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1991 Nonlinearity 4 543

(http://iopscience.iop.org/0951-7715/4/2/015)

Nonlinearity 4 (1991) 543–566. Printed in the UK

# The problem of quantum chaos in a kicked harmonic oscillator

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Received 28 June 1990
Accepted by Y Sinai

**Abstract.** Quantum chaos in a kicked harmonic oscillator is analysed. Under the condition of strong chaos of the classical limit, the time of classical description of quantum averages is shown to be of the order of $n_\hbar \sim \ln(1/\hbar)$. In the case of weak classical chaos this time considerably increases: $n_1 \sim 1/\hbar \gg n_\hbar$. The properties of symmetry for quasi-energy functions are discussed for different parameters of the system. The results of numerical analysis for the dependence of oscillator energy on time are given. The possibility of delocalization of quasi-energy eigenfunctions is discussed.

PACS numbers: 0365, 0545

## 1. Introduction

The problems of quantum chaos are widely discussed in modern literature. On the one hand, one often refers to the problems of quantum chaos as statistical properties of quantum systems interacting with a random field of photons. On the other hand the problems of quantum chaos arise in analogy with classical nonlinear systems where stochasticity (chaos) may not be due to the effect of random forces but to the peculiarities of complicated nonlinear dynamics in phase space. In the latter case the problem mentioned is usually formulated in terms of the peculiarities of quantum systems which, in the classical limit (i.e. at $\hbar = 0$), possess a property of mixing [1, 2]. Unfortunately, at present we cannot show exactly which of the properties of a classical system corresponds to quantum chaos. In classical systems chaos is associated with instability of mixing trajectories in phase space and destruction of independent integrals of motion. However in the quantum case there is no notion of trajectory, and in general there are no direct results on the properties of quantum systems with the number of independent integrals of motion (quantum numbers) less than the number of degrees of freedom. Therefore one should not ignore the possibility of the existence of not quite 'ordinary' properties in those quantum systems which have no classical limits at all. In particular, there is a suggestion that the destruction of quantum numbers might be accompanied by the appearance of statistical properties in the energy spectrum [3]. There are certain

0951-7715/91/010543 + 24$02.50  

quasiclassical estimates for statistical properties of these spectra. These properties manifest themselves in the appearance of repulsion between levels if there is chaos in the classical limit [2, 4–6]. An analogous suggestion has been also put forward in [1].

Another aspect of the problem of quantum chaos is related to investigation of the dynamics of quantum systems. The first model considered from this point of view was a kicked rotator [7] with the Hamiltonian

$$\hat{H} = -\frac{\gamma}{2}\frac{\partial^2}{\partial\vartheta^2} + \varepsilon\cos\vartheta \sum_{n=-\infty}^{\infty} (t - nT) \tag{1.1}$$

where $\gamma = \hbar^2/M_0$, $M_0$ is the moment of inertia of the rotator and $\varepsilon$ is a perturbation amplitude. Many papers deal with the investigation of the model because it represents a quantum analogue of the well known Chirikov–Taylor map.

An important result related to the problem of quantum chaos has been found already in [7]. This result showed the suppression of energy diffusion of the rotator in comparison with the corresponding classical limit. Later on, this effect was interpreted as the result of the localization of wavefunctions in quasi-energy space [8] which is similar to one-dimensional Anderson localization. The development of this idea and numerical analysis have proved the existence of a close relationship between saturation of the diffusion in chaotic quantum systems and localization length of wavefunctions [9–11].

However, the saturation of diffusion due to quantum effects is not apparently a universal manifestation of quantum chaos and may be absent in more complicated models than (1.1) [12]. The existence of a new timescale $\tau_\hbar$ may be a more universal effect under the conditions of quantum chaos. This timescale has been obtained in [13] and discussed in detail in [14–16]. Its meaning is as follows.

Consider a quantum nonlinear oscillator in the quasiclassical state where the wavefunction is localized in action $I$ in the region $\delta I$

$$\hbar \ll \delta I \ll I \tag{1.2}$$

and in phase $\vartheta$ in the region $\delta\vartheta$:

$$\delta\vartheta \ll 2\pi. \tag{1.3}$$

The existence of quantum dispersion leads to spreading of wavepackets for a time interval of the order

$$\tau_q = \left(\left|\frac{\mathrm{d}\omega}{\mathrm{d}I}\right| \delta I\right)^{-1} > \left(\hbar \left|\frac{\mathrm{d}\omega}{\mathrm{d}I}\right|\right)^{-1} \sim I/\hbar\omega \tag{1.4}$$

where $\omega = \omega(I)$ is a non-linear frequency of the oscillator. Due to perturbation the spreading may be much faster, especially when, in the classical limit, strong chaos occurs. Then for a shorter time $t_0$ the condition (1.3) is violated and the packet spreads over the phase to $\sim 2\pi$. Let us evaluate this time on the basis of the condition

$$\delta\vartheta(t_0) = \delta\vartheta_0 \exp(t_0\lambda) = 2\pi$$

where $\lambda$ is the characteristic Lyapunov exponent. Hence

$$t_0 = \frac{1}{\lambda}\ln(2\pi/\delta\vartheta_0).$$

But due to the uncertainty relation $\delta\vartheta_0\,\delta I_0 \gtrsim \hbar$. Therefore a quantum cell $\delta\vartheta_0\,\delta I_0 \sim \hbar$ spreads during the time

$$\tau_\hbar = \frac{1}{\lambda}\ln(2\pi\,\delta I_0/\hbar).$$

Owing to (1.2) the value $\delta I_0/\hbar \gg 1$. Due to the logarithm $\tau_\hbar$ is much less than $\tau_q$ in (1.4). It arises because of the simultaneous influence of the effects of quantum dispersion and the classical stochastic instability in phase space.

Of course, such a simple consideration is not enough, especially for the initial states, for which $\delta I_0 \ll \hbar$. In fact the existence of time

$$\tau_\hbar \propto \text{constant} \times \ln(\text{constant}/\hbar) \tag{1.5}$$

has been proved for some typical models of chaos in [13, 14]. Since the strong chaos in the classical dynamics often arises under perturbation larger than a certain critical value it means that the dynamics of the quantum system will be very sensitive to a perturbation amplitude. In particular, the dynamics of some quantum mechanical averages must sharply differ from the quasiclassical dynamics. This leads to the fact that the existence of the logarithmic time $\tau_\hbar$ may be one of the most universal and fundamental characteristics of quantum chaos accessible to experimental observation.

This important effect of quantum chaos has not been discussed so far. But recently more and more papers have appeared where the characteristic timescale (1.5) is described in some form [17–19]. Our paper deals with the investigation of quantum chaos in a non-stationary quantum system. As an example of such a system we have chosen a linear oscillator excited by a periodic sequence of $\delta$-pulses:

$$\hat{H} = \frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}m_0\omega_0^2 x^2 + \varepsilon\cos(k_0 x)\sum_{n=-\infty}^{\infty}\delta(t - nT_0). \tag{1.6}$$

The difference of (1.6) from (1.1) is in the fact that the unperturbed system ($\varepsilon = 0$) is linear. At the resonance

$$\omega_0 T_0 = 2\pi r/q \tag{1.7}$$

($r$, $q$ are integers) the effect of perturbation is especially strong. Recent investigations of the system (1.6) in the classical limit ($\hbar = 0$) have led to the discovery of an unlimited stochastic web with symmetry of order $q$ [20]. If one ignores the trivial cases $q = 1, 2$ at $q = 3, 4, 6$ the web will have a periodic structure with the symmetry corresponding to a crystalline lattice. For all other values of $q$ it will have a quasicrystalline structure. These unusual properties of classical chaos make the system (1.6) a new and interesting object for investigating quantum chaos, since the existence of the web affects quantum diffusion of particles.

In the present paper we mainly compare classical and quantum dynamics of the system (1.6) in the region of the parameters of quantum chaos. It is structured as follows. In section 2 we construct discrete quantum maps for a wavefunction of the system. Section 3 briefly describes the properties of the system (1.6) in the classical limit; here also some parameters are introduced which are necessary for further analysis. In section 4 we obtain a recurrence equation for a generating function which allows us to calculate arbitrary time averages. Section 5 deals with the method of determining parameters of the system (1.6) for which at finite times quantum dynamics is similar to classical dynamics. In section 6 the time of quasiclassical description is determined under the condition of strong chaos in the classical limit. This time is shown to be logarithmically small as a function of the quasiclassical parameter $n_\hbar \sim \ln(1/\hbar)$. Section 7 gives a comparative analysis of the classical and quantum dynamics under the condition of weak chaos in the classical limit. In this case the times of classical approach are shown to have a power dependence on the quasiclassical parameter and may considerably exceed $n_\hbar$. Section 8 contains an analysis of the symmetry of quasifunctions of the system (1.6) when the interaction between oscillator and external field is resonant. Section 9 gives a numerical analysis of the dynamics of the system (1.6) in a particular resonance case. In the last section we present some additional remarks concerning the main results and discuss the possibility of further investigations of the model (1.6).

## 2. Quantum maps

To investigate the properties of the system (1.6) it is convenient to pass from the Schrödinger equation to the finite difference equation for a wavefunction $\psi(t)$ by relating its values between two successive $\delta$-pulses, e.g. the $n$th and $(n+1)$th. The relation may be represented in the form

$$\psi_{n+1} = \hat{T}\psi_n \tag{2.1}$$

where

$$\psi_n = \psi(t_n - 0) \qquad \psi_{n+1} = \psi(t_{n+1} - 0) \qquad t_n = nT_0$$

and the operator $\hat{T}$ defines this map.

For convenience we introduce the following dimensionless variables:

$$\begin{aligned} &T = \omega_0 T_0 \qquad \varkappa = \varepsilon/\hbar \qquad k = k_0\tau \\ &\tau = (\hbar/m_0\omega_0)^{1/2} \qquad \eta = x/\tau \qquad \bar{t} = \omega_0 t. \end{aligned} \tag{2.2}$$

For these variables the Hamiltonian (1.6) takes the following dimensionless form:

$$\mathscr{H} = \hat{H}/\hbar\omega_0 = -\frac{1}{2}\frac{\partial^2}{\partial\eta^2} + \tfrac{1}{2}\eta^2 + \varkappa\cos(k\eta)\sum_{n=-\infty}^{\infty}\delta(\bar{t} - nT). \tag{2.3}$$

Now let us introduce operators of creation and annihilation $a^+$, $a([a, a^+] = 1)$ using the usual formulae:

$$\eta = \frac{1}{\sqrt{2}}(a^+ + a) \qquad p = \frac{\mathrm{i}}{\sqrt{2}}(a^+ - a) = -\mathrm{i}\,\partial/\partial\eta \tag{2.4}$$

Then the expression (2.3) takes the form

$$\mathscr{H} = a^+a + \tfrac{1}{2} + \varkappa\cos[(k/\sqrt{2})(a^+ + a)]\sum_{n=-\infty}^{\infty}\delta(\bar{t} - nT). \tag{2.5}$$

Using (2.5) we can easily find the evolution operator $\hat{T}$ introduced in (2.1) from the expression

$$\begin{aligned} \hat{T} &= \exp\left\{-\mathrm{i}\int_{\bar{t}_n+0}^{\bar{t}_{n+1}-0}\mathscr{H}\,\mathrm{d}\bar{t}\right\}\exp\left\{-\mathrm{i}\int_{\bar{t}_n-0}^{\bar{t}_n+0}\mathscr{H}\,\mathrm{d}\bar{t}\right\} \\ &= \exp\{-\mathrm{i}T(a^+a + \tfrac{1}{2})\}\exp\{-\mathrm{i}\varkappa\cos[(k/\sqrt{2})(a^+ + a)]\}. \end{aligned} \tag{2.6}$$

The operator $\hat{T}$ defines a quantum map. From its meaning the wavefunction $\psi_n$ at time $(\bar{t}_n - 0)$ is found from the equation

$$\psi_n = \hat{T}^n \psi_0 \tag{2.7}$$

where $\psi_0$ is the initial condition. The operator $\hat{T}$ can be also used to determine the properties of the quasi-energy eigenfunctions.

## 3. The classical limit

In what follows we need information on the properties of the system (2.5) or (1.6) in the classical limit $\hbar = 0$ [20].

We replace operators $a^+$, $a$ in (2.5) by their $c$-numbers:

$$\begin{aligned} a &\to \alpha = (I/\hbar)^{1/2} \exp(-\mathrm{i}\vartheta) \\ a^+ &\to \alpha^* = (I/\hbar)^{1/2} \exp(\mathrm{i}\vartheta) \end{aligned} \tag{3.1}$$

where $(I, \vartheta)$ are the classical action and oscillator phase. The expression (2.5) for the Hamiltonian takes the form:

$$\mathscr{H}_{\mathrm{cl}} = \alpha^* \alpha + \varkappa \cos[(k/\sqrt{2})(\alpha^* + \alpha)] \sum_{n=-\infty}^{\infty} \delta(\bar{t} - nT). \tag{3.2}$$

The map for the variables

$$\alpha_n = \alpha(\bar{t}_n - 0) \qquad \alpha_n^* = \alpha_n^*(\bar{t}_n - 0)$$

has the form

$$\alpha_{n+1} = \mathrm{e}^{-\mathrm{i}T}\{\alpha_n + \mathrm{i}\varkappa(k/\sqrt{2}) \sin[(k/\sqrt{2})(\alpha_n^* + \alpha_n)]\}. \tag{3.3}$$

The expression for $\alpha_{n+1}^*$ is obtained from (3.3) by means of complex conjugation. Though for convenience $\mathscr{H}_{\mathrm{cl}}$ is normalized to Planck's constant $\hbar$ it is absent in the finite expressions.

The analysis of equations (3.3) made in [20] has shown that for arbitrary values of the parameter

$$K_H = \varkappa k^2 = \Omega_0^2 T_0/\omega_0 \qquad \Omega_0^2 = \varepsilon k_0^2/m_0 T_0 \tag{3.4}$$

and under the condition of the resonance

$$T = \omega_0 T_0 = 2\pi r/q \tag{3.5}$$

($r$, $q$ are integers) there exists stochastic dynamics of particles in a certain region of phase space. The frequency $\Omega_0$ introduced in (3.4) means the frequency of small oscillations in a perturbed wave.

The regions of chaos exist at arbitrary small values of $\varepsilon$. In the phase space of the system they form a network of channels with the symmetry of the order $q$ which is called a stochastic web. An example of the web for $q = 4$ is shown in figure 1. Along the channels of the web there is unlimited transport of particles. This process, under the condition

$$\Delta I_n = |I_{n+1} - I_n| \ll I_n \tag{3.6}$$

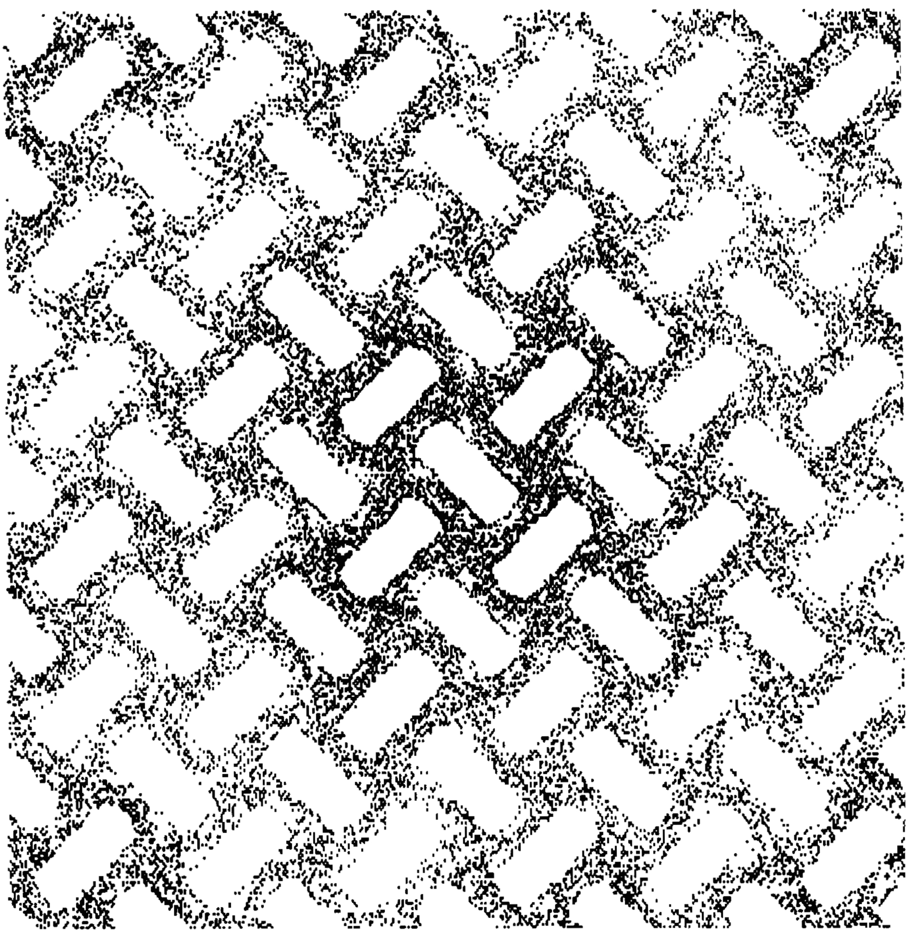

**Figure 1.** An example of the stochastic web generated by the classical map (3.3); $K_H = 2$; $T = \pi/2$ ($r = 1$, $q = 4$).

and $K_H \gg 1$, has the character of a usual diffusion. According to (3.3) the inequality (3.6) means that

$$\Omega_0^2 T_0/k_0 v_0 = K_H(\omega_0/k_0 v_0) \ll 1 \tag{3.7}$$

where $v_0$ is the characteristic velocity of a particle.

The existence of the stochastic web at arbitrary values of $\varepsilon$ must essentially influence the dynamic properties and the character of diffusion of quantum particles. This peculiarity of the quantum dynamics will be discussed below.

## 4. Equation for a generating function

The rather formal character of this section is related to constructing a recurrence relation for the following generating function

$$\mathcal{R}_n(\gamma \mid \beta) = \langle \psi_n | \, e^{\gamma a^+} e^{\beta a} \, | \psi_n \rangle \tag{4.1}$$

where the brackets $\langle \ldots \rangle$ mean a quantum mechanical average in the Schrödinger representation and $\gamma$ and $\beta$ are arbitrary values. Using (4.1) one can calculate an arbitrary average value at discrete time $n$:

$$\langle \psi_n | (a^+)^l a^s | \psi_n \rangle = \frac{\partial^l}{\partial \gamma^l} \frac{\partial^s}{\partial \beta^s} \mathcal{R}_n(\gamma \mid \beta)|_{\gamma=\beta=0} \tag{4.2}$$

Now we obtain a recurrence relation connecting functions $\mathcal{R}_n$ and $\mathcal{R}_{n-1}$. For this we use (2.1) and represent (4.1) as

$$\mathcal{R}_n(\gamma \mid \beta) = \langle \psi_{n-1} | \hat{T}^+ e^{\gamma a^+} e^{\beta a} \hat{T} | \psi_{n-1} \rangle. \tag{4.3}$$

By means of the explicit form of the operator $\hat{T}$ (2.6) and ordering of the operator functions based on the properties of non-commutating operators (see the appendix) we obtain for (4.1) the following recurrence relation:

$$\begin{aligned} \mathcal{R}_n(\gamma \mid \beta) = \sum_{m=-\infty}^{\infty} & J_m\{2\varkappa \sin[(k/\sqrt{2})(\beta e^{-iT} - \gamma e^{-iT})]\} \\ & \times \exp\left\{-\frac{m^2k^2}{4} + \frac{imk}{2\sqrt{2}}(\gamma e^{iT} + \beta e^{-iT})\right\} \\ & \times \mathcal{R}_{n-1}(\gamma e^{iT} + imk/\sqrt{2} \mid \beta e^{-iT} + imk/\sqrt{2}) \end{aligned} \tag{4.4}$$

where $J_m(z)$ is the Bessel function.

It is clear from (4.2) that the averages without phase correlation are obtained at $l = s \neq 0$. And, conversely, the average value taking into account the phase correlation can be found from (4.1), e.g. at $\beta = -\gamma^*$, i.e. from the expression:

$$\mathcal{R}_n(\gamma \mid -\gamma^*) = \langle \psi_n | e^{\gamma a^+} e^{-\gamma a} | \psi_n \rangle = \mathcal{D}_n(\gamma) e^{|\gamma|^2/2}. \tag{4.5}$$

Using (4.4), (4.5) we get a recurrence relation for the function $\mathcal{D}_n(\gamma)$:

$$\mathcal{D}_n(\gamma) = \sum_{m=-\infty}^{\infty} (-1)^m J_m\{2\varkappa \sin[(k/\sqrt{2})(\gamma e^{iT} + \gamma^* e^{-iT})]\} \mathcal{D}_{n-1}(\gamma e^{iT} + imk/\sqrt{2}). \tag{4.6}$$

Assuming that

$$\gamma = ik/\sqrt{2} \tag{4.7}$$

from (4.5) and definitions of (2.4), (2.2) we obtain

$$\mathcal{D}_n(ik/\sqrt{2}) = \langle \psi_n | e^{ik_0 x} | \psi_n \rangle. \tag{4.8}$$

Expression (4.8) corresponds to an average value of the simplest correlator characterizing phase properties of the wavepacket.

Now let us derive the classical limit of (4.6). For this we should take into account that according to (2.2), (2.4) and (2.7) for $k$, $\varkappa$, $a$, $a^+$ and $\gamma$ we have:

$$\gamma \sim k \sim \hbar^{1/2} \qquad \varkappa \sim 1/\hbar.$$

Using this and (4.6) we find at $\hbar \to 0$:

$$\mathcal{D}_n^{(cl)}(\gamma) = \sum_{m=-\infty}^{\infty} (-1)^m J_m\{(\varkappa k/\sqrt{2})(\gamma e^{iT} + \gamma^* e^{-iT})\} \mathcal{D}_{n-1}^{(cl)}(\gamma e^{iT} + imk/\sqrt{2}). \tag{4.9}$$

Below we shall give a quasiclassical estimate of the correlator $\mathcal{D}_n(ik/\sqrt{2})$ (4.8) and compare it with the classical limit $\mathcal{D}_n^{(cl)}(ik/\sqrt{2})$.

## 5. Region of parameters of the quasiclassical consideration

To write down the explicit expression (4.8) with the use of the recurrence formula (4.6) we shall introduce the following notation:

$$
\begin{aligned}
\gamma_0 &= \gamma = \mathrm{i}k/\sqrt{2} \\
\gamma_1 &= \gamma_0 \mathrm{e}^{\mathrm{i}T} + \mathrm{i}m_1 k/\sqrt{2} = \mathrm{i}(k/\sqrt{2})(\mathrm{e}^{\mathrm{i}T} + m_1); \\
&\vdots \\
\gamma_l &= \gamma_{l-1}\mathrm{e}^{\mathrm{i}T} + \mathrm{i}m_l k/\sqrt{2} \\
&= \mathrm{i}(k/\sqrt{2})(\mathrm{e}^{\mathrm{i}lT} + m_1 \mathrm{e}^{\mathrm{i}(l-1)T} + \ldots + m_l) \qquad (l = 1, \ldots, n)
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
\xi_1 &= -(k/2\sqrt{2})(\gamma_1 + \gamma_1^*) = (k^2/2)\sin T \\
&\vdots \\
\xi_l &= -(k/2\sqrt{2})(\gamma_l + \gamma_l^*) = (k^2/2)[\sin(lT) \\
&\quad + m_1 \sin(l-1)T + \ldots + m_{l-1}\sin T] \qquad (l = 1, \ldots, n; m_0 = 0).
\end{aligned}
\tag{5.2}
$$

Taking into account (4.6), (4.7), (5.1) and (5.2) the expression (4.8) can be represented in the following form:

$$
\mathscr{D}_n(\mathrm{i}k/\sqrt{2}) = \sum_{m_1,\ldots,m_n=-\infty}^{\infty} J_{m_1}(2\varkappa \sin \xi_1) J_{m_2}(2\varkappa \sin \xi_2) \ldots J_{m_n}(2\varkappa \sin \xi_n) \mathscr{D}_0(\gamma_n) \tag{5.3}
$$

where $\mathscr{D}_0(\gamma_n)$ is the initial condition. According to (4.5) it can be represented as:

$$
\mathscr{D}_0(\gamma_n) = \mathrm{e}^{-|\gamma_n|^2/2} \langle \psi_0 | \, \mathrm{e}^{\gamma_n a^+} \mathrm{e}^{-\gamma_n^* a} \, | \psi_0 \rangle \tag{5.4}
$$

where $\psi_0$ is an arbitrary initial state. Operators $a^+$, $a$ in (5.4) are determined at the initial time $n = 0$, and their action on $\psi_0$ is known.

In the classical limit it is necessary to replace $\sin \xi_l$ in (5.3) by $\xi_l$ and the expression (5.3) will be as follows (see also (4.9)):

$$
\begin{aligned}
\mathscr{D}_n^{(\mathrm{cl})}(\mathrm{i}k/\sqrt{2}) = \sum_{m_1,\ldots,m_n=-\infty}^{\infty} & J_{m_1}\{K_H \sin T\} J_{m_2} \\
& \times \{K_H[\sin(2T) + m_1 \sin T)\} \ldots J_{m_n} \\
& \times \{K_H[\sin(nT) + m_1 \sin(n-1)T + \ldots + m_{n-1}\sin T]\} \mathscr{D}_0^{(\mathrm{cl})}(\gamma_n)
\end{aligned}
\tag{5.5}
$$

The initial condition $\mathscr{D}_0^{(\mathrm{cl})}(\gamma_n)$ in the classical limit is determined by the following expression:

$$
\mathscr{D}_0^{(\mathrm{cl})} = \int_{-\infty}^{\infty} \mathrm{d}^2\alpha P(\alpha, \alpha^*) \mathrm{e}^{\gamma_n \alpha^* - \gamma_n^* \alpha} \qquad (\mathrm{d}^2\alpha = \mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha)) \tag{5.6}
$$

where $p(\alpha, \alpha^*)$ is the initial distribution function. According to (2.4) it may be expressed through the inital coordinates and oscillator momentum or, according to (3.1), through the initial condition and phase. Moreover, comparing quantum (5.3) and classical (5.5) expressions we need an additional requirement concerning the transformation of the initial condition (5.4) at $\hbar \to 0$ to the classical initial condition (5.6). This correspondence is known to be realized by different methods, and in what follows we shall consider the requirement to be satisfied.

It is seen from (5.3) and (5.2) that alongside with the initial condition (5.4) quantum dynamics is determined by three independent parameters $k$, $\varkappa$, $T$. It is convenient to choose the independent parameters

$$K_H = \varkappa k^2 \qquad \varkappa;\ T \tag{5.7}$$

where $T$ and $K_H$ are classical parameters, and $\varkappa$ is a quantum parameter. From (3.3) it follows that we can estimate the number of quanta absorbed by the oscillator during one kick as

$$\Delta I/\hbar \sim \varkappa\sqrt{Ik_0^2/\omega_0 m_0} \tag{5.8}$$

where $I$ is the characteristic action. The quasiclassical condition for perturbation means $\Delta I/\hbar \gg 1$.

Now let us introduce a 'quantum boundary' (QB) for parameters (5.7). It separates the region of parameters (5.7) where at least at finite intervals of time $n \geqslant 1$ the quantum dynamics can be approximated by the classical dynamics from the region of parameters with the purely quantum dynamics for all times $n \geqslant 1$. It is seen from (5.3) that the dynamics is essentially quantum at times larger than $T_0$ under the condition $|\chi_1| \geqslant 1$ which according to (5.2) gives

$$\frac{k^2}{2}|\sin T| = \frac{K_H}{2\varkappa}|\sin T| \geqslant 1. \tag{5.9}$$

In the region of parameters (5.7), where the inverse inequality

$$\frac{K_H}{2\varkappa}|\sin T| \ll 1 \tag{5.10}$$

is satisfied, the dynamics of the quantum system approximately coincides with that of the classical system. The time when the coincidence occurs will be obtained in the following section.

## 6. Time intervals of the quasiclassical description for strong chaos

Earlier we introduced a quantum correlator $\mathscr{D}_n$ in the formula (5.3) and its classical limit $\mathscr{D}_n^{(\mathrm{cl})}$ in formula (5.5). Comparison of both expressions allows us to find when the quantum dynamics approximates the classical one. The convenience of this approach is due to the fact that we analyse and compare not the finite solutions but their formal iteration representations. Such an approach makes the solution of the problem much simpler.

Consider first the case of strong chaos (in the classical limit), i.e. inequality $K_H \gg 1$. The applicability of the classical expression (5.5) for describing the quantum dynamics at finite times $n \gg 1$ means that in (5.3) at least the condition

$$|\chi_l| \ll 1 \qquad (l = 1, \ldots, n) \tag{6.1}$$

must be satisfied, and all the functions $\sin \xi_l$ in (5.3) may be approximately replaced by $\xi_l$. Assuming (6.1) to be satisfied and taking into account that at

$$|m_l| \gg 2\varkappa\, |\xi_l| = K_H\, |\sin(lT) + m_1 \sin(l-1)T + \ldots + m_{l-1} \sin T| \tag{6.2}$$

the Bessel functions $J_{m_l}(2\varkappa \sin \xi_l)$ are exponentially small, we have from (6.2) an estimate for the characteristic number of terms $m_l$ ($l = 1, \ldots, n$) in (5.3):

$$
\begin{aligned}
&|m_1| \sim 2\varkappa\,|\xi_1| = K_H\,|\sin T| \\
&|m_2| \sim 2\varkappa\,|\xi_2| = K_H\,|\sin 2T + m_1 \sin T| \\
&\vdots \\
&|m_n| \sim 2\varkappa\,|\xi_n| = K_H\,|\sin(nT) + m_1 \sin(n-1)T + \ldots + m_{n-1} \sin T|.
\end{aligned}
\tag{6.3}
$$

It is clear from (6.3) that the estimate of the number of terms in (5.3) as well as of the quantum dynamics of the correlator $\mathcal{D}_n(\mathrm{i}k/\sqrt{2})$ mostly depends not only on the parameter $K_H$ but also on the value $\sin T$. So we shall consider two limiting cases:

$$K = K_H\,|\sin T| \gg 1 \tag{6.4}$$

and weak chaos

$$K \ll 1. \tag{6.5}$$

First let us discuss (6.4). From (6.3) we have

$$
\begin{aligned}
&|m_1| \sim K_H\,|\sin T| = K \gg 1 \\
&|m_2| \sim K_H\,|\sin T|\,|m_1| \sim K^2 \\
&\vdots \\
&|m_n| \sim K_H\,|\sin T|\,|m_{n-1}| \sim K^n = \exp(n \ln k).
\end{aligned}
\tag{6.6}
$$

From (6.3), (6.6) one can estimate the value $|\xi_n|$:

$$|\xi_n| \sim |m_n|/2\varkappa \sim \frac{\exp(n \ln K)}{2\varkappa}. \tag{6.7}$$

From the condition (6.1) with an account of (6.7) one can estimate the time of the quasiclassical description $n_\hbar$ (i.e. the time of quasiclassical approach):

$$n \lesssim n_\hbar = \frac{\ln(2\varkappa)}{\ln K} \qquad (K \gg 1). \tag{6.8}$$

Thus, in the region of parameters (5.10) the dynamics of the quantum system at finite times $n$ approximately coincides with dynamics of the classical system, and under the additional condition (6.4) the characteristic time of the classical description $n_\hbar$ has the order of (6.8), that is in agreement with the estimate (1.5).

One should also note that in the case of (6.8) the quantum chaos in the initial system (1.6) is strong. Under the conditions of $K_H \gg 1$ and $|\sin T| \sim 1$ one can refer, to this case in particular, resonances (3.5) with $q = 3$–$7$ studied in detail in the classical limit in [20]. In this case the motion of a classical particle in the stochastic region of phase space is of diffusion character and the stochastic region is characterized by different symmetries depending on $q$. Now let us analyse (6.5).

## 7. Quantum dynamics in the region $K \ll 1$

For $K \ll 1$ chaos in the classical limit is realized in the region of phase space having many boundaries of the type 'order–chaos' and a considerable portion of the stable component. It will be shown below that due to this fact the difference between the quantum and classical dynamics is manifested at times considerably exceeding $n_\hbar$.

When analysing the dynamics of the correlator (5.3) for $K \ll 1$ we deal with the two limiting cases:

$$K_H \ll 1 \qquad \sin T \lesssim 1 \tag{7.1}$$

$$K_H \gtrsim 1 \qquad \sin T \ll 1. \tag{7.2}$$

At first we consider (7.1). For the sake of simplicity we shall confine ourselves to the resonance $T = \pi/2$ ($q = 4$). From (5.2) we have

$$\begin{aligned}
&\xi_1 = K_H/2\varkappa \\
&\xi_2 = (K_H/2\varkappa)m_1 \\
&\xi_3 = (K_H/2\varkappa)(-1 + m_2) \\
&\xi_4 = (K_H/2\varkappa)(-m_1 + m_3) \\
&\quad\vdots \\
&\xi_n = (K_H/2\varkappa)[\sin(n\pi/2) + m_1 \sin(n-1)/2 + \ldots + m_{n-1}].
\end{aligned} \tag{7.3}$$

Let the condition $|\xi_1| \ll 1$ hold which, according to the results of section 5, is the necessary condition for the classical description at finite times. From (5.3), taking into account (7.1) and (7.3), we estimate the maximum $m_l$ ($l = 1, \ldots, n$) in (5.3)

$$\begin{aligned}
&|m_1| \sim 2\varkappa\, |\xi_1| = K_H \ll 1 \\
&|m_2| \sim 2\varkappa\, |\xi_2| \sim K_H^2 \\
&|m_3| \sim 2\varkappa\, |\xi_3| \sim K_H + K_H^3 \\
&|m_4| \sim 2\varkappa\, |\xi_4| \sim 2K_H^2 + K_H^5 \\
&\vdots \\
&|m_{2n-1}| \sim K_H + (2n-3)K_H^3 + \mathrm{O}(K_H^5) \qquad (n = 2, \ldots) \\
&|m_{2n}| \sim nK_H^2 + \mathrm{O}(K_H^4) \qquad (n = 1, \ldots).
\end{aligned} \tag{7.4}$$

The estimate of the value $|\xi_n|$ from (7.4) gives

$$|\xi_m| \sim \begin{cases} nK_H^3/\varkappa & (m = 2n-1) \\ nK_H^2/2\varkappa & (m = 2n). \end{cases} \tag{7.5}$$

From the condition $|\xi_m| \ll 1$ we obtain that the timescale of the classical description of the correlator dynamics (5.3) is

$$n < n_1 = 2\varkappa/K_H^2. \tag{7.6}$$

Thus, in the case of (7.1) the times of classical approach for describing the dynamics of the correlator (5.3) appear to be of the order $1/\hbar$, and this essentially exceeds logarithmically small times $n_\hbar$ for the case of strong chaos (6.4).

Now let us analyse (7.2). From (5.2) we have

$$\begin{aligned}
&\xi_1 \sim K/2\varkappa \ll 1 \qquad |m_1| \sim K \ll 1 \\
&\xi_2 \sim (K/2\varkappa)[2 + \mathrm{O}(K)] \qquad |m_2| \sim 2K + \mathrm{O}(K^2) \\
&\vdots \\
&\xi_n \sim (K/2\varkappa)[n + \mathrm{O}(K)] \qquad |m_n| \sim nK + \mathrm{O}(K^2) \qquad (nT \ll 1).
\end{aligned} \tag{7.7}$$

We require that the condition of the classical description on times $n$ be satisfied, which leads to the inequality

$$\xi_n \sim nK/2\varkappa \ll 1. \tag{7.8}$$

Then we have

$$n < n_2 = 2\varkappa/K. \tag{7.9}$$

However, one should bear in mind that the estimates (7.7) are also obtained under the condition $nT \ll 1$, therefore the time of the classical description in the case (7.2) must be estimated based on

$$n_2 = \min(2\varkappa/K; 1/T). \tag{7.10}$$

From (7.10) it follows that in the semiclassical region

$$2\varkappa/K \gg 1/T \gg 1 \tag{7.11}$$

the semiclassical description of the dynamics of the quantum system is limited by the time $n_2 \sim 1/T$. When the semiclassical region is not very deep

$$1/T \gg 2\varkappa/K \gg 1 \tag{7.12}$$

the time $n_2$ is evaluated by (7.9).

The case (7.2) is apparently of no interest since the condition $nT \ll 1$ shows that the time under consideration is less than a period of oscillation:

$$t = nT_0 \ll 1/\omega_0.$$

In contrast, the case (7.1) is of great interest. It is known that for rather simple dynamical systems in the classical limit in the region of weak chaos correlation functions decay much more slowly due to the internal and external boundaries of the type 'order–chaos' [22–25 and references therein]. From the above analysis it follows that, due to a slow change of the correlation functions with time, the quantum dynamics differs from the classical dynamics only for comparatively large times $n_1 \sim 1/\hbar$.

We can draw the following conclusion: in the region of weak chaos the times of classical approach for describing the dynamics of quantum averages have a power dependence on the quasiclassical parameter $\varkappa \sim 1/\hbar$. A more detailed study of the quantum dynamics in the case of weak chaos needs a special investigation.

## 8. Symmetry in the presence of resonances

Let us consider the case of the resonance (3.5). It has been mentioned already that for (3.5) in the classical case the phase plane $(\eta, p)$ has a certain type of symmetry. Depending on the parameter $q$, stochastic layers form different structures with crystal and quasicrystal symmetries, which are more regular for smaller $K_H$ in the region $K_H < 1$ [20]. It is clear that in the quantum consideration these classical symmetries have to correspond to an operator determining the type of symmetry and commuting with the evolution operator $\hat{T}$ (2.6). In this section we construct such an operator of symmetry $\hat{U}_q$ for (1.6).

Consider now how the average of an arbitrary operator $f(a^+, a)$ varies in time:

$$\bar{f}_n \equiv \langle \psi_n | f(a^+, a) | \psi_n \rangle = \langle \psi_0 | (\hat{T}^+)^n f(a^+, a) \hat{T}^n | \psi_0 \rangle \tag{8.1}$$

where according to (2.6) and (3.5) the evolution operator $\hat{T}$ has the form

$$\hat{T} = \exp\left\{-2\pi i \frac{r}{q}(a^+ a + \tfrac{1}{2})\right\} \exp\{-i\varkappa \cos[(k/\sqrt{2})(a^+ + a)]\}. \tag{8.2}$$

The expression (8.1) with an account of (8.2) and ordering (A2) may be represented as

$$\bar{f}_n = \langle \psi_0 | (\hat{U}_q^+)^m f(a^+, a) \hat{U}_q^m | \psi_0 \rangle \tag{8.3}$$

where for convenience we put $n = mq$ ($m$ is an integer), and $\hat{U}_q$ is a unitary operator of the following form:

$$\begin{aligned}\hat{U}_q = \exp&\left[\!\left[-i\varkappa \cos\left\{(k/\sqrt{2})\left[a^+ \exp\left(\frac{2\pi i r(q-1)}{q}\right) + a \exp\left(-\frac{2\pi i r(q-1)}{q}\right)\right]\right\}\right]\!\right] \\ &\times \exp\left[\!\left[-i\varkappa \cos\left\{(k/\sqrt{2})\left[a^+ \exp\left(\frac{2\pi i r(q-2)}{q}\right)\right.\right.\right.\right. \\ &\left.\left.\left.\left. + a \exp\left(-\frac{2\pi i r(q-2)}{q}\right)\right]\right\}\right]\!\right] \ldots \exp\{-i\varkappa \cos[(k/\sqrt{2})(a^+ + a)]\}. \end{aligned} \tag{8.4}$$

We should note an important property of the operator $\hat{U}_q$ (8.4)—it commutes with the evolution operator $\hat{T}$ (8.2)

$$[\hat{T}, \hat{U}_q] = 0. \tag{8.5}$$

The expression (8.5) shows that under an additional condition of non-degeneracy the eigenfunctions of the operators $\hat{T}$ and $\hat{U}_q$ may be chosen to be coinciding. This means that in the presence of the resonance (3.5) the quasi-energy eigenfunctions of the evolution operator $\hat{T}$ have an additional symmetry ($U_q$-symmetry). It disappears when the condition (3.5) is violated. In a general case the symmetry defined by the operator (8.4) is rather complicated. This fact follows already from the structure of the classical phase space for different values of $q$. Let us describe some particular cases (below $\hat{\eta}$ and $\hat{p}$ are the operators determined in (2.4)):

(1) $q = 1$

$$\begin{aligned}\hat{U}_1(\eta) &= \exp\{-i\varkappa \cos[(k/\sqrt{2})(a^+ + a)]\} \\ &= \exp(-i\varkappa \cos k\eta) = \hat{U}_1(\eta + 2\pi/k)\end{aligned} \tag{8.6}$$

(2) $q = 2$

$$\begin{aligned}\hat{U}_2(\eta) &= \exp(-2i\varkappa \cos k\eta) = \hat{U}_1^2(\eta) \\ &= \hat{U}_2(\eta + 2\pi/k).\end{aligned} \tag{8.7}$$

In both cases the average momentum of oscillator $P(n)$ increases proportionally to $n$ (i.e. proportionally to time), and the average energy $E(n)$ grows proportionally to $n^2$:

$$\begin{aligned}p(n) &= \langle \psi_0 | (\hat{U}_{1,2}^+)^m (-i\, \partial/\partial\eta)(\hat{U}_{1,2})^m | \psi_0 \rangle \sim n \\ E(n) &= \tfrac{1}{2}\langle \psi_0 | (\hat{U}_{1,2})^m (-\partial^2/\partial\eta^2 + \eta^2)(\hat{U}_{1,2})^m | \psi_0 \rangle \sim n^2\end{aligned} \tag{8.8}$$

(3) $q = 3$

$$\begin{aligned}\hat{U}_3 = \exp\{-i\varkappa \cos[(k/\sqrt{2})(\hat{\eta} + \sqrt{3}\hat{p})\} \exp\{-i\varkappa \cos[(k/\sqrt{2})(\hat{\eta} - \sqrt{3}\hat{p})]\} \\ \times \exp\{-i\varkappa \cos k\hat{\eta}\}.\end{aligned} \tag{8.9}$$

In this case in the $\eta$-representation ($\hat{\eta}=\eta$; $\hat{p}=-\mathrm{i}\,\partial/\partial\eta$) there arises the translational invariance

$$\hat{U}_3(\eta)=\hat{U}_3(\eta+4\pi/k). \tag{8.10}$$

In the $p$-representation ($\hat{p}=p$; $\hat{\eta}=\mathrm{i}\,\partial/\partial p$) the operator $\hat{U}_3(p)$ (8.9) has the following translational symmetry:

$$\hat{U}_3(p)=\hat{U}_3(p+4\pi/k\sqrt{3}). \tag{8.11}$$

The operator $\hat{U}_6$ has the same symmetries as (8.10), (8.11).
(4) $q=4$

$$\hat{U}_4=[\exp(-\mathrm{i}\varkappa\cos k\hat{p})\exp(-\mathrm{i}\varkappa\cos k\hat{\eta})]^2. \tag{8.12}$$

In this case in both representations the translational invariance of the operator $\hat{U}_4$ is the same:

$$\hat{U}_4(\eta)=\hat{U}_4(\eta+2\pi/k) \qquad \hat{U}_4(p)=\hat{U}_4(p+2\pi/k). \tag{8.13}$$

In section 9 we give the results of numerical experiments on the dependence of the average energy of the system on time for this case. The properties of symmetry mentioned above automatically remain valid in the classical limit.

The presence of any of the four symmetries of crystal type leads to the possibility of the existence of delocalization of the quasi-energy eigenfunctions of the operator $\hat{U}_q$. In the case of symmetry of quasicrystal type (in particular, $q=5$ and 7) the properties (1)–(4) are absent. It is clear however, that in this case there also occurs a certain quasicrystal $U_q$-symmetry of the quasienergy eigenfunctions which is destroyed out of resonance (3.5).

Below we list some properties of the operator $\hat{U}_q$ (8.4) which hold for an arbitrary $q$ in (3.5).

1. Recall the problem of determining the eigenvalues for the unitary operator $\hat{U}_q$:

$$\hat{U}_q(a^+,a)\psi_\lambda=\mathrm{e}^{\mathrm{i}\lambda}\psi_\lambda \tag{8.14}$$

where the dependence of the operator $\hat{U}_q$ on $a^+$, $a$ is explicitly shown by (8.4). Replace the operators $a^+$, $a$ by the creation and annihilation operators $b^+$, $b$, using the formula:

$$B=a\mathrm{e}^{-2\pi\mathrm{i}/q} \qquad B^+=a^+\mathrm{e}^{2\pi\mathrm{i}/q}. \tag{8.15}$$

From (8.15) we have $[b,b^+]=[a,a^+]=1$. Using $b$, $b^+$ we can introduce new operators of the coordinate $\hat{\hat{\eta}}$ and the momentum $\hat{\hat{p}}$:

$$\begin{aligned}\hat{\hat{\eta}}&=\frac{1}{\sqrt{2}}(b^++b)=\hat{\eta}\cos(2\pi/q)+\hat{p}\sin(2\pi/q)\\ \hat{\hat{p}}&=\frac{\mathrm{i}}{\sqrt{2}}(b^+-b)=-\hat{\eta}\sin(2\pi/q))+\hat{p}\cos(2\pi/q).\end{aligned} \tag{8.16}$$

By means of the explicit form of the operator $\hat{U}_q(a^+,a)$ (8.4) and (8.15) we obtain from (8.14)

$$\hat{U}_q(b^+,b)\exp\{-\mathrm{i}\varkappa\cos[(k/\sqrt{2})(a^++a)]\}\psi_\lambda=\exp\{\mathrm{i}\lambda-\mathrm{i}\varkappa\cos[(k/\sqrt{2})(a^++a)]\}\psi_\lambda. \tag{8.17}$$

From (8.17) it follows that if the eigenfunction $\psi_\lambda$ in (8.14) is considered in the $\eta$-representation: $\psi_\lambda = \psi_\lambda(\eta)$, then in the $\bar{\eta}$-representation the eigenfunction $\psi_\lambda(\bar{\eta})$ will be related to $\psi_\lambda(\eta)$ by

$$\psi_\lambda(\bar{\eta}) = \mathrm{e}^{-\mathrm{i}\varkappa \cos k\eta}\psi_\lambda(\eta) \tag{8.18}$$

From (8.16) it follows that the transition from the $\eta$ to $\bar{\eta}$ representation is made by the rotation of the quantization axis in the classical phase space $(p, \eta)$ by the angle $\varphi = 2\pi/q$.

Let us expand the operator (8.4) into a series over Bessel functions:

$$\begin{aligned}\hat{U}_q = \sum_{m_1,\ldots,m_{q-1}=-\infty}^{\infty} & (-\mathrm{i})^{m_1+\ldots+m_{q-1}} J_{m_1}(\varkappa)\ldots J_{m_{q-1}}(\varkappa) \\ & \times \exp\left[\mathrm{i}km_1\left(\hat{\eta}\cos\frac{2\pi(q-1)}{q} + \hat{p}\sin\frac{2\pi(q-1)}{q}\right)\right] \\ & \times \exp\left[\mathrm{i}km_2\left(\hat{\eta}\cos\frac{2\pi(q-2)}{q} + \hat{p}\sin\frac{2\pi(q-2)}{q}\right)\right] \\ & \ldots \exp[-\mathrm{i}\varkappa\cos(k\eta)].\end{aligned} \tag{8.19}$$

From (8.19) it follows that $\hat{U}_q$ involves the displacement operator:

$$\hat{L}_{\{m\}} \equiv \exp\left[k\left(m_1\sin\frac{2\pi(q-1)}{q} + m_2\sin\frac{2\pi(q-2)}{q} + \ldots + m_{q-1}\sin\frac{2\pi}{q}\right)\right]\frac{\partial}{\partial\eta}. \tag{8.20}$$

Let the operator $\hat{U}_q$ (8.19) act on an arbitrary localized initial wavefunction $\psi_0(\eta)$. The operator $\hat{L}_{\{m\}}$ (8.20) shifts $\psi_0(\eta)$ to the point $\eta + \eta^{(q)}_{\{m\}}$. Since in (8.19) really $|m_i|_{\max} \sim \varkappa$ $(i = 1, \ldots, q-1)$, the maximum shift of the wavepacket $\psi_0(\eta)$ upon the action of the operator $\hat{U}_q$ is by the value $\Delta\eta$ having the order

$$\Delta\eta \sim \varkappa kq. \tag{8.21}$$

Thus, the maximal possible displacement of the centre of the wavepacket for a time $m$ is given by the estimate

$$\eta(m) \sim m\varkappa kq. \tag{8.22}$$

Hence we have

$$\overline{\eta^2}(n) \lesssim (\varkappa kn)^2 \qquad (n = mq \gg 1). \tag{8.23}$$

The estimate (8.23) shows that general dynamics of the wavepacket may involve such shifts of the wavepacket that the length of the displacement is proportional to time $t$ (and not $\sqrt{t}$ as occurs in the case of diffusion dynamics). These shifts ('flights') may contribute to anomalous evolution of the packet. However at present we do not know the measure of these flights and the distribution of their lengths.

## 9. Results of numerical experiments

Performing numerical simulations we calculated matrix elements of the evolution operator $\hat{T}$ (2.6) in the eigenrepresentation of the operator $\hat{H}_0 = (-\partial^2/\partial\eta^2 + \eta^2)/2$:

$$T_{m,n} = \langle m|\,\hat{T}\,|n\rangle \tag{9.1}$$

where the wavefunction $|n\rangle$ is taken from the equation:

$$\hat{H}_0 |n\rangle = (n + \tfrac{1}{2}) |n\rangle. \tag{9.2}$$

In what follows the matrix elements $T_{m,n}$ have been used to find a wavefunction of the complete system (2.3) at arbitrary time.

In the representation (9.2) the matrix elements have the form:

$$\begin{aligned} T_{m,n} &= \exp[-\mathrm{i}T(m + \tfrac{1}{2})]F_{m,n} \\ F_{m,n} &= \langle m| \exp(-\mathrm{i}\varkappa \cos k\eta) |n\rangle. \end{aligned} \tag{9.3}$$

Using the expansion

$$\exp(-\mathrm{i}\varkappa \cos k\eta) = \sum_{l=-\infty}^{\infty} (-\mathrm{i})^l J_l(\varkappa) \exp(\mathrm{i}lk\eta) \tag{9.4}$$

where $J_l(\varkappa)$ is the Bessel function and an explicit form of the function $|n\rangle$, one can represent $F_{m,n}$ in (9.3) in the following form:

$$\begin{aligned} F_{m,n} = J_0(\varkappa)\delta_{m,n} &+ 2 \sum_{l=1}^{\infty} (-\mathrm{i})^{l+n-m} J_l(\varkappa)\sqrt{n!/m!}\,(kl)^{m-n} 2^{-(m-n)/2} \\ &\times \exp[-(kl)^2/4] L_n^{m-n}[(kl)^2/2] \qquad (m \geqslant n) \end{aligned} \tag{9.5}$$

where $L_n^\alpha(x)$ are Laguerre polynomials.

The matrix elements $F_{m,n}$ are different from zero only for even $m - n$:

$$F_{m,n} = 0 \qquad (m - n = 2k + 1; k = 0, 1, \ldots). \tag{9.6}$$

Expression (9.5) is defined only for $m \geqslant n$. To calculate $F_{m,n}$ for $m < n$ one should use the property of symmetry:

$$F_{n,m} = F_{m,n}. \tag{9.7}$$

Computer calculations of Laguerre polynomials lead both to round-off errors and overflow of the computer register. Therefore to calculate the matrix elements of the evolution operator $\hat{T}$ we have used the recurrence relation for Laguerre polynomials [26]:

$$(n + 1)L_{n+1}^\alpha(x) - (2n + \alpha + 1 - x)L_n^\alpha(x) + (n + \alpha)L_{n-1}^\alpha(x) = 0. \tag{9.8}$$

The property (9.6) is in fact the selection rule caused by the spatial symmetry of the perturbation in (2.3): $V(-\eta) = V(\eta)$. Due to this property the states of the system fall into two independent subspaces:

$$\{|0\rangle, |2\rangle, |4\rangle, \ldots\} \tag{9.9}$$

$$\{|1\rangle, |3\rangle, |5\rangle, \ldots\} \tag{9.10}$$

and the behaviour of the system can be studied independently. No qualitative differences in the behaviour of the system have been observed in the numerical calculations for different subspaces.

Now we give the results of the numerical calculations for the dependence of the average energy on time $n$:

$$E(n) = \tfrac{1}{2}\langle \psi_n| (-\partial^2/\partial\eta^2 + \eta^2) |\psi_n\rangle \tag{9.11}$$

where $\psi_n$ coincides with (2.7), and the brackets $\langle \ldots \rangle$ mean quantum mechanical averaging. In the resonance case $T = \pi/2$ $(r = 1; q = 4)$ the dependence $E(n)$ is shown in figure 2. It is seen there that at the initial stage of evolution the

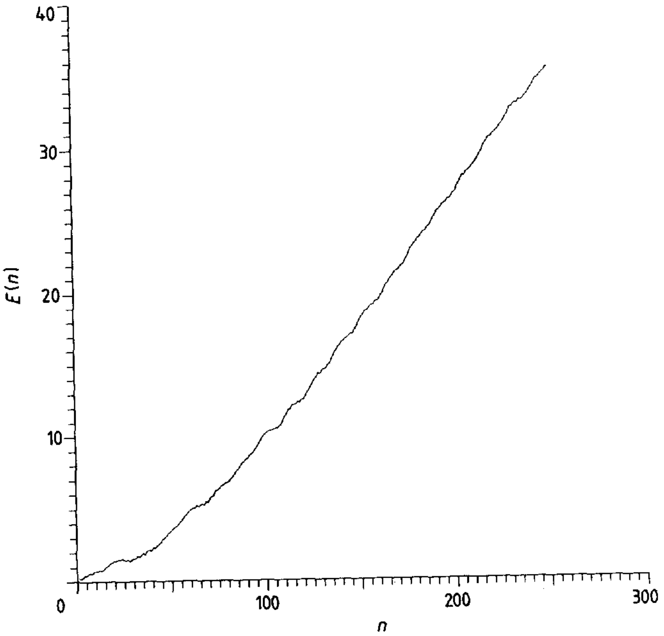


**Figure 2.** The dependence of the average energy $E(n)$ of the oscillator on dimensionless time $n$; $\varkappa = 1$; $k = 1.2$; $T = \pi/2$; $K_H = 1.44$. The initial condition: $\psi_0 = |0\rangle$ $(E(0) = 1/2;$ $\xi = 1.44)$.

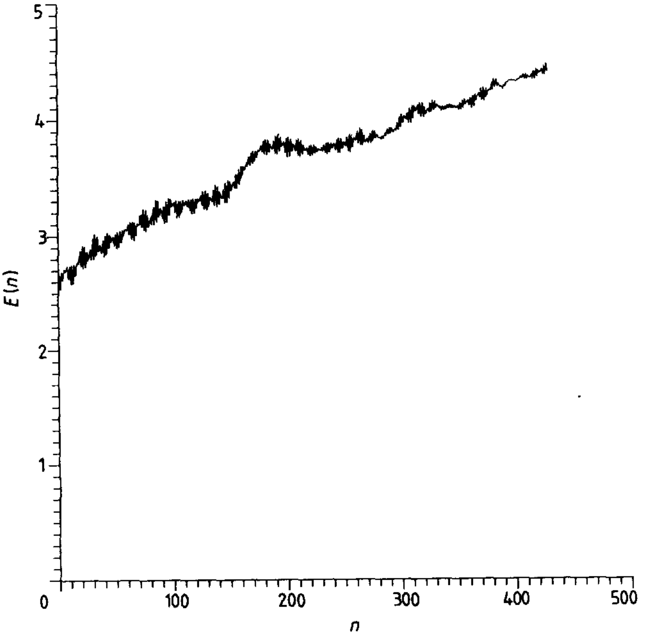


**Figure 3.** The dependence of $E(n)$ at $\varkappa = 1$; $k = 0.7$; $T = \pi/2$; $K_H = 0.49$; $\psi_0 = |2\rangle$ $(E(0) = 5/2;$ $\xi = 2.45)$.

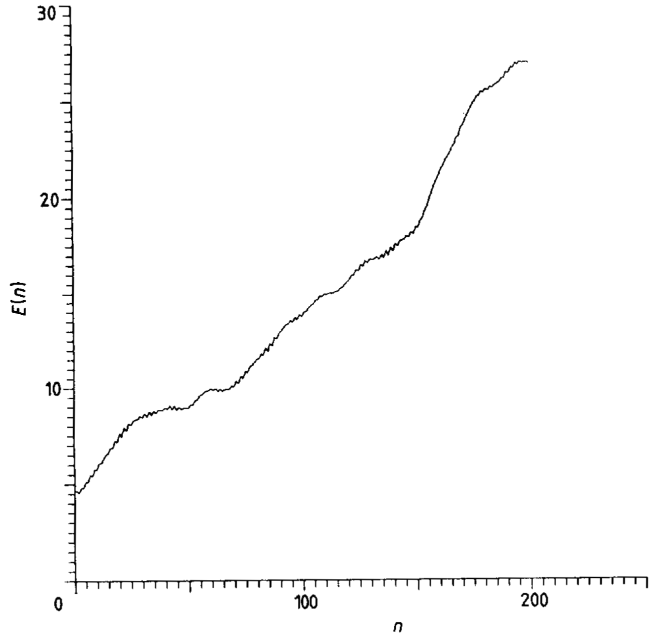


**Figure 4.** The dependence of $E(n)$ at $\varkappa=1$; $k=0.7$; $T=\pi/2$; $K_H=0.49$; $\psi_0=|4\rangle$ ($E(0)=9/2$; $\xi=4.41$).

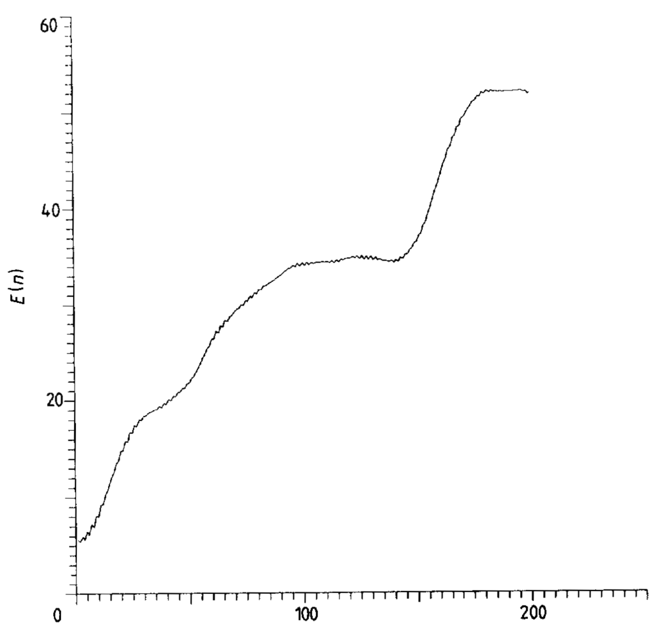


**Figure 5.** The dependence of $E(n)$ at $\varkappa=1$; $k=0.7$; $T=\pi/2$; $K_H=0.49$; $\psi_0 = 1/\sqrt{2}\,(|4\rangle + |6\rangle)$ ($E(0)=21/4$; $\xi=5.145$).

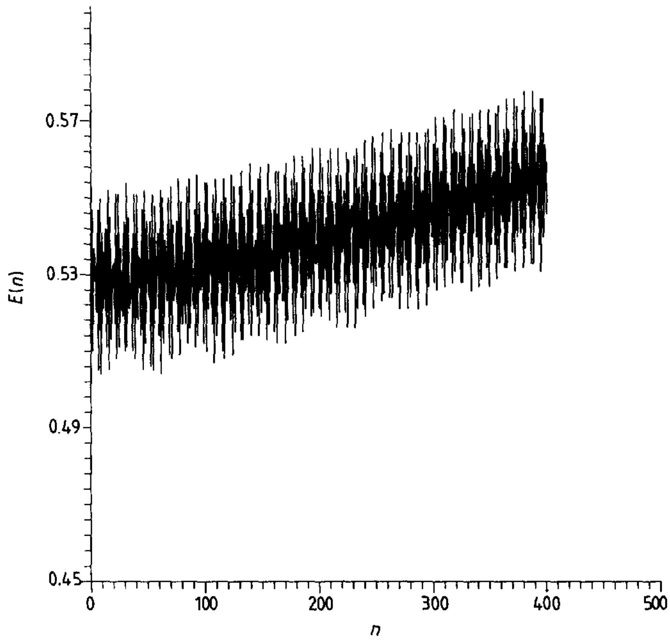


**Figure 6.** The dependence of $E(n)$ at $\varkappa = 1$; $k = 0.7$; $T = \pi/2$; $K_H = 0.49$; $\psi_0 = |0\rangle$ ($E(0) = 1/2$; $\xi = 0.49$).

dependence of the average energy growth in time is close to quadratic. Further increase in the energy goes according to the linear law at times $n < 250$.

Figures 3–6 show the results of calculations for the dependence $E(n)$ in the case of $T = \pi/2$ for different initial occupations of levels in the system (parameters $\varkappa$ and $k$ are chosen to be constant: $\varkappa = 1$; $k = 0.7$ that corresponds to $K_H = \varkappa k^2 = K = 0.49$). From figures 3–6 it is seen that the initial speed of growth for the average energy of the system depends much on the initial occupation. When only a zero level is initially occupied there is no visible growth of the average energy at times up to $n \approx 500$.

To explain how the initial speed of the average energy growth depends on the initial occupation of levels we use the recurrence expression for operators $a_n$ and $a_n^+$ in the Heisenberg representation. The operators may be obtained from (2.5) and have the form similar to (3.3):

$$a_{n+1} = \exp(-\mathrm{i}T)\{a_n + \mathrm{i}\varkappa(k/\sqrt{2})\sin[(k/\sqrt{2})(a_n^+ + a_n)]\} \tag{9.12}$$

where $a_n = a(nT_0 - 0)$. The expression for $a_{n+1}^+$ is obtained by Hermitian conjugation (9.12). In a general case the calculation of the average energy of the system by means of the recurrence relation (9.12) is impossible since they are nonlinear over operators $a_n$ and $a_n^+$. The situation is much simpler when

$$\sqrt{2}k\,|a_n| \ll 1 \tag{9.13}$$

where

$$|a_n| \sim \sqrt{E(n)} \tag{9.14}$$

is a characteristic value of the average value of $a_n$. Inequality (9.13) may be also represented in the form (we give it for $n = 0$):

$$\xi \equiv 2k^2 E(0) \tag{9.15}$$

In this case expanding (9.12) over a small parameter (9.13) (or $\xi$) we obtain in the linear approximation:

$$\begin{Vmatrix} a_{n+1}^+ \\ a_{n+1} \end{Vmatrix} = \begin{Vmatrix} (1 - iK_H/2)\exp(iT) & (iK_H/2)\exp(iT) \\ (iK_H/2)\exp(-iT) & (1 + iK_H/2)\exp(-iT) \end{Vmatrix} \begin{Vmatrix} a_n^+ \\ a_n \end{Vmatrix}. \tag{9.16}$$

An equation for the eigenvalues of (9.16) takes the form:

$$\mu^2 - (2\cos T + K_H \sin T)\mu + 1 = 0. \tag{9.17}$$

For $T = \pi/2$ we obtain from (9.17)

$$\mu_{1,2} = K_H/2 \pm \sqrt{K_H^2/4 - 1}. \tag{9.18}$$

It is clear from (9.18) that for

$$K_H < 2 \tag{9.19}$$

the system is stable.

It is easy to see that the condition (9.13) (or (9.15)) holds well for parameters of figure 6 ($\xi < 1$; $K_H = 0.49$), and does not hold for figures 2–5. Conditions (9.15) and (9.19) have a simple physical meaning, namely the classical trajectories corresponding to the initial state of the wavepacket must not get into a stochastic layer adjacent to the separatrix. In [20] it is shown that a characteristic cell size of the separatrix network is of the order

$$p_c \sim \eta_c \sim \pi/\sqrt{2}k.$$

So, to be stable the wavepacket must be localized at the initial moment of time in the region $\eta < \eta_c$. This corresponds to the limitation for the initial average energy of

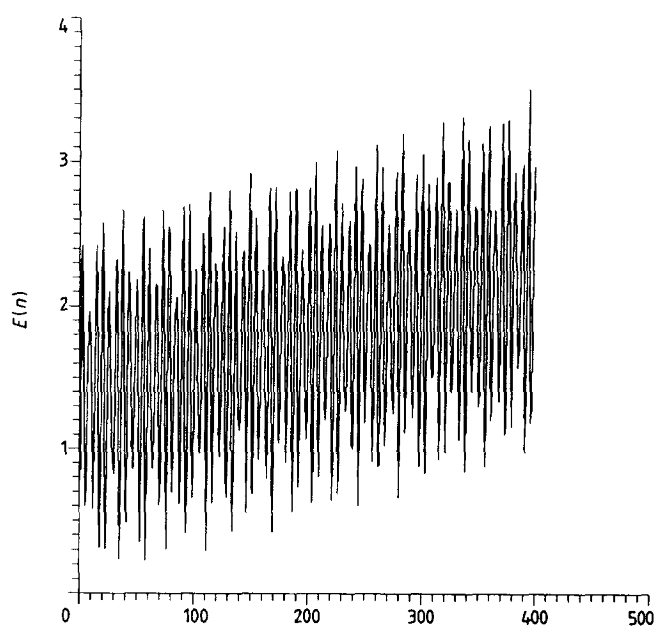


**Figure 7.** The dependence of $E(n)$ at $\varkappa = 12$; $k = 0.4$; $T = \pi/2$; $K_H = 0.16$; $\psi_0 = |0\rangle$ ($E(0) = 1/2$; $\xi = 0.16$).

the packet $E(0) = (\bar{\eta}^2 + \bar{p}^2)/2$:

$$E(0) < E_c \approx 5/k^2 \qquad (E_c \approx (\eta_c^2 + p_c^2)/2) \tag{9.20}$$

which coincides with (9.15) to an accuracy of a numerical factor ($\sim$10). One should note that for the initial state $|0\rangle$ the estimate (9.14) is rather high, and the estimates (9.15) and (9.20) in this case are in fact equivalent.

The condition (9.19) corresponds to the fact that there is no period-doubling bifurcation [20] when the point (0.0) becomes unstable, and a stochastic layer goes through this point of coordinates. In our numerical experiments the condition (9.19) always holds.

Figure 7 shows the results for $\varkappa \gg 1$. In this case the conditions (9.15) and (9.19) are satisfied. It is seen that there is no average growth at times under consideration.

For the cases when conditions (9.15) and (9.19) hold the average energy growth characteristic for resonance $T = \pi/2$ may be manifested both due to exponentially small occupation of a stochastic layer and due to purely quantum effects of tunnelling.

## 10. Concluding remarks

The quantum system (2.3) studied in our work possesses a number of interesting properties which allow us to consider it as a perspective object for further investigations in the region of quantum chaos. These properties are the following.

1. In the zero approximation (no perturbation: $\varkappa = 0$) the system (2.3) is linear, and even in the classical limit it should be investigated out of the framework of KAM theory [20]. There is no quantum consideration of these systems in the region of parameters of quantum chaos at present.

2. At resonances (3.5) the system (2.3) realizes crystal and quasicrystal symmetries in the classical phase space $(\eta, p)$. The problem is to study the properties of localization of quasi-energy eigenfunctions and dynamics of the observable values in the quantum consideration.

3. In the region of weak chaos ($K_H \ll 1$) stochastic layers (stochastic webs) are not localized in the classical phase space. It allows one to study the quantum diffusion and quantum correlation effects in stochastic layers in a purely quantum and quasiclassical limit.

4. Perturbation operator is both a function of the phase $\vartheta$ and action operator $\hat{n} = -\mathrm{i}\,\partial/\partial\vartheta$ (in contrast, e.g. to a model of an excited quantum rotator (1.1) and similar systems). In this case it is of great interest to find the relation between the properties of quasi-energy functions of the system (2.3) and the problem of Anderson localization. Determination of this relation is of great interest both in the region of resonances (3.5) with crystal and quasicrystal symmetry and in the case of absence of resonances (3.5).

The results obtained in our work allows us to draw the following conclusions concerning the behaviour of the system (2.3) for different regions of parameters.

1. There is a quantum boundary (QB): $K = 2\varkappa$ is such that for $K > 2\varkappa$ the motion of the system (2.3) at $n > 1$ is purely quantum; for $K < 2\varkappa$ quantum dynamics of the system (2.3) at finite times approximately coincides with classical dynamics.

2. Under the condition of strong chaos in the classical limit ($K \gg 1$) times of quasiclassical description $n_\hbar$ are logarithmically small over the quasiclassical

parameter:

$$n_\hbar = \ln(2\varkappa)/\ln K \sim \ln(1/\hbar) \qquad (2\varkappa \gg K \gg 1).$$

3. In the region of weak chaos in the classical limit the time of the validity of quasiclassical description has a power-law dependence on the quasiclassical parameter: $n_1 = 2\varkappa/K_H^2 \sim 1/\hbar \gg n_\hbar$.

4. Under the condition of resonances ($\omega_0 T_0 = 2\pi r/q$) quasi-energy functions have a certain $U_q$-symmetry which may become apparent in the character of quantum diffusion of the average energy of the system in time.

**Acknowledgment**

We are grateful to G K Sereyakov for his help in developing the numerical algorithm.

**Appendix**

From (4.3) and (2.6) we have

$$\mathcal{R}_n(\gamma \mid \beta) = \langle \psi_{n-1} | \exp\{i\varkappa \cos[(k/\sqrt{2})(a^+ + a)]\}(\gamma e^{iT} a^+) \times \exp(\beta e^{-iT} a) \exp\{-i\varkappa \cos[(k/\sqrt{2})(a^+ + a)]\} | \psi_{n-1} \rangle \tag{A1}$$

where the ordering [21]

$$e^{\tau a^+ a} f(a, a^+) e^{-\tau a^+ a} = f(a e^{-\tau}, a^+ e^{\tau}). \tag{A2}$$

is used. To make (A1) simpler let us do the following. Represent (A1) in the form:

$$\mathcal{R}_n(\gamma \mid \beta) = \langle \psi_{n-1} | \exp(\gamma e^{iT} a^+) \exp(-\gamma e^{iT} a^+) \exp\{i\varkappa \cos[(k/\sqrt{2})(a^+ + a)]\} \times \exp(\gamma e^{iT} a^+) \exp(\beta e^{-iT} a) \exp\{-i\varkappa \cos[(k/\sqrt{2})(a^+ + a)]\} \times \exp(-\beta e^{-iT} a) \exp(\beta e^{-iT} a) | \psi_{n-1} \rangle \tag{A3}$$

and use the ordering [21]:

$$\begin{aligned} e^{-\tau a^+} f(a, a^+) e^{\tau a^+} &= f(a + \tau, a^+) \\ e^{\tau a} f(a, a^+) e^{-\tau a} &= f(a, a^+ + \tau). \end{aligned} \tag{A4}$$

With an account of (A4) we obtain from (A3)

$$\mathcal{R}_n(\gamma \mid \beta) = \langle \psi_{n-1} | \exp(\gamma e^{iT} a^+) \exp\{i\varkappa \cos[(k/\sqrt{2})(a^+ + a + \gamma e^{iT})]\} \times \exp\{-i\varkappa \cos[(k/\sqrt{2})(a^+ + a + \beta e^{-iT})]\} \exp(\beta e^{-iT} a) | \psi_{n-1} \rangle. \tag{A5}$$

A significant point for further simplification is the fact that functions in (A5) having $\varkappa$ commute since the arguments of the corresponding cosines are different only in the $c$-number. It makes it possible to unite the exponents having $\varkappa$ in (A5) and to represent $\mathcal{R}_n(\gamma \mid \beta)$ in the form:

$$\mathcal{R}_n(\gamma \mid \beta) = \langle \psi_{n-1} | \exp(\gamma e^{iT} a^+) \exp\{2i\varkappa \sin[(k/\sqrt{2})(\beta e^{-iT} - \gamma e^{iT})] \times \sin\left[(k/\sqrt{2})\left(a^+ + a + \frac{\gamma e^{iT} + \beta e^{-iT}}{2}\right)\right]\right\} \exp(\beta e^{-iT} a) | \psi_{n-1} \rangle$$

$$= \sum_{m=-\infty}^{\infty} J_m\{2\varkappa \sin[(k/2\sqrt{2})(\beta e^{-iT} - \gamma e^{iT})]\} \times \exp\{im(k/2\sqrt{2})(\gamma e^{iT} + \beta e^{-iT})\} \langle \psi_{n-1} | \times \exp(\gamma e^{iT} a^+) \exp[im(k/\sqrt{2})(a^+ + a)] \exp(\beta e^{-iT} a) | \psi_{n-1} \rangle. \qquad \text{(A6)}$$

The average $\langle \psi_{n-1} | \ldots | \psi_{n-1} \rangle$ entering (A6) may be represented as

$$\langle \psi_{n-1} | \ldots | \psi_{n-1} \rangle = \exp(-m^2k^2/4) \langle \psi_{n-1} | \exp[(\gamma e^{iT} + imk/\sqrt{2}) a^+] \times \exp[(\beta e^{-iT} + imk/\sqrt{2}) a] | \psi_{n-1} \rangle$$

$$= \exp(-m^2k^2/4) \mathcal{R}_{n-1}(\gamma e^{iT} + imk/\sqrt{2} \mid \beta e^{-iT} + imk/\sqrt{2}). \qquad \text{(A7)}$$

Substituting (A7) into (A6) we obtain the recurrence relation (4.4).

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