Linear Programming in Game Theory – Part I

Introduction to Game Theory

What is a game?

You should ask a philosopher. That's a truly interesting and difficult question!

We will concentrate on *mathematical games*. This term has a precise meaning which does not always correspond to the common usage of the word "game".

1. Mathematical games

2. Two players zero sum game

3. Mixed strategies

1. Mathematical games

Example 1: finding your friend

Alice and Bob are supposed to meet on the UBC campus. They have forgotten their phone so they cannot communicate. Each of them must decide to go to a iconic place of the UBC campus:

- The nest,
- or the bus loop,
- or BUCH A201 (because they have a lecture there soon).

Alice and Bob win the game if they decide to go the same place, and they lose if they choose different places.

Here, you try to anticipate what the other will do.

Example 2: Morning commute

You have to decide at which time you leave your place in the morning to commute to work[1]. Your goal is to avoid traffic: you want the other “players” (that is the other commuters) to choose a different time than you.

So you’re taking a decision but “winning” or “loosing” (that is avoiding traffic) depends on the other players.

Comparison with Example 1: in Example 1 you want to do the same as the other player, in this example you want to do the opposite.

[1] Well in most cases you don’t have much choice...

Example 3: rock paper scissors

There are two players: each one chooses independently on the other Rock, Paper or Scissors. Then Rock beats Scissors, Scissors beats Paper and Paper beats Rock.

Comparison with Examples 1 and 2: here one player wins and the other loses, or it is a draw. In Example 1 and Example 2, it could happen that everybody wins or everybody loses.

Mathematical Game

Definition

In a game, there is a set of *players*. Each player has to make a decision[2] *without knowing* what the other players chose. Then each player gains (or loses) something called the *payoff*, and the payoff of each player depends not only of his/her chosen strategy, but what the others have done.

[2]Each possible choice is called a strategy.

A rich theory

This models a lot of situation, and it can lead to many interesting effects, for instance:

- There could be alliances, betrayals, etc.
- *Price of anarchy*: if every player follows its own interest, the global outcome could be worse than if a benevolent central planner had forced everybody's decision.

Game theory was developed from the 1920s and onward. Originally it was built against the classical theory of economics[3]. It provides a language to analyze mathematically the cases where *laisser faire* doesn't work. Nowadays, game theory is part of the standard cursus in economics.

With Linear Programming, we can analyze a simple case: the two players zero sum games.

[3] Roughly, the statement that everybody follows its own interest yields the best outcome.

2. Two players zero sum game

What is a two players zero sum game?

It is a mathematical game with the following features:

- only two players,
- what a player wins is what the other player loses[4].

In the previous examples, only Example 3 (Rock Paper Scissors) is a two players zero sum game, “Finding a friend” and “Morning commute” are not.

This situation is way simpler than general game theory because the two players have opposite goals.

[4]So the sum of what the gain of the two players is zero, hence “zero sum”.

Matrix description

Now some mathematics notations. The set of possible strategies[5] for the first player is $1, 2, \ldots, m$. The set of strategies for the second player is $1, 2, \ldots, n$.

If the Player 1 chooses $i \in \{1, 2, \ldots, m\}$ and the second player chooses (independently from Player 1) $j \in \{1, 2, \ldots, m\}$ the payoff of Player 1 is a_{ij} and the payoff of Player 2 is $-a_{ij}$.

Let's denote by $A = (a_{ij})$ the $m \times n$ matrix storing all the $a_i j$ coefficients called the *payoff* matrix.

So Player 1, the "row" player, chooses a row of the matrix, Player 2, the "column" player chooses a column, and the gain (resp. loss) of the row player (resp. the column player) is the coefficient at the intersection of the chosen row and column.

[5] That is what the player is allowed to choose.

Rock Paper Scissors

Let $+1$ be the reward of Player 1 if they win, 0 if it is a draw and -1 if they lose. The reward of Player 2 is negative the one of Player 1. The matrix in this case is

$$A = \begin{array}{c} \\ R \\ P \\ S \end{array} \begin{array}{c} \begin{array}{ccc} R & P & S \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

The rows of A are indexed by Rock, Paper ,Scissors the strategies of Player 1, while the columns are indexed by Rock, Paper ,Scissors the strategies of Player 2.

In this case the game is symmetric in the two players. At the matrix level it means $A^\top = -A$.

Another example

Any $m \times n$ matrix correspond to a two players zero sum game. For instance, you could take A as follows

$$A = \begin{pmatrix} 3 & -1/5 & -2 \\ -\sqrt{2} & 1 & 18 \end{pmatrix}$$

Here the first player has two strategies while the second player has three. Note also that the coefficients of A need not to be integers.

If the first player chooses the second row and the second player chooses the first column, then the payoff of Player 1 is $-\sqrt{2}$ while the one of Player 2 is $\sqrt{2}$.

3. Mixed strategies

Pure strategies

Let's go back to Rock Paper Scissors. A *pure strategy* for the first Player is for instance to play only Rock. Or to play only Paper. Or to play only Scissors.

Every pure strategy is easy to beat. If Player 1 chooses Rock, then by playing Paper the second player will always win.

So pure strategies are not enough to describe how the players should play.

Mixed strategies

Definition

A mixed strategy is a probability distribution over the different strategies.

That is, in Rock Paper Scissors, playing a mixed strategy is deciding at random whether you play Rock (with probability x_R), Paper (with probability x_P) or Scissors (with probability x_S).

In general, if the different strategies for Player 1 are $\{1, 2, \ldots, m\}$ then a mixed strategy is a vector $\mathbf{x} = (x_i)_{1 \leqslant i \leqslant m}$, where x_i is the probability to choose strategy i. In particular, $\mathbf{x}$ must satisfy

$$\mathbf{x} \geqslant \mathbf{0} \qquad \text{and} \qquad x_1 + x_2 + \ldots + x_m = 1.$$

Indeed, probabilities are non negative and they sum up to 1.

Example: Rock Paper Scissors

In Rock Paper Scissors, you can try to play indifferently Rock, Paper, Scissors. This corresponds to the strategy

$$\mathbf{x} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

As we will see later, this will be the best strategy for Player 1 (and Player 2).

You can also recover a pure strategy from a mixed strategy: for instance, you could consider

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

that is Player 1 plays only the second strategy (say Paper if the order of the index for $\mathbf{x}$ is R,P,S).

Payoff for a mixed strategy

Let's assume that Player 1 plays a mixed strategy described by $\mathbf{x} \in \mathbb{R}^m$. Let's also assume that Player 2 plays a mixed strategy described by $\mathbf{y} \in \mathbb{R}^n$. That is, $\mathbf{x}, \mathbf{y} \geqslant \mathbf{0}$ are vectors and

$$x_1 + x_2 + \ldots + x_m = 1 \qquad \text{and} \qquad y_1 + y_2 + \ldots + y_n = 1.$$

The two mixed strategies are played independently: Player 1 plays without knowning what Player 2 and *vice versa*.

Let's compute the *expected* payoff of Player 1. The probability that Player 1 plays strategy i is x_i, and the probability that Player 2 plays strategy j is y_j. *By independence*, the probability that Player 1 plays strategy i and Player 2 plays strategy j is $x_i y_j$. If this happens the payoff for Player 1 is a_{ij}.

Payoff for a mixed strategy (continued)

So if I sum over all possibilities, the expected payoff of Player 1 is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{a_{ij}}_{\text{payoff}} \underbrace{x_i y_j}_{\text{probability}} .$$

With matrix notations, this can be written $\mathbf{x}^\top A \mathbf{y}$.

To remember

Let's assume that Player 1 plays a mixed strategy described by $\mathbf{x} \in \mathbb{R}^m$. Let's also assume that Player 2 plays independently a mixed strategy described by $\mathbf{y} \in \mathbb{R}^n$. Then the expected payoff of Player 1 is $\mathbf{x}^\top A \mathbf{y}$ and the expected payoff of Player 2 is $-\mathbf{x}^\top A \mathbf{y}$ (the opposite of the one of Player 1).

Remarks

Now the problem that we want to solve is:

What is the best mixed strategy for Player 1?

Notice that we will define "best" by "best expected payoff".

The introduction of mixed strategies will enable to solve the game by Linear Programming. Notice however that we have transitioned from "payoff" to "expected payoff". If you are able to play a large number of games the two notions match by the law of large numbers. But sometimes you play only one game! Even if we won't come back to that in the sequel, I want to emphasize that sometimes from a modeling point of view the expected payoff is not the best notion.