How to check optimality of a basis?

In sensitivity analysis, one usually has to check if a given basis (that is set of basic variables) leads to an optimal feasible solution. There are two ways to do it:

1. check directly if the associated dictionary is optimal;

2. try to find the dual solution and use weak duality;

and these two ways are closely related. As a rule of thumb, the first one is usually enough and faster when the number of constraints and decision variables does not change, while the second one is needed when one adds constraints and/or decision variables.

We consider a LP in its standard inequality form:

$$\begin{array}{rrcl} \max & \mathbf{c}^\top \mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & \leqslant & \mathbf{b} \\ & \mathbf{x} & \geqslant & \mathbf{0} \end{array}$$

We assume that you are given a basis, and the question is about the optimality of the solution given by this basis (that is obtained by setting the non basic variables to 0). Once you have the basis, you can define the objects $B, A_N, \mathbf{c}_B, \mathbf{c}_N$ of the revised simplex formulas[1]

Checking if the dictionary is optimal

To check if the solution is optimal, the idea is the following: write the dictionary with the revised simplex formulas. Then optimality is reached if the dictionary is feasible *and* the coefficients in the z-row are negative.

The first condition imposes that the coefficients in the "constant" column of the dictionary (the first column of the right hand side) are all non negative. Equivalently, it means that setting the non basic variables to 0 yields a feasible solution, that is one with non negative entries. In both cases, using the revised simplex formulas it reads

$$B^{-1}\mathbf{b} \geqslant \mathbf{0}.$$

The second condition is about the coefficients in the z-row, which are directly given by the revised simplex formulas. These coefficients must be non positive at optimality, this reads

$$\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1} A_N \leqslant \mathbf{0}.$$

Putting these two pieces together we reach the following conclusion.

Let's consider a LP in standard inequality form with the usual notations. A given basis yields an optimal feasible solution of the problem if and only if

$$B^{-1}\mathbf{b} \geqslant \mathbf{0} \quad \text{and} \quad \mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1} A_N \leqslant \mathbf{0}.$$

[1] See `https://hugolav.github.io/teaching/revised_simplex_formula.pdf`.

Using duality theory

Thanks to the weak and strong duality theorems, we know that we can use the dual problem to provide certificates for optimality.

Recall (see for instance Quiz 3) that usually if we have a primal solution, we can try to use complementary slackness to guess a dual solution. Then, if the dual solution if feasible, we know that optimality is reached. This process can actually be performed with the revised simplex formulas.

Let's consider a LP in standard inequality form with the usual notations. Take a given basis and assume that it is non denegerate, that is no component of $B^{-1}\mathbf{b}$ (the basic variables) vanishes. Then the only solution of the dual problem which satisfies the complementary slackness conditions is

$$\mathbf{y} = B^{-\top}\mathbf{c}_B.$$

I won't do the proof: let's say that this is exactly what you were doing in Quiz 3 on some concrete cases. Why is there a non degeneracy assumption? To be sure that basic variables will be non zero. If you remember complementary slackness, what is important is which variable is non zero and which one is zero.

Then, if you remember Quiz 3 and the complementary slackness theorem, the values of the primal and the dual for $\mathbf{y} = B^{-\top}\mathbf{c}_B$ will always match, but $\mathbf{y}$ will not always be dual feasible. However, if it is then everything is optimal. As a conclusion, we can write the following.

Let's consider a LP in standard inequality form with the usual notations. Take a given basis and assume that it is non denegerate, that is no component of $B^{-1}\mathbf{b}$ (the basic variables) vanishes. Then it is an optimal feasible solution if and only if

$$B^{-1}\mathbf{b} \geqslant \mathbf{0} \quad \text{and, defining } \mathbf{y} = B^{-\top}\mathbf{c}_B \text{ there holds } \begin{cases} A^\top\mathbf{y} \geqslant \mathbf{c} \\ \mathbf{y} \geqslant \mathbf{0} \end{cases}.$$

Link between the two criteria

We have *a priori* two different criteria. However, we have seen in the proof of Strong duality that, if we define $\mathbf{y} = B^{-\top}\mathbf{c}_B$, there holds

$$\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1}A_N \leqslant \mathbf{0} \quad \text{if and only if} \quad \begin{cases} A^\top\mathbf{y} \geqslant \mathbf{c} \\ \mathbf{y} \geqslant \mathbf{0} \end{cases}.$$

So everything is consistent, the two citeria are in fact the same!

Actually, the condition $B^{-1}\mathbf{b} \geqslant \mathbf{0}$ can be interpreted as primal feasibility while $\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1}A_N \leqslant \mathbf{0}$ is about dual feasibility. And if you have both, you are optimal.

How to decide which criterion to use?

- If you start from an optimal solution and you only change $\mathbf{b}$ or $\mathbf{c}$, the first optimality criterion is the most straightforward.
- If you start from an optimal solution and you add or remove constraints (or decision variables), it may be better to compute both the primal and dual solution of the initial problem (via complementary slackness or reading the z-row in a dictionary), and then use weak duality as a crtierion for optimality.