigroup for which M is faithful.

EXERCISES FOR §3.6

1. For each element ω of an index set Ω, let $s \to M_\omega(s)$ be a representation of a semigroup S by $\Lambda_\omega \times \Lambda_\omega$ matrices over a group with zero G^0_ω. Assume that the sets Λ_ω are mutually disjoint, and let Λ be their union. Let G be any group containing all the groups G_ω, for example, their direct product. Then the direct sum $M(s)$ of all the $M_\omega(s)$ can be expressed as a $\Lambda \times \Lambda$ matrix over G^0.

2. Let S be a regular Rees matrix semigroup $\mathscr{M}^0(G; I, \Lambda; P)$. Let $s \to M_D(s)$ be the Schützenberger representation of S corresponding to the $\mathscr{D}$-class $D = S \setminus 0$. We say that the ith and jth columns of P are right

proportional if there exists c in G such that $p_{\lambda i} = p_{\lambda j}c$ for all λ in Λ. Then M_D is faithful if and only if no two columns of P are right proportional.

(By Theorem 3.21, $M_D \oplus M_D^*$ is faithful; this follows also from Exercise 3 of §3.5.)

3. Let $a = (1, 1)$, $b = (1, 2)$, $c = (2, 1)$, and $d = (2, 2)$ be the elements of the 2×2 rectangular band B_{22} (§1.8). Let $e, \bar{a}, \bar{b}, \bar{c}, \bar{d}$ be five elements distinct from each other and from a, b, c, d. Let

$$S = \{a, b, c, d, e, \bar{a}, \bar{b}, \bar{c}, \bar{d}\}.$$

Define product in S by the table, where x and y range over B_{22}. Then S is a band, in particular a regular semigroup, in which the converse of Lemma 3.20 fails if we take $s = a$ and $t = c$.

	y	e	$\bar{y}$
x	xy	e	$\bar{y}$
e	$\bar{y}$	e	$\bar{y}$
$\bar{x}$	$\overline{xy}$	e	$\bar{y}$

http://dx.doi.org/10.1090/surv/007.1/04

CHAPTER 4

DECOMPOSITIONS AND EXTENSIONS

The notion of a decomposition of a semigroup into the union of disjoint subsemigroups, in particular into the union of a band or semilattice of subsemigroups, was discussed briefly at the end of §1.8. In the first two sections of this chapter we treat the theory of decompositions of a semigroup into the union of simple semigroups of various sorts, including groups. §4.1 is a simplified version of the first part of Croisot's paper [1953], while §4.2 is taken from Clifford [1941].

§4.3 presents a modified version of a theory of decompositions of commutative semigroups due to Tamura and Kimura [1954] and to Hewitt and Zuckerman [1956], and used by the latter in their theory of characters of a commutative semigroup (§5.5).

The last two sections deal with extensions of one semigroup by another, somewhat analogous to the Schreier theory of group extensions. §4.4 is taken from Clifford [1950], and §4.5 from Munn [1955*a*], his results being published here for the first time.

4.1 Croisot's theory of decompositions of a semigroup

Croisot [1953] connects the matter of decompositions of a semigroup S with two other sets of conditions on S, regularity and semiprime conditions. This section is devoted to his theory.

As in §1.9, we say that a semigroup S is *regular* if, for any element a of S, there exists x in S such that $axa = a$. We say that a semigroup S is *left* [*right*] *regular* if, for any element a of S, there exists x in S such that $xa^2 = a$ [$a^2x = a$]. (Croisot uses the term "inversif" instead of "regular".) We shall say that S is *intra-regular* if, for any element a of S, there exist x and y in S such that $xa^2y = a$. These four conditions on S are expressible as follows ("for all a in S" understood): S is regular if $a \in aSa$; S is left regular if $a \in Sa^2$; S is right regular if $a \in a^2S$; S is intra-regular if $a \in Sa^2S$.

In terms of Green's equivalence relations (§2.1), S is left [right, intra-] regular if and only if, $a\mathscr{L}a^2$ [$a\mathscr{R}a^2$, $a\mathscr{J}a^2$] for every a in S. In case the last of these is not clear, $a\mathscr{J}a^2$ if and only if $a \in J(a^2) = a^2 \cup a^2S \cup Sa^2 \cup Sa^2S$. If, say, $a \in a^2S$, then $a^2 \in a \cdot a^2S$, and $a \in a \cdot a^2SS \subseteq Sa^2S$. We shall use these constantly without comment.

A subset X of a semigroup S is called *semiprime* if $a^2 \in X$, $a \in S$ imply $a \in X$.

Lemma 4.1. *A semigroup S is left [right, intra-] regular if and only if every left [right, two-sided] ideal of S is semiprime.*

PROOF. Let S be intra-regular, and let T be an ideal of S. Let $a^2 \in T$, $a \in S$. Then $a \in Sa^2S \subseteq STS \subseteq T$. Conversely, assume that every ideal of S is semiprime. Let $a \in S$. Then $a^2 \in J(a^2)$ implies $a \in J(a^2)$, hence $a \mathscr{J} a^2$, and S is intra-regular. The proof of the equivalence of left [right] regularity of S with the semiprimality of all left [right] ideals of S is similar.

THEOREM 4.2. *The following are equivalent conditions on a semigroup S.*

(A) *S is left regular.*
(B) *Every left ideal of S is semiprime.*
(C) *Every $\mathscr{L}$-class of S is a left simple subsemigroup of S.*
(D) *Every $\mathscr{L}$-class of S is a subsemigroup of S.*
(E) *S is a disjoint union of left simple subsemigroups.*
(F) *S is a union of left simple subsemigroups.*

PROOF. The equivalence of (A) and (B) follows from Lemma 4.1. Assume (A), so that $a \mathscr{L} a^2$ for every a in S. Let $a \mathscr{L} b$. Then $a^2 \mathscr{L} ba$, since $\mathscr{L}$ is a right congruence. But this implies $ba \mathscr{L} a$, and so the $\mathscr{L}$-class L_a containing a is a subsemigroup of S.

To show that L_a is left simple, let $b \in L_a$. We must show that $ca = b$ for some c in L_a. Now $ba \in L_a$, as we have shown, so $b = xba$ for some x in S^1. Let $c = xb$. We are to show that $c \in L_a$. Since S is left regular, we can solve $x = yx^2$ for y in S. Then

$$b = xba = yx^2ba = (yx)(xba) = yxb = yc.$$

From $b = yc$ and $c = xb$ follows $c \mathscr{L} b$, hence $c \in L_b = L_a$. This shows that (A) implies (C).

(C) implies (D) trivially. If we now assume (D), then $a^2 \in L_a$ for every a in S, since L_a is a subsemigroup of S; thus $a^2 \mathscr{L} a$, and S is left regular. Hence (D) implies (A), and we have shown the equivalence of (A), (B), (C), and (D).

Evidently (C) implies (E), and (E) implies (F) trivially. The proof will be finished when we show that (F) implies (A). Assume (F), and let $a \in S$. Then a belongs to some left simple subsemigroup T of S. Hence $a^2 \in T$, and $xa^2 = a$ is solvable for x in T.

THEOREM 4.3. *The following are equivalent conditions on a semigroup S.*

(A) *S is a union of groups.*
(B) *S is both left and right regular.*
(C) *Every left and every right ideal of S is semiprime.*
(D) *S is left regular and regular.*
(D′) *S is right regular and regular.*
(E) *Every $\mathscr{H}$-class of S is a group.*
(F) *S is a union of disjoint groups.*

PROOF. If (A) holds, then S is clearly left regular, right regular, and regular; for we may solve $xa^2 = a$, $a^2y = a$, $aza = a$ for x, y, z within a subgroup of S to which a belongs. Thus (A) implies (B), (D), and (D′). Moreover, (B) is equivalent to (C) by Lemma 4.1.

Now assume (B). Then $a\mathscr{L}a^2$ and $a\mathscr{R}a^2$, that is, $a\mathscr{H}a^2$, for every a in S. By Green's Theorem (2.16), this implies that the $\mathscr{H}$-class H_a containing a is a group. Hence (E) holds. (E) implies (F) since $\mathscr{H}$-classes are disjoint, and (F) implies (A) trivially. We have thus established the equivalence of (A), (B), (C), (E), and (F), and we have shown that (A) implies (D) and (D′).

By duality, the proof will be complete when we show that (D) implies (A). Let $a \in S$. By Theorem 4.2, L_a is a left simple subsemigroup of S. Since S is regular, $axa = a$ for some x in S. Now xa is an idempotent belonging to L_a. Thus L_a is a left simple semigroup containing an idempotent. By Theorem 1.27, L_a is the direct product of a group and a band, and so is a union of groups. Since S is the union of its $\mathscr{L}$-classes, we conclude that S is a union of groups.

The following theorem was found independently of Croisot by Olaf Andersen, and stated without proof in his thesis [1952]. It generalizes an earlier result (Theorem 4.6 below) due to Clifford [1941].

THEOREM 4.4. *The following four assertions concerning a semigroup S are equivalent.*

(A) *S is a union of simple semigroups.*

(B) *S is intra-regular.*

(C) *Every ideal of S is semiprime.*

(D) *The principal ideals of S constitute a semilattice Y under intersection; in fact $J(a) \cap J(b) = J(ab)$ for every a, b in S; furthermore, S is the union of the semilattice Y of simple semigroups S_α ($\alpha \in Y$), each S_α being a $\mathscr{J}$-class of S.*

PROOF. Assuming (A), let $a \in S$. Then a and a^2 both belong to the same simple subsemigroup T of S, whence $a \in Ta^2T \subseteq Sa^2S$. Thus (A) implies (B). (B) is equivalent to (C) by Lemma 4.1. Evidently (D) implies (A). The proof will be complete when we show that (D) follows from (B) and (C), and this we do in several steps.

(1) SaS is the principal ideal $J(a)$ generated by a. For $a \in Sa^2S \subseteq SaS$.

(2) $J(ab) = J(ba)$ for every a, b in S. For $(ab)^2 = a(ba)b \in SbaS = J(ba)$, and we infer from (C) that $ab \in J(ba)$. Hence $J(ab) \subseteq J(ba)$, and equality follows by symmetry.

(3) $J(ab) = J(a) \cap J(b)$ for every a, b in S. Clearly $J(ab) \subseteq J(a) \cap J(b)$. Conversely, let $c \in J(a) \cap J(b)$, say $c = uav = xby$ with u, v, x, y in S. Then $c^2 = xbyuav \in J(byua) = J(abyu)$ by (2). By (C), this implies $c \in J(abyu) \subseteq J(ab)$. Hence $J(a) \cap J(b) \subseteq J(ab)$, and equality follows.

(4) By (3), the set Y of principal ideals of S is a semilattice under intersection, and the mapping $a \to J(a)$ is a homomorphism of S upon Y. The

inverse image of the element $J(a)$ of Y is the set J_a of generators of $J(a)$, that is, the $\mathscr{J}$-class to which a belongs. In particular, J_a is a subsemigroup of S, and S is the semilattice Y of the mutually disjoint semigroups J_a. The proof of (D) will be complete when we show that each J_a is simple. By Lemma 2.39, the principal factor $J(a)/I(a) = J_a \cup 0$ is either 0-simple or a null semigroup. From this and the fact that J_a is closed under multiplication, it is clear that J_a must be simple.

Let m and n be non-negative integers. Croisot says that a semigroup S *satisfies the condition* (m, n) if, for each a in S, there exists x in S such that $a^m x a^n = a$. In this we agree that a^m $[a^n]$ is to be suppressed if $m = 0$ $[n = 0]$. Only the conditions (m, n) with $m + n > 1$ are considered. This apparently infinite set of conditions on S splits into four evidently inequivalent sets of logically equivalent conditions:

I. All $(m, 0)$ with $m \geqq 2$.
II. All $(0, n)$ with $n \geqq 2$.
III. $(1, 1)$.
IV. All (m, n) with $m \geqq 1$, $n \geqq 1$, and $m + n \geqq 3$.

To see that all the conditions in I are equivalent, we note that $(m, 0)$ implies $(2, 0)$ trivially. On the other hand, if we assume $(2, 0)$, and $a \in S$, then $a^2 x = a$ for some x in S. Multiplying successively by a on the left and x on the right, we obtain

$$a = a^2 x = a^3 x^2 = \cdots = a^m x^{m-1} = \cdots.$$

Thus $(m, 0)$ follows from $(2, 0)$, and all the conditions in I are equivalent. But $(2, 0)$ is just right regularity. Dually, all the members of class II are equivalent to left regularity. The lone member of class III is evidently regularity.

Finally, we show that all members of class IV are equivalent to the conjunction of I, II, and III; or, by Theorem 4.3 (B, D, D′), to the conjunction of any two of the three (the third being a consequence of the other two); or, again by Theorem 4.3, to the condition that S be a union of groups.

Let S satisfy (m, n) with $m \geqq 1$, $n \geqq 1$, and $m + n \geqq 3$. Then $(1, 1)$ follows trivially, and either $(0, 2)$ or $(2, 0)$. Hence either (D) or (D′) of Theorem 4.3 holds. By Theorem 4.3, both (D) and (D′) hold, likewise (B), and S is a union of groups. Conversely, if S is a union of groups, it evidently satisfies every condition (m, n), in particular those in IV.

We conclude this section with a statement, without proofs, of three further results from the same paper by Croisot [1953]. Several others will be found in the exercises below. A semigroup S is said to satisfy condition (m, n): (1) *with uniqueness* if (m, n) holds and $a^m x a^n = a^m y a^n$ implies $x = y$; (2) *with reciprocity* if (m, n) holds and $a^m x a^n = a$ implies $x^m a x^n = x$; (3) *with anti-reciprocity* if (m, n) holds and $a^m x a^n = a$ and $x^m a x^n = x$ imply $x = a$. A semigroup S satisfies (m, n) with uniqueness for some $m \geqq 1$, $n \geqq 1$ $[m \geqq 2,$

$n = 0$] if and only if S is a group [right group]. A semigroup S satisfies any one of the conditions $(0, 2)$, $(2, 0)$, or $(1, 1)$ with reciprocity if and only if it is completely simple. A semigroup S satisfies (m, n) with anti-reciprocity for some $m \geqq 1$, $n \geqq 1$, if and only if S is a union of disjoint groups whose identity elements commute, and with certain conditions on the orders of the elements of the groups.

Exercises for §4.1

1. An intra-regular semigroup is semisimple.

2. Call an element a of a semigroup S left regular if $xa^2 = a$ for some x in S. The set of left regular elements in an $\mathscr{L}$-class of S is either empty or a subsemigroup of S.

3. (a) Let Q and R be left simple subsemigroups of a semigroup S having an element a in common. Then the subsemigroup $T = \langle Q, R\rangle$ of S generated by $Q \cup R$ is also left simple. [Hint: we have $Ta = T$, $Tq = T$ for every q in Q, $Tr = T$ for every r in R, and finally $Tt = T$ for every t in T.]

(b) Every left simple subsemigroup of a semigroup S is contained in a maximal left simple subsemigroup of S, and any two distinct maximal left simple subsemigroups of S are disjoint. (Croisot [1953].)

4. (a) Every simple subsemigroup of a semigroup S is contained in a maximal simple subsemigroup of S.

(b) Let S be the Rees 2×2 matrix semigroup over a group with zero G^0 with sandwich matrix

$$P = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix},$$

where e is the identity element of G. In the notation introduced just before Lemma 3.2, L_1 and R_2 are distinct maximal simple subsemigroups of S with non-vacuous intersection.

5. If a semigroup S is a union of simple semigroups, then the maximal simple subsemigroups of S are disjoint. (Croisot [1953].)

6. If $|X| = 2$, $\mathscr{T}_X$ (§1.1) is a union of groups, but is not a band of groups.

7. If $|X| > 2$, $\mathscr{T}_X$ is not a union of simple semigroups. (Croisot [1953].)

8. If S is a regular and intra-regular semigroup, then S is a disjoint union of regular simple semigroups. (Croisot [1953].)

9. An ideal P of a semigroup S is called *prime* if $S \setminus P$ is a subsemigroup of S.

(a) Any intersection of prime ideals of S is either empty or a semiprime ideal of S.

(b) If A is a semiprime ideal of S, and $b \in S \setminus A$, there exists (by Zorn's Lemma) at least one ideal M of S which is maximal in the set of all ideals of S containing A and disjoint from $\langle b\rangle$. Any such ideal M is prime if S is commutative.

(c) Every semiprime ideal of a commutative semigroup is an intersection of prime ideals. (Schwarz [1954c], for finite semigroups; Iséki [1956c], in general. They use the term "closed" for what we, following Croisot, have called "semiprime". The corresponding theorem for rings goes back to W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*, Math. Ann. 101 (1929), 729–744.)

4.2 Semigroups which are unions of groups

In the previous section we considered decompositions of semigroups into disjoint unions of groups, simple semigroups, right simple and left simple semigroups. Conditions were given on a semigroup S under which such decompositions exist. In this section, we shall be interested in the most special of these: decomposition into groups. Theorem 4.3 gives a number of conditions on S each of which is equivalent to the assertion that S is a union of groups. These conditions, however, shed no light on the actual structure of S, and it is the purpose of this section to provide some illumination in this direction. The main results are taken from Clifford [1941].

THEOREM 4.5. *A simple semigroup is a union of groups if and only if it is completely simple.*

PROOF. It is immediate from Theorem 2.52 (i) that a completely simple semigroup (without zero) is a union of groups. The converse is just a specialization of Theorem 2.55.

THEOREM 4.6. *The following assertions concerning a semigroup S are mutually equivalent.*

(A) *S is a union of groups.*

(B) *S is a union of completely simple semigroups.*

(C) *S is a semilattice Y of completely simple semigroups S_α ($\alpha \in Y$), where Y is the semilattice of principal ideals of S, and each S_α is a $\mathscr{J}$-class of S.*

PROOF. (C) implies (B) trivially, and (B) implies (A) by Theorem 4.5. By Theorem 4.4, (A) implies (C) except for the complete simplicity of the simple subsemigroups S_α. But this feature is immediate from Theorem 4.5, since each $\mathscr{J}$-class S_α is a union of the $\mathscr{H}$-classes of S contained in it, while (A) implies that every $\mathscr{H}$-class of S is a group (Theorem 4.3).

Since, by the Rees Theorem (3.5), the structure of completely simple semigroups is known (as always, to within that of groups!), a semigroup which is a union of groups is a semilattice Y of semigroups S_α ($\alpha \in Y$) of known structure. Even if we also regard the structure of semilattices as known, we still do not know the structure of S. For, although we know that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$, we are not in a position to say just how the products $a_\alpha b_\beta$ (a_α in S_α, b_β in S_β) lie in $S_{\alpha\beta}$ when $\alpha \neq \beta$. This is in general a complicated problem. But if we make the further assumption that the idempotent elements of S commute with each other, then we can determine the structure of S, and the rest of this section is concerned with this determination. We observe, by Theorem 1.17, that S is now an inverse semigroup; we are thus dealing with inverse semigroups which are unions of groups.

We remark first that two distinct idempotents in a completely simple semigroup M never commute. For if $e_{i\lambda}$ is the idempotent element of the $\mathscr{H}$-class $H_{i\lambda}$, then, by Lemma 3.2, $e_{i\lambda}e_{j\mu} \in H_{i\mu}$ and $e_{j\mu}e_{i\lambda} \in H_{j\lambda}$. For these to be equal, we must have $i = j$ and $\lambda = \mu$, and so $e_{i\lambda} = e_{j\mu}$, since an $\mathscr{H}$-class contains only one idempotent. Hence if the idempotents of M commute, then M is a group. If S is a union of groups with commuting idempotents, it follows from Theorem 4.6 that S is a semilattice Y of groups. We also notice that in this case Y is isomorphic with E, the set of idempotents of S, which is clearly a semilattice. If $a \in S$, then $H_a = L_a = R_a = D_a = J_a$ (notation of §2.1), all being the maximal subgroup of S containing a. In other words, all the relations of Green are the same, and the equivalence classes are just the maximal subgroups of S.

In the next four lemmas we assume that S is an inverse semigroup which is a union of groups. The foregoing remarks and notation will be used without comment. Moreover, if $a \in S$, then a^{-1} will denote the inverse of a, which in this case is its group-inverse in H_a.

Lemma 4.7. *Let $e \leqq f$ $(e, f$ in $E)$, and let $a \in H_f$. Then $ea = ae$, and $ea \in H_e$.*

Proof. We have $e = fe = a^{-1}ae$, and so $ae \in L_e = H_e$. Similarly, $ea \in H_e$. Hence $ea = (ea)e = e(ae) = ae$.

Lemma 4.8. *Every idempotent of S is in the center of S.*

Proof. Let $g \in E$ and $a \in S$. Then $a \in H_f$ for some f in E. Let fg $(= gf) = e$. Then $e \leqq f$, and hence $ea = ae$ by Lemma 4.7. Hence $ga = gfa = ea = ae = afg = ag$.

Now let Y be a semilattice isomorphic with E (say that in Theorem 4.6). Let $\alpha \to e_\alpha$ be an isomorphism of Y upon E. Thus $e_\alpha e_\beta = e_{\alpha\beta} = e_{\beta\alpha} = e_\beta e_\alpha$, and $e_\alpha \leqq e_\beta$ if and only if $\alpha \leqq \beta$ in Y. We shall write G_α for H_{e_α}; thus $G_\alpha G_\beta \subseteq G_{\alpha\beta}$. Elements of G_α will be denoted by $a_\alpha, b_\alpha, \cdots$.

Lemma 4.9. *If $\alpha \geqq \beta$, the mapping $\phi_{\alpha,\beta}$ defined by*

$$a_\alpha\phi_{\alpha,\beta} = a_\alpha e_\beta \ (a_\alpha \in G_\alpha)$$

is a homomorphism of G_α into G_β. If $\alpha \geqq \beta \geqq \gamma$, then

$$\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}.$$

Finally, $\phi_{\alpha,\alpha}$ is the identity mapping of G_α.

Proof. That $\phi_{\alpha,\beta}$ maps G_α into G_β follows from Lemma 4.7. If $a_\alpha, b_\alpha \in G_\alpha$, then, using Lemma 4.8,

$$\begin{aligned}(a_\alpha\phi_{\alpha,\beta})(b_\alpha\phi_{\alpha,\beta}) &= (a_\alpha e_\beta)(b_\alpha e_\beta) = a_\alpha b_\alpha e_\beta \\ &= (a_\alpha b_\alpha)\phi_{\alpha,\beta}.\end{aligned}$$

Hence $\phi_{\alpha,\,\beta}$ is a homomorphism. If $\alpha \geqq \beta \geqq \gamma$, then, for any a_α in G_α,

$$(a_\alpha \phi_{\alpha,\,\beta})\phi_{\beta,\,\gamma} = (a_\alpha e_\beta)e_\gamma = a_\alpha e_{\beta\gamma} = a_\alpha e_\gamma = a_\alpha \phi_{\alpha,\,\gamma}.$$

Finally, $a_\alpha \phi_{\alpha,\,\alpha} = a_\alpha e_\alpha = a_\alpha$.

LEMMA 4.10. *If $a_\alpha \in G_\alpha$ and $b_\beta \in G_\beta$, then*

$$a_\alpha b_\beta = (a_\alpha \phi_{\alpha,\,\gamma})(b_\beta \phi_{\beta,\,\gamma})$$

where $\gamma = \alpha\beta$.

PROOF. Using Lemma 4.8,

$$\begin{aligned} a_\alpha b_\beta &= a_\alpha e_\alpha b_\beta e_\beta = a_\alpha b_\beta e_\alpha e_\beta = a_\alpha b_\beta e_\gamma \\ &= a_\alpha e_\gamma b_\beta e_\gamma = (a_\alpha \phi_{\alpha,\,\gamma})(b_\beta \phi_{\beta,\,\gamma}). \end{aligned}$$

From Lemma 4.10 it is clear that every product in S is known if we know the semilattice Y, each group G_α ($\alpha \in Y$), and the system of homomorphisms $\phi_{\alpha,\,\beta}$ ($\alpha \geqq \beta$ in Y).

THEOREM 4.11. *Let Y be any semilattice, and to each element α of Y assign a group G_α such that G_α and G_β are disjoint if $\alpha \neq \beta$ in Y. To each pair of elements α, β of Y such that $\alpha > \beta$, assign a homomorphism $\phi_{\alpha,\,\beta}$ of G_α into G_β such that if $\alpha > \beta > \gamma$ then*

$$\phi_{\alpha,\,\beta}\phi_{\beta,\,\gamma} = \phi_{\alpha,\,\gamma}. \tag{1}$$

Let $\phi_{\alpha,\,\alpha}$ be the identity automorphism of G_α. Let S be the union of all the groups G_α ($\alpha \in Y$), and define the product of any two elements a_α, b_β of S (a_α in G_α and b_β in G_β) by

$$a_\alpha b_\beta = (a_\alpha \phi_{\alpha,\,\gamma})(b_\beta \phi_{\beta,\,\gamma}), \tag{2}$$

where γ is the product $\alpha\beta$ of α and β in the semilattice Y.

Then S is a semigroup which is a union of groups and in which the idempotents commute, or (equivalently) an inverse semigroup which is a union of groups. Conversely, every such semigroup can be constructed in this manner.

PROOF. The converse statement has already been established by Lemmas 4.9 and 4.10.

Turning to the direct statement, we first need to prove associativity. Let a, b, c be arbitrary elements of S; then $a = a_\alpha \in G_\alpha$, $b = b_\beta \in G_\beta$, and $c = c_\gamma \in G_\gamma$, for some α, β, γ in Y. From (2) we have

$$(a_\alpha b_\beta)c_\gamma = [(a_\alpha \phi_{\alpha,\,\alpha\beta})(b_\beta \phi_{\beta,\,\alpha\beta})]\phi_{\alpha\beta,\,\alpha\beta\gamma}(c_\gamma \phi_{\gamma,\,\alpha\beta\gamma}),$$

since $a_\alpha b_\beta \in G_{\alpha\beta}$. Using the hypothesis that $\phi_{\alpha\beta,\,\alpha\beta\gamma}$ is a homomorphism, and then using (1) with $\alpha \geqq \alpha\beta \geqq \alpha\beta\gamma$, we get

$$(a_\alpha b_\beta)c_\gamma = [(a_\alpha \phi_{\alpha,\,\alpha\beta\gamma})(b_\beta \phi_{\beta,\,\alpha\beta\gamma})](c_\gamma \phi_{\gamma,\,\alpha\beta\gamma}).$$

Similarly, we find that

$$a_\alpha(b_\beta c_\gamma) = (a_\alpha \phi_{\alpha,\ \alpha\beta\gamma})[(b_\beta \phi_{\beta,\ \alpha\beta\gamma})(c_\gamma \phi_{\gamma,\ \alpha\beta\gamma})],$$

and the two are equal by associativity in $G_{\alpha\beta\gamma}$.

From (2) we get, for any elements e_α, e_β of E,

$$e_\alpha e_\beta = (e_\alpha \phi_{\alpha,\ \alpha\beta})(e_\beta \phi_{\beta,\ \alpha\beta}) = e_{\alpha\beta} e_{\alpha\beta} = e_{\alpha\beta},$$

and so $e_\alpha e_\beta = e_{\alpha\beta} = e_{\beta\alpha} = e_\beta e_\alpha$. Hence S is a semigroup with commuting idempotents, and is a union of groups.

Exercises for §4.2

1. Noting that a band (idempotent semigroup) is a union of one-element groups, it follows from Theorem 4.6 that every band is a semilattice of rectangular bands (§1.8). (McLean [1954].)

2. The condition that a semigroup be a semilattice of groups is equivalent to the conjunction of any two of the following conditions:
   (1) S is a union of groups.
   (2) S is an inverse semigroup.
   (3) Every one-sided ideal of S is a two-sided ideal.

3. A semigroup S is a band of groups if and only if:
   (1) S is both left and right regular (§4.1), and
   (2) for all a and b in S, $Sba = Sba^2$ and $abS = a^2bS$. (Clifford [1954].)

4. A rectangular band of [completely] simple semigroups is [completely] simple. (Note Exercise 4 of §2.7.) (Clifford [1954].)

5. Let $\mathscr{C}$ be a class of semigroups. If a semigroup S is a band of semigroups of type $\mathscr{C}$, then S is a semilattice of semigroups each of which is a rectangular band of semigroups of type $\mathscr{C}$. (Clifford [1954]. Exercise 6 of §4.1 shows that the converse is false.)

6. A semigroup S has the property that every subsemigroup of S has an identity element if and only if S is a semilattice Y of periodic groups, with Y well-ordered downward (i.e., every non-empty subset of Y has a greatest element). (Vorobev [1953*a*].)

7. A semigroup S has the three properties (i) S is a union of groups, (ii) S contains a group-ideal K, and (iii) the product of any two distinct idempotents of S is the identity element of K, if and only if S is a null semilattice Y of groups (i.e., Y has a zero element 0, and the product of any two distinct elements of Y is 0). (Thierrin [1955*c*].)

8. (a) If S is a finitely generated band such that $aba = ab$ for all a, b in S, then S is finite. (Schützenberger [1947].)
   (b) A band S has the property that $aba = ab$ for all a, b in S if and only if it is semilattice Y of left zero semigroups P_α $(\alpha \in Y)$.

9. Denote by $\mathscr{F}\mathscr{B}_n$ the free band (idempotent semigroup) on n generators. In other words, $\mathscr{F}\mathscr{B}_n$ is the semigroup generated by a set $X = \{x_1, \cdots, x_n\}$

of cardinal n subject to all relations of the form $[w(x)]^2 = w(x)$, where $w(x)$ is an arbitrary word in the free semigroup $\mathscr{F}_X$ on X (§1.12).

(a) The maximal semilattice homomorphic image of $\mathscr{F}\mathscr{B}_n$ is the free commutative band generated by X. It is isomorphic with the (finite) semilattice Y of subsets of X under union.

(b) $\mathscr{F}\mathscr{B}_n$ is the union of the semilattice Y of rectangular bands S_α ($\alpha \in Y$), where S_α consists of all words w such that the set of generators appearing in w is α. If $|\alpha| = |\beta|$, then $S_\alpha \cong S_\beta$. By symmetry, the number of minimal left ideals of S_α is equal to the number of minimal right ideals of S_α, i.e., the rectangular band S_α is "square". Denote this number (if finite) by N_r.

(c) Let S_r be the band S_α for $\alpha = \{x_1, \cdots, x_r\}$. Let R be a minimal right ideal of S_r, and let y be an element of minimal length in R. Let x_i be the last generator in y, when y is expressed in the shortest possible way. Then x_i cannot occur elsewhere in y, for otherwise we could exhibit an element in R shorter than y. Hence, we get all the minimal right ideals of S_r from among those generated by elements expressible in the form

$$w(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r)x_i. \tag{$*$}$$

If we assume that N_{r-1} is finite, then this shows that $N_r \leq rN_{r-1}^2$. By induction, each N_r ($r = 1, \cdots, n$) is finite, and hence so is $\mathscr{F}\mathscr{B}_n$.

(d) Any finitely generated band is a homomorphic image of some $\mathscr{F}\mathscr{B}_n$, and so must be finite.

(McLean [1954]; Green and Rees [1952]. The latter give without proof a formula for the order of $\mathscr{F}\mathscr{B}_n$. This can be derived by showing that no two distinct elements ($*$) belong to the same minimal right ideal of S_r, and hence that $N_r = rN_{r-1}^2$.)

10. The direct product of two semigroups is a union of groups if and only if each factor is a union of groups. (Ivan [1953].)

4.3 Decomposition of a commutative semigroup into its archimedean components; separative semigroups

The present section begins with a basic result (Theorem 4.12), due to Tamura and Kimura [1954], from which it follows easily (Theorem 4.13) that any commutative semigroup S is uniquely expressible as a semilattice of "archimedean" semigroups; the latter we call the "archimedean components" of S. Except for the uniqueness, this result is also essentially due to Thierrin [1954–b], who showed that any commutative semigroup S is a disjoint union of archimedean semigroups; one readily sees that any such partition of S must be a semilattice decomposition.

The rest of the section is due to Hewitt and Zuckerman [1956, §4]. This elegant theory was developed by them for their theory of characters, part of which will be given in §5.5. They show (Theorem 5.59 below) that the characters of a commutative semigroup S separate the elements of S if and only if S has the property that $ab = a^2 = b^2$ (a, b in S) implies $a = b$. For the present, we shall say that S is *separative* if it has this latter property. Combining Theorems 4.16 and 4.17, we see that the following are mutually equivalent: (1) S is separative; (2) the archimedean components of S are cancellative; (3) S can be embedded in a union of groups.

If a and b are elements of a commutative semigroup S, we say that a *divides* b, and write $a|b$, if there exists x in S^1 such that $ax = b$. Divisibility is a reflexive, transitive, and compatible (§1.5) relation on S. It is, of course, the usual divisibility relation if S is the multiplicative semigroup of a commutative ring. On the other hand, it is the usual order relation if S is the additive semigroup of positive real numbers. We have the latter in mind in introducing the term "archimedean". We shall say that a commutative semigroup S is *archimedean* if, for any two elements of S, each divides some power of the other. (Indeed, it is easily seen that an ordered abelian group G is archimedean, as customarily defined, if and only if the semigroup of positive elements of G is archimedean, as defined above.)

It follows from Proposition 1.7 that every semigroup S possesses a maximal semilattice homomorphic image. We now give an explicit form for this, due to Tamura and Kimura [1954], when S is commutative. For non-commutative S, see Yamada [1955*b*], and the review thereof (MR 17, 584).

If ρ is a congruence on a semigroup S such that S/ρ is idempotent, then we shall say that ρ *is idempotent.*

We define a relation η on any commutative semigroup S as follows: $a\,\eta\, b$ (a, b in S) if and only if each of the elements a and b divides some power of the other.

THEOREM 4.12. *The relation η on any commutative semigroup S is a congruence on S, and S/η is the maximal semilattice homomorphic image of S.*

PROOF. The relation η is evidently reflexive and symmetric. To show that it is transitive, let $a\,\eta\, b$ and $b\,\eta\, c$ (a, b, c in S). Then $a|b^m$ and $b|c^n$ for some positive integers m and n, and these clearly imply that $a|c^{mn}$. Similarly, c divides some power of a, and we conclude that $a\,\eta\, c$. To show that η is compatible, let $a, b, c \in S$, and let $a\,\eta\, b$. From $a|b^m$ we have $ac|b^m c$, and clearly $b^m c|(bc)^m$, whence $ac|(bc)^m$. Similarly, bc divides some power of ac, and we conclude that $ac\,\eta\, bc$. Hence, η is a congruence on S.

Evidently $a\,\eta\, a^2$ for any a in S, and, since S is commutative, it follows that S/η is a semilattice. The proof will be concluded when we show that η is contained in any idempotent congruence ρ on S. Let $a\,\eta\, b$ (a, b in S). Then there exist positive integers m and n, and elements x and y of S^1, such that $ax = b^m$ and $by = a^n$. Since $a\,\rho\, a^2$ and $b\,\rho\, b^2$, by assumption that ρ is idempotent, we infer that $ax\,\rho\, b$ and $by\,\rho\, a$. Hence

$$a\,\rho\,(by)\,\rho\,(b^2y)\,\rho\,(ba)\,\rho\,(a^2x)\,\rho\,(ax)\,\rho\, b.$$

Thus $a\,\rho\, b$, and we conclude that $\eta \subseteq \rho$.

THEOREM 4.13. *Every commutative semigroup S is uniquely expressible as a semilattice Y of archimedean semigroups S_α $(\alpha \in Y)$. The semilattice Y is isomorphic with the maximal semilattice homomorphic image S/η of S, and the S_α $(\alpha \in Y)$ are the equivalence classes of S* mod η.

PROOF. Let S be a commutative semigroup, and let η be the relation on S defined just before Theorem 4.12. By Theorem 4.12, S/η is a semilattice, and $S \sim S/\eta$. That S is a semilattice of archimedean semigroups will follow when we show that each equivalence class A of S mod η is an archimedean subsemigroup of S.

That A is a subsemigroup of S is clear, since S/η is idempotent. Let $a, b \in A$. Then $a \eta b$, so that $ax = b^m$ and $by = a^n$ for some x, y in S^1 and some positive integers m and n. Then $a(bx) = b^{m+1}$ and $b(ay) = a^{n+1}$. From $bx | b^{m+1}$ and $b | bx$ we have $bx \eta b$, and so $bx \in A$. Similarly $ay \in A$. Thus $a | b^{m+1}$ and $b | a^{n+1}$ relative to A, whence A is archimedean.

Turning now to the uniqueness, let S be a semilattice Y of archimedean semigroups S_α $(\alpha \in Y)$. Once we show that the S_α are the equivalence classes of S mod η, we are finished, since then $Y \cong S/\eta$ follows immediately. Let $a, b \in S$. We are to show that $a \eta b$ if and only if a and b belong to the same S_α. If a and b both belong to S_α, then each divides a power of the other, since S_α is archimedean, and this implies $a \eta b$ by definition. Conversely, let $a \eta b$, and let $a \in S_\alpha$, $b \in S_\beta$. Since $a \eta b$ we have $ax = b^m$ and $by = a^n$ for some x, y in S and some positive integers m and n. Let $x \in S_\gamma$. Then $ax \in S_{\alpha\gamma}$ and $b^m \in S_\beta$, so that $\alpha\gamma = \beta$. Hence $\alpha \geqq \beta$ in the semilattice Y. By symmetry, $\beta \geqq \alpha$, and hence $\alpha = \beta$.

We say that a congruence ρ on a commutative semigroup S is separative if S/ρ is separative, that is, if $ab \rho a^2 \rho b^2$ implies $a \rho b$. Clearly the intersection of any set of separative congruences on S is separative, and it follows from Proposition 1.7 that S has a maximal separative homomorphic image. We wish to get an explicit form for this.

We define a relation σ on any commutative semigroup S as follows: $a \sigma b$ (a, b in S) if and only if there exists a positive integer n such that $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$. We remark that *if there exist positive integers m and n such that $ab^m = b^{m+1}$ and $ba^n = a^{n+1}$, then $a \sigma b$.* For if, say, $m < n$, then we can multiply $ab^m = b^{m+1}$ by b^{n-m}, and obtain $ab^n = b^{n+1}$.

THEOREM 4.14. *The relation σ just defined on any commutative semigroup S is a congruence, and S/σ is the maximal separative homomorphic image of S.*

PROOF. The relation σ is evidently reflexive and symmetric. To show that it is transitive, assume $a \sigma b$ and $b \sigma c$ (a, b, c in S), so that positive integers m and n exist such that

$$ab^n = b^{n+1}, \qquad ba^n = a^{n+1},$$

$$bc^m = c^{m+1}, \qquad cb^m = b^{m+1}.$$

Let $k = (n + 1)(m + 1) - 1 = n(m + 1) + m$. Then

$$\begin{aligned} ac^k &= ac^{n(m+1)}c^m = a(bc^m)^n c^m \\ &= ab^n c^{m(n+1)} = b^{n+1}c^{m(n+1)} \\ &= (bc^m)^{n+1} = c^{(m+1)(n+1)} = c^{k+1}, \end{aligned}$$

and similarly $ca^k = a^{k+1}$. To show that σ is compatible, assume $a \,\sigma\, b$, so that

$$ab^n = b^{n+1}, \qquad ba^n = a^{n+1}$$

for some positive integer n, and let $c \in S$. Then

$$(ac)(bc)^n = ab^n c^{n+1} = b^{n+1}c^{n+1} = (bc)^{n+1},$$

and similarly $(bc)(ac)^n = (ac)^{n+1}$. Hence $ac \,\sigma\, bc$, and we have shown that σ is a congruence.

We show next that σ is separative. Let a and b be elements of S such that $ab \,\sigma\, a^2$ and $ab \,\sigma\, b^2$. Then there exist positive integers m and n such that $(ab)(a^2)^m = (a^2)^{m+1}$ and $(ab)(b^2)^n = (b^2)^{n+1}$. Thus $ba^{2m+1} = a^{2m+2}$ and $ab^{2n+1} = b^{2n+2}$. From the remark preceding the statement of the theorem, it follows that $a \,\sigma\, b$.

The proof will be concluded when we show that σ is contained in every separative congruence ρ on S. Let $a \,\sigma\, b$, say $ab^n = b^{n+1}$, $ba^n = a^{n+1}$. We are to show that $a \,\rho\, b$. Let k be any positive integer such that

$$ab^k \,\rho\, b^{k+1}, \qquad ba^k \,\rho\, a^{k+1}, \tag{1}$$

for example, $k = n$. Assume $k \geqq 2$. Construing ab^0 to mean a in the following (if $k = 2$), we have

$$\begin{aligned} (ab^{k-1})^2 = (ab^{k-2})(ab^k) \,\rho\, (ab^{k-2})b^{k+1} &= (ab^{k-1})b^k, \\ (ab^{k-1})b^k = (ab^k)b^{k-1} \,\rho\, b^{k+1}b^{k-1} &= (b^k)^2. \end{aligned}$$

Setting $x = ab^{k-1}$, $y = b^k$, we have $xy \,\rho\, x^2$ and $xy \,\rho\, y^2$, and hence $x \,\rho\, y$, since ρ is separative. Hence $ab^{k-1} \,\rho\, b^k$, and similarly we show that $ba^{k-1} \,\rho\, a^k$. Therefore (1) holds for $k - 1$. By induction down from $k = n$, it follows that (1) holds for $k = 1$. Hence, $ab \,\rho\, b^2$ and $ba \,\rho\, a^2$, whence $a \,\rho\, b$.

Corollary 4.15. *Let S be a separative commutative semigroup. If a and b are elements of S such that $ab^m = b^{m+1}$ and $ba^n = a^{n+1}$ for some positive integers m and n, then $a = b$.*

Proof. By the remark before Theorem 4.14, $a \,\sigma\, b$. Since S is separative, the identity relation ι on S is separative. By Theorem 4.14, $\sigma \subseteq \iota$, and hence $a = b$.

Theorem 4.16. *A commutative semigroup is separative if and only if its archimedean components are cancellative.*

PROOF. Let S be a separative commutative semigroup, and let S_α be an archimedean component of S. Clearly S_α is also separative, and we are to show that it is cancellative. Let a, b, c be elements of S_α such that $ac = bc$. Since S_α is archimedean, there exist elements x, y of S_α and positive integers m and n such that $cx = a^m$ and $cy = b^n$. Then

$$a^{m+1} = acx = bcx = ba^m,$$
$$b^{n+1} = bcy = acy = ab^n.$$

By Corollary 4.15, $a = b$.

Conversely, let S be a commutative semigroup such that every archimedean component S_α of S is cancellative. Let a and b be elements of S such that $a^2 = b^2 = ab$. If, say, $a \in S_\alpha$ and $b \in S_\beta$ (α, β in Y), then $a^2 \in S_\alpha$ and $b^2 \in S_\beta$, so that $\alpha = \beta$. We conclude that $a = b$ from cancellativity in S_α.

THEOREM 4.17. *A commutative semigroup S can be embedded in a semigroup which is a union of groups if and only if S is separative.*

PROOF. Suppose first that S can be embedded in a semigroup Q which is a union of groups. Let a and b be elements of S such that $a^2 = b^2 = ab$. If H_x denotes the maximal subgroup of Q containing x, then $a^2 \in H_a$ and $b^2 \in H_b$, so that $H_a = H_b$. But $a^2 = ab$ then implies $a = b$. Hence S is separative.

Assume conversely that S is separative. Let $S = \bigcup S_\alpha$ ($\alpha \in Y$) be the expression of S as a semilattice Y of its archimedean components S_α, as given by Theorem 4.13. By Theorem 4.16, each S_α is cancellative. Let G_α be the group of quotients of S_α (§1.10); we recall that G_α is a group containing S_α, that every element of G_α can be expressed in the form ab^{-1} with a and b in S_α, and that $ab^{-1} = cd^{-1}$ (a, b, c, d in S_α) if and only if $ad = bc$.

Since the S_α are mutually disjoint, we may assume that the G_α are mutually disjoint. Let T be their union. We define a product ($\circ$) in T as follows. Let $a, b \in T$, say $a \in G_\alpha$ and $b \in G_\beta$. Then $a = a_1a_2^{-1}$ with a_1, a_2 in S_α, and $b = b_1b_2^{-1}$ with b_1, b_2 in S_β. We define

$$a \circ b = (a_1b_1)(a_2b_2)^{-1}.$$

Since a_1b_1 and a_2b_2 both belong to $S_{\alpha\beta}$, this is an element of $G_{\alpha\beta}$. We must show that it is independent of the representation of a [b] as a quotient of elements of S_α [S_β]. Suppose $a = a_3a_4^{-1}$ with a_3, a_4 in S_α, and $b = b_3b_4^{-1}$ with b_3, b_4 in S_β. Then $a_1a_4 = a_2a_3$ and $b_1b_4 = b_2b_3$, whence $(a_1b_1)(a_4b_4) = (a_2b_2)(a_3b_3)$ in $S_{\alpha\beta}$, and consequently $(a_1b_1)(a_2b_2)^{-1} = (a_3b_3)(a_4b_4)^{-1}$ in $G_{\alpha\beta}$.

To prove associativity, let $a \in G_\alpha$, $b \in G_\beta$, and $c \in G_\gamma$. Then $a = a_1a_2^{-1}$ with a_1, a_2 in S_α; $b = b_1b_2^{-1}$ with b_1, b_2 in S_β; and $c = c_1c_2^{-1}$ with c_1, c_2 in S_γ. We have

$$\begin{aligned}(a \circ b) \circ c &= [(a_1b_1)(a_2b_2)^{-1}] \circ [c_1c_2^{-1}] \\ &= [(a_1b_1)c_1][(a_2b_2)c_2]^{-1} \\ &= [a_1(b_1c_1)][a_2(b_2c_2)]^{-1} \\ &= [a_1a_2^{-1}] \circ [(b_1c_1)(b_2c_2)^{-1}] = a \circ (b \circ c).\end{aligned}$$

Thus T is a semigroup which is a union of groups, and which contains S. We must show, however, that if a and b are elements of S then $a \circ b$ is the same as the original product ab of a and b in S. Let $a \in S_\alpha$, $b \in S_\beta$ (α, β in Y). Then $a = a^2a^{-1}$ and $b = b^2b^{-1}$, so that $a \circ b = (a^2b^2)(ab)^{-1}$. By a simple calculation within the commutative group $G_{\alpha\beta}$, we have $(a^2b^2)(ab)^{-1} = ab$, and so $a \circ b = ab$ as desired.

We did not include in the statement of the foregoing theorem any information about the nature of the union of groups in which we embedded the separative commutative semigroup S. We include this in the next theorem, which summarizes the results of this section. We note in Exercise 1(b) below that there is in general no unique minimal union of groups containing a given separative commutative semigroup.

THEOREM 4.18. *Any commutative semigroup S is uniquely expressible as a semilattice Y of archimedean semigroups S_α ($\alpha \in Y$). The semigroup S can be embedded in a semigroup T which is a union of groups if and only if S is separative, and this is so if and only if each S_α is cancellative. The semigroup T can be taken to be the union of the same semilattice Y of groups G_α, where G_α is the quotient group of S_α, for each α in Y.*

EXERCISES FOR §4.3

1. (a) Let S be the semigroup of positive integers under multiplication. Let α be a finite set of prime numbers, and let S_α consist of every positive integer n such that the set of prime factors of n is precisely α. Then the archimedean components of S are just the sets S_α. The maximal semilattice homomorphic image of S is isomorphic with the semilattice Y of all finite subsets of a countably infinite set under union.

(b) S can be embedded in the union T of the semilattice Y of groups G_α, each G_α being the direct product of $|\alpha|$ infinite cyclic groups. But T is not the only minimal semigroup which is a union of groups containing S, since S can be embedded in a single group.

2. An archimedean commutative semigroup A can contain at most one idempotent. If A contains the idempotent e, then every element a of A has an inverse a' with respect to e, that is, $aa' = a'a = e$. (Tamura and Kimura [1954].)

3. Let S be a commutative semigroup containing an idempotent element e. Then S is archimedean if and only if (1) H_e is the group of zeroids of S (§2.5, Exercises 4–7), and (2) some power of each element of S belongs to H_e.

4. If an archimedean component A of a separative commutative semigroup contains an idempotent, then A is a group. (Hewitt and Zuckerman [1956].)

5. Let S be a commutative semigroup such that some power of each element of S belongs to a subgroup of S. Then each archimedean component

S_α ($\alpha \in Y$) of S contains a unique idempotent e_α. The maximal subgroup H_α of S containing e_α is the group ideal (zeroid group) of S_α. The union H_S of all the H_α ($\alpha \in Y$) is a subsemigroup of S which is the union of the semilattice Y of groups H_α ($\alpha \in Y$). If $a \in S_\alpha$, then $a\,\sigma\,(ae_\alpha)$, where σ is the relation defined prior to Theorem 4.14. Moreover, if a and b are elements of S, then $a\,\sigma\,b$ if and only if (i) a and b belong to the same S_α, and (ii) $ae_\alpha = be_\alpha$. Hence the maximal separative homomorphic image S/σ of S is isomorphic with the "group part" H_S of S. (Schwarz [1954*b*] for periodic S; see also Exercise 5 of §5.5 below.)

6. (a) The semigroup generated by symbols x and y, subject to the generating relations $x^2 = y^2 = xy = yx$, is a union of cancellative semigroups, but is not separative.

(b) A commutative semigroup is a union of disjoint cancellative semigroups if and only if it is separative.

7. Let S be any semigroup, not necessarily commutative. Call S separative if $x^2 = xy = y^2$ (x, y in S) implies $x = y$.

(a) If a semigroup S is a union of disjoint cancellative subsemigroups, then S is separative.

(*b*) Let S be a periodic semigroup. For each idempotent e of S, let S_e be the set of all elements a of S such that $a^n = e$ for some positive integer n. If $a \in S_e$ then $ea = ae$, and $a \in H_e$ if and only if $ea = a$. If S is separative, then $a^2 \in H_e$ implies $a \in H_e$, so that $S_e = H_e$.

(c) Hence a semigroup S is a disjoint union of periodic groups if and only if it is periodic and separative. (Schwarz [1956].)

8. Let S be a cancellative, archimedean, commutative semigroup without idempotent. Let a be a fixed element of S, and let $T_n = a^nS \setminus a^{n+1}S$ ($n = 0, 1, 2, \cdots$), where we agree that a^0x means x for every x in S.

(a) $S = T_0 \cup T_1 \cup T_2 \cup \cdots$, and every element of T_n is uniquely expressible in the form a^nz with z in T_0.

(b) For x, y in S, define $x\,\rho\,y$ if there exists a non-negative integer n such that $x = a^ny$ or $y = a^nx$. Then ρ is a congruence, and S/ρ is a group, the identity element of which is the class ρa.

(c) Denote the elements of S/ρ by S_α, with α ranging over a group G isomorphic with S/ρ. For each α in G, the set $S_\alpha \cap T_0$ consists of a single element u_α of S, and T_0 serves as a set of representatives of the ρ-classes of S. For each pair of elements α, β of G, there is a uniquely determined non-negative integer $n = I(\alpha, \beta)$ such that $u_\alpha u_\beta = a^n u_{\alpha\beta}$, and the function I has the following properties:

(i) $I(\alpha, \beta) = I(\beta, \alpha)$ for all α, β in G;

(ii) $I(\alpha, \beta) + I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma) + I(\beta, \gamma)$ for all α, β, γ in G;

(iii) for each α in G there exists a positive integer m such that $I(\alpha^m, \alpha) > 0$;

(iv) $I(\epsilon, \epsilon) = 1$, where ϵ is the identity element of G.

(d) Conversely, if we start with an abelian group G and a mapping I of

$G \times G$ into the set N_0 of non-negative integers satisfying (i)–(iv) above, and if we define a product in $S = N_0 \times G$ by

$$(m, \alpha)(n, \beta) = (m + n + I(\alpha, \beta), \alpha\beta),$$

then S is a cancellative, archimedean, commutative semigroup without idempotent element. Moreover, if we take $a = (0, \epsilon)$, then S/ρ as constructed above is isomorphic with G, and we obtain the same mapping I on taking $u_\alpha = (0, \alpha)$.

(e) The original semigroup S is isomorphic with the semigroup $N_0 \times G$ constructed in (d), where $G \cong S/\rho$ and I is defined in (c).

(Tamura [1957]. In the terminology of Rédei [1952], S is a Schreier extension of N_0 by G.)

4.4 Extensions of semigroups

Let S and T be disjoint semigroups, T having a zero element 0. A semigroup Σ will be called an (*ideal*) *extension of S by T* if it contains S as an ideal, and if the Rees factor semigroup Σ/S (§1.5) is isomorphic with T. This type of extension arises naturally in the study of semigroups having a composition series (§2.6); we would know their structure if we knew that of all 0-simple semigroups, and if we had an effective solution of the extension problem. In this sense it is somewhat analogous to the Schreier theory of group extensions. An actual generalization of the latter to semigroups was given by Rédei [1952] and developed further by Wiegandt [1958*a*, *b*] and Hancock [1960*a*, *b*]. (As an example, see Exercise 8 of §4.3.) Lack of space forbade our giving both theories, and we chose that of ideal extensions because of the preceding remarks. The present section is taken from Clifford [1950].

In the Schreier theory, we can always find an extension of one group (or semigroup with identity) by another, for example their direct product, but for ideal extensions this is not always possible (note Exercise 1 below). Necessary and sufficient conditions on S and T for the existence of an ideal extension of S by T are not known.

We shall henceforth drop the modifier "ideal", since we do not deal with other types of extension. Throughout this section we shall adhere to the following notation. The letters S and T will denote disjoint semigroups, T having a zero element 0. Let $T^* = T \setminus 0$, and let $\Sigma = S \cup T^*$.

The letters $A, B, C \cdots [s, t, u, \cdots]$ will always denote elements of T^* $[S]$. Expressions like "for all A, B in T^* and all s, t in S" will usually be omitted. Since any extension of S by T is the union of S and T^*, it is clear that we shall get every possible extension $\Sigma(\circ)$ of S by T by finding all ways in which an associative binary operation $(\circ)$ can be defined in $\Sigma = S \cup T^*$ so that the following conditions hold:

(P1) $$A \circ B \begin{cases} = AB & \text{if } AB \neq 0, \\ \in S & \text{if } AB = 0; \end{cases}$$

(P2) $A \circ s \in S$; (P3) $s \circ A \in S$; (P4) $s \circ t = st$.

We begin with an especially simple method of constructing an extension. The notion of a partial homomorphism of one partial groupoid into another was defined in the Remark after Theorem 3.4. Thus a mapping $A \to \bar{A}$ of T^* into S is a partial homomorphism if and only if $\overline{AB} = \bar{A}\,\bar{B}$ whenever $AB \neq 0$.

THEOREM 4.19. *A partial homomorphism $A \to \bar{A}$ of the partial groupoid T^* into S determines an extension $\Sigma(\circ)$ of S by T as follows:*

$$\text{(M1)} \qquad A \circ B = \begin{cases} AB & \text{if } AB \neq 0, \\ \bar{A}\,\bar{B} & \text{if } AB = 0; \end{cases}$$

$$\text{(M2) } A \circ s = \bar{A}s; \qquad \text{(M3) } s \circ A = s\bar{A}; \qquad \text{(M4) } s \circ t = st.$$

If S has an identity element, then every extension of S by T is found in this fashion.

PROOF. The proof of the first half of the theorem is simply a routine verification of the associativity of $(\circ)$. There are eight cases, which we denote by SSS, SST^*, etc., according to where the three factors lie. The case SSS follows from associativity in S. Looking at SST^*,

$$(s \circ t) \circ A = (st) \circ A = (st)\bar{A} = s(t\bar{A}) = s(t \circ A) = s \circ (t \circ A).$$

The cases ST^*S, T^*SS, and T^*ST^* are similar. Looking at ST^*T^*,

$$(s \circ A) \circ B = (s\bar{A}) \circ B = (s\bar{A})\bar{B} = s(\bar{A}\,\bar{B}).$$

If $AB = 0$, then

$$s(\bar{A}\,\bar{B}) = s(A \circ B) = s \circ (A \circ B)$$

by (M1) and (M4). If $AB \neq 0$, then $\overline{AB} = \bar{A}\,\bar{B}$, and so

$$s(\bar{A}\,\bar{B}) = s(\overline{AB}) = s \circ (AB) = s \circ (A \circ B).$$

The case T^*T^*S is similar. Finally coming to the case $T^*T^*T^*$, we note that if $ABC \neq 0$ then $A \circ (B \circ C) = A(BC) = (AB)C = (A \circ B) \circ C$. We may therefore assume that $ABC = 0$; there are then four subcases according to whether AB and BC are $\neq 0$ or $= 0$. We have

$$(A \circ B) \circ C = \begin{cases} (AB) \circ C = (\overline{AB})\bar{C} = (\bar{A}\,\bar{B})\bar{C} & \text{if } AB \neq 0, \\ (\bar{A}\,\bar{B}) \circ C = (\bar{A}\,\bar{B})\bar{C} & \text{if } AB = 0; \end{cases}$$

$$A \circ (B \circ C) = \begin{cases} A \circ (BC) = \bar{A}(\overline{BC}) = \bar{A}(\bar{B}\,\bar{C}) & \text{if } BC \neq 0, \\ A \circ (\bar{B}\,\bar{C}) = \bar{A}(\bar{B}\,\bar{C}) & \text{if } BC = 0. \end{cases}$$

In all four subcases, associativity follows from that in S.

Assume conversely that S has an identity element 1. From (P2, 3, 4) and associativity in $\Sigma(\circ)$, we have

$$A \circ 1 = 1 \circ (A \circ 1) = (1 \circ A) \circ 1 = 1 \circ A.$$

Define $\bar{A} = A \circ 1$ $(= 1 \circ A)$. Then from the above and (P1), we have

$$\bar{A}\,\bar{B} = A \circ 1 \circ B \circ 1 = A \circ B \circ 1 \circ 1 = A \circ B \circ 1. \tag{1}$$

Denote product in S and T by juxtaposition. If $AB \neq 0$ in T, then $A \circ B = AB \in T^*$, and we have $\bar{A}\,\bar{B} = A \circ B \circ 1 = (AB) \circ 1 = \overline{AB}$. Hence the mapping $A \to \bar{A}$ is a partial homomorphism of T^* into S.

The first part of (M1) requires no proof. To show the second part, let $AB = 0$ in T, so that $A \circ B \in S$. Then, by (1), $\bar{A}\,\bar{B} = (A \circ B) \circ 1 = (A \circ B)1 = A \circ B$. To show (M2), $A \circ s = A \circ 1 \circ s = \bar{A} \circ s = \bar{A}s$. The proof of (M3) is dual, and (M4) requires no proof.

For the remainder of this section, we assume that S is weakly reductive (§1.3). A little reflection indicates that this is a relatively mild condition on S. By Lemma 1.2, the translational hull $\bar{S}$ of S is an extension of S.

THEOREM 4.20. *Let S be a weakly reductive semigroup, and let $\bar{S}$ be its translational hull. Let T be a semigroup with zero 0, and let $T^* = T \setminus 0$. Let $\Sigma = T^* \cup S$ and $\bar{\Sigma} = T^* \cup \bar{S}$. Let $\bar{\Sigma}(\circ)$ be an extension of $\bar{S}$ by T. Then $\Sigma(\circ)$ is an extension of S by T if and only if $\Sigma(\circ)$ is a subsemigroup of $\bar{\Sigma}(\circ)$, and this is the case if and only if $A \circ B \in S$ for every pair of elements A, B of T such that $AB = 0$ in T.*

Conversely, let $\Sigma(\circ)$ be an extension of S by T. Then there exists an extension $\bar{\Sigma}(\circ)$ of $\bar{S}$ by T such that $\Sigma(\circ)$ is a subsemigroup of $\bar{\Sigma}(\circ)$.

PROOF. Let $\bar{\Sigma}(\circ)$ be an extension of $\bar{S}$ by T. Since $\bar{S}$ contains an identity element, it follows from Theorem 4.19 that $(\circ)$ is determined by a partial homomorphism $\theta: A \to \bar{A}$ of T^* into $\bar{S}$. Let $A\theta = \bar{A} = (\lambda_A, \rho_A)$ for each A in T^*. By (M3) of Theorem 4.19, we have, for any element (λ, ρ) of $\bar{S}$,

$$(\lambda, \rho) \circ A = (\lambda, \rho)(\lambda_A, \rho_A) = (\lambda_A\lambda, \rho\rho_A).$$

In particular, for an element $s = (\lambda_s, \rho_s)$ of S $(= \bar{S}_0)$,

$$s \circ A = (\lambda_A\lambda_s, \rho_s\rho_A) = (\lambda_t, \rho_t),$$

where $t = s\rho_A$, by Lemma 1.1. Thus $S \circ T^* \subseteq S$, and dually $T^* \circ S \subseteq S$. Hence $S \circ \Sigma \subseteq S$ and $\Sigma \circ S \subseteq S$.

If Σ is a subsemigroup of $\bar{\Sigma}$, then the foregoing shows that S is an ideal of Σ, and consequently Σ is an extension of S by T. If, on the other hand, Σ is an extension of S, it must be a semigroup. It is also clear from the foregoing that Σ is a subsemigroup of $\bar{\Sigma}$ if and only if $T^* \circ T^* \subseteq \Sigma$, and this in turn is equivalent to the assertion that $AB = 0$ implies $A \circ B \in S$.

Passing to the converse, let $\Sigma(\circ)$ be an extension of S by T. For each A in T, define the transformations λ_A and ρ_A of S by

$$s\lambda_A = A \circ s, \qquad s\rho_A = s \circ A.$$

From the associative law in $\Sigma(\circ)$ we see that λ_A $[\rho_A]$ is a left [right] translation of S, and that λ_A and ρ_A are linked. Hence $A \to A\theta = \bar{A} = (\lambda_A, \rho_A)$, $(A \in T^*)$, defines a mapping θ of T^* into $\bar{S}$. Again using the associative law in $\Sigma(\circ)$, we see that, for A and B in T^* with $AB \neq 0$, $\lambda_{AB} = \lambda_B\lambda_A$ and $\rho_{AB} = \rho_A\rho_B$. Hence

$$\begin{aligned}(AB)\theta &= (\lambda_{AB}, \rho_{AB}) = (\lambda_B\lambda_A, \rho_A\rho_B) \\ &= (\lambda_A, \rho_A)(\lambda_B, \rho_B) = (A\theta)(B\theta),\end{aligned}$$

and so θ is a partial homomorphism of T^* into $\bar{S}$. By Theorem 4.19, θ determines an extension $\bar{\Sigma}(*)$ of $\bar{S}$ by T. The proof of Theorem 4.20 will be complete when we show that $\Sigma(\circ)$ is a subsemigroup of $\bar{\Sigma}(*)$, that is, that $(*)$ coincides with $(\circ)$ on $\Sigma \times \Sigma$.

This is clear for $S \times S$. As for $S \times T^*$, we have

$$\begin{aligned}s * A &= (\lambda_s, \rho_s) * A = (\lambda_s, \rho_s)(A\theta) \\ &= (\lambda_s, \rho_s)(\lambda_A, \rho_A) = (\lambda_t, \rho_t) = t,\end{aligned}$$

where $t = s\rho_A$, by Lemma 1.1. But $s\rho_A = s \circ A$, and so $s * A = s \circ A$. Dually, we can show that $(*)$ coincides with $(\circ)$ on $T^* \times S$.

As for $T^* \times T^*$, we have $A * B = AB = A \circ B$ if $AB \neq 0$. We may therefore assume that A and B are elements of T^* such that $AB = 0$. Then, by Theorem 4.19,

$$\begin{aligned}A * B &= (A\theta)(B\theta) = (\lambda_A, \rho_A)(\lambda_B, \rho_B) \\ &= (\lambda_B\lambda_A, \rho_A\rho_B).\end{aligned}$$

Since $AB = 0$, we have $A \circ B \in S$, and so

$$s(\rho_A\rho_B) = (s\rho_A)\rho_B = (s \circ A) \circ B = s \circ (A \circ B) = s\rho_{A \circ B}.$$

Thus $\rho_A\rho_B = \rho_{A \circ B}$ and, similarly, $\lambda_B\lambda_A = \lambda_{A \circ B}$. Hence

$$A * B = (\lambda_{A \circ B}, \rho_{A \circ B}) = A \circ B.$$

Theorem 4.20 reduces the problem of finding all possible extensions of a weakly reductive semigroup S by a semigroup T with zero to that of finding all partial homomorphisms θ of the partial groupoid $T^* = T \setminus 0$ into the translational hull $\bar{S}$ of S such that if A and B are elements of T^* such that $AB = 0$, then $(A\theta)(B\theta)$ shall fall into S (not just into $\bar{S}$). Theorem 4.21 expresses these conditions on θ in terms of mappings involving only T^* and S.

First, however, let us answer a question that may occur to the reader at this point. If S has an identity element, all its extensions by T are given, by Theorem 4.19, by means of a partial homomorphism of T^* into S; in this case $\bar{S} = S$. Now Theorem 4.20 enables us to apply the same Theorem 4.19 to $\bar{S}$, since $\bar{S}$ contains an identity element. But $S^1 = S \cup \{1\}$ also is a semigroup with identity containing S; why can't we use S^1 instead of $\bar{S}$?

The answer is that we would not thereby get all the extensions of S. In particular, we would not in general get the extension $\bar{S}$. In fact, if we could get $\bar{S}$, then $\bar{S} \simeq S^1$.

For suppose we could get $\bar{S}$ from an extension $\bar{\Sigma}(\circ) = T^* \cup S^1$ of S^1, where $T^* = \bar{S}\backslash S$. Let $A = (\lambda_A, \rho_A)$ be an element of T^*. Since $A \circ 1$ and $1 \circ A$ belong to S^1,

$$A \circ 1 = 1 \circ (A \circ 1) = (1 \circ A) \circ 1 = 1 \circ A.$$

We cannot have $A \circ 1 = a \in S$. For this would imply, for every s in S,

$$s\lambda_A = A \circ s = A \circ 1 \circ s = a \circ s = as = s\lambda_a,$$

hence $\lambda_A = \lambda_a$, and similarly $\rho_A = \rho_a$, contrary to $A \notin S$. Hence we must have $A \circ 1 = 1 \circ A = 1$. But then, for every s in S,

$$s\lambda_A = A \circ s = A \circ 1 \circ s = 1 \circ s = s,$$

hence $\lambda_A = \iota$, and similarly $\rho_A = \iota$. Hence $A = (\iota, \iota)$ is the only element of $\bar{S}$ outside of S, and so $\bar{S} \cong S^1$.

Now let S be any semigroup, T any semigroup with 0, and $T^* = T\backslash 0$. Let W be the set of all pairs (A, B) of elements A, B of T^* such that $AB = 0$. Any mapping ϕ of W into S will be called a *ramification of T into S*. If θ is a partial homomorphism of T^* into S, and we define ϕ by $(A, B)\phi = (A\theta)(B\theta)$, then we shall say that ϕ is the ramification of T into S *associated with θ*.

THEOREM 4.21. *Let S be a weakly reductive semigroup, and let T be any semigroup with zero 0. Let ϕ be a ramification of T into S, and let $A \to \lambda_A$ and $A \to \rho_A$ be mappings of $T^* = T\backslash 0$ into the semigroups Λ and P of left and right translations of S, respectively, such that the following conditions hold.*

(C1) $$\lambda_B\lambda_A = \begin{cases} \lambda_{AB} & \text{if } AB \neq 0, \\ \lambda_{(A,\,B)\phi} & \text{if } AB = 0; \end{cases}$$

(C2) $$\rho_A\rho_B = \begin{cases} \rho_{AB} & \text{if } AB \neq 0, \\ \rho_{(A,\,B)\phi} & \text{if } AB = 0; \end{cases}$$

(C3) $s(t\lambda_A) = (s\rho_A)t$; *that is, λ_A and ρ_A are linked.*

Let $\Sigma = T^* \cup S$, and let the binary operation $(\circ)$ be defined in Σ as follows:

(N1) $$A \circ B = \begin{cases} AB & \text{if } AB \neq 0, \\ (A, B)\phi & \text{if } AB = 0; \end{cases}$$

(N2) $A \circ s = s\lambda_A$; (N3) $s \circ A = s\rho_A$; (N4) $s \circ t = st$.

Then $\Sigma(\circ)$ is an extension of S by T, and every extension of S by T can be constructed in this way.

PROOF. If $\Sigma(\circ) = T^* \cup S$ is an extension of S by T, and we define the mappings ϕ, λ_A, ρ_A by (N1), (N2), (N3), then all the statements of the theorem are easy consequences of the associative law in $\Sigma(\circ)$.

Conversely, let ϕ, λ_A, ρ_A be mappings satisfying the hypotheses of the theorem. By (C3), the mapping θ defined by $A\theta = (\lambda_A, \rho_A)$ is a mapping of

T^* into the translational hull $\bar{S}$ of S. By (C1) and (C2), θ is a partial homomorphism of T^* into $\bar{S}$. Let $\bar{\Sigma}(\circ) = T^* \cup \bar{S}$ be the corresponding extension of $\bar{S}$ by T given by Theorem 4.19. The proof will be complete when we show that (N1–4) hold.

Since S is weakly reductive, we may identify it with $\bar{S}_0$, that is, each s in S with (λ_s, ρ_s). Let A and B be elements of T^* such that $AB = 0$. From (C1), (C2), and (M1) of Theorem 4.19,

$$\begin{aligned} A \circ B &= (A\theta)(B\theta) = (\lambda_A, \rho_A)(\lambda_B, \rho_B) \\ &= (\lambda_B\lambda_A, \rho_A\rho_B) \\ &= (\lambda_{(A,\, B)\phi}, \rho_{(A,\, B)\phi}) = (A, B)\phi. \end{aligned}$$

Hence (N1) holds. From (M2) of Theorem 4.19, and Lemma 1.2,

$$A \circ s = (A\theta)s = (\lambda_A, \rho_A)s = s\lambda_A.$$

Hence (N2) holds. The proof of (N3) is dual, and (N4) requires no proof.

EXERCISES FOR §4.4

1. Let S be a semigroup not containing a left identity element, and such that its semigroup Λ of left translations contains no idempotent other than the identity mapping. (For example, S could be an infinite cyclic semigroup.) Let T be a semigroup with 0 containing two non-zero idempotent elements E and F such that $EF = 0$. (For example, T could be the multiplicative semigroup of the ring of integers mod 6, and $E = 3$, $F = 4$.) Then there exists no extension of S by T.

2. An extension of S by T always exists in either of the following two cases: (1) T contains no proper divisors of zero; (2) S contains an idempotent element.

3. Let S be a semigroup which is the union of two disjoint groups, H_1 and H_2. Then S is a left group or a right group, or else either H_1 or H_2 is the kernel of S, let us say H_2, and the structure of S is determined by a homomorphism of H_1 into H_2 as described by Theorem 4.11 or Theorem 4.19. (Suschkewitsch [1937], Chapter 3, §29.)

4.5 EXTENSIONS OF A GROUP BY A COMPLETELY 0-SIMPLE SEMIGROUP; EQUIVALENCE OF EXTENSIONS

In the last section we gave general means for finding all possible extensions of a (weakly reductive) semigroup S by a semigroup T with zero. As with the Schreier theory of group extensions, these means are of a theoretical nature, and to carry them out for particular classes of semigroups is usually difficult. This has been done for two cases: (1) S completely simple, and T arbitrary; (2) S a group, and T a completely 0-simple semigroup. The results in case (1), due to Clifford [1941, §4; 1950, §5] are fairly involved to state, and so we shall omit them here. The results in case (2), however, due to Munn [1955*a*], are more satisfactory, and these we proceed to give. To these results of Munn, which are published here for the first time, we have added only the rather unsatisfactory converse half of Theorem 4.24.

We shall also discuss the equivalence of extensions, not only in case (2) but in the more general case when S is any semigroup with identity element.

Let G^* be a group and T a completely 0-simple semigroup. By Theorem 4.19, to find all the extensions of G^* by T, we need only find all the partial homomorphisms of $T \setminus 0$ into G^*. These in turn are supplied by Theorem 3.14. By the Rees Theorem (3.5), we represent T as a regular Rees $I \times \Lambda$ matrix semigroup over a group with zero G^0, with $\Lambda \times I$ sandwich matrix $P = (p_{\lambda i})$ over G^0. The semigroup S^* of Theorems 3.11 and 3.14 now reduces to the group G^*. We therefore regard I^* and Λ^* as one-element sets, and P^* as the 1×1 matrix with entry e^*, the identity element of G^*. Equations (1) and (2) of §3.4 reduce respectively to

$$(1) \qquad p_{\lambda i}\omega = v_\lambda u_i,$$

$$(2) \qquad (a;\, i,\, \lambda)\theta = u_i(a\omega)v_\lambda \quad (a \in G;\, i \in I;\, \lambda \in \Lambda).$$

Hence we have the following corollary of Theorem 3.14.

THEOREM 4.22. *With the above notation, let $i \rightarrow u_i$ and $\lambda \rightarrow v_\lambda$ be mappings of I into G^* and Λ into G^*, respectively, and let ω be a homomorphism of G into G^* such that* (1) *holds for all i in I and λ in Λ such that $p_{\lambda i} \neq 0$. Then* (2) *defines a partial homomorphism of $T \setminus 0$ into G^*, and every partial homomorphism of $T \setminus 0$ into G^* is obtained in this way.*

Let us call two extensions Σ and Σ' of a semigroup S *equivalent* if there exists an isomorphism of Σ upon Σ' mapping S upon itself. The mapping of S upon itself must evidently be an automorphism.

THEOREM 4.23. *Let S be a semigroup with identity and let T be a semigroup with zero. Let Σ and Σ' be extensions of S determined by the partial homomorphisms θ and θ' of $T \setminus 0$ into S. Then Σ and Σ' are equivalent if and only if there exist automorphisms ψ of T and α of S such that $\theta\alpha = \psi_1\theta'$, where ψ_1 is the restriction of ψ to $T \setminus 0$.*

PROOF. Assume first that Σ and Σ' are equivalent. Then there exists an isomorphism ϕ of Σ upon Σ' which induces an automorphism α of S. Evidently ϕ also induces an automorphism ψ of $T \cong \Sigma/S$, since ϕ maps $T \setminus 0$ upon $T \setminus 0$ and S upon S.

Let multiplication in Σ be denoted by $(\circ)$ and that in Σ' by $(*)$. By Theorem 4.19,

$$A \circ B = \begin{cases} AB & \text{if } AB \neq 0, \\ (A\theta)(B\theta) & \text{if } AB = 0; \end{cases}$$

$$A \circ s = (A\theta)s; \qquad s \circ A = s(A\theta); \qquad s \circ t = st.$$

Likewise,

$$A * B = \begin{cases} AB & \text{if } AB \neq 0, \\ (A\theta')(B\theta') & \text{if } AB = 0; \end{cases}$$

$$A * s = (A\theta')s; \qquad s * A = s(A\theta'); \qquad s * t = st.$$

Then, denoting by 1 the identity element of S,

$$A\theta\phi = (A \circ 1)\phi = (A\phi) * (1\phi) = (A\phi) * 1 = A\phi\theta'.$$

Since $\alpha = \phi|S$, $\psi_1 = \phi|(T \setminus 0)$, and $A\theta \in S$,

$$A\theta\alpha = A\theta\phi = A\phi\theta' = A\psi_1\theta'.$$

Since this holds for every A in $T \setminus 0$, we conclude that $\theta\alpha = \psi_1\theta'$.

Conversely, let ψ be an automorphism of T and α an automorphism of S such that $\theta\alpha = \psi_1\theta'$. Define ϕ by $s\phi = s\alpha$ (all s in S) and $A\phi = A\psi_1$ (all A in $T \setminus 0$). Then ϕ is clearly a permutation of $S \cup (T \setminus 0)$. To show that ϕ is an isomorphism of Σ upon Σ' we must show that

$$\begin{aligned} (A \circ B)\phi &= A\phi * B\phi, \\ (A \circ s)\phi &= A\phi * s\phi, \\ (s \circ A)\phi &= s\phi * A\phi, \\ (s \circ t)\phi &= s\phi * t\phi. \end{aligned}$$

Noting that $AB \neq 0$ if and only if $(A\psi_1)(B\psi_1) \neq 0$, we see that these are equivalent to

$$\begin{aligned} (AB)\psi_1 &= (A\psi_1)(B\psi_1) && \text{if } AB \neq 0 \text{ in } T, \\ [(A\theta)(B\theta)]\alpha &= (A\psi_1\theta')(B\psi_1\theta') && \text{if } AB = 0 \text{ in } T, \\ [(A\theta)s]\alpha &= (A\psi_1\theta')(s\alpha), \\ [s(A\theta)]\alpha &= (s\alpha)(A\psi_1\theta'), \\ (st)\alpha &= (s\alpha)(t\alpha). \end{aligned}$$

The first and last are immediate from the hypotheses that ψ is an automorphism of T and α an automorphism of S. The left-hand side of the second is $(A\theta\alpha)(B\theta\alpha)$, which equals the right-hand side since $\theta\alpha = \psi_1\theta'$. The left-hand side of the third is $(A\theta\alpha)(s\alpha)$, and this equals the right-hand side for the same reason. The fourth equation is proved similarly.

Since the isomorphism ϕ of Σ upon Σ' induces the automorphism α of S upon itself, it is an equivalence of Σ upon Σ'.

If Σ and Σ' are extensions of a group G^*, and if ϕ is an isomorphism of Σ upon Σ', then ϕ must map G^* upon itself; for G^* is the kernel of Σ and of Σ'. Hence *if* Σ *and* Σ' *are isomorphic, then they are equivalent extensions of* G^*. This was noted by Tamura [1954*b*]. We now combine Theorems 4.22 and 4.23 to obtain the following.

THEOREM 4.24. *Let G^* and T be as in Theorem 4.22. Let Σ and Σ' be extensions of G^* by T defined by the partial homomorphisms θ and θ', respectively, of $T\setminus 0$ into G^*. By Theorem 4.22, θ and θ' have the form*

$$(3)\qquad \begin{aligned}(a\,;i,\lambda)\theta &= u_i(a\omega)v_\lambda,\\ (a\,;i,\lambda)\theta' &= u_i'(a\omega')v_\lambda',\end{aligned}$$

where ω and ω' are homomorphisms of G into G^*, and u_i, v_λ, u_i', v_λ' are elements of G^* such that

$$(4)\qquad p_{\lambda i}\omega = v_\lambda u_i,\quad p_{\lambda i}\omega' = v_\lambda' u_i',$$

for every i in I and λ in Λ such that $p_{\lambda i} \neq 0$. If Σ and Σ' are isomorphic then there exist mappings $i \to x_i$ of I into G and $\lambda \to y_\lambda$ of Λ into G, permutations σ of I and τ of Λ, and automorphisms γ of G^* and δ of G, such that

$$(5)\qquad \delta\omega' = \omega\gamma$$

and

$$(6)\qquad p_{\lambda i}\delta = y_\lambda p_{\lambda\tau, i\sigma} x_i \quad (\textit{all } i \textit{ in } I,\ \lambda \textit{ in } \Lambda),$$

where we extend δ to G^0 by defining $0\delta = 0$.

The converse holds if T contains no proper divisors of zero.

REMARK. As in Corollary 3.12, we can also express (6) by saying that there exist an invertible $I \times I$ matrix X and an invertible $\Lambda \times \Lambda$ matrix Y such that $P\delta = YPX$.

Regarding the converse of Theorem 4.24, see Exercises 2 and 3 below.

PROOF. Assume that Σ and Σ' are isomorphic, and hence equivalent, by the remark preceding the theorem. By Theorem 4.23, there exist automorphisms ψ of T and α of G^* such that $\theta\alpha = \psi_1\theta'$, where $\psi_1 = \psi|(T\setminus 0)$. We now apply Theorem 3.11, taking S and S^* therein to be T, and θ to be ψ, noting (as in the proof of Corollary 3.12) that the mappings ϕ, ψ, and ω must now be one-to-one and upon. We conclude that there exist mappings $i \to x_i$ of I into G and $\lambda \to y_\lambda$ of Λ into G, permutations σ of I and τ of Λ, and an automorphism δ of G^0, together satisfying (6), and such that

$$(7)\qquad (a\,;i,\lambda)\psi_1 = (x_i(a\delta)y_\lambda\,; i\sigma, \lambda\tau)$$

for all a in G, i in I, and λ in Λ.

Applying α to the first equation in (3), we obtain

$$(8)\qquad (a\,;i,\lambda)\theta\alpha = (u_i\alpha)(a\omega\alpha)(v_\lambda\alpha).$$

Applying θ' to (7), and using the second equation in (3), we find

$$(9)\qquad (a\,;i,\lambda)\psi_1\theta' = u_{i\sigma}'(x_i\omega')(a\delta\omega')(y_\lambda\omega')v_{\lambda\tau}'.$$

Since $\theta\alpha = \psi_1\theta'$, these are equal for all $(a\,;i,\lambda)$ in $T\setminus 0$. Hence

$$(10)\qquad [(x_i\omega')^{-1}u_{i\sigma}'^{-1}(u_i\alpha)](a\omega\alpha) = (a\delta\omega')[(y_\lambda\omega')v_{\lambda\tau}'(v_\lambda\alpha)^{-1}]$$

for all a in G, i in I, and λ in Λ. Setting $a = e$, the identity element of G, the two bracketed expressions are equal for all i in I and λ in Λ. Since the one on the left is independent of λ, and the one on the right is independent of i, both must be equal (for all i and λ) to some fixed element b of G^*. Hence

$$b(a\omega\alpha) = (a\delta\omega')b \tag{11}$$

for all a in G. Let β be the inner automorphism $x\beta = bxb^{-1}$ of G^*. Then (11) states that $\omega\alpha\beta = \delta\omega'$. Setting $\gamma = \alpha\beta$, we obtain (5).

Assume conversely that there exist mappings $i \to x_i$, $\lambda \to y_\lambda$, σ, τ, γ, and δ satisfying the conditions stated in the theorem. Using (4), (5), (6), and the assumption that $p_{\lambda i} \neq 0$ for all λ in Λ and i in I, we obtain

$$\begin{aligned}(v_\lambda\gamma)(u_i\gamma) &= (v_\lambda u_i)\gamma = p_{\lambda i}\omega\gamma = p_{\lambda i}\delta\omega' \\ &= (y_\lambda p_{\lambda\tau, i\sigma} x_i)\omega' \\ &= (y_\lambda\omega')(p_{\lambda\tau, i\sigma}\omega')(x_i\omega') \\ &= (y_\lambda\omega')(v'_{\lambda\tau}u'_{i\sigma})(x_i\omega').\end{aligned}$$

Hence

$$(u_i\gamma)(x_i\omega')^{-1}u'^{-1}_{i\sigma} = (v_\lambda\gamma)^{-1}(y_\lambda\omega')v'_{\lambda\tau}.$$

This holds for all i in I and λ in Λ, and we conclude as before that both sides must be equal to some fixed element b of G^*. Then

$$\begin{aligned}u_i\gamma &= bu'_{i\sigma}(x_i\omega'), \\ v_\lambda\gamma &= (y_\lambda\omega')v'_{\lambda\tau}b^{-1}.\end{aligned}$$

Defining β by $x\beta = bxb^{-1}$ ($x \in G^*$) and $\alpha = \gamma\beta^{-1}$, these become

$$\begin{aligned}u_i\alpha &= b^{-1}(u_i\gamma)b = u'_{i\sigma}(x_i\omega')b, \\ v_\lambda\alpha &= b^{-1}(v_\lambda\gamma)b = b^{-1}(y_\lambda\omega')v'_{\lambda\tau}.\end{aligned}$$

From these we see that the two bracketed expressions in (10) are both equal to b, and since

$$b(a\omega\alpha)b^{-1} = a\omega\alpha\beta = a\omega\gamma = a\delta\omega'$$

by (5), it follows that (10) holds for all a in G, i in I, and λ in Λ. But this implies that the right-hand members of (8) and (9) are equal. Equation (7) serves to define a one-to-one mapping ψ_1 of $T\backslash 0$ upon itself, and ψ_1 has a unique extension ψ to T such that $0\psi = 0$. From (6) and Theorem 3.11 it follows easily that ψ is an automorphism of T. The equality of the right-hand members of (8) and (9) then gives $\theta\alpha = \psi_1\theta'$, and the equivalence of Σ and Σ' then follows from Theorem 4.23.

Exercises for §4.5

1. Let $G = \langle a, b\rangle$ be the direct product of two cyclic groups of order four ($a^4 = b^4 = e$, $ab = ba$). Let G^* be a cyclic group $\{e^*, a^*\}$ of order two. Let

T be the Rees 2×2 matrix semigroup without zero over G having sandwich matrix

$$P = \begin{pmatrix} e & e \\ e & a^2 \end{pmatrix}.$$

By Exercise 1 (a) of §3.4, every homomorphism θ of T into G^* has the form $(a; i, \lambda)\theta = a\omega$, where ω is any homomorphism of G into G^*. (The condition $p_{\lambda i}\omega = e^*$ holds automatically since G^* has order two.) There are four homomorphisms ω_i ($i = 1, 2, 3, 4$) of G into G^*, defined by

$$\begin{aligned} a\omega_1 &= a^*, & b\omega_1 &= a^*, \\ a\omega_2 &= a^*, & b\omega_2 &= e^*, \\ a\omega_3 &= e^*, & b\omega_3 &= a^*, \\ a\omega_4 &= e^*, & b\omega_4 &= e^*, \end{aligned}$$

and correspondingly four extensions Σ_i ($i = 1, 2, 3, 4$) of G^* by T^0. Then $\Sigma_1 \cong \Sigma_2$, while Σ_2, Σ_3, and Σ_4 are mutually non-isomorphic.

We note incidentally that there exist automorphisms δ of G satisfying $\delta\omega_2 = \omega_3$, but for no such δ do there exist x_i, y_λ, σ, and τ such that equation (6) of Theorem 4.24 holds.

2. Let T be the 2×2 Brandt semigroup $\mathscr{M}^0(E; I, I; \Delta)$ over the one-element group $E = \{e\}$, where $I = \{1, 2\}$ and Δ is the 2×2 identity matrix over E^0. Let $G^* = \{e^*, a\}$ be a cyclic group of order two. In the notation of Theorem 4.24, let $u_1 = v_1 = e^*$, $u_2 = v_2 = a$, and let $u_1' = u_2' = v_1' = v_2' = e^*$. Let σ and τ be the identity mapping of I onto itself. Then (5) and (6) hold, but the extensions of G^* by T determined by the partial homomorphisms θ and θ' given by (3) are not equivalent. (Munn, by letter to the authors.)

3. The converse of Theorem 4.24 would hold without the assumption that T contains no proper divisors of zero if we assume, in addition to (5) and (6), that

$$(v_\lambda u_i)\gamma = (y_\lambda\omega')v'_{\lambda\tau}u'_{i\sigma}(x_i\omega') \tag{12}$$

for all λ in Λ and i in I. This condition is also necessary, and hence we can assert that Σ and Σ' are isomorphic if and only if there exist mappings $i \to x_i$ of I into G and $\lambda \to y_\lambda$ of Λ into G, permutations σ of I and τ of Λ, and automorphisms γ of G^* and δ of G, such that (5), (6), and (12) hold.

http://dx.doi.org/10.1090/surv/007.1/05

CHAPTER 5

REPRESENTATION BY MATRICES OVER A FIELD

In Chapter 3, we considered representations of a semigroup by means of (possibly infinite) matrices over a group. In the present chapter, we consider representations by means of finite matrices over a field, and the term "representation" will have this meaning throughout the chapter. This extends to semigroups the classical theory of representations of groups. As usual, we regard the representations of a semigroup S as known if they can be constructed from representations of groups associated in some way with S.

Let Φ be a field. By $(\Phi)_n$, where n is a positive integer, we shall mean the algebra of all $n \times n$ matrices with entries in Φ. We may also regard $(\Phi)_n$ as the algebra of all linear transformations of an n-dimensional vector-space into itself; then the product AB of two elements A and B of $(\Phi)_n$ will mean their iteration, first A and then B.

Let S be a semigroup. By a *representation* Γ *of* S *of degree* n *over* Φ we mean a homomorphism of S into the multiplicative semigroup of $(\Phi)_n$. This of course means that to each element a of S corresponds an $n \times n$ matrix (or linear transformation) $\Gamma(a)$ such that $\Gamma(ab) = \Gamma(a)\Gamma(b)$ for all a, b in S. If Γ is an isomorphism of S upon a subsemigroup of $(\Phi)_n$, then Γ is said to be *faithful* or *true*. The objective of this chapter is to determine all the representations of several classes of semigroups.

If S is a finite semigroup, there is an obvious one-to-one correspondence between the representations of S and those of its algebra $\Phi[S]$ over Φ. This correspondence preserves reduction and decomposition, and hence full reducibility holds for the representations of S if and only if $\Phi[S]$ is semisimple. In §5.1 we review the classical theory of representations of semisimple algebras, since no existing account of this theory expresses the results in the form which we need. In §5.2 we give necessary and sufficient conditions, due to Munn [1955*b*], on a finite semigroup S in order that $\Phi[S]$ be semisimple; these were found independently by Ponizovsky [1956]. The representations of such a semigroup are given explicitly in Corollary 5.34, following Munn [1957*a*]; they are also given by Ponizovsky [1958]. This could be done in §5.2, but is postponed to §5.3 to save duplication.

We turn thereafter to semigroups which need not be finite. If S is a semigroup satisfying the minimal condition M_J for principal two-sided ideals, then (§5.3) we can construct all the irreducible representations of S from those of its principal factors. This is due to Munn [1960], the idea being based on Hewitt and Zuckerman's treatment [1957] of the full transformation semigroup on a finite set. In §5.4 (due to Suschkewitsch [1933] and

Clifford [1942]) we construct all representations of a completely 0-simple semigroup from those of its structure group. It will be shown in Chapter 6 that if a semigroup S satisfies both the minimal conditions M_R and M_L for right and left principal ideals, then S also satisfies M_J, and the non-null principal factors of S are completely 0-simple. Combining this with the results of §§5.3 and 5.4, it follows that we can construct all irreducible representations of a semigroup S satisfying both M_R and M_L from those of subgroups of S.

In the concluding §5.5, we give the theory (due independently to Schwarz [1954*a*, *b*, *c*] and to Hewitt and Zuckerman [1955, 1956]) of characters of a commutative semigroup S. Moreover, assuming only that the minimal condition holds for the maximal semilattice image of S (see §4.3), the structure of the character semigroup of S is determined. This section can be read independently of the rest of the chapter.

Lack of space prevents us from giving the theory of characters of not necessarily commutative semigroups. (By a character of a semigroup S is meant the trace function of an irreducible representation of S over the complex field.) For this the reader is referred to Munn [1957*a*].

5.1 Representations of semisimple algebras of finite order

By the term *algebra* we shall always mean linear, associative algebra. An algebra $\mathfrak{A}$ over a field Φ is at the same time a ring and a vector space over Φ, with ring addition the same as vector addition, and with ring multiplication and scalar-by-vector product satisfying the conditions

$$(\alpha a)b = a(\alpha b) = \alpha(ab) \quad (\text{all } a, b \text{ in } \mathfrak{A};\ \alpha \text{ in } \Phi).$$

By the *order* of $\mathfrak{A}$ we mean its dimension over Φ as a vector space, i.e., the cardinal number of a basis of $\mathfrak{A}$ over Φ. Throughout the present section, $\mathfrak{A}$ will denote an algebra of finite order over a field Φ.

By an *ideal* of $\mathfrak{A}$ we mean a subset of $\mathfrak{A}$ which is both a linear subspace and a ring ideal of $\mathfrak{A}$, in other words an "admissible ideal" if we regard $\mathfrak{A}$ as a ring with operator domain Φ. By the kth *power* (k a positive integer) of an ideal $\mathfrak{B}$ of $\mathfrak{A}$ we mean the linear subspace of $\mathfrak{A}$ spanned (generated) by the set of all products $b_1b_2\cdots b_k$ of k elements b_i of $\mathfrak{B}$ ($i = 1, \cdots, k$); it is also an ideal of $\mathfrak{A}$. An ideal is called *nilpotent* if some power of it is 0, and the *radical* $\mathfrak{N}$ of $\mathfrak{A}$ is the union of all the nilpotent ideals of $\mathfrak{A}$. The radical also contains every nilpotent left [right] ideal of $\mathfrak{A}$. Since $\mathfrak{A}$ has finite order, $\mathfrak{N}$ is itself nilpotent. An element a of $\mathfrak{A}$ is called *properly nilpotent* if ax (and hence also xa) is nilpotent for every x in $\mathfrak{A}$; then $\mathfrak{N}$ may be described as the set of all properly nilpotent elements of $\mathfrak{A}$.

$\mathfrak{A}$ is called *semisimple* if $\mathfrak{N} = 0$. The objective of the present section is to review the classical theory of the structure and representations of a semisimple algebra of finite order, which we shall need in the next two sections.

For details, the reader is referred to Chapters 15 and 16 of van der Waerden's *Modern Algebra* (English translation by T. J. Benac, Ungar, New York, 1950).

If $\mathfrak{A}$ is any algebra of finite order over Φ and $\mathfrak{N}$ is its radical, then the factor (or difference) algebra $\mathfrak{A}/\mathfrak{N}$ (or $\mathfrak{A}-\mathfrak{N}$) is semisimple. An algebra $\mathfrak{A}$ is called *simple* if it contains no proper ideal $\neq 0$, and is not the null algebra of order one.

WEDDERBURN'S FIRST THEOREM. *An algebra $\mathfrak{A}$ of finite order is semisimple if and only if it is a direct sum*

$$\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_c \tag{1}$$

of (two-sided) ideals $\mathfrak{A}_\sigma$ $(\sigma = 1, 2, \cdots, c)$ each of which is a simple algebra. These $\mathfrak{A}_\sigma$ are uniquely determined by $\mathfrak{A}$.

We note, moreover, that *every non-zero ideal of $\mathfrak{A}$ is the sum of one or more of the $\mathfrak{A}_\sigma$.* We call the $\mathfrak{A}_\sigma$ the *simple components* of $\mathfrak{A}$, and the number c of them we call the *class number* Cl $(\mathfrak{A})$ of $\mathfrak{A}$.

We remark that if $a_\sigma \in \mathfrak{A}_\sigma$, $b_\tau \in \mathfrak{A}_\tau$, and $\sigma \neq \tau$, then $a_\sigma b_\tau = 0$. For $\mathfrak{A}_\sigma \mathfrak{A}_\tau \subseteq \mathfrak{A}_\sigma \cap \mathfrak{A}_\tau = 0$. Hence

$$(a_1 + a_2 + \cdots + a_c)(b_1 + b_2 + \cdots + b_c) = a_1 b_1 + a_2 b_2 + \cdots + a_c b_c.$$

Let

$$\mathfrak{A} = \mathfrak{B}_1 \supset \mathfrak{B}_2 \supset \cdots \supset \mathfrak{B}_m \supset \mathfrak{B}_{m+1} = 0 \tag{2}$$

be a *relative ideal series* of $\mathfrak{A}$, i.e., each $\mathfrak{B}_{i+1}$ is an ideal of $\mathfrak{B}_i$ $(i = 1, \cdots, m)$. The factor algebras $\mathfrak{B}_i/\mathfrak{B}_{i+1}$ are called the *factors* of the series (2).

LEMMA 5.1. *Let (2) be a relative ideal series of an algebra $\mathfrak{A}$ of finite order over a field Φ. Then $\mathfrak{A}$ is semisimple if and only if each factor $\mathfrak{B}_i/\mathfrak{B}_{i+1}$ is semisimple. Moreover, if this is the case, then*

$$\mathrm{Cl}(\mathfrak{A}) = \sum_{i=1}^{m} \mathrm{Cl}(\mathfrak{B}_i/\mathfrak{B}_{i+1}).$$

PROOF. By an evident induction, it suffices to prove the lemma for the case $m = 2$: *if $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, then $\mathfrak{A}$ is semisimple if and only if both $\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{B}$ are semisimple, and then*

$$\mathrm{Cl}(\mathfrak{A}) = \mathrm{Cl}(\mathfrak{B}) + \mathrm{Cl}(\mathfrak{A}/\mathfrak{B}).$$

We can evidently assume that $\mathfrak{B}$ is a proper ideal $\neq 0$ of $\mathfrak{A}$.

Assume first that $\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{B}$ are semisimple. Let a be a properly nilpotent element of $\mathfrak{A}$. Then $a + \mathfrak{B}$ is a properly nilpotent element of $\mathfrak{A}/\mathfrak{B}$, and since the latter is semisimple by hypothesis, $a + \mathfrak{B} = \mathfrak{B}$, i.e., $a \in \mathfrak{B}$. But now a is a properly nilpotent element of $\mathfrak{B}$, and since $\mathfrak{B}$ is semisimple by hypothesis, $a = 0$. Hence 0 is the only properly nilpotent element of $\mathfrak{A}$, i.e., the radical of $\mathfrak{A}$ is 0, and so $\mathfrak{A}$ is semisimple.

Conversely, assume that $\mathfrak{A}$ is semisimple. By Wedderburn's First Theorem, $\mathfrak{A}$ is a direct sum (1) of simple algebras $\mathfrak{A}_\sigma$. As remarked after the statement of this theorem, every ideal of $\mathfrak{A}$ is the sum of one or more of the $\mathfrak{A}_\sigma$, and by suitable numeration we can assume that

$$\mathfrak{B} = \mathfrak{A}_1 \oplus \mathfrak{A}_2 \oplus \cdots \oplus \mathfrak{A}_k \quad (1 \leqq k < c).$$

Then $\mathfrak{A}/\mathfrak{B} \cong \mathfrak{A}_{k+1} \oplus \cdots \oplus \mathfrak{A}_c$. By Wedderburn's First Theorem, $\mathfrak{B}$ and $\mathfrak{A}/\mathfrak{B}$ are semisimple, and

$$\mathrm{Cl}(\mathfrak{B}) + \mathrm{Cl}(\mathfrak{A}/\mathfrak{B}) = k + (c - k) = c = \mathrm{Cl}(\mathfrak{A}).$$

Let $\mathfrak{A}$ be an algebra of order r over Φ, and let n be a positive integer. We shall denote by $(\mathfrak{A})_n$ *the algebra of all* $n \times n$ *matrices over* $\mathfrak{A}$, i.e., having entries in $\mathfrak{A}$, with the usual definitions of addition and multiplication of matrices, and of the multiplication of a matrix by a scalar in Φ. $(\mathfrak{A})_n$ is an algebra of order rn^2 over Φ. In particular, $(\Phi)_n$ will denote the *full matrix algebra of degree* n *over* Φ.

An algebra $\mathfrak{D}$ over Φ is called a *division algebra* if $\mathfrak{D}\setminus 0$ is a group under multiplication.

WEDDERBURN'S SECOND THEOREM. *An algebra* $\mathfrak{A}$ *of finite order over a field* Φ *is simple if and only if it is isomorphic with* $(\mathfrak{D})_n$ *for some division algebra* $\mathfrak{D}$ *over* Φ, *and some positive integer* n. *Moreover,* n *is uniquely determined by* $\mathfrak{A}$, *and* $\mathfrak{D}$ *is uniquely determined to within isomorphism.*

If u is the identity element of $\mathfrak{D}$, then $(\mathfrak{D})_n$ contains the *identity matrix* U_n having u on the main diagonal and 0 elsewhere. Thus a simple algebra has an identity element. Likewise, *every semisimple algebra* $\mathfrak{A}$ *has an identity element*; for if e_σ is the identity element of the simple component $\mathfrak{A}_\sigma$ of $\mathfrak{A}$ $(\sigma = 1, \cdots, c)$, then $e_1 + e_2 + \cdots + e_c$ is the identity element of $\mathfrak{A}$.

The set of all n-dimensional row vectors, i.e., $1 \times n$ matrices, over Φ is an n-dimensional vector space V over Φ. The *natural basis* of V consists of the n vectors $v_1, \cdots, v_n$, where v_i has the identity element 1 of Φ for its ith component, and has 0 for the remaining components. If $A \in (\Phi)_n$, then the transformation $x \to xA$ of V is a linear transformation $\mathbf{A}$ of V, and the mapping $A \to \mathbf{A}$ is an isomorphism of $(\Phi)_n$ upon the algebra $\mathscr{L}\mathscr{T}(V)$ of all linear transformations of V. The ith row of A is the vector $v_i\mathbf{A}$.

Conversely, if V is any n-dimensional vector space, and we choose a basis $v_1, \cdots, v_n$ of V, then each linear transformation $\mathbf{A}$ of V determines a matrix $A = (\alpha_{ij})$ from the expressions

$$v_i\mathbf{A} = \sum_{j=1}^{n} \alpha_{ij} v_j \quad (i = 1, \cdots, n),$$

for the n vectors $v_i\mathbf{A}$ as linear combinations of the basis vectors. The mapping $\mathbf{A} \to A$ is an isomorphism of $\mathscr{L}\mathscr{T}(V)$ upon $(\Phi)_n$.

Let $\mathfrak{A}$ be an algebra over Φ. By a *representation of* $\mathfrak{A}$ *of degree* n *over* Φ we shall mean a homomorphism Γ of $\mathfrak{A}$ into $(\Phi)_n$. In other words, to each element a of $\mathfrak{A}$ corresponds an $n \times n$ matrix $\Gamma(a)$ such that (for all a, b in $\mathfrak{A}$ and α in Φ)

$$\begin{cases} \Gamma(a+b) = \Gamma(a) + \Gamma(b), \\ \Gamma(ab) \quad = \Gamma(a)\Gamma(b), \\ \Gamma(\alpha a) \quad = \alpha\Gamma(a). \end{cases} \tag{3}$$

As discussed above, we may also think of the matrices $\Gamma(a)$ as linear transformations of a vector space V of dimension n over Φ. We call such a space V a *representation* (or *carrier*) *space for* Γ. It is sometimes convenient to write xa instead of $x\Gamma(a)$, where $x \in V$ and $a \in \mathfrak{A}$. We then regard V as a Φ-$\mathfrak{A}$-*module* (or *double module*) admitting both Φ and $\mathfrak{A}$ as operator domains, with Φ acting on the left and $\mathfrak{A}$ on the right. Conversely, a Φ-$\mathfrak{A}$-module V determines a representation of $\mathfrak{A}$ by linear transformations of V.

Let Γ and Γ' be two representations of $\mathfrak{A}$, and let V and V' be representation spaces for Γ and Γ', respectively. We say that Γ and Γ' are *equivalent* if there exists a non-singular linear transformation $x \to x'$ of V upon V' such that if $x \to x'$ then $x\Gamma(a) \to x'\Gamma'(a)$ for every a in $\mathfrak{A}$. The mapping $x \to x'$ is thus an operator isomorphism of the Φ-$\mathfrak{A}$-module V upon the Φ-$\mathfrak{A}$-module V'. If we write $x' = xC$, then $xC\Gamma'(a) = x\Gamma(a)C$ for all x in V and a in $\mathfrak{A}$, so that

$$\Gamma'(a) = C^{-1}\Gamma(a)C \quad (\text{all } a \text{ in } \mathfrak{A}). \tag{4}$$

Since C is non-singular, Γ and Γ' must have the same degree, i.e., V and V' have the same dimension.

Thinking of $\Gamma(a)$ and $\Gamma'(a)$ as matrices (by choosing bases in V and V'), the representations Γ and Γ' are equivalent if and only if (4) holds for some non-singular matrix C in $(\Phi)_n$. In particular, we may have $V = V'$. Then Γ and Γ' are equivalent if and only if they can be regarded as the same representation of $\mathfrak{A}$ by linear transformations of V referred to (perhaps) different bases of V.

In developing the representation theory, it is useful to think of $\Gamma(a)$ sometimes as a matrix and sometimes as a linear transformation. Our failure to indicate this by different notations should not lead to confusion.

Let Γ be a representation of $\mathfrak{A}$ of degree n over Φ, and let V be a representation space for Γ. A subspace W of V is said to be *invariant* under Γ if $w\Gamma(a) \in W$ for every w in W and every a in $\mathfrak{A}$. An invariant subspace of V is just an admissible subgroup of V regarded as a Φ-$\mathfrak{A}$-module. The mapping $w \to w\Gamma(a)$ is a linear transformation $\Gamma_1(a)$ of W, and $a \to \Gamma_1(a)$ is a representation of $\mathfrak{A}$ which we call the representation of $\mathfrak{A}$ *induced* in W by Γ. If we choose a basis $\{v_1, \cdots, v_n\}$ of V such that $\{v_1, \cdots, v_r\}$ is a basis of W, then the representing matrices $\Gamma(a)$ take the partitioned form

$$\Gamma(a) = \begin{pmatrix} \Gamma_1(a) & 0 \\ \Gamma_{21}(a) & \Gamma_2(a) \end{pmatrix} \tag{5}$$

where $\Gamma_1(a)$ is an $r \times r$ matrix, and $\Gamma_2(a)$ is an $(n-r) \times (n-r)$ matrix. Conversely, if $\Gamma(a)$ has this form, if V is the space of row vectors, and if $\{v_1, \cdots, v_n\}$ is the natural basis of V, then the space W spanned by $v_1, \cdots, v_r$ is an invariant subspace of V. By matrix calculation, we see that $a \to \Gamma_2(a)$ is also a representation of $\mathfrak{A}$. We may regard the difference space V/W as the carrier space of this representation Γ_2.

If V contains no proper invariant subspace $\neq 0$, the representation Γ, and the space V itself, are said to be *irreducible*.

Let Γ be a representation of $\mathfrak{A}$ of degree n over Φ and let V be a representation space for Γ. Since $\dim V (= n)$ is finite, there exists a finite sequence

$$(6) \qquad 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{m-1} \subset V_m = V$$

of invariant spaces V_i of V such that V_i properly contains V_{i-1}, and there is no invariant subspace of V strictly between them. Then the representations Γ_i of $\mathfrak{A}$ carried by the difference spaces V_i/V_{i-1} are all irreducible. If we adapt a basis of V to the series (6) in an obvious way, the representing matrices $\Gamma(a)$ take the partitioned form

$$(7) \qquad \Gamma(a) = \begin{pmatrix} \Gamma_1(a) & 0 & 0 & \cdots & 0 \\ \Gamma_{21}(a) & \Gamma_2(a) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \Gamma_{m1}(a) & \Gamma_{m2}(a) & \cdots & \cdots & \Gamma_m(a) \end{pmatrix}.$$

By the Jordan-Hölder Theorem for groups with operators, the (unordered) set of irreducible factors V_i/V_{i-1} of the series (6) is uniquely determined by V as Φ-$\mathfrak{A}$-modules to within operator isomorphism. Hence the irreducible representations Γ_i $(i = 1, \cdots, m)$ of $\mathfrak{A}$ are uniquely determined by Γ to within equivalence (except for their order). We call them the *irreducible constituents of* Γ, and we call (7) an *ultimate reduction* of Γ into its irreducible constituents.

Returning to (5) above, if there exists an invariant subspace W' of V complementary to W in V, i.e., such that $V = W \oplus W'$, then $\Gamma_{21}(a) = 0$ for all a in A. We then say that Γ *decomposes* into Γ_1 and Γ_2 and we write $\Gamma = \Gamma_1 \oplus \Gamma_2$. If every invariant subspace of V admits an invariant complement in V, then there exist irreducible invariant subspaces $W_1, \cdots, W_m$ of V such that

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_m.$$

Adapting a basis of V to $W_1, \cdots, W_m$, the representing matrices $\Gamma(a)$ of Γ take the "diagonal block" form

$$(8) \qquad \Gamma(a) = \begin{pmatrix} \Gamma_1(a) & 0 & \cdots & 0 \\ 0 & \Gamma_2(a) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \Gamma_m(a) \end{pmatrix}$$

and we write $\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_m$. We then say that Γ is *fully* (or *completely*) *reducible*.

A representation Γ of an algebra $\mathfrak{A}$ over a field Φ is said to be *absolutely irreducible* if it is not only irreducible over Φ but remains irreducible when we extend Φ to its algebraic closure.

SCHUR'S LEMMA. *Let Γ and Δ be irreducible representations of an algebra $\mathfrak{A}$. If there exists a constant matrix C such that $C\Gamma(a) = \Delta(a)C$ for all a in $\mathfrak{A}$, then either $C = 0$, or else C is non-singular and then Γ and Δ are equivalent.*

If Γ is absolutely irreducible, and $C\Gamma(a) = \Gamma(a)C$ for all a in $\mathfrak{A}$, then C is a scalar multiple of the identity matrix.

For each element a of an algebra $\mathfrak{A}$ over a field Φ we define ρ_a to be the mapping $x \to x\rho_a = xa$ of $\mathfrak{A}$ into itself. Clearly this is a linear transformation of the vector space $\mathfrak{A}$. Moreover, $\rho: a \to \rho_a$ satisfies (3), so that ρ is a representation of $\mathfrak{A}$; we call it the (*right*) *regular representation* of $\mathfrak{A}$. A right ideal $\mathfrak{R}$ of $\mathfrak{A}$ is just an invariant subspace of the vector space $\mathfrak{A}$ under ρ. $\mathfrak{R}$ is minimal if and only if it is irreducible. Two right ideals $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of $\mathfrak{A}$ are said to be *operator-isomorphic* if the representations of $\mathfrak{A}$ induced in them by ρ are equivalent.

MAIN REPRESENTATION THEOREM FOR SEMISIMPLE ALGEBRAS. *Let $\mathfrak{A}$ be a semisimple algebra of finite order over a field Φ. Then $\mathfrak{A}$ is the direct sum of a finite set of minimal right ideals, i.e., the right regular representation ρ of $\mathfrak{A}$ is fully reducible. In fact every representation of $\mathfrak{A}$ is fully reducible, and every non-null irreducible representation of $\mathfrak{A}$ is contained in ρ, i.e., is equivalent to that induced by ρ in some minimal right ideal of $\mathfrak{A}$.*

Each minimal right ideal of $\mathfrak{A}$ is contained in one of the simple components $\mathfrak{A}_\sigma$ of $\mathfrak{A}$, and two minimal right ideals of $\mathfrak{A}$ are operator-isomorphic if and only if they are contained in the same $\mathfrak{A}_\sigma$. Thus there is a one-to-one correspondence between the irreducible representations of $\mathfrak{A}$ over Φ and the simple components of $\mathfrak{A}$. If Γ_σ corresponds to $\mathfrak{A}_\sigma$, then Γ_σ is a faithful representation of $\mathfrak{A}_\sigma$, while $\Gamma_\sigma(a_\tau) = 0$ for all a_τ in $\mathfrak{A}_\tau$ with $\tau \neq \sigma$. If $c = \mathrm{Cl}(\mathfrak{A})$, then $\{\Gamma_1, \cdots, \Gamma_c\}$ is a complete set of mutually inequivalent irreducible representations of $\mathfrak{A}$.

In particular, every simple algebra $\mathfrak{A}$ of finite order over Φ has to within equivalence exactly one irreducible representation, and every representation of $\mathfrak{A}$ is just a multiple thereof. For a semisimple algebra $\mathfrak{A}$, the irreducible representation Γ_σ of $\mathfrak{A}$ corresponding to $\mathfrak{A}_\sigma$ is essentially the lone irreducible representation of $\mathfrak{A}_\sigma$; for it coincides with it on $\mathfrak{A}_\sigma$ and vanishes on every $\mathfrak{A}_\tau$ with $\tau \neq \sigma$.

If Φ is algebraically closed, then there are no division algebras over Φ other than Φ itself, and in this case Wedderburn's Second Theorem tells us that every simple algebra $\mathfrak{A}$ over Φ is isomorphic with the full matrix algebra $(\Phi)_n$ of degree n, for some n. Any isomorphism of $\mathfrak{A}$ upon $(\Phi)_n$ is a representation of $\mathfrak{A}$, and gives the lone irreducible representation of $\mathfrak{A}$.

Let $\mathfrak{A}$ be an algebra of order n over Φ, and let Γ be a representation of $\mathfrak{A}$ of degree r over Φ. Let m be a positive integer. For each element (a_{ij}) of $(\mathfrak{A})_m$ we construct the $mr \times mr$ matrix

$$\Gamma^m[(a_{ij})] = (\Gamma(a_{ij}))$$

by replacing each entry a_{ij} in the $m \times m$ matrix (a_{ij}) over $\mathfrak{A}$ by the $r \times r$ matrix $\Gamma(a_{ij})$ over Φ. Then Γ^m is a representation of $(\mathfrak{A})_m$. For if (a_{ij}) and (b_{ij}) belong to $(\mathfrak{A})_m$, and (c_{ij}) is their product, i.e., $c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$, then

$$\begin{aligned} \Gamma^m[(c_{ij})] &= (\Gamma(c_{ij})) = (\Gamma(\sum_k a_{ik}b_{kj})) \\ &= (\sum_k \Gamma(a_{ik})\Gamma(b_{kj})) \\ &= (\Gamma(a_{ij}))(\Gamma(b_{ij})) \\ &= \Gamma^m[(a_{ij})]\Gamma^m[(b_{ij})], \end{aligned}$$

and similarly for sum and scalar product. We call Γ^m the representation of $(\mathfrak{A})_m$ *associated* with the representation Γ of $\mathfrak{A}$.

The following is proved in §121 of van der Waerden's *Modern Algebra.*

LEMMA 5.2. *Let $\mathfrak{D}$ be a division algebra, and let m be a positive integer. The right regular representation Δ of $\mathfrak{D}$ is irreducible, and the (lone) irreducible representation of the simple algebra $(\mathfrak{D})_m$ is just the representation Δ^m of $(\mathfrak{D})_m$ associated with Δ.*

From this the following is evident.

THEOREM 5.3. *Let $\mathfrak{A}_\sigma$ $(\sigma = 1, \cdots, c)$ be the simple components of a semisimple algebra $\mathfrak{A}$. By Wedderburn's Second Theorem, each $\mathfrak{A}_\sigma$ may be regarded as a full matrix algebra $(\mathfrak{D}_\sigma)_{m_\sigma}$ of some degree m_σ over a division algebra $\mathfrak{D}_\sigma$. Let Δ_σ be the regular representation of $\mathfrak{D}_\sigma$, and $\Delta_\sigma^{m_\sigma}$ the representation of $\mathfrak{A}_\sigma$ associated with Δ_σ. Then $\Delta_\sigma^{m_\sigma}$ is the only irreducible representation of $\mathfrak{A}_\sigma$. Extend $\Delta_\sigma^{m_\sigma}$ to $\mathfrak{A}$ by defining $\Gamma_\sigma(a) = \Delta_\sigma^{m_\sigma}(a_\sigma)$ if $a = \sum_{\tau=1}^{c} a_\tau$ is the (unique) expression of the element a of $\mathfrak{A}$ as a sum of elements a_τ of the $\mathfrak{A}_\tau$. Then $\{\Gamma_1, \cdots, \Gamma_c\}$ is a complete set of inequivalent irreducible representations of $\mathfrak{A}$. If d_σ is the order of $\mathfrak{D}_\sigma$, then the degree of Γ_σ is $d_\sigma m_\sigma$.*

If Φ is algebraically closed, each $\mathfrak{D}_\sigma$ reduces to Φ and we may regard $\mathfrak{A}$ as a direct sum of full matrix algebras over Φ. The irreducible representations of $\mathfrak{A}$ are then just the projections of $\mathfrak{A}$ upon its various components.

Specialization of Theorem 8, p. 99, of N. Jacobson's *The Theory of Rings* (Mathematical Surveys, No. 2, American Mathematical Society, 1943) gives the following.

THEOREM 5.4. *Let $\mathfrak{A}$ be an algebra with identity element. If $\mathfrak{B}$ is an ideal of $\mathfrak{A}$, then $(\mathfrak{B})_n$ is an ideal of $(\mathfrak{A})_n$, and every ideal of $(\mathfrak{A})_n$ is of the form $(\mathfrak{B})_n$ for some ideal $\mathfrak{B}$ of $\mathfrak{A}$.*

THEOREM 5.5. *Let $\mathfrak{A}$ be an algebra with identity element, and of finite order over a field Φ. Let n be a positive integer. Then $(\mathfrak{A})_n$ is semisimple if and only if $\mathfrak{A}$ is semisimple. More specifically, if $\mathfrak{A}$ is semisimple and $\mathfrak{A}_\sigma$ $(\sigma = 1, \cdots, c)$ are the simple components of $\mathfrak{A}$, then $(\mathfrak{A}_\sigma)_n$ $(\sigma = 1, \cdots, c)$ are those of $(\mathfrak{A})_n$. If Γ_σ is the irreducible representation of $\mathfrak{A}$ corresponding to $\mathfrak{A}_\sigma$ then the associated representation Γ_σ^n is the irreducible representation of $(\mathfrak{A})_n$ corresponding to $(\mathfrak{A}_\sigma)_n$.*

PROOF. If $\mathfrak{A}$ is not semisimple, and $\mathfrak{N}$ is the radical of $\mathfrak{A}$, then $(\mathfrak{N})_n$ is a non-zero nilpotent ideal of $(\mathfrak{A})_n$, and so $(\mathfrak{A})_n$ is not semisimple.

Assume that $\mathfrak{A}$ is semisimple, and that $\mathfrak{A}_1, \cdots, \mathfrak{A}_c$ are its simple components. By Theorem 5.4, each $(\mathfrak{A}_\sigma)_n$ is a simple algebra, and clearly $(\mathfrak{A})_n$ is their direct sum. Thus $(\mathfrak{A})_n$ is semisimple.

If $\mathfrak{A}_\sigma = (\mathfrak{D}_\sigma)_{m_\sigma}$, as in Theorem 5.3, then $(\mathfrak{A}_\sigma)_n = (\mathfrak{D}_\sigma)_{m_\sigma n}$, and the irreducible representation of $\mathfrak{A}_\sigma$ is $\Delta_\sigma^{m_\sigma}$. By Lemma 5.2, the irreducible representation of $(\mathfrak{A}_\sigma)_n$ is $\Delta_\sigma^{m_\sigma n}$. Since $\Delta_\sigma^{m_\sigma n} = (\Delta_\sigma^{m_\sigma})^n$, it follows easily that the irreducible representation of $(\mathfrak{A})_n$ corresponding to $(\mathfrak{A}_\sigma)_n$ is Γ_σ^n.

LEMMA 5.6. *Let $\mathfrak{A}$ be an algebra of finite order over a field Φ, and let $\mathfrak{N}$ be its radical. Any non-null irreducible representation of $\mathfrak{A}$ maps $\mathfrak{N}$ into 0, and so is effectively a representation of the semisimple algebra $\mathfrak{A}/\mathfrak{N}$.*

PROOF. Let V be a representation space of an irreducible representation Γ of $\mathfrak{A}$. Then $V\mathfrak{N}$ is an invariant subspace of V, and so either $V\mathfrak{N} = 0$ or $V\mathfrak{N} = V$. But the latter is impossible, since it would imply $V\mathfrak{N}^k = V$ for every positive integer k, whereas $\mathfrak{N}^k = 0$ for some k.

THEOREM 5.7. *An irreducible algebra of linear transformations is simple.*

PROOF. Let $\mathfrak{A}$ be an irreducible algebra of linear transformations. The identity mapping ι of $\mathfrak{A}$ upon itself is then an irreducible representation of $\mathfrak{A}$. Since ι is faithful, the radical of $\mathfrak{A}$ must be 0, by Lemma 5.6. Thus $\mathfrak{A}$ is semisimple. Let $\mathfrak{A}_\sigma$ $(\sigma = 1, \cdots, c)$ be the simple components of $\mathfrak{A}$, and let Γ_σ be the irreducible representation of $\mathfrak{A}$ corresponding to $\mathfrak{A}_\sigma$. The irreducible representation ι of $\mathfrak{A}$ must be one of the Γ_σ, say Γ_1. Since Γ_1 maps every $\mathfrak{A}_\sigma$ with $\sigma \neq 1$ into 0, and yet is faithful, it follows that no such $\mathfrak{A}_\sigma$ can exist. Thus $c = 1$, $\mathfrak{A} = \mathfrak{A}_1$, and $\mathfrak{A}$ is simple.

We conclude this section with some elementary results which we shall need concerning matrices over an algebra of finite order.

An element a of an algebra $\mathfrak{A}$ is called a *right* [*left*] *divisor of zero* if there exists an element $b \neq 0$ of $\mathfrak{A}$ such that $ba = 0$ [$ab = 0$].

LEMMA 5.8. *Let $\mathfrak{A}$ be an algebra of finite order over a field Φ. If an element a of $\mathfrak{A}$ is not a right [left] divisor of zero, then $\mathfrak{A}$ contains a right [left] identity element e with respect to which a has a two-sided inverse x $(ax = xa = e)$.*

PROOF. Let n be the least positive integer such that the powers $a, a^2, \cdots, a^n$ of a are linearly dependent. (Clearly $n \geqq 2$.) Then

$$\alpha_1 a + \alpha_2 a^2 + \cdots + \alpha_n a^n = 0$$

with $\alpha_1, \cdots, \alpha_n$ in Φ, and $\alpha_n \neq 0$. Since a is not a right divisor of zero, and n is minimal, we see that $\alpha_1 \neq 0$. Let

$$e = -\alpha_1^{-1}(\alpha_2 a + \alpha_3 a^2 + \cdots + \alpha_n a^{n-1}). \tag{9}$$

Then $ea = a$. Let $b \in \mathfrak{A}$. Then

$$(be - b)a = bea - ba = b(ea - a) = 0,$$

whence $be - b = 0$. Thus e is a right identity element of $\mathfrak{A}$. From (9) we see that a has the inverse

$$x = -\alpha_1^{-1}(\alpha_2 e + \alpha_3 a + \cdots + \alpha_n a^{n-2})$$

with respect to e. (We take $a^{n-2} = e$ if $n = 2$.)

COROLLARY 5.9. *Let $\mathfrak{A}$ be an algebra of finite order over a field Φ. If an element a of $\mathfrak{A}$ is neither a left nor a right divisor of zero, then $\mathfrak{A}$ contains an identity element u, and a is a unit, i.e., $ax = xa = u$ for some x in $\mathfrak{A}$.*

Let $\mathfrak{A}$ be an algebra and m a positive integer. If the algebra $(\mathfrak{A})_m$ has an identity element E, then it is quickly seen that $\mathfrak{A}$ must have an identity element u, and that E is the identity matrix U_m having u on the main diagonal and 0 elsewhere. A matrix P in $(\mathfrak{A})_m$ is called *non-singular* if it is a unit in $(\mathfrak{A})_m$, i.e., if $PQ = QP = U_m$ for some Q in $(\mathfrak{A})_m$. From these remarks, and the observation that if $\mathfrak{A}$ has finite order r over Φ then $(\mathfrak{A})_m$ has finite order rm^2, the following is immediate from Corollary 5.9.

COROLLARY 5.10. *Let $\mathfrak{A}$ be an algebra of finite order over a field Φ and let m be a positive integer. Let $P \in (\mathfrak{A})_m$, i.e., P is an $m \times m$ matrix with entries in $\mathfrak{A}$. If P is neither a left nor a right divisor of zero in $(\mathfrak{A})_m$, then $\mathfrak{A}$ has an identity element and P is non-singular.*

THEOREM 5.11. *Let P be an $n \times m$ matrix over an algebra $\mathfrak{A}$ of finite order over a field Φ. If $n > m$, then there exists a non-zero $m \times n$ matrix X over $\mathfrak{A}$ such that $XP = 0$. If $m > n$, then there exists a non-zero $m \times n$ matrix Y over $\mathfrak{A}$ such that $PY = 0$.*

PROOF. We shall prove the case $n > m$, the other being similar. Let

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

with P_1 an $m \times m$ matrix and P_2 an $(n - m) \times m$ matrix over $\mathfrak{A}$. Suppose first that P_1 is a right divisor of zero in $(\mathfrak{A})_m$, so that there exists $X_1 \neq 0$ in $(\mathfrak{A})_m$ such that $X_1 P_1 = 0$. We may then take

$$X = (X_1 \quad 0).$$

Hence we may assume that P_1 is not a right divisor of zero in $(\mathfrak{A})_m$. By Lemma 5.8 applied to the algebra $(\mathfrak{A})_m$, $(\mathfrak{A})_m$ contains a right identity element E with respect to which P_1 has a two-sided inverse Q_1 in $(\mathfrak{A})_m: P_1Q_1 = Q_1P_1 = E$. We may now take

$$X = (-X_2P_2Q_1 \quad X_2),$$

where X_2 is any non-zero $m \times (n-m)$ matrix over $\mathfrak{A}$. For then

$$XP = -X_2P_2Q_1P_1 + X_2P_2 = -X_2P_2E + X_2P_2 = 0,$$

since $X_2P_2 \in (\mathfrak{A})_m$ and E is a right identity of $(\mathfrak{A})_m$.

The foregoing theorem is stated in the form we want for later use. But we remark that the number of rows of X (or columns of Y) does not really matter. For if x is a non-vanishing row of X then $xP = 0$; and if x is a non-zero row vector of dimension n such that $xP = 0$, then we can construct from x a non-zero $t \times n$ matrix X of any number t of rows such that $XP = 0$. The real significance of Theorem 5.11 is that if $n > m$ then the rows of P are left linearly dependent over $\mathfrak{A}$, and if $m > n$ then the columns of P are right linearly dependent over $\mathfrak{A}$.

5.2 Semigroup algebras

The classical approach to the theory of representations of a finite group G over a field Φ is through the algebra $\Phi[G]$ of G over Φ. There is a natural one-to-one correspondence between the representations of G over Φ and those of $\Phi[G]$ which preserves equivalence, reduction, and decomposition. Thus the problem of finding all the representations of G over Φ is transferred to the algebra $\Phi[G]$. If $\Phi[G]$ is semisimple, then, by the Main Representation Theorem for Semisimple Algebras (§5.1), every representation of $\Phi[G]$—and hence every representation of G—is fully reducible into irreducible ones, and the latter are given by Theorem 5.3. The following classical theorem tells when this is the case.

MASCHKE'S THEOREM. *Let G be a finite group and let Φ be a field. Then $\Phi[G]$ is semisimple if and only if the characteristic of Φ does not divide the order of G.*

(In §127 of his *Modern Algebra*, van der Waerden states Maschke's Theorem in the equivalent, and historically more correct form: every representation of a finite group G over a field Φ is fully reducible if the characteristic of Φ does not divide the order of G. Actually, Maschke's original statement (Math. Ann. 52 (1899), 363–368) pertains only to the complex number field. We are indebted to Professor Richard Brauer for this reference.)

The algebra $\Phi[S]$ of a finite semigroup S over Φ is defined just like $\Phi[G]$, and indeed we do not require that S be finite (applications in §5.3). For the same reasons as for groups, it is important to know when $\Phi[S]$ is semisimple, and the present section is largely devoted to this. The theory we present is due to Munn [1955b]. We include also two related results due to Hewitt and Zuckerman [1955], namely Theorems 5.30 and 5.31.

Let S be a semigroup, and Φ a field. By the *algebra* $\Phi[S]$ *of* S *over* Φ we mean an algebra $\mathfrak{A}$ over Φ containing a subset $\bar{S}$ which is at the same time a basis of $\mathfrak{A}$ and a multiplicative subsemigroup of $\mathfrak{A}$ isomorphic with S. If it exists, $\Phi[S]$ is clearly uniquely determined by S and Φ to within an isomorphism. We shall "identify" $\bar{S}$ with S in the usual way, and thus regard $\Phi[S]$ as the algebra over Φ containing S itself as basis and subsemigroup.

To show that $\Phi[S]$ always exists, let $\mathfrak{A}$ be the set of all mappings $a: s \to a(s)$ of S into Φ such that the "support" of a, i.e., the set of all s in S such that $a(s) \neq 0$, is finite or empty. We define the sum $a + b$ of two elements a and b of $\mathfrak{A}$ to be the mapping $s \to a(s) + b(s)$, and the product αa of an element α of Φ and an element a of $\mathfrak{A}$ to be the mapping $s \to \alpha a(s)$. Clearly $\mathfrak{A}$ becomes thereby a vector space over Φ. We now define the product ab of two elements a and b of $\mathfrak{A}$ to be the element $c: s \to c(s)$, where

$$(1) \qquad c(r) = \sum_{st=r} a(s)b(t).$$

Here r is an arbitrary element of S, and for each r in S the summation is taken over all pairs s, t of elements of S such that $st = r$. (c is the "convolution" of a and b.) It is readily verified that $\mathfrak{A}$ becomes thereby a linear associative algebra over Φ.

For each s in S, let $\bar{s}$ denote the "characteristic function" of s, i.e.,

$$\bar{s}(t) = \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{if } t \neq s. \end{cases}$$

Here 1 denotes the identity element of Φ. Let $\bar{S}$ be the set of all $\bar{s}$ as s ranges over S. Then $\bar{S} \subseteq \mathfrak{A}$, and the mapping $s \to \bar{s}$ is clearly an isomorphism of S upon $\bar{S}$. If $a \in \mathfrak{A}$, if $\{s_1, \cdots, s_n\}$ is the support of a, and if we write $\alpha_i = a(s_i)$ $(i = 1, \cdots, n)$, then

$$(2) \qquad a = \alpha_1\bar{s}_1 + \alpha_2\bar{s}_2 + \cdots + \alpha_n\bar{s}_n.$$

Since $\bar{S}$ is evidently linearly independent, it follows that $\bar{S}$ is a basis of $\mathfrak{A}$, and so $\mathfrak{A}$ qualifies as $\Phi[S]$.

We shall frequently write (2) in the form $a = \sum_{s \in S} a(s)\bar{s}$, or rather $a = \sum_{s \in S} a(s)s$, since we shall now identify $\bar{s}$ with s. This is a finite sum since only a finite number of coefficients $a(s)$ are $\neq 0$. If $b = \sum_{t \in S} b(t)t$ is another element of $\Phi[S]$, then

$$ab = \sum_{s \in S} \sum_{t \in S} a(s)b(t)st = \sum_{r \in S} c(r)r$$

with $c(r)$ given by (1).

Example 1. If S is the infinite cyclic semigroup generated by x, then $\Phi[S^1]$ is the ring of polynomials $\Phi[x]$ in x over Φ. (Amitsur [1951].)

Let S be a semigroup, Φ a field, and $\Phi[S]$ the algebra of S over Φ. If T is any subset of S, $\Phi[T]$ will denote the subspace of $\Phi[S]$ "spanned" by T, i.e., the set of all finite linear combinations of elements of T with coefficients in Φ. We agree that $\Phi[\square] = 0$. $\Phi[T]$ is a subalgebra [ideal] of $\Phi[S]$ if and only if T is a subsemigroup [ideal] of S.

Let S be a semigroup with zero element z. By the *contracted algebra* $\Phi_0[S]$ *of* S *over* Φ we mean an algebra over Φ containing a basis B such that $B \cup 0$ is a subsemigroup of $\Phi_0[S]$ isomorphic with S. The factor algebra $\Phi[S]/\Phi[z]$ is an instance of such an algebra, with

$$B = \{s + \Phi[z] : s \in S \setminus z\}.$$

$\Phi_0[S]$ is clearly uniquely determined by S and Φ to within an isomorphism, and we may (in the usual way) regard it as containing $S \setminus z$ as a basis. In fact we may regard it as containing all of S if we identify z with 0.

We note that if $S \cup z$ is the semigroup resulting from the adjunction of a zero element z to a semigroup S (whether or not S has a zero to begin with), then $\Phi_0[S \cup z] \cong \Phi[S]$. Thus any semigroup algebra can also be regarded as a contracted semigroup algebra.

EXAMPLE 2. Let S be the semigroup of $n \times n$ matrix units (§2.7, Exercise 7), i.e.,

$$S = \{e_{ij} : i, j = 1, \cdots, n\} \cup \{z\}$$

with multiplication defined by

$$e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k, \\ z & \text{if } j \neq k, \end{cases}$$

and with z acting as a zero element. Then $\Phi_0[S]$ is isomorphic with the full $n \times n$ matrix algebra $(\Phi)_n$ over Φ. (Amitsur [1951].)

LEMMA 5.12. *Let T be an ideal of S. Then $\Phi[S]/\Phi[T]$ is isomorphic with the contracted algebra $\Phi_0[S/T]$ of S/T over Φ.*

PROOF. The set

$$B = \{s + \Phi[T] : s \in S \setminus T\}$$

is a basis of $\Phi[S]/\Phi[T]$, and $B \cup \{\Phi[T]\} \cong S/T$.

LEMMA 5.13. *Let S be a finite semigroup with zero element z. Then $\Phi[S]$ is semisimple if and only if $\Phi_0[S]$ is semisimple.*

PROOF. $\Phi[z]$ is an algebra of order 1 over Φ isomorphic with Φ, and hence semisimple. Since $\Phi_0[S] \cong \Phi[S]/\Phi[z]$, the result is immediate from Lemma 5.1.

By a *representation* Γ of a semigroup S of degree n over a field Φ we mean a homomorphism Γ of S into the multiplicative semigroup of $(\Phi)_n$. In other words, to each element s of S corresponds an $n \times n$ matrix $\Gamma(s)$ over Φ such that

$$\Gamma(st) = \Gamma(s)\Gamma(t), \quad (\text{all } s, t \text{ in } S).$$

If Γ is a representation of S, then we can define a representation Γ^* of $\Phi[S]$ by the rule

$$\Gamma^*\left(\sum_{s \in S} a(s)s\right) = \sum_{s \in S} a(s)\Gamma(s). \tag{3}$$

The sum on the right makes sense, since only a finite number of coefficients $a(s)$ are $\neq 0$. Conversely, if Γ^* is any representation of $\Phi[S]$, its restriction Γ to S is a representation of S. The correspondence $\Gamma \leftrightarrow \Gamma^*$ is one-to-one and is easily seen to preserve equivalence, reduction, and decomposition. The theory of matrix representations of S over Φ is thus identical with that of $\Phi[S]$.

If S has a zero element z, then there is a similar one-to-one correspondence $\Gamma \leftrightarrow \Gamma^*$ between the representations Γ of S over Φ such that $\Gamma(z) = 0$ and the representations Γ^* of the contracted algebra $\Phi_0[S]$ of S over Φ. The connection (3) between Γ and Γ^* still holds, with the summation taken over $S \setminus z$ instead of S. Representations Γ of S for which $\Gamma(z) \neq 0$ differ only trivially from those for which $\Gamma(z) = 0$; see Exercise 1 below.

In both cases, the relation between Γ and Γ^* is so close that henceforth we shall not distinguish between them.

A finite semigroup possesses a principal series (§2.6)

$$S = S_1 \supset S_2 \supset S_3 \supset \cdots \supset S_n \supset S_{n+1} = \square. \tag{4}$$

We recall (Theorem 2.40) that the Rees quotients S_i/S_{i+1} $(i = 1, \cdots, n)$, with the convention that $S_n/\square$ means S_n, are the principal factors of S, and that these are the same (except for order) for any two principal series of S.

THEOREM 5.14. *The algebra $\Phi[S]$ of a finite semigroup S over a field Φ is semisimple if and only if the algebra $\Phi[S_i/S_{i+1}]$ of each of the principal factors of S over Φ is semisimple.*

PROOF. Corresponding to (4), we have a series of ideals of $\Phi[S]$, namely

$$\Phi[S] = \Phi[S_1] \supset \Phi[S_2] \supset \cdots \supset \Phi[S_n] \supset 0,$$

recalling the convention that $\Phi[\square] = 0$. By Lemma 5.12,

$$\Phi[S_i]/\Phi[S_{i+1}] \cong \Phi_0[S_i/S_{i+1}] \quad (i = 1, \cdots, n-1).$$

By Lemma 5.1, $\Phi[S]$ is semisimple if and only if each factor algebra $\Phi[S_i]/\Phi[S_{i+1}]$ is semisimple. The result then follows from Lemma 5.13.

We recall (Lemma 2.39) that any principal factor of a semigroup S is 0-simple, simple, or null, and that S is called semisimple if every principal factor of S is 0-simple or simple.

COROLLARY 5.15. *If $\Phi[S]$ is semisimple, then S is semisimple.*

PROOF. If S_i/S_{i+1} is a principal factor of S, then $\Phi_0[S_i/S_{i+1}]$ is semisimple by Theorem 5.14 and Lemma 5.13. Were S_i/S_{i+1} a null semigroup, then $\Phi_0[S_i/S_{i+1}]$ would be a null algebra, and so could not be semisimple.

COROLLARY 5.16. *If S has the principal series* (4), *and $\Phi[S]$ is semisimple, then*

$$\mathrm{Cl}(\Phi[S]) = \sum_{i=1}^{n} \mathrm{Cl}(\Phi_0[S_i/S_{i+1}]).$$

PROOF. This is immediate from Lemma 5.1 and the foregoing.

As a result of Theorem 5.14 and Corollary 5.15, in order to find necessary and sufficient conditions for $\Phi[S]$ to be semisimple, it suffices to consider the case in which S is 0-simple, and we now turn to this.

Let S be a finite 0-simple semigroup. Then S is completely simple by Corollary 2.56, and so, by the Rees Theorem (3.5), S is isomorphic with a regular Rees $m \times n$ matrix semigroup $\mathcal{M}^0(G; m, n; P)$ over a finite group G, with $n \times m$ sandwich matrix $P = (p_{\lambda i})$ $(\lambda = 1, \cdots, n; i = 1, \cdots, m)$. For the purpose of finding all representations of S, or of ascertaining whether $\Phi[S]$ is semisimple, there is no loss in generality if we assume $S = \mathcal{M}^0(G; m, n; P)$.

Let $G = \{g_\sigma : \sigma = 1, \cdots, r\}$. Then the non-zero elements of S are the matrices $(g_\sigma)_{i\lambda}$ having g_σ in the (i, λ)-position and zero elsewhere. They multiply as follows:

$$(g_\sigma)_{i\lambda} \circ (g_\tau)_{j\mu} = (g_\sigma)_{i\lambda} P (g_\tau)_{j\mu} = (g_\sigma p_{\lambda j} g_\tau)_{i\mu}.$$

Let $\mathfrak{A}$ be any algebra over Φ. Let m and n be positive integers, and let P be a fixed $n \times m$ matrix over $\mathfrak{A}$. Let $\mathcal{M}(\mathfrak{A}; m, n; P)$ be the vector space of all $m \times n$ matrices over $\mathfrak{A}$. Define a product $\circ$ in $\mathcal{M}$ by

$$A \circ B = APB \quad (A, B \text{ in } \mathcal{M}).$$

Then $\mathcal{M}$ is an algebra over Φ, which we call the *Munn $m \times n$ matrix algebra over $\mathfrak{A}$ with sandwich matrix P.*

LEMMA 5.17. *The contracted algebra $\Phi_0[S]$ of $S = \mathcal{M}^0(G; m, n; P)$ over the field Φ is isomorphic with the Munn algebra $\mathfrak{B} = \mathcal{M}(\Phi[G]; m, n; P)$.*

PROOF. Let us identify the zero element of G^0 with that of $\Phi[G]$, and consequently the zero element of S with that of $\mathfrak{B}$. Since $G \subseteq \Phi[G]$, it follows that $S \subseteq \mathfrak{B}$. Each element A of $\mathfrak{B}$ is an $m \times n$ matrix $(a_{i\lambda})$ with elements $a_{i\lambda}$ in $\Phi[G]$, say

$$a_{i\lambda} = \sum_{\sigma=1}^{r} \alpha_{\sigma;i,\lambda}\, g_\sigma \quad (\alpha_{\sigma;i,\lambda} \text{ in } \Phi).$$

Then

$$A = \sum_{\sigma=1}^{r} \sum_{i=1}^{m} \sum_{\lambda=1}^{n} \alpha_{\sigma;i,\lambda}\, (g_\sigma)_{i\lambda},$$

and so every element of $\mathfrak{B}$ is a linear combination of the non-zero elements $(g_\sigma)_{i\lambda}$ of S with coefficients in Φ. Since $S\setminus 0$ is clearly linearly independent over Φ, it is a basis of $\mathfrak{B}$ and the result follows by definition of $\Phi_0[S]$.

LEMMA 5.18. *The Munn algebra $\mathfrak{B} = \mathscr{M}(\mathfrak{A}; m, n; P)$ over an algebra $\mathfrak{A}$ of finite order over a field Φ contains an identity element if and only if* (i) *$\mathfrak{A}$ contains an identity element, and* (ii) *the sandwich matrix P is non-singular (in particular $m = n$). If this is the case, then the mapping $A \to AP$ is an isomorphism of $\mathfrak{B}$ upon the full matrix algebra $(\mathfrak{A})_n$ over $\mathfrak{A}$.*

PROOF. Assume first that $\mathfrak{B}$ contains an identity element E. Then

$$X \circ E = XPE = X,$$
$$E \circ Y = EPY = Y,$$

for every X, Y in $\mathfrak{B}$. Then $m = n$. For were $n > m$, then, by Theorem 5.11, there would exist a non-zero $m \times n$ matrix X over $\mathfrak{A}$ such that $XP = 0$, which contradicts $XPE = X \neq 0$. Similarly, $m > n$ leads to a contradiction with $EPY = Y$. It is now apparent that P is neither a left nor a right divisor of zero in $(\mathfrak{A})_n$. By Corollary 5.10, $\mathfrak{A}$ contains an identity element and P is non-singular.

Conversely, assume that (i) $\mathfrak{A}$ contains an identity element and (ii) P is non-singular. By definition of non-singularity, we must have $m = n$. If P^{-1} is the inverse of P in $(\mathfrak{A})_n$, then $E = P^{-1}$ is the identity element of $\mathfrak{B}$.

If (i) and (ii) hold, then $\mathfrak{B}$ is the same set as $(\mathfrak{A})_n$, and the mapping $A \to AP$ is an isomorphism of $\mathfrak{B}$ upon $(\mathfrak{A})_n$. For it is one-to-one and upon because P is non-singular, and if $A, B \in \mathfrak{B}$ then $A \to AP$, $B \to BP$, and

$$A \circ B \to (A \circ B)P = (APB)P = (AP)(BP).$$

THEOREM 5.19. *The Munn algebra $\mathfrak{B} = \mathscr{M}(\mathfrak{A}; m, n; P)$ over an algebra $\mathfrak{A}$ of finite order over a field Φ is semisimple if and only if* (i) *$\mathfrak{A}$ is semisimple, and* (ii) *P is non-singular. If this is the case, then $\mathfrak{B} \cong (\mathfrak{A})_n$.*

PROOF. Since (§5.1) a semisimple algebra of finite order contains an identity element, it follows that if we assume either that $\mathfrak{B}$ is semisimple or that (i) and (ii) hold, then the hypotheses of the last assertion in Lemma 5.18 are satisfied. Hence, in either case, $A \to AP$ is an isomorphism of $\mathfrak{B}$ upon $(\mathfrak{A})_n$. But by Theorem 5.5, $(\mathfrak{A})_n$ is semisimple if and only if $\mathfrak{A}$ is semisimple.

The following theorem is now immediate from Lemma 5.17, Theorem 5.19, and Maschke's Theorem.

THEOREM 5.20. *Let S be a finite 0-simple semigroup, and let it be represented (according to the Rees Theorem) as a regular Rees $m \times n$ matrix semigroup $\mathscr{M}^0(G; m, n; P)$ over a finite group G, with $n \times m$ sandwich matrix P. Let Φ be a field, and let the zero elements of G^0 and $\Phi[G]$ be identified. Then $\Phi[S]$ is semisimple if and only if* (i) *the characteristic of Φ does not divide the order of G, and* (ii) *P is non-singular (in particular $m = n$) regarded as a matrix over $\Phi[G]$.*

We digress for a moment to apply the foregoing to commutative semigroups.

THEOREM 5.21. *Let S be a finite commutative semigroup, and let Φ be a field. Then $\Phi[S]$ is semisimple if and only if S is a union of groups the orders of which are not divisible by the characteristic of Φ.*

PROOF. Let (4) be a principal series of S. Suppose $\Phi[S]$ is semisimple. By Corollary 5.15, S is semisimple. Each principal factor S_i/S_{i+1} of S is a commutative completely 0-simple semigroup, and so is just a group with zero G_i^0 (without zero for $i = n$). The corresponding $\mathscr{J}$-class is the group G_i, and hence $S = G_1 \cup G_2 \cup \cdots \cup G_n$. By Theorem 5.14 and Lemma 5.13, each $\Phi[G_i]$ is semisimple, and by Maschke's Theorem, the characteristic of Φ does not divide the order of G_i.

Conversely, assume that S is a union of groups whose orders are not divisible by the characteristic of Φ. Let E be the semilattice of idempotent elements of S, and for each e in E let H_e be the maximal subgroup of S containing e. We see at once (by virtue of the commutativity of S) that $H_e H_f \subseteq H_{ef}$ (e, f in E). Hence S is the union of the semilattice E of groups H_e (e in E). From this it is clear that each principal factor of S is a group with zero (except that the kernel is a group). Hence, referring to the principal series (4), $S_i \backslash S_{i+1}$ is a group G_i ($i = 1, \cdots, n$), and the G_i are the same as the H_e in some order. By Maschke's Theorem, and the hypothesis about the characteristic of Φ, each $\Phi[G_i]$ is semisimple. Clearly

$$\Phi[G_i] \cong \Phi_0[S_i/S_{i+1}],$$

and hence $\Phi[S]$ is semisimple by Lemma 5.13 and Theorem 5.14.

Because of Theorem 5.20, it is of interest to find conditions for the non-singularity of a square $n \times n$ matrix P over a semisimple algebra $\mathfrak{A}$. If Γ is any representation of $\mathfrak{A}$ of degree r over Φ, and $P = (p_{ij})$, then (§5.1) by $\Gamma^n(P)$ we mean the $nr \times nr$ matrix $(\Gamma(p_{ij}))$ over Φ obtained by replacing each entry p_{ij} of P by the $r \times r$ matrix $\Gamma(p_{ij})$ over Φ.

LEMMA 5.22. *Let Γ be any faithful representation of a semisimple algebra $\mathfrak{A}$. Let P be an $n \times n$ matrix over $\mathfrak{A}$. Then P is non-singular if and only if $\Gamma^n(P)$ is non-singular.*

PROOF. Let $A = (a_{ij}) \in (\mathfrak{A})_n$. If $\Gamma^n(A) = 0$, then $\Gamma(a_{ij}) = 0$ for every $i, j = 1, \cdots, n$. Since Γ is faithful, each $a_{ij} = 0$, i.e., $A = 0$. Hence the representation Γ^n of $(\mathfrak{A})_n$ associated with the representation Γ of $\mathfrak{A}$ is also faithful. The desired result then follows from Corollary 5.10.

THEOREM 5.23. *Let $\{\Gamma_\sigma : \sigma = 1, \cdots, c\}$ be a complete set of inequivalent irreducible representations of the semisimple algebra $\mathfrak{A}$ over Φ. Let $P \in (\mathfrak{A})_n$. Then P is non-singular if and only if each matrix $\Gamma_\sigma^n(P)$ is non-singular ($\sigma = 1, \cdots, c$).*

PROOF. Let Γ be the regular representation of $\mathfrak{A}$. In ultimately reduced form, Γ has a diagonal-block form, each block being one of the Γ_σ, and every Γ_σ occurs at least once in Γ, by the Main Representation Theorem for Semisimple Algebras (§5.1). It is readily seen that Γ^n can be transformed to a corresponding block form, each Γ_σ in Γ being replaced by Γ_σ^n. Hence $\Gamma^n(P)$ is non-singular if and only if each $\Gamma_\sigma^n(P)$ is non-singular, and the result follows from Lemma 5.22.

The following corollary is due to Marianne Teissier [1952*b*].

COROLLARY 5.24. *If the algebra* $\Phi[S]$ *of a finite simple semigroup* S *is semisimple, then* S *is a group.*

PROOF. By Corollary 2.56 and Theorem 3.5, S is isomorphic with an $m \times n$ Rees matrix semigroup without zero over a group G, and with $n \times m$ sandwich matrix P. Each entry in P is a (non-zero) element of G, and $m = n$ by Theorem 5.20. Let Γ_1 be the "unit representation" of $\Phi[S]$, that is, the representation of degree 1 obtained by letting every element of S correspond to the identity element 1 of Φ. Then $\Gamma_1^n(P)$ is the $n \times n$ matrix over Φ every entry of which is 1. But by Theorems 5.20 and 5.23, this matrix must be non-singular, which evidently requires $n = 1$. Hence $S \cong G$.

COROLLARY 5.25. *If the algebra* $\Phi[S]$ *of a finite semigroup* S *is semisimple, then the kernel of* S *is a group.*

The following theorem was also found by Oganesyan [1955].

THEOREM 5.26. *The algebra* $\Phi[S]$ *of a finite inverse semigroup* S *over a field* Φ *is semisimple if and only if the characteristic of* Φ *is zero or a prime not dividing the order of any subgroup of* S.

PROOF. Let J be a $\mathcal{J}$-class of S, and let $Q = J \cup z$ be the corresponding principal factor. If $a \in J$, then $a^{-1} \in J$ also, and so Q is an inverse semigroup. Hence Q is a 0-simple (or simple) inverse semigroup, and thus either a Brandt semigroup (Theorem 3.9) or the group-ideal (and kernel) of S. By Theorem 3.9, $Q \cong \mathcal{M}^0(G; n, n; \Delta_n)$ for some finite group G and some positive integer n, the sandwich matrix Δ_n being the $n \times n$ identity matrix over G^0. Since Δ_n is non-singular in $\Phi[G]$, it follows from Theorem 5.20 that $\Phi[Q]$ is semisimple if and only if the characteristic of Φ is 0 or a prime not dividing the order of G. The result then follows from Theorem 5.14.

LEMMA 5.27. *Let* $\{\Gamma_\sigma : \sigma = 1, \cdots, c\}$ *be a complete set of inequivalent irreducible representations of a semisimple algebra* $\mathfrak{A}$ *over the field* Φ, *and let* P *be a non-singular* $n \times n$ *matrix over* $\mathfrak{A}$. *For each element* A *of* $\mathfrak{B} = \mathcal{M}(\mathfrak{A}; n, n; P)$, *let* $\Gamma'_\sigma(A) = \Gamma_\sigma^n(AP)$. *Then* $\{\Gamma'_\sigma : \sigma = 1, \cdots, c\}$ *is a complete set of inequivalent irreducible representations of* $\mathfrak{B}$.

PROOF. By Theorem 5.5, $\{\Gamma_\sigma^n : \sigma = 1, \cdots, c\}$ is a complete set of inequivalent irreducible representations of $(\mathfrak{A})_n$. By Theorem 5.19, $\mathfrak{B} = \mathcal{M}(\mathfrak{A}; n, n; P)$

is semisimple, and the mapping $A \rightarrow AP$ is an isomorphism of $\mathscr{M}$ upon $(\mathfrak{A})_n$. Thus each Γ'_σ is a representation of $\mathfrak{B}$, and the irreducibility and mutual inequivalence of the Γ'_σ is immediate from the same for the Γ^n_σ.

The following theorem determines all the irreducible representations, and hence all representations, over a field Φ, of a finite 0-simple semigroup S such that $\Phi[S]$ is semisimple. A characterization of such a semigroup is given by Theorem 5.20.

THEOREM 5.28. *Let S be a Rees $n \times n$ matrix semigroup $S = \mathscr{M}^0(G; n, n; P)$ over a finite group G the order of which is not divisible by the characteristic of Φ, with a sandwich matrix P which is non-singular regarded as a matrix over $\Phi[G]$. Let $\{\Gamma_\sigma : \sigma = 1, \cdots, c\}$ be a complete set of inequivalent irreducible representations of G over Φ. For each element $(a)_{ij}$ of S, define*

$$(5) \qquad \Gamma'_\sigma((a)_{ij}) = \Gamma^n_\sigma((a)_{ij}P) = \sum_{k=1}^{n} \Gamma^n_\sigma((ap_{jk})_{ik}).$$

Then $\{\Gamma'_\sigma : \sigma = 1, \cdots, c\}$ is a complete set of inequivalent irreducible representations of S over Φ which map the zero element of S upon the zero matrix of appropriate degree over Φ.

REMARK. The only irreducible representation of S not included in the Γ'_σ is the *unit representation* of S under which every element of S is mapped into the identity element 1 of Φ. (See Exercise 1 below.)

PROOF. All but the calculation in (5) is immediate from Lemma 5.17 and Lemma 5.27. As for the calculation, $(a)_{ij}P$ is the matrix whose ith row is

$$ap_{j1}, \cdots, ap_{jk}, \cdots, ap_{jn},$$

and whose remaining rows vanish. This implies that, in $(\Phi[G])_n$, $(a)_{ij}P$ is the sum of the n Rees matrices $(ap_{jk})_{ik}$ $(k = 1, \cdots, n)$, and hence that

$$\Gamma^n_\sigma((a)_{ij}P) = \Gamma^n_\sigma\Big(\sum_{k=1}^{n} (ap_{jk})_{ik}\Big) = \sum_{k=1}^{n} \Gamma^n_\sigma((ap_{jk})_{ik}).$$

The point of this calculation is that $\Gamma'_\sigma((a)_{ij})$ is thereby expressed in terms of matrices representing elements of G itself, rather than of $\Phi[G]$.

THEOREM 5.29. *A simple algebra of finite order over a field Φ is a contracted semigroup algebra over Φ if and only if it is isomorphic with $(\Phi)_n$ for some positive integer n.*

PROOF. Example 2 above shows that $(\Phi)_n$ is a contracted semigroup algebra. Conversely, let S be a finite semigroup with zero such that $\Phi_0[S]$ is simple. Then S is clearly 0-simple, and so representable as a Rees matrix semigroup $\mathscr{M}^0(G; m, n; P)$. By Lemma 5.17, $\Phi_0[S] \cong \mathscr{M}(\Phi[G]; m, n; P)$. By Theorem 5.19, $\Phi_0[S] \cong (\Phi[G])_n$. By Theorem 5.4, the latter is simple if and only if $\Phi[G]$ is simple. But $\Phi[G]$ is simple if and only if G is a one-element group; for if w is the sum of the elements of G, $\Phi[w]$ is an ideal of $\Phi[G]$. Hence $\Phi[G] \cong \Phi$, and $\Phi_0[S] \cong (\Phi)_n$.

In particular, we note that *no non-commutative division algebra over Φ can be a contracted semigroup algebra over Φ.*

We now leave Munn's development, and conclude the section with two results of Hewitt and Zuckerman ([1955], Theorems 4.2 and 5.22), the first of which is closely related to Theorem 5.29. (We shall resume Munn's development in the next section.)

THEOREM 5.30. *Let $\mathfrak{A}$ be a semisimple algebra of finite order over a field Φ. If $\mathfrak{A} = \Phi[S]$ for some (finite) semigroup S, then one of the simple components of $\mathfrak{A}$ is of order* 1 *over Φ. The converse holds if Φ is algebraically closed.*

PROOF. Let $\mathfrak{A} = \Phi[S]$. Since $\mathfrak{A}$ is semisimple by hypothesis, it follows from Corollary 5.25 that the kernel K of S is a group. Let w be the sum in $\mathfrak{A}$ of the elements of K, and let e be the identity element of K. Let $s \in S$. Then $sw = s(ew) = (se)w = w$, since the element se of K merely permutes the elements of which w is the sum. Similarly, $ws = w$. Hence $\Phi[w]$ is an ideal of $\mathfrak{A}$ of order 1 over Φ.

Conversely, assume that Φ is algebraically closed, and that one of the simple components of $\mathfrak{A}$ has order 1. Then

$$\mathfrak{A} \cong \mathfrak{A}' = (\Phi)_{n_1} \oplus (\Phi)_{n_2} \oplus \cdots \oplus (\Phi)_{n_c}$$

where, say, $n_1 = 1$. Let $E_{ij}^{(\sigma)}$ $(i, j = 1, \cdots, n_\sigma)$ be the matrix units in $(\Phi)_{n_\sigma}$ $(\sigma = 1, 2, \cdots, c)$. Then the matrices

$$E_{11}^{(1)}, E_{11}^{(1)} + E_{ij}^{(\sigma)} \quad (i, j = 1, \cdots, n_\sigma; \sigma = 2, \cdots, c)$$

constitute at the same time a semigroup S and a basis of $\mathfrak{A}'$, so that $\mathfrak{A} \cong \mathfrak{A}' = \Phi[S]$.

Regarding the converse proposition in Theorem 5.30, see Exercise 8 below.

Let S be a finite commutative semigroup. If $a \in S$, then $a^k = e$ for exactly one idempotent element e of S, and some positive integer k; we say that *a belongs to the idempotent e.* The set S_e of all elements of S belonging to e is a subsemigroup of S containing e but no other idempotent. S_e contains the maximal subgroup H_e of S of which e is the identity element, and H_e is the group ideal (kernel) of S_e. Let E be the set of idempotents of S. Then S is the union of the semilattice E of semigroups S_e $(e \in E)$. We call $H = \bigcup_{e \in E} H_e$ the *group part* of S; H is a subsemigroup of S, and is the union of the semilattice E of groups H_e $(e \in E)$.

THEOREM 5.31. *Let S be a finite commutative semigroup, and let H be its group part. Let Φ be a field whose characteristic does not divide the order of any subgroup of S. Assume that $H \neq S$, and that $S \backslash H$ consists of the r elements*

$t_1, t_2, \cdots, t_r$. *Let e_i be the idempotent element to which t_i belongs. Then the r elements $t_i - t_i e_i$ $(i = 1, \cdots, r)$ constitute a basis for the radical $\mathfrak{N}$ of $\Phi[S]$.*

PROOF. If $t \in S$ and $e \in E$, then (since S is commutative)

$$(t - te)(t^m - t^m e) = t^{m+1} - t^{m+1}e.$$

By induction,

$$(t - te)^m = t^m - t^m e$$

for every positive integer m. If t belongs to e then $t^k = e$ for some k, and hence $t - te$ is nilpotent. The r elements $t_i - t_i e_i$ therefore belong to $\mathfrak{N}$.

We show next that they are linearly independent. Suppose that $\alpha_1, \cdots, \alpha_r$ are elements of Φ such that

$$\sum_{i=1}^{r} \alpha_i (t_i - t_i e_i) = 0.$$

Then

$$\sum_{i=1}^{r} \alpha_i t_i - y = 0,$$

where $y = \sum \alpha_i t_i e_i$ is a linear combination of elements $t_i e_i$ of H. Since the vector spaces $\Phi[H]$ and $\Phi[S \backslash H]$ are complementary, and since $\{t_1, \cdots, t_r\}$ is linearly independent, we conclude that each $\alpha_i = 0$ $(i = 1, \cdots, r)$.

By Theorem 5.21, $\Phi[H]$ is semisimple, so that $\Phi[H] \cap \mathfrak{N} = 0$. Hence the dimension of $\mathfrak{N}$ cannot exceed $(m + r) - m = r$, where m is the order of H (and so $m + r$ that of S). But $\mathfrak{N}$ contains the r linearly independent elements $t_i - t_i e_i$, and so the latter must constitute a basis of $\mathfrak{N}$.

In [1952*a*], Marianne Teissier determined the radical $\mathfrak{N}$ of $\mathfrak{A} = \Phi[S]$ when S is a finite left simple semigroup and Φ has characteristic 0; she showed that $\mathfrak{A}\mathfrak{N} = 0$ and $\mathfrak{A}/\mathfrak{N} \cong \Phi[G]$, where G is the structure group of S. In [1952*b*], she showed that if S is a finite simple semigroup, $\mathfrak{N}$ is the radical of $\mathfrak{A} = \Phi[S]$, and the characteristic of Φ is 0, then $\mathfrak{A}\mathfrak{N}\mathfrak{A} = 0$, and $\mathfrak{N} \neq 0$ if S is not a group (Corollary 5.24). This result was extended by Munn [1955a] to any finite 0-simple semigroup S, with $\mathfrak{A} = \Phi_0[S]$. If $S = \mathcal{M}(G; m, n; P)$ then $\mathfrak{A}/\mathfrak{N} \cong \Phi[G]$ in this case also provided every entry in P is the identity element of G; the determination of $\mathfrak{N}$ in the general case remains open.

EXERCISES FOR §5.2

1. Let S be a semigroup with zero element z. Let Γ be a representation of S of degree n over a field Φ. Let r be the rank of $\Gamma(z)$. Since $\Gamma(z)$ is idempotent, there exists a non-singular $n \times n$ matrix C such that

$$C\Gamma(z)C^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where I_r is the $r \times r$ identity matrix. For any a in S we then have

$$C\Gamma(a)C^{-1} = \begin{pmatrix} I_r & 0 \\ 0 & \Gamma_2(a) \end{pmatrix},$$

where $\Gamma_2(a)$ is some $(n-r) \times (n-r)$ matrix over Φ. If $r = n$ then $\Gamma(a) = I_n$ for all a in S. If $0 < r < n$, Γ decomposes into a representation Γ_1 of degree r such that $\Gamma_1(a) = I_r$ for every a in S, and a representation Γ_2 of degree $n - r$ such that $\Gamma_2(z) = 0$. If Γ is irreducible, then either $\Gamma(z) = 0$ or else Γ is the unit representation of S.

2. Let S be a finite band (idempotent semigroup). Then $\Phi[S]$ is semisimple if and only if S is commutative. (Hewitt and Zuckerman, [1955], Theorem 5.27.)

3. (a) Let P be a [non-]singular $n \times n$ matrix over an algebra $\mathfrak{A}$ over a field Φ. Let Φ^* be a field containing Φ, and let $\mathfrak{A}^*$ be the extension of $\mathfrak{A}$ to Φ^*. Then P remains [non-]singular in $\mathfrak{A}^*$.

(b) Let S be a finite 0-simple semigroup, and let Φ_1 and Φ_2 be two fields of the same characteristic. Then $\Phi_1[S]$ and $\Phi_2[S]$ are either both semisimple or both not semisimple. (Munn, [1955*b*], Lemma 6.3.)

4. Let $G = \{i, a\}$ be a cyclic group of order 2 ($a^2 = i$), and let the characteristic of Φ be different from 2. Let

$$P = \begin{pmatrix} 0 & i & i \\ i & 0 & a \\ i & i & 0 \end{pmatrix}.$$

Let Γ_1 be the unit representation of G, and let Γ_2 be the other irreducible representation of G ($\Gamma_2(i) = 1$, $\Gamma_2(a) = -1$). Then $\Gamma_1^3(P)$ is non-singular, but $\Gamma_2^3(P)$ is singular. Thus P is singular in $\Phi[G]$.

5. Let $Q = \mathscr{M}^0(G; n, n; \Delta_n)$ be a finite Brandt semigroup (§3.3). Let Φ be a field whose characteristic does not divide the order of G. Then $\Phi[Q]$ is semisimple. Let Γ be an irreducible representation of G of degree r over Φ, and let Γ' be the corresponding irreducible representation of Q, as given by Theorem 5.28. If $(a)_{ij} \in Q$, then $\Gamma'((a)_{ij})$ is the $n \times n$ block matrix obtained by replacing the element a in $(a)_{ij}$ by $\Gamma(a)$, and each 0 in $(a)_{ij}$ by the $r \times r$ zero matrix. (Clifford [1942], p. 342; Munn [1957*a*], Theorem 4.5.)

6. The algebra of quaternions is not the contracted algebra of any semigroup over the real field, but it is a contracted semigroup algebra over the complex field.

7. Let S be the infinite cyclic semigroup generated by x, and let Φ be a field. Then $\Phi[S^1]$ is the ring $\Phi[x]$ of polynomials in x over Φ. Let $\alpha \in \Phi$, $\alpha \neq 0$. With f' denoting the formal derivative of the polynomial f, let

$$\Gamma(f) = \begin{pmatrix} f(\alpha) & 0 \\ f'(\alpha) & f(\alpha) \end{pmatrix}.$$

Then Γ is a representation of $\Phi[x]$, hence also of S, which is not fully reducible. (Suggested by p. 214 of N. Jacobson, Trans. Amer. Math. Soc. 42 (1937), 206–224.)

8. Let $\mathfrak{A}$ be the semisimple algebra $R \oplus Q$ over the field R of real numbers, the simple components of which are R itself and the algebra Q of quaternions

over R. There exists no semigroup S such that $\mathfrak{A} \cong R[S]$. (Note Theorem 5.30.)

5.3 Principal irreducible representations of a semigroup

Let S be any semigroup and let Φ be a field. The main result of this section (Theorem 5.33) is the expression of certain irreducible representations of S (which we call "principal") in terms of irreducible representations of the principal factors of S. Moreover (Part (D) of Theorem 5.33), if S satisfies the minimal condition M_J for principal ideals, then every non-null irreducible representation of S is principal. In this case we have (essentially) a one-to-one correspondence between the non-null irreducible representations of S over Φ and those of the various principal factors of S.

In this generality, the theorem is due to Munn [1960]. The idea of proof has its origin in Hewitt and Zuckerman's treatment [1957] of the irreducible representations of the full transformation semigroup on a finite set. The result for the case of a finite semigroup S with $\Phi[S]$ semisimple (Corollary 5.34) was given by Munn [1955*a*], and independently by Ponizovsky [1956]. Taken in conjunction with Theorem 5.28, this completes the determination of all representations over Φ of such a semigroup S.

We recall (§2.1) that there is a natural partial ordering of the $\mathscr{J}$-classes of any semigroup S, namely $J_1 \leqq J_2$ (J_1 and J_2 being $\mathscr{J}$-classes of S) if $S^1J_1S^1 \subseteq S^1J_2S^1$. In fact, $J \leftrightarrow S^1JS^1$ is a one-to-one correspondence between the $\mathscr{J}$-classes of S and the principal ideals of S, and the relation $\leqq$ is just that corresponding to inclusion.

If $\mathscr{X}$ is a set of $\mathscr{J}$-classes, then a member J of $\mathscr{X}$ is called (i) *minimal in* $\mathscr{X}$ if $J' \in \mathscr{X}$ implies $J' \nless J$, and (ii) *universally minimal in* $\mathscr{X}$ if $J' \in \mathscr{X}$ implies $J \leqq J'$. S is said to satisfy M_J if every non-empty set of $\mathscr{J}$-classes of S contains at least one minimal member. S satisfies M_J if and only if every properly descending chain of principal ideals of S is finite, i.e., M_J is equivalent to the *descending chain condition for principal ideals.*

We recall also (§2.6) that there is a one-to-one correspondence between the $\mathscr{J}$-classes of S and the *principal factors* of S, whereby the principal factor $Q(J)$ corresponding to the $\mathscr{J}$-class J is the Rees factor $S^1JS^1/I(J)$, where $I(J) = S^1JS^1 \setminus J$ is an ideal of S. For a fixed $\mathscr{J}$-class J, we shall denote by $x \rightarrow \bar{x}$ the natural homomorphism of S^1JS^1 upon $Q(J)$, namely

$$\bar{x} = \begin{cases} x \text{ for all } x \text{ in } J, \\ z \text{ for all } x \text{ in } I(J), \end{cases} \tag{1}$$

where z is the zero element of $Q(J)$. We recall also (Lemma 2.39) that $Q(J)$ is either a null semigroup or else is 0-simple (or simple in case J is the kernel of S). We shall say that J is 0-simple if Q is either 0-simple or simple.

Now let Φ be a field, and let $\Phi[S]$ be the algebra of S over Φ. If $T \subseteq S$, we denote (as in §5.2) by $\Phi[T]$ the linear subspace of $\Phi[S]$ spanned by T.

We extend the natural homomorphism $x \to \bar{x}$ of S^1JS^1 upon $Q(J)$ to a homomorphism of $\Phi[S^1JS^1]$ upon $\Phi[Q(J)]$ as follows: if

$$x = \sum_{i=1}^{n} \alpha_i x_i \in \Phi[S^1JS^1],$$

that is, $\alpha_i \in \Phi$ and $x_i \in S^1JS^1$ ($i = 1, 2, \cdots, n$), then we define $\bar{x} = \sum_{i=1}^{n} \alpha_i \bar{x}_i$.

If Γ is a representation of S of degree n over Φ, then $\Gamma(S)$ is contained in the full matrix algebra $(\Phi)_n$ of degree n over Φ. If $T \subseteq S$, let $[\Gamma(T)]$ denote the subspace of $(\Phi)_n$ spanned by the subset $\Gamma(T)$ of $(\Phi)_n$. Clearly

$$[\Gamma(T)] = \Gamma(\Phi[T]).$$

As in §5.2, we do not distinguish between a representation of S over Φ and its extension to a representation of $\Phi[S]$. We note also that $[\Gamma(S)]$ is a subalgebra of $(\Phi)_n$.

Lemma 5.32. *Let Γ be an irreducible representation of S of degree n over Φ. Let T be a subset of S such that $[\Gamma(T)]$ is an irreducible subalgebra of $(\Phi)_n$. Then there exists an element e of $\Phi[T]$ such that $\Gamma(e) = I_n$, the $n \times n$ identity matrix over Φ.*

Proof. By Theorem 5.7, $[\Gamma(T)]$ is a simple algebra, and so contains an identity element E (as remarked in §5.1 after Wedderburn's Second Theorem). E commutes with every element of the irreducible set $[\Gamma(T)]$, and since $E \neq 0$ it must be non-singular by Schur's Lemma (§5.1). But I_n is the only non-singular idempotent in $(\Phi)_n$, and so $E = I_n$. Since $E \in [\Gamma(T)]$, there exist elements x_i of T and α_i of Φ, finite in number, such that $E = \sum_i \alpha_i \Gamma(x_i)$. Setting $e = \sum_i \alpha_i x_i$, we have $e \in \Phi[T]$ and $\Gamma(e) = E = I_n$.

If Γ is a representation of S over Φ, we denote by $\Gamma^{-1}(0)$ the set of all elements s of S such that $\Gamma(s) = 0$. $\Gamma^{-1}(0)$ is clearly an ideal of S, which we call the *vanishing ideal* of Γ. Γ will be called *principal* if the set of all $\mathscr{J}$-classes of S not contained in $\Gamma^{-1}(0)$ contains a universally minimal member J. J is evidently uniquely determined by Γ, and will be called the *apex* of Γ. Thus Γ is principal with apex J if and only if (i) $\Gamma(x) \neq 0$ for some—and hence for every—element x of J, and (ii) if s is an element of S such that $J_s \ngeq J$, then $\Gamma(s) = 0$. It follows that if $J_s \geqq J$, then $\Gamma(s) \neq 0$. (We remark that if J is any $\mathscr{J}$-class of S, then the union of all $\mathscr{J}$-classes J' of S such that $J' \ngeq J$ is an ideal of S.)

Let J be a $\mathscr{J}$-class of S, and let Γ be a representation of S over Φ such that $\Gamma(y) = 0$ for all y in $I(J)$. We may then define a representation Γ' of $Q(J)$ by

$$\Gamma'(\bar{x}) = \Gamma(x) \quad (\text{all } x \text{ in } S^1JS^1), \tag{2}$$

where $x \to \bar{x}$ is the natural homomorphism (1). We call Γ' the representation of $Q(J)$ *induced* by Γ, and we call Γ an *extension* of Γ' to S. We call Γ a *principal extension* of Γ' if Γ is a principal representation of S with apex J. If we think of Γ as a representation of $\Phi[S]$ then

$$\Gamma'(\bar{x}) = \Gamma(x) \quad (\text{all } x \text{ in } \Phi[S^1JS^1]) \tag{2'}$$

defines the representation of $\Phi[Q(J)]$ corresponding to Γ'; here $x \to \bar{x}$ is the extension of (1) to $\Phi[S^1JS^1]$.

The following theorem establishes a one-to-one correspondence between the irreducible principal representations Γ of S over Φ, and the non-null irreducible representations Γ' of the various 0-simple principal factors Q of S over Φ which vanish on the zero elements of the Q's.

THEOREM 5.33 (Munn). *Let S be a semigroup and Φ a field.*

(A) *Let Γ be an irreducible principal representation of S of degree n over Φ, and let J be the apex of Γ. Then J is 0-simple, and the representation Γ' of the principal factor $Q(J)$ induced by Γ is also irreducible and non-null. There exists an element e of $\Phi[J]$ such that $\Gamma'(e) = I_n$, and, for any such element e,*

$$\Gamma(s) = \Gamma'(\overline{se}) \quad (\textit{all } s \textit{ in } S). \tag{3}$$

Here $x \to \bar{x}$ denotes the natural homomorphism of $\Phi[S^1JS^1]$ upon $\Phi[Q(J)]$.

(B) *Let J be a 0-simple $\mathscr{J}$-class of S, and let Γ' be a non-null irreducible representation of $Q(J)$ of degree n over Φ such that $\Gamma'(z) = 0$, where z is the zero element of $Q(J)$. Then there exists an element e of $\Phi[J]$ such that $\Gamma'(e) = I_n$, and, for any such element e, equation* (3) *serves to define an irreducible principal extension Γ of Γ'.*

(C) *Two irreducible principal representations of S are equivalent if and only if* (i) *they have the same apex J, and* (ii) *they induce equivalent representations in $Q(J)$.*

(D) *If S satisfies M_J, then every non-null irreducible representation of S over Φ is principal.*

PROOF. (A) Since $[\Gamma(J)] = [\Gamma(S^1JS^1)]$, it is clear that $[\Gamma(J)]$ is a non-vanishing ideal of $[\Gamma(S)]$. But $[\Gamma(S)]$ is by hypothesis an irreducible algebra of matrices, and hence simple, by Theorem 5.7. Consequently,

$$[\Gamma(J)] = [\Gamma(S)].$$

Let us write Q for $Q(J)$. By (2), $[\Gamma'(Q)] = [\Gamma(J)]$, and hence Γ' is irreducible and non-null. Again by Theorem 5.7, $[\Gamma'(Q)]$ is a simple algebra, and hence Q is either 0-simple or simple, since it is certainly not null. Thus J is 0-simple, by definition.

By Lemma 5.32, with T replaced by J, there exists e in $\Phi[J]$ such that $\Gamma(e) = I_n$. For any such element e, and any s in S, we have $se \in \Phi[S^1JS^1]$. Hence, using (2′),

$$\Gamma(s) = \Gamma(s)I_n = \Gamma(s)\Gamma(e) = \Gamma(se) = \Gamma'(\overline{se}),$$

which proves (3).

(B) Assuming the hypotheses of Part (B), $[\Gamma'(J)]$ is an irreducible algebra of matrices. By Lemma 5.32, there exists e in $\Phi[J]$ such that $\Gamma'(e) = I_n$. We note that $\bar{e} = e$. We proceed to show that Γ defined by (3), for any such element e, is a representation of S.

First we observe that, for any element s of S,

$$\Gamma'(\overline{se}) = \Gamma'(\overline{ese}) = \Gamma'(\overline{es}).$$

For

$$\Gamma'(\overline{se}) = I_n\Gamma'(\overline{se}) = \Gamma'(e)\Gamma'(\overline{se}) = \Gamma'(\bar{e})\Gamma'(\overline{se})$$
$$= \Gamma'(\bar{e}\cdot\overline{se}) = \Gamma'(\overline{ese}),$$

and the other half is proved similarly. (We caution the reader that we cannot say that $\overline{se} = \bar{s}\,\bar{e}$ since $\bar{s}$ is defined only if $s \in S^1JS^1$.) We now have, for any s, t in S,

$$\Gamma(s)\Gamma(t) = \Gamma'(\overline{se})\Gamma'(\overline{te}) = \Gamma'(\overline{es})\Gamma'(\overline{te})$$
$$= \Gamma'(\overline{es}\cdot\overline{te}) = \Gamma'(\overline{este})$$
$$= \Gamma'(\overline{ste}) = \Gamma(st).$$

If $x \in S^1JS^1$, then

$$\Gamma(x) = \Gamma'(\overline{xe}) = \Gamma'(\bar{x}\,\bar{e}) = \Gamma'(\bar{x})\Gamma'(\bar{e}) = \Gamma'(\bar{x}).$$

This shows in the first place that Γ induces Γ', and so is an extension of Γ', and it also shows that $\Gamma(S) = \Gamma'(Q)$, so that Γ is irreducible.

Finally, we must show that Γ is principal with apex J. Clearly $\Gamma(J) \neq 0$, and what remains is to show that if s is an element of S such that $J_s \ngeq J$, then $\Gamma(s) = 0$. If $x \in J$ then $J_{sx} \leqq J_x$; moreover, $J_{sx} < J_x$, for, were $J_{sx} = J_x$, we would have $J_s \geqq J_{sx} = J_x = J$. Hence $sx \in I(J)$, and, by (3), $\Gamma(sx) = \Gamma'(\overline{sxe}) = \Gamma'(z)$. But $\Gamma'(z) = 0$ by hypothesis, and so $\Gamma(sx) = 0$.

Now $e = \sum_i \alpha_i x_i$ for some x_i in J and α_i in Φ. Hence

$$\Gamma(s) = \Gamma(se) = \sum_i \alpha_i \Gamma(sx_i) = 0.$$

(C) Let Γ_1 and Γ_2 be principal irreducible representations of S over Φ. If they are equivalent, there exists a non-singular constant matrix C over Φ such that

$$\Gamma_1(s) = C\Gamma_2(s)C^{-1} \quad (\text{all } s \text{ in } S).$$

Clearly $\Gamma_1^{-1}(0) = \Gamma_2^{-1}(0)$, and so Γ_1 and Γ_2 have the same apex J. Then, by (2), for any x in S^1JS^1,

$$\Gamma_1'(\bar{x}) = \Gamma_1(x) = C\Gamma_2(x)C^{-1} = C\Gamma_2'(\bar{x})C^{-1},$$

so that Γ_1' and Γ_2' are equivalent.

Conversely, assume that Γ_1 and Γ_2 have the same apex J, and that Γ_1' and Γ_2' are equivalent. Then there exists a constant non-singular matrix C such that

$$\Gamma_1'(\bar{x}) = C\Gamma_2'(\bar{x})C^{-1} \quad (\text{all } \bar{x} \text{ in } Q).$$

By (A), there exists e in $\Phi[J]$ such that $\Gamma_2'(e) = I_n$, and, for all s in S,

$\Gamma_2(s) = \Gamma_2'(\overline{se})$. Then $\Gamma_1'(e) = C\Gamma_2'(e)C^{-1} = I_n$. Since (A) asserts that (3) holds for "any such element e", we have $\Gamma_1(s) = \Gamma_1'(\overline{se})$. Hence

$$\Gamma_1(s) = \Gamma_1'(\overline{se}) = C\Gamma_2'(\overline{se})C^{-1} = C\Gamma_2(s)C^{-1},$$

so that Γ_1 and Γ_2 are equivalent.

(D) Assume now that S satisfies M_J. Let Γ be a non-null irreducible representation of S of degree n over Φ. Since Γ is non-null, the set of $\mathscr{J}$-classes of S not contained in $\Gamma^{-1}(0)$ is not empty. By M_J, it contains a minimal member J.

If $y \in I(J)$, i.e., $J_y < J$, then $\Gamma(y) = 0$ by definition of J. Hence $[\Gamma(J)] = [\Gamma(S^1JS^1)]$, and so is a non-vanishing ideal of $[\Gamma(S)]$. Since the latter is irreducible, by hypothesis, it is a simple algebra, by Theorem 5.7, and hence $[\Gamma(J)] = [\Gamma(S)]$. By Lemma 5.32, there exists e in $\Phi[J]$ such that $\Gamma(e) = I_n$. Then

$$\Gamma(s) = \Gamma(s)I_n = \Gamma(s)\Gamma(e) = \Gamma(se) \quad (\text{all } s \text{ in } S).$$

That Γ is principal with apex J now follows exactly as in the last paragraph of the proof of Part (B). The only slight change is that we infer $\Gamma(sx) = 0$ from $sx \in I(J)$ directly from the definition of J.

The following corollary is immediate from Theorem 5.33. It completes the investigation of the preceding section. The irreducible representations of each principal factor of S are given by Theorem 5.28.

COROLLARY 5.34. *Let Φ be a field, and let S be a finite semigroup such that $\Phi[S]$ is semisimple. Let J_i $(i = 1, \cdots, n)$ denote the $\mathscr{J}$-classes of S, and let $Q_i = J_i \cup z_i$ be the principal factor of S corresponding to J_i. Then the contracted algebra $\Phi_0[Q_i]$ is semisimple* (*Lemma* 5.13 *and Theorem* 5.14), *and so has an identity element e_i; clearly $e_i \in \Phi[J_i]$. Let $\{\Gamma_{i\sigma}' : \sigma = 1, \cdots, c_i\}$ be a complete set of inequivalent irreducible representations of Q_i over Φ vanishing at z_i $(i = 1, \cdots, n)$. Define $\Gamma_{i\sigma}$ by*

$$\Gamma_{i\sigma}(s) = \Gamma_{i\sigma}'(\overline{se_i}) \quad (\textit{all } s \textit{ in } S),$$

where $x \to \bar{x}$ is the natural homomorphism of $\Phi[S^1J_iS^1]$ upon $\Phi[Q_i]$. Then $\{\Gamma_{i\sigma} : \sigma = 1, \cdots, c_i;\ i = 1, \cdots, n\}$ is a complete set of inequivalent irreducible representations of S over Φ.

We conclude the section with a further result of Munn [1960] concerning full reducibility. First a lemma.

LEMMA 5.35. *Let J be a 0-simple $\mathscr{J}$-class of a semigroup S. Let Γ be a representation of S of degree m over Φ, and for each s in S let $\Gamma(s) = (\gamma_{ij}(s))$ $(i, j = 1, \cdots, m)$. If $\gamma_{ij}(x) = 0$ for all i, j such that $1 \leqq i \leqq j \leqq m$ and for all x in J, then $\Gamma(x) = 0$ for all x in J.*

PROOF. Let $x \in J$. Then there exist a and b in J such that $axb = x$.

Hence $a^m x b^m = x$. But $\Gamma(a)^m = \Gamma(b)^m = 0$, since $\Gamma(a)$ and $\Gamma(b)$ are triangular with zero on the main diagonal. Hence $\Gamma(x) = \Gamma(a)^m\Gamma(x)\Gamma(b)^m = 0$.

THEOREM 5.36. *Let S be a semisimple semigroup satisfying M_J, and let Φ be a field. If every representation over Φ of every principal factor of S is fully reducible, then every representation of S over Φ is fully reducible.*

PROOF. Let

$$\Gamma(s) = \begin{pmatrix} \Gamma_1(s) & 0 & \cdots & 0 \\ \Gamma_{21}(s) & \Gamma_2(s) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \Gamma_{n1}(s) & \Gamma_{n2}(s) & \cdots & \Gamma_n(s) \end{pmatrix} \quad (s \in S)$$

be a representation of S over Φ in ultimately reduced form (§5.1), the representations $s \to \Gamma_i(s)$ $(i = 1, \cdots, n)$ being its irreducible constituents. Since a null representation is evidently fully reducible, we can assume that Γ is not null. We shall prove the theorem by induction on the number $n = N(\Gamma)$ of irreducible constituents of Γ. The theorem is trivial for $N(\Gamma) = 1$, and we shall assume that every representation Γ' of S with $N(\Gamma') < n$ is fully reducible.

Each element s of S belongs to some $\mathcal{J}$-class of S, and every $\mathcal{J}$-class of S is 0-simple by hypothesis. Were $\Gamma_1, \cdots, \Gamma_n$ all null, it would follow from Lemma 5.35 that Γ is null, contrary to assumption. Hence not all the Γ_i are null. Each non-null Γ_i is principal by M_J and Theorem 5.33 (D); let J_i be the apex of Γ_i. Let J_k be minimal in the set of these J_i. In other words, Γ_k is non-null, and if Γ_i is also non-null, then $J_i \nless J_k$.

If x is an element of S such that $J_x < J_k$, then clearly $J_x \ngeq J_i$ for every i such that Γ_i is non-null. Hence $\Gamma_i(x) = 0$ for every such i, and consequently for $i = 1, \cdots, n$. We conclude from Lemma 5.35 that $\Gamma(x) = 0$.

Let $Q_k = J_k \cup z_k$ (or $Q_k = J_k$ for the kernel of S) be the principal factor of S associated with J_k. Since, as we have just shown, $\Gamma(x) = 0$ for x in $I(J_k)$, it follows that

$$\Gamma'(\bar{x}) = \Gamma(x) \quad (\text{all } x \text{ in } S^1 J_k S^1),$$

where $x \to \bar{x}$ is the natural homomorphism of $S^1 J_k S^1$ upon Q_k, is a representation Γ' of Q_k. Moreover, Γ' is not null since Γ_k is not null on J_k, and Γ' contains the representation Γ'_k of Q_k induced by Γ_k. By hypothesis, Γ' is fully reducible, and, by Theorem 5.33(A), Γ'_k is irreducible; hence there exists a constant non-singular matrix C such that

$$C\Gamma'(\bar{x})C^{-1} = \begin{pmatrix} \Gamma'_k(\bar{x}) & 0 \\ 0 & \Delta'(\bar{x}) \end{pmatrix} \quad (\text{all } \bar{x} \text{ in } Q_k),$$

where Δ' is some representation of Q_k.

If we define Δ by

$$\Delta(x) = \Delta'(\bar{x}) \quad (\text{all } x \text{ in } S^1 J_k S^1),$$

then Δ is a representation of $S^1J_kS^1$ vanishing on $I(J_k)$. Moreover,

$$(4) \qquad C\Gamma(x)C^{-1} = \begin{pmatrix} \Gamma_k(x) & 0 \\ 0 & \Delta(x) \end{pmatrix} \quad \text{(all } x \text{ in } S^1J_kS^1).$$

Let us partition $C\Gamma(s)C^{-1}$ in the same way:

$$(5) \qquad C\Gamma(s)C^{-1} = \begin{pmatrix} \Gamma_{11}^*(s) & \Gamma_{12}^*(s) \\ \Gamma_{21}^*(s) & \Gamma_{22}^*(s) \end{pmatrix} \quad \text{(all } s \text{ in } S),$$

where $\Gamma_{11}^*(s)$ has the same degree as $\Gamma_k(x)$.

Let $s \in S$ and let $x \in J_k$. Then $sx \in S^1J_kS^1$, and from

$$C\Gamma(sx)C^{-1} = C\Gamma(s)C^{-1} \cdot C\Gamma(x)C^{-1},$$

we have

$$\begin{pmatrix} \Gamma_k(sx) & 0 \\ 0 & \Delta(sx) \end{pmatrix} = \begin{pmatrix} \Gamma_{11}^*(s) & \Gamma_{12}^*(s) \\ \Gamma_{21}^*(s) & \Gamma_{22}^*(s) \end{pmatrix} \begin{pmatrix} \Gamma_k(x) & 0 \\ 0 & \Delta(x) \end{pmatrix}.$$

Hence $\Gamma_{21}^*(s)\Gamma_k(x) = 0$ for every s in S and x in J_k. By Theorem 5.33 (B), there exists e in $\Phi[J_k]$ such that $\Gamma_k(e) = I$. Since e is a linear combination of elements x of J_k, it follows that

$$\Gamma_{21}^*(s) = \Gamma_{21}^*(s)I = \Gamma_{21}^*(s)\Gamma_k(e) = 0 \quad \text{(all } s \text{ in } S).$$

Similarly, by considering xs instead of sx, we can show that $\Gamma_{12}^*(s) = 0$. Thus Γ decomposes into the two representations Γ_{11}^* and Γ_{22}^*. Since the latter both have degree less than that of Γ, they both have less than $N(\Gamma)$ irreducible constituents, and the result follows by induction.

Although not necessary for the proof, we note that Γ_{11}^* is equivalent to Γ_k. For if, in (5), we specialize s to an element x of $S^1J_kS^1$, and compare with (4), we see that $\Gamma_{11}^*(x) = \Gamma_k(x)$. From this we readily see that Γ_{11}^* has the same apex J_k as Γ_k, and induces the same representation Γ_k' in Q_k; the equivalence of Γ_{11}^* and Γ_k then follows from Theorem 5.33 (C).

Exercises for §5.3

1. Let S be a semigroup with kernel K. Then the extension to S of the unit representation of K is the unit representation of S.

2. Let S be a finite inverse semigroup. Let J_i $(i = 1, \cdots, n)$ be the $\mathscr{J}$-classes of S, and let $Q_i = J_i \cup z_i$ (or $Q_i = J_i$ for the kernel) be the corresponding principal factors of S. By Exercise 3 of §3.3, each Q_i is a Brandt semigroup (or a group). Let e_{ij} $(j = 1, \cdots, m_i)$ be the idempotent elements of J_i. Then the identity element of $\Phi_0[Q_i]$ is $e_i = \sum_{j=1}^{m_i} e_{ij}$. Assume that the characteristic of the field Φ does not divide the order of any subgroup of S. In the notation of Corollary 5.34, it follows that

$$\Gamma_{i\sigma}(s) = \sum_{j=1}^{m_i} \Gamma_{i\sigma}'(\overline{se_{ij}}).$$

(Munn [1957*a*], Theorem 4.7.)

3. Let S be the infinite cyclic semigroup $\{x, x^2, \cdots\}$, and let Φ be a field.

S does not satisfy M_J. To each element α of Φ corresponds a representation Γ_α of S of degree 1, namely $\Gamma_\alpha(x^n) = \alpha^n$. These Γ_α are the only absolutely irreducible representations of S over Φ, and none of them is principal.

5.4 Representations of completely 0-simple semigroups

The ideas and methods upon which this section is based, and the first part of Theorem 5.37, are due to Suschkewitsch [1933], and most of the remaining results to Clifford [1942] and [1960].

Let S be a completely 0-simple semigroup. By the Rees Theorem (3.5), S is isomorphic with (and hence we may assume it to be) a regular Rees $I \times \Lambda$ matrix semigroup $\mathscr{M}^0(G; I, \Lambda; P)$ over a group G, with $\Lambda \times I$ sandwich matrix $P = (p_{\lambda i})$.

We shall see that each representation Γ^* of S over a field Φ is in a natural sense an extension of a representation Γ of G over Φ. Some representations of G may not be extendible to S. If Γ is a representation of G that can be extended, then, among all its extensions, there is one of least possible degree; this one is uniquely determined by Γ to within equivalence, and we call it the *basic extension* Γ_0^* of Γ. Every other extension of Γ reduces into Γ_0^* and null representations. This correspondence between extendible representations of G and their basic extensions to S preserves decomposition (Theorem 5.50), and reduction in a limited sense (Theorem 5.51); in particular, we get all the irreducible representations of S as basic extensions of irreducible representations of G.

If we combine the foregoing with Munn's Theorem 5.33, we can say that *if S is a semigroup satisfying M_J, and such that every 0-simple principal factor of S is completely 0-simple, then all the irreducible representations of S can be expressed in terms of those of subgroups of S. As we shall see in Chapter 6, this class of semigroups includes the class of semigroups satisfying both M_R and M_L.*

A representation Γ of a semigroup S will be called *proper* if (i) $\Gamma(z) = 0$ if S has a zero z, and (ii) Γ is not decomposable into two representations one of which is null. There is no essential loss of generality in confining our attention to proper representations (see Exercise 1 of §5.2). A proper representation Γ of a group G is uniquely extendible to a proper representation of G^0 by defining $\Gamma(0) = 0$; *throughout this section we shall tacitly assume that this has been done.* We now turn to the consideration of proper representations of $S = \mathscr{M}^0(G; I, \Lambda; P)$.

As in §3.2, we can assume that the index classes I and Λ have an element 1 in common, and (as noted after the proof of Lemma 3.6) that the sandwich matrix P has been normalized so that $p_{11} = e$, where e is the identity element of G. The product of two elements $(a)_{i\lambda}$ and $(b)_{j\mu}$ of S is given by

$$(1) \qquad (a)_{i\lambda} \circ (b)_{j\mu} = (ap_{\lambda j}b)_{i\mu} \quad (a, b \in G;\ i, j \in I;\ \lambda, \mu \in \Lambda).$$

Because of $p_{11} = e$, we have the following, which we shall need later.

$$(a)_{11} \circ (b)_{11} = (ab)_{11}, \tag{2}$$

$$(e)_{11} \circ (e)_{i1} = (p_{1i})_{11}, \tag{3}$$

$$(e)_{i1} \circ (e)_{11} = (e)_{i1}, \tag{4}$$

$$(e)_{11} \circ (e)_{1\lambda} = (e)_{1\lambda}, \tag{5}$$

$$(e)_{1\lambda} \circ (e)_{11} = (p_{\lambda 1})_{11}, \tag{6}$$

$$(e)_{1\lambda} \circ (e)_{i1} = (p_{\lambda i})_{11}, \tag{7}$$

$$(e)_{i1} \circ (a)_{11} \circ (e)_{1\lambda} = (a)_{i\lambda}. \tag{8}$$

By (2), the set G_{11} of all elements $(a)_{11}$ of S (a in G) is a subgroup of S isomorphic with G, and we identify G_{11} with G.

Let Γ^* be a non-null representation of degree m of S. Then Γ^* induces a representation of the subgroup G of S. Let n be the rank of the idempotent matrix $\Gamma^*[(e)_{11}]$. By adapting a basis of a representation space of Γ^* to the range and null space of $\Gamma^*[(e)_{11}]$, we can assume that

$$\Gamma^*[(e)_{11}] = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}.$$

From (2) with $b = e$ and $a = e$ in turn, we find that

$$\Gamma^*[(a)_{11}] = \begin{pmatrix} \Gamma(a) & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{all } a \text{ in } G), \tag{9}$$

where $a \to \Gamma(a)$ is a proper representation of G of degree n. We note that $n > 0$ since otherwise (8) would imply that Γ^* is a null representation. We call Γ^* *an extension to S of the representation Γ of G.*

In what follows, we partition all the matrices $\Gamma^*[(a)_{i\lambda}]$ by $m = n + (m - n)$, as in (9). Let $i \in I$, and let

$$\Gamma^*[(e)_{i1}] = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}. \tag{10}$$

From (3), (4), (9), and (10) we have

$$\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Gamma(p_{1i}) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R_{11} & 0 \\ R_{21} & 0 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}.$$

Hence $R_{11} = \Gamma(p_{1i})$, $R_{12} = 0$, $R_{22} = 0$. Denoting R_{21} by R_i, we have

$$\Gamma^*[(e)_{i1}] = \begin{pmatrix} \Gamma(p_{1i}) & 0 \\ R_i & 0 \end{pmatrix}. \tag{11}$$

Similarly, from (5), (6), and (9) we find that

$$\Gamma^*[(e)_{1\lambda}] = \begin{pmatrix} \Gamma(p_{\lambda 1}) & Q_\lambda \\ 0 & 0 \end{pmatrix}, \tag{12}$$

where Q_λ is some $n \times (m - n)$ matrix over Φ. Finally, by (8), $\Gamma^*[(a)_{i\lambda}]$ is the product of three matrices (11), (9), and (12), in that order. Using $\Gamma(ab) = \Gamma(a)\Gamma(b)$, we find

$$\Gamma^*[(a)_{i\lambda}] = \begin{pmatrix} \Gamma(p_{1i}ap_{\lambda 1}) & \Gamma(p_{1i}a)Q_\lambda \\ R_i\Gamma(ap_{\lambda 1}) & R_i\Gamma(a)Q_\lambda \end{pmatrix}. \tag{13}$$

Let $t = m - n$. Then R_i is a $t \times n$ matrix, Q_λ is an $n \times t$ matrix, and we note that

$$Q_1 = 0, \quad R_1 = 0. \tag{14}$$

We have thus proved the last part of the following theorem.

THEOREM 5.37. *Let Γ be a proper representation of G. Then* (13) *and* (14) *define a representation Γ^* of S if and only if the matrices Q_λ and R_i satisfy*

$$Q_\lambda R_i = \Gamma(p_{\lambda i}) - \Gamma(p_{\lambda 1}p_{1i}) \tag{15}$$

for all $i \neq 1$ in I and all $\lambda \neq 1$ in Λ. Every non-null representation of S is equivalent to one of this form.

REMARK. Because of (14) and the assumption $p_{11} = e$, equations (15) hold automatically for $i = 1$ or $\lambda = 1$. Furthermore, (9) holds, and hence Γ^* is an extension of Γ.

PROOF. Let Γ be a proper representation of G, and let Γ^* be defined by (13) and (14). We note that if we write

$$\bar{R}_i = \begin{pmatrix} \Gamma(p_{1i}) \\ R_i \end{pmatrix}, \quad \bar{Q}_\lambda = (\Gamma(p_{\lambda 1}) \quad Q_\lambda), \tag{16}$$

where $\bar{R}_i$ is thus an $m \times n$ matrix, and $\bar{Q}_\lambda$ an $n \times m$ matrix, then (13) becomes

$$\Gamma^*[(a)_{i\lambda}] = \bar{R}_i\Gamma(a)\bar{Q}_\lambda, \tag{17}$$

and (15) becomes

$$\bar{Q}_\lambda\bar{R}_i = \Gamma(p_{\lambda i}). \tag{18}$$

Assume now that (15), and hence (18) holds. Then, for any two elements $(a)_{i\lambda}$ and $(b)_{j\mu}$ of S, we have

$$\begin{aligned} \Gamma^*[(a)_{i\lambda}]\Gamma^*[(b)_{j\mu}] &= \bar{R}_i\Gamma(a)\bar{Q}_\lambda\bar{R}_j\Gamma(b)\bar{Q}_\mu \\ &= \bar{R}_i\Gamma(a)\Gamma(p_{\lambda j})\Gamma(b)\bar{Q}_\mu \\ &= \bar{R}_i\Gamma(ap_{\lambda j}b)\bar{Q}_\mu \\ &= \Gamma^*[(ap_{\lambda j}b)_{i\mu}] = \Gamma^*[(a)_{i\lambda} \circ (b)_{j\mu}]. \end{aligned}$$

Γ^* is thus a representation of S.

On the other hand, assume that Γ^*, as defined by (13) and (14), is a representation of S. Because of (14) and $p_{11} = e$, (9), (11), and (12) hold. From these and (7) we have

$$\begin{pmatrix} \Gamma(p_{\lambda 1}) & Q_\lambda \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Gamma(p_{1i}) & 0 \\ R_i & 0 \end{pmatrix} = \begin{pmatrix} \Gamma(p_{\lambda i}) & 0 \\ 0 & 0 \end{pmatrix},$$

from which (15) follows.

Let Γ be a proper representation of G of degree n over Φ. When do there exist matrices Q_λ and R_i satisfying (15), and, if they exist, how do we find them?

As noted in the Remark after Theorem 5.37, we need concern ourselves only with λ in $\Lambda_1 = \Lambda \setminus 1$ and i in $I_1 = I \setminus 1$. Let

$$\Omega_{\lambda i} = \Gamma(p_{\lambda i}) - \Gamma(p_{\lambda 1} p_{1i}) \quad (i \text{ in } I_1,\ \lambda \text{ in } \Lambda_1), \tag{19}$$

and let Ω be the $\Lambda_1 \times I_1$ matrix of $n \times n$ matrices over Φ having $\Omega_{\lambda i}$ in the (λ, i)-position:

$$\Omega = \begin{pmatrix} & \vdots & \\ \cdots & \Omega_{\lambda i} & \cdots \\ & \vdots & \end{pmatrix}. \tag{20}$$

Ω is a $\Lambda' \times I'$ matrix over Φ, with $\Lambda' = n \times \Lambda_1$ and $I' = n \times I_1$. We call Ω the *extending matrix of* Γ *relative to* S. The equations (15) can be expressed as a single matrix equation

$$\Omega = QR \tag{21}$$

if we put

$$Q = \begin{pmatrix} \vdots \\ Q_\lambda \\ \vdots \end{pmatrix}, \qquad R = (\cdots R_i \cdots) \quad (\lambda \text{ in } \Lambda_1,\ i \text{ in } I_1). \tag{22}$$

Q is a $\Lambda' \times t$ matrix over Φ, and R is a $t \times I'$ matrix over Φ, for some as yet unspecified positive integer t. We say that Γ^* arises from the *factorization* (21) of the extending matrix Ω of Γ, and that Q and R are the *defining matrices* of Γ^*.

The question of the existence and construction of extensions Γ^* of Γ is thus reduced to a problem in pure matrix theory, namely to the factorization (21) of a given $\Lambda' \times I'$ matrix Ω into a product QR, where, for some finite positive integer t, Q is a $\Lambda' \times t$ matrix and R is a $t \times I'$ matrix. We shall call t the *width* of the factorization. The next two lemmas and corollaries thereof deal with this problem. To economize on notation, we drop the primes; since the results will be valid for arbitrary index sets Λ and I, they will be valid for Λ' and I'.

Let Ω be a given $\Lambda \times I$ matrix over the field Φ, where Λ and I are any two (index) sets. Let $\Phi[\Lambda]$ be the vector space over Φ consisting of all mappings of Λ into Φ. Each column vector of Ω can be regarded as an element of $\Phi[\Lambda]$. By the *column space* of Ω we mean the subspace of $\Phi[\Lambda]$ spanned by the columns of Ω. Similarly, by the *row space* of Ω we mean the subspace of $\Phi[I]$ spanned by the rows of Ω. The *column* [*row*] *rank* of Ω is defined (for present purposes) to be the dimension of the column [row] space of Ω if finite, and ∞ otherwise. By an easy extension of the method of proof for the finite case, one readily sees that the column and row ranks of Ω are equal, and this common number (or ∞) will be called the *rank* of Ω.

LEMMA 5.38. *A $\Lambda \times I$ matrix Ω can be expressed as a product $\Omega = QR$ of a $\Lambda \times t$ matrix Q and a $t \times I$ matrix R, for some positive integer t, if and only if it has finite rank $h \leqq t$.*

PROOF. If $\Omega = QR$, then the rows of Ω are linear combinations of those of R, which are t in number, and hence the dimension h of the row space of Ω cannot exceed t. Conversely, if $h \leqq t$, we may take for R any t rows of Ω of which all other rows are linear combinations. The λ-row of Q may then be taken to be the coefficients in any expression of the λ-row of Ω as a linear combination of the rows of R.

Two factorizations $\Omega = QR$ and $\Omega = Q'R'$ of Ω will be called *equivalent* if they have the same width t and there exists a non-singular $t \times t$ matrix C such that $Q' = QC^{-1}$ and $R' = CR$. A factorization of width h (the rank of Ω) will be called *basic*.

LEMMA 5.39. *Two factorizations $\Omega = QR = Q'R'$ of width t of a $\Lambda \times I$ matrix Ω are equivalent if and only if Q and Q' have the same column space, and R and R' the same row space.*

PROOF. The "only if" is evident. To prove the "if", assume that Q and Q' have the same column space $\mathfrak{Q}$ of dimension q, and that R and R' have the same row-space $\mathfrak{R}$ of dimension r. Then there exist non-singular $t \times t$ matrices C and C' such that

$$CR = C'R' = \begin{pmatrix} R^* \\ 0 \end{pmatrix},$$

where R^* is an $r \times I$ matrix of rank r, its rows forming a basis of $\mathfrak{R}$. Let

$$QC^{-1} = (Q^* \quad Q_0), \qquad Q'C'^{-1} = (Q'^* \quad Q_0'),$$

where Q^* and Q'^* are $\Lambda \times r$ matrices. From

$$\Omega = Q^*R^* = Q'^*R^*$$

we conclude $Q^* = Q'^*$ since the rows of Ω are uniquely expressible as linear combinations of those of R^*.

Since the matrices $(Q^* \quad Q_0)$ and $(Q^* \quad Q_0')$ have the same column space $\mathfrak{Q}$, we can reduce one to the other by elementary column transformations not affecting the columns of Q^*. In other words, there exists a non-singular matrix

$$D = \begin{pmatrix} I_r & A \\ 0 & B \end{pmatrix}$$

such that

$$(Q^* \quad Q_0)D = (Q^* \quad Q_0').$$

Evidently

$$D\begin{pmatrix} R^* \\ 0 \end{pmatrix} = \begin{pmatrix} R^* \\ 0 \end{pmatrix}.$$

Hence

$$QC^{-1}DC' = Q', \qquad C^{-1}DC'R' = R,$$

with non-singular $C^{-1}DC'$, i.e., the two factorizations are equivalent.

COROLLARY 5.40. *Any two basic factorizations of Ω are equivalent.*

PROOF. If $\Omega = QR = Q'R'$ are two basic factorizations of Ω, they have the same width h (the rank of Ω), and Q and Q' [R and R'] have the same column [row] space, namely that of Ω.

COROLLARY 5.41. *Let $\Omega = Q^0R^0$ be any basic factorization of Ω, and let $\Omega = QR$ be any factorization of Ω of width t. Let h, q, r be the ranks of Ω, Q, R, respectively. Let Q^1 be a $\Lambda \times (q-h)$ matrix such that the $\Lambda \times q$ matrix $(Q^0 \quad Q^1)$ has the same column space as Q. Let R^1 be an $(r-h) \times I$ matrix such that the $r \times I$ matrix*

$$\begin{pmatrix} R^0 \\ R^1 \end{pmatrix}$$

has the same row space as R. Then the factorization $\Omega = QR$ is equivalent to the factorization

$$\Omega = (Q^0 \quad 0 \quad Q^1 \quad 0)\begin{pmatrix} R^0 \\ R^1 \\ 0 \\ 0 \end{pmatrix}, \tag{23}$$

where the width is t, and the dimensions of the blocks are, in order, as follows:

$$\begin{array}{ll} Q^0 : \Lambda \times h & R^0 : h \times I \\ 0 : \Lambda \times (r-h) & R^1 : (r-h) \times I \\ Q^1 : \Lambda \times (q-h) & 0 : (q-h) \times I \\ 0 : \Lambda \times (t+h-r-q) & 0 : (t+h-r-q) \times I. \end{array}$$

PROOF. This will follow immediately from Lemma 5.39 once we show that (23) can indeed have width t; in other words, that $t + h - r - q \geqq 0$.

By elementary row transformations on R we can find a non-singular $t \times t$ matrix C such that

$$CR = \begin{pmatrix} R^* \\ 0 \end{pmatrix},$$

where R^* is an $r \times I$ matrix the rows of which form any basis we like of the row-space $\mathfrak{R}$ of R, in particular

$$R^* = \begin{pmatrix} R^0 \\ R^1 \end{pmatrix}.$$

Let

$$QC^{-1} = (Q^* \; A)$$

where Q^* is some $\Lambda \times r$ matrix, and A is some $\Lambda \times (t - r)$ matrix. Then

$$\Omega = QC^{-1} \cdot CR = Q^* R^*.$$

Since the rows of Ω are vectors in $\mathfrak{R}$, they are unique linear combinations of the rows of R^*, and so Q^* is unique. From this and

$$(Q^0 \; 0)\begin{pmatrix} R^0 \\ R^1 \end{pmatrix} = Q^0 R^0 = \Omega,$$

we conclude that $Q^* = (Q^0 \; 0)$.

Since $(Q^* \; A) = QC^{-1}$ has rank q, and $Q^* = (Q^0 \; 0)$ has rank h, the rank of A must be at least $q - h$. Since A has $t - r$ columns, we conclude that $t - r \geqq q - h$, yielding the desired inequality $t + h - r - q \geqq 0$.

We record this inequality for later use in the following.

Corollary 5.42. *With the hypotheses of Corollary* 5.41, *we have*

$$t \geqq q \geqq h, \qquad t \geqq r \geqq h, \qquad t \geqq q + r - h.$$

We remark incidentally that this reduces to Sylvester's Law of Nullity for $t \times t$ matrices; for the inequalities may be written

$$t - h \geqq t - q, \qquad t - h \geqq t - r, \qquad t - h \leqq (t - q) + (t - r).$$

The following theorem is immediate from Theorem 5.37 and Lemma 5.38.

Theorem 5.43. *A given proper representation Γ of G of degree n over Φ has an extension Γ^* to S of finite degree over Φ if and only if the rank h of the extending matrix Ω of Γ is finite. The defining matrices Q_λ, R_i of any extension Γ^* of Γ of degree $m = n + t$ over Φ are obtained from a factorization of Ω. This is possible if and only if $t \geqq h$, and so every extension of Γ has degree at least $n + h$.*

Theorem 5.44. *Let Γ^* and Γ'^* be extensions to S of proper representations Γ and Γ', respectively, of G. Let Γ^* be defined by* (13), *and Γ'^* analogously by*

$$\Gamma'^*[(a)_{i\lambda}] = \begin{pmatrix} \Gamma'(p_{1i} a p_{\lambda 1}) & \Gamma'(p_{1i} a) Q'_\lambda \\ R'_i \Gamma'(a p_{\lambda 1}) & R'_i \Gamma'(a) Q'_\lambda \end{pmatrix}. \tag{24}$$

Then Γ^* *and* Γ'^* *are equivalent if and only if there exist non-singular constant matrices* C_1 *and* C_2 *such that*

(25) $$\Gamma'(a) = C_1\Gamma(a)C_1^{-1} \qquad (\textit{all } a \textit{ in } G),$$

(26) $$Q'_\lambda = C_1Q_\lambda C_2^{-1} \qquad (\textit{all } \lambda \neq 1 \textit{ in } \Lambda),$$

(27) $$R'_i = C_2R_iC_1^{-1} \qquad (\textit{all } i \neq 1 \textit{ in } I).$$

PROOF. Suppose that Γ^* and Γ'^* are equivalent. Then there exists a constant non-singular matrix C such that

(28) $$\Gamma'^*[(a)_{i\lambda}] = C\Gamma^*[(a)_{i\lambda}]C^{-1}.$$

Setting $i = \lambda = 1$, we see that $\Gamma'^*[(a)_{11}]$ and $\Gamma^*[(a)_{11}]$ have the same rank, say n, and hence, since Γ and Γ' are proper, Γ' has the same degree n as Γ. Let $n + t$ be the degrees of Γ^* and Γ'^*, and let C be partitioned accordingly:

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where C_{11} is an $n \times n$ matrix, C_{12} an $n \times t$ matrix, etc. Then

$$\begin{pmatrix} \Gamma'(a) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} \Gamma(a) & 0 \\ 0 & 0 \end{pmatrix},$$

that is,

$$\begin{pmatrix} \Gamma'(a)C_{11} & \Gamma'(a)C_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11}\Gamma(a) & 0 \\ C_{21}\Gamma(a) & 0 \end{pmatrix}.$$

Setting $a = e$ (the identity element of G) we obtain $C_{12} = 0$, $C_{21} = 0$. Hence

(29) $$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix},$$

where $C_1 = C_{11}$ and $C_2 = C_{22}$ must be non-singular. Moreover,

$$\Gamma'(a)C_1 = C_1\Gamma(a),$$

so that (25) holds, and we see that Γ and Γ' are equivalent. Now

(30) $$C\Gamma^*[(a)_{i\lambda}] = \begin{pmatrix} C_1\Gamma(p_{1i}ap_{\lambda 1}) & C_1\Gamma(p_{1i}a)Q_\lambda \\ C_2R_i\Gamma(ap_{\lambda 1}) & C_2R_i\Gamma(a)Q_\lambda \end{pmatrix},$$

(31) $$\Gamma'^*[(a)_{i\lambda}]C = \begin{pmatrix} \Gamma'(p_{1i}ap_{\lambda 1})C_1 & \Gamma'(p_{1i}a)Q'_\lambda C_2 \\ R'_i\Gamma'(ap_{\lambda 1})C_1 & R'_i\Gamma'(a)Q'_\lambda C_2 \end{pmatrix}.$$

These are equal by (28). Setting $a = e$ and $i = 1$, we obtain $C_1Q_\lambda = Q'_\lambda C_2$, giving (26). Setting $a = e$ and $\lambda = 1$, we obtain $C_2R_i = R'_iC_1$, giving (27).

Assume conversely that there exist non-singular matrices C_1 and C_2 satisfying (25), (26), and (27). Define the non-singular matrix C by (29). Then (30) and (31) hold. The equivalence of Γ^* and Γ'^* will follow if we show that the right-hand members of (30) and (31) are equal. The equality of their northwest corners is immediate from (25). As for the northeast corners, we have, using (25) and (26),

$$\Gamma'(p_{1i}a)Q'_\lambda C_2 = C_1\Gamma(p_{1i}a)C_1^{-1}Q'_\lambda C_2 = C_1\Gamma(p_{1i}a)Q_\lambda.$$

The southwest and southeast corners are handled in a similar manner.

COROLLARY 5.45. *Let Γ be a proper representation of G, and let Ω be the extending matrix of Γ. Then two extensions Γ^* and Γ'^* of Γ to S arising from two equivalent factorizations $\Omega = QR = Q'R'$ of Ω are equivalent.*

PROOF. By definition of equivalent factorizations, there exists a nonsingular matrix C_2 such that

$$Q' = QC_2^{-1}, \qquad R' = C_2R.$$

By (22), these are equivalent to

$$Q'_\lambda = Q_\lambda C_2^{-1}, \qquad R'_i = C_2R_i,$$

for all $\lambda \neq 1$ in Λ and all $i \neq 1$ in I. Taking C_1 to be the identity matrix, and $\Gamma' = \Gamma$, equations (25), (26), and (27) hold, whence Γ^* and Γ'^* are equivalent.

THEOREM 5.46. *A proper representation Γ of G of degree n over Φ, with extending matrix Ω of finite rank h, possesses to within equivalence exactly one extension Γ_0^* to S of degree $n + h$ over Φ; Γ_0^* is obtained from a basic factorization of Ω.*

PROOF. By Theorem 5.43, a basic factorization of Ω leads to an extension Γ_0^* of Γ of degree $n + h$, so that there is at least one such. Any two extensions of Γ of degree $n + h$ must arise from factorizations of Ω of width h, i.e., from basic factorizations, and so must be equivalent by Corollary 5.40 and Corollary 5.45.

We call Γ_0^* the *basic extension* of Γ to S. Any representation of S which is the basic extension to S of some proper representation of G will be called a *basic representation* of S. The following corollary establishes a one-to-one correspondence between the basic representations of S and the extendible representations of G.

COROLLARY 5.47. *Let Γ and Γ' be proper representations of G having extending matrices of finite rank, and let Γ_0^* and $\Gamma_0'^*$ be their respective basic extensions to S. Then Γ_0^* and $\Gamma_0'^*$ are equivalent if and only if Γ and Γ' are equivalent.*

PROOF. If Γ and Γ' are equivalent, we can choose a new basis in the representation space of $\Gamma_0'^*$ so that Γ' becomes identical with Γ. The equivalence of Γ_0^* and $\Gamma_0'^*$ then follows from Theorem 5.46. Conversely, if Γ_0^* and $\Gamma_0'^*$ are equivalent, then Γ and Γ' are equivalent by Theorem 5.44.

Theorem 5.48. *Every proper extension Γ^* of a proper representation Γ of G can be transformed into the form*

$$
\Gamma^*[(a)_{i\lambda}] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \Gamma(p_{1i}a)Q^1_\lambda & \Gamma(p_{1i}ap_{\lambda 1}) & \Gamma(p_{1i}a)Q^0_\lambda & 0 \\ R^0_i\Gamma(a)Q^1_\lambda & R^0_i\Gamma(ap_{\lambda 1}) & R^0_i\Gamma(a)Q^0_\lambda & 0 \\ R^1_i\Gamma(a)Q^1_\lambda & R^1_i\Gamma(ap_{\lambda 1}) & R^1_i\Gamma(a)Q^0_\lambda & 0 \end{pmatrix}. \tag{32}
$$

The diagonal block inside the dotted lines is the basic extension Γ^*_0 of Γ to S. If n is the degree of Γ, $n + t$ that of Γ^*, h the rank of the extending matrix Ω of Γ, and q and r the respective ranks of the defining matrices Q and R of Γ^*, then $t = q + r - h$. The two zero matrices on the diagonal are square (of dimension $q - h$ and $r - h$), so that Γ^*_0 is a component of Γ^*.

Proof. By Theorem 5.43, Γ^* arises from a factorization $\Omega = QR$ of Ω of width t. By Corollary 5.41, this factorization is equivalent to the factorization

$$
\Omega = (Q^0 \;\; 0 \;\; Q^1 \;\; 0)\begin{pmatrix} R^0 \\ R^1 \\ 0 \\ 0 \end{pmatrix}, \tag{33}
$$

where $\Omega = Q^0R^0$ is the basic factorization of Ω. By Corollary 5.45, the representation of S arising from the factorization (33) is equivalent to Γ^*. We may therefore assume that Γ^* itself arises from (33).

Translating back to the Q_λ and R_i of (22), we have

$$
Q_\lambda = (Q^0_\lambda \;\; 0 \;\; Q^1_\lambda \;\; 0), \qquad R_i = \begin{pmatrix} R^0_i \\ R^1_i \\ 0 \\ 0 \end{pmatrix}. \tag{34}
$$

Substituting (34) into (13) we obtain

$$
\Gamma^*[(a)_{i\lambda}] = \begin{pmatrix} \Gamma(p_{1i}ap_{\lambda 1}) & \Gamma(p_{1i}a)Q^0_\lambda & 0 & \Gamma(p_{1i}a)Q^1_\lambda & 0 \\ R^0_i\Gamma(ap_{\lambda 1}) & R^0_i\Gamma(a)Q^0_\lambda & 0 & R^0_i\Gamma(a)Q^1_\lambda & 0 \\ R^1_i\Gamma(ap_{\lambda 1}) & R^1_i\Gamma(a)Q^0_\lambda & 0 & R^1_i\Gamma(a)Q^1_\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{35}
$$

Since Γ^* is proper, the last row and last column of zeros must be absent, and hence $t + h - r - q = 0$. After deleting the last row and last column of (35), and then moving the fourth row and fourth column into first position, we obtain (32). The two (non-absent) zero matrices on the diagonal in (35) are easily seen to be square of dimension $q - h$ and $r - h$, and these become the diagonal zero matrices in (32).

If we do not worry about whether Γ^* is proper or not, then (32) defines an extension Γ^* of Γ for entirely arbitrary matrices R_i^1 and Q_λ^1. Thus, once we have calculated the basic representation Γ_0^* of Γ, all other extensions Γ^* of Γ can be written down immediately from (32). If Γ^* is not proper, we need not have $t = q + r - h$, and the diagonal zero matrices need not be square.

Corollary 5.49. *The non-null irreducible constituents of* Γ^* *are the same as those of* Γ_0^*, *even as to multiplicities.*

Proof. From the form of (32) it is clear that an ultimate reduction of Γ_0^* into triangular block form with irreducible diagonal blocks produces an ultimate reduction of Γ^*.

Theorem 5.50. *A proper representation* Γ *of* G *is extendible to* S *if and only if each of its indecomposable constituents is extendible. If* Γ *is extendible, then the indecomposable constituents of the basic extension* Γ_0^* *of* Γ *are the basic extensions of the indecomposable constituents of* Γ. *In particular,* Γ *is indecomposable if and only if* Γ_0^* *is indecomposable; in fact any proper extension to* S *of an indecomposable representation* Γ *of* G *is also indecomposable.*

Proof. First let Γ be an indecomposable representation of G which is extendible to S, and let Γ^* be any proper extension of Γ to S. Suppose that Γ^* decomposed into two representations, Δ and Δ', each of lower degree than Γ^*. The restrictions of Δ and Δ' to G cannot share the indecomposable representation Γ of G. We may suppose, then, that the restriction of Δ to G contains Γ. But then the restriction of Δ' to G is a null representation of G. As remarked after (9), this would imply that Δ' is a null representation of S, which is contrary to the assumption that Γ^* is proper. Hence Γ^* is indecomposable. This proves the last assertion in the theorem, and in particular that if Γ is indecomposable, so is its basic extension Γ_0^*. That the indecomposability of Γ_0^* implies that of Γ will follow when we show that a decomposition of Γ leads to a decomposition of Γ_0^*.

Let Γ be a representation of G which decomposes into the direct sum $\Gamma' \oplus \Gamma''$ of two representations Γ' and Γ'' of G each of degree less than that of Γ. We may assume that a basis of the representation space of Γ has been chosen so that

$$\Gamma(a) = \begin{pmatrix} \Gamma'(a) & 0 \\ 0 & \Gamma''(a) \end{pmatrix} \quad (\text{all } a \text{ in } G).$$

We proceed to show that Γ is extendible to S if and only if both Γ' and Γ'' are extendible, and that, if this is the case, the basic extension Γ_0^* of Γ decomposes into the basic extensions $\Gamma_0'^*$ of Γ' and $\Gamma_0''^*$ of Γ''. The first two assertions of the theorem will then follow by an evident induction on the number of indecomposable components of Γ.

Let the degrees of Γ, Γ', and Γ'' be n, n', and n'', respectively; clearly $n = n' + n''$. Let their respective extending matrices Ω, Ω', and Ω'' have ranks h, h', and h''. In (19), each $\Omega_{\lambda i}$ has the block form

$$\Omega_{\lambda i} = \begin{pmatrix} \Omega'_{\lambda i} & 0 \\ 0 & \Omega''_{\lambda i} \end{pmatrix},$$

and we can permute blocks of rows and columns in Ω so that

$$\Omega \to \begin{pmatrix} \Omega' & 0 \\ 0 & \Omega'' \end{pmatrix}.$$

Hence h is finite if and only if both h' and h'' are finite, and then $h = h' + h''$. By Theorem 5.43, Γ is extendible if and only if both Γ' and Γ'' are extendible.

Now assume that Γ, and hence also Γ' and Γ'', are extendible to S. By Theorem 5.46, the basic extensions $\Gamma_0'^*$ of Γ' and $\Gamma_0''^*$ of Γ'' have degrees $n' + h'$ and $n'' + h''$, respectively. Hence $\Gamma_0'^* \oplus \Gamma_0''^*$ has degree $(n' + h') + (n'' + h'') = n + h$, and so, by Theorem 5.46, is equivalent to the basic extension Γ_0^* of Γ.

THEOREM 5.51. *Let Γ be an extendible representation of G, and let Γ^* be any extension of Γ to S. Then the non-null irreducible constituents of Γ^* are the basic extensions of the non-null irreducible constituents of Γ. The basic extension Γ_0^* of Γ is irreducible if and only if Γ is irreducible; thus we get all the irreducible representations of S as the basic extensions to S of the extendible irreducible representations of G.*

PROOF. Assume first that Γ is an extendible, irreducible representation of G, and let Γ_0^* be its basic extension to S. Suppose that Γ_0^* reduced into representations Δ and Δ' of S each of degree less than that of Γ_0^*. Then either Δ or Δ' would be an extension to S of the irreducible representation Γ of G. But, by Theorems 5.43 and 5.46, no extension of Γ to S can have degree less than that of the basic extension Γ_0^*. Hence Γ_0^* is irreducible.

That conversely the irreducibility of Γ_0^* implies that of Γ will follow when we show below that a reduction of Γ leads to a reduction of Γ_0^*, or indeed of any extension Γ^* of Γ to S. The last assertion of the theorem then follows at once, since, by Theorem 5.48, every irreducible representation of S is necessarily basic.

Let Γ be an extendible, reducible representation of G, and let Γ reduce into representations Γ' and Γ'' of G, each of degree less than that of Γ. We may then assume that a basis of the representation space of Γ has been chosen so that

$$\Gamma(a) = \begin{pmatrix} \Gamma'(a) & 0 \\ * & \Gamma''(a) \end{pmatrix} \quad (\text{all } a \text{ in } G),$$

where the star indicates a block which may contain non-zero entries. Let the degrees of Γ, Γ', and Γ'' be n, n', and n'', respectively; then $n = n' + n''$, $n > n'$, $n > n''$.

Let Γ^* be any extension of Γ to S, and let the degree of Γ^* be $n + t$. Let V be the representation space of Γ^*, and let $V = V_1 \oplus V_2$, where V_1 is the representation space of Γ and V_2 that of the null representation of G, in accordance with (9). The dimensions of V_1 and V_2 are n and t, respectively. By Theorem 5.37, the representing matrices $\Gamma^*[(a)_{i\lambda}]$ of Γ^* have the form (13).

Let W_1 be the invariant subspace of V_1 carrying the representation Γ', so that Γ'' is carried by the factor space V_1/W_1. Let W be the subspace of V consisting of all vectors w of V having the form

$$w = x + \sum_{\mu \in \Lambda} x_\mu Q_\mu$$

with x and the x_μ in W_1, and where the sum is finite, i.e., all but a finite number of the x_μ are the zero vector of W_1. The Q_μ (μ in Λ) are the $n \times t$ matrices defined by (12), and appearing in (13), and may be regarded as linear transformations of V_1 into V_2. Thus $w = x + y$ with x in W_1 and $y = \sum x_\mu Q_\mu$ in V_2. It will be convenient in what follows to write w in partitioned form $(x \quad y)$ corresponding to (13), with x in V_1 (in W_1 actually) and y in V_2. Since $W \subseteq W_1 \oplus V_2$, it is a proper subspace of V, and we proceed to show that it is invariant under Γ^*.

By direct calculation from (13) we see that

$$(x \quad y)\Gamma^*[(a)_{i\lambda}] = (x' \quad y'),$$

where, using (15),

$$\begin{aligned} x' &= x\Gamma(p_{1i}ap_{\lambda 1}) + \sum_\mu x_\mu Q_\mu R_i \Gamma(ap_{\lambda 1}) \\ &= x\Gamma(p_{1i}ap_{\lambda 1}) + \sum_\mu x_\mu[\Gamma(p_{\mu i}) - \Gamma(p_{\mu 1}p_{1i})]\Gamma(ap_{\lambda 1}) \\ &= x\Gamma(p_{1i}ap_{\lambda 1}) + \sum_\mu x_\mu[\Gamma(p_{\mu i}ap_{\lambda 1}) - \Gamma(p_{\mu 1}p_{1i}ap_{\lambda 1})] \end{aligned}$$

and

$$\begin{aligned} y' &= x\Gamma(p_{1i}a)Q_\lambda + \sum_\mu x_\mu Q_\mu R_i \Gamma(a) Q_\lambda \\ &= x\Gamma(p_{1i}a)Q_\lambda + \sum_\mu x_\mu[\Gamma(p_{\mu i}a) - \Gamma(p_{\mu 1}p_{1i}a)]Q_\lambda. \end{aligned}$$

Since x and all the x_μ belong to W_1, and W_1 is invariant under $\Gamma(b)$ for every b in G, it is clear that $x' \in W_1$. Since y' is seen to be of the form $x'_\lambda Q_\lambda$ with x'_λ in W_1, it follows that $(x' \quad y') \in W$, and so W is invariant under Γ^*.

Let Δ' be the representation of S carried by the invariant subspace W of V constructed above, and let Δ'' be that carried by the factor space V/W. Now $V = V_1 \oplus V_2$ and $W = W_1 \oplus W_2$, where $W_2 = W \cap V_2$. It is clear from this that Δ' is an extension of Γ' to S. Furthermore,

$$V/W \cong V_1/W_1 \oplus V_2/W_2,$$

whence Δ'' is an extension of Γ'' to S.

We have thus shown that Γ^* reduces into two representations Δ' and Δ'' of S such that Δ' $[\Delta'']$ is an extension of Γ' $[\Gamma'']$ to S. By an evident induction on the number r of irreducible constituents Γ_i of Γ it is clear that Γ^* reduces into r representations Δ_i such that Δ_i is an extension of Γ_i $(i = 1, \cdots, r)$. If Γ_i is not null, then, by Theorem 5.48, Δ_i reduces into the basic extension Γ_{i0}^* of Γ_i and (possibly) null representations. We showed in the first paragraph that the basic extension of an irreducible representation is irreducible, and hence each such Γ_{i0}^* is irreducible. Hence the non-null irreducible constituents of Γ^* are precisely those Γ_{i0}^* for which Γ_i is not null.

REMARKS. (i) That it is possible for the basic extension Γ_0^* of Γ to have a null constituent in addition to these Γ_{i0}^*, even when all the Γ_i are non-null, is shown by Exercise 8 below.

(ii) If Γ is an extendible representation of G, Theorem 5.51 shows that its irreducible constituents are also extendible to S. Exercise 9 below shows that the converse does not hold.

THEOREM 5.52. *Full reducibility holds for the representations of S over the field Φ if and only if* (i) *full reducibility holds for the extendible representations of G over Φ, and* (ii) *the only proper extension to S of a proper representation of G is the basic extension.*

PROOF. Evidently full reducibility holds for the representations of S over Φ if and only if it holds for the proper representations of S over Φ.

Assume (i) and (ii), and let Γ^* be any proper representation of S. Then Γ^* is the extension to S of a proper representation Γ of G. By (i), Γ is fully reducible, say $\Gamma \sim \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_r$ ($\sim$ meaning equivalence), with each Γ_i irreducible and non-null. By Theorem 5.50, $\Gamma_0^* \sim \Gamma_{10}^* \oplus \cdots \oplus \Gamma_{r0}^*$. By Theorem 5.51, each Γ_{i0}^* is irreducible, and hence Γ_0^* is fully reducible. But $\Gamma^* \sim \Gamma_0^*$ by (ii).

Conversely, assume that every representation of S over Φ is fully reducible. Let Γ be an extendible proper representation of G, and let $\Gamma \sim \Gamma_1 \oplus \cdots \oplus \Gamma_r$ be its decomposition into (non-null) indecomposable representations Γ_i of G. By Theorem 5.50, $\Gamma_0^* \sim \Gamma_{10}^* \oplus \cdots \oplus \Gamma_{r0}^*$, and each Γ_{i0}^* is an indecomposable representation of S. But, by hypothesis, each Γ_{i0}^* is fully reducible, and so must be irreducible. By Theorem 5.51, each Γ_i is irreducible, and so Γ is fully reducible. This proves (i).

From Theorem 5.48, it is clear that a non-basic representation of S can be fully reducible only if it decomposes into the corresponding basic representation and a null representation. Hence every proper representation of S over Φ must be basic, which establishes (ii).

COROLLARY 5.53. *Let S be finite, and assume that the characteristic of Φ does not divide the order of G. Then $\Phi[S]$ is semisimple if and only if the only proper representation of S extending any given proper representation of G is its basic extension.*

The proof is immediate from Theorem 5.52 and Maschke's Theorem (§5.2). Corollary 5.53 is due to Munn [1955*b*]. In his proof, he shows that the representation Γ'_σ of S given by equation (5) in Theorem 5.28 must be the basic extension of the representation Γ_σ of G by calculating the rank of the extending matrix of Γ_σ. No expression is known which exhibits the equivalence of (5) in Theorem 5.28 and (13) of the present section.

Exercises for §5.4

1. Let Γ^* be an extension to $S=\mathscr{M}^0(G;\ I, \Lambda;\ P)$ of a proper representation Γ of G. Let the numbers h, q, r, and t be defined as in Theorem 5.48. Then Γ^* is proper if and only if $t = q + r - h$.

2. Let S be a left group. By Theorem 1.27 (or the Rees Theorem), $S = G \times I$, where G is a group and I is a left zero semigroup. The elements of S may be expressed uniquely in the form $(a)_i$ with a in G and i in I, multiplying as follows:

$$(a)_i(b)_j = (ab)_i \quad (a, b \text{ in } G;\ i, j \text{ in } I).$$

Let Φ be a field, and Γ a proper representation of G of degree n over Φ. Then Γ is extendible to S, and its basic extension can be considered to be

$$\Gamma_0^*[(a)_i] = \Gamma(a).$$

Any extension Γ^* of Γ has the form

$$\Gamma^*[(a)_i] = \begin{pmatrix} \Gamma(a) & 0 \\ R_i\Gamma(a) & 0 \end{pmatrix},$$

with arbitrary $t \times n$ matrices R_i (i in I), t being any fixed positive integer.

3. (a) The following assertions concerning an $m \times m$ matrix C of rank n over a field Φ are equivalent.

(i) C belongs to some multiplicative subgroup of $(\Phi)_m$.

(ii) For every (basic) factorization $C = AB$ of C into an $m \times n$ matrix A and an $n \times m$ matrix B, the $n \times n$ matrix BA is non-singular.

(iii) C^2 has rank n.

(b) Some power of any matrix in $(\Phi)_m$ has the above properties. (Suschkewitsch [1933].)

4. Let $S = \mathscr{M}^0(G;\Lambda, \Lambda;\Delta)$ be a Brandt semigroup (§3.3), Δ being the $\Lambda \times \Lambda$ identity matrix over G^0. Let Φ be a field.

(a) If Γ is a proper representation of G over Φ, the extending matrix Ω of Γ is the identity matrix.

(b) S admits a proper representation of finite degree over Φ if and only if the index class Λ is finite.

(c) The defining matrices of the basic extension Γ_0^* of Γ can be chosen so that Γ_0^* has the form stated in Exercise 5 of §5.2.

5. Two extensions to $S = \mathscr{M}^0(G; I, \Lambda; P)$ of the same absolutely irreducible representation Γ of G over Φ are equivalent if and only if they arise from equivalent factorizations of the extending matrix of Γ. (Clifford [1942].)

6. Let $\sigma = (123)$ and $\tau = (12)$ be generators, written in cyclic form, of the symmetric group G_3 on the set $\{1, 2, 3\}$. Let Φ be a field of characteristic $\neq 3$. Then

$$\Gamma(\sigma) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad \Gamma(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

determine an absolutely irreducible representation of G_3 of degree 2.

Let

$$S = \mathscr{M}^0(G_3; 2, 2; P), \quad \text{where } P = \begin{pmatrix} \iota & \iota \\ \iota & \iota \end{pmatrix},$$

ι being the identity element of G_3. Then the extending matrix Ω of Γ is zero, and a factorization $\Omega = QR$ of Ω is obtained by taking

$$Q = Q(\alpha) = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix}, \qquad R = R(\beta) = \begin{pmatrix} \beta & 1 \\ 0 & 0 \end{pmatrix}.$$

Let $\Gamma^*_{\alpha,\beta}$ be the corresponding extension of Γ to S. Then $\Gamma^*_{\alpha;\beta}$ is equivalent to $\Gamma^*_{\alpha',\beta'}$ if and only if $\alpha = \alpha'$ and $\beta = \beta'$.

If Φ is an infinite field, this shows that a finite semigroup (or a finite algebra) may have an infinite number of inequivalent indecomposable representations of the same degree.

7. (a) If a [0-] simple semigroup of matrices of finite degree over a field contains a non-zero idempotent, it is completely [0-] simple. Hence a non-completely [0-] simple semigroup containing an idempotent possesses no faithful proper representation.

(b) The set of all matrices

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$$

with a and b positive real numbers is a simple semigroup without idempotent. Hence non-completely simple semigroups without idempotents can possess faithful proper representations.

8. Let G be the cyclic group $\{e, a\}$ of order two. Let S be the Rees 2×2 matrix semigroup over G with sandwich matrix

$$P = \begin{pmatrix} e & e \\ e & a \end{pmatrix}.$$

Let Φ be the integers mod 2. Let

$$\Gamma(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma(a) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The extending matrix of Γ is

$$\Omega = \Omega_{22} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We can take

$$R_2 = (1 \quad 0), \qquad Q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The basic extension Γ_0^* of Γ to S is of degree 3. It reduces (not fully) into two unit representations and one null representation. (See Remark (i) after proof of Theorem 5.51.)

9. Let G, Φ, and Γ be as in Exercise 8. Let N be the set of natural numbers, and let S be the Rees $N \times N$ matrix semigroup over G with sandwich matrix $P = (p_{ij})$ given by

$$p_{ij} = \begin{cases} a & \text{if } i = j > 1, \\ e & \text{otherwise.} \end{cases}$$

Then, for all i, j in $N\backslash 1$,

$$\Omega_{ii} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Omega_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{if} \quad i \neq j.$$

The extending matrix Ω of Γ has infinite rank, and so Γ is not extendible to S. But the irreducible constituents of Γ are extendible to S. (See Remark (ii) after proof of Theorem 5.51.)

10. Under the operation $\oplus$ of forming the direct sum, the set of basic representations of S and the set of extendible representations of G are isomorphic semigroups. The relation of equality is here understood to be equivalence, and we deal only with proper representations.

5.5 Characters of a commutative semigroup

In this section (which does not depend on §§5.1–5.4) we follow Š. Schwarz [1954*a*, *b*, *c*] and Hewitt and Zuckerman [1955, §3; 1956, §5], who developed the theory of characters of a commutative semigroup independently. This section has been written so as to allow two interpretations of the term "character".

Let S be a commutative semigroup with identity element. (We shall see presently that the latter requirement is not an essential limitation of the theory.) Explicitly, by a *character* of S we mean a mapping χ of S into the field of complex numbers, not identically zero, satisfying

(C1) $$\chi(ab) = \chi(a)\chi(b) \quad (\text{all } a, b \text{ in } S).$$

It is easily verified, however, that *all the results of this section hold verbatim if we incorporate into the definition of the term "character" the condition that the mapping χ also satisfies*

(C2) $$|\chi(a)| = 0 \text{ or } 1 \quad (\text{all } a \text{ in } S).$$

If S is a group, (C2) requires $|\chi(a)| = 1$. We observe in particular that the classical extension and separation theorems for group characters (Theorems 5.57 and 5.58) are proved with this restriction in mind, and that as a result the separation and extension theorems for semigroups (Theorems 5.59 and 5.65) hold for characters satisfying (C2).

Hewitt and Zuckerman [1956] call a mapping χ satisfying (C1) a "multiplicative function"; if, moreover, χ is bounded and not identically zero, they call it a "semicharacter". We note that condition (C2) is equivalent to boundedness if S is a union of groups, but not in general. Without some such restriction on S, the results of this section do not hold for semicharacters.

The set S^* of all characters of S becomes a (commutative) semigroup if we define the product of two characters χ and ψ of S by

$$(\chi\psi)(a) = \chi(a)\psi(a) \quad \text{(all } a \text{ in } S).$$

We call S^* the *character semigroup of* S. The identity element of S^* is the *unit character* 1^* of S defined by $1^*(a) = 1$ for all a in S.

If S is a commutative semigroup without identity, it is desirable to modify the above definition of "character" by allowing χ to be identically zero, since otherwise the characters of S would not in general form a semigroup. Every character χ of S can be uniquely extended to a character of $S^1 = S \cup 1$ by defining $\chi(1) = 1$, and we see that *the mapping* $\chi \to \chi|S$ *is an isomorphism of* $(S^1)^*$ *upon* S^*. Hence there is no loss in generality if we restrict ourselves to semigroups with identity, and this we henceforth do.

An ideal P of S is called *prime* if $S \setminus P$ is a subsemigroup of S. We shall find it convenient to include the empty set $\square$, but not S itself, as a prime ideal of S. The union of two prime ideals of S is a prime ideal, but the intersection need not be (see Exercise 1 below). Thus the set Y^* of all prime ideals of S is a semilattice under union.

Let $\chi \in S^*$. Then

$$V_\chi = \{a : a \in S, \chi(a) = 0\}$$

is a prime ideal of S. For if $\chi(a) = 0$ then $\chi(ab) = \chi(a)\chi(b) = 0$ for every b in S; and if $\chi(a) \neq 0$, $\chi(b) \neq 0$, then $\chi(ab) = \chi(a)\chi(b) \neq 0$. We call V_χ the *vanishing ideal* of χ.

For a given prime ideal P of S, define ϵ_P by

$$\epsilon_P(a) = \begin{cases} 0 & \text{if } a \in P, \\ 1 & \text{if } a \in S \setminus P. \end{cases}$$

In other words, ϵ_P is the characteristic function of the set $S \setminus P$. Then $\epsilon_P(ab) = \epsilon_P(a)\epsilon_P(b)$ for all a, b in S, since both sides are 0 or both sides are 1. Moreover, $\epsilon_P^2 = \epsilon_P$. Thus ϵ_P is an idempotent element of S^*, and clearly its vanishing ideal is P. On the other hand, if ϵ is an idempotent element of S^*, then $\epsilon(a) = 0$ or 1 for every a in S, and we see at once that $\epsilon = \epsilon_P$ with $P = V_\epsilon$.

We note that if P and P' are prime ideals of S, then

$$\epsilon_{P \cup P'} = \epsilon_P \epsilon_{P'}.$$

For an element of S belongs to $S \setminus (P \cup P')$ if and only if it belongs to both $S \setminus P$ and $S \setminus P'$. We have therefore proved the following lemma, due to Schwarz [1954*a*].

Lemma 5.54. *There is an isomorphism between the semilattice E^* of idempotents of S^* and the semilattice Y^* of prime ideals of S such that if ϵ in E^* and P in Y^* correspond thereunder, then P is the vanishing ideal V_ϵ of ϵ, and ϵ is the characteristic function ϵ_P of $S \setminus P$.*

For a given prime ideal P of S, we define

$$H_P^* = \{\chi : \chi \in S^*,\ V_\chi = P\}.$$

In other words, H_P^* consists of all characters on S vanishing precisely on P. Clearly H_P^* is a subsemigroup of S^* containing ϵ_P, and ϵ_P is the identity element of H_P^*. If $\chi \in H_P^*$, and we define

$$\chi^{-1}(a) = \begin{cases} 0 & \text{if } a \in P, \\ 1/\chi(a) & \text{if } a \in S \setminus P, \end{cases}$$

then $\chi^{-1} \in H_P^*$, and $\chi\chi^{-1} = \epsilon_P$. Hence H_P^* is a subgroup of S^*. Every χ in S^* belongs to some H_P^*, namely that with $P = V_\chi$. The subgroups H_P^* of S^* are evidently mutually disjoint. If $P, P' \in Y^*$, then

$$H_P^* H_{P'}^* \subseteq H_{P \cup P'}^*.$$

For if $\chi \in H_P^*$ and $\chi' \in H_{P'}^*$, then $(\chi\chi')(a) = 0$ if and only if $\chi(a) = 0$ or $\chi'(a) = 0$, that is, $\chi\chi'$ vanishes precisely on $P \cup P'$.

We have thus shown the following theorem, proved for finite S by Schwarz ([1954*a*], Theorem 2), for periodic S with finite Y^* by Hewitt and Zuckerman ([1955], Theorem 3.13), and for inverse S by Warne and Williams [1961].

Theorem 5.55. *The character semigroup S^* of a commutative semigroup S with identity is the union of the semilattice Y^* of groups H_P^* ($P \in Y^*$), where Y^* is the semilattice of prime ideals of S, and H_P^* consists of all characters of S vanishing precisely on P.*

Let T be a commutative semigroup which is the union of a semilattice Y of groups G_α, $\alpha \in Y$ (§1.8). Let e_α be the identity element of G_α. A character χ of T vanishes on G_α if and only if $\chi(e_\alpha) = 0$, and otherwise $\chi(e_\alpha) = 1$. A character χ of T is called *principal* if there is an element β of Y such that $\chi(e_\alpha) = 1$ if and only if $\alpha \geqq \beta$. The element β of Y is evidently unique, and β (or G_β) is called the *apex* of χ.

The following theorem was found by Schwarz ([1954*a*], p. 230) and by Hewitt and Zuckerman ([1955], Theorems 3.2 and 3.3) for finite T, and by Warne and Williams [*c.* 1961] assuming the minimal condition for Y.

THEOREM 5.56. *Let T be a commutative semigroup with identity which is the union of a semilattice Y of groups G_α, $\alpha \in Y$. Let e_α be the identity element of G_α.*

(A) *Let χ be a principal character of T, and let β be the apex of χ (β in Y). The restriction χ' of χ to G_β is a character of G_β, and, for every s in T,*

$$\chi(s) = \begin{cases} \chi'(se_\beta) & \text{if } se_\beta \in G_\beta, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(B) *Let $\beta \in Y$, and let χ' be any character of G_β. Then (1) defines a principal character χ of T with apex β and coinciding with χ' on G_β.*

(C) *If the minimal condition holds for Y, then every character of T is principal.*

PROOF. This theorem is a direct consequence of Theorem 5.33, but we give the proof in order to keep the present section independent of the foregoing sections of this chapter.

(A) If $se_\beta \in G_\beta$, then

$$\chi(s) = \chi(s)1 = \chi(s)\chi(e_\beta) = \chi(se_\beta) = \chi'(se_\beta).$$

If $se_\beta \notin G_\beta$, and if $s \in G_\alpha$, then $se_\beta \in G_{\alpha\beta}$ with $\alpha\beta < \beta$. Hence $\chi(se_\beta) = 0$ by definition of β, and $\chi(s) = \chi(se_\beta)$ as before.

(B) If se_β and te_β are both in G_β, then so is $(se_\beta)(te_\beta) = (st)e_\beta$, and hence

$$\begin{aligned}\chi(st) &= \chi'((st)e_\beta) = \chi'((se_\beta)(te_\beta)) \\ &= \chi'(se_\beta)\chi'(te_\beta) = \chi(s)\chi(t).\end{aligned}$$

If they are not both in G_β, say $se_\beta \notin G_\beta$, and $s \in G_\alpha$, then $se_\beta \in G_{\alpha\beta}$ with $\alpha\beta < \beta$. Clearly, then, $ste_\beta \notin G_\beta$, and so

$$\chi(st) = 0 = 0\chi(t) = \chi(s)\chi(t).$$

(C) Let χ be a character of T. By hypothesis, there exists a minimal element β of Y such that χ does not vanish on G_β, that is, $\chi(e_\beta) = 1$. If χ does not vanish on G_α, then

$$\chi(e_{\alpha\beta}) = \chi(e_\alpha e_\beta) = \chi(e_\alpha)\chi(e_\beta) = 1 \cdot 1 = 1.$$

Since $\alpha\beta \leqq \beta$, and β is minimal, we conclude that $\alpha\beta = \beta$, and so $\alpha \geqq \beta$. Conversely, if $\alpha \geqq \beta$, then $e_\alpha e_\beta = e_\beta$, whence $\chi(e_\alpha) = 1$. Hence χ is principal with apex β.

We now take up the question of when the characters of a commutative semigroup S separate the elements of S. By this we mean that if a and b are any two distinct elements of S, there exists a character χ of S such that $\chi(a) \neq \chi(b)$. We shall prove the well-known theorem that this is true for commutative groups. But first we must prove a classical theorem concerning the extension of a character from a subgroup to the whole group. This will be necessary later for a similar result for semigroups (Theorem 5.65). For the sake of completeness, we give the proof.

THEOREM 5.57. *Let H_0 be a subgroup of a commutative group G, and let χ_0 be a character of H_0. Then there exists a character χ of G coinciding with χ_0 on H_0.*

PROOF. Let $\mathscr{P}$ be the set of all pairs (H, χ) with H a subgroup of G, and χ a character of H. We partially order $\mathscr{P}$ by defining $(H, \chi) \leqq (H', \chi')$ if $H \subseteq H'$ and χ' coincides with χ on H. If $\{(H_\lambda, \chi_\lambda) : \lambda \in \Lambda\}$ is a tower (chain) of elements of $\mathscr{P}$, then it has a supremum (H, χ) in $\mathscr{P}$, namely $H = \bigcup_{\lambda \in \Lambda} H_\lambda$ and χ defined (unambiguously) by $\chi(a) = \chi_\lambda(a)$ if $a \in H_\lambda$. Let $\mathscr{P}_0$ be the set of all pairs in $\mathscr{P}$ containing (H_0, χ_0). By Zorn's Lemma, $\mathscr{P}_0$ contains a maximal member (H, χ). The theorem will be established when we show that $H = G$.

Assume by way of contradiction that $a \in G \backslash H$. Let H' be the subgroup of G generated by H and a. We proceed to show that we can construct a character χ' of H' coinciding with χ on H, contrary to the maximality of (H, χ) in $\mathscr{P}_0$.

If $a^n \in H$ only for $n = 0$, then define

$$\chi'(ha^k) = \chi(h) \quad (h \in H;\, k = 0, \pm 1, \pm 2, \cdots).$$

Otherwise, let n be the least positive integer such that $a^n \in H$, and let $b = a^n$. Let ξ be any nth root of $\chi(b)$, and define

$$\chi'(ha^k) = \xi^k \chi(h) \quad (h \in H;\, k = 0, 1, \cdots, n-1).$$

In either case, χ' is easily seen to be a character of H' coinciding with χ on H.

THEOREM 5.58. *Let a and b be distinct elements of a commutative group G. Then there exists a character χ of G such that $\chi(a) \neq \chi(b)$.*

PROOF. Let $c = ab^{-1}$. It suffices to show that there exists a character χ of G such that $\chi(c) \neq 1$. Let H_0 be the cyclic subgroup of G generated by c. If H_0 has infinite order, let

$$\chi_0(c^k) = (-1)^k \quad (k = 0, \pm 1, \pm 2, \cdots).$$

If H_0 has finite order n, let ω be any nth root of unity other than 1, and define

$$\chi_0(c^k) = \omega^k \quad (k = 0, 1, \cdots, n-1).$$

In either case χ_0 is a character of H_0 such that $\chi_0(c) \neq 1$. The theorem then follows immediately from Theorem 5.57.

The following theorem, due to Hewitt and Zuckerman [1956, §5], answers the question raised above as to when the characters of a commutative semigroup S separate the elements of S. It explains the term "separative", which we introduced in §4.3. We recall that S is called *separative* if $a^2 = b^2 = ab$ (a, b in S) imply $a = b$.

THEOREM 5.59. *The characters of a commutative semigroup S with identity separate the elements of S if and only if S is separative.*

PROOF. Assume first that the characters of S separate the elements of S. Let a and b be elements of S such that $a^2 = b^2 = ab$. Then $\chi(a)^2 = \chi(b)^2 = \chi(a)\chi(b)$, and hence $\chi(a) = \chi(b)$, for any character χ of S. Hence $a = b$, since otherwise there would exist a character χ of S such that $\chi(a) \neq \chi(b)$. Hence S is separative.

Now assume that S is separative. By Theorem 4.18, S is the union of a semilattice Y of cancellative archimedean semigroups S_α ($\alpha \in Y$), and S can be embedded in a semigroup T which is the union of the same semilattice Y of groups G_α, where (for each α in Y) G_α is the quotient group of S_α.

Let a and b be distinct elements of S. Then there exist α and β in Y such that $a \in S_\alpha$ and $b \in S_\beta$. First suppose that $\alpha \neq \beta$. Then either $\alpha \nleqq \beta$ or $\beta \nleqq \alpha$, and we may assume by symmetry that $\beta \nleqq \alpha$. Let χ' be any character of G_β. Applying Theorem 5.56 (B) to T, it follows that equation (1) defines a character χ of T. Since $ae_\beta \in G_{\alpha\beta} \neq G_\beta$, it follows that $\chi(a) = 0$. Since $b \in G_\beta$, we have $\chi(b) \neq 0$. The restriction of χ to S is evidently a character of S separating a and b.

Now suppose that $\alpha = \beta$. Then a and b are distinct elements of the commutative group G_β. By Theorem 5.58, there exists a character χ' of G_β separating a and b. Defining χ by (1) above, the restriction of χ to S is again seen to be a character of S separating a and b. This concludes the proof of Theorem 5.59.

In §4.3 we introduced the congruence σ on any commutative semigroup S as follows: $a \sigma b$ if there exists a positive integer n such that $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$. By Theorem 4.14, $S' = S/\sigma$ is the maximal separative homomorphic image of S. Let θ denote the natural homomorphism of S upon S'. If χ is any character of S', then χ_θ defined by

$$\chi_\theta(a) = \chi(a\theta) \quad (\text{all } a \text{ in } S) \tag{2}$$

is clearly a character of S.

COROLLARY 5.60. *If a and b are elements of S, then $a \sigma b$ if and only if $\psi(a) = \psi(b)$ for every character ψ of S. The mapping $\chi \to \chi_\theta$ defined by (2) above is an isomorphism of the character semigroup S'^* of S' upon the character semigroup S^* of S.*

PROOF. Let $a, b \in S$, and let $\psi \in S^*$. If $a \sigma b$ then there exists a positive integer n such that

$$\psi(a)\psi(b)^n = \psi(b)^{n+1} \quad \text{and} \quad \psi(b)\psi(a)^n = \psi(a)^{n+1},$$

whence $\psi(a) = \psi(b)$. Conversely, if $(a, b) \notin \sigma$, then $a\theta \neq b\theta$, and, by Theorem 5.59, there exists χ in S'^* such that $\chi(a\theta) \neq \chi(b\theta)$. Hence $\psi = \chi_\theta$ is a character of S such that $\psi(a) \neq \psi(b)$.

The mapping $\chi \to \chi_\theta$ is evidently an isomorphism of S'^* into S^*. To see that it maps S'^* upon S^*, let $\psi \in S^*$. We may then define χ by $\chi(a\theta) = \psi(a)$; for if $a\theta = b\theta$, (a, b in S), then $a\sigma b$ and hence $\psi(a) = \psi(b)$, as shown above. Evidently $\chi \in S'^*$ and $\psi = \chi_\theta$.

The next theorem is due to Hewitt and Zuckerman [1956, §5].

THEOREM 5.61. *Let S be a separative commutative semigroup with identity. In accordance with Theorem 4.18, let S be expressed as the union of a semilattice Y of cancellative archimedean semigroups S_α ($\alpha \in Y$), and let T be the union of the semilattice Y of the quotient groups G_α of the S_α ($\alpha \in Y$). Then each character χ of S can be obtained as the restriction to S of a unique character χ^* of T. The mapping $\chi \to \chi^*$ is an isomorphism of the character semigroup S^* of S upon the character semigroup T^* of T.*

PROOF. Let χ be a character of S. Let a and b be elements of S belonging to the same archimedean component S_α of S. Then either $\chi(a) = \chi(b) = 0$, or else both $\chi(a) \neq 0$ and $\chi(b) \neq 0$. For, by the definition of "archimedean" (§4.3), each of a and b divides a power of the other, say $ax = b^m$ and $by = a^n$ (x, y in S; m and n positive integers). Then $\chi(a)\chi(x) = \chi(b)^m$ and $\chi(b)\chi(y) = \chi(a)^n$. Clearly these imply that $\chi(a) = 0$ if and only if $\chi(b) = 0$.

Each element of T belongs to some G_α, with α in Y, and as such is expressible in the form ab^{-1} with a and b in S_α. We define

$$\chi^*(ab^{-1}) = \begin{cases} \chi(a)/\chi(b) & \text{if } \chi(a) \neq 0 \text{ and } \chi(b) \neq 0, \\ 0 & \text{if } \chi(a) = \chi(b) = 0. \end{cases}$$

To see that this is single valued, let a, b, c, d be elements of S_α such that $ab^{-1} = cd^{-1}$. The complex numbers $\chi(a), \chi(b), \chi(c), \chi(d)$ are either all $= 0$ or all $\neq 0$. In the former case,

$$\chi^*(cd^{-1}) = 0 = \chi^*(ab^{-1}).$$

In the latter case, we infer $\chi(a)\chi(d) = \chi(b)\chi(c)$ from $ad = bc$, and hence

$$\chi^*(cd^{-1}) = \chi(c)/\chi(d) = \chi(a)/\chi(b) = \chi^*(ab^{-1}).$$

We show that χ^* is a character of T. Let s and t be elements of T. Then $s = ab^{-1}$ for some a, b in S_α (α in Y), and $t = cd^{-1}$ for some c, d in S_β (β in Y). By definition of product in T, $(ab^{-1})(cd^{-1}) = (ac)(bd)^{-1}$, and so

$$\chi^*(st) = \begin{cases} \chi(ac)/\chi(bd) & \text{if } \chi(ac) \neq 0 \text{ and } \chi(bd) \neq 0, \\ 0 & \text{if } \chi(ac) = \chi(bd) = 0. \end{cases}$$

Now $\chi(ac) \neq 0$ if and only if both $\chi(a)$ and $\chi(c)$ are $\neq 0$, and this is so if and only if both $\chi(b)$ and $\chi(d)$ are $\neq 0$. In this event

$$\begin{aligned} \chi^*(st) &= \chi(a)\chi(c)/\chi(b)\chi(d) \\ &= \chi^*(ab^{-1})\chi^*(cd^{-1}) = \chi^*(s)\chi^*(t). \end{aligned}$$

But if either $\chi(a) = 0$ or $\chi(c) = 0$, then

$$\chi^*(st) = 0 = \chi^*(ab^{-1})\chi^*(cd^{-1}) = \chi^*(s)\chi^*(t).$$

To show that the restriction of χ^* to S is χ, let $a \in S$. Then $a \in S_\alpha$ for some α in Y. Since $a = a^2a^{-1}$,

$$\chi^*(a) = \begin{cases} \chi(a^2)/\chi(a) & \text{if } \chi(a) \neq 0, \\ 0 & \text{if } \chi(a) = 0. \end{cases}$$

In either event $\chi^*(a) = \chi(a)$.

To show that χ^* is the only character of T that reduces to χ on S, suppose ψ^* is another such. Let $t \in T$. Then $t = ab^{-1}$ with a and b in S_α, for some α in Y. Assume first that $\chi(a) \neq 0$ and $\chi(b) \neq 0$. Since ψ^* induces in G_α an ordinary group character, we have

$$\psi^*(ab^{-1}) = \psi^*(a)/\psi^*(b) = \chi(a)/\chi(b) = \chi^*(ab^{-1}).$$

If $\chi(a) = \chi(b) = 0$, then $\psi^*(a) = 0$, and it follows that ψ^* must vanish on the whole group G_α. Hence

$$\psi^*(ab^{-1}) = 0 = \chi^*(ab^{-1}).$$

Thus $\psi^* = \chi^*$.

That the mapping $\chi \to \chi^*$ is an isomorphism of S^* upon T^* is now evident.

Having concluded the proof of Theorem 5.61, let us reflect a moment upon where we stand in the problem of determining the structure of the character semigroup S^* of an arbitrary commutative semigroup S with identity.

Let $S' = S/\sigma$ be the maximal separative homomorphic image of S, and let T be the union of the semilattice Y of groups G_α ($\alpha \in Y$) in which S' is embedded by Theorem 4.18. By Corollary 5.60, S^* and S'^* are isomorphic, and, by Theorem 5.61, replacing S therein by S', S'^* and T^* are in turn isomorphic. Thus $S^* \cong T^*$. Moreover, the method of obtaining the characters of S from those of T is clear. Starting with a character χ^* of T, take first its restriction χ to S', and then take the character χ_θ of S defined by (2) preceding Corollary 5.60. Hence the problem is reduced to describing T^*, to which we now turn. We are able to do this at present only under the assumption that Y satisfies the minimal condition (which is equivalent to M_J for T).

We shall need the concept of adjoint of a homomorphism. This is the analogue of the notion of adjoint of a linear transformation. Let G and H be commutative groups, and let their respective character groups be G^* and H^*. Let ϕ be a homomorphism of G into H. For each χ in H^* define a mapping $\chi\phi^*$ of G into the complex field by

$$(3) \qquad (\chi\phi^*)(a) = \chi(a\phi) \quad (\text{all } a \text{ in } G).$$

It is readily verified that $\chi\phi^* \in G^*$, and that $\chi \to \chi\phi^*$ is a homomorphism ϕ^* of H^* into G^*. We call ϕ^* the *adjoint* of the homomorphism ϕ of G into H.

LEMMA 5.62. *Let G and H be commutative groups, and let G^* and H^* be their respective character groups. Let ϕ be a homomorphism of G into H, and let ϕ^* be its adjoint. Then $H^*\phi^*$ is isomorphic with the character group of $G\phi$.*

PROOF. Let Ψ be the subgroup of G^* consisting of all characters ψ of G such that if $a\phi = b\phi$ then $\psi(a) = \psi(b)$, i.e., such that $\psi(c) = 1$ for every element c of the kernel of ϕ. Each element ψ of Ψ determines a character χ_0 of $G\phi$ by the rule

$$\chi_0(a\phi) = \psi(a) \quad (\text{all } a \text{ in } G). \tag{4}$$

Conversely, each character χ_0 of $G\phi$ determines an element ψ of Ψ by (4), and the mapping $\psi \rightarrow \chi_0$ is evidently an isomorphism of Ψ upon the character group of $G\phi$. The result will follow when we show that $\Psi = H^*\phi^*$.

Let $\psi \in H^*\phi^*$, and let $a\phi = b\phi$. Then $\psi = \chi\phi^*$ for some χ in H^*, and so

$$\psi(a) = (\chi\phi^*)(a) = \chi(a\phi) = \chi(b\phi) = (\chi\phi^*)(b) = \psi(b).$$

Conversely, let $\psi \in \Psi$, and let χ_0 be defined by (4). By Theorem 5.57, χ_0 can be extended to a character χ of H. Let $a \in G$. Then

$$(\chi\phi^*)(a) = \chi(a\phi) = \chi_0(a\phi) = \psi(a),$$

whence $\psi = \chi\phi^* \in H^*\phi^*$.

Let T be a commutative semigroup which is the union of a semilattice Y of (commutative) groups G_α ($\alpha \in Y$). By Theorem 4.11, the structure of T is determined by the system of homomorphisms $\phi_{\alpha,\beta}$ ($\alpha \geqq \beta$ in Y) of G_α into G_β defined by

$$a_\alpha\phi_{\alpha,\beta} = a_\alpha e_\beta \quad (\text{for every } a_\alpha \text{ in } G_\alpha), \tag{5}$$

where e_β is the identity element of G_β. These homomorphisms $\phi_{\alpha,\beta}$ satisfy the consistency condition

$$\phi_{\alpha,\beta}\,\phi_{\beta,\gamma} = \phi_{\alpha,\gamma} \quad (\alpha \geqq \beta \geqq \gamma \text{ in } Y), \tag{6}$$

and $\phi_{\alpha,\alpha}$ is the identity automorphism of G_α. Then the product of two arbitrary elements a_α and b_β of T (a_α in G_α and b_β in G_β) is given by

$$a_\alpha b_\beta = (a_\alpha\phi_{\alpha,\gamma})(b_\beta\phi_{\beta,\gamma}), \tag{7}$$

where $\gamma = \alpha\beta$ and the product on the right is carried out in the group G_γ.

The following theorem describes T^* completely, when the minimal condition holds for Y. It was found by Schwarz [1954a] for finite T, and in general by Warne and Williams [1961], but without specification of the system of homomorphisms $\theta_{\beta,\alpha}$. Schwarz showed essentially that $G_\beta^*\theta_{\beta,\alpha} \cong G_\alpha\phi_{\alpha,\beta}$ for finite T (see Corollary 5.64 below). The desirability of specifying these $\theta_{\beta,\alpha}$ to get the full structure of T^* was mentioned in the review of his paper.

THEOREM 5.63. *Let T be a commutative semigroup with identity which is the union of a semilattice Y of groups G_α $(\alpha \in Y)$, such that Y satisfies the minimal condition. Let $\phi_{\alpha,\beta}$ $(\alpha \geqq \beta$ in $Y)$ be the homomorphism of G_α into G_β defined by* (5) *above. Then Y is a lattice. Let Y^* be the dual of Y, and let T^* be the character semigroup of T. Then T^* is a semigroup isomorphic with the union of the semilattice Y^* of the character groups G_α^* of the groups G_α $(\alpha \in Y)$ determined by the system of homomorphisms $\theta_{\beta,\alpha} = \phi_{\alpha,\beta}^*$ of G_β^* into G_α^* $(\beta \geqq \alpha$ in Y^*, i.e., $\alpha \geqq \beta$ in $Y)$, where $\phi_{\alpha,\beta}^*$ is the adjoint of $\phi_{\alpha,\beta}$.*

PROOF. A semilattice with greatest element and satisfying the minimal condition is a lattice, and this applies to Y. Denote the join of two elements α and β of Y by $\alpha \vee \beta$.

Let χ_β' be a character of the group G_β. For each s in T, let

$$\chi_\beta(s) = \begin{cases} \chi_\beta'(se_\beta) & \text{if } se_\beta \in G_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 5.56 (B), χ_β so defined is a principal character of T with apex G_β, and such that χ_β' is just the restriction of χ_β to G_β. Since the minimal condition for Y is assumed as a hypothesis, it follows from Parts (A) and (C) of Theorem 5.56 that every character of T is obtained in this fashion. As χ_β' ranges over the character group G_β^* of G_β, χ_β ranges over a subgroup H_β of T^* isomorphic with G_β^*. The identity element ϵ_β of H_β is given by

$$\epsilon_\beta(s) = \begin{cases} 1 & \text{if } se_\beta \in G_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently the various H_β with β in Y are disjoint, and their union is T^*.

Let Y^* be the dual of Y. It will then follow that T^* is the union of the semilattice Y^* of groups H_β $(\beta \in Y^*)$ if we show that $\epsilon_\alpha\epsilon_\beta = \epsilon_{\alpha \vee \beta}$. If $s \in T$, $(\epsilon_\alpha\epsilon_\beta)(s) = \epsilon_\alpha(s)\epsilon_\beta(s)$, and this has the value 1 if and only if both $se_\alpha \in G_\alpha$ and $se_\beta \in G_\beta$. But clearly $se_\alpha \in G_\alpha$ if and only if $s \in G_\gamma$ with $\gamma \geqq \alpha$. Thus $(\epsilon_\alpha\epsilon_\beta)(s) = 1$ if and only if both $\gamma \geqq \alpha$ and $\gamma \geqq \beta$, i.e., if and only if $\gamma \geqq \alpha \vee \beta$. But $\epsilon_{\alpha \vee \beta}(s)$, with s in G_γ, has the value 1 if and only if $\gamma \geqq \alpha \vee \beta$, from which we conclude that $(\epsilon_\alpha\epsilon_\beta)(s) = \epsilon_{\alpha \vee \beta}(s)$. Since s is arbitrary, we have $\epsilon_\alpha\epsilon_\beta = \epsilon_{\alpha \vee \beta}$.

To conclude the proof of the theorem, it will save us some complexity if we identify the isomorphic groups H_α and G_α^*, and this we do.

The system of homomorphisms $\theta_{\beta,\alpha}$ $(\beta \geqq \alpha$ in Y^*, i.e., $\alpha \geqq \beta$ in $Y)$ which determines the structure of T^* is defined by

$$\chi_\beta\theta_{\beta,\alpha} = \chi_\beta\epsilon_\alpha \quad (\text{for every } \chi_\beta \text{ in } G_\beta^*),$$

analogous to (5). The adjoint $\phi_{\alpha,\beta}^*$ of $\phi_{\alpha,\beta}$ $(\alpha \geqq \beta)$ is defined by (3) above, which becomes

$$(\chi_\beta\phi_{\alpha,\beta}^*)(a_\alpha) = \chi_\beta(a_\alpha\phi_{\alpha,\beta}) \quad (\text{all } a_\alpha \text{ in } G_\alpha). \tag{8}$$

All that remains is to show that $\theta_{\beta,\alpha} = \phi^*_{\alpha,\beta}$ For this we have, for each element a_α of G_α,

$$\begin{aligned}(\chi_\beta\theta_{\beta,\alpha})(a_\alpha) &= (\chi_\beta\epsilon_\alpha)(a_\alpha) = \chi_\beta(a_\alpha)\epsilon_\alpha(a_\alpha)\\ &= \chi_\beta(a_\alpha)\cdot 1 = \chi_\beta(a_\alpha)\chi_\beta(e_\beta)\\ &= \chi_\beta(a_\alpha e_\beta) = \chi_\beta(a_\alpha\phi_{\alpha,\beta}).\end{aligned}$$

Comparing this with (8), we conclude that $\chi_\beta\theta_{\beta,\alpha} = \chi_\beta\phi^*_{\alpha,\beta}$. Since this holds for every χ_β in G^*_β, it follows that $\theta_{\beta,\alpha} = \phi^*_{\alpha,\beta}$.

As remarked above, the following is due to Schwarz [1954*a*] for finite T.

COROLLARY 5.64. *With the notation and hypotheses of Theorem* 5.63, *if* α *and* β *are elements of* Y *such that* $\alpha \geqq \beta$, *then* $G^*_\beta\theta_{\beta,\alpha}$ $(= G^*_\beta\epsilon_\alpha)$ *is isomorphic with the character group of* $G_\alpha\phi_{\alpha,\beta}$ $(= G_\alpha e_\beta)$. *If* T *is finite, then* $G^*_\beta\epsilon_\alpha \cong G_\alpha e_\beta$.

PROOF. The first assertion is immediate from Lemma 5.62 and Theorem 5.63. From the definition of $\phi_{\alpha,\beta}$ and $\theta_{\beta,\alpha}$ we have $G_\alpha\phi_{\alpha,\beta} = G_\alpha e_\beta$ and $G^*_\beta\theta_{\beta,\alpha} = G^*_\beta\epsilon_\alpha$. The second assertion then follows from the well-known theorem that a finite abelian group is isomorphic with its character group.

We conclude this section with a theorem on extending a character from a subsemigroup to the whole semigroup. It was proved for inverse commutative semigroups by Warne and Williams [1961] using the following more general theorem due to Ross [1959]. *A semicharacter* χ *of a subsemigroup* S *of a commutative semigroup* T *can be extended to a semicharacter of* T *if and only if it satisfies the following condition: if* a *and* b *are elements of* S *such that* a *divides* b *relative to* T, *then* $|\chi(a)| \geqq |\chi(b)|$. (The necessity is immediate; for $at = b$ with t in T, and $|\chi(t)| \leqq 1$.) As for all of the results of this section, Theorem 5.65 holds with the term "character" interpreted either with or without the unimodularity condition (C2). For recent related work, see Comfort [1960] and Ross [1961].

THEOREM 5.65. *Let* T *be a commutative semigroup with identity and such that the minimal condition holds for the maximal semilattice homomorphic image of* T. *Let* S *be any subsemigroup of* T *such that, for each archimedean component* T_α *of* T, $S \cap T_\alpha$ *is either empty or archimedean.*[2] *Then any character of* S *can be extended to a character of* T.

PROOF. Assume first that T is the union of a semilattice Y of groups G_α $(\alpha \in Y)$. Let χ be a character of S. Since χ can be extended to $S \cup 1$ by defining $\chi(1) = 1$, we may assume that $1 \in S$.

Let $S_\alpha = S \cap G_\alpha$, and let Y' be the set of all α in Y such that S_α is not empty. Clearly Y' is a subsemilattice of Y. For each α in Y', let G'_α be the subgroup of G_α generated by S_α. Let S' be the union of all the G'_α with α in Y'. By Theorem 5.61, χ can be extended (uniquely) to S'.

Since the minimal condition holds for Y, by hypothesis, it also holds for Y'. By Theorem 5.56, χ is principal, and is given by

[2] This hypothesis is due to R. O. Fulp, Proc. Edinburgh Math. Soc. **15** (1967), 199–202, who showed that the theorem is false without it.

$$\chi(s) = \begin{cases} \chi'(se_\beta) & \text{if } se_\beta \in G'_\beta, \\ 0 & \text{otherwise,} \end{cases}$$

in terms of a character χ' of its apex G'_β; here $\beta \in Y'$ and $s \in S'$. By Theorem 5.57, χ' can be extended to a character ψ' of G_β. But then, by Theorem 5.56 again,

$$\psi(s) = \begin{cases} \psi'(se_\beta) & \text{if } se_\beta \in G_\beta, \\ 0 & \text{otherwise,} \end{cases}$$

defines a character ψ of T coinciding with χ on S', and hence on S. For $e_\beta \in G'_\beta \subseteq S'$, and hence, if $s \in S'$, then $se_\beta \in G_\beta$ if and only if $se_\beta \in G'_\beta$.

Assume next that T is separative. Then, by Theorem 4.18, T is the union of a semilattice Y of cancellative semigroups T_α ($\alpha \in Y$), and can be embedded in a semigroup T' which is the union of the same semilattice Y of the quotient groups G_α of the T_α. But then S is also a subsemigroup of T', and from the foregoing, χ can be extended to a character ψ' of T'. The restriction ψ of ψ' to T is evidently an extension of χ to T.

Finally, assume only that T is a commutative semigroup with identity. Let σ be the congruence on T defined, as in §4.3 and just before Corollary 5.60, as follows: $a \sigma b$ if $ab^n = b^{n+1}$ and $ba^n = a^{n+1}$ for some n. Then $T' = T/\sigma$ is the maximal separative homomorphic image of T, and, by Corollary 5.60, the equation

$$\psi_\theta(a) = \psi(a\theta) \quad (\text{all } a \text{ in } T)$$

sets up a one-to-one correspondence between the characters ψ of T' and those (ψ_θ) of T, where θ is the natural homomorphism of T upon T'.

But, from the definition of σ, it is clear that its restriction to $S \times S$ is the same as the σ-relation for S itself, so that the maximal separative homomorphic image S' of S can be regarded as a subsemigroup of T'. The restriction θ' of θ to S is then the natural homomorphism of S upon S', and

$$\chi_{\theta'}(a) = \chi(a\theta') \quad (\text{all } a \text{ in } S)$$

sets up a one-to-one correspondence between the characters χ of S' and those ($\chi_{\theta'}$) of S.

Now let $\chi_{\theta'}$ be the given character of S. By what has already been proved, the corresponding character χ of S' can be extended to a character ψ of T'. Clearly the character ψ_θ of T is then an extension of $\chi_{\theta'}$.

EXERCISES FOR §5.5

1. Let S be the multiplicative semigroup of positive integers. Let Ω be a subset of the set Π of all prime numbers. Let P_Ω be the set of all positive integers divisible by at least one member of Ω. Then P_Ω is a prime ideal of S, and every prime ideal of S is a P_Ω for some $\Omega \subseteq \Pi$. We have $P_{\Omega_1} \cup P_{\Omega_2} = P_{\Omega_1 \cup \Omega_2}$, but $P_{\Omega_1} \cap P_{\Omega_2}$ is not in general prime. Nonetheless,

the set Y^* of prime ideals of S is a lattice under inclusion, with the meet of P_{Ω_1} and P_{Ω_2} being $P_{\Omega_1 \cap \Omega_2}$. Thus Y^* is isomorphic with the lattice of all subsets of the countably infinite set Π. The maximal subgroup $H^*_{P_\Omega}$ of the character semigroup S^* of S is isomorphic with the direct product (*i.e.*, unrestricted direct product) of $|\Pi \setminus \Omega|$ groups each isomorphic with the multiplicative group of the non-zero complex numbers [or those of modulus 1, if we require (C2)].

2. Let S be a finite commutative semigroup such that if $a \neq b$ in S there exists a character χ of S vanishing nowhere on S such that $\chi(a) \neq \chi(b)$. Then S is a group. (Hewitt and Zuckerman [1955], §3.1.6.)

3. The non-zero characters of a finite commutative semigroup S form a linearly independent set of functions on S. (Hewitt and Zuckerman [1955], §3.3.1.)

4. If H_S is the union of the maximal subgroups of a finite commutative semigroup S, then the number of non-zero characters of S is equal to $|H_S|$. (Hewitt and Zuckerman [1955], §3.6.2.)

5. Let S be a commutative semigroup such that some power of each element of S lies in a subgroup of S. As in Exercise 5 of §4.3, let $\{S_\alpha : \alpha \in Y\}$ be its archimedean components, H_α the kernel of S_α, and e_α the identity element of the group H_α. Let $H_S = \bigcup\{H_\alpha : \alpha \in Y\}$ be the "group part" of S.

(a) Let $a, b \in S$. Then $\chi(a) = \chi(b)$ for every character χ of S if and only if (i) a and b belong to the same archimedean component S_α of S, and (ii) $ae_\alpha = be_\alpha$.

(b) Let χ be a character of S, and let χ' be its restriction to H_S. Then for each α in Y, and each a_α in S_α, $\chi(a_\alpha) = \chi'(a_\alpha e_\alpha)$. Conversely, given a character χ' of H_S, χ so defined is a character of S. The mapping $\chi \to \chi'$ is an isomorphism of the character semigroup S^* of S upon the character semigroup H^*_S of H_S.

(Schwarz, [1954*b*], for periodic S.)

6. Continuing with the situation in Exercise 5, assume in addition that the minimal condition holds for Y and that S has an identity element. (This is not needed for parts (a) and (g) below.) Recall (§4.1) that an ideal A of S is called semiprime if $a \in S$ and $a^2 \in A$ imply $a \in A$. For present purposes, include $\square$ but exclude S as semiprime ideals of S.

(a) A character χ of S vanishes on S_α if and only if $\chi(e_\alpha) = 0$, and otherwise $\chi(e_\alpha) = 1$. Call χ principal if there exists β in Y such that $\chi(e_\alpha) = 1$ if and only if $\alpha \geqq \beta$, and call β the apex of χ. Then Theorem 5.56 holds with T replaced by S and (for each α in Y) G_α by S_α.

(b) There is a one-to-one correspondence between the prime ideals P of S and the elements β of Y such that if P and β correspond, then $P = \bigcup \{S_\alpha : \alpha \ngeqq \beta\}$ and β is the least element of Y such that $P \cap S_\beta = \square$.

(c) If P and β correspond as in (b), then the set H^*_β of all characters of S with apex β is the same as the group H^*_P of all characters of S vanishing precisely on P (Theorem 5.55). The semilattice Y is actually a lattice, and if Y^* denotes the semilattice of Y under join, then $H^*_\alpha H^*_\beta \subseteq H^*_{\alpha\vee\beta}$, and S^* is the union of the semilattice Y^* of groups $H^*_\alpha \quad (\alpha \in Y^*)$.

(d) Every proper ideal of S^* is semiprime. If A [A^*] is an ideal of S [S^*], let $A\pi$ [$A^*\pi^*$] be the set of all χ in S^* [all a in S] such that $\chi(a) = 0$ for all a in A [all χ in A^*]. Then π and π^* are mutually inverse anti-isomorphisms of the lattice of all semiprime ideals of S and the lattice of all ideals of S^* upon each other.

(e) The lattice of all prime ideals of S and the lattice of all principal ideals of S^* correspond to each other under π and π^*.

(f) If A is any ideal of S, the character semigroup of the semigroup A is isomorphic with $S^*\backslash A\pi$ if A has an identity element, and with $S^*/A\pi$ otherwise.

(g) The character semigroup of S/A, with the unit character removed, is isomorphic with $A\pi$.

(Schwarz [1954*c*]; Iséki [1957] for periodic S with a finite number of idempotents. They use the term "closed" for what we call "semiprime".)

7. (a) If S is separative, then S is isomorphic with a subsemigroup of the character semigroup S^{**} of the character semigroup S^* of S.

(b) If S is a finite commutative semigroup with identity element, and is a union of groups, then $S^{**} \cong S$.

APPENDIX A

A BRIEF ACCOUNT OF THE 1928 PAPER OF SUSCHKEWITSCH

Starting with an arbitrary finite semigroup S, he considers subsets of S of the form Sa having the least possible number of elements. These are evidently just the minimal left ideals of S, and we shall use the current terminology. He shows that each minimal left ideal of S is a left group, and (without using the expression "direct product") shows that it is the direct product of a group and a left zero semigroup. This is, of course, our Theorem 1.27 for finite semigroups. Moreover, any two minimal left ideals of S are isomorphic, and in particular are unions of the same number r of isomorphic groups.

He calls the union K of all the minimal left ideals of S the kernel ("Kerngruppe") of S. If s is the number of distinct minimal left ideals of S, then K is the union of rs mutually isomorphic groups. He shows that these can be arranged in a rectangular array as follows:

$$\begin{array}{c|cccc} K & L_1 & L_2 & \cdots & L_s \\ \hline R_1 & H_{11} & H_{12} & \cdots & H_{1s} \\ R_2 & H_{21} & H_{22} & \cdots & H_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ R_r & H_{r1} & H_{r2} & \cdots & H_{rs} \end{array}$$

(This is the source of our "eggbox picture", described in §2.1.) The union of the groups $H_{1\lambda}, \cdots, H_{r\lambda}$ in the λth column is the minimal left ideal L_λ ($\lambda = 1, \cdots, s$). Let $e_{i\lambda}$ be the identity element of the group $H_{i\lambda}$. He shows that the $H_{i\lambda}$ can be arranged so that each $e_{i\lambda}$ acts as a left identity on all the $H_{i\lambda}$ in the same row. When this is done, the union R_i of the groups $H_{i1}, \cdots, H_{is}$ in the ith row ($i = 1, \cdots, r$) is a minimal right ideal of S. Moreover, every minimal right ideal of S is one of the R_i. Hence:

Every finite semigroup has a kernel K which is the union of all the minimal left ideals of S and also of all the minimal right ideals of S. The intersection of a minimal left ideal and a minimal right ideal is a (maximal) subgroup of S. These results were subsequently extended to infinite semigroups having minimal left ideals and minimal right ideals (Exercise 13 of §2.7). We know now that it is simpler to introduce the L's and R's independently, and the H's as intersections thereof.

Suschkewitsch goes on to show by quite an involved argument that K *is uniquely determined by* (1) *the abstract group* H *to which each* $H_{i\lambda}$ *is isomorphic,* (2) *the numbers* r *and* s, *and* (3) *the* $(r-1)(s-1)$ *products* $e_{11}e_{i\lambda}$ $(i = 2, \cdots, r;$ $\lambda = 2, \cdots, s)$. He shows conversely that *the group* H, *the numbers* r *and* s, *and the* $e_{11}e_{i\lambda}$ *can be given arbitrarily.* This is done by means of transformations of a finite set. Thus he succeeds in determining the structure of the most general finite simple semigroup, but yet not (as came later with the Rees Theorem) in a readily usable form.

This theory also occupies the greater part of Chapter 3 of Suschkewitsch's book [1937].

BIBLIOGRAPHY

AMITSUR, S.

1951 *Semi-group rings.* Riveon Lematematika 5(1951) 5–9 (MR 13, 7).

ANDERSEN, O.

1952 *Ein Bericht über die Struktur abstrakter Halbgruppen.* Thesis (Staatsexamensarbeit), Hamburg, 1952.

BAER, R. AND LEVI, F.

1932 *Vollständige irreduzibele Systeme von Gruppenaxiomen.* (Beiträge zur Algebra No. 18.) Sitzber. Heidelberger Akad. Wiss., Abh. 2(1932) 1–12.

BALLIEU, R.

1950 *Une relation d'équivalence dans les groupoïdes et son application à une classe de demi-groupes.* III^e Congrès National des Sciences, Bruxelles 2(1950) 46–50 (MR 17, 126).

BRANDT, H.

1927 *Über eine Verallgemeinerung des Gruppenbegriffes.* Math. Ann. 96(1927) 360–366.

1940 *Über die Axiome des Gruppoids.* Vierteljschr. Naturforsch. Ges. Zürich 85(1940) 95–104 (MR 2, 218).

BRUCK, R. H.

1958 *A Survey of Binary Systems.* Ergebnisse der Math., Heft 20, Springer, Berlin, 1958 (185 pp.).

CARMAN, K. S.

1949 *Semigroup ideals.* M.A. Thesis, directed by D. D. Miller, University of Tennessee, 1949.

CLIFFORD, A. H.

1933 *A system arising from a weakened set of group postulates.* Annals of Math. 34(1933) 865–871.

1941 *Semigroups admitting relative inverses.* Annals of Math. 42(1941) 1037–1049 (MR 3, 199).

1942 *Matrix representations of completely simple semigroups.* Amer. J. Math. 64(1942) 327–342 (MR 4, 4).

1948 *Semigroups containing minimal ideals.* Amer. J. Math. 70(1948) 521–526 (MR 10, 12).

1949 *Semigroups without nilpotent ideals.* Amer. J. Math. 71(1949) 834–844 (MR 11, 327).

1950 *Extensions of semigroups.* Trans. Amer. Math. Soc. 68(1950) 165–173 (MR 11, 499).

1953 *A class of d-simple semigroups.* Amer. J. Math. 75(1953) 547–556 (MR 15, 98).

1954 *Bands of semigroups.* Proc. Amer. Math. Soc. 5(1954) 499–504 (MR 15, 930).

1960 *Basic representations of completely simple semigroups.* Amer. J. Math. 82(1960) 430–434.

CLIFFORD, A. H. AND MILLER, D. D.

1948 *Semigroups having zeroid elements.* Amer. J. Math. 70(1948) 117–125 (MR 9, 330).

CLIMESCU, A. C.

1946 *Sur les quasicycles.* Bull. École Polytech. Jassy 1(1946) 5–14 (MR 8, 134).

COMFORT, W. W.

1960 *The isolated points in the dual of a commutative semigroup.* Proc. Amer. Math. Soc. 11(1960) 227–233.

CONRAD, P. F.

1957 *Generalized semigroup rings.* J. Indian Math. Soc. (N.S.) 21(1957) 73–95 (MR 20, 142).

CROISOT, R.

1948a *Une interprétation des relations d'équivalence dans un ensemble.* C. R. Acad. Sci. Paris 226(1948) 616–617 (MR 9, 406).

1948b *Condition suffisante pour l'égalité des longueurs de deux chaînes de mêmes extrémités dans une structure. Application aux relations d'équivalence et aux sous-groupes.* C. R. Acad. Sci. Paris 226(1948) 767–768 (MR 9, 406).

1953 *Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples.* Ann. Sci. École Norm. Sup. (3) 70(1953) 361–379 (MR 15, 680).

DICKSON, L. E.

1905 *On semi-groups and the general isomorphism between infinite groups.* Trans. Amer. Math. Soc. 6(1905) 205–208.

DOSS, C. G.

1955 *Certain equivalence relations in transformation semigroups.* M.A. Thesis, directed by D. D. Miller, University of Tennessee, 1955.

DUBREIL, P.

1941 *Contribution à la théorie des demi-groupes.* Mém. Acad. Sci. Inst. France (2)63, no. 3 (1941), 52 pp. (MR 8, 15).

1943 *Sur les problèmes d'immersion et la théorie des modules.* C. R. Acad. Sci. Paris 216(1943) 625–627 (MR 5, 144).

GLUSKIN, L. M. (ГЛУСКИН, Л.М.)

1957 *Элементарные обобщенные группы.* (*Elementary generalized groups.*) Mat. Sbornik 41(1957) 23–36 (MR 19, 836).

GOOD, R. A. AND HUGHES, D. R.

1952 *Associated groups for a semigroup.* Bull. Amer. Math. Soc. 58(1952) 624–625 (Abstract).

GREEN, J. A.

1951 *On the structure of semigroups.* Annals of Math. 54(1951) 163–172 (MR 13, 100).

GREEN, J. A. AND REES, D.

1952 *On semigroups in which $x^r = x$.* Proc. Cambr. Phil. Soc. 48(1952) 35–40 (MR 13, 720).

GRIMBLE, HELEN B.

1950 *Prime ideals in semigroups.* M.A. Thesis, directed by D. D. Miller, University of Tennessee, March 1950.

HANCOCK, V. R.

1960a *On complete semimodules.* Proc. Amer. Math. Soc. 11(1960) 71–76.

1960b *Commutative Schreier extensions of semigroups.* Dissertation, The Tulane University of Louisiana, 1960.

HASHIMOTO, H.

1954 *On a generalization of groups.* Proc. Japan Acad. 30(1954) 548–549 (MR 16, 564).

HEWITT, E. AND ZUCKERMAN, H. S.

1955 *Finite dimensional convolution algebras.* Acta Math. 93(1955) 67–119 (MR 17, 1048).

1956 *The l_1-algebra of a commutative semigroup.* Trans. Amer. Math. Soc. 83(1956) 70–97 (MR 18, 465).

1957 *The irreducible representations of a semigroup related to the symmetric group.* Illinois J. Math. 1(1957) 188–213 (MR 19, 249).

ISÉKI, K.

1956*a* *Contribution to the theory of semigroups.* I. Proc. Japan Acad. 32(1956) 174–175 (MR 17, 1184).

1956*b* *Contributions to the theory of semi-groups.* III. Proc. Japan Acad. 32(1956) 323–324 (MR 18, 282).

1956*c* *Contributions to the theory of semigroups.* IV. Proc. Japan Acad. 32(1956) 430–435 (MR 18, 282).

1956*d* *A characterisation of regular semi-group.* Proc. Japan Acad. 32(1956) 676–677 (MR 18, 717).

1957 *Contributions to the theory of semigroups.* VI. Proc. Japan Acad. 33(1957) 29–30 (MR 19, 1158).

IVAN, J.

1953 *On the direct product of semigroups.* (Slovakian, Russian summary.) Mat.-Fyz. Časopis. Slovensk. Akad. Vied 3(1953) 57–66 (MR 16, 9).

1954 *On the decomposition of simple semigroups into a direct product.* (Slovakian, Russian summary.) Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4(1954) 181–202 (MR 17, 346).

KIMURA, N.

1954 *Maximal subgroups of a semigroup.* Kōdai Math. Sem. Rep. 1954(1954) 85–88 (MR 16, 443).

1957 *Note on idempotent semigroups.* I. Proc. Japan Acad. 33(1957) 642–645 (MR 20, 762).

1958*a* *Note on idempotent semigroups.* III. Proc. Japan Acad. 34(1958) 113–114 (MR 20, 762).

1958*b* *Note on idempotent semigroups.* IV. Proc. Japan Acad. 34(1958) 121–123 (MR 20, 762).

1958*c* *The structure of idempotent semigroups.* I. Pacific J. Math. 8(1958) 257–275 (MR 21, 253).

1958*d* *On some existence theorems on multiplicative systems.* I. *Greatest quotient.* Proc. Japan Acad. 34(1958) 305–309 (MR 21, 254).

KLEIN-BARMEN, F.

1940 *Über eine weitere Verallgemeinerung des Verbandsbegriffes.* Math. Z. 46(1940) 472–480 (MR 1, 327).

KOCH, R. J.

1953 *On topological semigroups.* Dissertation, The Tulane University of Louisiana, 1953.

LIBER, A. E. (ЛИБЕР, А. Е.)

1954 *К теории обобщенных групп.* (*On the theory of generalized groups.*) Doklady Akad. Nauk SSSR 97(1954) 25–28 (MR 16, 9).

LOEWY, A.

1927 *Über abstrakt definierte Transmutationssysteme oder Mischgruppe.* J. Reine Angew. Math. 157(1927) 239–254.

LYAPIN, E. S. (ЛЯПИН, Е. С.)

1953*a* *Ассоциативные системы всех частичных преобразований.* (*Associative systems of all partial transformations.*) Doklady Akad. Nauk SSSR 88(1953) 13–15 (MR 15, 395) (errata 92 (1953) 692).

1953*b* *Канонический вид элементов одной ассоциативной системы, заданной определяющими соотношениями.* (*Canonical form of elements of an associative system given by defining relations.*) Leningrad Gos. Ped. Inst. Uch. Zap. 89(1953) 45–54 (MR 17, 942).

1953c *Увеличительные элементы ассоциативных систем.* (*Increasing elements of associative systems.*) Leningrad Gos. Ped. Inst. Uch. Zap. 89(1953) 55–65 (MR 17, 942).

1954 *Полугруппы, во всех представлениях которых операторы имеют неподвижные точки, I.* (*Semigroups in all of whose representations the operators have fixed points, I.*) Mat. Sbornik 34(1954) 289–306 (MR 15, 850).

MALCEV, A. I. (МАЛЬЦЕВ, А. И.)

1939 *О включении ассоциативных систем в группы.* (*On the embedding of associative systems in groups.*) Mat. Sbornik 6(1939) 331–336 (MR 2, 7).

1952 *Симметрические группоиды.* (*Symmetric groupoids.*) Mat. Sbornik 31(1952) 136–151 (MR 14, 349).

MANN, H. B.

1944 *On certain systems which are almost groups.* Bull. Amer. Math. Soc. 50(1944) 879–881 (MR 6, 147).

McLEAN, D.

1954 *Idempotent semigroups.* Amer. Math. Monthly 61(1954) 110–113 (MR 15, 681).

MILLER, D. D. AND CLIFFORD, A. H.

1956 *Regular $\mathscr{D}$-classes in semigroups.* Trans. Amer. Math. Soc. 82(1956) 270–280 (MR 17, 1184).

MUNN, W. D.

1955a *Semigroups and their algebras.* Dissertation, Cambridge University, 1955.

1955b *On semigroup algebras.* Proc. Cambridge Phil. Soc. 51(1955) 1–15 (MR 16, 561).

1957a *Matrix representations of semigroups.* Proc. Cambridge Phil. Soc. 53(1957) 5–12 (MR 18, 489).

1957b *The characters of the symmetric inverse semigroup.* Proc. Cambridge Phil. Soc. 53(1957) 13–18 (MR 18, 465).

1960 *Irreducible matrix representations of semigroups.* Quarterly J. Math. Oxford, Ser. (2) 11 (1960) 295–309.

1961 *Pseudo-inverses in semigroups.* Proc. Cambridge Phil. Soc. 57 (1961) 247–250.

MUNN, W. D. AND PENROSE, R.

1955 *A note on inverse semigroups.* Proc. Cambridge Phil. Soc. 51(1955) 396–399 (MR 17, 10).

NUMAKURA, K.

1954 *A note on the structure of commutative semigroups.* Proc. Japan Acad. 30(1954) 262–265 (MR 16, 214).

OGANESYAN, V. A. (ОГАНЕСЯН, В.А.)

1955 *О полупростоте системной алгебры.* (*On the semisimplicity of a system algebra.*) Akad. Nauk Armyan. SSR. Dokl. 21(1955) 145–147 (MR 18, 465).

PONIZOVSKY, I. S. (ПОНИЗОВСКИЙ, И. С.)

1956 *О матричных представлениях ассочиативных систем.* (*On matrix representations of associative systems.*) Mat. Sbornik 38(1956) 241–260 (MR 18, 378).

1958 *О матричных неприводимых представлениях конечных полугрупп.* (*On irreducible matrix representations of finite semigroups.*) Uspehi Mat. Nauk 13(1958) 139–144 (MR 21, 383).

POOLE, A. R.

1937 *Finite ova.* Amer. J. Math. 59(1937) 23–32.

POSEY, E. E.

1949 *Endomorphisms and translations of semigroups.* M.A. Thesis, directed by D. D. Miller, University of Tennessee, 1949.

PRACHAR, K.

1947 *Zur Axiomatik der Gruppen.* Akad. Wiss. Wien. S.-B. II a. 155(1947) 97–102 (MR 9, 491).

PRESTON, G. B.

1954a *Inverse semi-groups.* J. London Math. Soc. 29(1954) 396–403 (MR 16, 215).

1954b *Inverse semi-groups with minimal right ideals.* J. London Math. Soc. 29(1954) 404–411 (MR 16, 215).

1954c *Representations of inverse semi-groups.* J. London Math. Soc. 29(1954) 411–419 (MR 16, 216).

1957 *A note on representations of inverse semigroups.* Proc. Amer. Math. Soc. 8(1957) 1144–1147 (MR 19, 941).

1958 *Matrix representations of semigroups.* Quarterly J. Math. Oxford Ser. (2) 9(1958) 169–176 (MR 20, 960).

RÉDEI, L.

1952 *Die Verallgemeinerung der Schreierschen Erweiterungstheorie.* Acta Sci. Math. Szeged 14(1952) 252–273 (MR 14, 614).

REES, D.

1940 *On semi-groups.* Proc. Cambridge Phil. Soc. 36(1940) 387–400 (MR 2, 127).

1941 *Note on semi-groups.* Proc. Cambridge Phil. Soc. 37(1941) 434–435 (MR 3, 199).

1948a *On the ideal structure of a semi-group satisfying a cancellation law.* Quarterly J. Math. Oxford Ser. 19(1948) 101–108 (MR 9, 567).

1948b *On the group of a set of partial transformations.* J. London Math. Soc. 22 (1948) 281–284 (MR 9, 568).

RICH, R. P.

1949 *Completely simple ideals of a semigroup.* Amer. J. Math. 71(1949) 883–885 (MR 11, 327).

ROSS, K. A.

1959 *A note on extending semicharacters on semigroups.* Proc. Amer. Math. Soc. 10(1959) 579–583 (MR 22, 12).

1961 *Extending characters on semigroups.* Proc. Amer. Math. Soc. 12 (1961).

SCHÜTZENBERGER, M. P.

1947 *Sur certains treillis gauches.* C. R. Acad. Sci. Paris 224(1947) 776–778 (MR 8, 432).

1956a *Sur une représentation des demi-groupes.* C. R. Acad. Sci. Paris 242(1956) 2907–2908 (MR 18, 13).

1956b *Sur deux représentations des demi-groupes finis.* C. R. Acad. Sci. Paris 243 (1956) 1385–1387 (MR 18, 282).

1957a *$\bar{\mathfrak{D}}$ représentation des demi-groupes.* C. R. Acad. Sci. Paris 244(1957) 1994–1996 (MR 19, 249).

1957b *Applications des $\bar{\mathfrak{D}}$ représentations à l'étude des homomorphismes des demi-groupes.* C. R. Acad. Sci. Paris 244(1957) 2219–2221 (MR 19, 249).

1958 *Sur la représentation monomiale des demi-groupes.* C. R. Acad. Sci. Paris 246(1958) 865–867 (MR 20, 394).

SCHWARZ, Š. (ШВАРЦ, Ш.)

1943 *Zur Theorie der Halbgruppen.* (Slovakian, German summary.) Sborník prác Prírodovedekej Fakulty Slovenskej Univerzity v Bratislave No. 6(1943) 64 pp. (MR 10, 12).

1951 *On the structure of simple semigroups without zero.* Czechoslovak Math. J. 1(1951) 41–53 (MR 14, 12).

1953a *К теории периодических полугрупп.* (*Contribution to the theory of torsion semigroups.*) Czechoslovak Math. J. 3(1953) 7–21 (MR 15, 850).

1953b *О максимальных идеалах в теории полугрупп.* (*On maximal ideals in the theory of semigroups.*) I. Czechoslovak Math. J. 3(1953) 139–153 (MR 15, 850).

1953c *О максимальных идеалах в теории полугрупп.* (*On maximal ideals in the theory of semigroups.*) II. Czechoslovak Math. J. 3(1953) 365–383 (MR 15, 850).

1954a *Теория характеров коммутативных полугрупп.* (*The theory of characters of finite commutative semigroups.*) Czechoslovak Math. J. 4(1954) 219–247 (MR 16, 1085).

1954b *Характеры коммутативных полугрупп как функции классов.* (*Characters of commutative semigroups as class functions.*) Czechoslovak Math. J. 4(1954) 291–295 (MR 16, 1086).

1954c *О некоторой связи Галуа в теории характеров полугрупп.* (*On a Galois connection in the theory of characters of commutative semigroups.*) Czechoslovak Math. J. 4(1954) 296–313 (MR 16, 1086).

1956 *Semigroups satisfying some weakened forms of the cancellation law.* (Slovakian, Russian and English summaries.) Mat.-Fyz. Časopis Slovensk. Akad. Vied 6(1956) 149–158 (MR 19, 940).

SKOLEM, T.

1951 *Some remarks on semi-groups.* Norske Vid. Selsk. Forh., Trondheim 24(1951) 42–47 (MR 13, 906).

STEINFELD, O.

1956 *Über die Quasiideale von Halbgruppen.* Publ. Math. Debrecen 4(1956) 262–275 (MR 18, 790).

1957 *Über die Quasiideale von Halbgruppen mit eigentlichem Suschkewitsch-Kern.* Acta Sci. Math. Szeged 18(1957) 235–242 (MR 20, 282).

STOLL, R. R.

1944 *Representations of finite simple semigroups.* Duke Math. J. 11(1944) 251–265 (MR 5, 229).

1951 *Homomorphisms of a semigroup onto a group.* Amer. J. Math. 73 (1951) 475–481 (MR 12, 799).

STOLT, B.

1956 *Über eine besondere Halbgruppe.* Ark. Mat. 3(1956) 275–286 (MR 17, 942).

SUSCHKEWITSCH, A. (СУШКЕВИЧ, А. К.)

1928 *Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit.* Math. Ann. 99(1928) 30–50.

1933 *Über die Matrizendarstellung der verallgemeinerte Gruppen.* Com. Soc. Math. Kharkow 6(1933) 27–38.

1937 *Теория обобщенных групп.* (*Theory of Generalized Groups*). Gos. Nauk.-Tekh. Izd. Ukranii, Kharkow, 1937 (176 pp.). (Out of print.)

1940a *Исследования о бесконечных подстановках.* (*Investigations on infinite substitutions.*) Memorial volume dedicated to D. A. Grave, Moscow (1940) 245–253 (MR 2, 217).

1940b *Исследования о бесконечных подстановках.* (*Investigations on infinite substitutions.*) Comm. Inst. Sci. Math. Mec. Univ. Kharkoff (4) 18(1940) 27–37 (MR 3, 99).

SZÉP, J.

1956 *Zur Theorie der Halbgruppen.* Publ. Math. Debrecen 4(1956) 344–346. (MR 18, 110).

TAMARI, DOV.

1948 *On a certain classification of rings and semigroups.* Bull. Amer. Math. Soc. 54(1948) 153–158 (MR 9, 490).

TAMURA, T.

1950 *Characterization of groupoids and semilattices by ideals in a semigroup.* J. Sci. Gakugei Fac. Tokushima Univ. 1(1950) 37–44 (MR 13, 430).

1954*a* *On finite one-idempotent semigroups.* I. J. Gakugei, Tokushima Univ. (Nat. Sci.) 4(1954) 11–20 (MR 15, 850).
1954*b* *Note on unipotent inversible semigroups.* Kōdai Math. Sem. Rep. 1954(1954) 93–95 (MR 16, 443).
1955*a* *One-sided bases and translations of a semigroup.* Math. Japonica 3(1955) 137–141 (MR 17, 1184).
1955*b* *On translations of a semigroup.* Kōdai Math. Sem. Rep. 7(1955) 67–70 (MR 18, 318).
1957 *Commutative nonpotent archimedean semigroup with cancellation law.* I. J. Gakugei Tokushima Univ. 8(1957) 5–11 (MR 20, 533).
1958 *Notes on translations of a semigroup.* Kōdai Math. Sem. Rep. 10(1958) 9–26 (MR 20, 1066).

TAMURA, T. AND KIMURA, N.
1954 *On decompositions of a commutative semigroup.* Kōdai Math. Sem. Rep. 1954(1954) 109–112 (MR 16, 670).
1955 *Existence of greatest decomposition of a semigroup.* Kōdai Math. Sem. Rep. 7(1955) 83–84 (MR 18, 192).

TEISSIER, MARIANNE
1952*a* *Sur l'algèbre d'un demi-groupe fini simple.* C. R. Acad. Sci. Paris 234(1952) 2413–2414 (MR 14, 10).
1952*b* *Sur l'algèbre d'un demi-groupe fini simple.* II. *Cas général.* C. R. Acad. Sci. Paris 234(1952) 2511–2513 (MR 14, 10).

THIERRIN, G.
1951 *Sur une condition nécessaire et suffisante pour qu'un semi-groupe soit un groupe.* C. R. Acad. Sci. Paris 232(1951) 376–378 (MR 12, 389).
1952*a* *Sur les éléments inversifs et les éléments unitaires d'un demi-groupe inversif.* C. R. Acad. Sci. Paris 234(1952) 33–34 (MR 13, 621).
1952*b* *Sur une classe de demi-groupes inversifs.* C. R. Acad. Sci. Paris 234(1952) 177–179 (MR 13, 621).
1954*a* *Sur quelques classes de demi-groupes possédant certaines propriétés des semi-groupes.* C. R. Acad. Sci. Paris 238(1954) 1765–1767 (MR 15, 849).
1954*b* *Sur quelques propriétés de certaines classes de demi-groupes.* C. R. Acad. Sci. Paris 239(1954) 1335–1337 (MR 16, 443).
1955*a* *Demi-groupes inversés et rectangulaires.* Acad. Roy. Belg. Bull. Cl. Sci. (5) 41(1955) 83–92 (MR 17, 10).
1955*b* *Sur une propriété caractéristique des demi-groupes inversés et rectangulaires.* C. R. Acad. Sci. Paris 241(1955) 1192–1194 (MR 17, 459).
1955*c* *Contribution à la théorie des équivalences dans les demi-groupes.* Bull. Soc. Math. France 83(1955) 103–159 (MR 17, 584).
1956 *Sur quelques decompositions des groupoïdes.* C. R. Acad. Sci. Paris 242(1956) 596–598 (MR 17, 825).

TULLY, E. J.
1960 *Representation of a semigroup by transformations of a set.* Dissertation, The Tulane University of Louisiana, 1960.

VAGNER, V. V. (ВАГНЕР, В. В.)
1952*a* *К теории частичных преобразований.* (*On the theory of partial transformations.*) Doklady Akad. Nauk SSSR (N.S.) 84(1952) 653–656 (MR 14, 10).
1952*b* *Обобщенные группы.* (*Generalized groups.*) Doklady Akad. Nauk SSSR (N.S.) 84(1952) 1119–1122 (MR 14, 12).

VANDIVER, H. S.
1940 *On the imbedding of one semi-group in another, with application to semi-rings.* Amer. J. Math. 62(1940) 72–78 (MR 1, 105).

VOROBEV, N. N. (ВОРОБЬЕВ, Н. Н.)

1953a *Ассоциативные системы, всякая подсистема которых имеет единицу.* (*Associative systems of which every subsystem has a unity.*) Doklady Akad. Nauk SSSR 88(1953) 393–396 (MR 14, 718).

1953b *О симметрических ассоциативных системах.* (*On symmetric associative systems*). Leningrad Gos. Ped. Inst. Uch. Zap. 89(1953) 161–166 (MR 17, 943).

WALLACE, A. D.

1953 *A note on mobs.* II. Anais Acad. Brasil Ci. 25(1953) 335–336 (MR 15, 854).

1955 *The structure of topological semigroups.* Bull. Amer. Math. Soc. 61(1955) 95–112 (MR 16, 796).

1957 *Retractions in semigroups.* Pacific J. Math. 7(1957) 1513–1517 (MR 20, 396).

WARNE, R. J. AND WILLIAMS, L. K.

1961 *Characters on inverse semi-groups.* Czechoslovak Math. J. 11 (1961) 150–155.

WIEGANDT, R.

1958a *On complete semi-groups.* Acta Sci. Math. Szeged 19(1958) 93–97 (MR 20, 394).

1958b *On complete semi-modules.* Acta Sci. Math. Szeged 19(1958) 219–223 (MR 20, 1066).

YAMADA, M.

1955a *A note on middle unitary semigroups.* Kōdai Math. Sem. Rep. 7(1955) 49–52 (MR 17, 585).

1955b *On the greatest semilattice decomposition of a semigroup.* Kōdai Math. Sem. Rep. 7(1955) 59–62 (MR 17, 584).

YAMADA, M. AND KIMURA, N.

1958 *Note on idempotent semigroups.* II. Proc. Japan Acad. 34(1958) 110–112 (MR 20, 762).

AUTHOR INDEX

Page numbers which include a reference to the exercises are printed in italics.

This page intentionally left blank

INDEX

Terms are listed primarily under the broad concept involved, such as *algebra*, *group*, *ideal*, *matrix*, *relation*, *representation*, and *semigroup*. One-sided concepts are listed under the stem word.
Page numbers which include a reference to the exercises are printed in italics. The dots and dashes stand for previous italicised terms (possibly of several words), the dashes being used for the earlier, and the dots for the later, terms.
For pure symbols, see the list of notation on page xiii.

ISBN 978-0-8218-0271-7

9 780821 802717

SURV/7.1