

# Constrained portfolio optimization

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## 1 Mathematical description

We need to solve the minimization problem at a given time  $t$

$$\min_{x_i \in \{-1, 0, 1\}} \left[ \lambda \sum_{ij=1}^n A_{ij}(t) x_i x_j - (1 - \lambda) \sum_{i=1}^n b_i x_i \right] = \min_{x \in \{-1, 0, 1\}^n} [\lambda x^T A x - (1 - \lambda) b \cdot x] \quad (1)$$

with the additional constraint

$$\sum_{i=1}^n x_i = K . \quad (2)$$

The variables encode the following information

- Each asset  $i$  has an associated position  $x_i \in \{-1, 0, 1\}$ , corresponding to short, hold or buy.
- $\lambda \in [0, 1]$  is the risk aversion. We can set it to  $1/2$  such that the problem becomes  $\min_{x \in \{-1, 0, 1\}^n} (x^T A x - b \cdot x)$ .
- $A_{ij}$  is the covariance describing the correlation between assets  $i$  and  $j$ .
- $b_i$  are the expected returns of asset  $i$ .  $b_i$  can be simply modeled as  $b_i = \bar{r}_i(t) = \sum_{\tau=1}^t r_i(\tau)/t$ , the time averaged historical returns, with  $r_i(t) \in \mathbb{R}$  the historical return of asset  $i$  at time  $t$ .
- $K \in \mathbb{Z}$  is the net position of the portfolio, interpreted as the net confidence in the stock market.

The returns can be calculated as

$$r_i(t) = \frac{p_i(t) - p_i(t-1)}{p_i(t-1)} , \quad (3)$$

where  $p_i(t)$  is the price of asset  $i$  at time  $t \in \{0, 1, 2, \dots\}$ . The covariance matrix can be derived from historical data as

$$A_{ij}(t) = \frac{1}{t-1} \sum_{\tau=1}^t [r_j(\tau) - \bar{r}_j(t)] [r_i(\tau) - \bar{r}_i(t)] , \quad (4)$$

## 2 Finding a corresponding dataset

We will consider the following use cases:

- Yearly time series of publicly traded companies in Qatar

## 3 Binary encoding

The ternary variables  $x_i$  can be encoded into two binary variables  $u_i^1, u_i^2 \in \{-1, 1\}$  in the following way

$x_i$	$u_i^1$	$u_i^2$
-1	1	-1
0	1	1
0	-1	-1
1	-1	1

Table 1:

Equation (1) becomes then

$$\min_{x \in \{-1,0,1\}^n} (x^T A x + b \cdot x) = \min_{u \in \{-1,1\}^{2n}} (u^T B u + c \cdot u) \quad (5)$$

with  $u = [u_1^1 \ u_1^2 \ u_2^1 \ u_2^2 \dots]$  and

$$B = \frac{A}{4} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad , \quad c = \frac{b}{2} \otimes \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (6)$$