Universidade Vigo

Máster en Enxeñaría de Telecomunicación

Signal Processing for Communications

Sampling and Quantization

Signal Processing in Communications Group (GPSC)

atlanTTic

E.E. Telecomunicación

Contents

- Sampling and reconstruction of analog signals
  - Sampling techniques
  - Reconstruction
  - Sample rate conversion
- Practical Analog-to-Digital Converters
  - Quantization
  - ADC parameters and distortion types

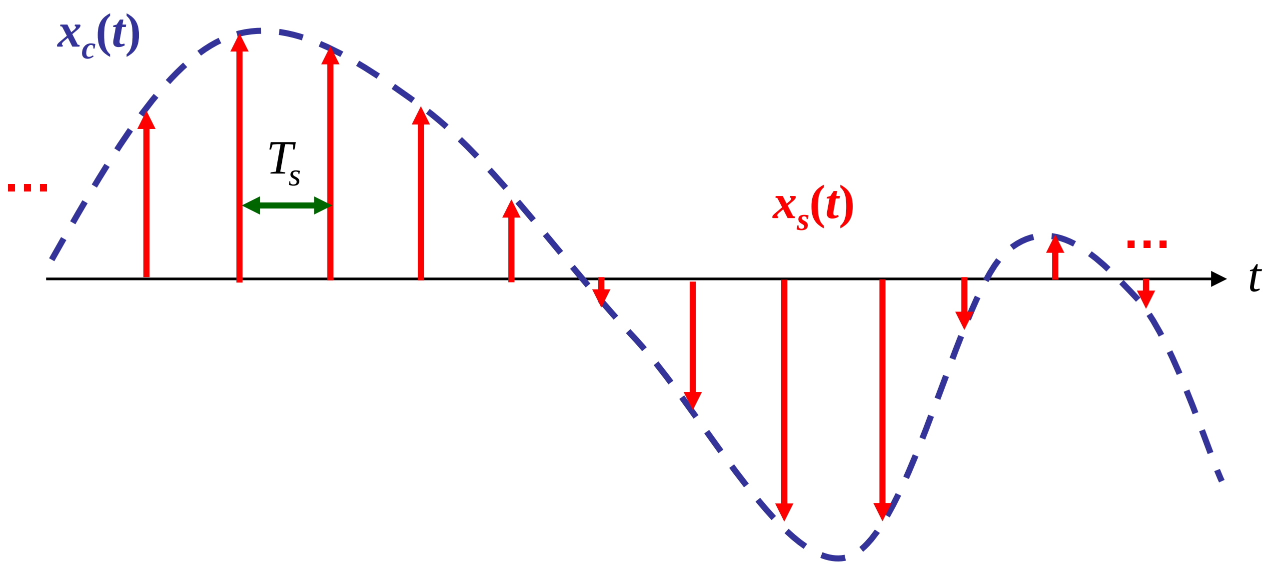
Impulse sampling

- **Analog signal** $x_c(t)$, with Fourier Transform $X_c(f)$,

$$X_c(f) = \int_{-\infty}^{\infty} x_c(t) e^{-j2\pi f t} dt$$

- **Sampled signal** (at $f_s = 1/T_s$ samples/s), $x_s(t)$, with Fourier Transform $X_s(f)$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\delta(t - nT_s)$$

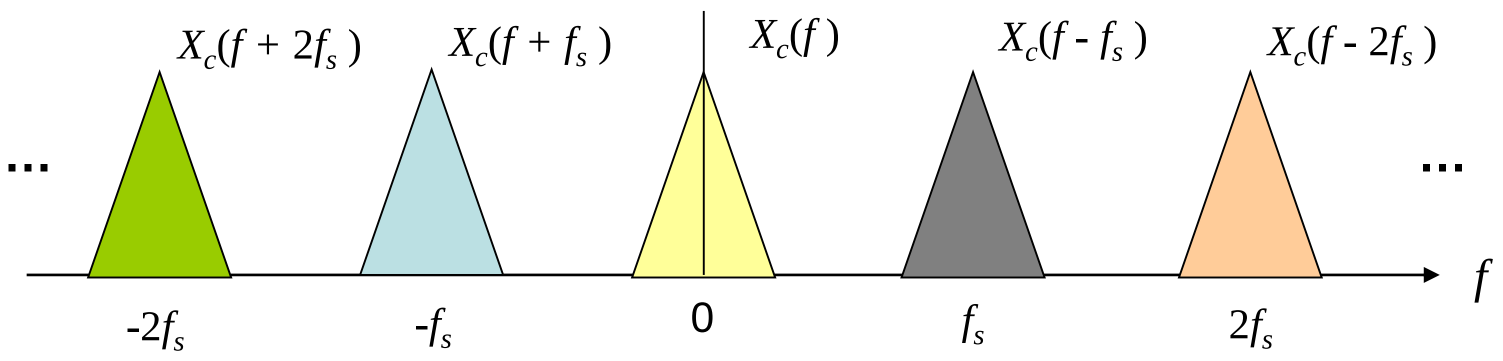


Impulse sampling in the frequency domain

- How are the spectra $X_c(f)$ and $X_s(f)$ related?

$$X_s(f) = f_s \cdot \sum_{k=-\infty}^{\infty} X_c\left(f - k \cdot f_s\right)$$

- We have replicas of the original spectrum located at integer multiples of the sampling rate
- The spectrum of the sampled signal $X_s(f)$ is periodic with period f_s.



Proof

- Write $x_s(t) = x_c(t) \cdot p(t)$, where $p(t)$ is an impulse train with period T_s:

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

- The Fourier Transform of $p(t)$ is also an impulse train in the frequency domain:

$$P(f) = f_s \cdot \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

- Multiplication in time domain $\Leftrightarrow$ convolution in frequency domain:

$$X_s(f) = X_c(f) \star P(f) = f_s \cdot \sum_{k=-\infty}^{\infty} X_c(f - kf_s)$$

Do not get confused…

- Sometimes we may work with the discrete-time sequence $x_d[n] = x_c(nT_s)$, with Discrete-Time Fourier Transform $X_d(e^{j\omega})$

- So what is the relation between $X_d(e^{j\omega})$ and $X_s(f)$?

- By definition,

$$\begin{aligned} X_d(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x_c(nT_s) e^{-j\omega n} \end{aligned}$$

- On the other hand, since $\delta(t-nT_s) \leftrightarrow e^{-j2\pi f T_s n}$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT_s)\delta(t-nT_s) \quad \leftrightarrow \quad X_s(f) = \sum_{n=-\infty}^{\infty} x_c(nT_s) e^{-j2\pi f T_s n}$$

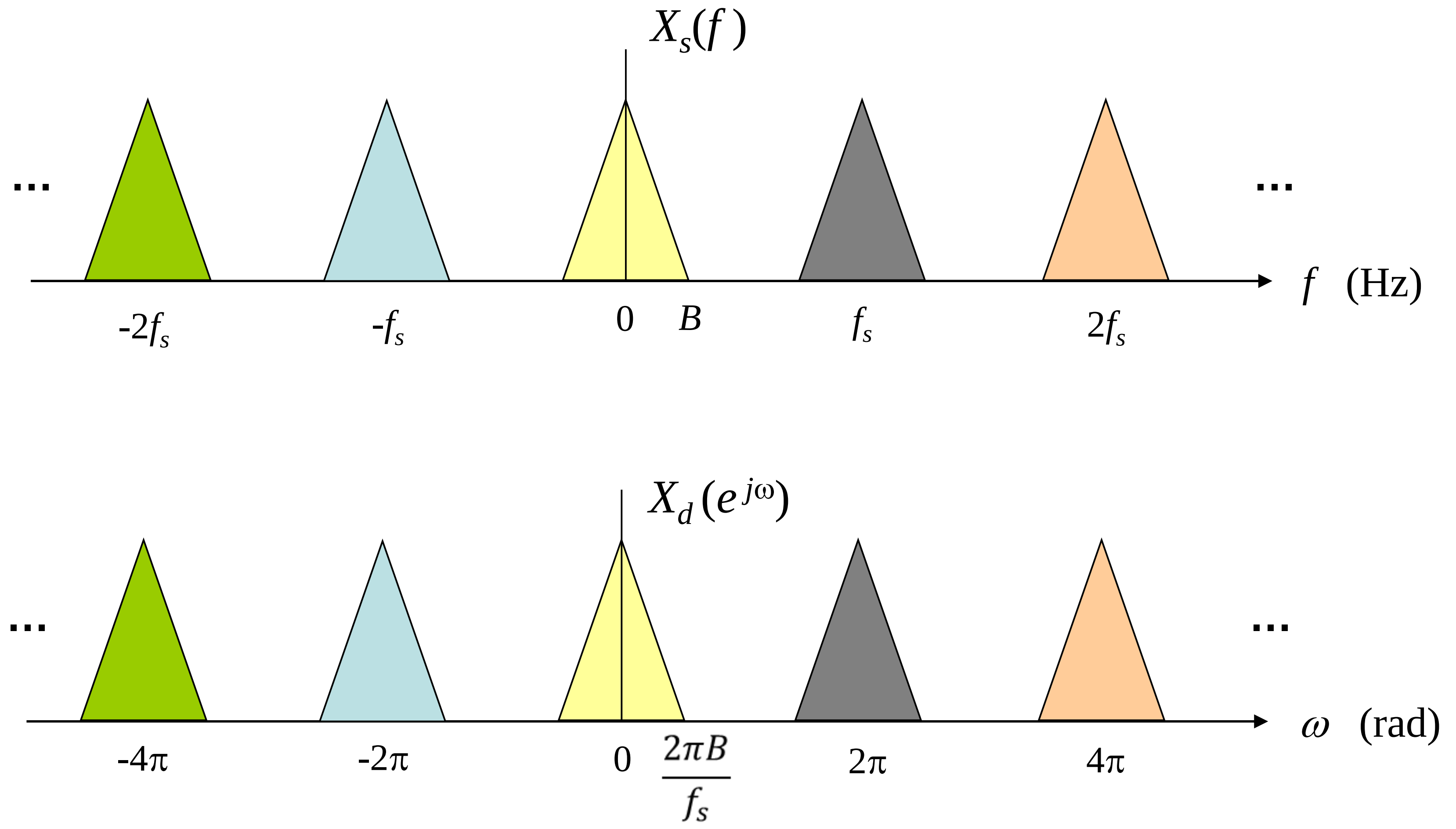
- Therefore:

$$X_s(f) = X_d(e^{j2\pi f T_s})$$

which means that $X_s(f)$ is obtained by evaluating $X_d(e^{j\omega})$ at $\omega = 2\pi f T_s$

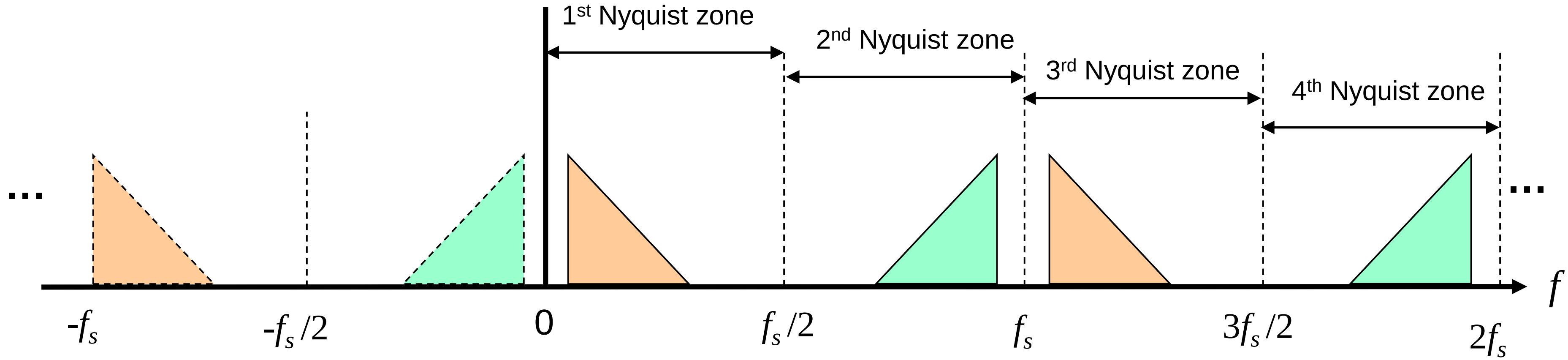
Relation between $X_d(e^{j\omega})$ and $X_s(f)$

$$X_s(f) = X_d(e^{j2\pi f T_s})$$



Nyquist Zones: mirroring

- The positive f axis is divided in segments of length $f_s/2$ termed “Nyquist zones”
- First NZ: from 0 to $f_s/2$ Second NZ: from $f_s/2$ to f_s , etc
- Each NZ contains one replica of the original spectrum
- *Frequency mirroring* occurs in *even* Nyquist Zones 2, 4, 6, …
- Remember that real signals have Hermitian spectra: $X_c(f) = X_c^*(-f)$



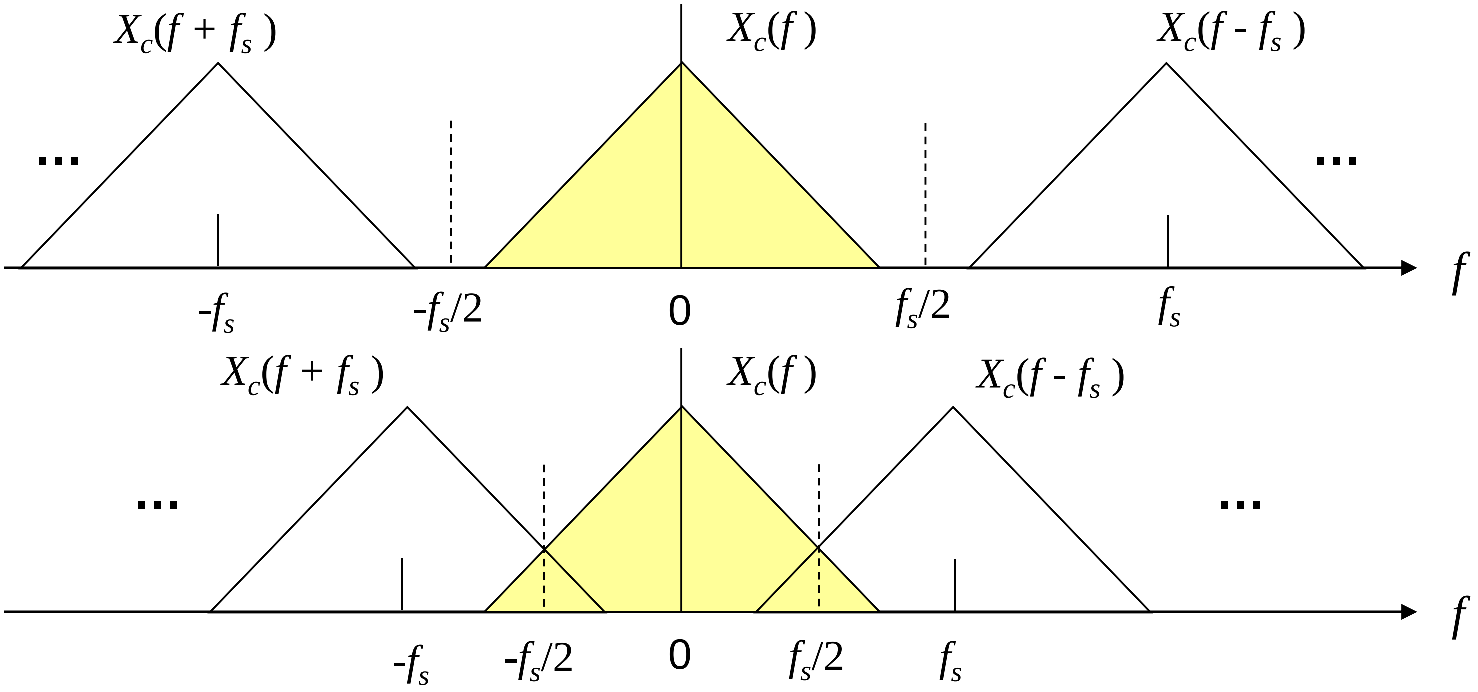
Aliasing

- If the original signal is not confined to the first Nyquist zone, the replicas may overlap, and we have **aliasing**

- **Sampling theorem:**

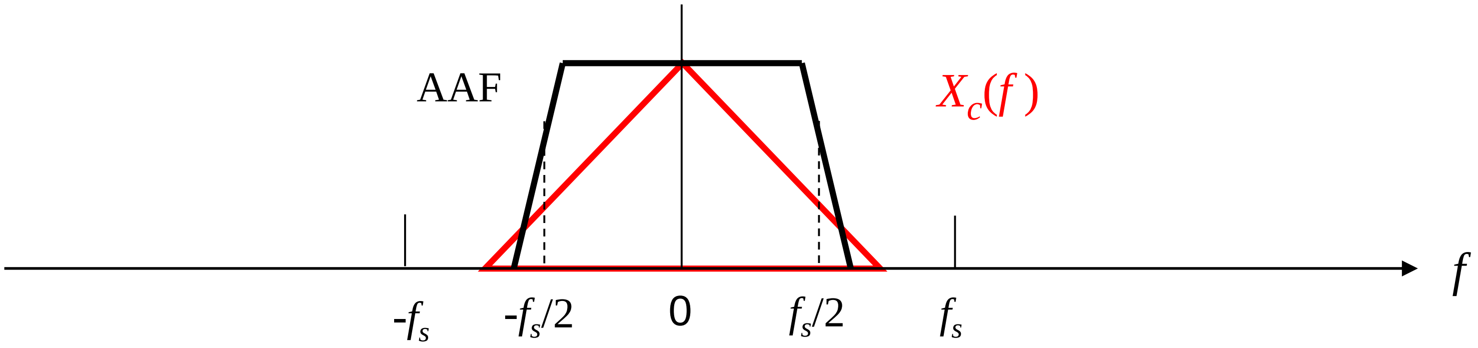
  If a bandlimited real-valued signal is sampled at a rate at least twice the value of its highest frequency component, then the signal can be reconstructed from its samples.

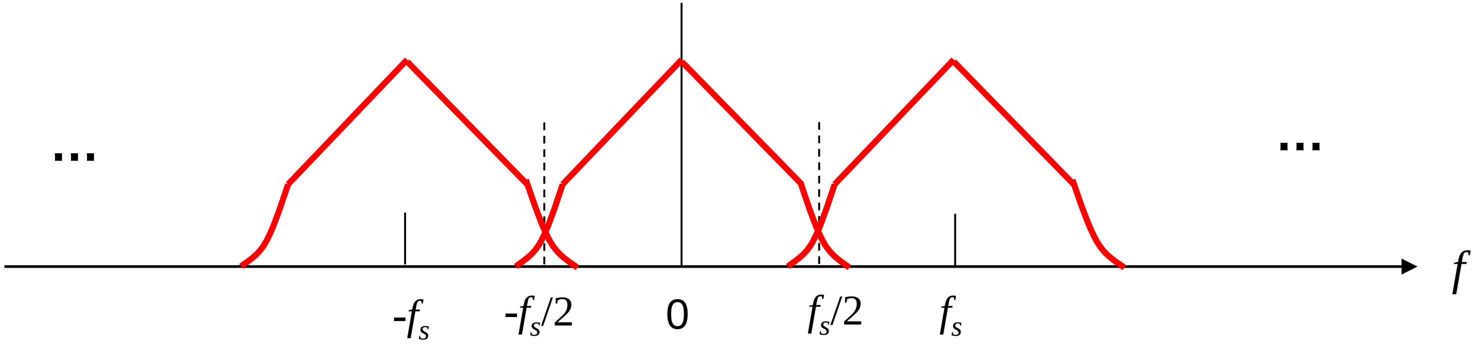
- Note that this is a sufficient condition. For some kinds of signals, it may not be necessary!



Anti-aliasing filter

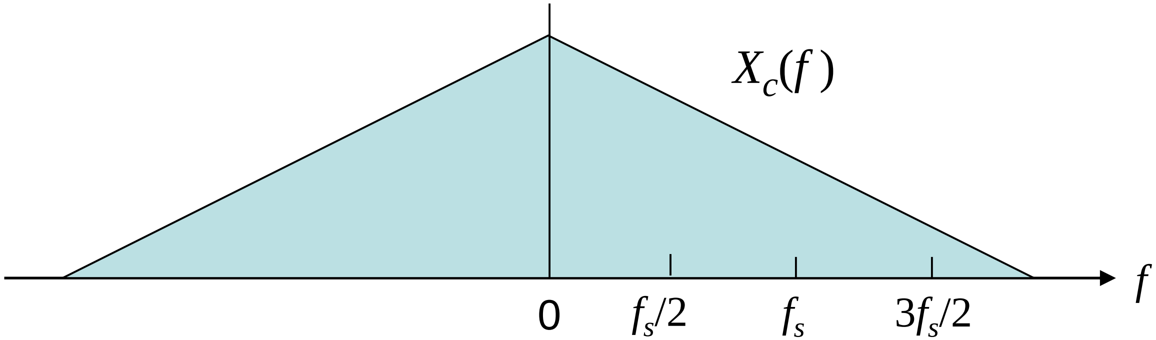
- Use an analog antialiasing filter (AAF) before sampling
- Better to filter out high frequencies than to have aliasing distortion
- Additionally, AAF avoids out-of-band noise aliasing into the signal band

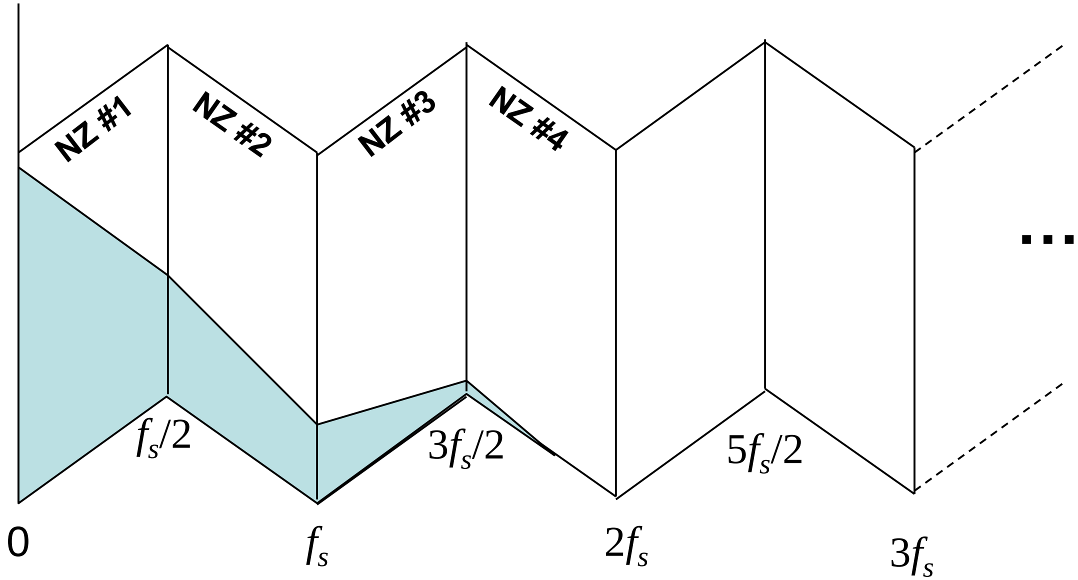


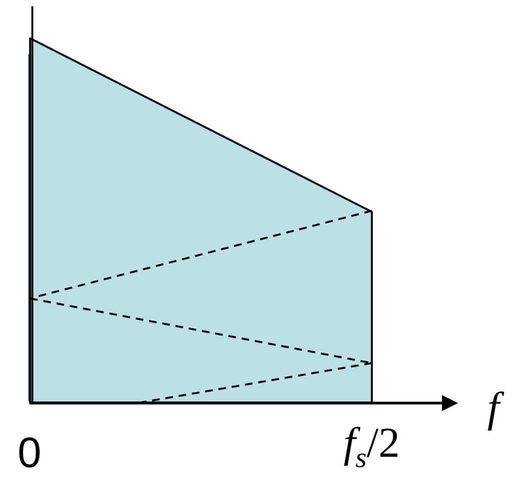


Folding

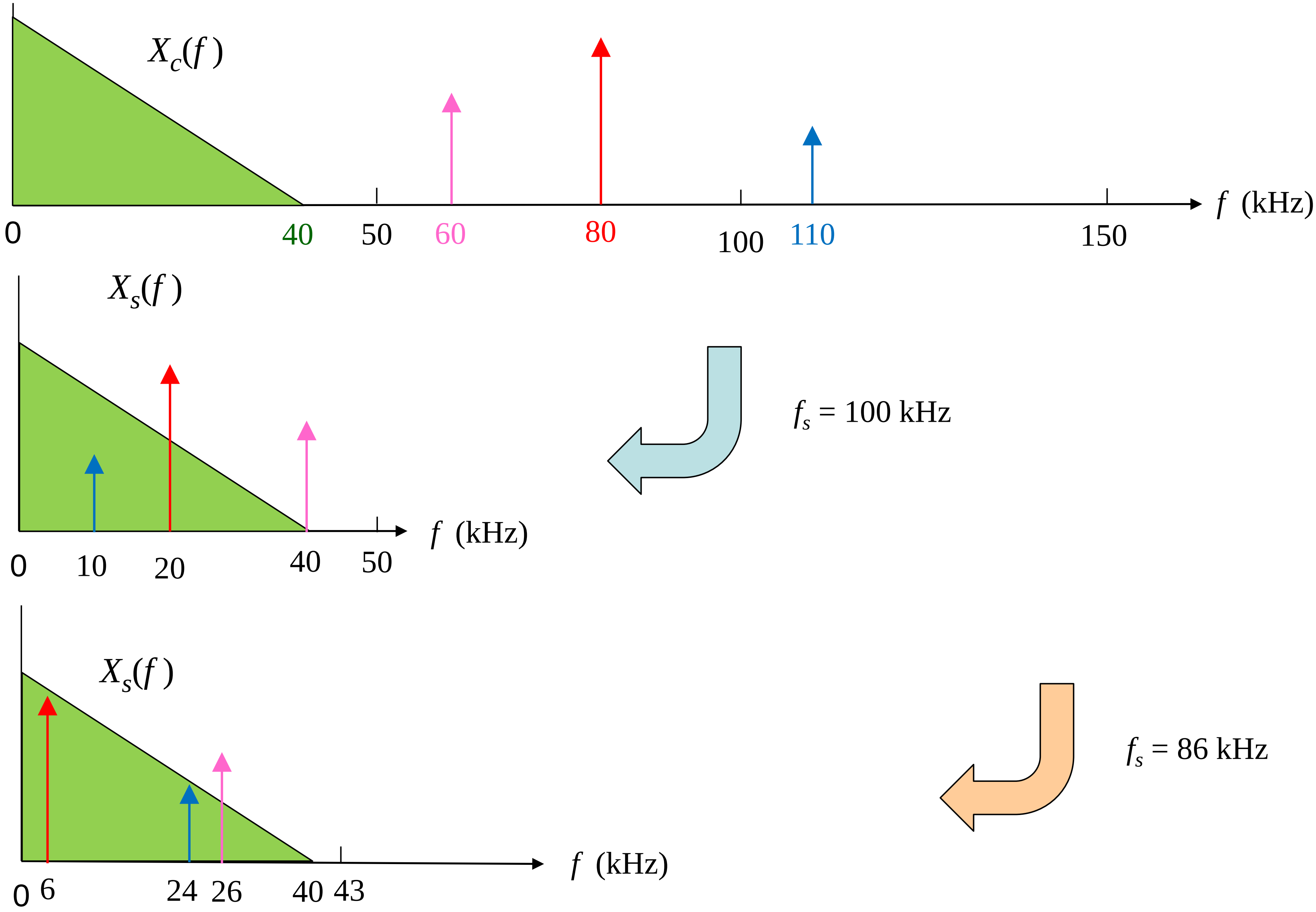
- Another way to visualize the effect of sampling in the frequency domain
- Sampling folds the *f* axis so that all the Nyquist zones overlap on top of the first one.





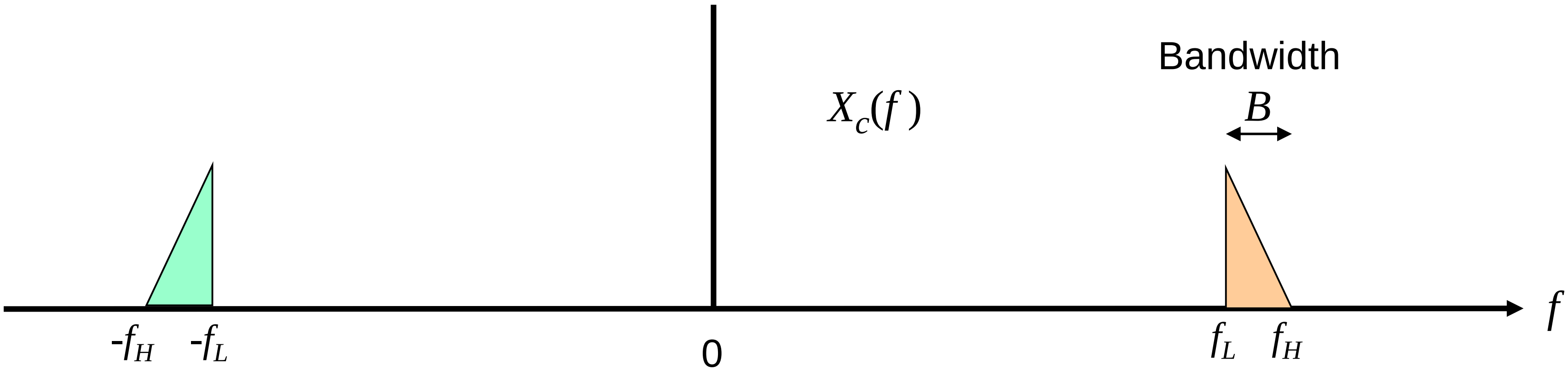


Folding: an example



Bandpass sampling

- Replicas = frequency translated versions of original spectrum
- This can be used to shift the spectral content through sampling
- We can sample bandpass signals with content in $f \in [f_L, f_H]$ at a rate much smaller than $2 \cdot f_H$
- Take care: spectral content from different Nyquist zones will overlap

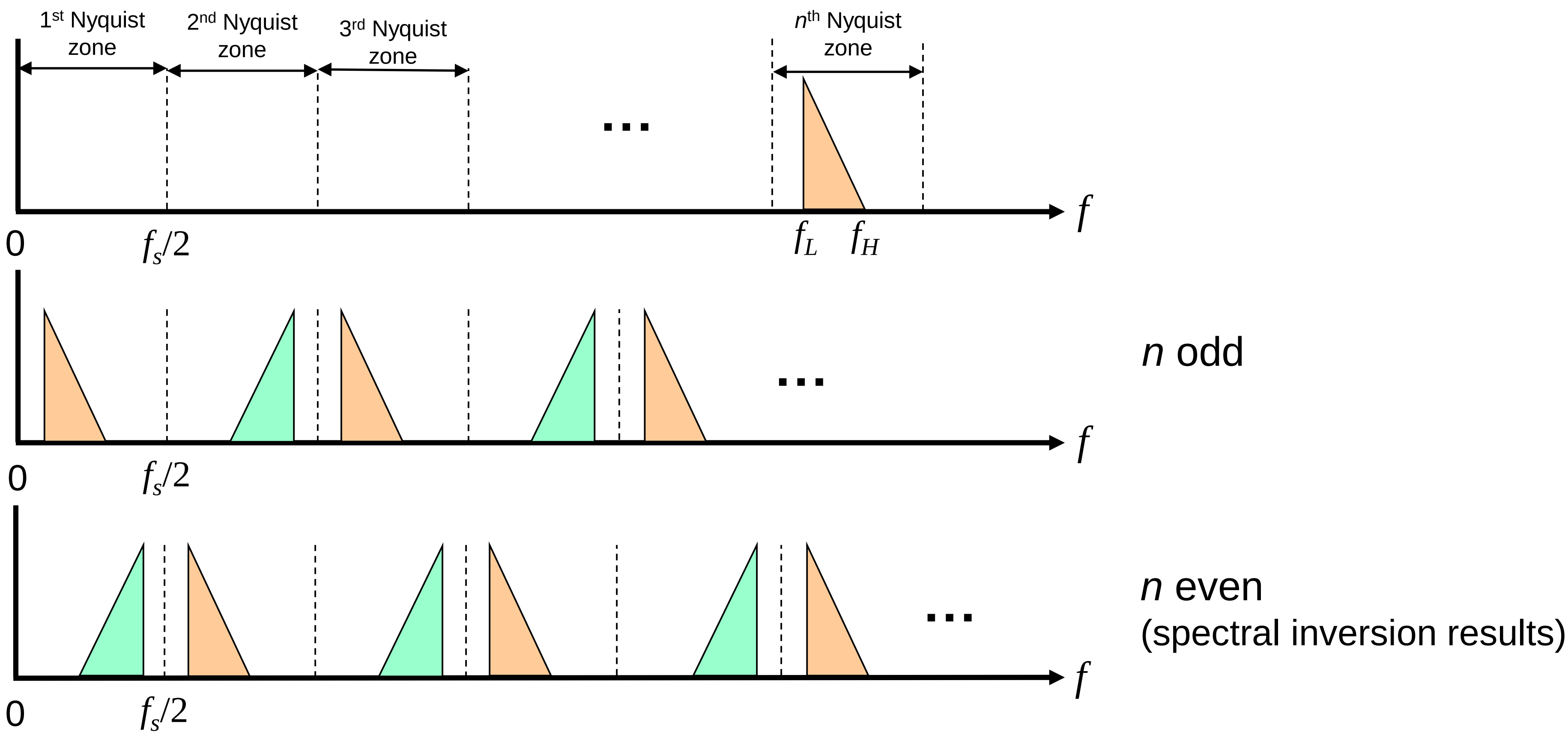


The signal is termed “bandpass” if

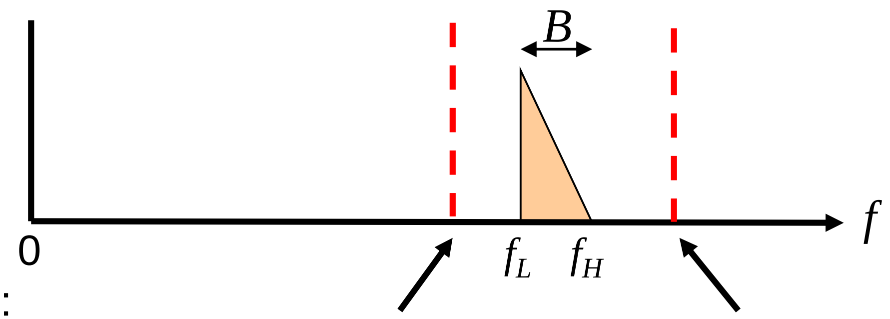
$$B = f_H - f_L << f_H$$

Bandpass sampling (II)

- If we choose f_s so that the original signal is confined to <u>a single</u> Nyquist zone, the replicas due to folding will not overlap!
- Otherwise, they will overlap and we will have **aliasing**



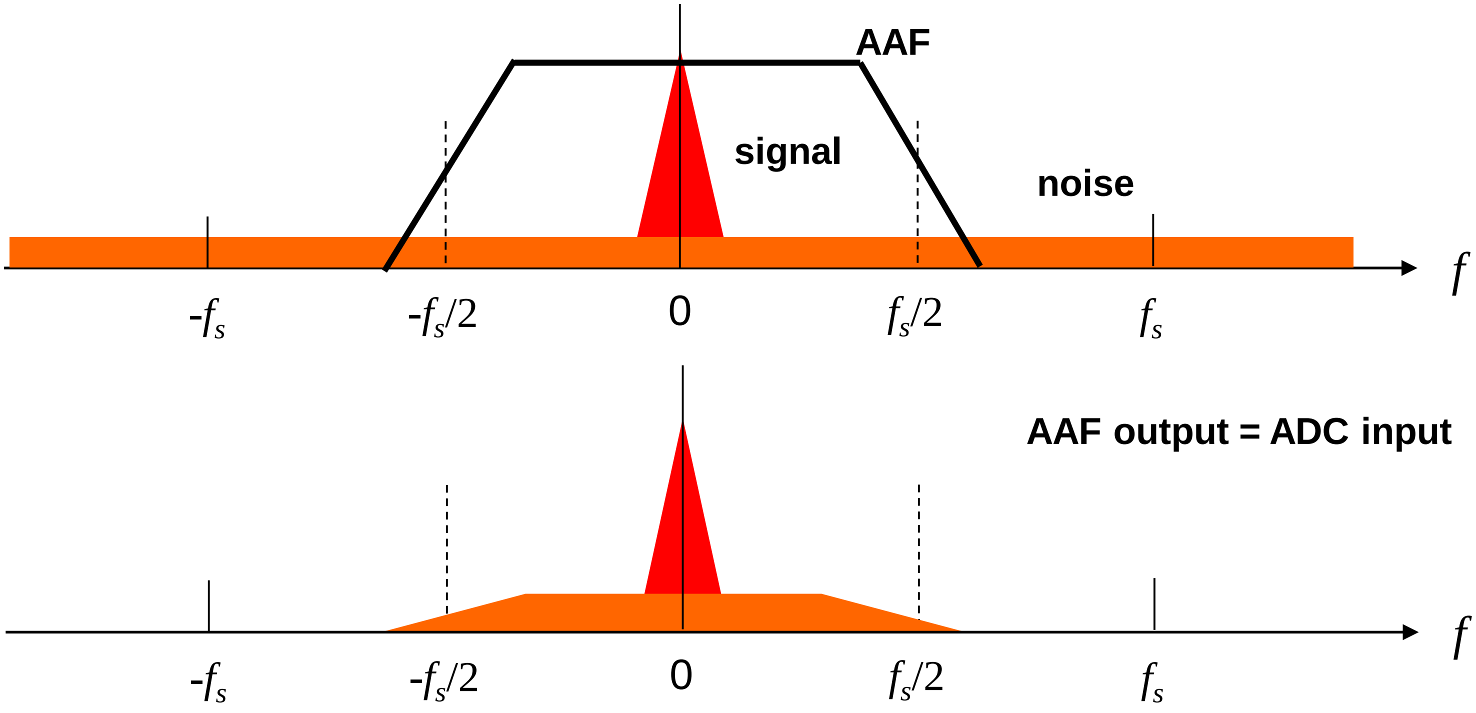
Bandpass sampling (III)



- Conditions on f_s to avoid aliasing?
- Spectrum must fit within a single Nyquist zone: for some integer *n*, we must have $n \cdot \frac{f_s}{2} < f_L \quad f_H < (n+1) \cdot \frac{f_s}{2}$
- From these, the sampling rate must satisfy $$\frac{2f_H}{n+1} < f_s < \frac{2f_L}{n} \quad \text{for some integer } n$$
- Not all integers *n* are valid, since it must hold that $$\frac{2f_H}{n+1} < \frac{2f_L}{n} \quad \Rightarrow \quad n < \frac{f_L}{B}$$
- Note that at the very least we must have $f_s \geq 2B$
- If we want to avoid inversion, it must be an *odd* Nyquist zone, so *n* must be *even*
- Popular in communication RX to demodulate IF signals directly via sampling.
- Caution #1: The ADC *input analog bandwidth* and *distortion* must be adequate at IF, and not just in the 1st Nyquist zone (performance ↓ when input frequency ↑)
- Caution #2: Use a bandpass filter before sampling to reduce *noise folding*.
- Example: f_L = 170 MHz, f_H = 185 MHz $\Rightarrow$ $n \leq 11$. Smallest f_s possible is f_s = 30.8333 MHz.
  If spectral inversion is to be avoided, smallest f_s possible is f_s = 33.6363 MHz.

Oversampling of baseband signals: $f_s >> 2B$

- Practical filters: finite transition bands, finite stopband attenuation
- Oversampling: relaxes analog AAF design;   increases SQNR (we'll see later)

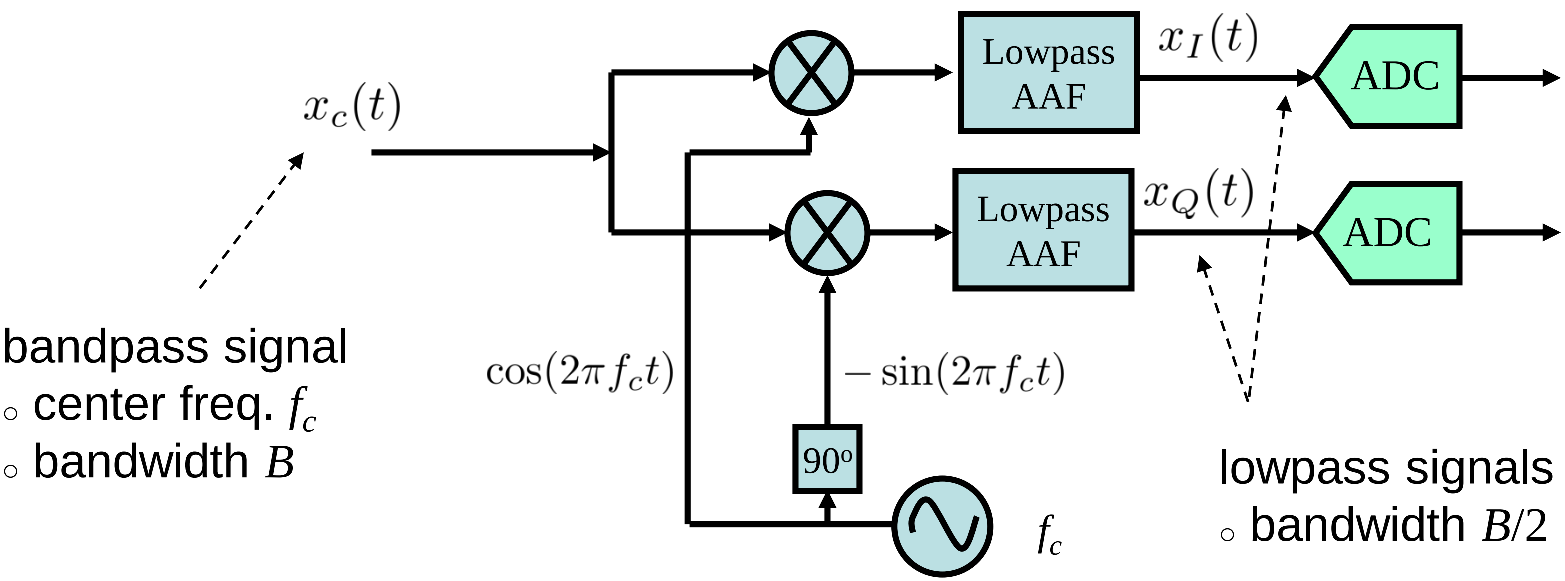


- Tradeoff: sampling rate vs AAF cost
- Sampling rate reduction in the digital domain: *Digital* Antialiasing Filters
- Any aliased components must be attenuated below the quantization noise floor (ADC resolution)

Quadrature sampling

- Recall that for bandpass signals, to avoid aliasing we need $f_s \geq 2B$
- Quadrature Sampling reduces required sampling rate by a factor of 2, and alleviates requirements on ADC analog bandwidth
- But it requires an analog quadrature demodulator and two synchronized ADCs
- If $x_c(t)$ is bandpass, it can be written in terms of its I/Q components:

$$x_c(t) = x_I(t)\cos(2\pi f_c t + \theta) - x_Q(t)\sin(2\pi f_c t + \theta)$$



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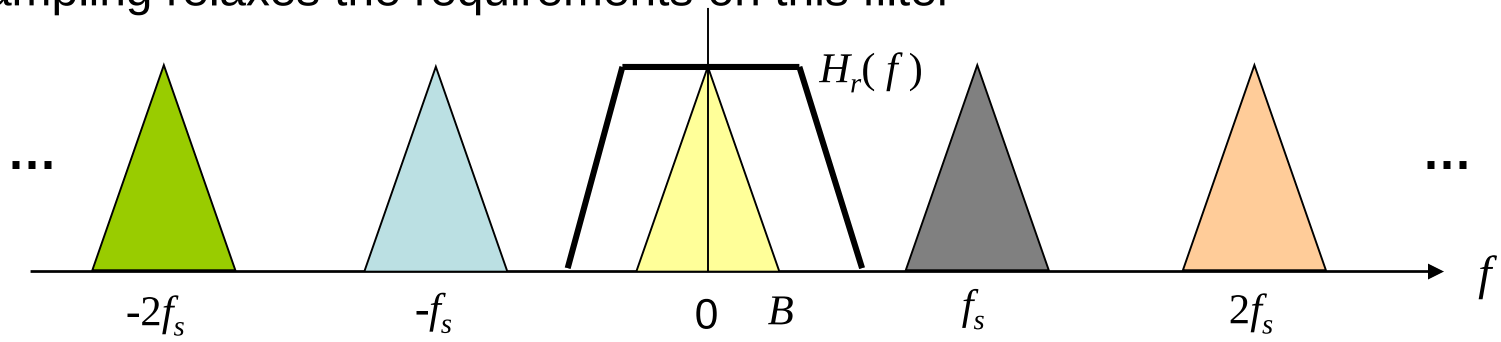
Digital to Analog Conversion

- Conceptually, if there is no aliasing, an LPF (*reconstruction filter*) $H_r(f)$ recovers the original analog signal $x_c(t)$ from the sampled signal $x_s(t) = x_c(t) \cdot p(t)$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \qquad X_s(f) = f_s \cdot \sum_{k=-\infty}^{\infty} X_c(f - k \cdot f_s)$$

- Oversampling relaxes the requirements on this filter



- Ideal reconstruction filter transfer function:

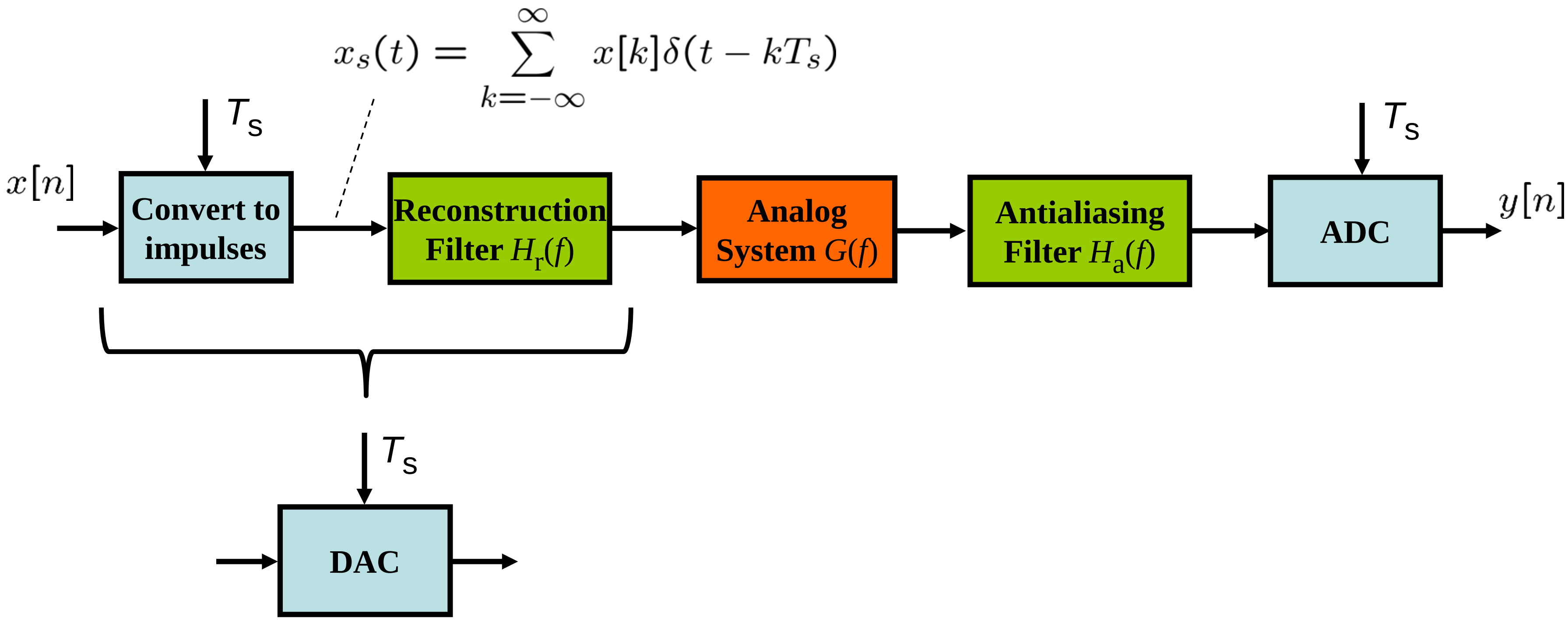
$$H_r(f) = \begin{cases} 1, & |f| < B & \text{(passband)} \\ ?, & B < |f| < f_s - B & \text{(transition band)} \\ 0, & |f| > f_s - B & \text{(stopband)} \end{cases}$$

- Reconstruction filter output:

$$x_r(t) = x_s(t) \star h_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h_r(t - nT_s)$$

Exercise: Analog processing of a digital signal

Show that the following scheme is equivalent to a discrete-time LTI system with impulse response $q[n]$, and find $q[n]$.

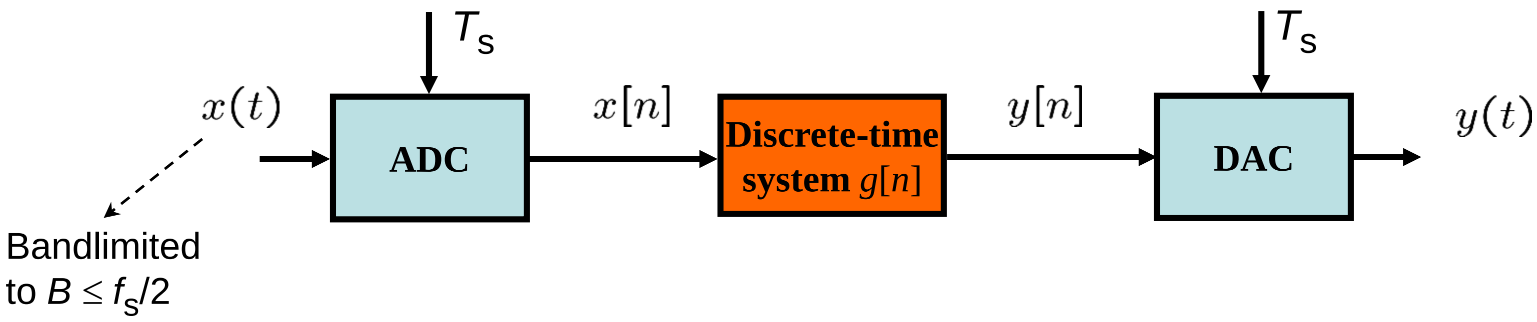


We must show that, for some sequence $q[n]$,

$$y[n] = x[n] \star q[n] = \sum_{k=-\infty}^{\infty} x[k]q[n-k]$$

Exercise: Digital processing of an analog signal

Show that the following scheme is equivalent to a continuous-time LTI system with impulse response $q(t)$ bandlimited to $f_s/2$, and find $q(t)$.



We must show that, for some bandlimited $q(t)$,

$$y(t) = x(t) \star q(t) \qquad \Leftrightarrow \qquad Y(f) = X(f)Q(f)$$

Let $H_r(f)$ be the reconstruction filter of the DAC:

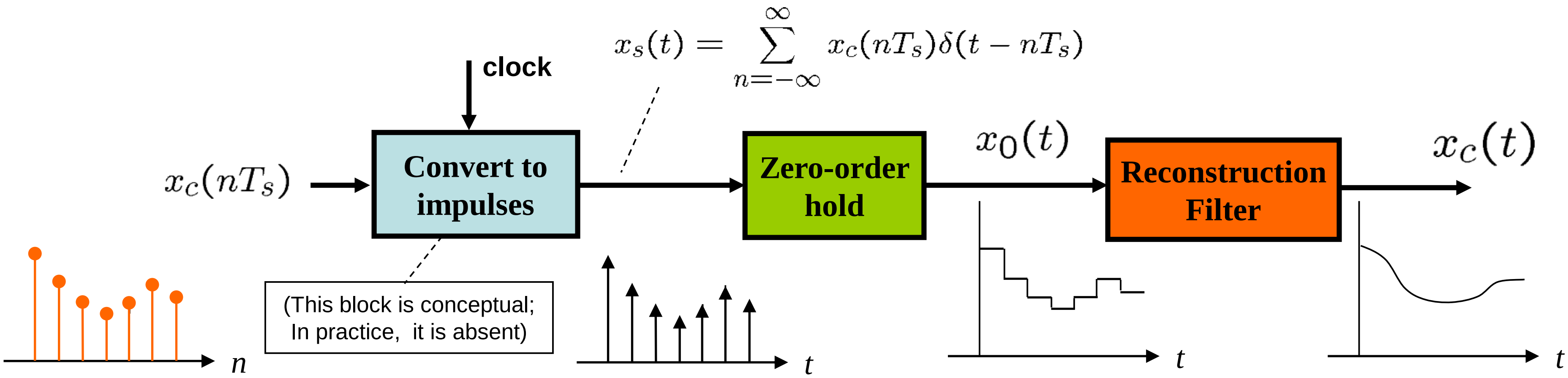
$$y(t) = \sum_{n=-\infty}^{\infty} y[n] h_r(t - nT_s)$$

Also note that, because $x(t)$ is bandlimited and $x[n] = x(nT_s)$:

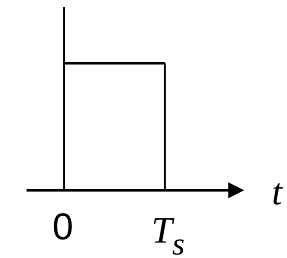
$$x(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT_s)$$

Mathematical model of a DAC

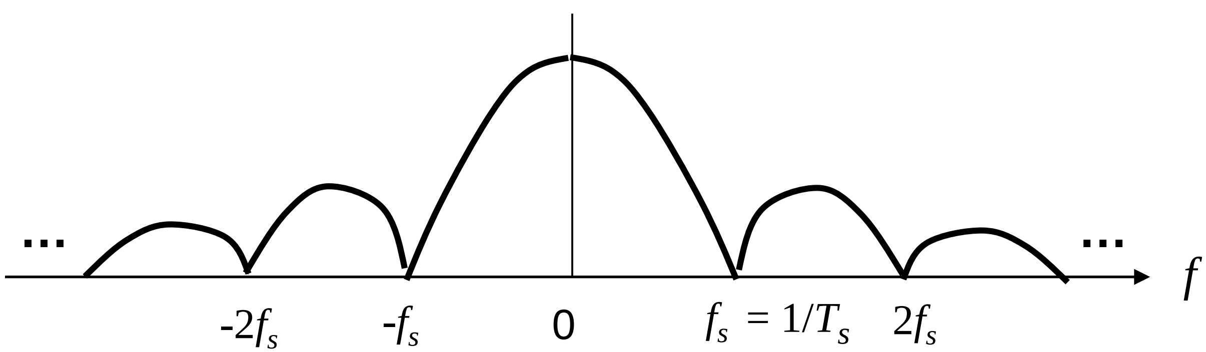
- Practical DACs: *Zero-Order Hold* followed by reconstruction filter



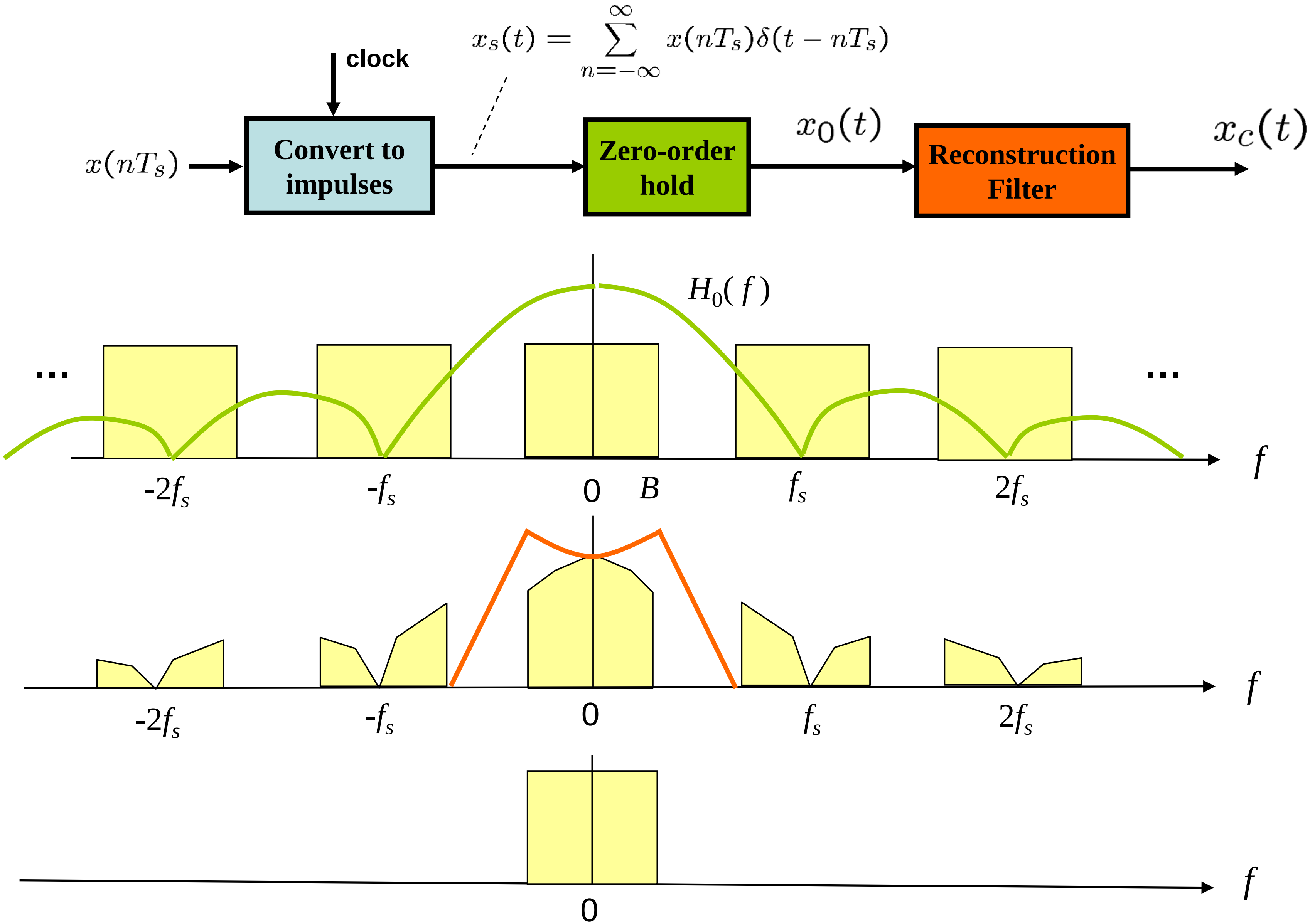
- Zero-Order Hold: LTI system with impulse response $h_0(t) = \begin{cases} 1, & 0 \le t < T_s \\ 0, & \text{else} \end{cases}$



- Transfer function of ZOH: $|H_0(f)| \propto |\mathrm{sinc}(T_s \cdot f)|$



Compensated reconstruction filter



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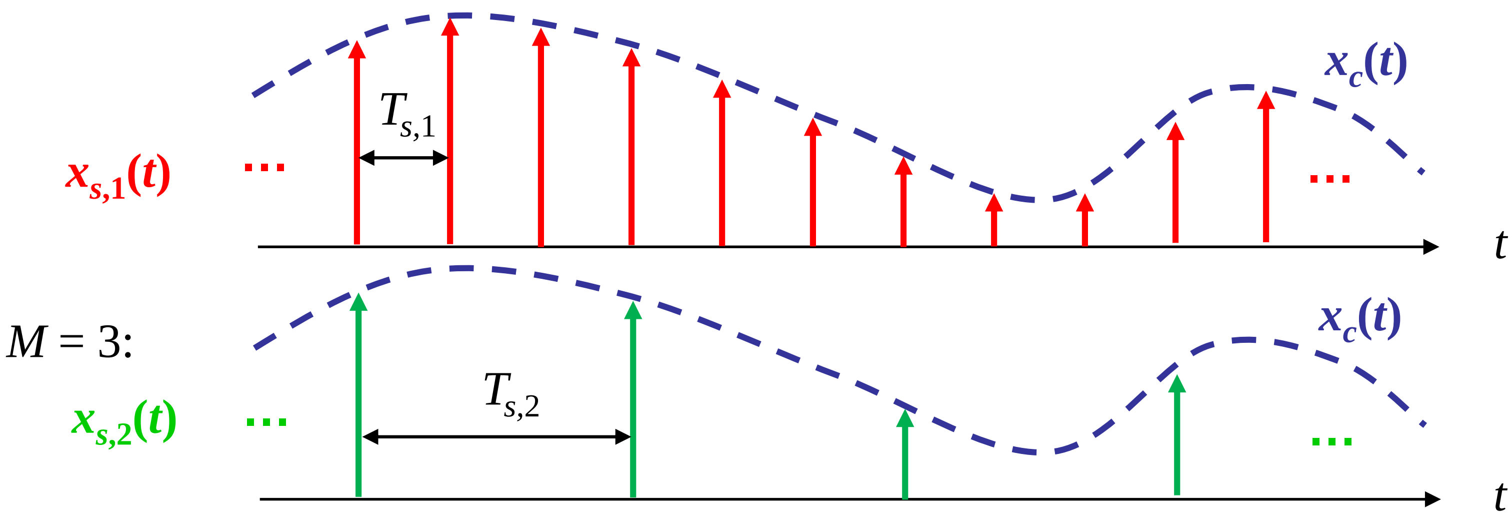
Sample Rate Reduction

- Suppose that we have sampled $x_c(t)$ at rate $f_{s,1}$, so that

$$X_{s,1}(f) = f_{s,1} \sum_{k=-\infty}^{\infty} X_c\left(f - k f_{s,1}\right)$$

- Then we decimate the set of samples, by keeping only 1 out of every M of them.



- The result is the same as if we had sampled $x_c(t)$ at rate $f_{s,2} = f_{s,1}/M$, thus:

$$X_{s,2}(f) = \frac{f_{s,1}}{M} \cdot \sum_{m=-\infty}^{\infty} X_c\left(f - m \cdot \frac{f_{s,1}}{M}\right)$$

Sample Rate Reduction (2)

$$X_{s,1}(f) = f_{s,1} \sum_k X_c\left(f - k f_{s,1}\right)$$

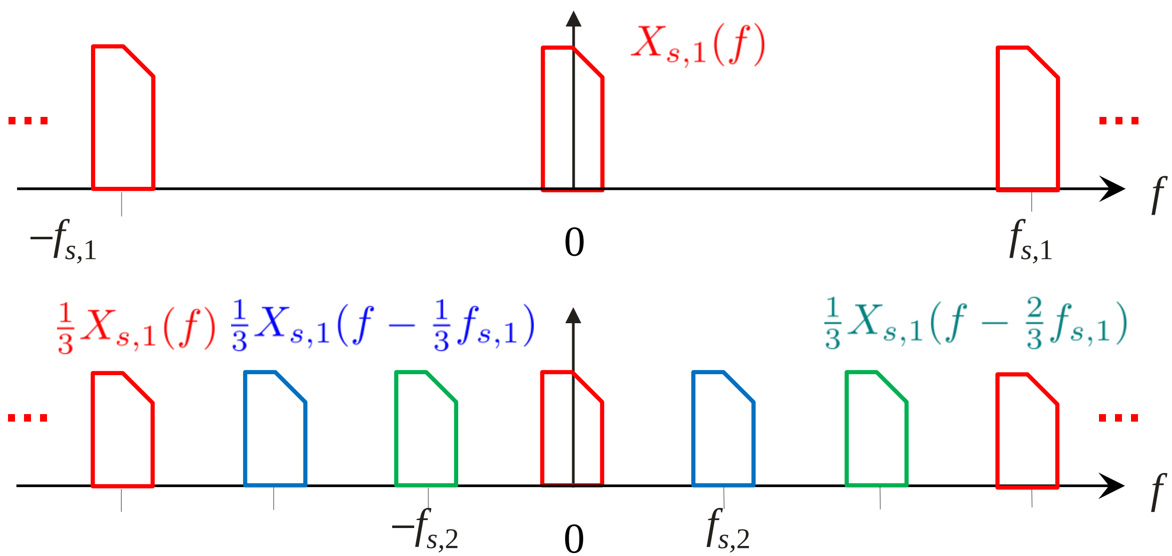
$$X_{s,2}(f) = \frac{f_{s,1}}{M} \sum_m X_c\left(f - m\frac{f_{s,1}}{M}\right)$$

- Suppose we only know $X_{s,1}(f)$ and not $X_c(f)$ (*e.g.*, there was aliasing).
- How can we find $X_{s,2}(f)$?

Any integer m can be written as $m = kM + l$ for some integers k, l and $l \in \{0, 1, \ldots, M\text{-}1\}$ ($k = \lfloor m/M \rfloor$, and l is the remainder).

$$X_{s,2}(f) = \frac{1}{M} \sum_{\ell=0}^{M-1} X_{s,1}\left(f - \ell\frac{f_{s,1}}{M}\right)$$

$M = 3$:



Note: aliasing may occur after decimation!

Use a digital AAF: LPF with cutoff freq. corresponding to $f_{s,2}/2 = f_{s,1}/(2M)$ Hz

$$\rightarrow \boxed{h[n]} \rightarrow \boxed{\downarrow M} \rightarrow$$

Sample Rate Increase

- Suppose that we have sampled $x_c(t)$ at rate $f_s = 1/T_s$, but we want to obtain the samples at a higher rate $L{\cdot}f_s$ (where L is an integer larger than 1)

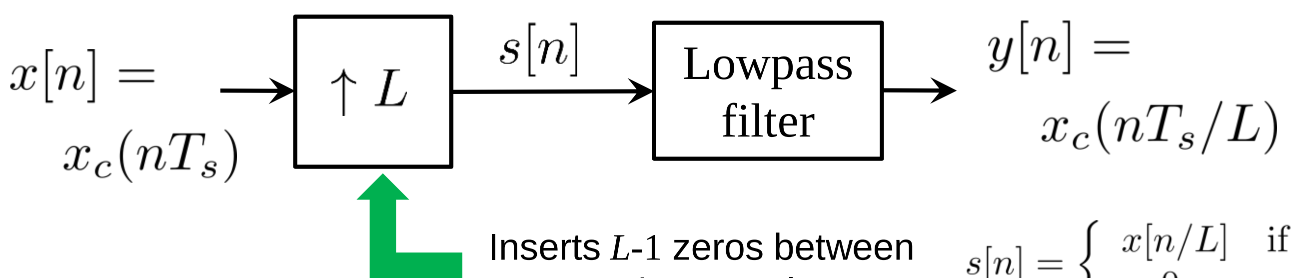
- If there was no aliasing, $x_c(t)$ can be recovered from its samples for all t by lowpass filtering (cutoff freq. $f_s/2$):

$$\begin{aligned} x_c(t) &= x_s(t) \star h_r(t) \\ &= \sum_k x_c(kT_s) h_r(t - kT_s) \end{aligned}$$

- In particular, for $t = nT_s/L$:

$$x_c\left(n\frac{T_s}{L}\right) = \sum_k x_c(kT_s) h_r\left((n - kL)\frac{T_s}{L}\right)$$

- This shows that we can obtain the samples at the higher rate by directly operating on the samples at the lower rate (only digital processing involved).



$$s[n] = \begin{cases} x[n/L] & \text{if } n/L \text{ integer} \\ 0, & \text{else} \end{cases}$$

Sample Rate Increase (2)

$$x_c\left(n\frac{T_s}{L}\right) = \sum_k x_c(kT_s)h_r\left((n-kL)\frac{T_s}{L}\right)$$

$$x[n] \rightarrow \boxed{\uparrow L} \xrightarrow{s[n]} \boxed{h[n]} \rightarrow y[n]$$

Any integer m can be written as $m = kL + l$ for some integers k, l and $l \in \{0,1,\ldots,L\text{-}1\}$

In particular, $k = \lfloor m/L \rfloor$, and l is the remainder.

$$s[n] = \begin{cases} x[n/L], & \text{if } n/L \text{ integer,} \\ 0, & \text{else.} \end{cases}$$

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} s[m]h[n-m] \\ &= \sum_{k=-\infty}^{\infty} \sum_{\ell=0}^{L-1} s[kL+\ell]h[n-kL-\ell] \\ &= \sum_{k=-\infty}^{\infty} s[kL]h[n-kL] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-kL] \end{aligned}$$

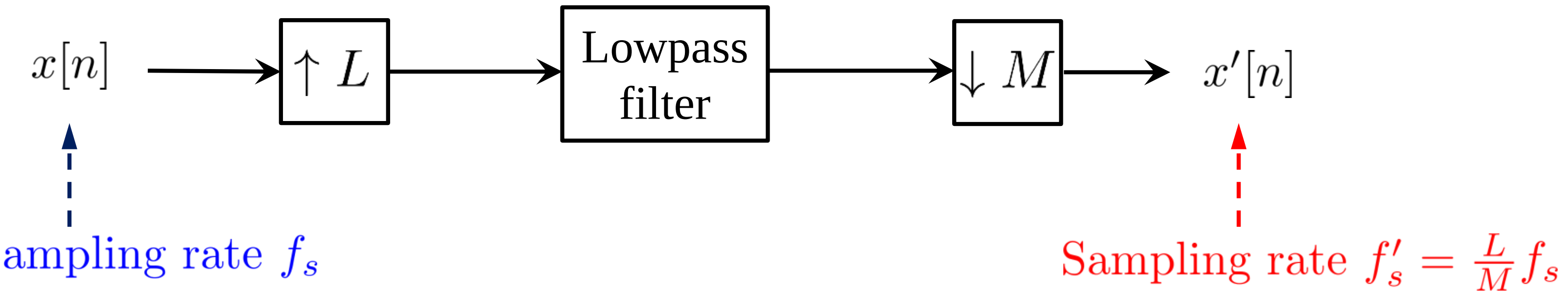
- Hence, if we take $h[n] = h_r(nT_s/L)$ and $x[n] = x_c(nT_s)$, we get $y[n] = x_c(nT_s/L)$.
- In general $h[n]$ is a digital LPF operating at L/T_s samples/s, with cutoff frequency corresponding to $1/(2T_s)$ Hz
- Known as *interpolation filter*

General sample rate change

- Suppose that we have sampled $x_c(t)$ at rate f_s , but we want to obtain the samples at a rate $f_s' = L{\cdot}f_s /M$ (where L, M are integers larger than 1)

- Note that the new sample rate may be higher or lower than the original one

- First we increase the sample rate by a factor of L (interpolation)
- Then we reduce the sample rate by a factor of M (decimation)
- The order is important to avoid aliasing!



- The LPF operates at Lf_s samples/s, acts as reconstruction filter for the interpolation stage and as AAF for the decimation stage
- Its cutoff freq. must correspond to the minimum of $f_s/2$ and $f_s'/2 = Lf_s /(2M)$
- If L is large, the LPF works at a very high rate. More efficient implementations are possible, based on *polyphase structures*

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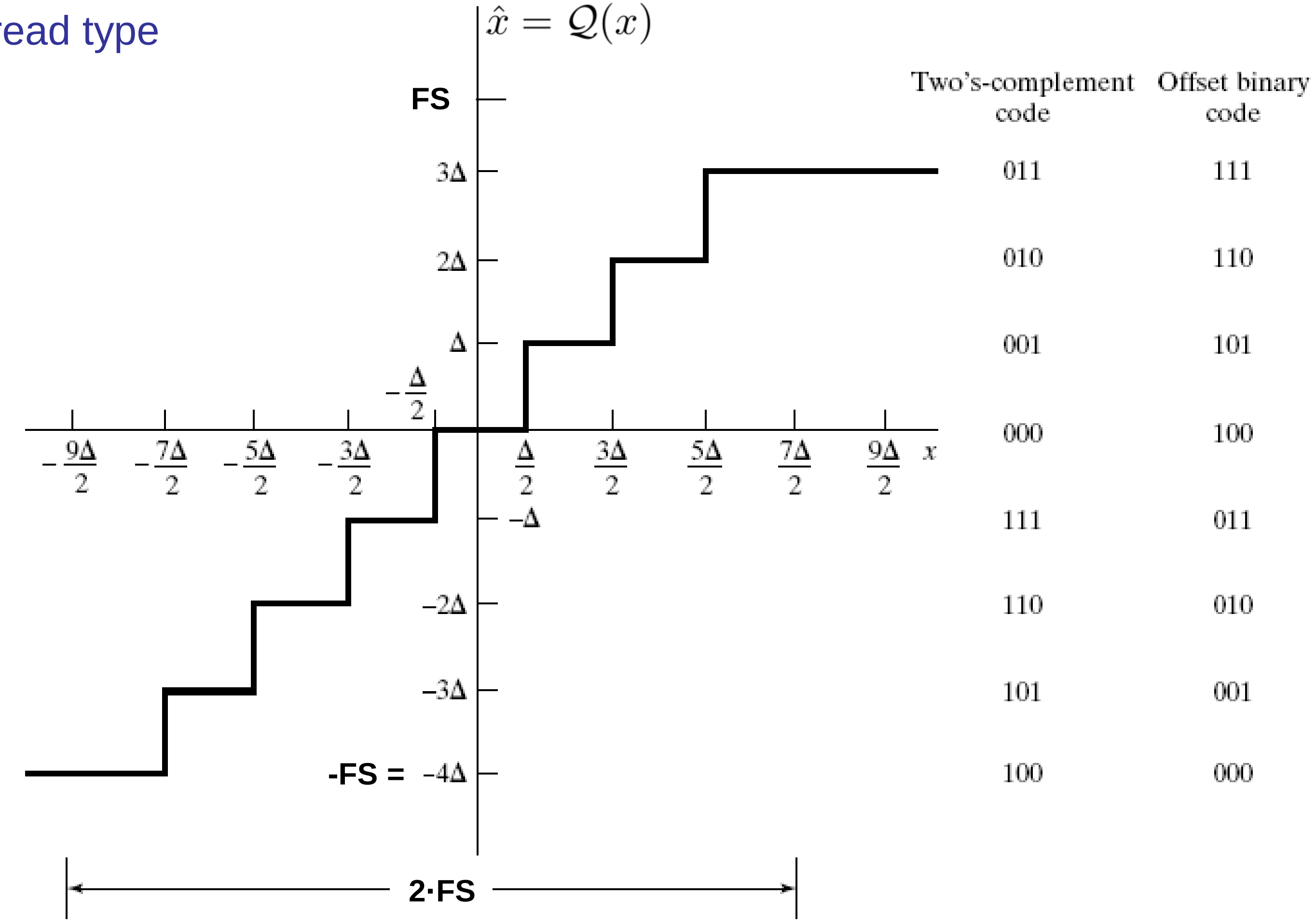
Uniform quantizer

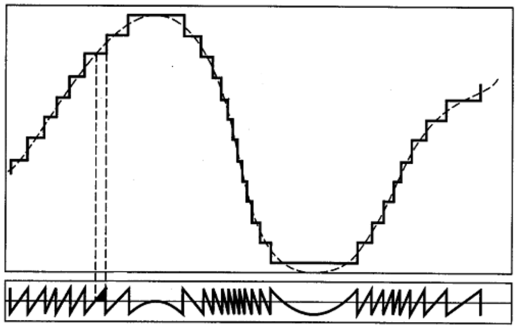
- Example: N = 3 bits, midtread type



- N = total number of bits (including sign bit)
- Full Scale level = FS
- Quantization step Δ = weight of the LSB = (2·FS) / 2^N
- 2^N quantized levels, from –FS to (FS – Δ)
- Weight of the MSB = FS/2

MSB: *Most Significant Bit*
LSB: *Least Significant Bit*

Quantization error



- Difference between original and quantized value:

$$v_q[n] = x_c(nT_s) - \mathcal{Q}\{x_c(nT_s)\}$$

- As long as the input <u>does not get clipped</u>, the quantization error will satisfy

$$-\frac{\Delta}{2} \le v_q[n] \le \frac{\Delta}{2}$$

- If the input signal is "spectrally rich", we can assume that the quantization error:
  - is uncorrelated with the signal
  - is uniformly distributed in [-Δ/2, Δ/2]
  - is a white process: $\mathrm{E}\{v_q[n]v_q[m]\} = 0$ for $n \neq m$
- The mean and variance of the quantization error are then zero and $\sigma_q^2 = \frac{\Delta^2}{12}$
- Signal-to-Quantization Noise Ratio (SQNR) (we assume a zero-mean signal, so that its power equals its variance):

$$\begin{aligned} \mathrm{SQNR} &= 10\log_{10}\frac{\sigma_x^2}{\sigma_q^2} \\ &= 6.02N + 4.77 - 20\log_{10}\frac{\mathrm{FS}}{\sigma_x} \quad \text{(dB)} \end{aligned}$$

where we have used $\sigma_q^2 = \frac{\Delta^2}{12}$ and $\Delta = \frac{2\cdot\mathrm{FS}}{2^N}$

Full-Scale Range Usage

$$\mathrm{SQNR} = 6.02N + 4.77 - 20\log_{10}\frac{\mathrm{FS}}{\sigma_x} \quad (\mathrm{dB})$$

- This assumes there is no clipping.

- For a sinusoid of amplitude $A = \alpha\cdot\mathrm{FS}$, $\alpha < 1$ (no clipping), we have $\sigma_x = A/\sqrt{2} = \alpha\,\mathrm{FS}/\sqrt{2}$ and

$$\mathrm{SQNR} = 6.02N + 1.76 + 20\log_{10}\alpha \quad (\mathrm{dB})$$

- When input occupies only a fraction $\alpha < 1$ of the full-scale range (FSR), resolution is lost (SQNR degrades).

- For example, if only 25% of the FSR is occupied, then the SQNR is reduced by 12 dB !

- To maintain signal quality, signal gain must be *adaptively* adjusted *previously* to A/D conversion:

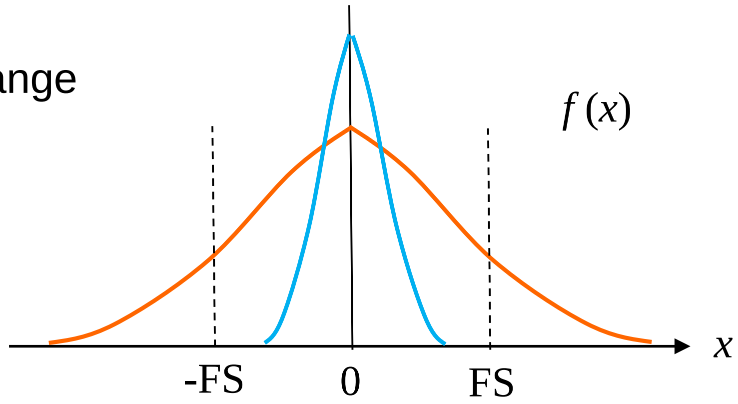
    - too little gain $\Rightarrow$ $\alpha \ll 1$ $\Rightarrow$ large degradation

    - too much gain $\Rightarrow$ clipping $\Rightarrow$ large degradation

- FSR utilization is determined in order to keep the probability of clipping at some tolerable level

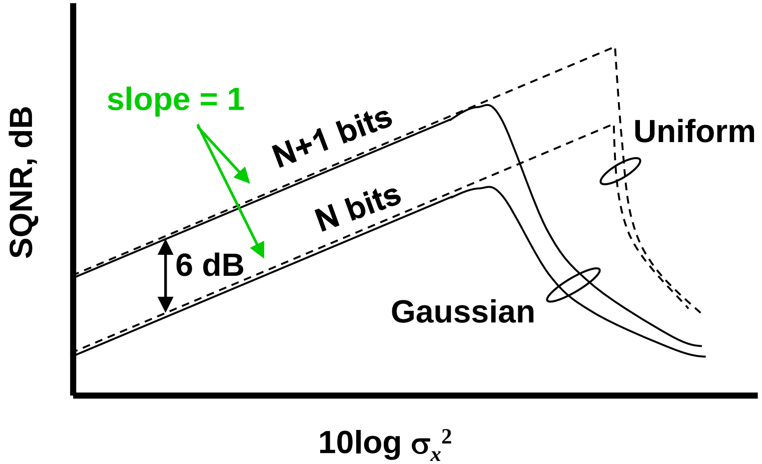
- In practice, it is common to back off to 50% of FSR utilization (1 bit loss, or 6 dB) to make room for sudden power jumps.

Overload distortion (clipping)



- Clipping occurs when the ADC input exceeds the FS range
- Let $f(x)$ be the probability density function (pdf) of the unquantized samples $x[n] = x_c(nTs)$
- The quantization error power is then

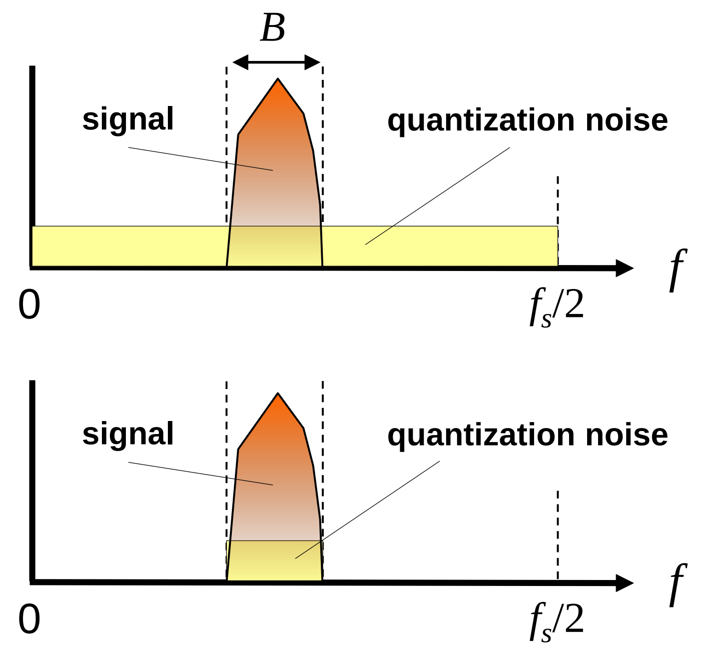
$$
\begin{aligned}
\sigma_q^2 &= E\{(x - \mathcal{Q}\{x\})^2\} \\
&= \int_{-\infty}^{\infty} (x - \mathcal{Q}\{x\})^2 f(x) dx \\
&= \int_{-\infty}^{-\mathrm{FS}} (x - (-\mathrm{FS}))^2 f(x) dx + \int_{-\mathrm{FS}}^{\mathrm{FS}} (x - \mathcal{Q}\{x\})^2 f(x) dx + \int_{\mathrm{FS}}^{\infty} (x - \mathrm{FS})^2 f(x) dx
\end{aligned}
$$



- Clipping is unavoidable in practice
- We can control how often clipping occurs via AGC
- Tradeoff: avoiding clipping vs. quantization error
- Optimal operation point depends on application (signal pdf)

AGC: Automatic Gain Control

Oversampling



- Quantization error power is uniformly distributed from 0 to $f_s/2$
- If the signal of interest occupies a smaller bandwidth B we can apply a digital filter to remove out-of-band noise
- This assumes that the wordlength of the DSP system is (significantly) larger than *N* bits
- Doing this, the SQNR is improved by a gain factor $(f_s/2)/B$:

$$\mathrm{SQNR} = 6.02N + 4.77 - 20\log_{10}\frac{\mathrm{FS}}{\sigma_x} + 10\log_{10}\frac{f_s}{2B} \quad (\mathrm{dB})$$

- Oversampling + filtering improves SQNR !
- But caution! The quantization noise becomes more correlated (i.e., our assumption $\mathrm{E}\{v_q[n]v_q[m]\} = 0$ for $n \neq m$ starts to break down), and harmonics tend to appear.

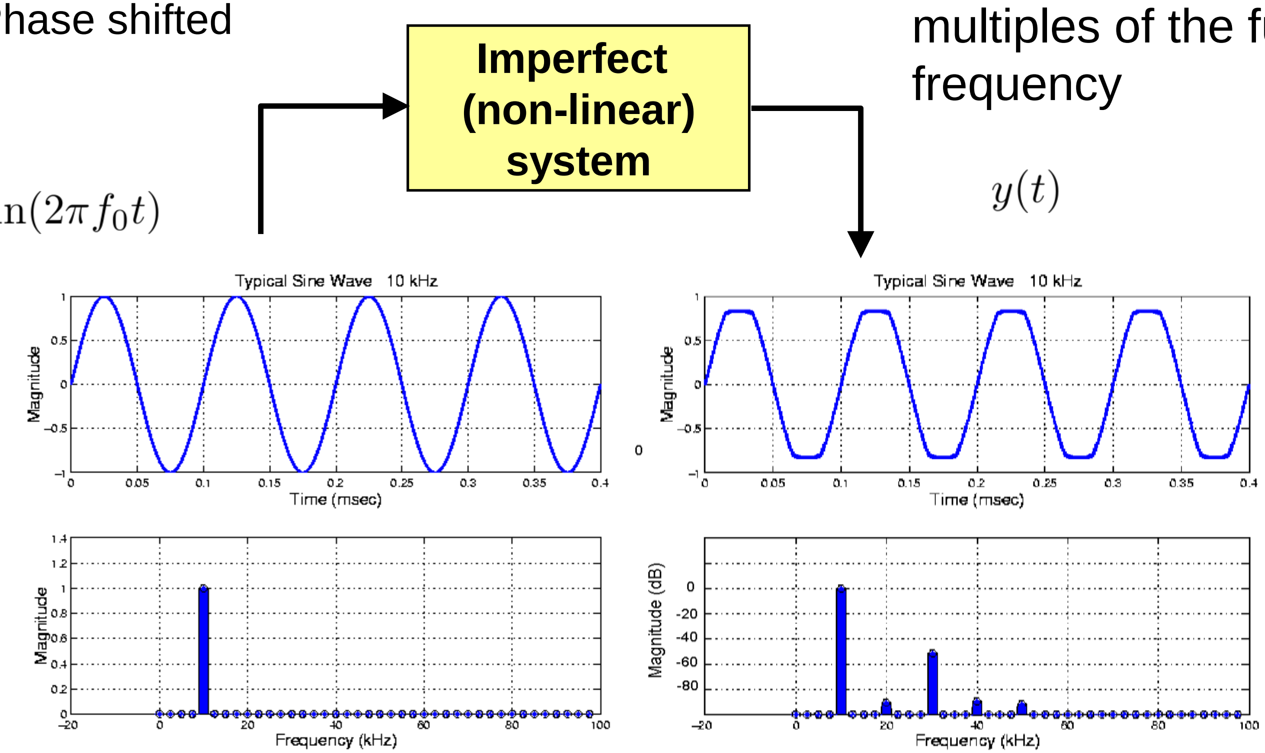
Contents

- Sampling and reconstruction of analog signals
  - Sampling techniques
  - Reconstruction
  - Sample rate conversion
- Practical Analog-to-Digital Converters
  - Quantization
  - ADC parameters and distortion types

Nonlinear distortion

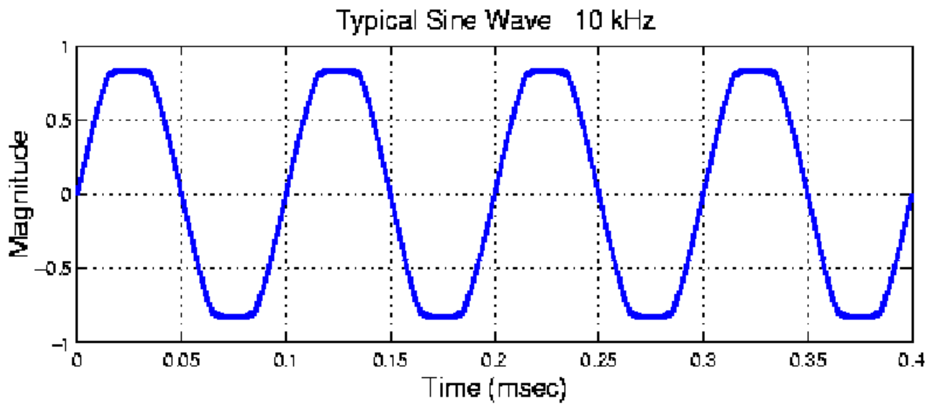
- A **Linear System**:
  - Sine wave in = Sine wave out
    - Amplitude may be reduced or increased
    - Phase shifted
- A **Non-Linear System** will **distort** a signal
  - Sine wave in = Sine wave out + Harmonics at integer multiples of the fundamental frequency

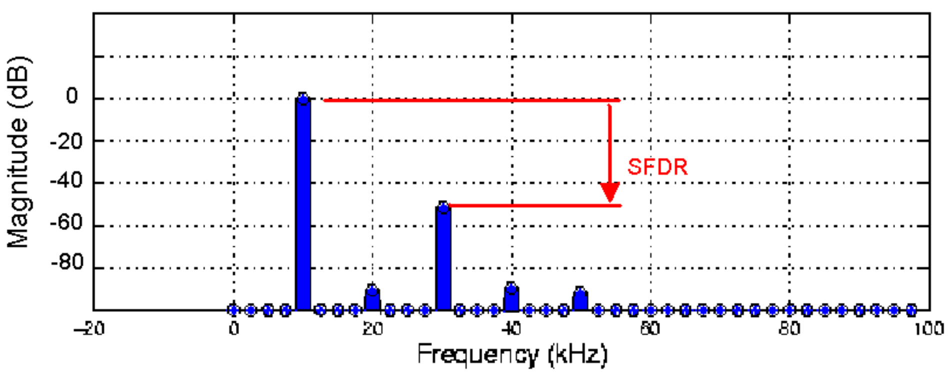
$$x(t) = A\sin(2\pi f_0 t)$$



- This **THD** (Total Harmonic Distortion) of the signal is a measure of system **linearity**

Nonlinear distortion metrics





- **THD = Total Harmonic Distortion**

  Measure of the power of all harmonics relative to the fundamental (usually an FS sinusoidal input signal)

$$\text{THD} = \frac{P_{\text{harmonics}}}{P_{\text{fundamental}}}$$

$$\text{THD} = 10\log\frac{A_2^2 + A_3^2 + A_3^2 + \cdots}{A_1^2} \quad \text{dB}$$

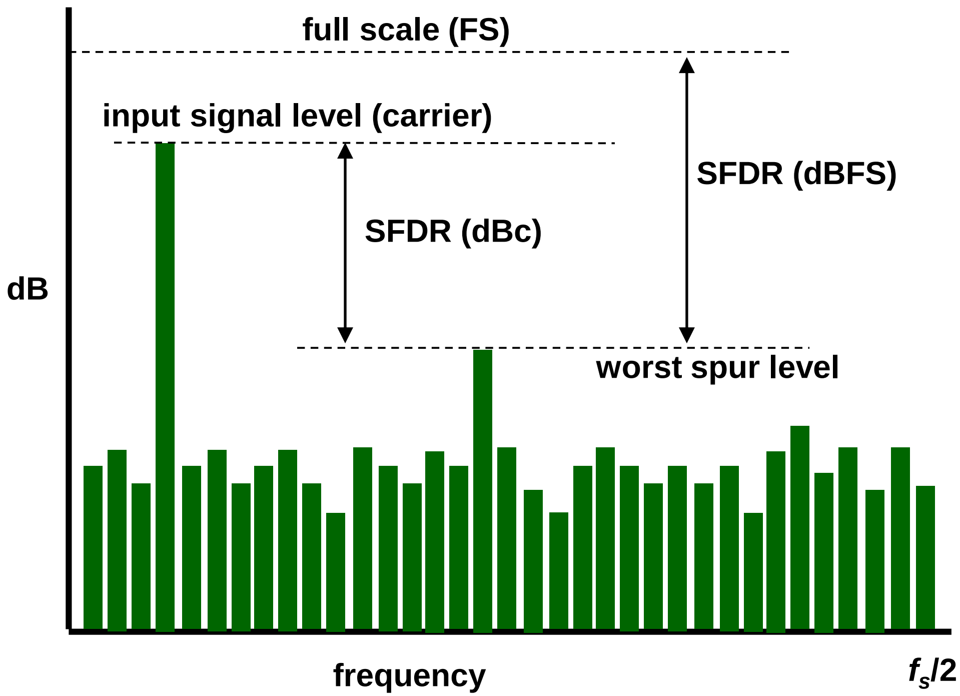
$$\text{THD} = 100 \times \frac{\sqrt{A_2^2 + A_3^2 + A_3^2 + \cdots}}{A_1} \quad \%$$

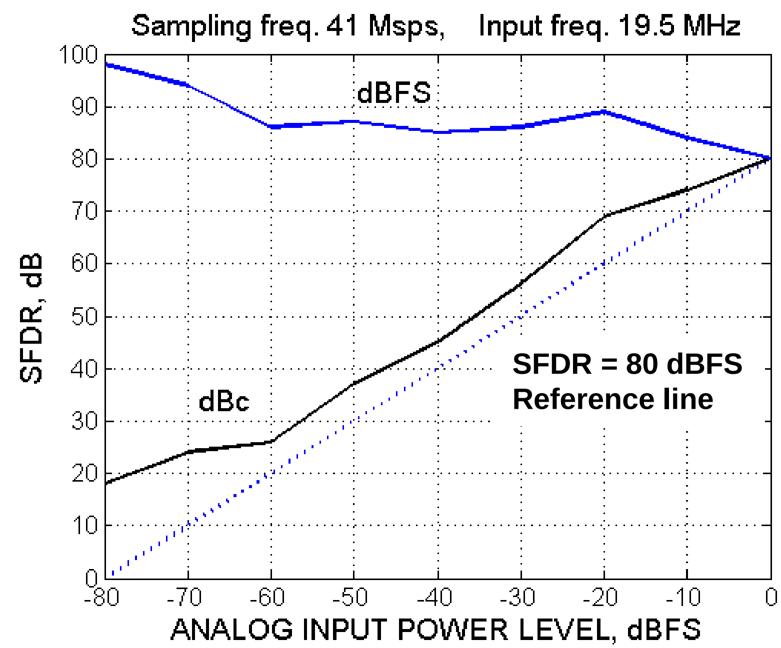
- **SFDR = Spurious-Free Dynamic Range**
  - Magnitude of largest harmonic relative to the fundamental
  - In this example, about 50 dB = 0.32%

- **IEEE 1241-2000 THD Specification**: compute for first 9 harmonics (A_2 through A_{10})

Spurious-Free Dynamic Range

- SFDR depends on input frequency and amplitude
- Can be specified in dBFS or dBc
- Increasing the resolution of an ADC will improve SQNR, but it may or may not improve SFDR





- SFDR (dBFS) = FS (dB) – WSL (dB)
- SFDR (dBc) = CL (dB) – WSL (dB)

WSL: *Worst Spur Level*
CL: *Carrier Level*

- Therefore : SFDR (dBc) = SFDR (dBFS) + (CL – FS)
  = SFDR (dBFS) + CL (dBFS)

FFT analysis

- Assume input = FS sinewave with freq $f_0 = (k_0 / M)\, f_s$
- M = FFT size
- Then the largest DFT coefficient $X[k_0]$ takes the value $$|X[k_0]|^2 = \left(\frac{\mathrm{FS}\cdot M}{2}\right)^2$$
- Quantization noise $v_q[n]$ has variance $\sigma^2_q = \Delta^2/12$ and is uncorrelated
- DFT of quantization noise, $V_q[k]$, is also noise:
  - Its mean is zero
  - Its autocorrelation:

$$\begin{aligned} E\{V_q[k]V_q^*[\ell]\} &= E\left\{\sum_{n=0}^{M-1} v_q[n]e^{-j\frac{2\pi}{M}kn} \cdot \sum_{m=0}^{M-1} v_q[m]e^{j\frac{2\pi}{M}\ell m}\right\} \\ &= \sum_{n=0}^{M-1}\sum_{m=0}^{M-1} E\{v_q[n]v_q[m]\}e^{j\frac{2\pi}{M}(\ell m - kn)} \\ &= \sum_{n=0}^{M-1} \sigma_q^2 e^{j\frac{2\pi}{M}(\ell-k)n} = M\cdot\frac{\Delta^2}{12}\delta[\ell-k] \end{aligned}$$

FFT analysis (II)

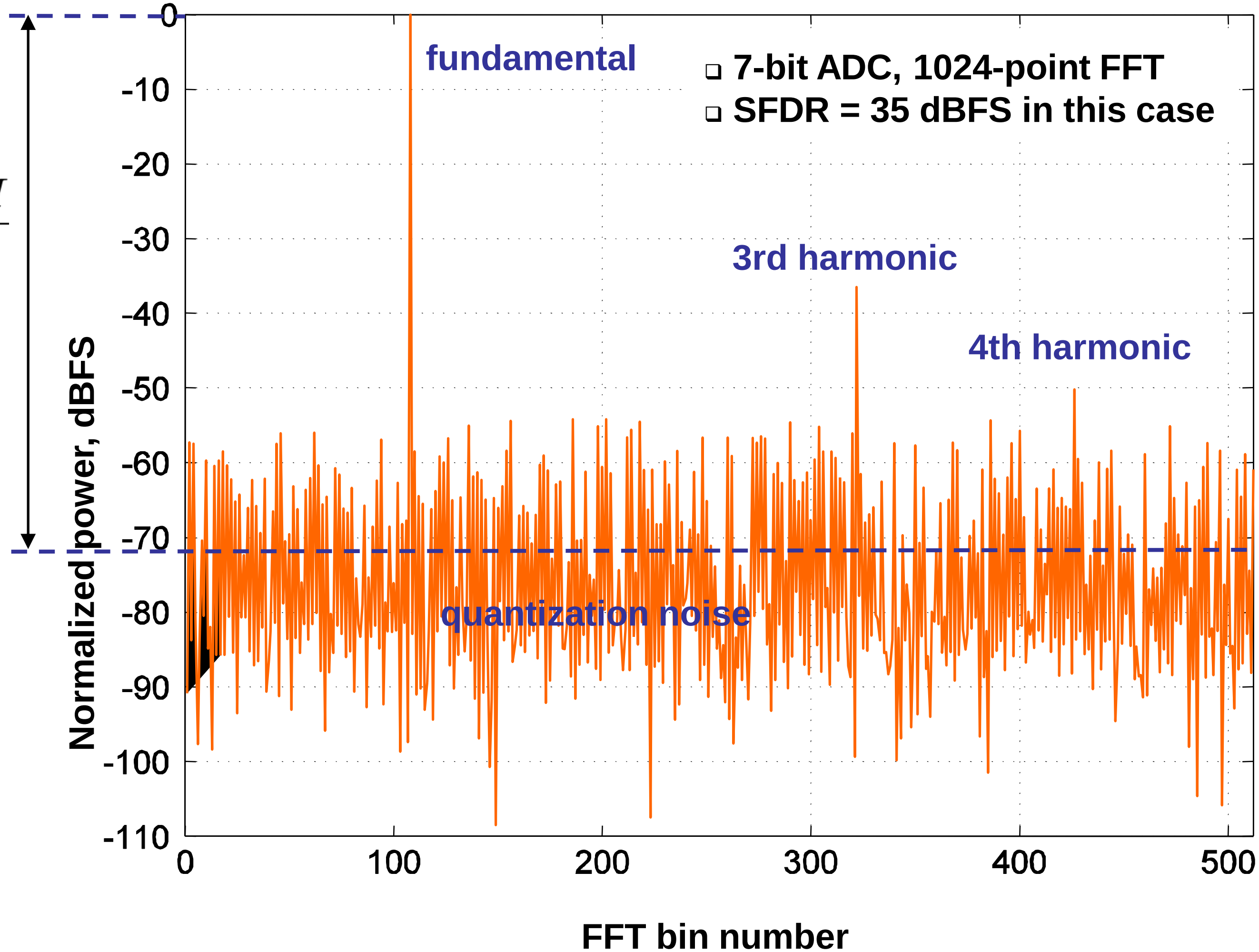
$$10\log\frac{|X[k_0]|^2}{\sigma_V^2} = $$

$$= \quad 6.02N + 1.76 + 10\log\frac{M}{2}$$

$$= \quad 71 \text{ dB}$$

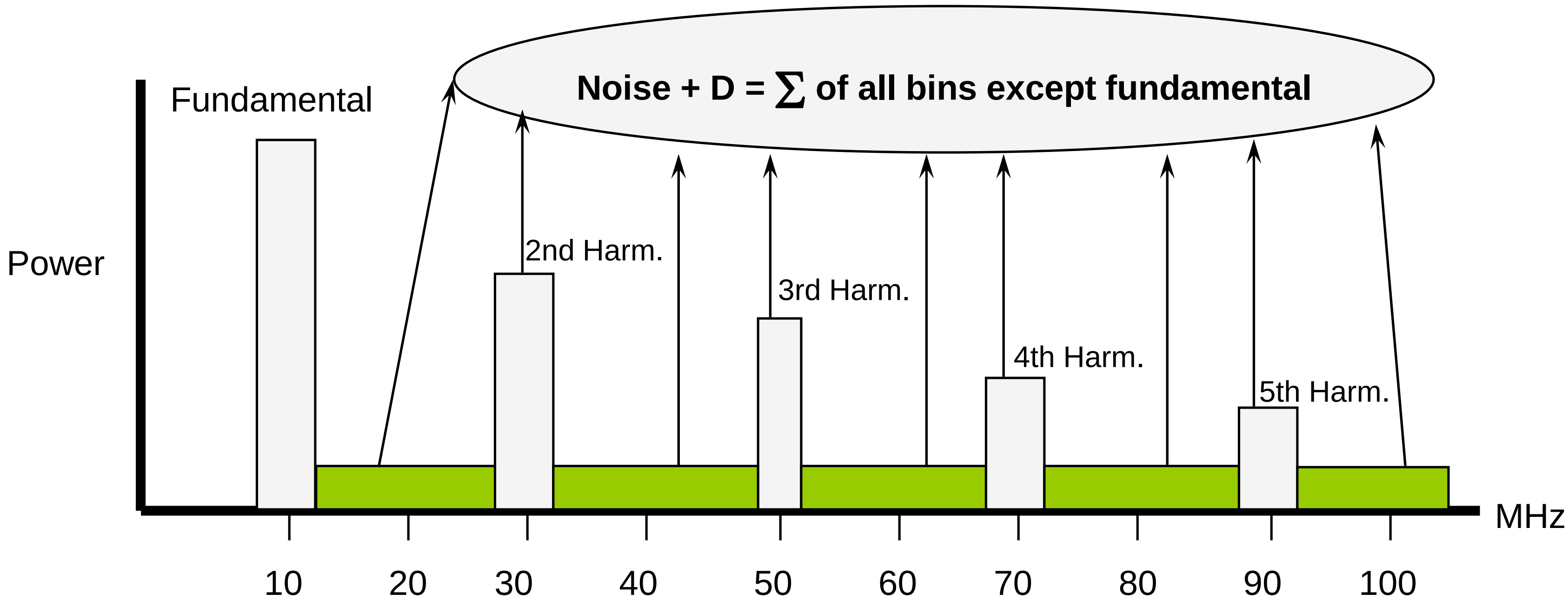
$$|X[k_0]|^2 = \left(\frac{\text{FS}\cdot M}{2}\right)^2$$

$$\sigma_V^2 = M \cdot \frac{\Delta^2}{12}$$



- Increasing FFT size “pushes down” the perceived noise floor
- $10\ \log_{10}(M/2)$ is the *processing gain*
- Of course, the true SQNR does not depend on M

Signal to Noise + Distortion (SINAD)



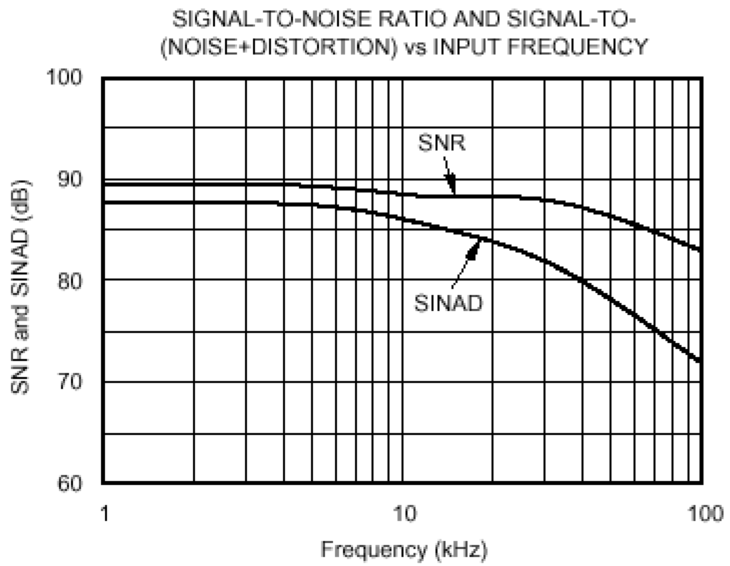
$$\text{SNR (dB)} = 10\log\frac{P_{\text{Fundamental}}}{P_{\text{noise}}}$$

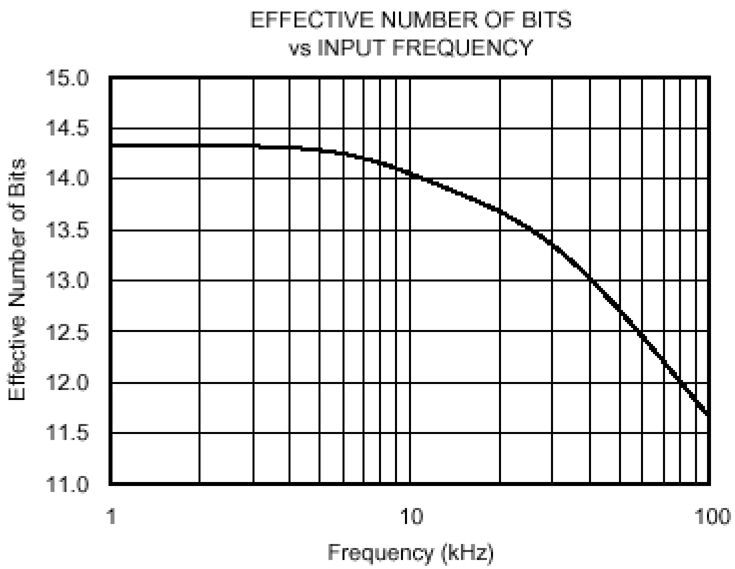
- Does not include harmonics
- Indication of Converter noise floor
- No indication of dynamic range

$$\text{SINAD (dB)} = 10\log\frac{P_{\text{Fundamental}}}{P_{\text{noise}} + P_{\text{harmonics}}}$$

- Includes all error components
- Indication of Converter useful dynamic range

Example: ADS8344 (16-bit, 100 ksps)





- Performance degrades with input frequency
- Ideal SNR is 98 dB for 16-bit resolution
- ENOB is always below nominal resolution

$$\text{ENOB} = \frac{\text{SINAD} - 1.76}{6.02}$$

Summary of distortion metrics

- Signal to Noise Ratio (SNR)
- Total Harmonic Distortion (THD)
  - Signal to harmonics
- Signal to Noise + Distortion Ratio (SINAD)
  - Signal to everything else
- Spurious-Free Dynamic Range (SFDR)
  - Signal to largest spectral spur
- Effective number of bits (ENOB)

Dynamic Range considerations

- Wireless communications: antenna output includes desired signal & adjacent channel interference (ACI)

- ADC samples a signal comprising several channels. The desired channel is extracted in the digital domain (digital filtering)

- What if the ACI/desired signal ratio is larger than the SFDR of the ADC?

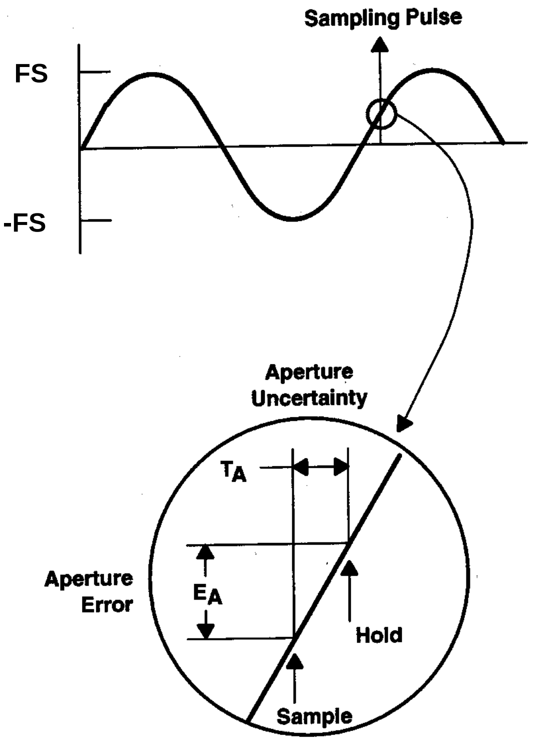
- **Dynamic range is extremely important!**

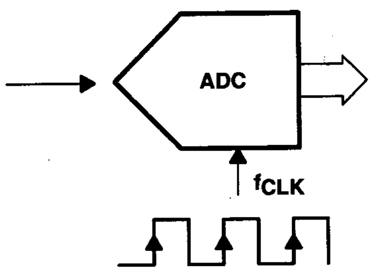
- **Example:** GSM-900 specs for receivers

  - Noise floor ~ -107 dBm
  - Recover a -101 dBm signal in the presence of a -13 dBm ACI
  - 88 dB requirement
  - At the very least we need $6.02\,N + 1.76 > 88$, i.e. 15 bits or more (treating signals as tones)
  - In practice even more (Nonlinear distortion & ADC noise)

Aperture jitter

- RMS variation in time of the sampling instant
- Caused by jitter in the clock timing
- Places an upper limit on input frequency to maintain resolution
- Differences in sampling time (aperture jitter) alter the sampled value (aperture error)





$$v_i(t) = \mathsf{FS} \sin 2\pi f_c t$$

$$\frac{dv_i(t)}{dt} = 2\pi f_c \mathsf{FS} \cos 2\pi f_c t$$

$$\max \left| \frac{dv_i(t)}{dt} \right| = 2\pi f_c \mathsf{FS}$$

- For the same aperture jitter, the aperture error increases with increasing input frequency
- Very important issue when sampling bandpass signals!!

Aperture jitter: SNR analysis

- Let $x(t) = A \cdot \cos(2\pi f_c t + \theta)$ be the analog input signal
- Model θ as a uniform random variable in $(-\pi, \pi)$
- Let t_0 = correct sampling instant, and $x(t_0 - \tau)$ = actual sample acquired, due to jitter τ
- Model τ as a uniform random variable independent of θ, with zero mean and RMS value σ_τ
- Compute the signal power $E\{x^2(t_0)\}$:

$$E\{x^2(t_0)\} = \int_{-\infty}^{\infty} x^2(t_0) f_\theta(\theta) d\theta = \int_{-\pi}^{\pi} A^2 \cos^2(2\pi f_c t_0 + \theta) \frac{d\theta}{2\pi} = \frac{A^2}{2}$$

- Compute the error power:

$$\begin{aligned} E\{[x(t_0) - x(t_0 - \tau)]^2\} &= \int_{-\sqrt{3}\sigma_\tau}^{\sqrt{3}\sigma_\tau} \int_{-\pi}^{\pi} [x(t_0) - x(t_0 - \tau)]^2 \frac{d\theta}{2\pi} \frac{d\tau}{2\sqrt{3}\sigma_\tau} \\ &= A^2[1 - \mathrm{sinc}(2\sqrt{3} f_c \sigma_\tau)] \approx A^2 \cdot 2\pi^2 f_c^2 \sigma_\tau^2 \end{aligned}$$

where we used the 2nd-order approximation $\mathrm{sinc}(u) \approx 1-(\pi u)^2/6$ for small $|u|$. Therefore,

$$\mathrm{SNR} = 10 \log \frac{E\{x^2(t_0)\}}{E\{[x(t_0) - x(t_0 - \tau)]^2\}} \approx 20 \log \frac{1}{2\pi f_c \sigma_\tau} \quad \mathrm{dB}$$

- Note that this SNR only takes into account errors due to aperture jitter, but not quantization
- Quantization errors are the main source of distortion for *low* signal frequencies
- As the signal frequency is increased, errors due to jitter eventually dominate over quantization errors