

# SWERC NoteBook

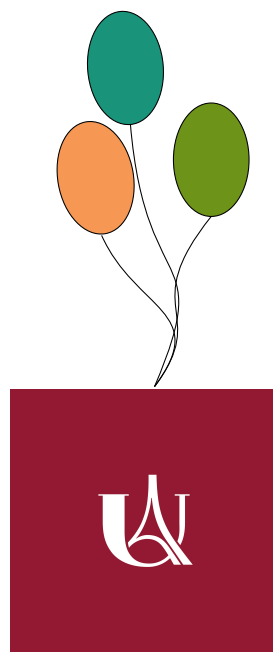
Équipe B.I.O.S

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## Ensemble d'algorithmes et techniques de programmation

*Et quelques notions de Mathématiques*



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# 1 Configuration

## 1.1 C/C++

### 1.1.1 Header

---

```
1 #include <iostream>
2 #include <stdlib.h>
3 #include <stdio.h>
```

---

### 1.1.2 Vecteur

---

```
1 #include <vector>
2 //Declaration :
3 std::vector<int> name;
4 //Debut /Fin (element)
5 name.begin();
6 name.back();
7 //Debut / Fin
8 name.front();
9 name.end();
10 //Ajout /Deletion (fin)
11 name.push_back(val);
12 name.pop_back();
13 //Ajout /Suppression(index)
14 name.insert(index, val);
15 name.erase(index);
16 name.erase(index1,index2);
17 //Taille
18 name.size();
19 //Destruction
20 name.clear();
```

---

### 1.1.3 Deque

---

```
1 #include <deque>
2 //meme methodes que vector, plus :
3 name.pop_front();
4 name.push_front();
5 name.emplace_front();
6 name.emplace_back();
```

---

### 1.1.4 Stack

---

```
1 #include <stack>
2 std::stack<int> pile;
3 pile.push(val);
4 pile.pop();
5 pile.size();
6 pile.empty();
7 pile.top();
```

---

### 1.1.5 Tri

---

```
1 sort(v.begin(),v.end());
```

---

## 1.2 Python

### 1.2.1 Numpy

---

```
1 import numpy as np
2 #creation array
3 a = np.array([1,2,3])
4 b = np.array([(1.5,2,3), (4,5,6)], dtype = float)
5 c = np.array([[(1.5,2,3), (4,5,6)],[(3,2,1), (4,5,6)]], dtype = float)
6 #Fonction de bases
7 np.zeros((3,4)) #Create an array of zeros
8 np.ones((2,3,4),dtype=np.int16) #Create an array of ones
9 d = np.arange(10,25,5)#Create an array of evenly spaced values (step value)
10 np.linspace(0,2,9) #Create an array of evenly spaced values (number of samples)
11 e = np.full((2,2),7)#Create a constant array
12 f = np.eye(2) #Create a 2X2 identity matrix
13 np.random.random((2,2)) #Create an array with random values
14 np.empty((3,2)) #Create an empty array
15 #Aide
16 np.info(np.ndarray.dtype)
17 #Informations sur l'array
18 a.shape #Array dimensions
19 len(a)#Length of array
20 b.ndim #Number of array dimensions
21 e.size #Number of array elements
22 b.dtype #Data type of array elements
23 b.dtype.name #Name of data type
24 b.astype(int) #Convert an array to a different type
25 #Datatype
26 np.int64 #Signed 64-bit integer types
27 np.float32 #Standard double-precision floating point
28 np.complex #Complex numbers represented by 128 floats
29 np.bool #Boolean type storing TRUE and FALSE values
30 np.object #Python object type
31 np.string_ #Fixed-length string type
32 np.unicode_ #Fixed-length unicode type
33 #Maths
34 #operations arithmetiques
35 g = a - b #Subtraction
36 np.subtract(a,b) #Subtraction
37 b + a #Addition
38 np.add(b,a) #Addition
39 a/b #Division
40 np.divide(a,b) #Division
41 a * b #Multiplication
42 np.multiply(a,b) #Multiplication
43 np.exp(b) #Exponentiation
44 np.sqrt(b) #Square root
45 np.sin(a) #Print sines of an array
46 np.cos(b) #Elementwise cosine
47 np.log(a)#Elementwise natural logarithm
48 e.dot(f) #Dot product
49 #Comparaisons
50 a == b #Elementwise comparison
51 a < 2 #Elementwise comparison
52 np.array_equal(a, b) #Arraywise comparison
53 #Copies
54 h = a.view()#Create a view of the array with the same data
55 np.copy(a) #Create a copy of the array
56 h = a.copy() #Create a deep copy of the array
57 #Tri
58 a.sort() #Sort an array
59 c.sort(axis=0) #Sort the elements of an array's axis
60 #Manipulation
61 #Transposee
```

---

```

62 i = np.transpose(b) #Permute array dimensions
63 i.T #Permute array dimensions
64 #Changer la forme
65 b.ravel() #Flatten the array
66 #Modifier des elements
67 h.resize((2,6)) #Return a new arraywith shape(2,6)
68 np.append(h,g) #Append items to an array
69 np.insert(a,1,5) #Insert items in an array
70 np.delete(a,[1]) #Delete items from an array
71 #Combiner les array
72 np.concatenate((a,d),axis=0) #Concatenate arrays
73 np.vstack((a,b)) #Stack arrays vertically(row wise)
74 np.r_[e,f] #Stack arrays vertically(row wise)
75 np.hstack((e,f)) #Stack arrays horizontally(column wise)
76 np.column_stack((a,d)) #Create stacked column wise arrays
77 np.c_[a,d] #Create stacked column wise arrays
78 #Splitter les array
79 np.hsplit(a,3) #Split the array horizontally at the 3rd index
80 np.vsplit(c,2) #Split the array vertically at the 2nd index
81 #Statistics and probabilities
82 np.std(my_array) #ecart type
83 my_array.corrcoef() #coefficient de correlation
84 np.median(my_array) #mediane
85 np.mean(my_array) #moyenne empirique

```

---

## 2 Chaînes de caractères

### 2.1 Anagramme

$\mathcal{O}(nk \log(k))$  en moyenne

---

```

1 def anagrams(w):
2     w = list(set(w)) #retire les doublons
3     d = {}
4     for i in range(len(w)):
5         s = ''.join(sorted(w[i]))
6         if s in d:
7             d[s].append(i)
8         else:
9             d[s] = [i]
10    answer = []
11    for s in d:
12        if len(d[s]) > 1:
13            answer.append([w[i] for i in d[s]])
14    return answer

```

---

### 2.2 Plus long palindrome d'une chaîne

$\mathcal{O}(n)$

---

```

1 def manacher(s):
2     assert set.isdisjoint({'$', '^', '#'}, s) # Forbidden letters
3     if s == "":
4         return (0, 1)
5     t = "^#" + "#".join(s) + "$"
6     c = 1
7     d = 1
8     p = [0] * len(t)
9     for i in range(2, len(t) - 1):
10        # -- reflect index i with respect to c
11        mirror = 2 * c - i # = c - (i-c)
12        p[i] = max(0, min(d - i, p[mirror]))
13        # -- grow palindrome centered in i
14        while t[i + 1 + p[i]] == t[i - 1 - p[i]]:

```

---

```

15         p[i] += 1
16         #                                     -- adjust center if necessary
17         if i + p[i] > d:
18             c = i
19             d = i + p[i]
20         (k, i) = max((p[i], i) for i in range(1, len(t) - 1))
21         return ((i - k) // 2, (i + k) // 2) # extract solution

```

---

## 3 Séquences

### 3.1 Distance de Levenshtein

$\mathcal{O}(nm)$

---

```

1 def levenshtein(x,y):
2     n = len(x)
3     m = len(y)
4     A = [[i+j for j in range(m+1)] for i in range(n+1)]
5     for i in range(n):
6         for j in range(m):
7             A[i+1][j+1] = min(A[i][j+1] + 1, A[i+1][j] + 1, A[i][j] + int(x[i] != y[j]))
8     return A[n][m]

```

---

### 3.2 Plus grand facteur commun

$\mathcal{O}(nm)$

---

```

1 def longest_common_subsequence(x,y):
2     n = len(x)
3     m = len(y)
4     A = [[0 for j in range(m+1)] for i in range(n+1)]
5     for i in range(n):
6         for j in range(m):
7             if x[i] == y[j]:
8                 A[i+1][j+1] = A[i][j] + 1
9             else:
10                 A[i+1][j+1] = max(A[i][j+1], A[i+1][j])
11     sol = []
12     i, j = n, m
13     while A[i][j] > 0:
14         if A[i][j] == A[i-1][j]:
15             i -= 1
16         elif A[i][j] == A[i][j-1]:
17             j -= 1
18         else:
19             i -= 1
20             j -= 1
21         sol.append(x[i])
22     return ''.join(sol[::-1])

```

---

### 3.3 Plus longue sous-séquence croissante

$\mathcal{O}(|x| \times \log(|y|))$

---

```

1 from bisect import bisect_left
2
3
4 def longest_increasing_subsequence(x):
5     n = len(x)
6     p = [None] * n
7     h = [None]
8     b = [float('-inf')] # - infinity
9     for i in range(n):
10         if x[i] > b[-1]:

```

```

11         p[i] = h[-1]
12         h.append(i)
13         b.append(x[i])
14     else:
15         # -- binary search: b[k - 1] < x[i] <= b[k]
16         k = bisect_left(b, x[i])
17         h[k] = i
18         b[k] = x[i]
19         p[i] = h[k - 1]
20     # extract solution in reverse order
21     q = h[-1]
22     s = []
23     while q is not None:
24         s.append(x[q])
25         q = p[q]
26     return s[::-1] # reverse the list to obtain the solution

```

---

### 3.4 Rabin-Karp — Recherche d'un pattern

$\mathcal{O}(\text{len}(s) + \text{len}(t))$  en temps amortit

```

1  #a mettre en C++ !
2  PRIME = 72057594037927931 # < 2^{56}
3  DOMAIN = 128
4  def roll_hash(old_val, out_digit, in_digit, last_pos):
5      """roll_hash """
6      val = (old_val - out_digit * last_pos + DOMAIN * PRIME) % PRIME
7      val = (val * DOMAIN) % PRIME
8      return (val + in_digit) % PRIME
9  def matches(s, t, i, j, k):
10     for d in range(k):
11         if s[i + d] != t[j + d]:
12             return False
13     return True
14  def rabin_karp_matching(s, t):
15     hash_s = 0
16     hash_t = 0
17     len_s = len(s)
18     len_t = len(t)
19     last_pos = pow(DOMAIN, len_t - 1) % PRIME
20     if len_s < len_t: # substring too long
21         return -1
22     for i in range(len_t): # preprocessing
23         hash_s = (DOMAIN * hash_s + ord(s[i])) % PRIME
24         hash_t = (DOMAIN * hash_t + ord(t[i])) % PRIME
25     for i in range(len_s - len_t + 1):
26         if hash_s == hash_t: # hashes match
27             # check character by character
28             if matches(s, t, i, 0, len_t):
29                 return i
30         if i < len_s - len_t:
31             # shift window and calculate new hash on s
32             hash_s = roll_hash(hash_s, ord(s[i]), ord(s[i + len_t]),
33                                last_pos)
34     return -1 #no match
35  def rabin_karp_factor(s, t, k):
36     last_pos = pow(DOMAIN, k - 1) % PRIME
37     pos = {}
38     assert k > 0
39     if len(s) < k or len(t) < k:
40         return None
41     hash_t = 0
42     # First calculate hash values of factors of t
43     for j in range(k):
44         hash_t = (DOMAIN * hash_t + ord(t[j])) % PRIME

```

```

45     for j in range(len(t) - k + 1):
46         # store the start position with the hash value
47         if hash_t in pos:
48             pos[hash_t].append(j)
49         else:
50             pos[hash_t] = [j]
51         if j < len(t) - k:
52             hash_t = roll_hash(hash_t, ord(t[j]), ord(t[j + k]), last_pos)
53     hash_s = 0
54     # Now check for matching factors in s
55     for i in range(k): # preprocessing
56         hash_s = (DOMAIN * hash_s + ord(s[i])) % PRIME
57     for i in range(len(s) - k + 1):
58         if hash_s in pos: # is this signature in s?
59             for j in pos[hash_s]:
60                 if matches(s, t, i, j, k):
61                     return (i, j)
62         if i < len(s) - k:
63             hash_s = roll_hash(hash_s, ord(s[i]), ord(s[i + k]), last_pos)
64     return None

```

---

## 4 Parcours de graphes

### 4.1 DFS - Depth First Search

$\mathcal{O}(|V| + |E|)$

```

1 #version iterative pour eviter la recursion limit de python
2 def dfs_iterative(graph, start, seen):
3     seen[start] = True
4     to_visit = [start]
5     while to_visit:
6         node = to_visit.pop()
7         for neighbour in graph[node]:
8             if not seen[neighbour]:
9                 seen[neighbour] = True
10                to_visit.append(neighbour)

```

---

### 4.2 BFS - Breadth First Search

$\mathcal{O}(|V| + |E|)$  (Liste adjacence)  $\mathcal{O}(|V|^2)$  (Matrice adjacence)

```

1 from collections import deque
2 def bfs(graph, start=0):
3     to_visit = deque()
4     dist = [float('inf')] * len(graph)
5     prec = [None] * len(graph)
6     dist[start] = 0
7     to_visit.appendleft(start)
8     while to_visit: #evaluate a faux si vide
9         node = to_visit.pop()
10        for neighbour in graph[node]:
11            if dist[neighbour] == float('inf'):
12                dist[neighbour] = dist[node] + 1
13                prec[neighbour] = node
14                to_visit.appendleft(neighbour)
15    return dist, prec

```

---

### 4.3 Topological Sort

$\mathcal{O}(|V| + |E|)$

```

1 def topological_order(graph):
2     V = range(len(graph))

```

---

---

```

3     indeg = [0 for _ in V]
4     for node in V:           # compute indegree
5         for neighbor in graph[node]:
6             indeg[neighbor] += 1
7     Q = [node for node in V if indeg[node] == 0]
8     order = []
9     while Q:
10        node = Q.pop()        # node without incoming arrows
11        order.append(node)
12        for neighbor in graph[node]:
13            indeg[neighbor] -= 1
14            if indeg[neighbor] == 0:
15                Q.append(neighbor)
16    return order

```

---

## 4.4 Composantes connexes

$\mathcal{O}(|N| + |E|)$

---

```

1  #connex : 4-connex
2  connex = [(i,j+1),(i,j-1),(i+1,j),(i-1,j)]
3  #connex : 8-connex
4  connex = [(i,j+1),(i,j-1),(i+1,j),(i-1,j),(i+1,j+1),(i-1,j-1),(i+1,j-1),(i-1,j+1)]
5  def dfs_grid(grid, i, j, mark, free):
6      grid[i][j] = mark
7      height = len(grid)
8      width = len(grid[0])
9      for ni, nj in connex:
10         if 0 <= ni < height and 0 <= nj < width:
11             if grid[ni][nj] == free:
12                 dfs_grid(grid, ni, nj, mark, free)
13  def nb_connected_components(grid, free='#'):
14      nb_components = 0
15      height = len(grid)
16      width = len(grid[0])
17      for i in range(height):
18          for j in range(width):
19              if grid[i][j] == free:
20                  nb_components += 1
21                  dfs_grid(grid, i, j, str(nb_components), free)
22  return nb_components

```

---

## 4.5 Composantes bi-connexe

$\mathcal{O}(m\alpha(m,n))$  avec  $\alpha(x,y)$  la réciproque de la fonction de Ackermann

---

```

1  def cut_nodes_edges(graph):
2      n = len(graph)
3      time = 0
4      num = [None] * n
5      low = [n] * n
6      parent = [None] * n      # parent[v] = None if root else parent of v
7      critical_children = [0] * n # cc[u] = #{children v | low[v] >= num[u]}
8      times_seen = [-1] * n
9      for start in range(n):
10         if times_seen[start] == -1:           # init DFS path
11             times_seen[start] = 0
12             to_visit = [start]
13             while to_visit:
14                 node = to_visit[-1]
15                 if times_seen[node] == 0:      # start processing
16                     num[node] = time
17                     time += 1
18                     low[node] = float('inf')
19                 children = graph[node]

```



```

20         if times_seen[node] == len(children): # end processing
21             to_visit.pop()
22             up = parent[node]                # propagate low to parent
23             if up is not None:
24                 low[up] = min(low[up], low[node])
25                 if low[node] >= num[up]:
26                     critical_children[up] += 1
27             else:
28                 child = children[times_seen[node]] # next arrow
29                 times_seen[node] += 1
30                 if times_seen[child] == -1: # not visited yet
31                     parent[child] = node      # link arrow
32                     times_seen[child] = 0
33                     to_visit.append(child)    # (below) back arrow
34                 elif num[child] < num[node] and parent[node] != child:
35                     low[node] = min(low[node], num[child])
36     cut_edges = []
37     cut_nodes = []                                # extract solution
38     for node in range(n):
39         if parent[node] is None:                  # characteristics
40             if critical_children[node] >= 2:
41                 cut_nodes.append(node)
42         else:                                     # internal nodes
43             if critical_children[node] >= 1:
44                 cut_nodes.append(node)
45             if low[node] >= num[node]:
46                 cut_edges.append((parent[node], node))
47     return cut_nodes, cut_edges

```

---

## 4.6 Composantes fortement connexe

$\mathcal{O}(|V| + |E|)$

### 4.6.1 Kosaraju

---

```

1 def kosaraju_dfs(graph, nodes, order, sccp):
2     times_seen = [-1] * len(graph)
3     for start in nodes:
4         if times_seen[start] == -1:
5             to_visit = [start]
6             times_seen[start] = 0
7             sccp.append([start])
8             while to_visit:
9                 node = to_visit[-1]
10                children = graph[node]
11                if times_seen[node] == len(children):
12                    to_visit.pop()
13                    order.append(node)
14                else:
15                    child = children[times_seen[node]]
16                    times_seen[node] += 1
17                    if times_seen[child] == -1:
18                        times_seen[child] = 0
19                        to_visit.append(child)
20                        sccp[-1].append(child)
21 def reverse(graph):
22     rev_graph = [[] for node in graph]
23     for node in range(len(graph)):
24         for neighbour in graph[node]:
25             rev_graph[neighbour].append(node)
26     return rev_graph
27 def kosaraju(graph):
28     n = len(graph)
29     order = []

```

---

```

30     sccp = []
31     kosaraju_dfs(graph, range(n), order, [])
32     kosaraju_dfs(reverse(graph), order[::-1], [], sccp)
33     return sccp[::-1]

```

---

## 4.7 2-SAT

Complexité linéaire

---

```

1 def vertex(lit):
2     if lit > 0:
3         return 2 * (lit - 1)
4     else:
5         return 2 * (-lit - 1) + 1
6 def two_sat(formula):
7     n = max(abs(clause[p]) for p in (0,1) for clause in formula)
8     graph = [[] for node in range(2*n)]
9     for x,y in formula:
10         graph[vertex(-x)].append(vertex(y))
11         graph[vertex(-y)].append(vertex(x))
12     sccp = kosaraju(graph)
13     comp_id = [None] * (2*n)
14     affectations = [None] * (2*n)
15     for component in sccp:
16         rep = min(component)
17         for vtx in component:
18             comp_id[vtx] = rep
19             if affectations[vtx] == None:
20                 affectations[vtx] = True
21                 affectations[vtx ^ 1] = False
22     for i in range(n):
23         if comp_id[2*i] == comp_id[2*i+1]:
24             return None
25     return affectations[::2]

```

---

## 4.8 Cycle le plus court

Powergraph :  $\mathcal{O}(n^3)$ , Shortest Cycle :  $\mathcal{O}(|V| \times |E|)$  Path :  $\iota(V)$

---

```

1 def path(tree, v):
2     P = []
3     while not P or P[-1] != v:
4         P.append(v)
5         v = tree[v]
6     return P
7 def shortest_cycle(graph):
8     best_cycle = float('inf')
9     best_u = None
10    best_v = None
11    best_tree = None
12    V = list(range(len(graph)))
13    for root in V:
14        tree, cycle_len, u, v = bfs(graph, root, best_cycle // 2)
15        if cycle_len < best_cycle:
16            best_cycle = cycle_len
17            best_u = u
18            best_v = v
19            best_tree = tree
20    if best_cycle == float('inf'):
21        return None # no cycle found
22    Pu = path(best_tree, best_u) # combine path to make a cycle
23    Pv = path(best_tree, best_v)
24    cycle = Pu[::-1] + Pv # last vertex equals first vertex
25    return cycle[1:] # remove duplicate vertex
26 def powergraph(graph, k):

```

---

```

27     V = range(len(graph))
28     # create weight matrix for paths of length 1
29     M = [[float('inf') for v in V] for u in V]
30     for u in V:
31         for v in graph[u]:
32             M[u][v] = M[v][u] = 1
33     M[u][u] = 0
34     floyd_warshall(M)
35     return [[v for v in V if M[u][v] <= k] for u in V]

```

---

## 4.9 Chemin eulérien

$\mathcal{O}(|V| + |E|)$  (Pour les deux)

### 4.9.1 Dirigé

---

```

1 def eulerian_tour_directed(graph):
2     P = []
3     Q = [0]
4     R = []
5     next = [0] * len(graph)
6     while Q:
7         node = Q.pop()
8         P.append(node)
9         while next[node] < len(graph[node]):
10             neighbour = graph[node][next[node]]
11             next[node] += 1
12             R.append(neighbour)
13             node = neighbour
14         while R:
15             Q.append(R.pop())
16     return P

```

---

### 4.9.2 Non Dirigé

---

```

1 def eulerian_tour_undirected(graph):
2     P = []
3     Q = [0]
4     R = []
5     next = [0] * len(graph)
6     seen = [set() for _ in graph]
7     while Q:
8         node = Q.pop()
9         P.append(node)
10        while next[node] < len(graph[node]):
11            neighbour = graph[node][next[node]]
12            next[node] += 1
13            if neighbour not in seen[node]:
14                seen[neighbour].add(node)
15                R.append(neighbour)
16                node = neighbour
17        while R:
18            Q.append(R.pop())
19    return P

```

---

## 4.10 Chemin le plus court

### 4.10.1 Poids positif ou nul - Dijkstra

$\mathcal{O}(|V|^2)$

---

```

1 from heapq import heappop, heappush
2 def dijkstra(graph, weight, source=0, target=None):

```

---

```

3     """single source shortest paths by Dijkstra
4     :complexity:  $O(|V| + |E|\log|V|)$ """
5     n = len(graph)
6     assert all(weight[u][v] >= 0 for u in range(n) for v in graph[u])
7     prec = [None] * n
8     black = [False] * n
9     dist = [float('inf')] * n
10    dist[source] = 0
11    heap = [(0, source)]
12    while heap:
13        dist_node, node = heappop(heap)    # Closest node from source
14        if not black[node]:
15            black[node] = True
16            if node == target:
17                break
18            for neighbor in graph[node]:
19                dist_neighbor = dist_node + weight[node][neighbor]
20                if dist_neighbor < dist[neighbor]:
21                    dist[neighbor] = dist_neighbor
22                    prec[neighbor] = node
23                    heappush(heap, (dist_neighbor, neighbor))
24    return dist, prec
25 def dijkstra_update_heap(graph, weight, source=0, target=None):
26     """single source shortest paths by Dijkstra
27     :complexity:  $O(|V| + |E|\log|V|)$ """
28     n = len(graph)
29     assert all(weight[u][v] >= 0 for u in range(n) for v in graph[u])
30     prec = [None] * n
31     dist = [float('inf')] * n
32     dist[source] = 0
33     heap = OurHeap([(dist[node], node) for node in range(n)])
34     while heap:
35         dist_node, node = heap.pop()    # Closest node from source
36         if node == target:
37             break
38         for neighbor in graph[node]:
39             old = dist[neighbor]
40             new = dist_node + weight[node][neighbor]
41             if new < old:
42                 dist[neighbor] = new
43                 prec[neighbor] = node
44                 heap.update((old, neighbor), (new, neighbor))
45     return dist, prec

```

---

#### 4.10.2 Poids arbitraire - Bellman-Ford

$\mathcal{O}(|V| \times |E|)$

---

```

1
2 def bellman_ford2(graph, weight, source):
3     """    :complexity:  $O(|V|*|E|)$ """
4     n = len(graph)
5     dist = [float('inf')] * n
6     prec = [None] * n
7     dist[source] = 0
8
9     def relax():
10        for nb_iterations in range(n-1):
11            for node in range(n):
12                for neighbor in graph[node]:
13                    alt = dist[node] + weight[node][neighbor]
14                    if alt < dist[neighbor]:
15                        dist[neighbor] = alt
16                        prec[neighbor] = node
17    relax()

```

```

18     intermediate = dist[:] # is fixpoint in absence of neg cycles
19     relax()
20     for node in range(n):
21         if dist[node] < intermediate[node]:
22             dist[node] = float('-inf')
23     return dist, prec, min(dist) == float('-inf')

```

---

### 4.10.3 Floyd-Warshall

$\mathcal{O}(n^3)$

```

1 def floyd_warshall(weight):
2     """ $\mathcal{O}(|V|^3)$ """
3     for k, Wk in enumerate(weight):
4         for _, Wu in enumerate(weight):
5             for v, Wuv in enumerate(Wu):
6                 alt = Wu[k] + Wk[v]
7                 if alt < Wuv:
8                     Wu[v] = alt
9     for v, Wv in enumerate(weight):
10        if Wv[v] < 0: # negative cycle found
11            return True
12    return False

```

---

### 4.10.4 A\* - Chemin le moins coûteux

```

1 Input : graph G(V,E) avec node start et node end
2 Output : Chemin le moins couteux de start a end
3 ALGORITHME :
4 Initialisation :
5     to_visit = {start}
6     visited = {}
7     g(start) = 0
8     h_cost(start) = heuristic_function(start, end)
9     f(start) = g(start) + h(start)
10 While to_visit not empty:
11     m = noeud on top of to_visit with least f
12     if m == end :
13         return
14     remove m from to_visit
15     add m to visited
16     for each n in child(m):
17         if n in visited:
18             continue
19         cost = g(m) + distance(m,n)
20         if n in to_visit and cost < g(n):
21             remove n from to_visit if better path
22         if n in visited and cost < g(n):
23             remove n from visited
24         if n not in to_visit and n not in visited:
25             add n to to_visit
26             g(n) = cost
27             h(n) = heuristic_function(n,end)
28             f(n) = g(n) + h(n)

```

---

## 4.11 Couplages

### 4.11.1 Bipartite Matching

$\mathcal{O}(|U| \times |E|)$

```

1 def augment(u, bigraph, visit, match):
2     for v in bigraph[u]:
3         if not visit[v]:

```

```

4         visit[v] = True
5         if match[v] is None or augment(match[v], bigraph,
            visit, match):
6             match[v] = u      # found an augmenting path
7             return True
8     return False
9 def max_bipartite_matching(bigraph):
10     n = len(bigraph)          # same domain for U and V
11     match = [None] * n
12     for u in range(n):
13         augment(u, bigraph, [False] * n, match)
14     return match

```

---

#### 4.11.2 Biparti avec poids - Kuhn-Munkres

$\mathcal{O}(|V|^3)$

---

```

1 def kuhn_munkres(G, TOLERANCE=1e-6):
2     nU = len(G)
3     U = range(nU)
4     nV = len(G[0])
5     V = range(nV)
6     assert nU <= nV
7     mu = [None] * nU          # empty matching
8     mv = [None] * nV
9     lu = [max(row) for row in G] # trivial labels
10    lv = [0] * nV
11    for root in U:              # build an alternate tree
12        au = [False] * nU      # au, av mark nodes...
13        au[root] = True        # ... covered by the tree
14        Av = [None] * nV       # Av[v] successor of v in the tree
15        # for every vertex u, slack[u] := (val, v) such that
16        # val is the smallest slack on the constraints (*)
17        # with fixed u and v being the corresponding vertex
18        slack = [(lu[root] + lv[v] - G[root][v], root) for v in V]
19        while True:
20            (delta, u), v = min((slack[v], v) for v in V if Av[v] is None)
21            assert au[u]
22            if delta > TOLERANCE: # tree is full
23                for u0 in U:      # improve labels
24                    if au[u0]:
25                        lu[u0] -= delta
26                for v0 in V:
27                    if Av[v0] is not None:
28                        lv[v0] += delta
29                else:
30                    (val, arg) = slack[v0]
31                    slack[v0] = (val - delta, arg)
32            assert abs(lu[u] + lv[v] - G[u][v]) <= TOLERANCE # equality
33            Av[v] = u            # add (u, v) to A
34            if mv[v] is None:
35                break            # alternating path found
36            u1 = mv[v]
37            assert not au[u1]
38            au[u1] = True        # add (u1, v) to A
39            for v1 in V:
40                if Av[v1] is None: # update margins
41                    alt = (lu[u1] + lv[v1] - G[u1][v1], u1)
42                    if slack[v1] > alt:
43                        slack[v1] = alt
44            while v is not None:  # ... alternating path found
45                u = Av[v]         # along path to root
46                prec = mu[u]
47                mv[v] = u         # augment matching
48                mu[u] = v

```

```

49         v = prec
50     return (mu, sum(lu) + sum(lv))

```

---

#### 4.11.3 Biparti avec préférence - Gale-Shapley

$\mathcal{O}(|V|^2)$

```

1 def gale_shapley(men, women):
2     n = len(men)
3     assert n == len(women)
4     current_suitor = [0] * n
5     spouse = [None] * n
6     rank = [[0] * n for j in range(n)] # build rank
7     for j in range(n):
8         for r in range(n):
9             rank[j][women[j][r]] = r
10    singles = deque(range(n)) # all men are single and get in the queue
11    while singles:
12        i = singles.popleft()
13        j = men[i][current_suitor[i]]
14        current_suitor[i] += 1
15        if spouse[j] is None:
16            spouse[j] = i
17        elif rank[j][spouse[j]] < rank[j][i]:
18            singles.append(i)
19        else:
20            singles.put(spouse[j]) # sorry for spouse[j]
21            spouse[j] = i
22    return spouse

```

---

#### 4.11.4 Couverture par sommet minimum

$\mathcal{O}(|U| \times |E|)$

```

1
2 def _alternate(u, bigraph, visitU, visitV, matchV):
3     """extend alternating tree from free vertex u.
4     visitU, visitV marks all vertices covered by the tree.
5     """
6     visitU[u] = True
7     for v in bigraph[u]:
8         if not visitV[v]:
9             visitV[v] = True
10            assert matchV[v] is not None # otherwise match is not maximum
11            _alternate(matchV[v], bigraph, visitU, visitV, matchV)
12 def bipartite_vertex_cover(bigraph):
13     """Bipartite minimum vertex cover by Koenig's theorem
14     :param bigraph: adjacency list, index = vertex in U,
15                     value = neighbor list in V
16     :assumption: U = V = {0, 1, 2, ..., n - 1} for n = len(bigraph)
17     :returns: boolean table for U, boolean table for V
18     :comment: selected vertices form a minimum vertex cover,
19               i.e. every edge is adjacent to at least one selected vertex
20               and number of selected vertices is minimum
21     """
22     V = range(len(bigraph))
23     matchV = max_bipartite_matching(bigraph)
24     matchU = [None for u in V]
25     for v in V: # -- build the mapping from U to V
26         if matchV[v] is not None:
27             matchU[matchV[v]] = v
28     visitU = [False for u in V] # -- build max alternating forest
29     visitV = [False for v in V]
30     for u in V:
31         if matchU[u] is None: # -- starting with free vertices in U

```

```

32         _alternate(u, bigraph, visitU, visitV, matchV)
33     inverse = [not b for b in visitU]
34     return (inverse, visitV)

```

---

## 4.12 Coupes & Flots

## 4.13 Dilworth

Complexité : comme Bipartite Matching  
Le graphe doit être acyclique

```

1 def dilworth(graph):
2     n = len(graph)
3     match = max_bipartite_matching(graph) # maximum matching
4     part = [None] * n                    # partition into chains
5     nb_chains = 0
6     for v in range(n - 1, -1, -1):      # in inverse topological order
7         if part[v] is None:              # start of chain
8             u = v
9             while u is not None:          # follow the chain
10                part[u] = nb_chains        # mark
11                u = match[u]
12            nb_chains += 1
13     return part

```

---

# 5 Points et polygones

## 5.1 Points

### 5.1.1 Points

```

1 point = [x,y]

```

---

### 5.1.2 Cross-product

```

1 def cross_product(p1, p2):
2     return p1[0] * p2[1] - p2[0] * p1[1]

```

---

### 5.1.3 Direction

```

1 def left_turn(a,b,c):
2     return (a[0]-c[0]) * (b[1]-c[1]) - (a[1]-c[1]) * (b[0]-c[0]) > 0
3     # If floats are used, instead of 0 test if in [0-10E-7,0+10E-7]

```

---

## 5.2 Enveloppe convexe

Complexité :  $\mathcal{O}(n \log(n))$

```

1 def andrew(S):
2     S.sort()
3     top = []
4     bot = []
5     for p in S:
6         while len(top) >= 2 and not left_turn(p,top[-1],top[-2]):
7             top.pop()
8         top.append(p)
9         while len(bot) >= 2 and not left_turn(bot[-2],bot[-1],p):
10            bot.pop()
11        bot.append(p)
12    return bot[:-1] + top[:0:-1]

```

---



### 5.3 Points les plus proches

---

```
1 def dist(p, q):
2     return hypot(p[0] - q[0], p[1] - q[1]) # Euclidean dist.
3 def cell(point, size):
4     """ returns the grid cell coordinates containing the given point.
5     size is the side length of a grid cell
6     beware: in other languages negative coordinates need special care
7     in C++ for example int(-1.5) == -1 and not -2 as we need
8     hence we need floor(x / pas) in C++ using #include <cmath>
9     """
10    x, y = point # size = grid cell side length
11    return (int(x // size), int(y // size))
12 def improve(S, d):
13     G = {} # maps grid cell to its point
14     for p in S: # for every point
15         a, b = cell(p, d / 2) # determine its grid cell
16         for a1 in range(a - 2, a + 3):
17             for b1 in range(b - 2, b + 3):
18                 if (a1, b1) in G: # compare with points
19                     q = G[a1, b1] # in surrounding cells
20                     pq = dist(p, q)
21                     if pq < d: # improvement found
22                         return pq, p, q
23     G[a, b] = p
24     return None
25 def closest_points(S):
26     shuffle(S)
27     assert len(S) >= 2
28     p = S[0] # start with distance between
29     q = S[1] # first two points
30     d = dist(p, q)
31     while d > 0: # distance 0 cannot be improved
32         r = improve(S, d)
33         if r: # distance improved
34             d, p, q = r
35         else: # r is None: could not improve
36             break
37     return p, q
```

---

### 5.4 Aire d'un polygone

Complexité linéaire *Uniquement pour les polygones simples. Réduire à des composantes simples sinon.*  
Voir partie Mathématiques.

---

```
1 def area(p):
2     A = 0
3     for i in range(len(p)):
4         A += p[i-1][0] * p[i][1] - p[i][0] * p[i-1][1]
5     return A/2
```

---

### 5.5 Polygone simple

$\mathcal{O}(n \log(n))$

---

```
1 def is_simple(polygon):
2     """complexity:  $\mathcal{O}(n \log n)$  for  $n=\text{len}(\text{polygon})$ """
3     n = len(polygon)
4     order = list(range(n))
5     order.sort(key=lambda i: polygon[i]) # lexicographic order
6     rank_to_y = list(set(p[1] for p in polygon))
7     rank_to_y.sort()
8     y_to_rank = {y: rank for rank, y in enumerate(rank_to_y)}
9     S = RangeMinQuery([0] * len(rank_to_y)) # sweep structure
10    last_y = None
```

---

```

11     for i in order:
12         x, y = polygon[i]
13         rank = y_to_rank[y]
14         right_x = max(polygon[i - 1][0], polygon[(i + 1) % n][0])
15         left = x < right_x
16         below_y = min(polygon[i - 1][1], polygon[(i + 1) % n][1])
17         high = y > below_y
18         if left:                                # y does not need to be in S yet
19             if S[rank]:
20                 return False                    # two horizontal segments intersect
21                 S[rank] = -1                    # add y to S
22             else:
23                 S[rank] = 0                    # remove y from S
24         if high:
25             lo = y_to_rank[below_y] # check S between [lo + 1, rank - 1]
26             if (below_y != last_y or last_y == y or
27                 rank - lo >= 2 and S.range_min(lo + 1, rank)):
28                 return False                # horiz. & vert. segments intersect
29             last_y = y                      # remember for next iteration
30     return True

```

---

## 5.6 Rectangle avec des points

$\mathcal{O}(n^2)$

Combien de rectangles peut-on former dans un ensemble de points

---

```

1 def rectangles_from_points(S):
2     """
3     :param S: list of points, as coordinate pairs
4     :returns: the number of rectangles
5     """
6     answ = 0
7     pairs = {}
8     for j, _ in enumerate(S):
9         for i in range(j):    # loop over point pairs (p,q)
10             px, py = S[i]
11             qx, qy = S[j]
12             center = (px + qx, py + qy)
13             dist = (px - qx) ** 2 + (py - qy) ** 2
14             signature = (center, dist)
15             if signature in pairs:
16                 answ += len(pairs[signature])
17                 pairs[signature].append((i, j))
18             else:
19                 pairs[signature] = [(i, j)]
20     return answ

```

---

## 5.7 Plus grand rectangle dans un histogramme

Complexité linéaire

---

```

1 def rectangles_from_histogram(H):
2     """Largest Rectangular Area in a Histogram
3     :param H: histogram table
4     :returns: area, left, height, right, rect. is [0, height] * [left, right]
5     """
6     best = (float('-inf'), 0, 0, 0)
7     S = []
8     H2 = H + [float('-inf')] # extra element to empty the queue
9     for right, _ in enumerate(H2):
10         x = H2[right]
11         left = right
12         while len(S) > 0 and S[-1][1] >= x:
13             left, height = S.pop()
14             # first element is area of candidate

```

---

```

15         rect = (height * (right - left), left, height, right)
16         if rect > best:
17             best = rect
18         S.append((left, x))
19     return best

```

---

## 5.8 Rectangle sur une grille

$\mathcal{O}(n)$

---

```

1 def rectangles_from_grid(P, black=1):
2     """Largest area rectangle in a binary matrix
3     :param P: matrix
4     :param black: search for rectangles filled with value black
5     :returns: area, left, top, right, bottom of optimal rectangle
6               consisting of all (i,j) with
7               left <= j < right and top <= i <= bottom
8     """
9     rows = len(P)
10    cols = len(P[0])
11    t = [0] * cols
12    best = None
13    for i in range(rows):
14        for j in range(cols):
15            if P[i][j] == black:
16                t[j] += 1
17            else:
18                t[j] = 0
19            (area, left, height, right) = rectangles_from_histogram(t)
20            alt = (area, left, i, right, i-height)
21            if best is None or alt > best:
22                best = alt
23    return best

```

---

## 6 Ensembles

### 6.1 Rendu de monnaie

Problème NP-Complet.

---

```

1 def coin(x, R):
2     b = [False] * (R+1)
3     b[0] = True
4     for xi in x:
5         for s in range(xi, R+1):
6             b[s] |= b[s - xi]
7     return b[R]

```

---

### 6.2 Sac à dos

Problème NP-Complet.

---

```

1 def knapsack(p, v, cmax):
2     n = len(p)
3     Opt = [[0] * (cmax + 1) for _ in range(n+1)]
4     Sel = [[False] * (cmax + 1) for _ in range(n+1)]
5     #cas de base
6     for cap in range(p[0], cmax + 1):
7         Opt[0][cap] = v[0]
8         Sel[0][cap] = True
9     # cas d'induction
10    for i in range(1,n):
11        for cap in range(cmax+1):
12            if cap >= p[i] and Opt[i-1][cap - p[i]] + v[i] > Opt[i-1][cap]:
13                Opt[i][cap] = Opt[i-1][cap-p[i]] + v[i]

```

---

```

14         Sel[i][cap] = True
15     else:
16         Opt[i][cap] = Opt[i-1][cap]
17         Sel[i][cap] = False
18     cap = cmax
19     sol = []
20     for i in range(n-1, -1, -1):
21         if Sel[i][cap]:
22             sol.append(i)
23             cap -= p[i]
24     return (Opt[n-1][cmax], sol)

```

---

### 6.3 k-somme

$\mathcal{O}(nR)$

```

1 def subset_sum(x, R):
2     b = [False] * (R + 1)
3     b[0] = True
4     for xi in x:
5         for s in range(R, xi - 1, -1):
6             b[s] |= b[s - xi]
7     return b[R]

```

---

### 6.4 Valeurs les plus proches

```

1 def closest_values(L):
2     assert len(L) >= 2
3     L.sort()
4     valmin, argmin = min((L[i] - L[i - 1], i) for i in range(1, len(L)))
5     return L[argmin - 1], L[argmin]

```

---

### 6.5 Union d'intervalles

$\mathcal{O}(n \log(n))$

```

1 def intervals_union(S):
2     E = [(low, -1) for (low, high) in S]
3     E += [(high, +1) for (low, high) in S]
4     nb_open = 0
5     last = None
6     retval = []
7     for x, _dir in sorted(E):
8         if _dir == -1:
9             if nb_open == 0:
10                 last = x
11                 nb_open += 1
12             else:
13                 nb_open -= 1
14                 if nb_open == 0:
15                     retval.append((last, x))
16     return retval

```

---

### 6.6 MiniMax

$\mathcal{O}(b^m)$

```

1 minimax(state, depth, player)
2     if (player == max) then
3         best = [null, -infinity]
4     else
5         best = [null, +infinity]
6     if (depth == 0 or gameover) then
7         score = evaluate this state for player

```

```

8     return [null, score]
9 for each valid move m for player in state s do
10     execute move m on s
11     [move, score] = minimax(s, depth - 1, -player)
12     undo move m on s
13     if (player = max) then
14         if score > best.score then best = [move, score]
15     else
16         if score < best.score then best = [move, score]
17     return best
18 end

```

---

## 7 Calculs

### 7.1 PGCD

```

1 def pgcd(a,b):
2     return a if b == 0 else pgcd(b,a%b)

```

---

### 7.2 Coefficients de Bézout

```

1 def bezout(a,b):
2     if b == 0:
3         return (1,0)
4     else:
5         u,v = bezout(b,a%b)
6         return (v, u - (a//b) *v)
7 def inv(a,p):
8     return bezout(a,p)[0]%p

```

---

### 7.3 Coefficients binomiaux

```

1 def binom(n,k):
2     prod = 1
3     for i in range(k):
4         prod = (prod * (n-i)) // (i+1)
5     return prod
6 def binom_modulo(n,k,p):
7     prod = 1
8     for i in range(k):
9         prod = (prod * (n-i) * inv(i+1,p)) %p
10    return prod

```

---

### 7.4 Inverse

```

1 def inv(a,p):
2     return bezout(a,p)[0] %p

```

---

### 7.5 Nombres premiers

Crible d'Eratosthène :  $\mathcal{O}(n \log(\log(n)))$   
 Gries Misra :  $\mathcal{O}(n)$

```

1 def eratosthene(n):
2     P = [True] * n
3     answ = [2]
4     for i in range(3, n, 2):
5         if P[i]:
6             answ.append(i)
7             for j in range(i * i, n, i):

```

```

8         P[j] = False
9     return answ
10 def gries_misra(n):
11     primes = []
12     factor = [0] * n
13     for x in range(2, n):
14         if not factor[x]:      # no factor found
15             factor[x] = x      # meaning x is prime
16             primes.append(x)
17         for p in primes:      # loop over primes found so far
18             if p > factor[x] or p * x >= n:
19                 break
20             factor[p * x] = p # p is the smallest factor of p * x
21     return primes, factor

```

---

## 8 Mathématiques

### 8.1 Mesure

#### 8.1.1 Distance

Définition :

- $\delta(x, y) \geq 0, \delta(x, y) = 0$  ssi  $x = y$
- $\delta(x, y) = \delta(y, x)$  (Symétrique)
- Satisfait l'inégalité triangulaire :  $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$

Distance de Manhattan :  $\delta(a, b) = |x_b - x_a| + |y_b - y_a|$

### 8.2 Géométrie

#### 8.2.1 3D

Caractéristique d'Euler  $\chi$  :

$\chi = s - a + f$ , si  $\chi = 2$  alors le polyèdre est de rang 0 (pas de trou)

Formule usuelles :

- Sphère : Volume :  $\frac{4}{3}\pi r^3$  — Surface :  $4\pi r^2$
- Cylindre droit : Volume  $\pi r^2 h$  — Surface :  $2\pi r(r + h)$
- Cone circulaire droit : Volume  $\frac{1}{3}\pi r^2 h$  — Surface :  $\pi r(r + s)$
- Prisme triangulaire : Volume  $Al$  ou  $\frac{1}{2}bhl$  — Surface :  $bh + 2ls + lb$
- Prisme : Volume  $Ah$  — Surface :  $2A + (h \times p)$
- Pyramide : Volume :  $\frac{1}{3}Ah$
- Tétraèdre : Volume :  $\frac{b^3}{6\sqrt{2}}$  — Surface :  $\sqrt{3}b^2$
- Pyramide carré : Volume :  $\frac{1}{3}s^2 \times h$  — Surface :  $s^2 + 2sh$
- Cuboïde : Volume :  $l \times w \times h$  — Surface :  $2lh + 2lw + 2wh$  (Cube :  $6s^2$ )

#### 8.2.2 2D

- Polygone simple : Aire :  $A = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - x_{i+1} y_i)$
- Cercle : Aire :  $\pi r^2$  — Périmètre :  $2 \times \pi \times r$
- Losange : Aire :  $\frac{D \times d}{2}$
- Trapèze : Aire :  $\frac{(B+b) \times h}{2}$
- Parralélogramme : Aire :  $B \times h$
- Ellipse : Aire :  $a \times b \times \pi$   
Périmètre approximation :  
 $p = \pi(a + b) \left(1 + \frac{1}{4}h + \frac{1}{64}h^2 + \frac{1}{256}h^3 + \dots\right)$

### 8.2.3 Points entiers dans un polygone

Sur le contour :

Dans le polygone :

Théorème de Pick :  $P = n_i + \frac{n_b}{2} - 1$

### 8.2.4 Théorème de la galerie d'art

Pour garder un polygone simple à  $n$  sommets,  $\lfloor \frac{n}{3} \rfloor$  gardiens suffisent.

## 8.3 Approximations

### 8.3.1 Méthode de Newton

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

### 8.3.2 Méthode de la sécante

Cette méthode est à appliquer quand le calcul de la dérivée est couteux  $\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$

### 8.3.3 Plus forte pente — Méthode du gradient

Cette méthode peut être assez couteuse (ZigZag)

**Algorithme du gradient** — On se donne un point initial  $x_0 \in \mathbb{E}$  et un seuil de tolérance  $\epsilon \geq 0$ . L'algorithme du gradient donne une suite d'itérés  $x_1, x_2, \dots \in \mathbb{E}$ , jusqu'à ce qu'un test d'arrêt soit satisfait. Il passe de  $x_k$  à  $x_{k+1}$  par les étapes suivantes :

- 1. *Simulation* : Calcul de  $\nabla f(x_k)$
- 2. *Test d'arrêt* : Si  $\|\nabla f(x_k)\| \leq \epsilon$ , arrêt
- 3. *Calcul du pas* :  $\alpha_k > 0$  par un règle de recherche linéaire sur  $f$  en  $x_k$  le long de la direction  $-\nabla f(x_k)$
- 4. *Nouvel itéré* :  $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$

Note : plus le pas est petit, plus la convergence est lente, mais précise, et plus il est grand, plus la convergence est forte, mais imprécise : **il faut donc choisir un pas adapté**

## 8.4 Probabilités et Statistiques

### 8.4.1 Loïs de probabilités

Discrètes :

- Poisson :  $\mathbb{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$ ,  $\mathbb{E}[X] = \lambda$ ,  $\mathbb{V}[X] = \lambda$
- Binomiale :  $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ ,  $\mathbb{E}[X] = np$ ,  $\mathbb{V}[X] = np(1-p)$
- Géométrique :  $\mathbb{P}(X = k) = (1-p)^{k-1} p$ ,  $\mathbb{E}[X] = \frac{1}{p}$ ,  $\mathbb{V}[X] = \frac{1-p}{p^2}$
- Uniforme :  $\mathbb{P}(X = k) = \frac{1}{n}$ ,  $\mathbb{E}[X] = \frac{1}{n} \sum_{k=1}^n x_k$

### 8.4.2 Techniques statistiques

Théorème de la limite centrale :

Soit  $X_1, X_2, \dots$  une suite de variable aléatoires réelles définies sur le même espace de probabilités, i.i.d et suivant la même loi  $\mathcal{L}$ . De plus, l'espérance  $\mu$  et l'écart-type  $\sigma$  de  $\mathcal{L}$  existent et soient finis avec  $\sigma \neq 0$ .

Soit la somme  $S_n = X_1 + X_2 + \dots + X_n$

Alors l'espérance de  $S_n$  est  $n\mu$  et l'écart-type est  $\sigma\sqrt{n}$

Quand  $n$  est assez grand, la Loi Normale  $\mathcal{N}(n\mu, n\sigma^2)$  est une bonne approximation de  $S_n$

On pose  $\overline{X_n} = \frac{S_n}{n}$  et  $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\overline{X_n} - \mu}{\sigma/\sqrt{n}}$

## 8.5 Analyse

### 8.5.1 Sommes

$$\sum_{k=0}^n a^k = \frac{1 - a^{n+1}}{1 - a}, a \neq 1 \quad (1)$$

$$\sum_{k=0}^n k a^k = \frac{a}{(1 - a)^2} [1 - (n + 1)a^n + n a^{n+1}], a \neq 1 \quad (2)$$

$$\sum_{k=0}^n k^2 a^k = \frac{a}{(1 - a)^3} [(1 + a) - (n + 1)^2 a^n + (2n^2 + 2n - 1)a^{n+1} - n^2 a^{n+2}], a \neq 1 \quad (3)$$

$$\sum_{k=0}^n k = \frac{n(n + 1)}{2} \quad (4)$$

$$\sum_{k=0}^n k^2 = \frac{n(n + 1)(2n + 1)}{6} \quad (5)$$

$$\sum_{k=0}^n k^3 = \left( \frac{n(n + 1)}{2} \right)^2 \quad (6)$$

$$\sum_{k=0}^n k^3 = \frac{n}{30} (n + 1)(2n + 1)(3n^2 + 3n - 1) \quad (7)$$

### 8.5.2 Dérivée et primitives usuelles

Fonction	Domaine de dérivabilité	Dérivée
$\ln(x)$	$\mathbb{R}^{+*}$	$\frac{1}{x}$
$e^x$	$\mathbb{R}$	$e^x$
$\frac{1}{x}$	$\mathbb{R}^*$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\mathbb{R}^{+*}$	$\frac{1}{2\sqrt{x}}$
$x^\alpha, \alpha \in \mathbb{R}$	$\mathbb{R}^{+*}$	$\alpha x^{\alpha-1}$
$\cos(x)$	$\mathbb{R}$	$-\sin(x)$
$\sin(x)$	$\mathbb{R}$	$\cos(x)$
$\tan(x)$	$]-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi[, k \in \mathbb{Z}$	$1 + \tan^2(x) = \frac{1}{\cos^2(x)}$
$\arccos(x)$	$] -1; 1[$	$\frac{-1}{\sqrt{1-x^2}}$
$\arcsin(x)$	$] -1; 1[$	$\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\mathbb{R}$	$\frac{1}{1+x^2}$
$\cosh(x)$	$\mathbb{R}$	$\sinh(x)$
$\sinh(x)$	$\mathbb{R}$	$\cosh(x)$
$\tanh(x)$	$\mathbb{R}$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$



Fonction	Intervalle d'intégration	Primitive
$(x-a)^n, n \in \mathbb{N}, a \in \mathbb{R}$	$\mathbb{R}$	$\frac{1}{n+1}(x-a)^{n+1}$
$\frac{1}{x-a}, a \in \mathbb{R}$	$] -\infty; a[ \cup ]a; +\infty[$	$\ln( x-a )$
$\frac{1}{(x-a)^n}, a \in \mathbb{R}, n \geq 2$	$] -\infty; a[ \cup ]a; +\infty[$	$-\frac{1}{(n-1)(x-a)^{n-1}}$
$\cos(ax), a \in \mathbb{R}^*$	$\mathbb{R}$	$\frac{1}{a} \sin(ax)$
$\sin(ax), a \in \mathbb{R}^*$	$\mathbb{R}$	$-\frac{1}{a} \cos(ax)$
$\tan(x)$	$]-\frac{\pi}{2} + k\pi; \frac{\pi}{2} + k\pi[, k \in \mathbb{Z}$	$-\ln( \cos(x) )$
$\ln(x)$	$\mathbb{R}^{+*}$	$x \ln(x) - x$
$e^{ax}, a \in \mathbb{R}^*$	$\mathbb{R}$	$\frac{1}{a} e^{ax}$
$(x-a)^\alpha, a \in \mathbb{R}, \alpha \in \mathbb{R} \setminus \{-1\}$	$]a; +\infty[$	$\frac{1}{\alpha+1}(x-a)^{\alpha+1}$
$a^x, a > 0$	$\mathbb{R}$	$\frac{1}{\ln(a)} a^x$
$\frac{1}{x^2+1}$	$\mathbb{R}$	$\arctan(x)$
$\sqrt{x-a}, a \in \mathbb{R}$	$]a; +\infty[$	$\frac{3}{2}(x-a)^{\frac{3}{2}}$
$\frac{1}{\sqrt{x-a}}, a \in \mathbb{R}$	$]a; +\infty[$	$2\sqrt{x-a}$
$\frac{1}{1-x^2}$	$] -1; 1[$	$\arcsin(x)$
$\cosh(x)$	$\mathbb{R}$	$\sinh(x)$
$\sinh(x)$	$\mathbb{R}$	$\cosh(x)$
$\tanh(x)$	$\mathbb{R}$	$\ln(\cosh(x))$

Opération	Dérivée
$f + g$	$f' + g'$
$f \cdot g$	$f' \cdot g + f \cdot g'$
$\frac{f}{g}$	$\frac{f' \cdot g - f \cdot g'}{g^2}$
$g \circ f$	$f' \times g' \circ f$
$\frac{1}{u}$	$-\frac{u'}{u^2}$
$u^n$	$nu'u^{n-1}$
$\sqrt{u}$	$\frac{u'}{2\sqrt{u}}$
$e^u$	$u'e^u$
$\ln(u)$	$\frac{u'}{u}$
$\sin(u)$	$u' \cos(u)$
$\cos(u)$	$-u' \sin(u)$

## 8.6 Trigonométrie

Soient  $a, b, x \in \mathbb{R}$  :

$$\cos^2(x) + \sin^2(x) = 1 \quad (8)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b) \quad (9)$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \quad (10)$$

$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \quad (11)$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \quad (12)$$

$$\sin(2x) = 2\sin(x)\cos(x) \quad (13)$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} \quad (14)$$

## 9 Techniques de programmation

### 9.1 Programmation dynamique

Résoudre le problème en le divisant en sous-problèmes, résoudre les sous-problèmes, stocker les résultats intermédiaires ("mémoïsation")

### 9.2 Diviser pour régner

Diviser un problème en sous-problèmes; Résoudre les sous-problèmes; Combiner : calculer la solution grâce aux solutions des sous-problèmes.

### 9.3 Floyd's Hare and Tortoise

L'objectif de cette méthode est de détecter des cycles. L'idée est de parcourir la liste chaînée avec deux pointeurs : un lent (tortoise) et un deux fois plus rapide (hare). Si les deux pointeurs s'intersectent, il y a un cycle.

## 10 Expressions régulières

Utilité	Python & C++
Tous les caractères	.
Début	^
Fin	\$
Bordure du mot	\b
non(Bordure)	\B
Opérateur "ou"	
Un des caractères inclus	[abc]
Sauf les caractères inclus	[^abc]
Un des caractères dans l'intervalle	[a-z]
Un des caractères dans les intervalles	[a-zA-Z]
$\geq 0$	*
$\geq 1$	+
$\leq 1$	?
$n$	{n}
$\geq n$	{n, }
$\geq n$ ET $\leq m$	{n, m}