Wasserstein distance between Lévy measures with applications to Bayesian nonparametrics

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Joint work with:

Marta Catalano

Antonio Lijoi

Igor Prünster

Joint work with:

WARWICK
THE UNIVERSITY OF WARWICK

Marta Catalano

Antonio Lijoi

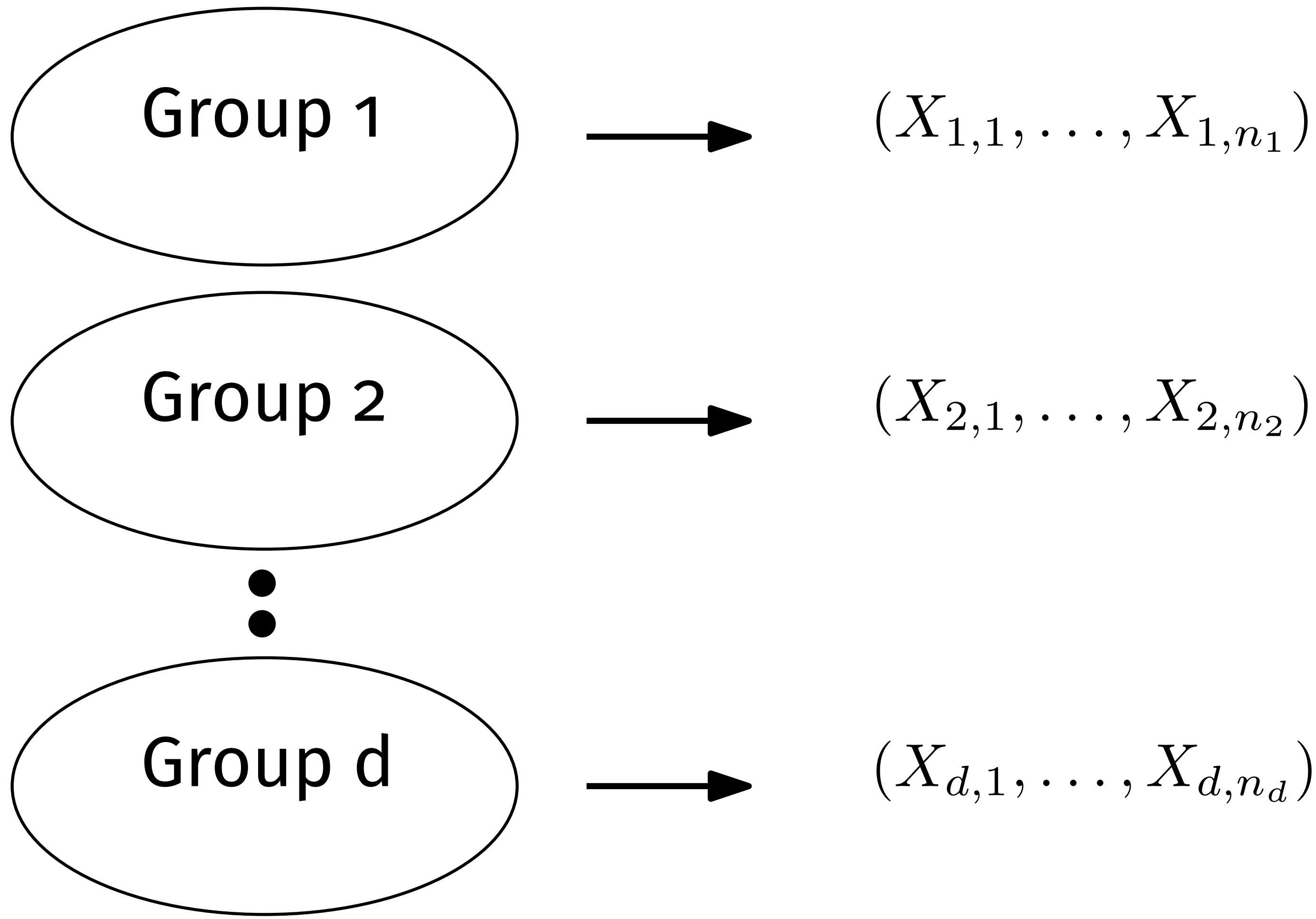
Igor Prünster

Disclaimer

I am not a (Bayesian) statistican.

My background: mathematical analysis, optimal transport.

Quantifying dependence



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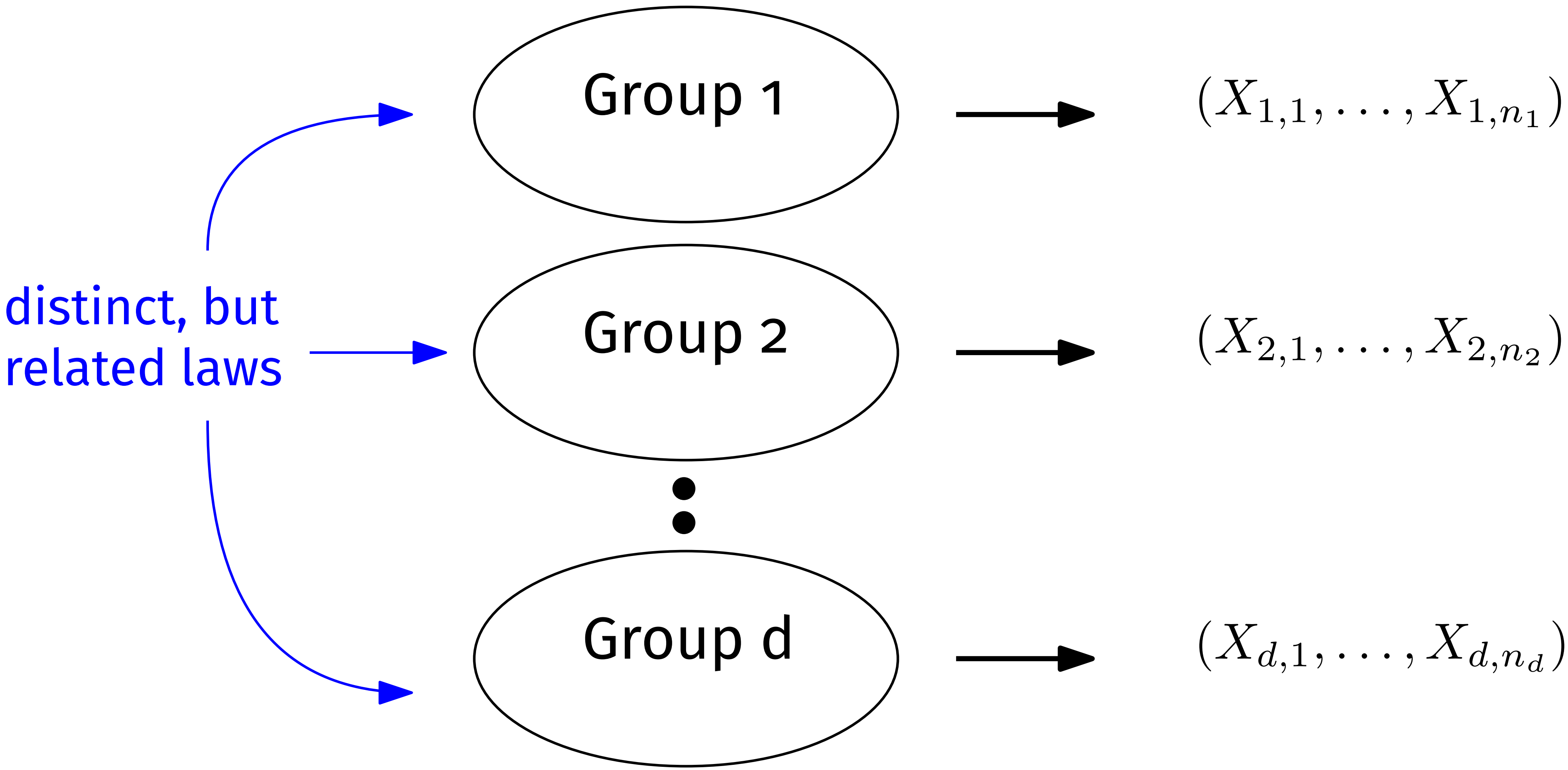
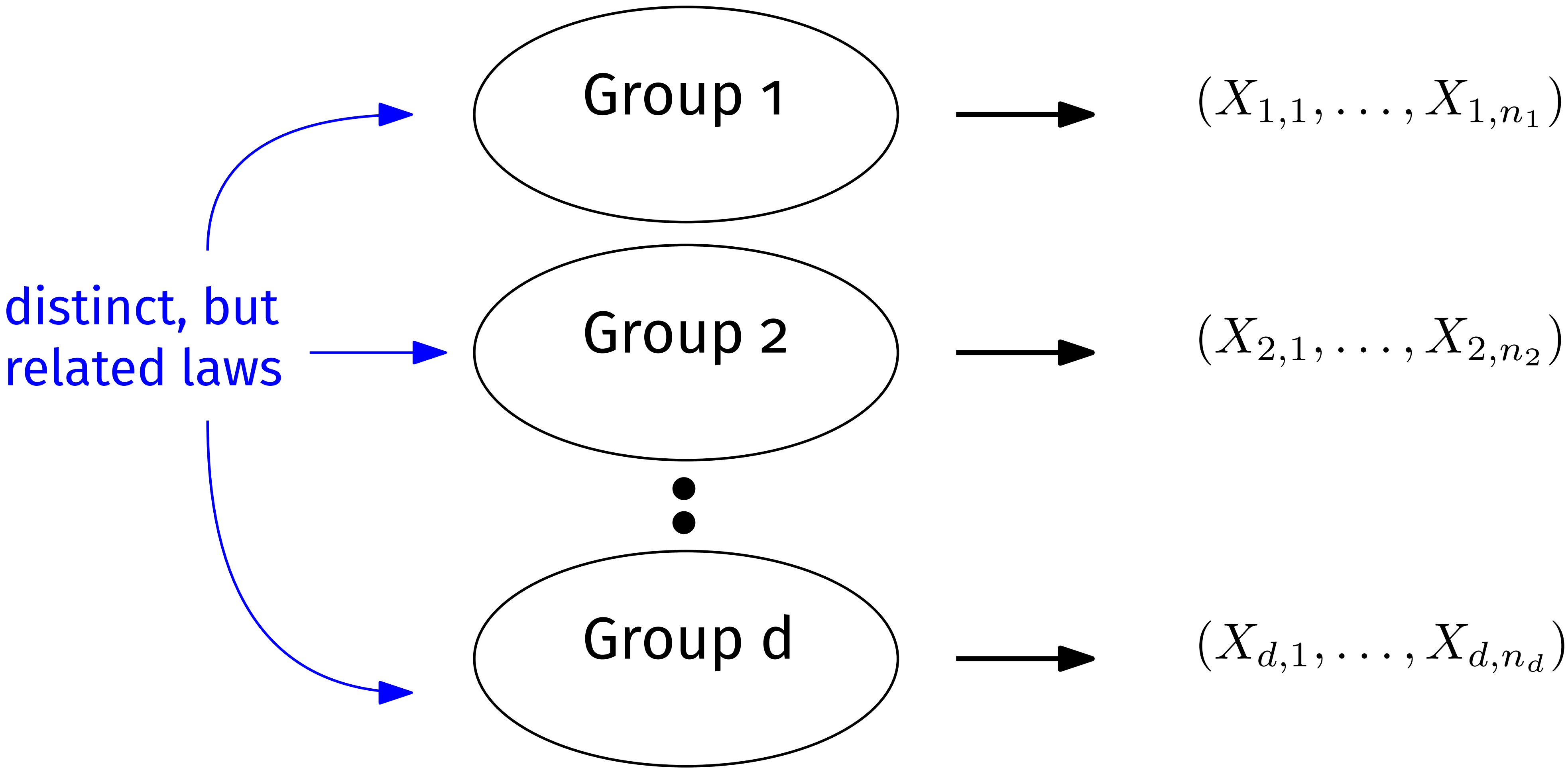


Bayesian inference allows for borrowing of information

Quantifying dependence



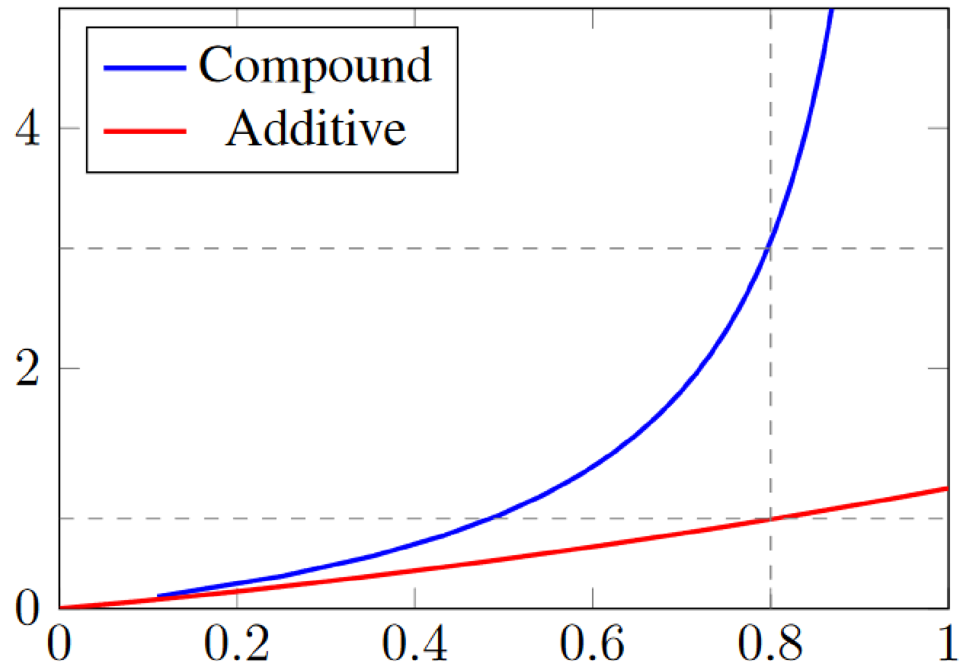
Bayesian inference allows for borrowing of information

Goal: quantifying the amount of **dependence** between groups already present in the **prior**

Snapshot of the final result

Our contribution: an index of dependence quantifying dependence in the prior

Different parametrized models of prior

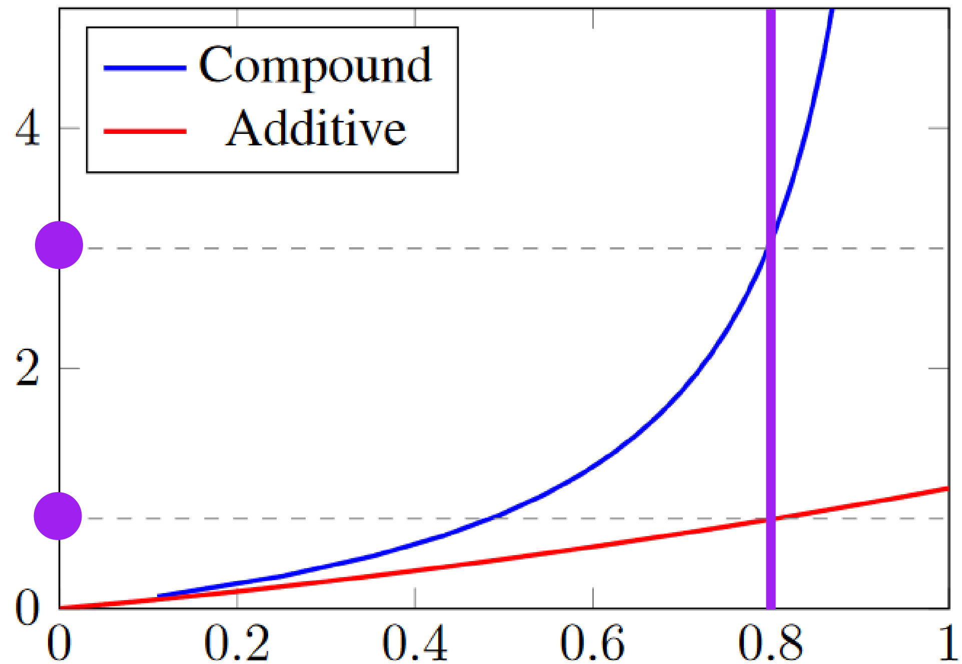


Value of the index

Snapshot of the final result

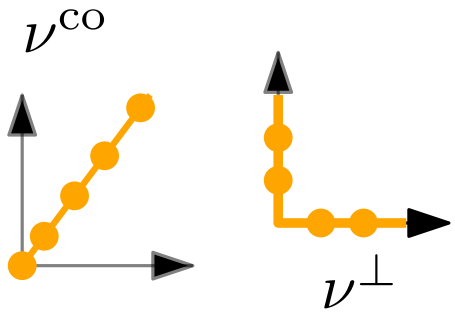
Our contribution: an index of dependence quantifying dependence in the prior

"Compound" with parameter 3 has same dependence as "Additive" with parameter 0.75.



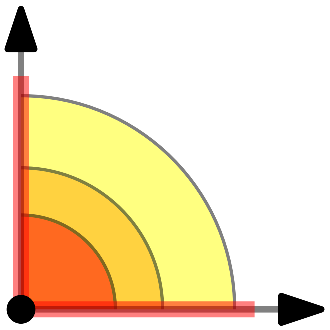
Value of the index

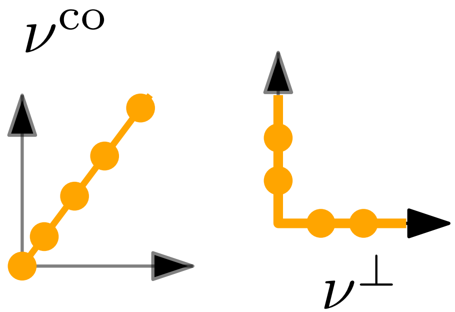
Allow for comparision between **different** priors



1 - Context, general strategy

2 - Building the index with optimal transport





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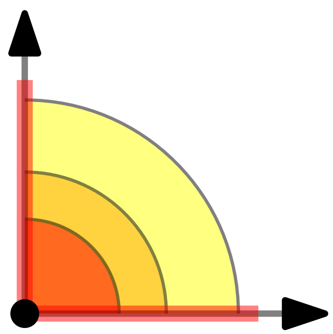

Bayesian Non Parametrics

$\tilde{p}$ random probability measure on $\mathbb{X}$

$$X_1, X_2, \ldots, X_n | \tilde{p} \overset{\text{i.i.d.}}{\sim} \tilde{p}$$

(justified by exchangeability)

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Group 1 $\tilde{p}_1$ $\longrightarrow$ $X_{1,1}, \ldots, X_{1,n_1} \,|\, (\tilde{p}_1, \ldots, \tilde{p}_d) \overset{\text{i.i.d.}}{\sim} \tilde{p}_1$

Group 2 $\tilde{p}_2$ $\longrightarrow$ $X_{2,1}, \ldots, X_{2,n_2} \,|\, (\tilde{p}_1, \ldots, \tilde{p}_d) \overset{\text{i.i.d.}}{\sim} \tilde{p}_2$

$\vdots$

Group d $\tilde{p}_d$ $\longrightarrow$ $X_{d,1}, \ldots, X_{d,n_d} \,|\, (\tilde{p}_1, \ldots, \tilde{p}_d) \overset{\text{i.i.d.}}{\sim} \tilde{p}_d$

(justified by partial exchangeability)

Specific setting: Completely Random Vectors

$\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_d)$ Completely Random Vector

$$X_{1,1}, X_{1,2}, \ldots, X_{1,n_1} \,|\, \tilde{\boldsymbol{\mu}} \overset{\text{i.i.d.}}{\sim} \frac{\tilde{\mu}_1}{\tilde{\mu}_1(\mathbb{X})}$$

$$X_{2,1}, X_{2,2}, \ldots, X_{2,n_2} \,|\, \tilde{\boldsymbol{\mu}} \overset{\text{i.i.d.}}{\sim} \frac{\tilde{\mu}_2}{\tilde{\mu}_2(\mathbb{X})}$$

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Definition (CRV). For all $A_1, \ldots, A_n \subseteq \mathbb{X}$ disjoints, the vectors $\tilde{\boldsymbol{\mu}}(A_1), \ldots, \tilde{\boldsymbol{\mu}}(A_n)$ are independent random vectors in $\mathbb{R}_+^d$.

Specific setting: Completely Random Vectors

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Contains all dependence in the prior

$$X_{1,1}, X_{1,2}, \ldots, X_{1,n_1} \mid \tilde{\boldsymbol{\mu}} \overset{\text{i.i.d.}}{\sim} \frac{\tilde{\mu}_1}{\tilde{\mu}_1(\mathbb{X})}$$

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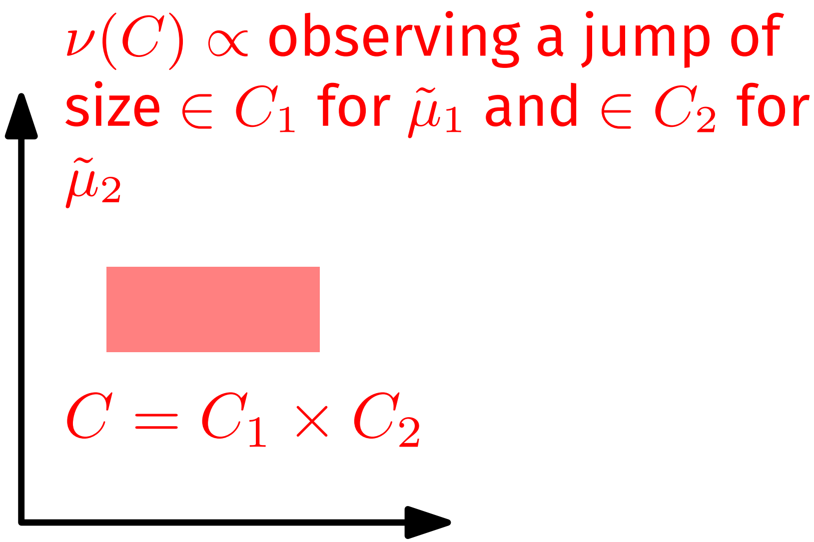
For $A \subseteq \mathbb{X}$, the random variables $\tilde{\mu}_1(A), \ldots, \tilde{\mu}_d(A)$ may be dependent.

Lévy measure of a Completely Random Vector

Assumptions of **homogeneity** and no fixed atoms:

$$\tilde{\boldsymbol{\mu}} = \sum_{i=1}^{\infty} \tilde{\mathbf{J}}_i \delta_{Y_i}$$

where $(Y_i)_i \in \mathbb{X}$ (**atoms**) follow base measure P_0; and $(\tilde{\mathbf{J}}_i)_i$ (**jumps**) independent from $(Y_i)_i$ follow Poisson point cloud on $\mathbb{R}_+^d$ with intensity measure ν (**Lévy measure**).



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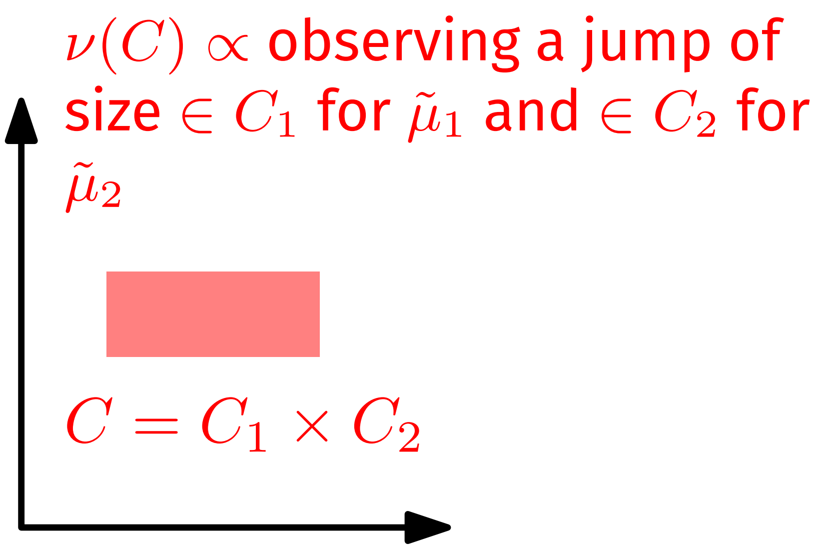

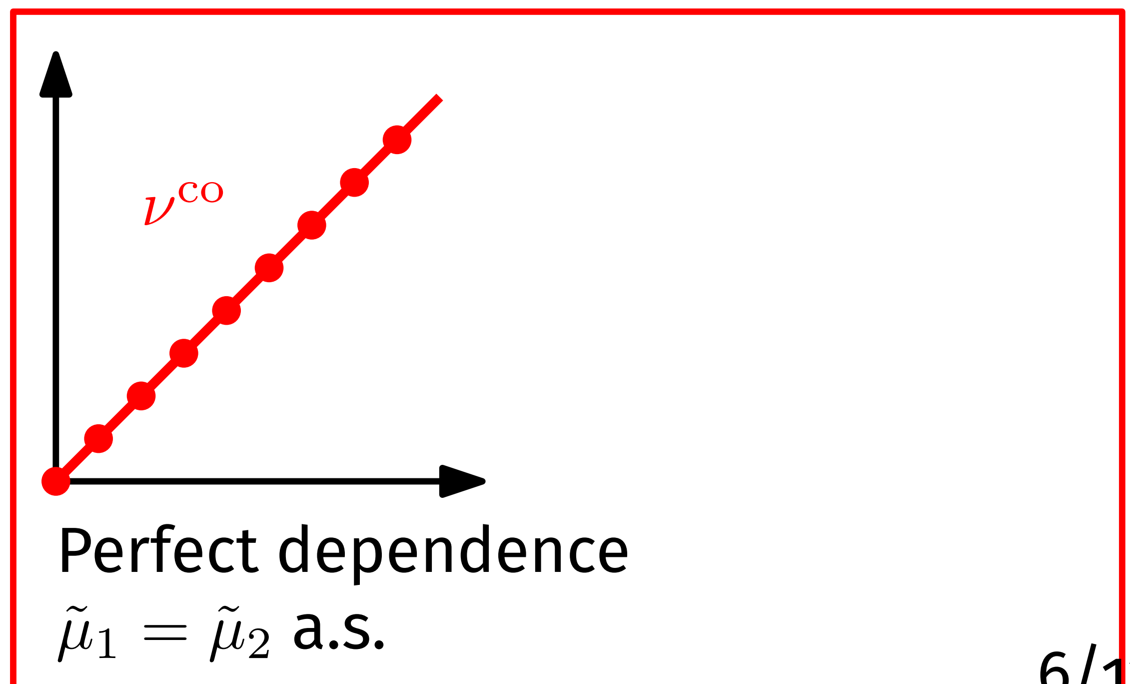

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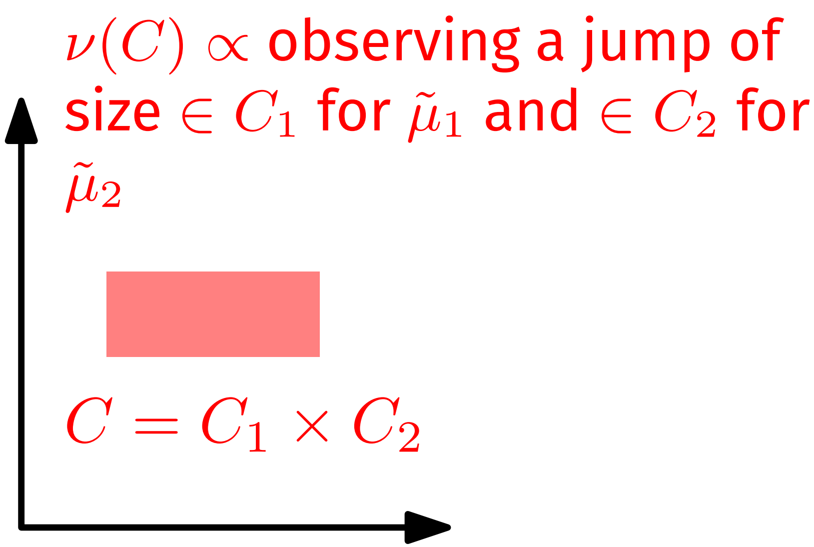
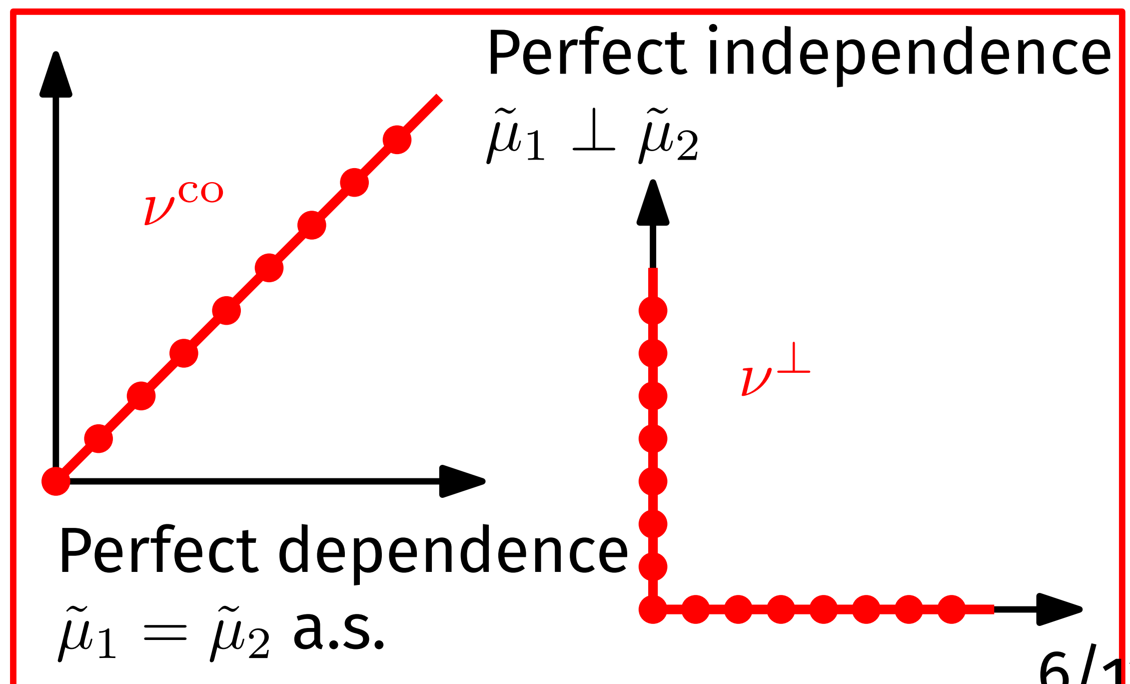


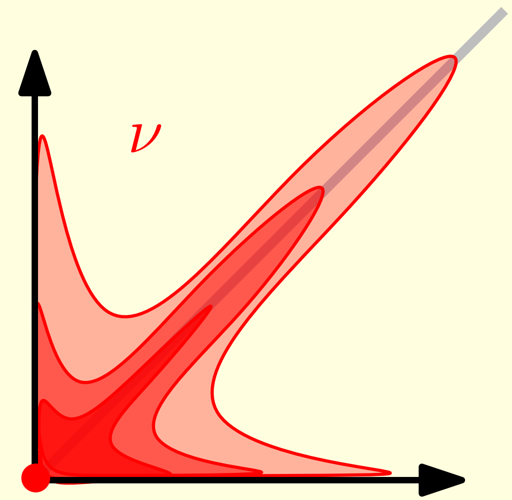


Lévy measure of a Completely Random Vector

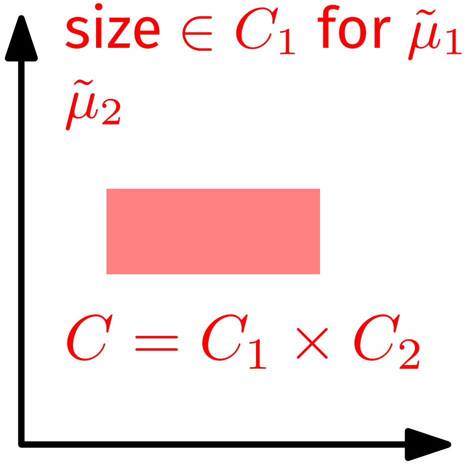
Assumpti

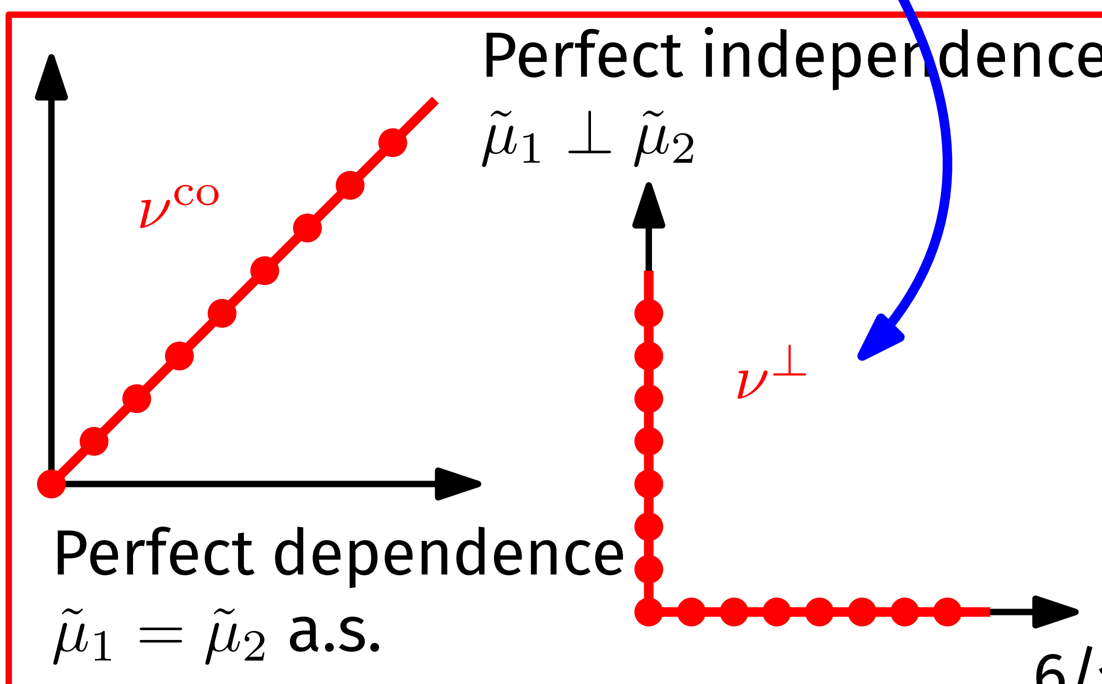
where $(Y$ $(\tilde{\mathbf{J}}_i)_i$
(jumps) i cloud
on $\mathbb{R}_+^d$ wi

Goal: distinguish between these two cases.



$\nu(C) \propto$ observing a jump of size $\in C_1$ for $\tilde{\mu}_1$ and $\in C_2$ for $\tilde{\mu}_2$

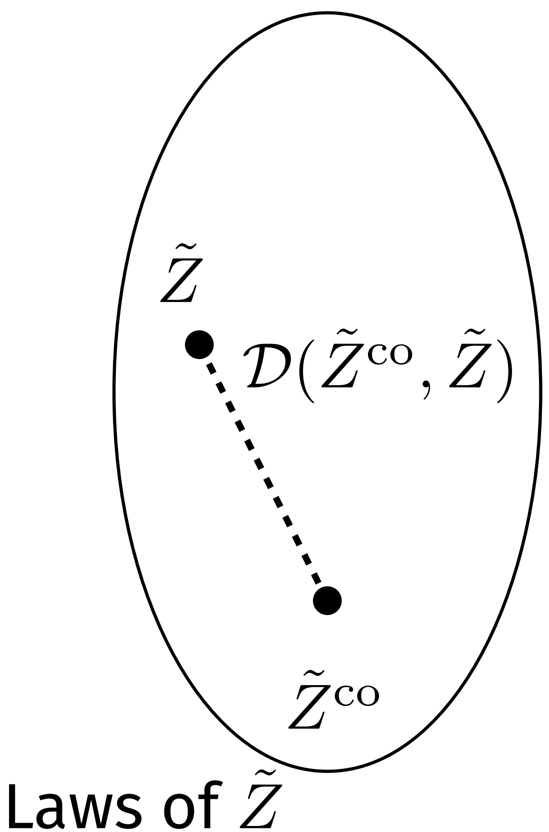




A general method to construct an index

Ingredients:

- $\tilde{Z}$ random object, $\tilde{Z}^{\mathrm{co}}$ "most dependent".
- $\mathcal{D}$ "discrepancy" between laws of random objects.



Móri, and Székely (2020). The Earth Mover's correlation.

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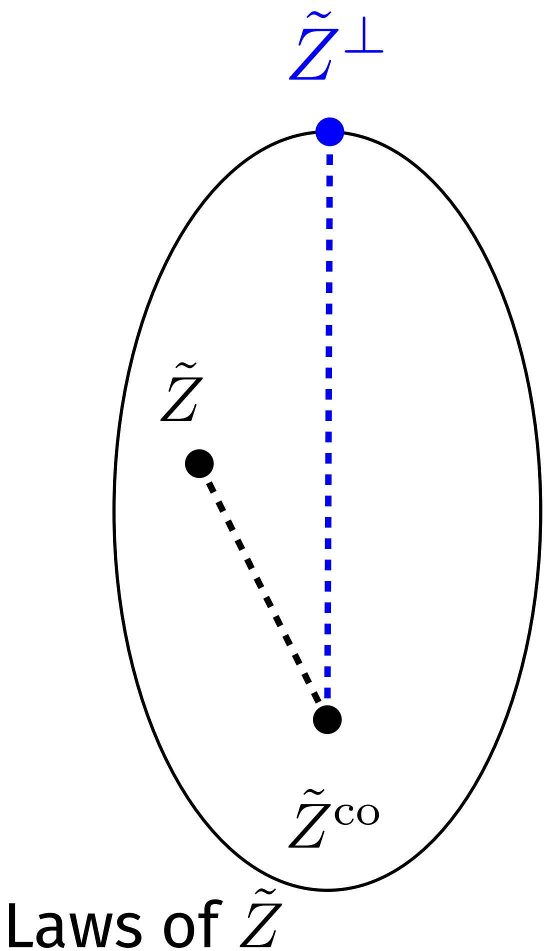

To check: $\mathcal{D}(\tilde{Z}^{\mathrm{co}}, \tilde{Z})$ is maximized when $\tilde{Z} = \tilde{Z}^{\perp}$ the most independent structure.

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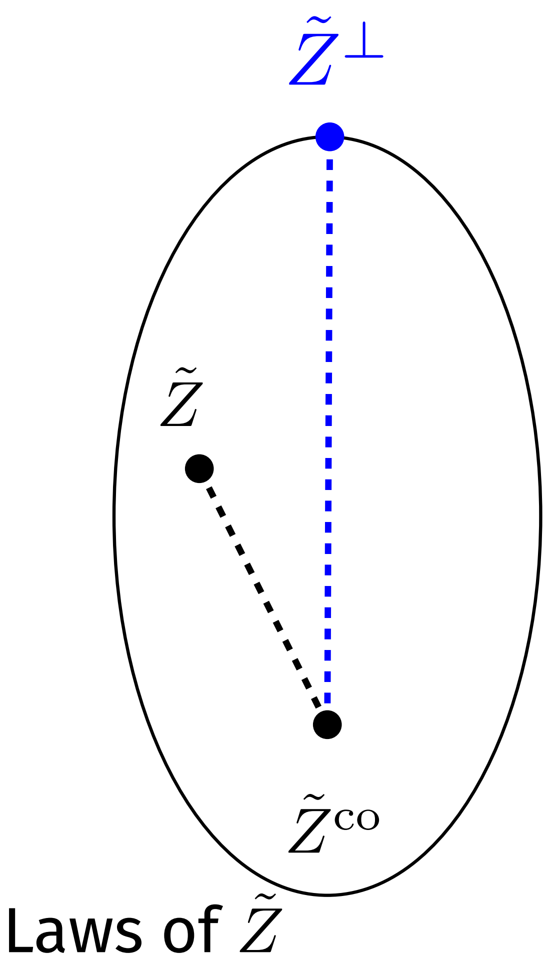

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Then **define**:

$$\mathcal{I}(\tilde{Z}) = 1 - \frac{\mathcal{D}(\tilde{Z}^{\mathrm{co}}, \tilde{Z})}{\mathcal{D}(\tilde{Z}^{\mathrm{co}}, \tilde{Z}^{\perp})}.$$

It belongs to $[0,1]$ and satisfies:

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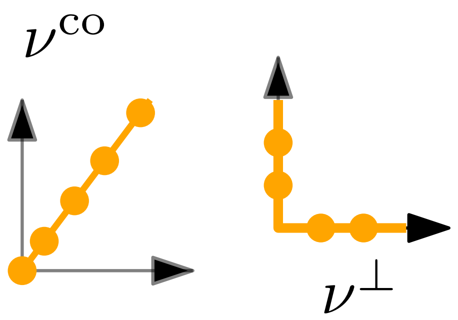

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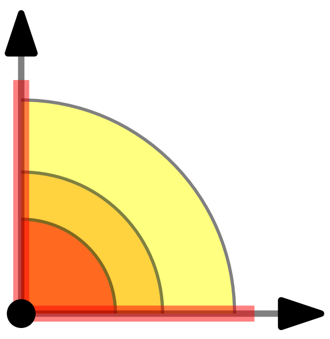

How to measure discrepancy between Completly Random Vectors?

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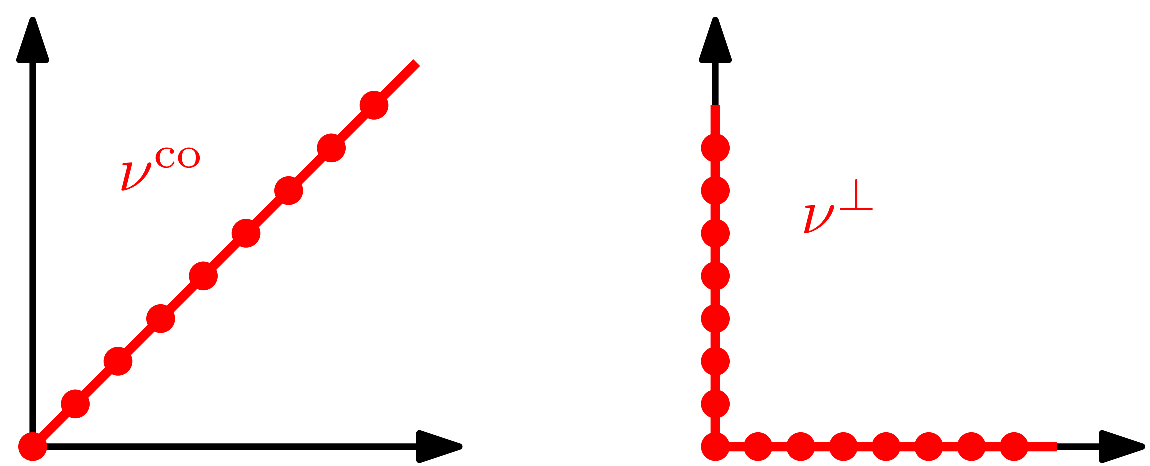

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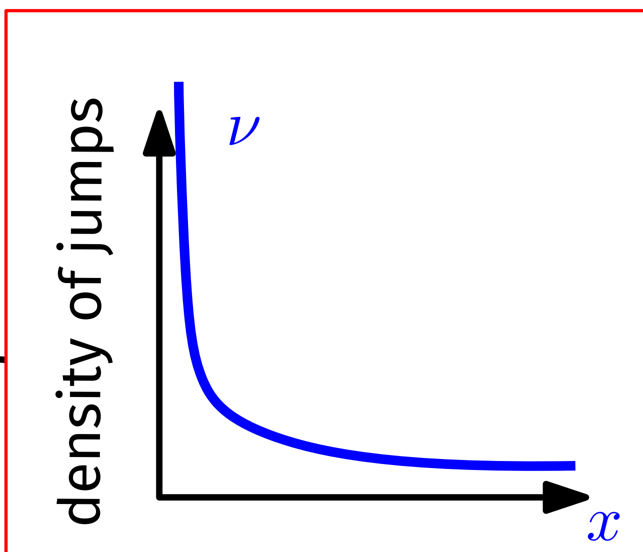

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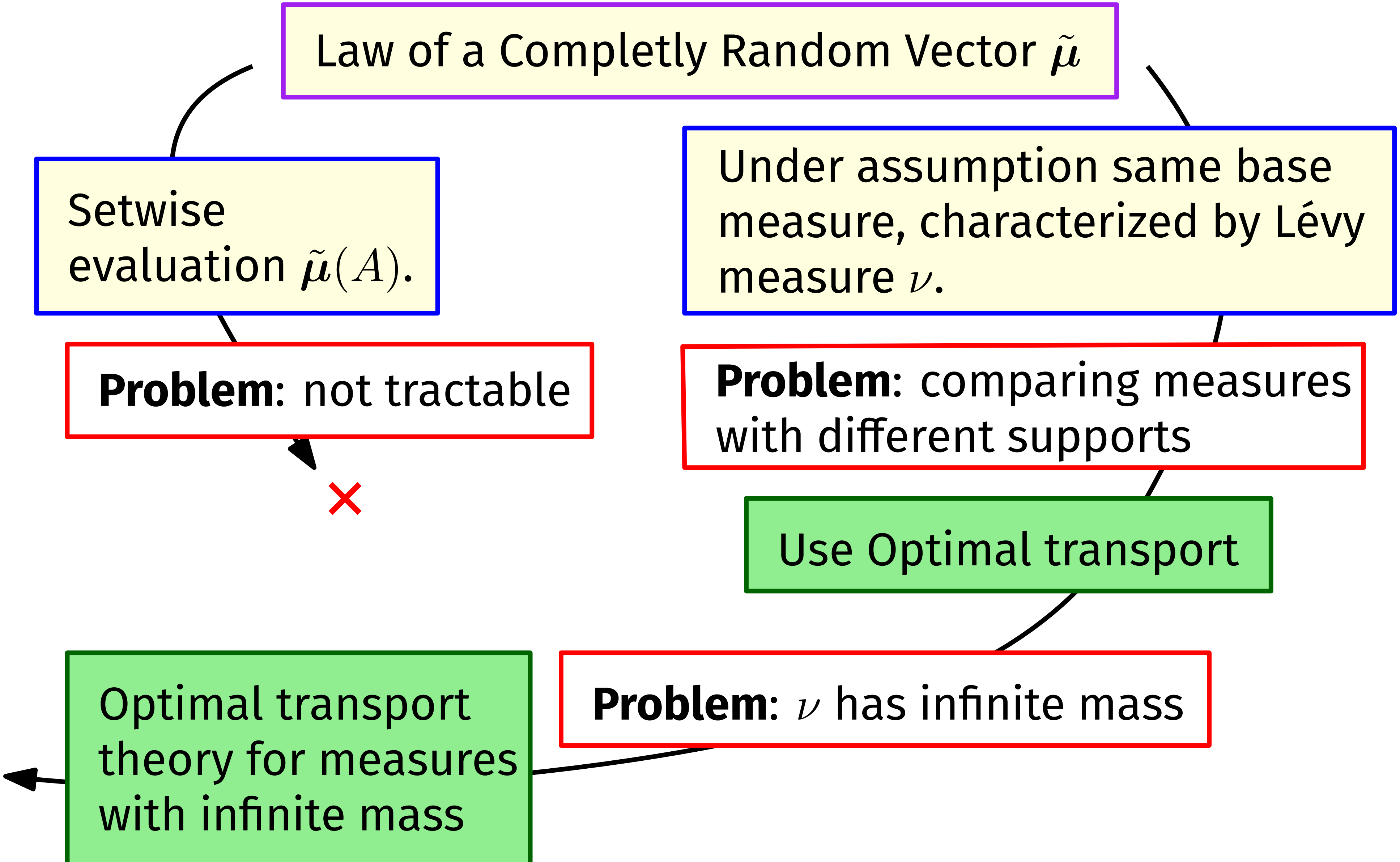

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Optimal transport theory for measures with infinite mass

Upper bound

(Classical) optimal transport

Definition. If ν^1, ν^2 probability distributions, the Wasserstein distance is

$$\mathcal{W}(\nu^1, \nu^2)^2 = \min_{(X,Y)} \left\{ \mathbb{E}\left[\|X - Y\|^2 \right] \ : \ X \sim \nu^1 \text{ and } Y \sim \nu^2 \right\}$$

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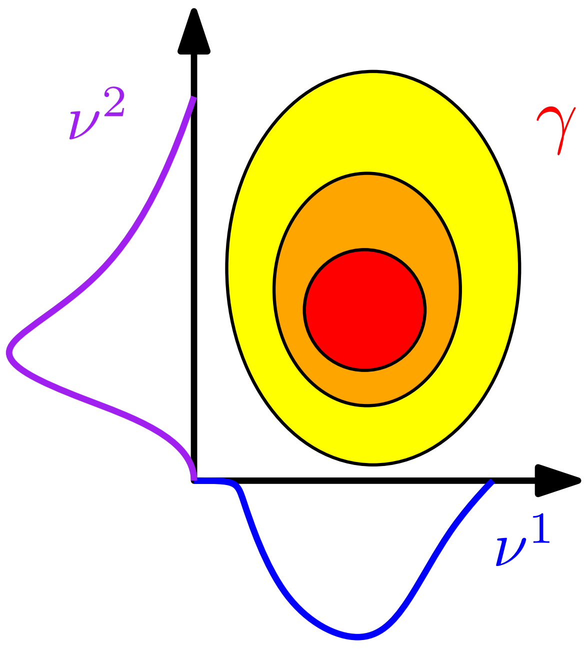
$$= \min_{\gamma} \left\{ \iint \|x-y\|^2 \mathrm{d}\gamma(x,y) \;:\; \pi_1\#\gamma = \nu^1 \text{ and } \pi_2\#\gamma = \nu^2 \right\}$$



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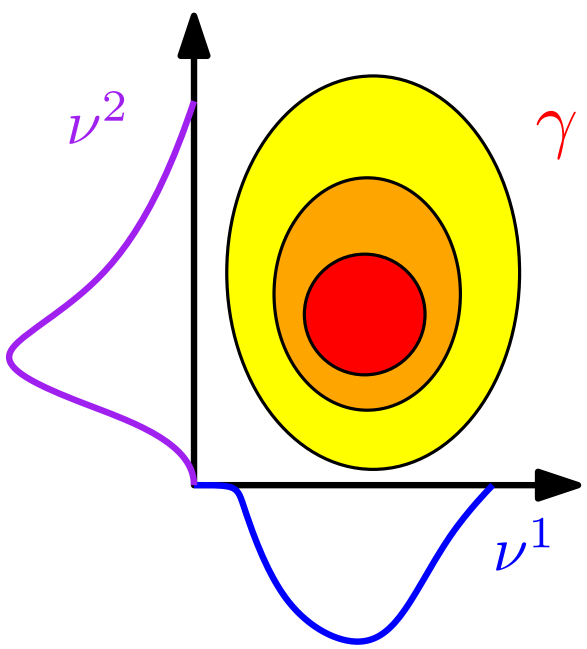
$$\leq \int \|x\|^2 \mathrm{d}\nu^1(x) + \int \|y\|^2 \mathrm{d}\nu^2(y)$$



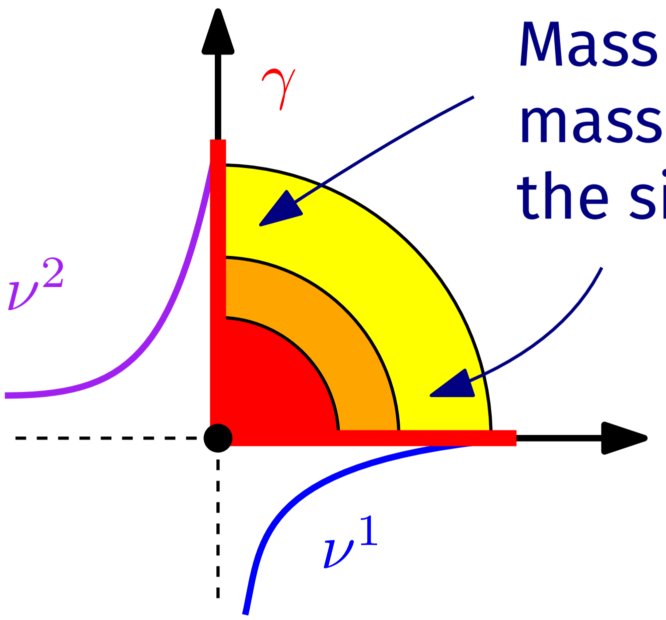
Observation. Naively, makes sense if ν^1, ν^2 have infinite mass but **finite** second moment.

Extended Wasserstein distance

Definition. If ν^1, ν^2 positive measures on $\mathbb{R}^d_+ \setminus \{0\}$ with **finite second moments**, the Wasserstein distance is

$$\mathcal{W}_*(\nu^1,\nu^2)^2 = \min_\gamma \left\{ \iint \|x-y\|^2 \mathrm{d}\gamma(x,y) \ : \ \begin{array}{l} \pi_1 \# \gamma|_{\mathbb{R}^d_+\setminus\{0\}} = \nu^1 \\ \text{and} \ \ \pi_2 \# \gamma|_{\mathbb{R}^d_+\setminus\{0\}} = \nu^2 \end{array} \right\}$$

with γ measure on $\mathbb{R}^{2d}_+ \setminus \{(0,0)\}$.



Mass on $\mathbb{R}^d_{+,*} \times \{0\}$ and $\{0\} \times \mathbb{R}^d_{+,*}$: mass "destroyed" or "created" from the sink/reservoir $(0,0)$.

Figalli and Gigli (2010). A new transportation distance between non-negative measures, with applications to gradients flows with Dirichlet boundary conditions.

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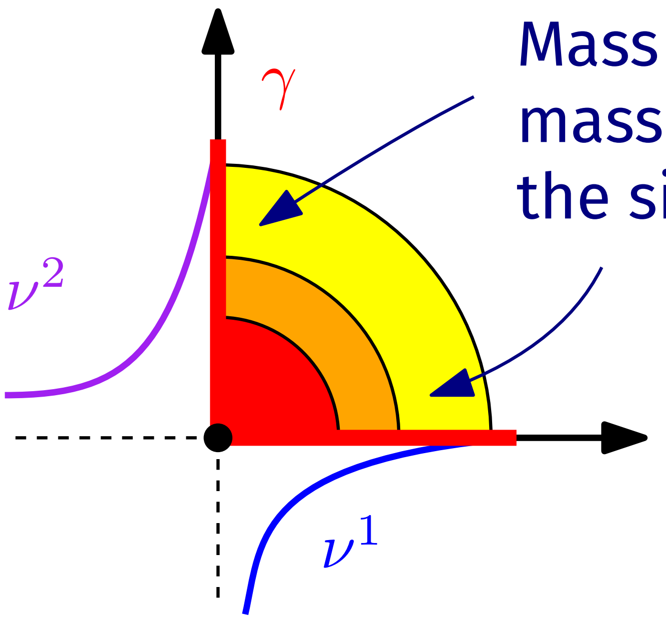

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In progress: extending this idea to couple also the law of atoms for inhomogeneous CRV

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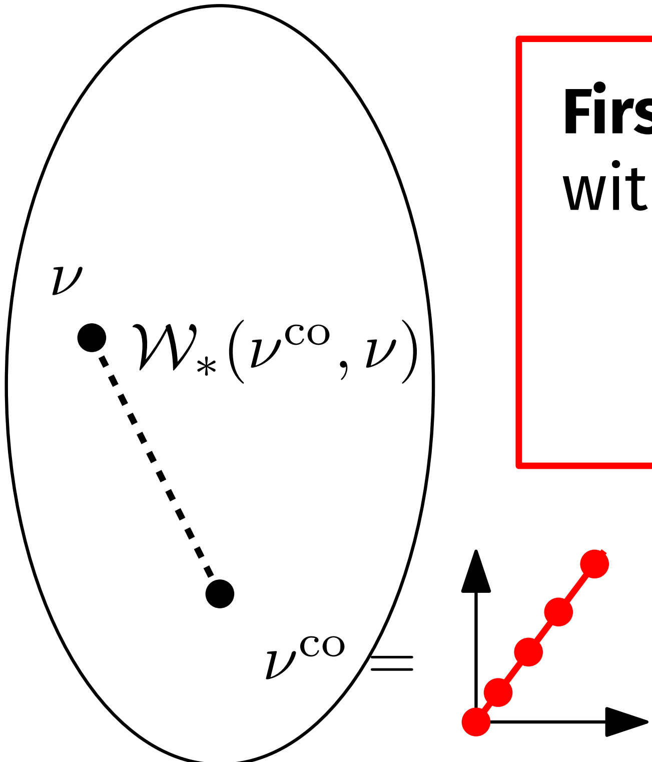
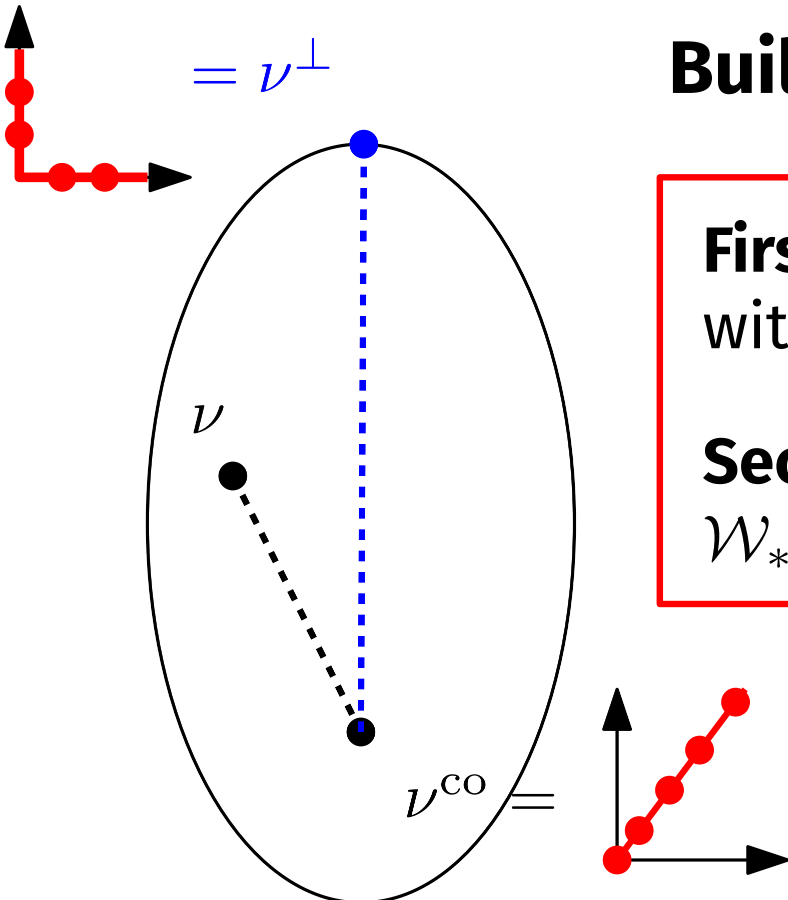
Building the index



Space of Lévy measure over $\mathbb{R}_+^d$ having same marginals

First result. $\mathcal{W}_*(\nu^{\text{co}}, \nu)$ can be computed with 1d integrals of tail functions.

Catalano, Lavenant, Lijoi, Prünster (2022+). A Wasserstein index of dependence.

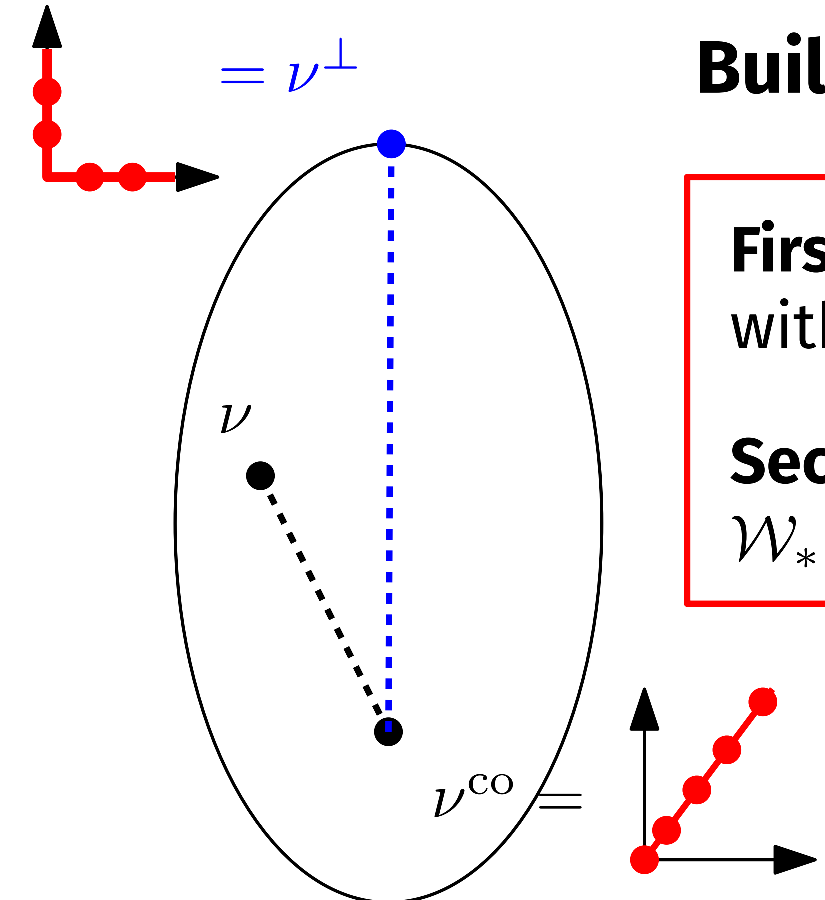


Space of Lévy measure over $\mathbb{R}_+^d$ having same marginals

Building the index

First result. $\mathcal{W}_*(\nu^{\mathrm{co}}, \nu)$ can be computed with 1d integrals of tail functions.

Second result. If ν^{co} has infinite mass, $\mathcal{W}_*(\nu^{\mathrm{co}}, \nu)$ is maximized exactly for $\nu = \nu^{\perp}$.



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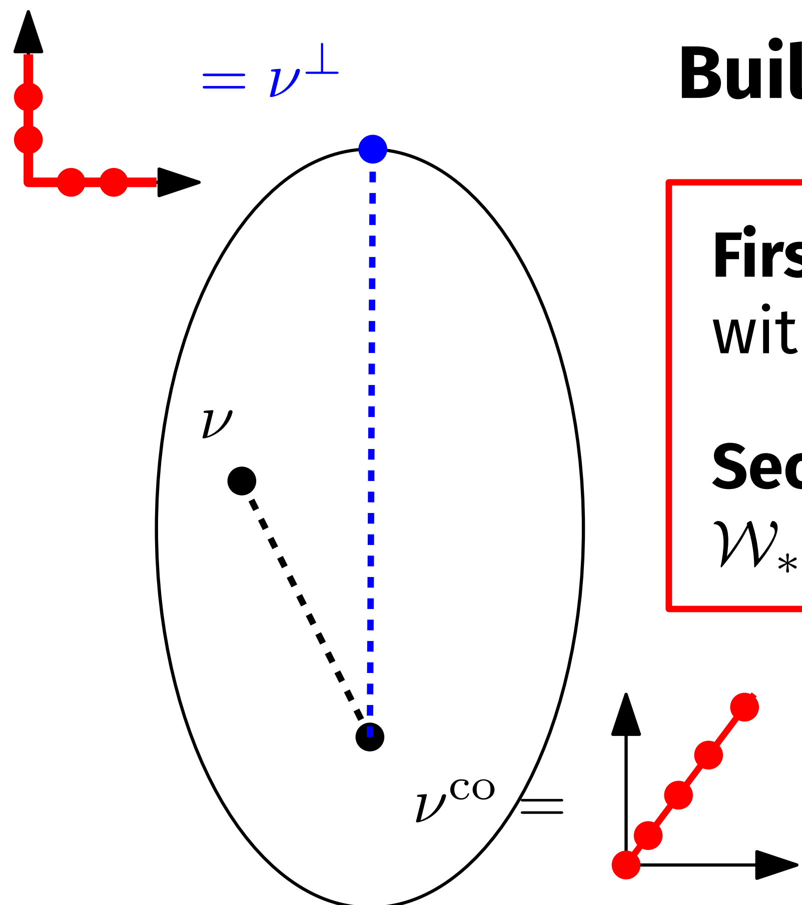

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$$\mathcal{I}(\nu^{\perp}) = 0, \qquad \mathcal{I}(\nu^{\mathrm{co}}) = 1.$$

Consequence. We have an index of dependence for homogeneous infinitely active completely random vectors without fixed atoms, with equal marginals and finite second moments.

Catalano, Lavenant, Lijoi, Prünster (2022+). A Wasserstein index of dependence.

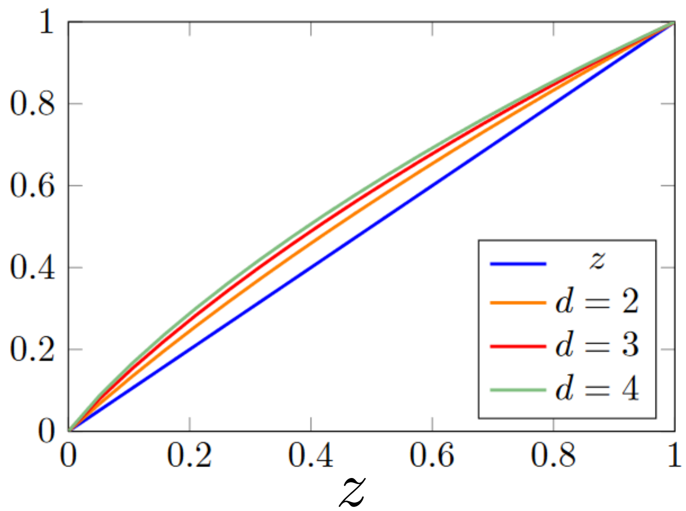
Examples

Additive model

Parameter $z \in [0,1]$,

$$\nu = (1-z)\nu^{\perp} + z\nu^{\text{co}}$$

$\mathcal{I}(z)$



$\mathcal{I}(z) \geq z$ [$=$ Covariance if $d=2$]

Lijoi, Nipoti and Prünster (2014). Bayesian inference with dependent normalized completely random measures.

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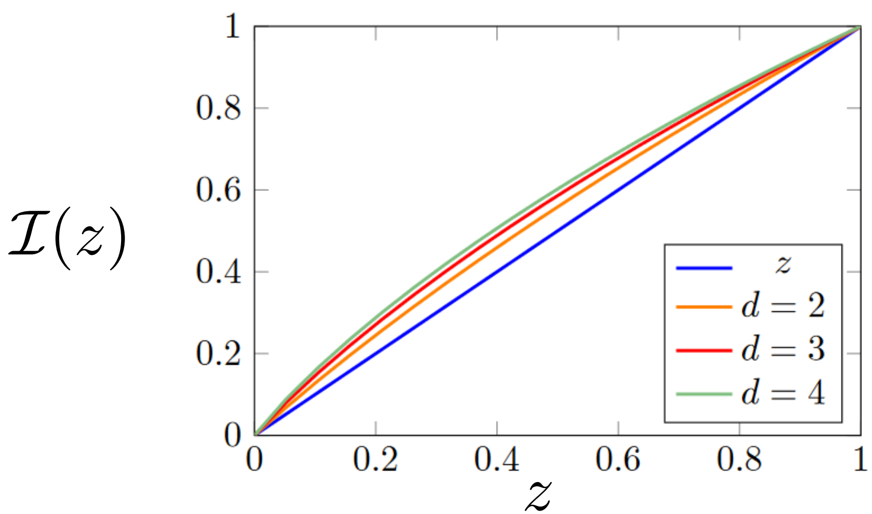

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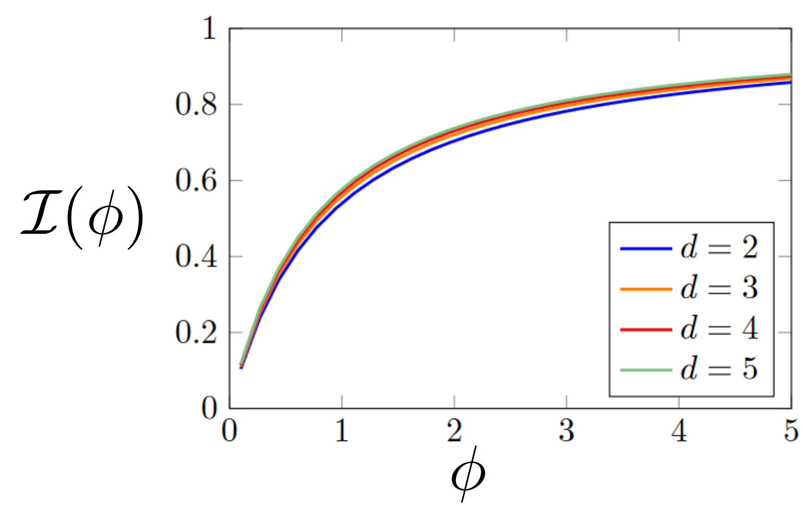
Compound random measures

Parameter ϕ measures dependence

$$\nu(s_1,\ldots,s_d) = \int_0^{+\infty} h^{\phi}\left(\frac{s_1}{u},\ldots,\frac{s_d}{u}\right) \mathrm{d}\nu_*^{\phi}(u)$$

for well chosen h^{ϕ}, ν_*^{ϕ}.



Lijoi, Nipoti and Prünster (2014). Bayesian inference with dependent normalized completely random measures.
Griffin and Leisen (2017). Compound random measures and their use in bayesian non-parametrics.

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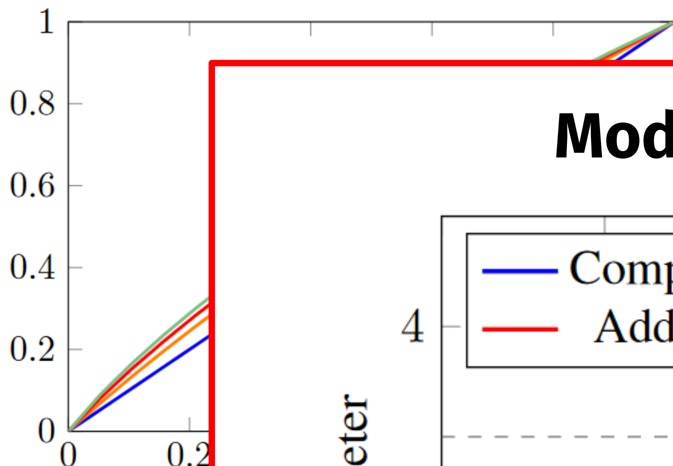

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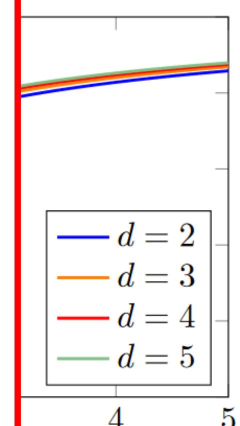
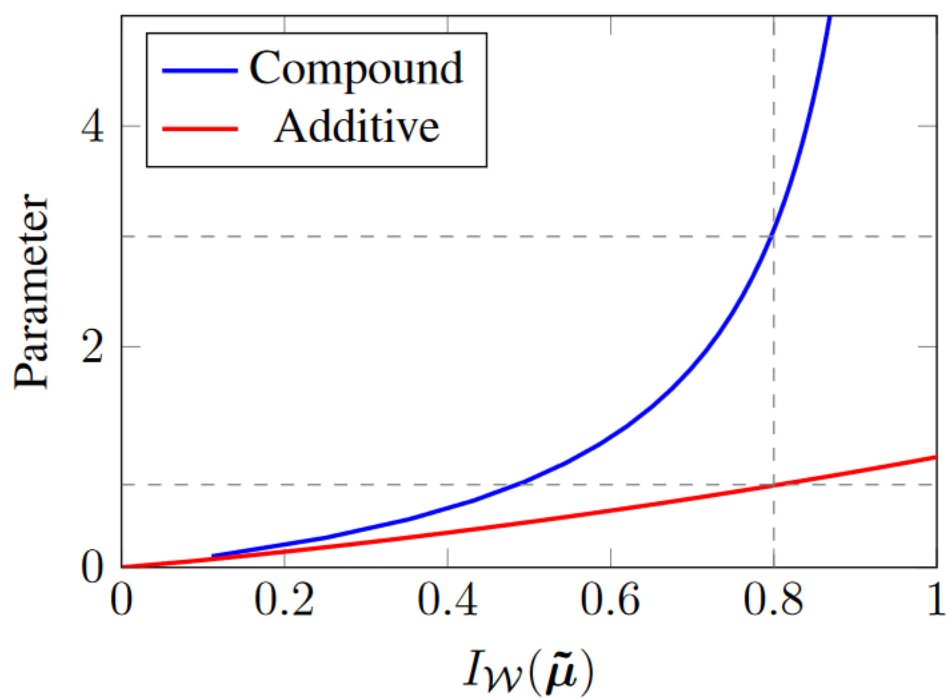


Model comparision



Lijoi, Nipoti and Prünste
Griffin and Leisen (2017)

Conclusion

What is done:

- Wasserstein distance between Lévy measures.
- Index of dependence between Completely Random Vectors.

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- Study dependence in the posterior.
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Thank you for your attention