The revised simplex formulas and their proof

The revised simple formulas are the matrix description of dictionaries. We will use them in the rest of the class, so **you must know them**.

Let's consider a LP in standard inequality form with n decision variables and m constraints, where the decision variables are denoted by $\mathbf{x}_D$.

$$\begin{array}{lrcl} \max & \mathbf{c}^\top \mathbf{x}_D & & \\ \text{s.t.} & A\mathbf{x}_D & \leqslant & \mathbf{b} \\ & \mathbf{x}_D & \geqslant & 0 \end{array}$$

In order to write a dictionary, the slack variables $\mathbf{x}_S$ are introduced. Then, assume that the variables $(\mathbf{x}_D, \mathbf{x}_S)$ are now split in basic variables $\mathbf{x}_B$ and non-basic variables $\mathbf{x}_N$. We denote by B the matrix whose columns are the ones of

$$\tilde{A} = \begin{pmatrix} A & \mathrm{Id}_m \end{pmatrix}$$

corresponding to the basic variables, while A_N is the matrix whose columns are the ones of $\tilde{A}$ indexed by the non-basic variables. Notice that B is of size $m \times m$ while A_N is of size $m \times n$. Similarly, let $\mathbf{c}_B$ be the vector whose components are the ones of

$$\tilde{\mathbf{c}} = \begin{pmatrix} \mathbf{c} \\ 0_m \end{pmatrix}$$

corresponding to the basic variables, while $\mathbf{c}_N$ is the vector whose components are the ones of $\tilde{\mathbf{c}}$ corresponding to the non-basic variables. Notice that $\mathbf{c}_B$ is of size $m \times 1$ while $\mathbf{c}_N$ is of size $n \times 1$.
Then the dictionary with basic variables $\mathbf{x}_B$ and non-basic variables $\mathbf{x}_N$ is

$$\begin{array}{rcll} \mathbf{x}_B & = & B^{-1}\mathbf{b} & -B^{-1}A_N\mathbf{x}_N \\ z & = & \mathbf{c}_B^\top B^{-1}\mathbf{b} & +(\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1}A_N)\mathbf{x}_N. \end{array} \tag{1}$$

Let's make a few remarks.

- As in a dictionary, in the revised simplex formulas (1) the basic variables and z are a function of the non-basic variables.
- Setting the non-basic variables to 0 yields $\mathbf{x}_B = B^{-1}\mathbf{b}$ with $z = \mathbf{c}_B^\top B^{-1}b$.
- For (1) to represent a "feasible" dictionary, you should get a feasible solution when setting the non-basic variables to 0: this leads to the constraint $B^{-1}\mathbf{b} \geqslant 0$.
- Note that B is a square matrix. Not every set of basic variables corresponds to a dictionary obtained by pivoting from the initial dictionary. Actually, a set of basic variables leads to a legit dictionary if and only if B is invertible and $B^{-1}\mathbf{b} \geqslant 0$.

Proof

I will do the proof with abstract matrix notation but I advise that you first look at the proof on a concrete example below to understand what's going on.

Th idea of the proof is to write the first dictionary (the one where slack variables are introduced) with the help of the matrix $\tilde{A}$. Indeed, it reads

$$\begin{array}{rclcl} \tilde{A}\mathbf{x} & = & A\mathbf{x}_D & +\mathbf{x}_S & = & \mathbf{b} \\ \tilde{\mathbf{c}}^\top\mathbf{x} & = & \mathbf{c}^\top\mathbf{x}_D & & = & z \end{array} \tag{2}$$

Now, it is where it becomes a little bit complicated. In the first row, I keep the same equations but I just split the variables differently: I go from the division decision/slack to basic/non-basic. Hence the first row *exactly* reads

$$B\mathbf{x}_B + A_N\mathbf{x}_N = \mathbf{b}. \tag{3}$$

For the second row I can do exactly the same thing and it reads

$$\mathbf{c}_B^\top\mathbf{x}_B + \mathbf{c}_N^\top\mathbf{x}_N = z. \tag{4}$$

Now that I have done that, it's only linear algebra to isolate $\mathbf{x}_B$ and express it as a function of $\mathbf{x}_N$. Indeed, from (3) I can write

$$B\mathbf{x}_B = \mathbf{b} - A_N\mathbf{x}_N.$$

Then, if the matrix B is invertible (and this is the case if the set of basic variables actually come from a dictionary), I multiply on the left by B^{-1} to get

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N.$$

This is the first row of the revised simplex formulas (1). For the second row, the one with z I start from (4). This equation features $\mathbf{x}_B$ but thanks to the first row of the revised simplex formulas (1) I can substitute $\mathbf{x}_B$ by its expression. I end up with

$$\begin{aligned} z &= \mathbf{c}_B^\top\mathbf{x}_B + \mathbf{c}_N^\top\mathbf{x}_N \\ &= \mathbf{c}_B^\top(B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N) + \mathbf{c}_N^\top\mathbf{x}_N \\ &= \mathbf{c}_B^\top B^{-1}\mathbf{b} + \mathbf{c}_N^\top\mathbf{x}_N - \mathbf{c}_B^\top B^{-1}A_N\mathbf{x}_N \\ &= \mathbf{c}_B^\top B^{-1}\mathbf{b} + (\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1}A_N)\mathbf{x}_N. \end{aligned}$$

This is the second row of the revised simplex formulas (1), hence the proof is finished.

The proof on a concrete example

Let's perform the steps of the proof of an example so that you understand what's going on. It will also make concrete the notations $\tilde{A}, \tilde{\mathbf{c}}, B, A_N, \mathbf{c}_B$ and $\mathbf{c}_N$. I start with the LP

$$\begin{array}{lrrrcr} \max & 7x_1 & & -2x_3 & & \\ & 3x_1 & +x_2 & -x_3 & \leqslant & -2 \\ & x_1 & -x_2 & -2x_3 & \leqslant & -3 \\ & x_1 & & & \leqslant & 2 \end{array} \qquad x_1, x_2, x_3 \geqslant 0.$$

I introduce the slack variables x_4, x_5, x_6 and I write the first dictionary. I write it not in the standard way, that is I put all the variables (decision and slack) on the left hand side:

$$\begin{array}{rrrrrrcr} 3x_1 & +x_2 & -x_3 & +x_4 & & & = & -2 \\ x_1 & -x_2 & -2x_3 & & +x_5 & & = & -3 \\ x_1 & & & & & +x_6 & = & 2 \\ 7x_1 & & -2x_3 & & & & = & z \end{array} \tag{5}$$

This equation is exactly the equation (2) with here

$$\tilde{A} = \begin{pmatrix} 3 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \tilde{\mathbf{c}}^\top = \begin{pmatrix} 7 & 0 & -2 & 0 & 0 & 0 \end{pmatrix}$$

Now assume that you are told that $\{x_3, x_5, x_1\}$ are the basic variables in a dictionary[1] and you want to write it without going through the pivoting process.

How to do it? The goal is to express $\{x_3, x_5, x_1\}$ in terms of the other variables. So in (5) let's isolate the basic variables by reordering the "columns":

$$\begin{array}{rrrrrrcr} -x_3 & & +3x_1 & +x_2 & +x_4 & & = & -2 \\ -2x_3 & +x_5 & +x_1 & -x_2 & & & = & -3 \\ & & +x_1 & & & +x_6 & = & 2 \end{array} \tag{6}$$

What we have above is exactly equation (3) in the proof of the revised simplex formulas with

$$B = \begin{pmatrix} -1 & 0 & 3 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A_N = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we isolate the basic variables by putting the non-basic variables in the right hand side:

$$\begin{array}{rrrcrrrr} -x_3 & & +3x_1 & = & -2 & -x_2 & -x_4 & \\ -2x_3 & +x_5 & +x_1 & = & -3 & +x_2 & & \\ & & +x_1 & = & 2 & & & -x_6 \end{array}$$

If $\{x_3, x_5, x_1\}$ are the unknowns, this is a linear system whose right hand side depends on $\{x_2, x_4, x_6\}$. We solve this linear system, for instance by multiplying the right hand side by the inverse of the matrix B, which a computer tell us is

$$B^{-1} = \begin{pmatrix} -1 & 0 & 3 \\ -2 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we do that, after reordering the terms we get

$$\begin{array}{rcrrrr} x_3 & = & 8 & +x_2 & +x_4 & -3x_6 \\ x_5 & = & 11 & +3x_2 & +2x_4 & -5x_6 \\ x_1 & = & 2 & & & -x_6 \end{array}$$

This is a dictionary: the basic variables as linear combinations of the non-basic variables! And the right hand side is given by $B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N$, that is we obtained the first row of the revised simplex formulas (1).

Eventually, we have to deal with the z-row. Recall that $z = 7x_1 - 2x_3$. Now in this expression we substitute the basic variables by the linear combination of the non-basic variables we obtained above to end up with

$$z = -2 - x_6 - 2x_2 - 2x_4.$$

What we have obtained is the second row of the revised simplex formulas (1).

So if I summarize, what I did are the following steps:

1. Introduce slack variables.
2. Isolate the basic variables and put the non-basic variables on the right hand side.
3. I end up with a linear system whose unknowns are the basic variables and the right hand side depends on the non-basic variables I. solved this system to get basic variables as a function of the non-basic ones.
4. For the z-row, I substituted the basic variables by the linear combination of the non-basic variables that I just obtained.

Now if you perform these steps for a generic LP in standard inequality form, you retrieve the proof of the revised simplex formulas.

[1] Note that the basic variables are not ordered. This is because the pivoting process sometimes breaks the natural ordering of the variables.