The Markowitz Portfolio Theory

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Abstract

In this paper we present the Markowitz Portfolio Theory for portfolio selection. There is also a reading guide for those who wish to dug deeper into the world of portfolio optimization.

Both of us have contributed to all parts of the report.

1   Markowitz Portfolio Theory

Which portfolio is the best? This question is probably as old as the stock-market itself. However when Markowitz published his paper on portfolio selection in 1952 he provided the foundation for modern portfolio theory as a mathematical problem [2].

The return R_t of a portfolio at time t can be defined to be the total value T_t of the portfolio divided by the total value at an earlier time $t-1$, i.e.

$$R_t = \frac{T_t}{T_{t-1}} - 1, \tag{1}$$

hence its simply the percentally change in the value from one time to another. Markowitz portfolio theory provides a method to analyse how good a given portfolio is based on only the means and the variance of the returns of the assets contained in the portfolio. An investor is supposed to be risk-averse, hence he/she wants a small variance of the return (i.e. a small risk) and a high expected return[3].

Consider a portfolio with n different assets where asset number i will give the return R_i. Let μ_i and σ_i^2 be the corresponding mean and variance and let $\sigma_{i,j}$ be the covariance between R_i and R_j. Suppose the the relative amount of the value of the portfolio invested in asset i is x_i. If R is the return of the whole portfolio then:

$$\mu = \mathrm{E}[R] = \sum_{i=1}^{n} \mu_i x_i \tag{2}$$

$$\sigma^2 = \mathrm{Var}[R] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j} x_i x_j \tag{3}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{4}$$

$$x_i \geq 0, \ i = 1, 2, ..., n \tag{5}$$

Condition (5) is the same as saying that only long positions are allowed, hence if short sale should be included in the model this condition should be omitted[3]. For different choices of $x_1, ..., x_n$ the investor will get different combinations of μ and σ^2. The set of all possible (σ^2, μ) combinations is called the attainable set. Those (σ^2, μ) with minimum σ^2 for a given μ or more and maximum μ for a given σ^2 or less are called the efficient set (or efficient frontier). Since an investor wants a high profit and a small risk he/she wants to maximize μ and minimize σ^2 and therefore he/she should choose a portfolio which gives a (σ^2, μ) combination in the efficient set. In figure 1 the attainable set is the interior of the ellipse and the efficient set is the upper left part of its boundary [1].

2   Special case: Portfolio with tree assets

Given a portfolio containing only three assets, x_1, x_2 and x_3 the problem has a clear geometrical interpretation [1]. We get the expected return to be $\mu = \sum_{i=1}^{3} \mu_i x_i$, the variance of the return $\sigma^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{i,j} x_i x_j$ with the constraints that $1 = \sum_{i=1}^{3} x_i$ and $x_i \geq 0, i = 1, 2, 3$. Using the relations given above we can express x_3 as $x_3 =$
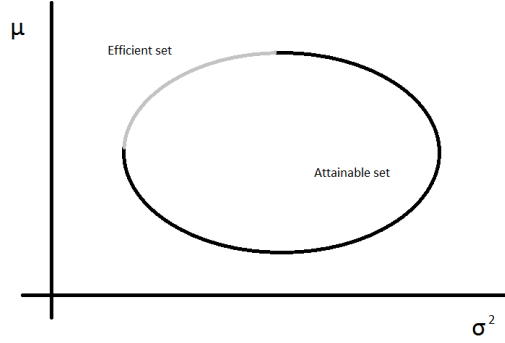
Figure 1: The efficient set in the $\mu\sigma^2$ plane.

$1 - x_1 - x_2$, yielding a two-dimensional problem with the constraints $x_1 \geq 0, x_2 \geq 0$ and $1 - x_1 - x_2 \geq 0$. Hence μ and σ are now functions of x_1 and x_2. It may be shown that the isoclines of μ are parallel straight lines and the isoclines of σ are concentric ellipses in the $x_1 x_2$ plane [1].

What is the efficient set in this case? Fixing a μ and minimizing the variance, we easily observe that the point yielding the minimal variance and hence the most efficient portfolio, is where the isomean line is parallel to the isovariance ellipses. Now varying μ gives rise to a set of efficient portfolios i.e. the efficient set. This set forms a straight line, called the critical line denoted by l [1].

In figure 2 (figure borrowed from [1]) the graphical representation of the problem can be seen. The point in this plane giving the least variance is denoted by x. The set of efficient portfolios starts at the point x and then goes in the direction of increasing expected return until it hits the border of the attainable set, then continuing along the border in the direction of growing expectation. At any moment x_1 and x_2 can easily be obtained from the picture (and hence x_3). In the case of a portfolio containing four assets we might analogously give a three dimensional graphical representation. However, for portfolios with more than four assets this can not be done.

3 Portfolio selection as optimization problem

As mentioned in section 1 the Markowitz portfolio theory states that an investor should choose a portfolio from the efficient set, depending on how risk averse he/she is. One way to handle this is to consider the optimization problem

$$\min(\sigma^2 - A\mu) \tag{6}$$

$$\text{subject to: } \sum_{i=1}^{n} x_i = 1 \tag{7}$$
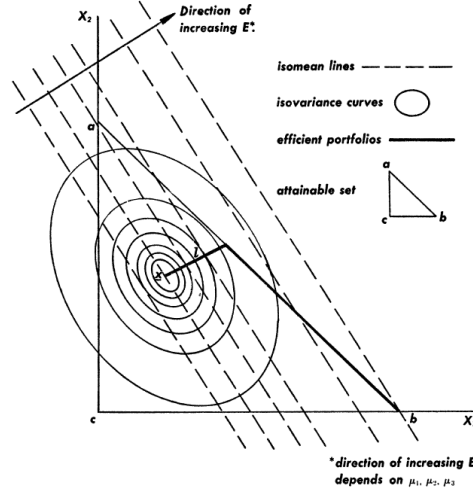
$$x_i \geq 0, \ i = 1, 2, ..., n \tag{8}$$

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Figure 2: Graphical representation of the problem.

A increasing A corresponds to the investor becoming more willing to take a bigger risk to get a higher expected return and $A = \infty$ corresponds to the investor only caring about getting a large expected return no matter what the risk is [2].

4 Risk free lending and borrowing

Consider the case where it is possible to lend out money without risk. This can be seen as another asset with $\sigma^2 = 0$ and return r and if it is possible to borrow money to the same interest rate than it means that this asset may be hold in short position. Let x be the proportion of the portfolio invested in risky assets with the mean μ_p and the variance σ_p^2 and let $1 - x$ be the proportion put in the risk free asset (i.e. lending or borrowing). Then, for the whole portfolio:

$$\mu = \mathrm{E}[R] = (1 - x)r + x\mu_p = r + x(\mu_p - r) \tag{9}$$
$$\sigma^2 = \mathrm{V}[R] = x^2\sigma_p^2 \tag{10}$$

Combining equation (9) and equation (10) gives:

$$\mu = \mathrm{E}[R] = r + \frac{\mu_p - r}{\sigma_p}\sigma \tag{11}$$

which is a straight line going through the point $(0, r)$ and the slope $\frac{\mu_p - r}{\sigma_p}$. (With σ on the x-axis and μ on the y-axis) Since the investor wants maximum return for a fixed level of risk he/she wants this line to be as steep as possible[3]. The portfolio giving the μ_p and σ_p that maximizes $\frac{\mu_p - r}{\sigma_p}$ is called the Optimal Portfolio of Risky Assets (OPRA). Hence the OPRA is the solution to the maximization problem:

$$\max \frac{\mu_p - r}{\sigma_p} = \mathrm{E}[R] = \sum_{i=1}^{n} \mu_i x_i \tag{12}$$

$$\text{subject to: } \sigma_p^2 = \mathrm{Var}[R] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j} x_i x_j \tag{13}$$

$$\sum_{i=1}^{n} x_i = 1 \tag{14}$$

The OPRA will generate a line that has the steepest gradient of all lines going through the attainable set and the point $(0, r)$ (See figure 3). This line is called the capital market line, cml. Investors can place themselves anywhere on this line through lending/borrowing an appropriate amount and buying OPRA for the rest [3].
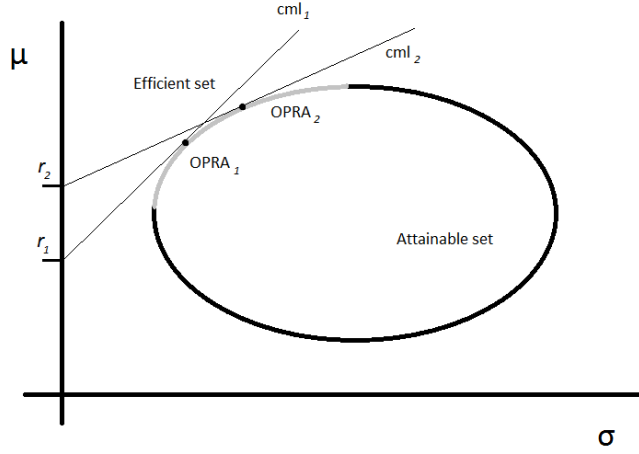

Figure 3: Two examples of the cml and the OPRA given an interest rate r.

5 Impact on portfolio theory

The Markowitz portfolio selection model laid the foundation for modern portfolio theory but it is not used in practice[2].The main reason for this is that it requires a huge amount of data (if n assets are considered then the model needs $2n + \binom{n}{2}$ parameters). More useful models have however been developed from the Markowitz model by use of approximation. The first model that reduced the data requirements was the model of Sharpe which only need $3n + 2$ parameters. This model does not require estimations of the pairwise correlation between the assets but only estimations of how the asset depend on the behavior of the market [3]. The model of Sharpe led to Markowitz and Sharpe winning The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel in 1990 [4].

6   Our thoughts

One drawback with the Markowitz model is that the variance of a portfolio is not a complete measure of the risk taken by the investor. What is the Value at Risk for a given portfolio? That is impossible to answer if only the variance and mean and not the distribution is known. Hence the Markowitz model do not tell an investor which portfolio he/she can afford to buy if he/she is willing to take a certain risk to get bankrupted. If an investor wants to use the Markowitz model to choose a suitable portfolio then it is probably a good idea to do some complementary calculations of the risk of extreme events using extreme value statistics (like PoT modelling for the tails). However this is not really an issue since nobody uses the model in practice.

7   Reading guide

Here follows a short reading guide for those who want to know more about portfolio optimization theory:

- An easy introduction to the concept Efficient Portfolio is given in the video *Efficient Portfolio Frontier*
  (http://www.youtube.com/watch?v=3ntwyjXZdS0&feature=related).

- In [3] the Markowitz portfolio theory, together with more involved models (the model of Sharpe, the Sharpe-Lintner-Mossin CAP-M, the Arbitrage Pricing Theory, APT and the Black-Litterman model) arising from the MPT are briefly discussed.

- The interested reader can also find a complete book in the subject by Harry. M. Markowitz by the name *Portfolio Selection: Efficient diversification of investments* [1]. Its first parts cover elementary probability theory and can be skipped if wanted.

References

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