What does complementary slackness look like in practice ?

If you have a primal feasible solution as well as a dual feasible solution, and if you have equality of the objective functions, then not only the solutions are respectively optimal for the primal and the dual, but in addition it means that some "complementary slackness" conditions are satisfied (and conversely). It means that there is a strong relation between which inequalities in the primal and the dual are in fact equalities.

Abstractly, if you have a LP

$$\begin{array}{lrcl} \max & \mathbf{c}^\top \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & \leqslant & \mathbf{b} \\ & \mathbf{x} & \geqslant & 0 \end{array}$$

whose dual is

$$\begin{array}{lrcl} \min & \mathbf{b}^\top \mathbf{y} & & \\ \text{s.t.} & A^\top \mathbf{y} & \geqslant & \mathbf{c} \\ & \mathbf{y} & \geqslant & 0 \end{array}$$

then for $\mathbf{x}$ primal feasible and $\mathbf{y}$ dual feasible there holds $\mathbf{c}^\top \mathbf{x} = \mathbf{b}^\top \mathbf{y}$ if and only if

- for all j, there holds $\mathbf{x}_j = 0$ or $\sum_{i=1}^{m} A_{ij}\mathbf{y}_i = \mathbf{c}_j$;
- and for all i, there holds $\mathbf{y}_i = 0$ or $\sum_{i=j}^{m} A_{ij}\mathbf{x}_j = \mathbf{b}_i$;

In other words, what **cannot** happen is that a primal variable is strictly positive and the inequality associated to it in the dual is also a strict inequality ; or that a dual variable is strictly positive and the inequality associated to it in the primal is also a strict inequality.

An example Let us write the example we did in class. The primal LP is

$$\begin{array}{lrrrcr} \max & 7x_1 & & -2x_3 & & \\ & 3x_1 & +x_2 & -x_3 & \leqslant & -2 \\ & x_1 & -x_2 & -2x_3 & \leqslant & -3 \\ & x_1 & & & \leqslant & 2 \end{array} \qquad x_1, x_2, x_3 \geqslant 0.$$

To write the dual, roughly the columns of the primal become the rows of the dual and vice versa. Specifically, one gets

$$\begin{array}{lrrrcr} \min & -2y_1 & -3y_2 & +2y_3 & & \\ & 3y_1 & +y_2 & +y_3 & \geqslant & 7 \\ & y_1 & -y_2 & & \geqslant & 0 \\ & -y_1 & -2y_2 & & \geqslant & -2 \end{array} \qquad y_1, y_2, y_3 \geqslant 0.$$

How does complementary slackness look like ? For $\mathbf{x} = (x_1, x_2, x_3)^\top$ primal feasible and $\mathbf{y} = (y_1, y_2, y_3)^\top$ dual feasible to satisfy $7x_1 - 2x_3 = -2y_1 - 3y_2 + 2y_3$, it is necessary and sufficient that

- Complementary slackness between variables in the primal and constraints in the dual :
  - $x_1 = 0$ or $3y_1 + y_2 + y_3 = 7$,
  - $x_2 = 0$ or $y_1 - y_2 = 0$,
  - $x_3 = 0$ or $-y_1 - 2y_2 = -2$;

- and complementary slackness between variables in the dual and constraints in the primal :
  - $y_1 = 0$ or $3x_1 + x_2 - x_3 = -2$,
  - $y_2 = 0$ or $x_1 - x_2 - 2x_3 = -3$,
  - $y_3 = 0$ or $x_1 = 2$.

Finding the dual solution from the primal one Now, let's assume that you **are told** that $(x_1^*, x_2^*, x_3^*) = (2, 0, 8)$ is an optimal solution of the primal problem. How do you find the dual solution ? For that, you just use the complementary slackness conditions. Namely, as x_1^* and x_3^* are strictly positive, it means that the first and third constraint of the dual are equalities. That is, we should have :

$$\begin{array}{cccccc} 3y_1 & +y_2 & +y_3 & = & 7 \\ -y_1 & -2y_2 & & = & -2 \end{array}$$

For the second constraint we just know that $y_1 - y_2 \geqslant 0$ (and there could be equality still), so we drop it for the moment. Moreover, as $x_1^* - x_2^* - 2x_3^* = -14 < -3$, for complementary slackness to hold we need to impose $y_2 = 0$. The two other constraints of the primal are equalities and not inequalities, so we can't say much about y_1 and y_3.

We have collected all the information that we can. Using $y_2 = 0$ and the system above, we get

$$\begin{array}{cccc} 3y_1 & +y_3 & = & 7 \\ -y_1 & & = & -2 \end{array}$$

This can be solved easily to get $(y_1, y_3) = (2, 1)$. Hence, if we believe that $(x_1^*, x_2^*, x_3^*) = (2, 0, 8)$ is optimal for the primal, **the only reasonable candidate** for the solution of the dual is

$$\begin{pmatrix} y_1^* \\ y_2^* \\ y_3^* \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

But now we can check that (y_1^*, y_2^*, y_3^*) is indeed a feasible solution for the dual problem (all components are non-negative and you can check that the three inequalities in the dual are be satisfied). Moreover, as

$$7x_1^* - 2x_3^* = -2 = -2y_1^* - 3y_2^* + 2y_3^*$$

we conclude that (x_1^*, x_2^*, x_3^*) is an optimal solution for the primal problem and (y_1^*, y_2^*, y_3^*) is an optimal solution for the dual problem.

What if we are not given an optimal solution of the primal ? Let's assume that someone claims that $(x_1^*, x_2^*, x_3^*) = (0, 0, 2)$ is a solution of the primal. It is indeed primal feasible. What happens if we look for a solution of the dual by complementary slackness ?

As only $x_3^* > 0$, we only keep the last constraint of the dual, that is $-y_1 - 2y_2 = -2$. Moreover, as the second and third constraints of the primal are strict inequalities, we would have $y_2 = y_3 = 0$. Hence we get $y_1 = 2$. So the only reasonable candidate is

$$\begin{pmatrix} y_1^* \\ y_2^* \\ y_3^* \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

However, **this solution is not dual feasible** as the first constraint of the dual is violated :

$$3y_1^* + y_2^* + y_3^* = 6 < 7.$$