equation system 2D phytoplankton & benthic algae V4

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Version 4 is mathematically identical to version 3, the only change being the enabling of variable diffusion coefficients in the implementation.

Version 3 is based on version 2, with an added state variable (D) representing dead detritus. The detritus diffuses and sinks the same way as live plankton, but they do not respire since they are dead. The remineralization rate of the detritus in the water is the same as the remineralization rate of the sedimented nutrients. The unit of D is [mg P m^{-3}].

The loss of nutrients of the live algae is split up into two factors, where the death rate is governed by l_d and the respiration losses is l_r.

The nutrient losses of the sediment is treated analogously.

The detritus locked in the sediment is resuspended with a constant rate r_{resus}. (In the future the resuspension rate could be dependent on the diffusion coefficients at the bottom).

1 Interior of the lake

Equations governing dynamics in the interior of the lake.

$$\frac{\partial A}{\partial t} = (G(R,I) - (l_d + l_r))A - v_A \frac{\partial A}{\partial z} + d_x \frac{\partial^2 A}{\partial x^2} + d_z \frac{\partial^2 A}{\partial z^2} \tag{1}$$

$$\frac{\partial R_d}{\partial t} = -q_A\big((G(R,I) - l_r)A\big) + r_D D + d_x \frac{\partial^2 R}{\partial x^2} + d_z \frac{\partial^2 R}{\partial z^2} \tag{2}$$

$$\frac{\partial D}{\partial t} = q_A l_d A - r_D D - v_D \frac{\partial D}{\partial z} + d_x \frac{\partial^2 D}{\partial x^2} + d_z \frac{\partial^2 D}{\partial z^2} \tag{3}$$

$$I(x,z) = I_0 \ exp\left(-k_{bg} z - \int_0^z k_A A(x,s) + k_D D(x,s) ds\right) \tag{4}$$

$$G(R,I) = G_{max} \cdot min\left(\frac{R}{M_L + R}, \frac{I}{H_L + I}\right) \tag{5}$$

k_A and k_D are the light attenuation coefficients of phytoplankton and ditritus respectively. r_D is the mineralization rate of the detritus.

2 Top, left, and right boundaries

At the top, left, and right boundaries, the flow of algae and dissolved nutrients in direction parallel to the normal is zero (i.e. no flow in our out of the lake at these boundaries).

$$\nabla A \cdot \hat{n} = 0, \tag{6}$$

$$\nabla D \cdot \hat{n} = 0, \tag{7}$$

$$\nabla R \cdot \hat{n} = 0. \tag{8}$$

3 Bottom boundary

At the bottom border there is a sediment layer of nutrients, as well as a layer of benthic algae. Starting off with the benthic algae, their growth is either limited by light or nutrients. the light limited growth rate is given by

$$g_L = \frac{G_{max,benth}}{k_B} \log\left(\frac{H + I}{H + Ie^{-k_B B}}\right) \tag{9}$$

which is derived by assuming that the benthic algae is a uniform layer with some thickness, and integrating the growth over that layer. H is the half-saturation constant of light dependent benthic algal production, and I is the light intensity at the surface of the benthic layer. k_B is the specific light attenuation coefficient of the benthic algae.

The nutrient limited growth rate of benthic algae is given by

$$g_N = G_{max,benth} \cdot B \frac{R_d}{\mu + R_d} \tag{10}$$

where μ is the half saturation constant for the nutrient limited growth, l_B is the death rate of benthic algae.

The growth of the benthic algae is given by

$$\frac{dB}{dt} = min(g_N, g_L) - l_B B. \tag{11}$$

When the phytoplankton sink to the bottom they die and become sedimented nutrients. The rate of change of the sediment nutrients is given by

$$\frac{\partial R_s}{\partial t} = -rR_s + (1 - r_B)l_B q_B B + q_A v A(z_{max}(x)), \tag{12}$$

r is the remineralization rate of the sedimented nutrients, R_s the sediment nutrient concentration, $\{z_{max}(x) = max(z)|_x\}$ and q_B is the benthic algae phosphorus-carbon ratio (stochiometry).

The rate of change of the dissolved nutrients on the bottom boundary is given below. nutrient limited flux:

$$NlimFlux = q_B g_N = q_B G_{max,Benth} B \cdot \frac{Rd}{Rd + \mu} \tag{13}$$

Light limited flux:

$$LlimFlux = q_B g_L = q_B \frac{G_{max,benth}}{k_B} \log\left(\frac{H + I}{H + Ie^{-k_B B}}\right) \tag{14}$$

the rate of change of the dissolved nutrients at the bottom boundary is given by

$$\frac{dR_d}{dt}|_{z=z_{max}(x)} = r_B l_B q_B B + rR_s - min(LlimFlux, NlimFlux). \tag{15}$$

The rate of change of the suspended dead detritus D at the bottom is given by

$$\frac{dD}{dt}|_{z=z_{max}(x)} = r_{resus} R_s. \tag{16}$$