1. The three inequalities $-x+2y \le -2$, $2x+y \ge 1$, $-3x+y \ge -4$ have no solution x, y with $x, y \ge 0$:

   We first transform the inequalities to standard form $-x+2y \le -2$, $-2x-y \le -1$, $+3x-y \le 4$. Then apply Phase One:

$$\begin{array}{rcrrrr} x_1 &=& -2 & +x & -2y & +x_0 \\ x_2 &=& -1 & +2x & +y & +x_0 \\ x_3 &=& 4 & -3x & +y & +x_0 \\ w &=& & & & -x_0 \end{array}$$

   Fake pivot to feasibility, x_0 enters and x_1 leaves.

$$\begin{array}{rcrrrr} x_0 &=& 2 & -x & +2y & +x_1 \\ x_2 &=& 1 & +x & +3y & +x_1 \\ x_3 &=& 6 & -4x & +3y & +x_1 \\ w &=& -2 & +x & +y & -x_1 \end{array}$$

   (I used Anstee's rule with x before y) x enters and x_3 leaves.

$$\begin{array}{rcrrrr} x_0 &=& 1/2 & +(1/4)x_3 & +(5/4)y & +(3/4)x_1 \\ x_2 &=& 5/2 & -(1/4)x_3 & +(15/4)y & +(5/4)x_1 \\ x &=& 3/2 & -(1/4)x_3 & +(3/4)y & +(1/4)x_1 \\ w &=& -1/2 & -(1/4)x_3 & -(5/4)y & -(3/4)x_1 \end{array}$$

   We are at optimality with $w = -1/2$. Thus we cannot drive x_0 to 0. Using the magic coefficients, the negatives of the coefficients of the slack variables x_1, x_2, x_3, namely $\frac{3}{4}, 0, \frac{1}{4}$ we apply these to the three inequalities:

$$\frac{3}{4}(-x+2y \le -2) + 0(-2x-y \le -1) + \frac{1}{4}(+3x-y \le 4)$$

   to get the contradiction to $y \ge 0$:

$$(5/4)y \le \frac{-1}{2}.$$

   Thus the three inequalities only yield solutions with $y < 0$.

   We have discovered that we need only two inequalities, the first and third, to get a contradiction.

2.

a) First we transform LP3 into standard inequality form and call it LP7. We must replace the equality by two inequalities and substitute $x_2 = x_2' - x_2''$ where $x_2', x_2'' \ge 0$.

$$\begin{array}{llrrrcr} LP7 & \max & 2x_1 & +4x_2' & -4x_2'' & & \\ & & 3x_1 & -7x_2' & +7x_2'' & \le & 6 \\ & & -3x_1 & +7x_2' & -7x_2'' & \le & -6 \\ & & 6x_1 & +2x_2' & -2x_2'' & \le & 5 \end{array} \qquad x_1, x_2', x_2'' \ge 0$$

The dual of an LP7 is:

$$\begin{array}{llrrrcrl} LP8 & \min & 6z_1 & -6z_2 & +5z_3 & & & \\ & & 3z_1 & -3z_2 & +6x_3 & \geq & 2 & \\ & & -7z_1 & +7z_2 & +2z_3 & \geq & 4 & z_1, z_2, z_3 \geq 0 \\ & & 7z_1 & -7z_2 & -2z_3 & \geq & -4 & \end{array}$$

Now we can see that LP8 equivalent to LP4 by combining the last two inequalities into the equivalent equality $-7z_1 + 7z_2 + 2z_3 = 4$ and using the variable substitutions $z_1 - z_2 = y_1$ and $z_3 = y_2$ so that y_1 is unconstrained and $y_2 \geq 0$.

In general our transformations may have affected the value of the objective function when we replace z by $-z$ (going from a min to a max) or when we delete a constant. But this does not occur in this problem.

b) We proceed as above transforming LP5 into LP9

$$\begin{array}{llrrrcrl} LP9 & \max & \mathbf{a}\cdot\mathbf{x} & +\mathbf{b}\cdot\mathbf{y}' & -\mathbf{b}\cdot\mathbf{y}'' & & & \\ & & A\mathbf{x} & +B\mathbf{y}' & -B\mathbf{y}'' & \leq & \mathbf{c} & \\ & & -A\mathbf{x} & -B\mathbf{y}' & +B\mathbf{y}'' & \leq & -\mathbf{c} & \mathbf{x}, \mathbf{y}', \mathbf{y}'' \geq \mathbf{0} \\ & & C\mathbf{x} & +D\mathbf{y}' & -D\mathbf{y}'' & \leq & \mathbf{d} & \end{array}$$

The dual LP10 is

$$\begin{array}{llrrrcrl} LP10 & \min & \mathbf{c}\cdot\mathbf{z}' & -\mathbf{c}\cdot\mathbf{z}'' & +\mathbf{d}\cdot\mathbf{t} & & & \\ & & A^T\mathbf{z}' & -A^T\mathbf{z}'' & +C^T\mathbf{t} & \geq & \mathbf{a} & \\ & & B^T\mathbf{z}' & -B^T\mathbf{z}'' & +D^T\mathbf{t} & \geq & \mathbf{b} & \mathbf{z}', \mathbf{z}'', \mathbf{t} \geq 0 \\ & & -B^T\mathbf{z}' & +B^T\mathbf{z}'' & -D^T\mathbf{t} & \geq & -\mathbf{b} & \end{array}$$

We can see that LP10 can be obtained from LP6 by realizing that the inequalities $B^T\mathbf{z}' - B^T\mathbf{z}'' + D^T\mathbf{t} \geq \mathbf{b}$ and $-B^T\mathbf{z}' + B^T\mathbf{z}'' - D^T\mathbf{t} \geq -\mathbf{b}$ yields the equalities $B^T\mathbf{z}' - B^T\mathbf{z}'' + D^T\mathbf{t} = \mathbf{b}$ and then we replace $\mathbf{z}' - \mathbf{z}'' = \mathbf{z}$ and have that each variable in $\mathbf{z}$ is free.

3. Consider $F = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}\}$. Given a vector $\mathbf{z} \in F$, define the set of *feasible directions* $F_{\mathbf{z}}$ *at* $\mathbf{z}$ as follows

$$F_{\mathbf{z}} = \{\mathbf{y} : \text{ there exists some constant } c_{\mathbf{y}} > 0 \text{ with } \mathbf{z} + t\mathbf{y} \in F \text{ for all } t \in [0, c_{\mathbf{y}}]\}$$

These are the directions you can go in at least a small amount and still be feasible.

a) Show that if $\mathbf{u} \in F_{\mathbf{z}}$, then for any $c > 0$, it is also true that $c\mathbf{u} \in F_{\mathbf{z}}$.

We simple note that if $z + c_u\mathbf{u} \in F$ then that is the same as saying $z + (c_u/c)(c\mathbf{u}) \in F$ from which it follows $c\mathbf{u} \in F_{\mathbf{z}}$.

b) Consider $\mathbf{u}, \mathbf{v} \in F_{\mathbf{z}}$ where we have positive non zero constants $c_{\mathbf{u}}$ and $c_{\mathbf{v}}$ with $\mathbf{z} + c_{\mathbf{u}}\mathbf{u} \in F$ and $\mathbf{z} + c_{\mathbf{v}}\mathbf{v} \in F$ (which show that $\mathbf{u}, \mathbf{v} \in F_{\mathbf{z}}$). Show that $\frac{1}{2}c_{\mathbf{u}}\mathbf{u} + \frac{1}{2}c_{\mathbf{v}}\mathbf{v} \in F_{\mathbf{z}}$.

Because $\mathbf{z} + c_{\mathbf{u}}\mathbf{u} \in F$ and $\mathbf{z} + c_{\mathbf{v}}\mathbf{v} \in F$ we have

$$A(\mathbf{z} + \mathbf{c_u}\mathbf{u}) \leq \mathbf{b}, \mathbf{z} + \mathbf{c_u}\mathbf{u} \geq \mathbf{0}, \qquad A(\mathbf{z} + \mathbf{c_v}\mathbf{v}) \leq \mathbf{b}, \mathbf{z} + \mathbf{c_v}\mathbf{v} \geq \mathbf{0}$$

We compute using linearity of matrix multiplication

$$A(\mathbf{z} + \frac{1}{2}c_{\mathbf{u}}\mathbf{u} + \frac{1}{2}c_{\mathbf{v}}\mathbf{v}) = A(\frac{1}{2}\mathbf{z} + \frac{1}{2}c_{\mathbf{u}}\mathbf{u} + \frac{1}{2}\mathbf{z} + \frac{1}{2}c_{\mathbf{v}}\mathbf{v})$$

$$\frac{1}{2}A(\mathbf{z}+\mathbf{c_u}\mathbf{u})+\frac{1}{2}A(\mathbf{z}+\mathbf{c_v}\mathbf{v}) \geq \frac{1}{2}\mathbf{b}+\frac{1}{2}\mathbf{b}=\mathbf{b}$$

Similarly

$$\begin{aligned}\mathbf{z}+\frac{1}{2}c_{\mathbf{u}}\mathbf{u}+\frac{1}{2}c_{\mathbf{v}}\mathbf{v} &= \frac{1}{2}\mathbf{z}+\frac{1}{2}c_{\mathbf{u}}\mathbf{u}+\frac{1}{2}\mathbf{z}+\frac{1}{2}c_{\mathbf{v}}\mathbf{v}\\ &= \frac{1}{2}(\mathbf{z}+c_{\mathbf{u}}\mathbf{u})+\frac{1}{2}(\mathbf{z}+c_{\mathbf{v}}\mathbf{v}) \geq \frac{1}{2}\mathbf{0}+\frac{1}{2}\mathbf{0}=\mathbf{0}\end{aligned}$$

Thus $\frac{1}{2}c_{\mathbf{u}}\mathbf{u}+\frac{1}{2}c_{\mathbf{v}}\mathbf{v} \in F_{\mathbf{z}}$.

c) Now show that $\frac{1}{2}e\mathbf{u}+\frac{1}{2}e\mathbf{v} \in F_{\mathbf{z}}$ by replacing our choices of $c_{\mathbf{u}}$ and $c_{\mathbf{v}}$ by $e=\min\{c_{\mathbf{u}}, c_{\mathbf{v}}\}$.

The idea is we can replace our choices of $c_{\mathbf{u}}$ and $c_{\mathbf{v}}$ by any values smaller (or the same) and so with $e=\min\{c_{\mathbf{u}}, c_{\mathbf{v}}\}$ where we have $e>0$ and $\mathbf{z}+e\mathbf{u} \in F$ and $\mathbf{z}+e\mathbf{v} \in F$.

d) Then show that $\mathbf{u}+\mathbf{v} \in F_{\mathbf{z}}$. Result a) should help.

Our result in a) shows that since $\frac{1}{2}e\mathbf{u}+\frac{1}{2}e\mathbf{v} \in F_{\mathbf{z}}$ we must have $(2/e)\frac{1}{2}e\mathbf{u}+\frac{1}{2}e\mathbf{v}=\mathbf{u}+\mathbf{v} \in F_{\mathbf{z}}$. Note that we need $e>0$.

e) Finally show that for any positive constants a, b, $a\mathbf{u}+b\mathbf{v} \in F_{\mathbf{z}}$.

With $\mathbf{u} \in F_{\mathbf{z}}$, we use a) with $c=a$, to obtain $a\mathbf{u} \in F_{\mathbf{z}}$ for any $a>0$. Similarly $b\mathbf{v} \in F_{\mathbf{z}}$. Now we appeal to our result of d), with $\mathbf{u}$ replaced by $a\mathbf{u}$ and $\mathbf{v}$ replaced by $b\mathbf{v}$.

4. Consider an LP: max $\mathbf{c}\cdot\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Assume that the LP has a feasible solution $\mathbf{u}$ and there exists a vector $\mathbf{v}$ with $\mathbf{v} \geq 0$, $A\mathbf{v} \leq \mathbf{0}$ and $\mathbf{c}\cdot\mathbf{v}>0$. Show that the LP is unbounded. Hint: Consider $\mathbf{u}+t\mathbf{v}$ as $t \to \infty$.

   We compute $A(\mathbf{u}+t\mathbf{v})=A\mathbf{u}+tA\mathbf{v}$. Now $A\mathbf{v} \leq \mathbf{0}$ and so $tA\mathbf{v} \leq \mathbf{0}$ for $t>0$. Also $A\mathbf{u} \leq \mathbf{b}$ and so $A(\mathbf{u}+t\mathbf{v})=A\mathbf{u}+tA\mathbf{v} \leq \mathbf{b}+\mathbf{0}=\mathbf{b}$. Also $\mathbf{u} \geq \mathbf{0}$ and $\mathbf{v} \geq \mathbf{0}$ so that $t\mathbf{v} \geq \mathbf{0}$ for $t>0$. Then $\mathbf{u}+t\mathbf{v} \geq \mathbf{0}+\mathbf{0}=\mathbf{0}$. Thus $\mathbf{u}+t\mathbf{v}$ is a feasible solution to the LP for any $t>0$.

   The objective function is $\mathbf{c}\cdot(\mathbf{u}+t\mathbf{v})=\mathbf{c}\cdot\mathbf{u}+t\mathbf{c}\cdot\mathbf{v}$ (linearity again!) . Let $e=\mathbf{c}\cdot\mathbf{u}$ and $f=\mathbf{c}\cdot\mathbf{v}$. One of our hypotheses is $f>0$. We have $f>0$ and so $\lim_{t\to\infty}\mathbf{c}\cdot(\mathbf{u}+t\mathbf{v})=\lim_{t\to\infty}(e+tf)=\infty$ since $f>0$.

5. Our simplex algorithm pivots from basic feasible solution to basic feasible solution, namely solutions depending on a basis so that the variables with non zero values index a linearly independent set of columns. We consider the following
   LP: max $\mathbf{c}\cdot\mathbf{x}$ such that $A\mathbf{x}=\mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
   that has a feasible solution $\mathbf{u}=(u_1, u_2, \ldots, u_{n+m})^T$ (i.e. $A\mathbf{u}=\mathbf{b}$ and $u \geq \mathbf{0}$). We wish have you give a step in the proof to show the LP has a basic feasible solution *without using the simplex algorithm*. You may imagine we have included the slack variables as well as the original variables in th evector $\mathbf{u}$.

   Let A_i denote the ith column of A. Let $P=\{i : u_i>0\}$, namely the indices for which $\mathbf{u}$ is non zero (strictly positive) . Assume $\{A_i : i \in P\}$ is a linearly dependent set of columns such that $\sum_{i\in I} a_iA_i=\mathbf{0}$. Thus we can find $\mathbf{a}=(a_1, a_2, \ldots, a_{n+m})^T$ so that $A\mathbf{a}=\mathbf{0}$ and $\mathbf{a} \neq \mathbf{0}$ where $a_i=0$ for $i \notin P$.

   We note that $\mathbf{u}+e\mathbf{a}$ satisfies $A(\mathbf{u}+e\mathbf{a})=A\mathbf{u}=\mathbf{b}$. For your assignment, indicate how to choose e so that $\mathbf{u}+e\mathbf{a} \geq 0$ such that there are fewer non zero entries in $\mathbf{u}+e\mathbf{a}$ than in $\mathbf{u}$ and at the same time $\mathbf{u}+e\mathbf{a}$ is a feasible solution to the LP. We already have $A(\mathbf{u}+e\mathbf{a})=\mathbf{b}$.

Solution: The idea is that if we choose e too large (in absolute value) we risk getting an infeasible solution $\mathbf{u} + e\mathbf{a}$ which has negative entries (we expect $\mathbf{a}$ to have both positive and negative entries). But we wish to take e large enough (in absolute value) to create some new 0's. We have $\mathbf{u} + e\mathbf{a} = (u_1 + ea_1, u_2 + ea_2, \ldots, u_{n+m} + ea_{n+m})^T$. SInce $\mathbf{a} \neq \mathbf{0}$, we can split or analysis into two cases. First $\mathbf{a}$ has a strictly positive entry $a_k > 0$. Secondly $\mathbf{a} \leq \mathbf{0}$ with a strictly negative entry $a_\ell < 0$.

Case 1. $\mathbf{a}$ has an entry $a_k > 0$.

In this case we take e to be negative driving some entry to 0. We must have $u_i + ea_i \geq 0$ for $i = 1, 2, \ldots, n + m$. With $e < 0$ and $a_i \leq 0$, we have $ea_i \geq 0$ and so $u_i + ea_i \geq 0$. For $a_i > 0$, we have that $u_i + ea_i \geq 0$ yields $u_i/a_i \geq -e$ or $e \geq -u_i/a_i$. So let $e = \max\{-u_i/a_i \,:\, a_i > 0\}$. This is the maximum of a set of negative numbers. With this value of e, there will be some entry driven to 0, namely the pth entry if $e = -u_p/a_p$ (with $a_p > 0$).

Case 2. $\mathbf{a} \leq \mathbf{0}$ has an entry $a_\ell < 0$.

We must have $u_i + ea_i \geq 0$ for $i = 1, 2, \ldots, n + m$ for those i with $a_i < 0$. Thus $u_i/a_i + e \leq 0$ and so $e \leq -u_i/a_i$. Let $e = \min\{-u_i/a_i \,:\, a_i < 0\}$. This is the minimum of a set of negative numbers. Then $u_i + ea_i \geq 0$ for $i = 1, 2, \ldots, n + m$ for those i with $a_i < 0$ and moreover there will be some entry driven to 0 namely the qth entry when $e = -u_q/a_q$ (with $a_q < 0$).

It is indeed easier to use some illustrative examples to see what is happening. The writeup can be annoying.

e.g for $\mathbf{u} = (0, 1, 2, 3, 4)^T$, $a = (0, 0, 3, 3, -3)^T$, take $e = -2/3$.

e.g. for $\mathbf{u} = (0, 1, 2, 3, 4)^T$, $a = (0, 0, -3, -3, -3)^T$, take $e = 2/3$.

We would repeat this procedure until there are fewer and fewer 0's in $\mathbf{u}$ so that the columns of A indexed by the non zero entries of $\mathbf{u}$ are linearly independent.