

## How to perform a pivot?

A **pivot** is an operation which takes you from one dictionary to another one. A **dictionary** is a set of equations expressing the **basic** variables and the value of the objective function  $z$  as a function of the **non-basic** variables. Recall that all the variables (except  $z$ ) are constrained to be non-negative. The convention is that the basic variables and  $z$  are on the left hand side while the non-basic variables are on the right hand side. Moreover, **if the non-basic variables are set to 0 then one must obtain non-negative values for basic variables**. The example of the dictionary that we saw in class is

$$\begin{array}{rcllcl} x_5 & = & 3 & -x_1 & -2x_2 & & +x_4 \\ x_6 & = & 2 & -2x_1 & -x_2 & +x_3 & -x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & & 4x_1 & +3x_2 & +x_3 & +x_4 \end{array}$$

## Method

The pivoting process amounts to perform the following task :

1. Choose an **entering** variable. It must be a non-basic variable whose coefficient in the  $z$ -row is **strictly positive**. In case where there are more than one choice : choose the one with the **largest** coefficient in the  $z$ -row (this way of choosing in case of ties is called **Anstee's rule**).
2. Choose a **leaving** variable (it is a basic variable). To find it, set all the non-basic variable except the entering one to 0, and find the first basic variable which is driven to 0 when one increases the entering variable. In case of ties, choose the variable with the smallest subscript.
3. Transform the dictionary in such a way that the entering variable becomes basic while the leaving one becomes non-basic. To do that, add or subtract suitable multiples of the row of the leaving variable to the other rows, including the  $z$  one.

The pivoting process must be repeated until all the coefficients of the non-basic variables in the  $z$  row are non-positive. Indeed, we will see later in class that it means that we have reached optimality. **Once optimality is reached to get the optimal solution, one must set the non-basic variables to 0 and compute the basic variables according to the dictionary.**

Usually the first step is very fast, the second one is easy, and the third one is tedious and often generates errors.

## One detailed example

Let me know detail the steps in the example above.

**First step.** All the non-basic variables in the  $z$  row have positive coefficients. Using Anstee's rule we choose the one with the largest coefficient, that is  $x_1$ . So  $x_1$  is the **entering variable**.

**Second step.** In the dictionary we set all the other non-basic variables except  $x_1$  to 0 and look at the basic variables. This reads

$$\begin{aligned}x_5 &= 3 - x_1 \\x_6 &= 2 - 2x_1 \\x_7 &= 2\end{aligned}$$

When we increase  $x_1$  the first variable to reach 0 is  $x_6$  for  $x_1 = 1$ . Indeed,  $x_7$  is never driven to 0 while  $x_5$  is set to 0 when  $x_1 = 3$  but  $3 > 1$ . Hence **the leaving variable is  $x_6$** .

**Third step.** Now we must transform the dictionary in such a way that  $x_1$  is a basic variable and  $x_6$  a non-basic variable. So in each row  $x_1$  must not be in the right hand side while  $x_6$  can. And the row of  $x_6 = \dots$  must be replaced by a row of the type  $x_1 = \dots$ . For the first row, we add to it  $-1/2$  times the second row to eliminate  $x_1$  (we always use the row of the leaving variable for the elimination). This reads

$$x_5 - 1/2 x_6 = 2 - 3/2 x_2 - 1/2 x_3 + 3/2 x_4.$$

Moving the (now non-basic) variable  $x_6$  to the right hand side, it reads

$$x_5 = 2 + 1/2 x_6 - 3/2 x_2 - 1/2 x_3 + 3/2 x_4.$$

For the second row, the one with  $x_6$  (the leaving variable), it is simple. Just rewrite it with  $x_1$  on the left hand side and  $x_6$  on the right hand side. In this case, dividing the row by 2 and moving the variables we get

$$x_1 = 1 - 1/2 x_6 - 1/2 x_2 + 1/2 x_3 - 1/2 x_4.$$

As  $x_1$  is already not present in the third row, we do need to change it. Eventually, we need to remove  $x_1$  in the z-row. To do that we add twice the second row to the last one and we get

$$z + 2x_6 = 4 + x_2 + 3x_3 - x_4,$$

which can be written once we move the  $x_6$  variable around

$$z = 4 - 2x_6 + x_2 + 3x_3 - x_4.$$

Once we have performed these operations on all these rows, we collect the result in the new dictionary which becomes

$$\begin{aligned}x_5 &= 2 + 1/2 x_6 - 3/2 x_2 - 1/2 x_3 + 3/2 x_4 \\x_1 &= 1 - 1/2 x_6 - 1/2 x_2 + 1/2 x_3 - 1/2 x_4 \\x_7 &= 2 - x_2 - x_3 \\z &= 4 - 2x_6 + x_2 + 3x_3 - x_4\end{aligned}$$

As there are still positive coefficients in the z-row we keep going and do a new pivot.

**First step.** The non-basic variable with the largest (positive) coefficient in the z-row is  $x_3$  hence  $x_3$  **is the entering variable**.

**Second step.** We set all the non-basic variables to 0 except  $x_3$  in the dictionary to get

$$\begin{aligned}x_5 &= 2 - 1/2 x_3 \\x_1 &= 1 + 1/2 x_3 \\x_7 &= 2 - x_3.\end{aligned}$$

As  $x_3$  starts from 0 and increases, the first basic variable to reach 0 is  $x_7$  (for  $x_3 = 2$ ) and not  $x_5$ . Hence **the leaving variable is  $x_7$** .

**Third step.** To remove the entering variable  $x_3$  from the first row we add to it  $-1/2$  times the third row (which is the row of the leaving variable). It reads

$$x_5 - 1/2 x_7 = 1 + 1/2 x_6 - x_2 + 3/2 x_4$$

which is rewritten

$$x_5 = 1 + 1/2 x_6 - x_2 + 1/2 x_7 + 3/2 x_4.$$

We add  $1/2$  times the third row to the second one to eliminate  $x_3$  in the second row : it reads

$$x_1 + 1/2 x_7 = 2 - 1/2 x_6 - x_2 - 1/2 x_4,$$

which reads once we put  $x_7$  on the right hand side :

$$x_1 = 2 - 1/2 x_6 - 1/2 x_7 - 1/2 x_4.$$

For the third row, we just move  $x_3$  to the left hand side and  $x_7$  to the right hand side : it reads

$$x_3 = 2 - x_2 - x_7.$$

Eventually, we add three times the  $x_7 = \dots$  row to the z-row to eliminate  $x_3$ , this reads

$$z + 3x_7 = 10 - 2x_6 - 2x_2 - x_4,$$

which can be rewritten

$$z = 10 - 2x_6 - 2x_2 - 3x_7 - x_4.$$

We collect all these computations in the new dictionary which reads

$$\begin{array}{rcllclcl} x_5 & = & 1 & +1/2 x_6 & -x_2 & +1/2 x_7 & +3/2 x_4 \\ x_1 & = & 2 & -1/2 x_6 & -x_2 & -1/2 x_7 & -1/2 x_4 \\ x_3 & = & 2 & & -x_2 & -x_7 & \\ z & = & 10 & -2x_6 & -2x_2 & -3x_7 & -x_4 \end{array}$$

**As in the dictionary all the coefficients in the z-row are non-negative it means that optimality is reached.**

To get an optimal feasible solution, one should set all the values of the non-basic variables to 0 and compute the values of the basic variables with the dictionary. In this case it means that the optimal solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and the value of the objective function is  $z = 10$ . As a safety check, this set of values is also a solution of the equations of the first dictionary.

## Some remarks

For the quizz all you need to know is how to perform the computations above, the theoretical justification will come later in the lectures. Below are some remarks to help you detect mistakes.

1. It helps to perform the computations if there is only one non-basic variable per column in the right hand side. Notice how the leaving variable took the "column" of the entering variable in the example above. This is just a convention that you should follow (mainly because it is easier to grade if all computations are organized in the same way as mine).
2. The value of the objective function  $z$  (when non-basic variables are set to 0) must increase at each iteration. If it doesn't, it means that you have chosen the wrong entering variable.
3. Remember that if you set the non-basic variables to 0 then the basic variables should be non-negative. If this property is lost after one pivoting operation it means that you choose the wrong leaving variable.

Note that on your quizz sheet there is no need to detail the first two steps of the pivoting process, the entering and leaving variables can be given without justification. Also you don't need to detail the rows addition and subtraction as much as above.

## Practice examples

In addition to the practice quizz which is on Richard Anstee's website, below are two examples with solution which can be found in the exercise part of Chapter 2 of the textbook. The difficulty of the quizz will be similar to the first example. If you succeed to do the second one you are more than ready.

### First example

A rather simple example where you need only one pivot to reach optimality.

$$\begin{array}{rcl} x_1 & = & 3 - 2x_5 - 3x_6 \\ x_2 & = & 1 - x_5 - 5x_6 \\ x_3 & = & 4 - 2x_5 - x_6 \\ x_4 & = & 5 - 4x_5 - x_6 \\ z & = & 2x_5 + x_6 \end{array}$$

Following Anstee's rule, one chooses  $x_5$  as the entering variable. Easy computations yields that  $x_2$  is the leaving variable. Once all computations are performed the new dictionary is

$$\begin{array}{rcl} x_1 & = & 1 + 2x_2 + 7x_6 \\ x_5 & = & 1 - x_2 - 5x_6 \\ x_3 & = & 2 + 2x_2 + 9x_6 \\ x_4 & = & 1 + 4x_2 + 19x_6 \\ z & = & 2 - 2x_2 - 9x_6 \end{array}$$

All the variables in the  $z$ -row have negative coefficients hence optimality is reached. We set the non-basic variables to 0 to get the optimal solution :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

with the optimal value of the objective function being  $z = 2$ .

### Second example

Here is a second example, where the dictionary you should start with is

$$\begin{array}{rcl} x_1 & = & 4 - x_4 - x_5 - 2x_6 \\ x_2 & = & 5 - 2x_4 - 3x_6 \\ x_3 & = & 7 - 2x_4 - x_5 - 3x_6 \\ z & = & 3x_4 + 2x_5 + 4x_6 \end{array}$$

Following Anstee's rule, you need 3 pivots to get to optimality and computations are messier.

First  $x_6$  is entering while  $x_2$  is leaving the basis. The new dictionary is

$$\begin{array}{rcl} x_1 & = & 2/3 + 1/3 x_4 - x_5 + 2/3 x_2 \\ x_6 & = & 5/3 - 2/3 x_4 - 1/3 x_2 \\ x_3 & = & 2 - x_5 + x_2 \\ z & = & 20/3 + 1/3 x_4 + 2x_5 - 4/3 x_2 \end{array}$$

Still following Anstee's rule  $x_5$  enters while  $x_1$  leaves. It gives the following dictionary :

$$\begin{array}{rcl} x_5 & = & 2/3 + 1/3 x_4 - x_1 + 2/3 x_2 \\ x_6 & = & 5/3 - 2/3 x_4 - 1/3 x_2 \\ x_3 & = & 4/3 - 1/3 x_4 + x_1 + 1/3 x_2 \\ z & = & 24/3 + x_4 - 2x_1 \end{array}$$

Only one choice for the entering variable : it is  $x_4$ . Then the leaving variable is  $x_6$ . The dictionary one gets is

$$\begin{array}{rcllcl} x_5 & = & 3/2 & -1/2 x_6 & -x_1 & +1/2 x_2 \\ x_4 & = & 5/2 & -3/2 x_6 & & -1/2 x_2 \\ x_3 & = & 1/2 & +1/2 x_6 & +x_1 & +1/2 x_2 \\ z & = & 21/2 & -3/2 x_6 & -2x_1 & -1/2 x_2 \end{array}$$

All the variables in the z-row have negative coefficients hence optimality is reached. We set the non-basic variables to 0 to get the optimal solution :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 5/2 \\ 3/2 \\ 0 \end{pmatrix}$$

with the optimal value of the objective function being  $z = 21/2$ .