CHAPTER 4

The Church-Pythagoras Thesis

It is my contention that mathematics took a disastrous wrong turn some time in the sixth century B.C. This wrong turn can be expressed as an ongoing attempt, since then, to identify effectiveness with computability. That identification is nowadays associated with the name of Alonzo Church and is embodied in Church's Thesis. But Church was only among the latest in a long line going back to the original culprit. And that was no less than Pythagoras himself.

From that original mistake, and the attempts to maintain it, have grown a succession of foundation crises in mathematics. The paradoxes of Zeno (described later in this chapter), which it has been said retarded the development of the calculus by a thousand years, were an immediate outgrowth of the thesis. The plague of divergent sequences, which nearly wrecked mathematics in the eighteenth century, constituted the next outbreak. We are presently mired in yet another one, ironically the one that Church was provoked to help resolve by rearticulating that original mistake in modern dress.

The impact of that wrong turn, made so long ago, has spread far beyond mathematics. It has entangled itself into our most basic notions of what science is. It might seem a far cry from the ultimately empirical concerns of contemporary science to the remote inner world of mathematics, but at root it is not; they both, in their different ways, rest on processes of measuring things, and on searching for relations ("laws") between what it is that they measure. From this common concern with measurement, concepts pertaining to mathematics have seeped into epistemology, becoming so basic a part of the adjective *scientific* that most people are quite unaware they are even there.

From earliest times, mathematics has had an invincible aura of objectivity. Mathematical truth was widely regarded as the best truth—independent of the mathematician, independent of the external world, unchangeable even by God himself, beyond the scope of miracle in a way that the material world never was. Science has always craved and coveted that kind of objectivity. That is what the thesis seems to offer. But it has been a dangerous gift.

Pythagoras

The name of Pythagoras is commonly associated not only with the history of mathematics but with its involvement in a form of occult numerical mysticism, which purely mathematical considerations were employed to justify. There is certainly a cosmogonic aspect to Pythagorean doctrine, in addition to a religious aspect. Over against these received notions, consider the following, from W. K. C. Guthrie (1962):

> [Pythagoras'] father Mnesarchos of Samos . . . is described as a gem-engraver, and it would be in accordance with regular Greek custom for Pythagoras to be trained in his father's craft. (p. 173)
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> Aristoxenus, the friend of fourth-century Pythagoreans, wrote in his treatise on arithmetic that Pythagoras derived his enthusiasm for the study of number from its *practical applications in commerce.* This is by no means an improbable supposition. The impact of monetary economy, as a relatively recent phenomenon, on a thoughtful citizen of mercantile Samos might well have been to implant the idea that *the one constant factor by which things were related was the quantitative.* A fixed numerical *value* in drachmas or minas may "represent" things as widely different in quality as a pair of oxen, a cargo of wheat, and a gold drinking-cup. (p. 221; *emphases added*)

Even more surprising, perhaps, is the following:

> Pythagoras may have both introduced and designed the unique incuse coinage which was the earliest money of Croton and the neighboring South Italian cities under her influence. This is a coinage which excites the enthusiasm of numismatists by its combination of a remarkable and difficult technique with outstanding beauty of design, and Seltman

> claims its sudden appearance with no evolutionary process behind it postulates a genius of the order of Leonardo da Vinci. . . . As the son of an engraver, he would himself have been a practising artist, and of his genius there can be no doubt. One begins to appreciate the dictum of Empedocles that he was "skilled in all manner of cunning works."
>
> (p. 176)

These intriguing possibilities cast an entirely new light on the personality behind the adjective *Pythagorean.* It is also curious to note that, some two millennia later, Isaac Newton was to concern himself with coinage as well.

Thus there was a motivating concern for *quantitation* and the treatment of numbers as *values* (i.e., as a way of dealing with qualities). It is only a short conceptual step from this concern to the more modern concept of *measurement,* and to the idea of a (numerical) value as something inherent in what is being measured, not merely a convenient convention.

The Pythagorean concept of harmony was an essential step in developing relations between numbers and qualities:

> The word *harmonia* . . . meant primarily the joining or fitting of things together, even the material peg with which they were joined . . ., then especially the stringing of an instrument with strings of different tautness (perhaps thought of as a method of *joining* the arms of a lyre . . .), and so a musical scale. (Guthrie 1962:220)

In such a context, there is an essential connotation of design, or craft, in the word *harmony,* a connotation much stronger than the word has in its present usage.

To Pythagoras has always gone the credit for discovering that the euphonies, or concords, of Greek music were associated with particular ratios of lengths, especially the octave (1:2), the fifth (3:2), and the fourth (4:3). Moreover, in Greek music, these three primary concords provided the basic elements out of which *any* musical scale or composition was built. To perceive that these generating concords were associated with simple numerical ratios (in this case, of *lengths*) must have been a mind-boggling discovery in itself. For what can be more qualitative than a musical pitch or sound-quality? And what can be more subjective than a euphony?

What we can do with tone and sound, we can surely do elsewhere. This insight provided the springboard from *harmonia* to *kosmos,* the elevation of what was true in music to something encompassing the entire universe, and with it, the ineffable role of number, of measure, of value, as its manifestation. As an integrative world-view, this Pythagorean conception, intertwining as it does the good, the true, and the beautiful, the world of perceptual qualities and the world of number, still manifests its grandeur. Especially so in our world of grubbing empiricists and niggling disputators, in whose efforts we find no such qualities.

Indeed, a great deal of the Pythagorean vision clearly survives in what we call modern science, filtered in various ways through the years, and through people such as Pascal, even Gauss in later years, through Plato and Aristotle and many others in ancient times, to the latest unification schemes in theoretical physics. It is not surprising that Pythagoras became a legendary figure, even in his own time.

Commensurability

The Pythagorean story is an outgrowth of the eternal interplay between extension and enumeration, between quality and quantity, between geometry and arithmetic.

Length is one of the most immediate qualities of a geometric line segment. Measuring it, however, is a matter of enumeration, of counting. Associating a number, a length, with a particular extension or line segment is one of the most primitive examples of an *effective* process.

The first step of an effective process is to pick an arbitrary line segment A, which will serve as our unit of length and is hence arbitrarily assigned the number 1. The number assigned to any other line segment B we wish to measure has to do with how many times we can lay the unit A on the segment B; it is thus a matter of mere counting.

At this point, the Pythagoreans made an assumption: that *any* two line segments, say A and B, are *commensurable.* This amounted to supposing that, given A and B, there was another line segment U, which could be "counted off" an *integral* number of times on both A and B. This is, of course, a kind of quantization hypothesis, with U as a quantum, and as we shall see, it was wrong.

But if this is so, then we can write (committing a forgivable *abus de langage* in the process),

$$A = mU, \ B = nU,$$

where m, n are now *integers.* Then we can further write,

$$A/B = m/n,$$

independent of U. Hence finally,

$$B = (n/m)A,$$

and hence the *rational* number (n/m) is the length of the segment B, as measured in units of A.

According to these ideas, the scheme (or software) I have outlined serves either to *measure* B given A, or to *construct* B given A. Herein lie the seeds of another fateful confusion.

But for the moment, I will restrict myself to observing that commensurability implies that, whatever the unit A, any other length B will be a *rational* number relative to it. That is, it will be the ratio of two integers.

To Pythagoras, who associated lengths with qualities such as musical tones and pitches, and ratios of integral lengths with subjectivities such as musical euphonies, such arguments must have been profoundly satisfying. Indeed, once we have posited commensurability, all this forms an incredibly tidy, closed, consistent, self-contained system, from which there is no exit *from within.* But on the other hand, there is also no way from within it to *test* the basic assumption of commensurability, no way to violate the basic link it establishes between counting and extension. And moreover, it is precisely on this notion that the effectiveness of the entire procedure rests.

"Everything Is Number"

Aristotle referred to "the Pythagoreans" often in the course of his extant writings, although his book devoted to them has been lost. In general, he was rather disapproving of them; he felt that they did not properly understand causality (and I agree). He credited them with putting forth the doctrine that "things are numbers."

Aristotle was not quite sure what was meant by this doctrine, and he gave three different interpretations of it. The first, most literally, was that real things were actually *composed* of numbers, in much the same sense that the Atomists later claimed that things were composed of atoms. The second, that things take their being from the "imitation" (*mimesis*) of numbers. And finally, that the "elements" of number, from which numbers themselves arise, were simultaneously the elements of everything.

These three statements of the Pythagorean doctrine are not necessarily inconsistent with one another and may have appeared much less so in ancient times. The following quotation from Guthrie (1962) is instructive:

> The fact that Aristotle was able to equate Pythagorean *mimesis* with Plato's notion of physical objects as "sharing in" the Ideas (which Plato himself elsewhere describes as "patterns" for the world of sense) should put us on our guard against the simple translation "imitation." The fact is, of course, that even Plato, and still more the Pythagoreans, were struggling to express new and difficult conceptions within the compass of an inadequate language. We may take a hint first from K. Joel . . . who points to the trouble that the Pythagoreans must have experienced in clearly differentiating the concepts of similarity and identity, "a defect which still plagued Sophistic thought and which Plato's Ideal theory and Aristotle's logic only overcame with difficulty because it is rooted deep in the Greek consciousness: even their language has only one word for 'same' and 'similar' (*homoios*). . . . Are things imitations of numbers or numbers themselves? Aristotle ascribes both views to the Pythagoreans, and whoever is alive to the mind of Greece will also credit them with both and agree that for them numbers served alike as real and as ideal principles." (p. 230)

It is in fact astonishing that so much of this still lives on with us today.

The concept that "everything is number" in fact lies directly behind the attempt to subordinate geometry to arithmetic, and quality to quantity, which we have been discussing. If everything is number, then in particular the extended geometric line segments, the things endowed with length, and measured as we have described, must also be number as well. Let us pause now to express where 2,500 years have taken us in this direction; our modern language is still inadequate, but it is better than it was, and I will use it.

We start with a given set of line segments: call it L. We also presume an effective process of measurement, based on commensurability (a hypothesis about L), whereby we associate with each segment in L a (rational) number, its length relative to a preselected unit A. That is, we presuppose a mapping,

$$\lambda_A : L \to \mathfrak{R},$$

which is also (in the modern sense) effective. The (rational) numbers $\mathfrak{R}$ show up here as the range of this mapping; they comprise the set of values that length may assume, in terms of a process of counting.

But if "everything is number," we must extract these values of length that are only the range of λ_A and put them somehow into the domain; we must get them into L itself. This is in fact a very Gödelian thing to try to do, and we do it by trying to regard a line segment, an element of L, as made up out of the *values* of lengths. In particular, we go from $\mathfrak{R}$ to $2^{\mathfrak{R}}$, the set of all subsets of $\mathfrak{R}$, and try to find an image, or model, of L in there. Thus in effect we rebuild the domain of λ_A out of its range. If we can do this, we can simply throw L away and quite correctly assert that "everything *is* number" of what is left.

This is embodied in the diagram shown in figure 4.1. Here, F is part of a functor, the one that associates with every set its power set. The identification Φ is a code, or a dictionary, relating L and $2^{\mathfrak{R}}$ by identifying a line segment $\theta \in L$ with an interval of *values* of length in $\mathfrak{R}$; that is, with a set of the form

$$\begin{tikzcd} L \arrow[r, "\lambda_A"] \arrow[d, "\Phi"'] & \mathfrak{R} \arrow[dl, dashed, "F"] \\ 2^{\mathfrak{R}} \arrow[r, "\bar{\lambda}_A"'] & \mathfrak{R} \end{tikzcd}$$

FIGURE 4.1

$$I_{ab} = \{r : a \leq r \leq b\}.$$

Now, the length $\lambda_A(\theta)$ of a segment $\theta \in L$ relative to a unit A rests on a process of counting (i.e., of counting off A on B), and the answer takes the form of an integer plus a fractional part. In $2^{\mathfrak{R}}$, on the other hand, we do not have to count any more. Indeed, if $\Phi(\theta) = I_{ab}$, and $\Phi(A) = I_{01}$, we can replace the original λ_A, defined on L, with a new map defined on $2^{\mathfrak{R}}$, where

$$\bar{\lambda}(I_{ab}) = b - a.$$

This is certainly effective, in the new universe, where "everything" is indeed number.

Notice that the new map $\bar{\lambda}$ behaves nicely with respect to set-theoretic operations in $2^{\mathfrak{R}}$. In particular, it is finitely additive on disjoint unions of intervals: if I_i, $i = 1, \ldots, N$ are disjoint intervals, then

$$\bar{\lambda}\Big(\bigcup_{i=1}^{N} I_i\Big) = \sum_{i=1}^{N} \bar{\lambda}(I_i) \tag{4.1}$$

i.e., $\bar{\lambda}$ is *finitely additive.* And this much good behavior seems to be good enough.

The trouble is that there are more things in L than can be accommodated in these diagrams. The first trouble will be that the set of *values* we are using, the rational numbers $\mathfrak{R}$, is in some sense too small. That will give trouble with the *range* of λ_A. If we try to enlarge that range, we will find trouble with the power set we are using to "arithmetize" L (i.e., trouble with the *domain* of $\bar{\lambda}$). In fact, these innocent-looking diagrams are going to get very muddy indeed.

Incommensurability

In light of the preceding, it was a supreme and doubtless most painful irony that it fell to Pythagoras himself to discover the falsity of commensurability. This discovery was a corollary of his most celebrated mathematical result, the one always called, since his time, the Pythagorean Theorem. And it came about by departing from the province of a single geometrical dimension, by leaving the one-dimensional world of line-segments and going to the geometric plane.

In the two-dimensional world of the plane, ratios of lengths of line segments now become associated with measuring a new kind of thing, a new kind of extension or quality called angle. The Pythagorean Theorem dealt with the angles in a right triangle. And it was an immediate corollary of that theorem that, in an isosceles right triangle, the hypotenuse *could not* be commensurable with the side. Stated another way, the sine of 45° is not a rational number.

In two dimensions, the construction of such incommensurable line segments was as effective as anything. In *one* dimension, however, it was quite impossible. Clearly, as a line segment, the hypotenuse of an isosceles right triangle had to have a length, and the Pythagorean Theorem even told us what that length must be. But how was it to be measured, as a length in itself? How could the process of its measurement be related to counting? How could we then reinterpret that measurement process as an alternate means for constructing it?

The discovery of incommensurability thus posed a number of lethal problems. It drove a wedge between the notions of measurement and construction, hitherto assumed identical, because it produced something that could be *effectively* constructed but not measured thereby. Further, it separated the notion of measurement from the notion of counting itself, which had hitherto been its basis and foundation. What, indeed, becomes of the proposition that everything is number, embodied in the tidy diagrams of the preceding section?

It was precisely at this point that mistakes began to be made, or more accurately, wrong choices taken, which have haunted us ever since. The decision made at that time was to try to *enlarge arithmetic*. In a nutshell, it was decided that we needed more numbers—irrational numbers, things to be the values of lengths such as the hypotenuse of an isosceles right triangle. That is, we must enlarge the range of the mappings λ. But that is not enough. We must also relate these new numbers back to counting, so that we may effectively measure and construct segments with these new irrational lengths. That is, we must find a way to reidentify measurement with construction in this new, larger realm.

The trouble with this is that such a program requires excursions into the realm of the arithmetic infinite. We suddenly have to count infinite numbers of things, add infinitely many summands together. The Greeks traditionally hated to do this, but at the time it must have seemed the lesser evil. However, once embarked on this path, there turned out to be (pardon the pun) no limit.

The Paradoxes of Zeno

The first hints of trouble from these procedures were not long in coming. Indeed, within a century, Zeno of Elea, a disciple of Parmenides, had put forward his devastating paradoxes. (Parmenides, by the way, argued that there was no such thing as becoming, that change and motion were illusion, and hence that there was only being.)

The problem faced by the Pythagoreans was to enlarge the rational numbers $\Re$, the set of values of λ, by including "irrationalities" such as $\sqrt{2}$. Only in this way could line segments such as the hypotenuse of an isosceles right triangle be assigned a length at all. Plane geometry gave us effective procedures for producing line segments that had to have such irrational lengths (and in fact for producing them in the greatest profusion); what was lacking was a corresponding *arithmetic* procedure for measuring and constructing them the way rational lengths could be.

The answer was to stray from the finite and admit *infinitely* many repetitions of precisely the same procedures that had worked for rational lengths. At each finite step, only rationalities would be involved, and only at the end, in the *limit,* would we actually meet our new irrational. This was the method of exhaustion, based on the observations that

$$1 < \sqrt{2} < 2,$$

$$1.4 < \sqrt{2} < 1.5,$$

$$1.41 < \sqrt{2} < 1.42, \qquad \text{and so on.}$$

In modern language, we are seeking to partition a line segment of irrational length, such as the hypotenuse θ of a right triangle of unit sides, into a *countable* union

$$\bigcup_{i=1}^{\infty} \theta_i$$

of disjoint subintervals θ_i, each of *rational* length, in a particular way, and then adding up all these rational lengths. In the terminology of equation 4.1, we are seeking to write

$$\lambda(\theta) = \lambda(\bigcup_{i=1}^{\infty} \theta_i) = \sum_{i=1}^{\infty} (\theta_i) \tag{4.2}$$

That is, we must now require that λ be *countably additive,* not just finitely additive, as sufficed before. Under these conditions, we could write, for example,

$$\lambda(\theta) = \sqrt{2} = 1 + .4 + .01 + .004 + \ldots. \tag{4.3}$$

What Zeno pointed out is that this same line segment θ, via the identification in equation 4.3, is also *the countable union of its constituent points.* And each of *them* is of length *zero.* Hence, repeating precisely the above argument under these conditions leads to the paradoxical conclusion

$$\sqrt{2} = 0.$$

In fact, I could make the "length" of θ be anything I want (i.e., assume *any* value between 0 and √2) depending on how I partition it into disjoint subsets θ_i, $i = 1, 2, \ldots, \infty$.

But this is a truly devastating blow. The essence of length of a line segment is that it inheres in, and depends *only* on, the segment, not on how that segment is measured or constructed. It must in fact be independent of these; it should not depend on the specific process by which it is evaluated. What the Zeno paradoxes do, at root, is to call into question the very objectivity of the entire picture, by making the *value* of a length be determined by something extrinsic, by a particular context in which that value was determined. The problem raised by these paradoxes is thus closely akin to such things as "the measurement problem," which underlies the quantum physics of 2,500 years later, in which the observer gets causally entangled in the results of his own measurements.

Notice that these paradoxes rest upon only the *countability* of the set of *values* of length. In particular, the same argument would hold if we enlarged this set of values from the rationals $\mathfrak{R}$ to any set of computable numbers, numbers that arise from finite or countable repetitions of rote arithmetic operations (i.e., from algorithms) starting from integers.

The devastating paradoxes of Zeno could be interpreted in two ways, however. The first, argued tacitly by Zeno himself, is that the entire Pythagorean program to maintain the primacy of arithmetic over geometry (i.e., the identification of effectiveness with computation) and the identification of measurement with construction is inherently flawed and must be abandoned. That is, there are procedures that are perfectly effective but that cannot be assigned a computational counterpart. In effect, he argued that what we today call Church's Thesis must be abandoned, and, accordingly, that the concepts of measurement and construction with which we began were inherently far too narrow and must be extended beyond any form of arithmetic or counting.

The other interpretation, a much more conservative and tactical one, is that the thesis is sound, but we simply have not yet carried it out properly. In this light, the paradoxes merely tell us that we *still need more points,* more values for λ, that we have not yet sufficiently enlarged arithmetic. So let us add more points and see what happens.

Nonmeasurability

The ultimate enlargement of arithmetic, the transition from the original rational numbers $\mathfrak{R}$ to the full real number system $\mathbb{R}$, was not in fact achieved until the latter part of the nineteenth century, in the work of Dedekind and Weierstrass, among others. But long before then, concepts such as length, and higher-dimensional counterparts such as area and volume, had been vastly generalized, transmuted to analytic domains involving integration and associated concepts, soon to be collected into a distinguishable part of mathematics today known as Measure Theory.

In the context I have developed here, the transition from rationals to reals gets around the original Zeno paradoxes, mainly because the reals are *uncountable,* shocking as this word might be to Pythagoras. In particular, a line segment is no longer a countable union of its constituent points, and hence, as long as we demand only *countable* additivity of our measure λ, we seem to be safe. We are certainly so as long as we do not stray too far from line segments (or intervals); i.e., we limit ourselves to a universe populated only by things built from intervals by at most countable applications of set-theoretic operations (unions, intersections, complementations). This is the universe of the *Borel sets.*

But if $\mathbb{R}$ is now the *real* numbers, figure 4.1 tells us that we are now playing with $2^{\mathbb{R}}$. Thus we would recapture the Pythagorean ideals, and fully embody the primacy of arithmetic expressed in figure 4.1, if it were the case that everything in $2^{\mathbb{R}}$ were a Borel set. We could then say that we had enlarged arithmetic "enough" to be able to stop.

Unfortunately, that is not the way it is. Not only is there a set in $2^{\mathbb{R}}$ that is not a Borel set, but it turns out that there are *many* of them; it turns out further that there are *so* many that a Borel set is of the greatest rarity among them. It is in fact *nongeneric* for something in $2^{\mathbb{R}}$ to be Borel. We shall discuss the significance of these facts, and some of the ironies embodied in them, in a moment.

To construct a set in $2^{\mathbb{R}}$ that is not a Borel set (or to "exhibit" one) means that we have to get far away from intervals and their syntactic, set-theoretic progeny. Let us then briefly describe how such a set can be built. It is in fact quite simple. We start with the integers Z, and a single irrational number, say our old friend $\sqrt{2}$. We form the set G of all (real) numbers of the form

$$m \div n\sqrt{2},$$

where m, n are integers. Then it is easy to show that this countable set G is everywhere dense on the line.

We will say that two *real* numbers are *equivalent* if their difference lies in G (i.e., if they are congruent *modulo* G). Again, it is easy to verify that this notion of congruence is a true equivalence relation on the real numbers $\mathbb{R}$, and hence it partitions $\mathbb{R}$ into a family of disjoint equivalence classes whose union is $\mathbb{R}$. Since each equivalence *class* is countable (being in fact an orbit of the countable group G, of the form $r + G$, where r is a given real number), there must be *uncountably* many such equivalence classes.

Picking an element out of each equivalence class, we form a set $E \subset \mathbb{R}$. This set *cannot be measurable;* it can have no objective length at all. Indeed, if it did have a length, that length would have to be zero. But then we are in a Zeno-like situation, for the whole set of reals $\mathbb{R}$ is in effect a countable union of disjoint copies of sets like E. Thus countable additivity prevails, and we come up with the familiar paradox that the length of $\mathbb{R}$ would itself be zero.

We can see the genericity of sets like E by, in effect, bumping everything up into a higher dimension. That is what we have tacitly already done in the course of the preceding argument. We have expressed the real numbers $\mathbb{R}$, a one-dimensional thing, as a Cartesian product of two subsets—basically the countable "fiber" G and the uncountable quotient set $\mathbb{R}$ *modulo* G, the "base space." Any cross section, any set of the form E, fails to be measurable as a subset of $\mathbb{R}$.

Doing this more systematically clearly reveals the nongenericity of measurability. Following an example in Halmos (1950), we can take the unit interval $I = \{r : 0 \leq r \leq 1\}$ and form the unit square $I \times I$. For any subset $U \subset I$, we can associate with U the "cylinder" $U \times I$. Define a *measure* in $I \times I$ by saying the measure of such a cylinder is the ordinary measure of its base U. Clearly, most subsets of $I \times I$ are not cylinders and hence fail to be measurable. Hence, most sets of I itself fail to be measurable.

Prodded in large part by these considerations, people quite early began looking for a way to redefine *measure* in such a way that *every* subset was measurable. By *measure,* of course, was meant a countably additive, real-valued measure, such that individual points are of measure zero. We have just seen that ordinary (Lebesque) measure, tied ineluctably to intervals, is very far from this, and accordingly, if such a measure even exists, it must take us far from our intuitive ideas going back to Pythagoras.

Indeed, a set S that could support such a measure must, by any criterion, be a very strange set. Among other things, it must be very large indeed—large enough so that, roughly, 2^S is not much bigger than S. Such an S is called a *measurable cardinal.* The existence of such a measurable cardinal turns out to contradict the "axiom of constructibility," which is in itself a direct descendant (in fact, a close cousin) of the ancient attempt to subordinate quality to quantity manifested in what we have discussed.

In another direction, these ideas lead to such things as the Banach-Tarski paradox: a measurable set can be dissected into a small number of *non*measurable subsets (i.e., it can be expressed as their disjoint union). These nonmeasurable fragments can then be moved around by the most benign transformations (e.g., rigid motions, which preserve measure, if there is a measure to be preserved) and thereby reassembled into a different measurable set, of arbitrarily different measure than the one we started with. In a way, this is the other side of the Zeno paradox; similarly, this one exhibits the ultimate in context-dependence of length.

General Discussion

There are many ramifications indeed of the ideas we have been discussing. As we cannot pursue most of them here, we will concentrate on those connected with Church's Thesis. I shall argue that the thesis rests, at root, on an attempt to reimpose a restriction such as commensurability, and to deny existence, or reality, to whatever fails to satisfy that condition.

In an old film called *Body and Soul,* the racketeer Roberts represents the best possible expression of the thesis. He says, "Everything is either addition or subtraction. The rest is conversation."

As we have seen, commensurability was the original peg, the *harmonia,* that tied geometry to arithmetic. It said that we could express a geometric quality, such as the length of an extended line segment, by a number. Moreover, this number was obtained by repetition of a rote process, basically *counting.* Finitely many repetitions of this rote process would *exhaust* the quality. But more than this, it asserted that we could replace the original geometric *referents* of these numbers by things generated *from* the numbers themselves, and hence we could completely dispense with these referents. As I described in *Life Itself,* the elimination of (external) referents amounts to the replacement of semantics by syntax.

The repetition of rote operations is the essence of *algorithm.* The effect of commensurability was to assert an all-sufficiency of algorithm. In such an algorithmic universe, as we have seen, we could always equate quality with quantity, and construction with computation, and effectiveness with computability.

Once inside such a universe, however, we cannot get out again, because all the original external *referents* have presumably been pulled inside with us. The thesis in effect assures us that we never *need* to get outside again, that all referents have indeed been internalized in a purely syntactic form.

But commensurability was false. Therefore the Pythagorean formalizations did not in fact pull qualities inside. The successive attempts to maintain the nice *consequences* of commensurability, in the face of the falsity of commensurability itself, have led to ever-escalating troubles in the very foundations of mathematics, troubles that are nowhere near their end at the present time.

What we today call Church's Thesis began as an attempt to internalize, or formalize, the notion of effectiveness. It proceeded by *equating* effectiveness with what could be done by iterating rote processes that were already inside—i.e., with algorithms based entirely on syntax. That is exactly what *computability* means. But it *entails commensurability.* Therefore it too is false.

This is, in fact, one way to interpret the Gödel Incompleteness Theorem. It shows the inadequacy of repetitions of rote processes in general. In particular, it shows the inadequacy of the rote metaprocess of adding more rote processes to what is already inside.

Formalization procedures, such as the one contemplated by Pythagoras so long ago and relentlessly pursued since then, create worlds without external referents; from within such a world, one cannot entail the *existence* of anything outside. The idea behind seeking such a formalized universe is that, if it is big enough, then everything originally outside will, in some sense, have an exact image inside. Once inside, we can claim objectivity; we can claim independence of any external context, because there is no external context anymore. Inside such a world, we can literally claim that "everything is number" in the strongest sense, in the sense of *identity.*

But if we retain an exterior perspective, what goes on within any such formalized universe is at best *mimesis* of what is outside. It is simulation. Hence, if we cannot eliminate the outside completely, cannot claim to have fully internalized it, the next best thing is to claim that whatever is not thus internalized can be simulated by what has been, that is, simulated by the repetition of rote processes (algorithms), and rest content with these approximations.

But once we have admitted an outside at all, we can no longer equate effectiveness and computation, or equate construction and measurement, or in general equate *anything* that happens outside with the algorithms inside. Mimesis does not extend that far; the geometric construction of $\sqrt{2}$ from the Pythagorean Theorem has nothing to do with the arithmetic algorithm that computes it, or measures it, or "constructs" it. In short, mimesis is not science, it only mimics science.

Throughout the foregoing discussion, and indeed throughout the history of mathematics, we have found ourselves being relentlessly pushed from small and tidy systems to large ones. We have been pushed from integers to rationals to reals; from sets of elements to sets of sets. We have repeatedly tried to pull the bigger sets back into the smaller ones. We keep finding that they do not generally fit.

Perhaps the main reason they do not fit is that, while we are (grudgingly) willing to enlarge the sets of *elements* we deal with, we are unwilling to correspondingly enlarge the class of *operations* through which we allow ourselves to manipulate them. In particular, we have found ourselves repeatedly going from a set S to a power set 2^S. But operations on S become highly nongeneric in 2^S. Or stated another way, we cannot expect to do much in 2^S if we only permit ourselves those operations inherited from S. We do not have enough *entailment* to deal with 2^S properly under these circumstances.

Klein's famous Erlangen Programm in geometry gives an example. In his efforts to redefine geometry, Klein posited that it is the study of pairs (S, G), where S is an arbitrary set of *points,* and G is a *transitive* group of transformations (permutations) of S. But the set of geometric *figures* is the set of subsets, 2^S, a much bigger set. We can make the original group G operate on 2^S in the obvious way, but on 2^S, G has become highly *nontransitive.* Whereas G had only one orbit (S itself) while operating on S, it now has many orbits on 2^S; these are the congruence classes. Geometry is hard, in this light, precisely because the operations in G are rare, considered in the context of 2^S, whereas they were not rare in the original context of S. And, of course, we can iterate the procedure indefinitely, in the course of which the operations from G become rarer and rarer.

Computability is just like this. That is precisely why, as I have said in *Life Itself,* formalizable systems are so incredibly feeble in entailment. They attempt to do mathematics in large realms with only those procedures appropriate to small ones. As the realms themselves enlarge, those procedures get more and more special and, at the same time, less and less objective. That is why formalizable systems themselves are so nongeneric among mathematical systems.

From Pythagoras to Church, the attempt to preserve smallness in large worlds has led to a succession of disastrous paradoxes, and it is past time to forget about it. At the outset, I said that the Pythagorean program had seeped insidiously into epistemology, into how we view science and the scientific enterprise, through common concerns with measurement and the search for laws relating them. Pythagorean ideas are found at the root of the program of reduction, the attempt to devolve what is large exclusively on what is small and, conversely, to generate everything large by syntactic processes entirely out of what is small. Another manifestation is the confusion of science with mimesis, through the idea that whatever is, is simulable. This is embodied in the idea that *every model of a material process must be formalizable.* I have discussed the ramifications of these ideas, and especially their impacts on biology, extensively in *Life Itself.*

In particular, these ideas have become confused with *objectivity,* and hence with the very fabric of the scientific enterprise. Objectivity is supposed to mean observer independence, and more generally, context independence. Over the course of time, this has come to mean only building from the smaller to the larger, and reducing the larger to the smaller. As we have seen, this can be true only in a very small world. In any large world, such as the one we inhabit, this kind of identification is in fact a mutilation, and it serves only to estrange most of what is interesting from the realm of science itself.

Epilog: A Word about Fractals

For the Greeks, the length of a curve, like an arc of a circle, was the length of a line segment obtained when the curve was straightened out. It was presumed that this process of straightening could always be performed without changing that length. This is another facet of commensurability.

An integration process, such as a line integral, basically embodies such a straightening process, approximating the curve by segments of smaller and smaller lengths. These approximations distort the curve, but in the limit, when the unit is infinitely small, the curve is presumably effectively straightened without distortion, and the length of the resulting line segment is assigned to the original curve.

Ideas of this kind underlay Mandelbrot's (1977) original conception of fractals. From this point of view, a fractal is a curve that cannot be straightened out into a line segment without distortion, without stretching some parts of it and compressing others. If we try to measure the length of such a thing by any process of unrolling, the number we get will depend entirely on how we straightened it, not on the curve itself.

Indeed, it is in some sense nongeneric for a curve to be thus straightened without distorting it. Stated another way, there are no algorithms for straightening arbitrary curves of this type, no algorithms for computing their lengths. We can, in a sense, effectively generate (some of) them, albeit via an infinite process. But their *analysis* immediately violates Church's Thesis. Zeno would have loved them.