Math 340 Assignment #1 Due Friday January 31 at the beginning of class.

1. Show that the three inequalities

$$-x + 2y \le -2 \qquad 2x + y \ge 1 \qquad -3x + y \ge -4$$

have no solution x, y with $x, y \ge 0$ by using our two phase method (not using LINDO; you need the practice! Fractions are good for you!). In addition, try to show infeasibility by finding a (positive) linear combination of the three inequalities which provides an 'obvious' contradiction to $x, y \ge 0$. Remember that it can be done by using the coefficients in front of the slack variables in the final w row.

2. The dual of an LP in standard inequality form can be defined as follows:

$$\text{(primal)}LP: \quad \begin{array}{rc} \max & \mathbf{c}\cdot\mathbf{x} \\ & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array} \qquad\qquad \text{dual } LP: \quad \begin{array}{rc} \min & \mathbf{b}\cdot\mathbf{y} \\ & A^T\mathbf{y} \ge \mathbf{c} \\ & \mathbf{y} \ge \mathbf{0} \end{array}$$

where A is an $m \times n$ matrix, $\mathbf{c}$ is $n \times 1$, $\mathbf{b}$ is $m \times 1$, $\mathbf{x}$ is $n \times 1$, $\mathbf{y}$ is $m \times 1$. The notation A^T denotes the transpose of the matrix A while $\mathbf{c}\cdot\mathbf{x} = \mathbf{c}^T\mathbf{x}$ is the scalar product between $\mathbf{c}$ and $\mathbf{x}$. Note how inequalities in the primal become variables in the dual and how variables in the primal correspond to inequalities in the dual. We will be discussing this in class as well. As an example, the dual of

$$LP1 \quad \begin{array}{rrrrcl} \max & x_1 & +2x_2 & +3x_3 & & \\ & 4x_1 & +5x_2 & +6x_3 & \le & 7 \\ & 8x_1 & +9x_2 & +10x_3 & \le & 11 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

is

$$LP2 \quad \begin{array}{rrrcl} \min & 7y_1 & +11y_2 & & \\ & 4y_1 & +8y_2 & \ge & 1 \\ & 5y_1 & +9y_2 & \ge & 2 \\ & 6y_1 & +10y_2 & \ge & 3 \end{array} \qquad y_1, y_2 \ge 0$$

a) Show that the dual of LP3 is equivalent to LP4. To compute the dual of LP3 you will have to transform to standard inequality form. In the resulting dual LP you will transform to LP4. Note that your transformations do not affect the objective function values.

$$LP3 \quad \begin{array}{rrrcl} \max & 2x_1 & +4x_2 & & \\ & 3x_1 & -7x_2 & = & 6 \\ & 6x_1 & +2x_2 & \le & 5 \end{array} \qquad\qquad LP4 \quad \begin{array}{rrrcl} \min & 6y_1 & +5y_2 & & \\ & 3y_1 & +6y_2 & \ge & 2 \\ & -7y_1 & +2y_2 & = & 4 \end{array}$$

$$x_1 \ge 0, x_2 \text{ unconstrained} \qquad\qquad y_1 \text{ unconstrained }, y_2 \ge 0$$

b) Generalize a) to an arbitrary number of equations and free variables and show that the dual of LP5 is equivalent to LP6. (We assume that $\mathbf{x} \in \mathbf{R}^{n_1}$, $\mathbf{y} \in \mathbf{R}^{n_2}$, $\mathbf{z} \in \mathbf{R}^{m_1}$, $\mathbf{t} \in \mathbf{R}^{m_2}$ so that for example D is $m_2 \times n_2$).

$$LP5 \quad \begin{array}{rrrcl} \max & \mathbf{a}\cdot\mathbf{x} & +\mathbf{b}\cdot\mathbf{y} & & \\ & A\mathbf{x} & +B\mathbf{y} & = & \mathbf{c} \\ & C\mathbf{x} & +D\mathbf{y} & \le & \mathbf{d} \end{array} \qquad\qquad LP6 \quad \begin{array}{rrrcl} \min & \mathbf{c}\cdot\mathbf{z} & +\mathbf{d}\cdot\mathbf{t} & & \\ & A^T\mathbf{z} & +C^T\mathbf{t} & \ge & \mathbf{a} \\ & B^T\mathbf{z} & +D^T\mathbf{t} & = & \mathbf{b} \end{array}$$

$$\mathbf{x} \ge \mathbf{0}, \mathbf{y} \text{ all unconstrained} \qquad\qquad \mathbf{z} \text{ all unconstrained}, \mathbf{t} \ge \mathbf{0}$$

The general rule is that equalities in one LP transform to free (unconstrained) variables in its dual and free (unconstrained) variables in one LP transform to equalities in its dual.

3. Let A be a $m \times n$ matrix and $\mathbf{b}$ a $m \times 1$ vector. Consider $F = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}\}$: it can be interpreted as the set of feasible solutions for a certain LP. Given a vector $\mathbf{z} \in F$, define the set of *feasible directions* $F_{\mathbf{z}}$ *at* $\mathbf{z}$ as follows

$$F_{\mathbf{z}} = \{\mathbf{y} : \text{ there exists some constant } c_{\mathbf{y}} > 0 \text{ with } \mathbf{z} + t\mathbf{y} \in F \text{ for all } t \in [0, c_{\mathbf{y}}]\}$$

These are the directions $\mathbf{y}$ you can go in from $\mathbf{z}$ at least a small amount and still remain feasible. Note the difference between F and $F_{\mathbf{z}}$. The choice of the constant $c_{\mathbf{y}}$ would depend on $\mathbf{y}$ but if a constant $k = c_{\mathbf{y}}$ works then so does other choices such as $(1/2)k$.

a) Show that if $\mathbf{u} \in F_{\mathbf{z}}$, then for any $c > 0$, it is also true that $c\mathbf{u} \in F_{\mathbf{z}}$

b) Consider $\mathbf{u}, \mathbf{v} \in F_{\mathbf{z}}$ where we have positive non zero constants $c_{\mathbf{u}}$ and $c_{\mathbf{v}}$ with $\mathbf{z} + c_{\mathbf{u}}\mathbf{u} \in F$ and $\mathbf{z} + c_{\mathbf{v}}\mathbf{v} \in F$ (which show that $\mathbf{u}, \mathbf{v} \in F_{\mathbf{z}}$). Show that $\frac{1}{2}c_{\mathbf{u}}\mathbf{u} + \frac{1}{2}c_{\mathbf{v}}\mathbf{v} \in F_{\mathbf{z}}$.

c) Now show that $\frac{1}{2}e\mathbf{u} + \frac{1}{2}e\mathbf{v} \in F_{\mathbf{z}}$ by replacing our choices of $c_{\mathbf{u}}$ and $c_{\mathbf{v}}$ by $e = \min\{c_{\mathbf{u}}, c_{\mathbf{v}}\}$.

d) Then show that $\mathbf{u} + \mathbf{v} \in F_{\mathbf{z}}$. Result a) should help.

e) Finally show that for any positive constants a, b, $a\mathbf{u} + b\mathbf{v} \in F_{\mathbf{z}}$.

(This shows that $F_{\mathbf{z}}$ is what is called a *cone*. We shall see cones later in the course.)

4. Consider an LP: max $\mathbf{c} \cdot \mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Assume that the LP has a feasible solution $\mathbf{u}$ and there exists a vector $\mathbf{v}$ with $\mathbf{v} \geq 0$, $A\mathbf{v} \leq \mathbf{0}$ and $\mathbf{c} \cdot \mathbf{v} > 0$. Show that the LP is unbounded. (Hint: consider $\mathbf{u} + t\mathbf{v}$).

5. Our simplex algorithm pivots from basic feasible solution to basic feasible solution, namely solutions depending on a basis so that the variables with non zero values index a linearly independent set of columns. We consider the following
LP: max $\mathbf{c} \cdot \mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$
that has a feasible solution $\mathbf{u} = (u_1, u_2, \ldots, u_{n+m})^T$ (i.e. $A\mathbf{u} = \mathbf{b}$ and $\mathbf{u} \geq \mathbf{0}$). We have included the slack variables in our original dictionary formulation so that this LP is in equality (and not inequality) form. We wish to give you a step in the proof to show the LP has a basic feasible solution *without using the simplex algorithm.*

Let A_i denote the ith column of A. Let $P = \{i : u_i > 0\}$, namely the indices for which $\mathbf{u}$ is non zero (strictly positive) . Assume $\{A_i : i \in P\}$ is a linearly *dependent* set of columns: it means that there exist choices for a_i, not all 0, so that $\sum_{i \in P} a_i A_i = \mathbf{0}$. Thus we can find $\mathbf{a} = (a_1, a_2, \ldots, a_n)^T$ so that $A\mathbf{a} = \mathbf{0}$ and $\mathbf{a} \neq \mathbf{0}$ where we set $a_i = 0$ for $i \notin P$.

We note that $\mathbf{u} + e\mathbf{a}$ satisfies $A(\mathbf{u} + e\mathbf{a}) = A\mathbf{u} = \mathbf{b}$. For your assignment, indicate how to choose e so that $\mathbf{u} + e\mathbf{a} \geq 0$ such that there are fewer non zero entries in $\mathbf{u} + e\mathbf{a}$ than in $\mathbf{u}$ and at the same time $\mathbf{u} + e\mathbf{a}$ is a feasible solution to the LP. We already have $A(\mathbf{u} + e\mathbf{a}) = \mathbf{b}$.

(With a bit more work this yields the result we gave at the beginning but you can stop at this point).