Methods for sensitivity analysis

I will try to describe the different methods to tackle the sensitivity analysis questions. I will *not* provide examples: you can take a look at the truck company example[1] or the practice for Quiz 5[2].

Reading information on the final dictionary

Optimal solution An optimal solution is obtained by setting the non basic variables to 0. If all the coefficients in the z row are strictly negative, this is the unique optimal solution.

Basis The basis is the list of basic variables, which are the variables in the left hand side of the dictionary. The variables in the right hand side are the non basic variables.

Marginal values This has to do with the constraints. The marginal value of a constraint is:

- 0 if the slack variable associated to it is basic,
- negative the coefficient in front of this variable in the z row if the slack variable associated to it is a non basic variable.

At optimality, the marginal values are always non negative. The marginal values will always give you an upper bound in the change of the value of the problem if the right hand side of the constraints change. The marginal values provide you with an optimal solution of the dual problem.

Reduced costs This has to do with the decision variables. The reduced cost of a decision variable is:

- 0 if the decision variable is basic,
- the coefficient in front of this variable in the z row if it is a non basic variable.

At optimality, the reduced cost are always non positive. They tell you by how much the profit would change if you try to increase a decision variable which is non basic.

Reading B^{-1} It is possible to read B^{-1} from a dictionary. The matrix B^{-1} has columns indexed by slack variables and rows indexed by the basic variables (in the order in which they appear in the dictionary). The matrix B^{-1} should be filled column by column.

- If a slack variable is basic, in the matrix B^{-1} its column is 0 everywhere except for the row indexed by the same slack variable.
- If a slack variable is non basic, then put negative the coefficients which appear in the column of this variable in the dictionary.

[1] https://www.math.ubc.ca/~anstee/math340/340trucksensitivity.pdf
[2] https://www.math.ubc.ca/~anstee/math340/340pracquiz5.pdf

General strategy for handling changes

When changing parameters, the first attempt is that the current basis stays optimal. For the general criteria to check optimality of a basis, see the notes on *How to check optimality of a basis?*[3].

If the basis is no longer optimal, write the new dictionary and pivot to optimality with the primal or dual simplex. Below I will try to describe all possible changes, but with a good understanding of the general strategy, you should be able to guess what is written below by yourself.

Changing the right hand side of the constraints

If one changes the vector $\mathbf{b}$, that is the right hand side of the constraints, there are tow cases.

- If the new $B^{-1}\mathbf{b}$ is still a vector with non negative components, then the current basis remains optimal. The new values of the basic variables are given by $B^{-1}\mathbf{b}$ while the non basic variables are still set to 0. Optimal solution(s) of the dual problem do not change.
- If at least one component of $B^{-1}\mathbf{b}$ is strictly negative then the current basis is no longer optimal. You should write the new dictionary: only the first column of the right hand side changes and is replaced by the new $B^{-1}\mathbf{b}$. The new dictionary is *dual* feasible (the coefficients in the z row are negative) and you should apply the dual simplex algorithm.

Changing the vector c of the objective function

If one changes the vector $\mathbf{c}$, that is the objective function, there are tow cases.

- If the new $\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1}A_N$ is still a vector with non positive components, then the current basis remains optimal. Optimal solution(s) of the primal remain the same. Optimal solution(s) of the dual problem change, they will correspond to negative the coefficients indexed by the slack variables in $\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1}A_N$.
- If at least one component of $\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1}A_N$ is strictly positive then the current basis is no longer optimal. You should write the new dictionary: only the coefficients in the z row change and are replaced by the new $\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1}A_N$. The new dictionary is *primal* feasible and you should apply the simplex algorithm.

Removing or adding a decision variable

The dichotomy is about the decision variable is basic or not. Indeed, if a decision variable is non basic, the basically it already doesn't play any role.

Specifically, to remove a variable.

- If the variable is non basic, then the current optimal solution will stay optimal. The fastest way to check that is to see that the optimal solution with the variable removed and the dual optimal solution (with no change) will both stay feasible and yield the same value of the objective function, hence are optimal by weak duality.
- If the variable is basic, then the optimal solution will change. Let's say x_1 is basic and to be removed. One option is to introduce the additional constraint $x_1 \leqslant 0$. Namely, define a new slack variable $x_s = -x_1 \geqslant 0$, and add an additional row $x_s = \dots$ in the dictionary, where $\dots$ have to expressed in terms of the non basic variables. The dictionary that you obtain is no longer primal feasible but is dual feasible: you can use the dual simplex. Once you reach a new optimal solution, $x_1 = x_s = 0$ and you can remove it.

[3] `https://hugolav.github.io/teaching/optimality_certificates.pdf`

On the other hand, to add a variable.

- The first attempt should be to see what happens if you set this new variable to 0. To check if the solution remains optimal, you just have to check that the dual optimal solution remains feasible. Indeed, adding a decision variable is equivalent to adding a constraint in the dual.
- If not, just add the new variable as a non basic one, that is add a new column in the left hand side of the dictionary. You end up with a dictionary which is *primal* feasible hence you can use the standard simplex.

Removing or adding a constraint

These are operations which are dual to the previous ones as decisions variables are associated to constraints in the dual.

Specifically, to remove a constraint.

- If the constraint is non binding, that is is a strict inequality then basically it is already removed. Indeed, it means that the dual variable associated to this constraint is equal to 0 by complementary slackness. So you can remove this dual variable from the dual optimal solution: you still get a pair of primal/dual feasible solutions which yield the same value of the objective function, hence are optimal by weak duality.
- If the constraint is an equality, then removing it will likely change the solution. Note that it means that the slack variable associated to it is 0 so[4] it is a non basic variable. One option, if I call this slack variable x_1 is to remove the constraint $x_1 \geqslant 0$ by writing $x_1 = x_1' - x_1''$ with $x_1', x_1'' \geqslant 0$. That amounts to introduce two new non basic variables and remove one. The new dictionary that one gets has one additional column and is still primal feasible: one can use the standard simplex.

Eventually, to add a constraint.

- If the new constraint is already satisfied by the current optimal solution, then it stays optimal. To prove it, one can for instance look at the dual, set to 0 the new dual variable associated to this constraint and use weak duality to conclude to optimality.
- It the new constraint is violated by the current optimal solution, then it will no longer stay optimal. You should introduce a new slack variable defined by this constraint and consider as basic, that is you're adding a row in the dictionary. To be put in the dictionary, this new slack variable must be expressed in terms of the non basic variables. Then you get a dictionary which is no longer primal feasible but is dual feasible: you have to pivot with the dual simplex.

Changing the matrix A

If you change the columns of the non basic variables, then A_N changes but B stays the same. Using the revised simplex formulas, you can see that $\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1} A_N$ the coefficients in the z row will change: this is something that can be handled.

If you change the columns of the basic variables, this is much more of a mess because B changes, dut B^{-1} depends on B in a highly non linear way. Writing the changes of B^{-1} in terms of the ones of B is involved, and you will not be asked that in a quiz or the final.

[4] except in case of a degenerate primal solution.