Math 340 Assignment #3 Due Friday March 20, 2020 at the beginning of class

You can submit either a paper version or an electronic version on Canvas (but not both). For an electronic version, pictures or scan of a paper version are accepted **but** you should submit **a single pdf file**.

1. Give an example of a primal LP which is infeasible while simultaneously its dual LP is infeasible.

   Hint. Either work with the simplest possible LP. Or you can also take for the A matrix

   $$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

   and adjust $\mathbf{b}$ and $\mathbf{c}$ to get both primal and dual infeasibility.

2. Let us consider our usual LP in standard inequality form $\max \mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ with n decision variables and m constraints. We introduce slack variables, that is we rewrite it

   $$\text{(Primal)} \qquad \begin{array}{lcl} \max & \mathbf{c} \cdot \mathbf{x}_D & \\ & (A|\mathrm{I}_m) \begin{pmatrix} \mathbf{x}_D \\ \mathbf{x}_S \end{pmatrix} & = \mathbf{b} \end{array} \qquad \mathbf{x}_D, \mathbf{x}_S \geq \mathbf{0},$$

   with $\mathbf{x}_D$ the decision variables and $\mathbf{x}_S$ the slack variables, while I_m is the identity matrix. We assume that this LP has an optimal solution, we denote by $\mathbf{c}_B, \mathbf{c}_N, B, A_N$ the usual objects of the revised simplex formulas for an optimal dictionary of (Primal).

   (1) Write the dual of the LP (Primal). What is the number of variables in the dual?

   (2) Write the system that you obtain if in the dual you transform the constraints associated to the basic variables of the primal into equalities and you drop the other ones. Justify that this system is well posed.

   (3) Prove that the method that we learned to determine an optimal dual solution from an optimal primal one yields a well posed system if and only if the primal optimal solution is non degenerate (that is all the basic variables are strictly positive when non basic variables are set to 0).

3. A question using LINDO[1] (or other software). Consider the following LP:

   $$\begin{array}{lccccl} \max & c_1x_1 & +7x_2 & +7x_3 & +11.1x_4 & \\ & (10+d_1)x_1 & +4.5x_2 & +1.2x_3 & +9x_4 & \leq 100 \\ & (10+d_2)x_1 & +4x_2 & +1x_3 & +8.2x_4 & \leq 100 \\ & (10+d_3)x_1 & +3x_2 & +3x_3 & +3x_4 & \leq 100 \\ & (10+d_4)x_1 & +2x_2 & +4x_3 & +x_4 & \leq 100 \\ & (10+d_5)x_1 & +x_2 & +5x_3 & +x_4 & \leq 100 \end{array} \qquad x_1, x_2, x_3, x_4 \geq 0$$

   where $d_1d_2d_3d_4d_5$ are the first 5 digits of your student number.

[1] To download LINDO, you can go at `https://www.lindo.com/index.php/ls-downloads` and take the "Classic LINDO". As written in the outline you also are supposed to have LINDO on the computer lab in LSK 310. The syntax in LINDO is simple: write `maximize` then your mathematical expression, then `subject to` and the one constraint per line. The symbol `<` means $\leq$, and LINDO automatically adds non negativity constraints on decision variables.

a) Graph (sketch) the optimal value of the objective function as a function of c_1 for all $c_1 \in (-\infty, \infty)$. Use the LINDO package or LINGO and provide a printout of at least one input file and one output file. The syntax for LINGO is a bit annoying in for such a simple case (all those semicolons) while LINDO is easier. Even getting the report on ranging is a little more difficult (you must enter the Options window then General Solver Tab and click on Dual Computaions asking for prices and ranges. Then while using solver tab, click on Range. At least LINGO is easy to download onto a Mac and LINGO is much easier dealing with bigger input files.

Begin with $c_1 = 0$ and use ranging to determine an interval for c_1 for which you know the answer. Ranging gives you the interval in which the optimal basis B is unchanged and hence the value for x_1 in the optimal solution gives the slope of the objective function value as a function of c_1 in that interval (Why?). Now choose a c_1 outside this interval and repeat. Continue until you know the optimal values for all possible c_1. You might need as many as nine intervals or as few as two intervals depending on your student number.

b) Consider the optimal value of the objective function as a function of the value c_1, say $f(c_1)$. Show that $f(c_1)$ is a concave upwards function by showing that for each interval from a), in which the objective function takes the value $ac_1 + b$ where a, b are constants, then $f(c_1) \geq ac_1 + b$ for all choices c_1 not just in the interval..

4.

a) Show there is an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$ if and only if there is an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} \leq -\mathbf{1}$.

Note: we use the definition $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ if and only if $x_1 < y_1$, $x_2 < y_2$, ... **and** $x_n < y_n$. This is the standard notation in matrix theory for matrix or vector inequalities. This may be contrary to your expectations. Mathematically speaking, the symbol $>$ would generally mean $\geq$ and $\neq$ but this is not true for matrices or vectors. A vector $\mathbf{x}$ might satisfy $x \geq 0$ and also $\mathbf{x} \neq \mathbf{0}$. If $x \geq 0$ and yet $\mathbf{x}$ has still has some 0 entries then $\mathbf{x} \not> \mathbf{0}$.

b) Let A be an $m \times n$ matrix. Prove that either:

i) there exists an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$ or

ii) there exists $\mathbf{y} \geq \mathbf{0}$ with $A^T\mathbf{y} \geq 0$ and $\mathbf{y} \neq \mathbf{0}$

but not both.

Hint: Extend the idea in a) and use it in setting up a primal dual pair.

5. (from an old exam) We seek a minimum cost diet selected from the following three foods.

	food 1	food 2	food 3
vitamins/100gms	13.23	18.4	36
calories/100gms	100	125	139
minimum (100gms)	10	10	8
cost \$/100gm	3.00	5.00	8.00

We require a diet that has at least 760 units of vitamins and at least 3500 calories. The minimums are stated in units of 100gms. We let the variable food$i$ refer to the amount of food i purchased in units of 100gms. The input to LINDO is:

min 3food1+5food2+8food3

subject to

13.23food1+18.4food2+36food3>760

100food1+125food2+139food3>3500

food1>10

food2>10

food3>8

end

The following is the output from LINDO:

OBJECTIVE FUNCTION VALUE

1) 178.6000

VARIABLE	VALUE	REDUCED COST
$FOOD1$	10.000000	0.000000
$FOOD2$	10.000000	0.000000
$FOOD3$	12.325000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	0.000000	−0.222222
3)	463.174988	0.000000
4)	0.000000	−0.060000
5)	0.000000	−0.911111
6)	4.325000	0.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
$FOOD1$	3.000000	$INFINITY$	0.06000
$FOOD2$	5.000000	$INFINITY$	0.911111
$FOOD3$	8.000000	0.163265	8.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	760.000000	$INFINITY$	119.958984
3	3500.000000	463.174988	$INFINITY$
4	10.000000	11.768706	9.468493
5	10.000000	8.461956	8.584380
6	8.000000	4.325000	$INFINITY$

a) There is a special on food 2 reducing the price to $4.10/100gms. Would this change your purchase strategy? What about a price reduction to $3.10?

b) What is the marginal cost of 10 units of vitamins; namely what is the cost of increasing the vitamin requirement by 10? Considering the chosen diet as a whole, what is the dollar cost (approximately only) of the whole diet per 10 units of vitamins obtained. Which cost is cheaper?

c) Is integrality important in diet problems such as this? Note that integrality refers to requiring the variables to be integers.

d) Give a linear inequality that expresses the requirement that at least 20% of the weight of the purchased diet comes from food 2.