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Optimal Portfolios with Downside Risk

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Abstract

We show, first, that minimization of downside risk for portfolios with pre-specified expected returns leads to the same solution as minimization of the variance. Hence such optimal portfolios are defined by the Markovitz optimal solution. If the expected returns are not pre-specified, we show that the problem of minimization of downside risk has an analytical solution and we present this solution together with several illustrative numerical examples.

Key-words: Downside Risk, Optimization, Portfolio Management, Downside Loss Aversion.

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1 Introduction

Markowitz optimal portfolio theory (Markowitz (1987)), also known as the Mean-Variance theory, has had a tremendous impact and hundreds of papers are devoted to this topic. This theory addresses the question of minimizing risk for a given expected return and the optimal solution is found under one of the two assumptions: the distribution of the portfolio is normal, or the utility function is quadratic. In this theory, investor's decision formulates a trade-off between the return and the risk, in which the risk is measured by the variance of the returns. However, it has also been noted numerously in the past, starting with Markowitz himself, that the investors are more concerned with downside risk, i.e. the possibility of returns falling short of a specified target, rather than the variance, which takes into account both the favorable upside deviations as well as the averse downside parts. Moreover, such classic Mean-Variance framework does not consider investor's individual preference. Thus, alternatives are proposed in the literature in the form of downside risk measures, such as *target shortfall* and *semivariance*, or more generally, the so-called lower partial moments; see, for example, Harlow & Rao (1989).

The lower partial moments for stochastic returns, as downside risk measures, are defined as the expectation of the n-th power of the return's deviation below a pre-specified target which depends on investor's preference. The first and the second order lower partial moments are usually called *target shortfall* and *below target variance,* respectively. Intuitively, these risk measures are asymmetric and focuses on the left tail of the returns below a given target rate rather than the entire domain. The target shortfall is the expectation of portfolio returns dropping below the given target rate or benchmark return. By contrast, the below target variance (it is often called the semivariance in case the target is set as the expectation of the return) considers the dispersion of return below the target rate (Fishburn (1977)). Both aim to measure the extent that the portfolio fails to reach its manager's target or benchmark return. In this regard, the downside risk measures are more appropriate for investment risk because investors are often more interested in losses relative to a target returns. Moreover, unlike the variance, semivariance could remain the same with higher "upside potentials". For more details on the downside risk measures, we refer to Harlow (1991), Nawrocki (1999) and Chapter 2 of McNeil et al. (2005).

A manager who does not wish the return of his portfolio to fail the target rate would tend to compose portfolios minimizing downside risk measures, which is the so-called *downside optimization* (Harlow (1991)). Such portfolios with optimal downside risks are more attractive as they may lower the risks while maintaining or improving the expected returns in the classic mean-variance framework. As a result, by considering downside risk measures, portfolio managers are usually able to search for more profits in the trade-off between risk and returns, Empirical evidences also show that downside measures are more efficient than Mean-Variance measures in this sense. Moreover, in assets pricing models, downside risk framework could provide less downside exposure and reserve the same or a greater level of expected return, see Harlow and Rao (1989). In fact, despite of the complexity of the computation, it is clear from the literature that the lower partial moments, especially the target shortfall and below target variance, are not simply adhoc measures but grounded in the capital market theory with both appealing theoretical and intuitive features. For instance, Markowitz (2010) accords support to semivariance as a comparable risk measure with variance and there are mounting empirical evidences showing the superiority of the downside risk measures (Jarrow and Zhao (2006)).

Due to the formulation of partial moments, downside optimization is naturally connected with utility theory and stochastic dominance. Bawa (1978) used lower partial moment with α-degree and τ-threshold model to capture investor's risk aversion. More recently, Cumova and Nawrocki (2014) proposed a ratio of upside partial moment and lower partial moment to study investors' behavior. In contrast to the classical Mean-Variance model, which is consistent with a quadratic utility function, the downside optimization is a more general framework as it allows investors to consider different order n and choose a favorable target. Hence, there is no doubt that the downside-risk framework should provide a useful set of tools for portfolio managers considering a broad set of problems.

In addition to the literature mentioned above, there are ongoing efforts devoted to various applications of downside risks and related risk measures in finance. We list a few of them. Zhu et al. (2009) discuss robust portfolio selection with respect to parameter uncertainty under the downside risks framework. Moreover, the target shortfall is directly related to the Conditional Value-at-Risk (CVaR), which has been frequently applied as the optimizing objective in many portfolio selection studies. For more details on the risk measure CVaR, we refer to Artzner et al. (1999) and Rockafellar and Uryasev (2002). Mansini et. al. (2007) considers optimal portfolio with respect to the CVaR and obtained the solution by linear programming. Sawik (2008) formulates the portfolio optimization problem as multi-objective mixed integer program. Ogryczak and Sliwinski (2011a) and Ogryczak and Sliwinski (2011b) use duality to improve the efficiency of linear programming in portfolio selection. Sawik (2012a) compares three different bi-criteria portfolio optimization models based on Value-at-Risk, CVaR and variance. Sawik (2012b) studies multi-objective portfolio optimization with downside risk approaches and Sawik (2016) proposed further work in multi-objective models. Cumova and Nawrocki (2014) use both upside and downside partial moments to account for investor's utility. A literature review on downside risk measures can be found in Nawrocki (1999) and more account on robust portfolio optimization with respect to various risk measures is in Gabrel et al.

(2014).

However, most aforementioned works have to rely on numerical method to find the solutions to the optimization problem. In fact, as Jarrow and Zhao (2006) commented, analytical solutions to optimal portfolio's weights are generally out of reach for such downside measures. One could either employ numerical techniques to search for the optimal solution or implement optimization based on empirical estimations and simulations, see, for instance, Nawrocki (1991) and Cumova and Nawrocki (2014). Nevertheless, in the absence of massive computational efforts such as in simulation and numerical optimization, analytical solutions are always preferable in both theoretical studies and practical applications, and furthermore, they do not suffer from computational errors either.

Landsman (2008) finds an analytical solution to optimal portfolio's weight in the classical Mean-Variance framework with additional constraints on the returns. Based on his results, we adopt a similar approach and derive the analytical solutions to the downside optimization in the context of normally distributed returns. This work offers several novel contributions. Firstly, we show that optimal portfolios with respect to downside risk are the same as those generated by the classic Markowitz Mean-Variance analysis in such a framework where the expected portfolio return is pre-specified. Thus, in this case the downside risk framework could not improve the classic results and investors can not expect additional profits from the downside approach assuming normal distributions of the returns. Particularly, Jarrow and Zhao (2006) empirically observed that optimal portfolios with respect to mean-variance and downside risk frameworks are much alike. Our findings provide a theoretical proof of their results.

Secondly, we consider a more general portfolio selection with downside risk measure when the expected return of the portfolio is not pre-specified. We show that such downside optimization is a convex optimization and has a unique solution. We further derive analytical formulae for the solution and obtain optimal weights. Consequently, we can easily select optimal portfolios with respect to downside risk measures for any inputs of means and covariances. As mentioned above, such results are useful both theoretically and practically.

Thirdly, we present numerical examples to illustrate our results. We first observe that the general principal that an asset with higher return should have higher risk holds true in the downside risk measure framework. Moreover, we notice that a portfolio with minimal target shortfall can have very different weights from the one with minimal below target variance and it is plausible to argue that one could always outperform the other. Hence, instead of optimizing expected shortfall or below target variance separately, we propose a new downside optimization that targets a combination of the two downside risk measures. In this regard, investor could choose his/her preference between the downwards deviation from the target and the risk to drop below the target and the corresponding analytical weights are still available. Similarly to the classic results, we also show that one can obtain an analogue to the Mean-Variance efficient frontier based on the Mean-Downside-risk framework.

The rest of the paper is organized as follows. In Section 2, we present the formulae of the risk associated with downside risk measures and state some useful properties. Section 3 provides the downside optimization and finds an analytical solution. We offer numerical examples in Section 4. We discuss further potential developments and conclude our paper in Section 5.

2 Downside Risk

To facilitate the expressions of the equations and formulae in the sequel, we first present a short list of the notions and notations that are useful as below.

- $X \sim N(\mu, \sigma^2)$. X denotes stochastic return of an asset that is normally distributed with mean μ and variance σ
- $\mathbf{X} \sim N(\boldsymbol{\mu}, \Sigma)$. $\mathbf{X}$ denotes stochastic returns of multiple assets that are multi-normally distributed with means $\boldsymbol{\mu}$ and covariance matrix Σ.
- $\Phi(x)$ and $\varphi(x)$ are the cumulative distribution function and density function of the standard normal random variable respectively.
- For a fixed target K, we consider the downside risks below K

$$R_i(\mu, \sigma) := E[((X-K)^-)^i] = \sigma^i \int_{-\infty}^{\frac{K-\mu}{\sigma}} (\frac{K-\mu}{\sigma} - x)^i \varphi(x) dx$$

  for $i = 1$ and $i = 2$, i.e., $R_1(\mu, \sigma)$ is the *target shortfall* and $R_2(\mu, \sigma)$ is the *below target variance.*
- $A_K = (\mathbf{1}, \boldsymbol{\mu} - K\mathbf{1})^T$ where $\mathbf{1}$ is a n-dimenional vector with identical unit components. $\delta = \left| A_K \Sigma^{-1} A_K^T \right|$ is the determinant of $A_K \Sigma^{-1} A_K^T$.
- $q_1 = \delta^{-1} \mathbf{1}^T \Sigma^{-1} \mathbf{1},$ $q_2 = \delta^{-1} (\boldsymbol{\mu} - K\mathbf{1})^T \Sigma^{-1} \mathbf{1},$ $q_3 = \delta^{-1} (\boldsymbol{\mu} - K\mathbf{1})^T \Sigma^{-1} (\boldsymbol{\mu} - K\mathbf{1})$.

In the context of normality, we compute the downside risk functions as the follows.

Proposition 1.

$$
\begin{aligned}
R_1(\mu,\sigma) &= E[(X-K)^-] \\
&= \sigma\varphi(\frac{K-\mu}{\sigma}) + (K-\mu)\Phi(\frac{K-\mu}{\sigma}), \qquad (1)\\
R_2(\mu,\sigma) &= E[((X-K)^-)^2] \\
&= (\sigma^2+(K-\mu)^2)\Phi(\frac{K-\mu}{\sigma}) + \sigma(K-\mu)\varphi(\frac{K-\mu}{\sigma}) \qquad (2)
\end{aligned}
$$

Proof. Let $X = \sigma Z + \mu$, where $Z \sim N(0,1)$.

$$
\begin{aligned}
R_1(\mu,\sigma) &= E[(\sigma Z + \mu - K)^-] = \int_{-\infty}^{\frac{K-\mu}{\sigma}} \sigma \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz + (\mu - K)\int_{-\infty}^{\frac{K-\mu}{\sigma}} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \\
&= \sigma\varphi(\frac{K-\mu}{\sigma}) + (K-\mu)\Phi(\frac{K-\mu}{\sigma}).
\end{aligned}
$$

$$
\begin{aligned}
R_2(\mu,\sigma) &= E[((X-K)^-)^2] \\
&= \sigma^2 E[((Z+\frac{\mu-K}{\sigma})^-)^2] \\
&= \sigma^2 E[((Z+\frac{\mu-K}{\sigma})^2 I_{Z<\frac{K-\mu}{\sigma}})] \\
&= \sigma^2\Big(E[Z^2 I_{Z<\frac{K-\mu}{\sigma}}] - 2\frac{K-\mu}{\sigma}E[Z I_{Z<\frac{K-\mu}{\sigma}}] + \frac{(K-\mu)^2}{\sigma^2}P(Z<\frac{K-\mu}{\sigma})\Big).
\end{aligned}
$$

Using $d\varphi(z) = -z\varphi(z)dz$ and integration by parts, we have

$$
R_2(\sigma) = (\sigma^2+(K-\mu)^2)\Phi(\frac{K-\mu}{\sigma}) + \sigma(K-\mu)\varphi(\frac{K-\mu}{\sigma}).
$$

∎

Theorem 2. *Downside risk functions (1) and (2) are increasing functions of σ.*

Proof. Assume now that μ is given. We show that $R_1'(\sigma) > 0$ and $R_2'(\sigma) > 0$. Direct calculations using the property $\phi'(x) = -x\phi(x)$ gives

$$
R_1'(\sigma) = \varphi(\frac{K-\mu}{\sigma}) > 0,
$$

and similarly,

$$
R_2'(\sigma) = 2\sigma\Phi(\frac{K-\mu}{\sigma}) > 0.
$$

∎

Obviously, the conclusion of this theorem is that for a pre-specified mean of return, minimizing each of these risk measures is equivalent to minimizing the variance. Hence there is a clear equivalence to the mean-variance principle.

3 Downside Risk for Portfolios

In this section we establish optimal portfolios with minimal downside risk measures. We assume that short selling is permitted. Let $\mathbf{X}$ denote the vector of returns in the portfolio and $\boldsymbol{\alpha}$ be the vector of corresponding weights that adds up to unity, then $\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}$ is the return of portfolio. Note that in our setting the components of $\boldsymbol{\alpha}$ are not necessarily positive. Assuming multivariate normal distribution for $\mathbf{X}$, $N(\boldsymbol{\mu}, \Sigma)$, we have that $\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} \sim N(\boldsymbol{\alpha}^T\boldsymbol{\mu}, \boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha})$.

Theorem 2 indicates that finding a portfolio that minimizes $E[(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-]$ or $E[((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-)^2]$ is equivalent to finding $\boldsymbol{\alpha}$ that minimize $\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}$. Formally, given any expected return c on a portfolio, the solutions to

$$\min_{\boldsymbol{\alpha}} E[(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-] \text{ subject to } B\boldsymbol{\alpha} = \mathbf{c}, \tag{3}$$

and

$$\min_{\boldsymbol{\alpha}} E[((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-)^2] \text{ subject to } B\boldsymbol{\alpha} = \mathbf{c}, \tag{4}$$

are the same as

$$\min_{\boldsymbol{\alpha}} \boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha} \text{ subject to } B\boldsymbol{\alpha} = \mathbf{c}, \tag{5}$$

where

$$B = \begin{pmatrix} 1, 1, ..., 1 \\ \mu_1, \mu_2, ..., \mu_n \end{pmatrix}_{2\times n}, \ \mathbf{c} = \begin{pmatrix} 1 \\ c \end{pmatrix}.$$

The constant B allows us, in addition to requiring all weights to sum to 1, to put a constraint (c) on the expected return of the portfolio. We now present the following proposition and corollary.

Proposition 3 (Landsman (2008))**.** *A portfolio with return $\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}$, $\mathbf{X} \sim N(\boldsymbol{\mu},\Sigma)$, that minimizes $\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}$ with given expected return $\boldsymbol{\alpha}^{\mathbf{T}}\boldsymbol{\mu} = c$ is defined by weights*

$$\boldsymbol{\alpha} = \Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}\mathbf{c}.$$

Corollary 4. *A portfolio with return $\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}$, $\mathbf{X} \sim N(\boldsymbol{\mu},\Sigma)$, that minimizes either $E(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}-$*

$K)^-$ or $E((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}-K)^-)^2$ with given expected return $\boldsymbol{\alpha}^{\mathbf{T}}\boldsymbol{\mu}$ =c is defined by weights

$$\boldsymbol{\alpha}=\Sigma^{-1}B^T(B\Sigma^{-1}B^T)^{-1}\mathbf{c}.$$

Note that in Proposition 3 and Corollary 4, we tacitly assume that the means of $\mathbf{X}$ are inhomogeneous. In case that all X_i, $i=1,2,...,n$ have a common mean μ, i.e., $\boldsymbol{\mu}=\mu\mathbf{1},$ where $\mathbf{1}$ is a n-dimenional vector with identical unit components, $B\Sigma^{-1}B^T$ becomes a singular matrix which is not invertible. Also note that $\boldsymbol{\alpha}^T\boldsymbol{\mu}\equiv\mu$ due to the constraint $\boldsymbol{\alpha}^T\mathbf{1}$ =1 for homogeneous means. Thus c can only be μ and (5) reduces to the classic quadratic optimization:

$$\min_{\boldsymbol{\alpha}}\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}\text{ subject to }\boldsymbol{\alpha}^T\mathbf{1}=1,$$

whose solution is well-known as

$$\boldsymbol{\alpha}=\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}},$$

see Landsman (2008) and Luenberger (1984). Without loss of generality, we shall further assume that the means of $\mathbf{X}$ are inhomogeneous.

Let us further consider a more general optimization problem for downside risk portfolios where the expected return on the portfolio is not pre-specified. That is, we drop off the second row in B and only reserve the unity constraint,

$$\min_{\boldsymbol{\alpha}}E[(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}-K)^-],\text{ subject to }\boldsymbol{\alpha}^T\mathbf{1}=1. \tag{6}$$

We obtain the following results.

Theorem 5. *The downside risk function (6) of a portfolio with return $\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}$, $\mathbf{X}\sim N(\boldsymbol{\mu},\boldsymbol{\Sigma})$, subject to $\boldsymbol{\alpha}^T\mathbf{1}=1$, is minimized at*

$$\boldsymbol{\alpha}^*=\Sigma^{-1}A_K^T(A_K\Sigma^{-1}A_K^T)^{-1}\mathbf{c}^*,$$

where $\mathbf{c}^*=(1,c^*)^T$ and c^* is the solution to the equation

$$\varphi\left(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}}\right)(q_1c-q_2)=\sqrt{q_1c^2-2q_2c+q_3}\,\Phi\left(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}}\right).$$

c^* exists and is unique.

Proof. Note that (6) can be recasted as

$$\min_{\boldsymbol{\alpha}} E[(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{Y})^{-}], \text{ subject to } \boldsymbol{\alpha}^T\mathbf{1} = 1 \text{ and } \mathbf{Y} = \mathbf{X} - K\mathbf{1},\ \mathbf{Y} \sim N(\boldsymbol{\mu}_{\mathbf{Y}}, \Sigma),\ \boldsymbol{\mu}_{\mathbf{Y}} = \boldsymbol{\mu} - K\mathbf{1}. \tag{7}$$

We know that for any given $\boldsymbol{\alpha}^{\mathbf{T}}\boldsymbol{\mu}_{\mathbf{Y}} = c$, $E(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{Y})^{-}$ is an increasing function at $\boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha}$ and is minimized at

$$\boldsymbol{\alpha}_c = \arg\min_{\boldsymbol{\alpha}} \boldsymbol{\alpha}^T\Sigma\boldsymbol{\alpha} = \Sigma^{-1}A_K^T(A_K\Sigma^{-1}A_K^T)^{-1}\mathbf{c}, \tag{8}$$

where $A_K^T = (\mathbf{1}, \boldsymbol{\mu} - K\mathbf{1})^T$ and $\mathbf{c} = (1, c)^T$. It is straightforward to show that

$$\begin{aligned}
A_K\Sigma^{-1}A_K^T &= \begin{pmatrix} \mathbf{1}^T\Sigma^{-1}\mathbf{1} & \mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}_{\mathbf{Y}} \\ \boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\mathbf{1} & \boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\boldsymbol{\mu}_{\mathbf{Y}} \end{pmatrix}, \\
(A_K\Sigma^{-1}A_K^T)^{-1} &= \frac{1}{\delta}\begin{pmatrix} \boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\boldsymbol{\mu}_{\mathbf{Y}} & -\mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}_{\mathbf{Y}} \\ -\boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\mathbf{1} & \mathbf{1}^T\Sigma^{-1}\mathbf{1} \end{pmatrix}, \\
\boldsymbol{\alpha}_c^T\Sigma\boldsymbol{\alpha}_c &= (\mathbf{1}^T\Sigma^{-1}\mathbf{1}c^2 - 2\boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\mathbf{1}c + \boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\boldsymbol{\mu}_{\mathbf{Y}})\delta^{-1}, \qquad (9)
\end{aligned}$$

where $\delta = \mathbf{1}^T\Sigma^{-1}\mathbf{1} \times \boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\boldsymbol{\mu}_{\mathbf{Y}} - \mathbf{1}^T\Sigma^{-1}\boldsymbol{\mu}_{\mathbf{Y}} \times \boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\mathbf{1}$ is the determinant of $A_K\Sigma^{-1}A_K^T$. Thus, the minimum of (7), say $\boldsymbol{\alpha}^{*\mathbf{T}}\mathbf{Y}$, should always have mean c and standard deviation that satisfy (9). Consequently, taking (1) into account, (7) is equivalent to the univariate minimization,

$$\begin{aligned}
\min_c E(\boldsymbol{\alpha}_c^T\mathbf{Y})^{-} \quad &: \;= \min_c f(c), \text{ where} \qquad (10) \\
f(c) \quad &= \sqrt{q_1c^2 - 2q_2c + q_3}\,\varphi\Big(\frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}}\Big) - c\Phi\Big(\frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}}\Big)
\end{aligned}$$

with $q_1 = \delta^{-1}\mathbf{1}^T\Sigma^{-1}\mathbf{1},\ q_2 = \delta^{-1}\boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\mathbf{1},\ q_3 = \delta^{-1}\boldsymbol{\mu}_{\mathbf{Y}}^T\Sigma^{-1}\boldsymbol{\mu}_{\mathbf{Y}}$. We compute the first and second order derivatives of $f(c)$:

$$\begin{aligned}
f'(c) &= \frac{\varphi\big(\frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}}\big)(q_1c - q_2)}{\sqrt{q_1c^2 - 2q_2c + q_3}} - \Phi\Big(\frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}}\Big), \qquad (11) \\
f''(c) &= \varphi\Big(\frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}}\Big)(q_1c^2 - 2q_2c + q_3)^{-5/2}g(c), \\
g(c) &= c^2(q_2^2 - q_1q_2^2 + q_1^2q_3) + 2c(q_2^3 - q_3q_2 - q_1q_2q_3) + q_3^2 - q_2^2q_3 + q_1q_3^2.
\end{aligned}$$

Due to the positive definity of the covariance matrices, we have $q_1 > 0, q_3 > 0$ and $q_1c^2 - 2q_2c + q_3 > 0$. Also note that $q_1c^2 - 2q_2c + q_3 > 0$ implies that $q_1q_3 > q_2^2$. Then, we have

$$\begin{aligned} g(c) &= c^2(q_2^2 - q_1 q_2^2) + 2c(q_2^3 - q_3 q_2) + q_3^2 - q_2^2 q_3 + q_1 q_3(q_1 c^2 - 2q_2 c + q_3) \\ &> c^2(q_2^2 - q_1 q_2^2) + 2c(q_2^3 - q_3 q_2) + q_3^2 - q_2^2 q_3 + q_2^2(q_1 c^2 - 2q_2 c + q_3) \\ &= c^2 q_2^2 - 2c q_3 q_2 + q_3^2 = (c q_2 - q_3)^2 \geq 0. \end{aligned}$$

Thus, we see that $f(c)$ is convex and has unique minimum point c^*, which is the solution to the equation $f'(c^*) = 0$. ∎

Following Theorem 5, we are able to establish that the optimal portfolio which solves the following minimization,

$$\min_{\boldsymbol{\alpha}} E[((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-)^2], \text{ subject to } \boldsymbol{\alpha}^T\mathbf{1} = 1. \tag{12}$$

However, to make the proof easier, we would first present the following lemma.

Lemma 6. *The function*

$$h(t) := (1+t)\Phi(\frac{-1}{\sqrt{t}}) - \sqrt{t}\varphi(\frac{-1}{\sqrt{t}}) \tag{13}$$

is increasing and positive for all $t > 0$.

Proof. Taking the derivative of $h(t)$,we have

$$h'(t) = \Phi(\frac{-1}{\sqrt{t}}) > 0$$

Thus, for $t > 0$, $h(t)$ are non-decreasing and attain the minimum at 0 :

$$\lim_{t \to 0} h(t) = 0,$$

which concludes the proof. ∎

The following theorem presents the optimal portfolio that minimizes (2).

Theorem 7. *The downside risk function (12) of a portfolio $\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}$ with $\mathbf{X} \sim N(\mu, \Sigma)$, subject to $\boldsymbol{\alpha}^T\mathbf{1} = 1$, is minimized at*

$$\boldsymbol{\alpha}^* = \Sigma^{-1} A_K^T (A_K \Sigma^{-1} A_K^T)^{-1} \mathbf{c}^*,$$

where $\mathbf{c}^* = (1, c^*)^T$, c^* *is the solution to the equation*

$$\varphi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}})\sqrt{q_1c^2-2q_2c+q_3} = ((q_1+1)c-q_2)\Phi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}}).$$

c^* *exists and is unique.*

Proof. Similarly to Theorem 5, we first recast (12) as

$$\min_{\boldsymbol{\alpha}} E[((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{Y})^-)^2], \text{ subject to } \boldsymbol{\alpha}^T\mathbf{1} = 1 \text{ and } \mathbf{Y} = \mathbf{X}-K\mathbf{1},\ \mathbf{Y} \sim N(\boldsymbol{\mu}_{\mathbf{Y}}, \Sigma),\ \boldsymbol{\mu}_{\mathbf{Y}} = \boldsymbol{\mu}-K\mathbf{1}, \tag{14}$$

and transform it to an equivalent univariate minimization according to (2), (9) and Corollary 4, i.e.,

$$\begin{aligned}
\min_c E[((\boldsymbol{\alpha}_c^T\mathbf{Y})^-)^2] \quad &: \quad = \min_c f(c), \text{ where} \\
f(c) \quad &= \quad ((q_1+1)c^2-2q_2c+q_3)\Phi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}}) \\
& \quad\quad -c\sqrt{q_1c^2-2q_2c+q_3}\,\varphi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}}).
\end{aligned}$$

We compute the first three order derivatives of $f(c)$:

$$\begin{aligned}
\frac{f'(c)}{2} &= ((q_1+1)c-q_2)\Phi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}}) - \varphi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}})\sqrt{q_1c^2-2q_2c+q_3}, \\
\frac{f''(c)}{2} &= (q_1+1)\Phi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}}) - (\varphi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}})(cq_1-q_2)\times \\
& \quad (c^2q_1-3cq_2+2q_3)(q_1c^2-2q_2c+q_3)^{-3/2}, \\
\frac{f'''(c)}{2} &= \varphi(\frac{-c}{\sqrt{q_1c^2-2q_2c+q_3}})(q_1c^2-2q_2c+q_3)^{-7/2}(cq_2-q_3)g(c), \text{ where} \\
g(c) &= c^2(q_2^2-3q_1q_2^2+3q_1^2q_3)+2c(3q_2^3-q_3q_2-3q_1q_2q_3)+q_3^2-3q_2^2q_3+3q_1q_3^2.
\end{aligned}$$

Due to the fact that $q_1 > 0$, $q_3 > 0$, $q_1c^2 - 2q_2c+q_3 > 0$ and $q_1q_3 > q_2^2$, we have

$$\begin{aligned}
g(c) &= 3q_1q_3(q_1c^2-2cq_2+q_3) - 3q^2(q_1c^2-2cq_2+q_3) + c^2q_2^2 - 2cq_2q_3 + q_3^2 \\
&> c^2q_2^2 - 2cq_2q_3 + q_3^2 = (cq_2-q_3)^2 \geq 0.
\end{aligned}$$

Then, we establish the positivity of $f''(c)$ via the following statement.

1. $q_2 = 0$; $f''(c)$ is decreasing w.r.t. c and has minimum at $+\infty$;

$$\lim_{c\to+\infty} \frac{f''(c)}{2} = (1+q_1)\Phi(\frac{-1}{\sqrt{q_1}}) - \sqrt{q_1}\varphi(\frac{-1}{\sqrt{q_1}}), q_1 > 0. \tag{15}$$

It is easy to see that (15) is of the form of (13). Thus, by Lemma 6, we have $f''(c) > 0$.

2. $q_2 < 0$; $f''(c)$ is increasing for $c < q_3/q_2$ and decreasing for $c > q_3/q_2$. Thus, $f''(c)$ has maximum at $c = q_3/q_2$ and minimum at $+\infty$ or $-\infty$. But

$$\lim_{c\to-\infty} \frac{f''(c)}{2} = (1+q_1)\Phi(\frac{1}{\sqrt{q_1}}) + \sqrt{q_1}\varphi(\frac{1}{\sqrt{q_1}}) > \lim_{c\to+\infty} \frac{f''(c)}{2}.$$

Thus, similar to the case $q_2 = 0$, we see that $f''(c)$ is positive according to (15) and Lemma 6.

3. $q_2 > 0$; $f''(c)$ is decreasing for $c < q_3/q_2$ and increasing for $c > q_3/q_2$. Thus, $f''(c)$ has minimum at $c = q_3/q_2$, i.e.,

$$\frac{f''(q_3/q_2)}{2} = (1+q_1)\Phi(\frac{-\sqrt{q_3}}{\sqrt{q_1q_3 - q_2^2}}) - \varphi(\frac{-\sqrt{q_3}}{\sqrt{q_1q_3 - q_2^2}})\frac{\sqrt{q_1q_3 - q_2^2}}{\sqrt{q_3}}.$$

Moreover, note that $q_1 > (q_1q_3 - q_2^2)/q_3$, we have

$$\frac{f''(q_3/q_2)}{2} > (1 + \frac{q_1q_3 - q_2^2}{q_3})\Phi(\frac{-\sqrt{q_3}}{\sqrt{q_1q_3 - q_2^2}}) - \varphi(\frac{-\sqrt{q_3}}{\sqrt{q_1q_3 - q_2^2}})\frac{\sqrt{q_1q_3 - q_2^2}}{\sqrt{q_3}},$$

which is again of the form of (13) with $t = (q_1q_3 - q_2^2)/q_3 > 0$. Thus, $f''(c) \geq f''(q_3/q_2) > 0$.

We therefore conclude that $f(c)$ is convex and has unique minimum c^*, which is the solution to the equation $f'(c) = 0$. ∎

With Theorem 5 and Theorem 7, we are able to find the optimal portfolios with respect to the downside-risk approach. Such methods are obviously more appropriate to the objectives of the downside-risk philosophy as compared with the classic mean-variance framework. However, one may consider the two downside risk functions simultaneously by using a linear combination of the risk measures as a new goal function. More specifically, for any given $0 < \lambda < 1$, we aim to solve

$$\min_{\boldsymbol{\alpha}} E[\lambda((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-)^2 + (1-\lambda)(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-], \text{ subject to } \boldsymbol{\alpha}^T\mathbf{1} = 1. \tag{16}$$

According to the proofs of Theorem 5 and Theorem 7, we can see that the analytical solution to the optimal portfolio of (16) is again available. We formulate this in the following corollary.

Corollary 8. *The mixed downside risk function (16) of a portfolio with return* $\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X}$*,* $\mathbf{X} \sim N(\mu, \Sigma)$*, subject to* $\boldsymbol{\alpha}^T\mathbf{1} = 1$*, is minimized at*

$$\boldsymbol{\alpha}^* = \Sigma^{-1}A_K^T(A_K\Sigma^{-1}A_K^T)^{-1}\mathbf{c}^*,$$

where $\mathbf{c}^* = (1, c^*)^T$*,* c^* *is the solution to the equation*

$$\begin{aligned} & \varphi\Big(\frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}}\Big)\Big(\frac{(\lambda-1)(q_1c - q_2)}{\sqrt{q_1c^2 - 2q_2c + q_3}} + 2\lambda\sqrt{q_1c^2 - 2q_2c + q_3}\Big) \\ = \ & \Phi\Big(\frac{-c}{\sqrt{q_1c^2 - 2q_2c + q_3}}\Big)(\lambda(2q_1c + 2c - 2q_2 + 1) - 1). \end{aligned}$$

c^* *exists and is unique.*

Proof. The proof is similar to that of Theorem 5 and Theorem 7. ∎

Clearly, λ reflects the manager's preference in the choice of downside risk measures, i.e., the target shortfall and below target variance. As aforementioned, by choosing $\lambda = 1$, the manager's only concern is the variance of the portfolio below the threshold. The portfolio is, therefore, expected to be more "stable" in the target zone. On the other hand, for $\lambda = 0$, the manager's portfolio has the minimal expected shortfall in mind. We, therefore, anticipate its return to be closer to the target. Intuitively, portfolios with minimal expected shortfall should outperform the ones with minimal below target variances in the sense of the expected returns for the same target. In contrast, portfolios with minimal below target variances should generally be less risky in the sense of lower variability. In next section, we demonstrate these two alternatives using numerical examples.

4 Numerical illustration

In this section, we present a numerical example to illustrate our the results. We consider a portfolio of 10 stocks from the NASDAQ stock exchanges (ADOBE Sys. Inc., Compuware Corp., NVIDIA Corp., Starles Inc.,Verisign Inc., Sandisk Corp, Microsoft Corp., Symantec Corp., Citrix Sys Inc., Intuit Inc.) for the year 2005, and denote by $\mathbf{X} = (X_1, ..., X_n)^T$, $n = 10$, stocks' weekly returns. We implement the Kolmogorov–Smirnov test to test the normality assumption and since the statistics is 0.15486 with p-value $= 0.1484$, we do not reject the hypothesis of the normality.

Morover, our approach could also provide an analytical approximation for non-normal data. For example, in the case that the stochastic return r is modelled by log-normal distribution, we could take the well-known approximations that $\log(1+r) \approx r$, $r \ll 1$ into account. Then, we have approximately a normal distribution again. In fact, such an approximation is pretty precise when r is small, which is particularly true for short-term returns such as in daily and weekly trades.

We present the corresponding means and covariance in Table 1 and Table 2. Optimal weights that minimize R_1 and R_2 with respect to different targets K are presented in Table 3. Table 4 presents the results regarding the portfolios that minimize the combined objectives[1].

According to Table 3 and Table 4, we can see that for the same given target K, portfolios that have minimal expected shortfall have higher expected returns as well as higher variances, compared with the ones that minimize the below target variances. Moreover, we observe that both the expected return and variance of the optimal portfolio rise with the target K. On the one hand, minimal expected shortfall keep the expected return of the portfolio close to the target; on the other hand, higher target results in an increase in volatility. After all, such a scenario agrees with the common sense that higher returns go hand in hand with higher risks.

Furthermore, the consistency between the expected return and the variance also implies a Mean-Variance efficient frontier for the optimal portfolio selections. The Mean-Variance efficient frontier is as important as it is useful for establishing the Capital Asset Pricing Model and for measuring the performance of the portfolio, see for example Sharpe (1964) and Joro and Na (2006). In contrast to the classic Mean-Variance efficient frontier, we can work out the Mean-Downside risks efficient frontiers, see Figure 1. Thus, we may also use such Mean-Downside risks efficient frontiers to measure the performance of the portfolios.

[1] For the computational convenience, we consider the initial constraint on the weight as $\boldsymbol{\alpha}^T\mathbf{1} = 10$. In fact, for any initial wealth $W_0 \neq 0$, we are to solve

$$\min_{\boldsymbol{\alpha}} E[(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-], \text{subject to } \boldsymbol{\alpha}^T\mathbf{1} = W_0.$$

We can rewrite it as

$$\min_{\boldsymbol{\beta}} E[(\boldsymbol{\beta}^{\mathbf{T}}\widehat{\mathbf{X}} - K)^-], \text{subject to } \boldsymbol{\beta}^T\mathbf{1} = 1,\ \widehat{\mathbf{X}} = W_0\mathbf{X},$$

then it is easy to see that the results obtained in Section 3 are applicable as $\widehat{\mathbf{X}}$ is again multivariate normal distributed.

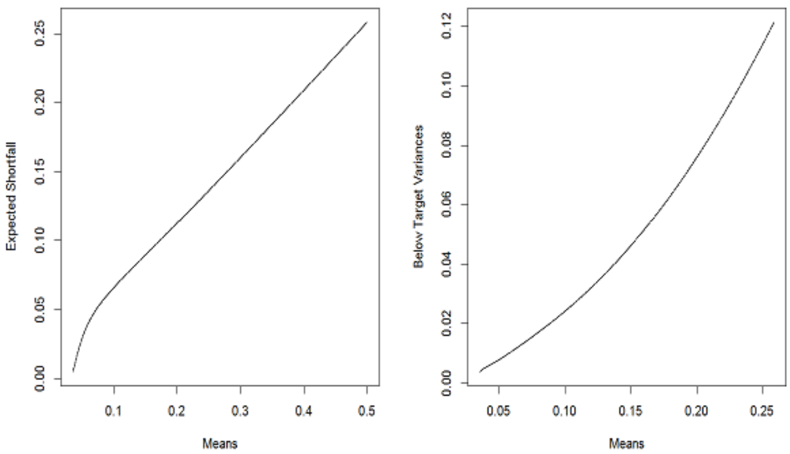

Figure 1: Examples of Mean-Downside risk efficient frontier.

Adobe	Compuware	NVDIA	Staples	VeriSign
-0.0061022899	0.0081117935	0.0095500842	-0.0057958520	-0.0064370369
Sandisk	Microsoft	Citrix	Intuit	Symantec
0.0198453317	-0.0001754431	0.0037884516	0.0041225491	-0.0061032938

Table 1: Expected returns (means) of the 10 stocks.

	Adobe	Compuware	NVDIA	Staples	VeriSign	Sandisk	Microsoft	Citrix	Intuit	Symantec
Adobe	0.006101557	0.001173171	0.000118111	0.000512857	0.000120697	0.001004334	-0.000119954	0.00039590	0.000134857	0.00053654
Compuware	0.001173172	0.003310202	0.001047261	0.000498469	0.000847448	0.000428135	0.000309324	0.00067374	0.000550476	0.00080746
NVDIA	0.000118111	0.001047261	0.002145104	0.000122035	0.000771984	0.000468733	0.000255908	0.00055918	0.000479347	0.00026985
Staples	0.000512858	0.000498469	0.000122035	0.002940476	-0.00054654	0.001053215	0.000028702	0.00039449	0.000254851	0.00043542
VeriSign	0.000120698	0.000847448	0.000771984	-0.00054654	0.003486011	0.00013176	0.000115362	0.00079373	0.000665291	0.00090585
Sandisk	0.001004335	0.000428135	0.000468733	0.001053215	0.00013176	0.004012978	-3.26811E-05	0.00084372	0.000131403	8.3298E-05
Microsoft	-0.00011995	0.000309324	0.000255908	2.87029E-05	0.000115362	-3.26811E-05	0.000485266	0.00021972	0.000167364	6.18376E-05
Citrix	0.000395904	0.000673744	0.000559185	0.000394499	0.000793735	0.000843728	0.00021972	0.00136509	0.000397118	0.00044450
Intuit	0.000134857	0.000550476	0.000479347	0.000254851	0.000665291	0.000131403	0.000167364	0.00039711	0.000876343	2.71609E-05
Symantec	0.000536543	0.000807469	0.000269851	0.000435426	0.000905851	8.3298E-05	6.18376E-05	0.00044450	2.71609E-05	0.00254227

Table 2: Covariance Matrix of the 10 stocks.

	$K=-0.05$		$K=-0.01$		$K=0$		$K=0.01$		$K=0.05$	
	$\min R_1$	$\min R_2$	$\min R_1$	$\min R_2$	$\min R_1$	$\min R_2$	$\min R_1$	$\min R_2$	$\min R_1$	$\min R_2$
Adobe	0.0646365	0.1819154	-0.001813	0.1524191	-0.020668	0.1444455	-0.0404826	0.1362223	-0.129726	0.1007518
Compuware	-0.078993	-0.237032	0.0105504	-0.197284	0.0359588	-0.186539	0.06265868	-0.1754587	0.1829182	-0.127660
NVDIA	0.7040149	0.5598366	0.7857061	0.5960982	0.8088861	0.6059006	0.83324444	0.6160099	0.9429572	0.6596161
Staples	-0.521878	-0.216597	-0.694849	-0.293377	-0.743930	-0.314132	-0.7955065	-0.335538	-1.027810	-0.427869
VeriSign	-0.766799	-0.485576	-0.926140	-0.556305	-0.971353	-0.575425	-1.0188648	-0.595144	-1.232862	-0.680198
Sandisk	1.5692155	1.2622548	1.7431389	1.3394571	1.7924901	1.3603267	1.84434969	1.3818498	2.0779319	1.4746889
Microsoft	4.7980679	5.0722589	4.6427118	5.0032984	4.5986292	4.9846566	4.5523059	4.9654313	4.3436598	4.8825033
Citrix	-0.042999	-0.082311	-0.020724	-0.072424	-0.014404	-0.069751	-0.0077627	-0.066995	0.0221520	-0.055105
Intuit	3.4131130	3.058321	3.6141371	3.1475537	3.6711781	3.1716753	3.7311185	3.1965521	4.0010974	3.3038573
Symantec	0.8616230	0.886930	0.8472839	0.8805655	0.8432151	0.8788449	0.8389395	0.8770704	0.8196818	0.8694163
Portfolio mean	0.052597	0.037737	0.061017	0.041475	0.063406	0.042485	0.065917	0.0435273	0.077224	0.048021
Portfolio variance	0.038591	0.034286	0.041687	0.03523	0.042651	0.035501	0.043706	0.0357880	0.048980	0.037108

Table 3: Optimal portfolios with respect to Optimization (6) and Optimization (12) for different targets. Sum of the weights equals to 10. Expected returns and variances of the portfolio are reported correspondingly in the last two rows.

	$K = -0.01$			$K = 0.01$		
	$\lambda = .05$	$\lambda = .5$	$\lambda = .95$	$\lambda = .05$	$\lambda = .5$	$\lambda = .95$
Adobe	0.00147659	0.0428046	0.1342750	-0.0364065	0.0134727	0.1167599
Compuware	0.00611682	-0.049574	-0.172834	0.05716594	-0.010048	-0.149232
NVDIA	0.78166133	0.7308542	0.6184039	0.82823342	0.7669136	0.6399363
Staples	-0.6862854	-0.578707	-0.340607	-0.7848963	-0.655059	-0.386199
VeriSign	-0.9182508	-0.819150	-0.599813	-1.0090907	-0.8894852	-0.641813
Sandisk	1.73452743	1.6263573	1.3869466	1.83368104	1.7031292	1.432789
Microsoft	4.65040405	4.7470263	4.9608786	4.56183564	4.6784503	4.9199293
Citrix	-0.0218276	-0.035681	-0.066342	-0.0091291	-0.025848	-0.060471
Intuit	3.60418379	3.4791586	3.2024431	3.71878747	3.5678930	3.2554297
Symantec	0.84799390	0.8569120	0.8766502	0.83981915	0.8505825	0.8728707
Portfolio mean	0.06060	0.05536	0.04377	0.06540	0.05908	0.04599
Portfolio variance	0.04152	0.03955	0.03585	0.04348	0.04093	0.03649

Table 4: Optimal portfolios with respect to Optimization (16) for different λ. Sum of the weights equals to 10. Expected returns and variances of the portfolio are reported correspondingly in the last two rows.

5 Conclusions and Further Discussion

In this paper, we discuss optimal portfolios with respect to the downside risks, i.e., $E((X-K)^-)^\beta$, where $\beta = 1, 2$, in the context of multivariate normal distribution. We show that the two downside risks are monotonic increasing in σ, which immediately suggests that the solutions to the optimal minimization problems

$$\min_{\boldsymbol{\alpha}} E(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^- \text{ subject to } \mathbf{1}^T\boldsymbol{\alpha} = 1,\ \boldsymbol{\mu}^T\boldsymbol{\alpha} = c$$

and

$$\min_{\boldsymbol{\alpha}} E((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-)^2 \text{ subject to } \mathbf{1}^T\boldsymbol{\alpha} = 1,\ \boldsymbol{\mu}^T\boldsymbol{\alpha} = c,$$

coincides with ones to the classical mean-variance optimization. Hence, considering downside risk, though it appears to be more attractive for many investors compared with the classical variance, cannot improve the optimal gain when the expected return of the portfolio is fixed. This proffers theoretical supports to the empirical findings in literature such as Jarrow and Zhao (2006).

Moreover, we drop off the assumption of pre-specified return, i.e. we consider the following optimization problems

$$\min_{\boldsymbol{\alpha}} E(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^- \text{ subject to } \mathbf{1}^T\boldsymbol{\alpha} = 1,$$

and

$$\min_{\boldsymbol{\alpha}} E((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-)^2 \text{ subject to } \mathbf{1}^T\boldsymbol{\alpha} = 1,$$

then the optimal solutions differ. In the context of normality, we obtain the analytical solutions to the two general problems and show that the solution to each optimization exists and is unique. We also provide a numerical illustration of the results. According to the numerical results, we find that the optimal portfolios with respect to the two minimizations can be very different. Hence, we further propose a new downside optimization that targets the combination of the two downside risk measures as follows,

$$\min_{\boldsymbol{\alpha}} E[\lambda((\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-)^2 + (1-\lambda)(\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{X} - K)^-], \text{ subject to } \boldsymbol{\alpha}^T\mathbf{1} = 1,$$

where λ reflects the investors' preference between the two downside risk measures. Again, we obtain an analytical solution to this combined optimizations and show it exists and is unique.

As a matter of fact, there is a vast number of references concerning optimal portfolio. Due to the complexity of the objective function, most of them rely on numerical techniques to find the optimal solutions. This is particularly the case for downside risks, which has been often discussed in the literature. In contrast to a numerical solution, analytical solution guarantees the existence and the uniqueness of the solution. On the one hand, it is easy to calculate and to implement; and on the other hand, it does not suffer from the unavoidable computational errors nor the great computational efforts involved. Clearly, the analytical solution is simply superior. In this regard, our results provide useful methodology and new insights to the literature on optimal portfolio with respect to downside risks, and it gives both theoretical and practical contributions to such problems.

We rely on the assumptions of normality to set up our results. Although the normal distribution is probably one of the most frequently applied probability laws in modeling the stochastic returns of assets, empirical evidences sometimes suggest the non-normality of such data, especially for short-term returns. While our methodology offers a sound approximation to non-normal returns, it is our aim to further extend our results to the case where the stochastic returns are modelled by non-normal distributions in general, and the elliptical distribution (Fang et. al. (1990)) in particular, which is much more flexible and richer than the normal distribution.

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