Constrained portfolio optimization

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1 Mathematical description

We need to solve the minimization problem at a given time t

$$\min_{x_i\in\{-1,0,1\}}\left[\lambda\sum_{ij=1}^{n}A_{ij}(t)x_ix_j-(1-\lambda)\sum_{i=1}^{n}b_ix_i\right]=\min_{x\in\{-1,0,1\}^n}\left[\lambda x^TAx-(1-\lambda)b\cdot x\right] \tag{1}$$

with the additional constraint

$$\sum_{i=1}^{n}x_i=K\ . \tag{2}$$

The variables encode the following information

- Each asset i has an associated position $x_i\in\{-1,0,1\}$, corresponding to short, hold or buy.
- $\lambda\in[0,1]$ is the risk avertion. We can set it to 1/2 such that the problem becomes $\min_x\left(x^TAx-bx\right)$.
- A_{ij} is the covariance describing the correlation between assets i and j.
- b_i are the expected returns of asset i. b_i can be simply modeled as $b_i=\overline{r_i}(t)=\sum_{\tau=1}^{t}r_i(\tau)/t$, the time averaged historical returns, with $r_i(t)\in\mathbb{R}$ the historical return of asset i at time t.
- $K\in\mathbb{Z}$ is the net position of the portfolio, interpreted as the net confidence in the stock market.

The returns can be calculated as

$$r_i(t)=\frac{p_i(t)-p_i(t-1)}{p_i(t-1)}\ , \tag{3}$$

where $p_i(t)$ is the price of asset i at time $t\in\{0,1,2,\dots\}$. The covariance matrix can be derived from historical data as

$$A_{ij}(t)=\frac{1}{t-1}\sum_{\tau=1}^{t}\left[r_j(\tau)-\overline{r_j}(t)\right]\left[r_i(\tau)-\overline{r_i}(t)\right]\ , \tag{4}$$

2 Finding a corresponding dataset

We will consider the following use cases:

- Yearly time series of publicly traded companies in Qatar

3 Binary encoding

The ternary varibles x_i can be enconded into two binary variables $u_i^1,u_i^2\in\{-1,1\}$ in the following way

x_i	u_i^1	u_i^2
-1	1	-1
0	1	1
0	-1	-1
1	-1	1

Table 1:

Equation (1) becomes then

$$\min_{x \in \{-1,0,1\}^n} \left(x^T A x + b \cdot x \right) = \min_{u \in \{-1,1\}^{2n}} \left(u^T B u + c \cdot u \right) \tag{5}$$

with $u = [u_1^1 \ u_1^2 \ u_2^1 \ u_2^2 \ldots]$ and

$$B = \frac{A}{4} \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad , \quad c = \frac{b}{2} \otimes \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{6}$$