How to perform a pivot ?

A **pivot** is an operation which takes you from one dictionary to another one. A **dictionary** is a set of equations expressing the **basic** variables and the value of the objective function z as a function of the **non-basic** variables. Recall that all the variables (expect z) are constrained to be non-negative. The convention is that the basic variables and z are on the left hand side while the non-basic variables are on the right hand side. Moreover, **if the non-basic variables are set to** 0 **then one must obtain non-negative values for basic variables**. The example of the dictionary that we saw in class is

$$\begin{array}{rcrrrrr} x_5 & = & 3 & -x_1 & -2x_2 & & +x_4 \\ x_6 & = & 2 & -2x_1 & -x_2 & +x_3 & -x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & & 4x_1 & +3x_2 & +x_3 & +x_4 \end{array}$$

Method

The pivoting process amounts to perform the following task :

1. Choose an **entering** variable. It must be a non-basic variable whose coefficient in the z-row is **strictly positive**. In case where there are more than one choice : choose the one with the **largest** coefficient in the z-row (this way of choosing in case of ties is called **Anstee's rule**).
2. Choose a **leaving** variable (it is a basic variable). To find it, set all the non-basic variable except the entering one to 0, and find the first basic variable which is driven to 0 when one increases the entering variable. In case of ties, choose the variable with the smallest subscript.
3. Transform the dictionary in such a way that the entering variable becomes basic while the leaving one becomes non-basic. To do that, add or subtract suitable multiples of the row of the leaving variable to the other rows, including the z one.

The pivoting process must be repeated until all the coefficients of the non-basic variables in the z row are non-positive. Indeed, we will see later in class that it means that we have reached optimality. **Once optimality is reached to get the optimal solution, one must set the non-basic variables to** 0 **and compute the basic variables according to the dictionary**.

Usually the first step is very fast, the second one is easy, and the third one is tedious and often generates errors.

One detailed example

Let me know detail the steps in the example above.

First step. All the non-basic variables in the z row have positive coefficients. Using Anstee's rule we choose the one with the largest coefficient, that is x_1. So x_1 **is the entering variable**.

Second step. In the dictionary we set all the other non-basic variables except x_1 to 0 and look at the basic variables. This reads

$$\begin{array}{rcll} x_5 & = & 3 & -x_1 \\ x_6 & = & 2 & -2x_1 \\ x_7 & = & 2 & \end{array}$$

When we increase x_1 the first variable to reach 0 is x_6 for $x_1 = 1$. Indeed, x_7 is never driven to 0 while x_5 is set to 0 when $x_1 = 3$ but $3 > 1$. Hence **the leaving variable is** x_6.

Third step. Now we must transform the dictionary in such a way that x_1 is a basic variable and x_6 a non-basic variable. So in each row x_1 must not be in the right hand side while x_6 can. And the row of $x_6 = \cdots$ must be replaced by a row of the type $x_1 = \cdots$. For the first row, we add to it $-1/2$ times the second row to eliminate x_1 (we always use the row of the leaving variable for the elimination). This reads

$$x_5 - 1/2\, x_6 = 2 - 3/2\, x_2 - 1/2\, x_3 + 3/2\, x_4.$$

Moving the (now non-basic) variable x_6 to the right hand side, it reads

$$x_5 = 2 + 1/2\, x_6 - 3/2\, x_2 - 1/2\, x_3 + 3/2\, x_4.$$

For the second row, the one with x_6 (the leaving variable), it is simple. Just rewrite it with x_1 on the left hand side and x_6 on the right hand side. In this case, dividing the row by 2 and moving the variables we get

$$x_1 = 1 - 1/2\, x_6 - 1/2\, x_2 + 1/2\, x_3 - 1/2\, x_4.$$

As x_1 is already not present in the third row, we do need to change it. Eventually, we need to remove x_1 in the z-row. To do that we add twice the second row to the last one and we get

$$z + 2x_6 = 4 + x_2 + 3x_3 - x_4,$$

which can be written once we move the x_6 variable around

$$z = 4 - 2x_6 + x_2 + 3x_3 - x_4.$$

Once we have performed these operations on all these rows, we collect the result in the new dictionary which becomes

$$\begin{array}{rcllllll} x_5 & = & 2 & +1/2\, x_6 & -3/2\, x_2 & -1/2\, x_3 & +3/2\, x_4 \\ x_1 & = & 1 & -1/2\, x_6 & -1/2\, x_2 & +1/2\, x_3 & -1/2\, x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & 4 & -2x_6 & +x_2 & +3x_3 & -x_4 \end{array}$$

As there are still positive coefficients in the z-row we keep going and do a new pivot.

First step. The non-basic variable with the largest (positive) coefficient in the z-row is x_3 hence x_3 **is the entering variable**.

Second step. We set all the non-basic variables to 0 except x_3 in the dictionary to get

$$\begin{array}{rcll} x_5 & = & 2 & -1/2\, x_3 \\ x_1 & = & 1 & +1/2\, x_3 \\ x_7 & = & 2 & -x_3. \end{array}$$

As x_3 starts from 0 and increases, the first basic variable to reach 0 is x_7 (for $x_3 = 2$) and not x_5. Hence **the leaving variable is** x_7.

Third step. To remove the entering variable x_3 from the first row we add to it $-1/2$ times the third row (which is the row of the leaving variable). It reads

$$x_5 - 1/2\, x_7 = 1 + 1/2\, x_6 - x_2 + 3/2\, x_4$$

which is rewritten

$$x_5 = 1 + 1/2\, x_6 - x_2 + 1/2\, x_7 + 3/2\, x_4.$$

We add $1/2$ times the third row to the second one to eliminate x_3 in the second row : it reads

$$x_1 + 1/2\, x_7 = 2 - 1/2\, x_6 - x_2 - 1/2\, x_4,$$

which reads once we put x_7 on the right hand side :

$$x_1 = 2 - 1/2\, x_6 - 1/2\, x_7 - 1/2\, x_4.$$

For the third row, we just move x_3 to the left hand side and x_7 to the right hand side : it reads

$$x_3 = 2 - x_2 - x_7.$$

Eventually, we add three times the $x_7 = \cdots$ row to the z-row to eliminate x_3, this reads

$$z + 3x_7 = 10 - 2x_6 - 2x_2 - x_4,$$

which can be rewritten

$$z = 10 - 2x_6 - 2x_2 - 3x_7 - x_4.$$

We collect all these computations in the new dictionary which reads

$$\begin{array}{rclccccc}
x_5 & = & 1 & +1/2\, x_6 & -x_2 & +1/2\, x_7 & +3/2\, x_4 \\
x_1 & = & 2 & -1/2\, x_6 & -x_2 & -1/2\, x_7 & -1/2\, x_4 \\
x_3 & = & 2 & & -x_2 & -x_7 & \\
z & = & 10 & -2x_6 & -2x_2 & -3x_7 & -x_4
\end{array}$$

As in the dictionary all the coefficients in the z-row are non-negative it means that optimality is reached. To get an optimal feasible solution, one should set all the values of the non-basic variables to 0 and compute the values of the basic variables with the dictionary. In this case it means that the optimal solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and the value of the objective function is $z = 10$. As a safety check, this set of values is also a solution of the equations of the first dictionary.

Some remarks

For the quizz all you need to know is how to perform the computations above, the theoretical justification will come later in the lectures. Below are some remarks to help you detect mistakes.

1. It helps to perform the computations if there is only one non-basic variable per column in the right hand side. Notice how the leaving variable took the "column" of the entering variable in the example above. This is just a convention that you should follow (mainly because it is easier to grade if all computations are organized in the same way as mine).
2. The value of the objective function z (when non-basic variables are set to 0) must increase at each iteration. If it doesn't, it means that you chose the wrong entering variable.
3. Remember that if you set the non-basic variables to 0 then the basic variables should be non-negative. If this property is lost after one pivoting operation it means that you chose the wrong leaving variable.

Note that on your quizz sheet there is no need to detail the first two steps of the pivoting process, the entering and leaving variables can be given without justification. Also you don't need to detail the rows addition and subtraction as much as above.

Practice examples

In addition to the practice quizz which is on Richard Anstee's website, below are tree examples with solution which can be found in the exercise part of Chapter 2 of the textbook. The difficulty of the quizz will be similar to the first example. If you succeed to do the second one you are more than ready. The third one tests perseverance as 4 pivots are needed to reach optimality.

First example

A rather simple example where you need only one pivot to reach optimality.

$$\begin{array}{rcrrr} x_1 & = & 3 & -2x_5 & -3x_6 \\ x_2 & = & 1 & -x_5 & -5x_6 \\ x_3 & = & 4 & -2x_5 & -x_6 \\ x_4 & = & 5 & -4x_5 & -x_6 \\ z & = & & 2x_5 & +x_6 \end{array}$$

Following Anstee's rule, one chooses x_5 as the entering variable. Easy computations yields that x_2 is the leaving variable. Once all computations are performed the new dictionary is

$$\begin{array}{rcrrr} x_1 & = & 1 & +2x_2 & +7x_6 \\ x_5 & = & 1 & -x_2 & -5x_6 \\ x_3 & = & 2 & +2x_2 & +9x_6 \\ x_4 & = & 1 & +4x_2 & +19x_6 \\ z & = & 2 & -2x_2 & -9x_6 \end{array}$$

All the variables in the z-row have negative coefficients hence optimality is reached. We set the non-basic variables to 0 to get the optimal solution :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

with the optimal value of the objective function being $z = 2$.

Second example

Here is a second example, where the dictionary you should start with is

$$\begin{array}{rcrrrr} x_1 & = & 4 & -x_4 & -x_5 & -2x_6 \\ x_2 & = & 5 & -2x_4 & & -3x_6 \\ x_3 & = & 7 & -2x_4 & -x_5 & -3x_6 \\ z & = & & 3x_4 & +2x_5 & +4x_6 \end{array}$$

Following Anstee's rule, you need 3 pivots to get to optimality and computations are messier.

First x_6 is entering while x_2 is leaving the basis. The new dictionary is

$$\begin{array}{rcrrrr} x_1 & = & 2/3 & +1/3\,x_4 & -x_5 & +2/3\,x_2 \\ x_6 & = & 5/3 & -2/3\,x_4 & & -1/3\,x_2 \\ x_3 & = & 2 & & -x_5 & +x_2 \\ z & = & 20/3 & +1/3\,x_4 & +2x_5 & -4/3\,x_2 \end{array}$$

Still following Anstee's rule x_5 enters while x_1 leaves. It gives the following dictionary :

$$\begin{array}{rcllll} x_5 & = & 2/3 & +1/3\,x_4 & -x_1 & +2/3 x_2 \\ x_6 & = & 5/3 & -2/3\,x_4 & & -1/3\,x_2 \\ x_3 & = & 4/3 & -1/3\,x_4 & +x_1 & +1/3\,x_2 \\ z & = & 24/3 & +x_4 & -2x_1 & \end{array}$$

Only one choice for the entering variable : it is x_4. Then the leaving variable is x_6. The dictionary one gets is

$$\begin{array}{rcllll} x_5 & = & 3/2 & -1/2\,x_6 & -x_1 & +1/2\,x_2 \\ x_4 & = & 5/2 & -3/2\,x_6 & & -1/2\,x_2 \\ x_3 & = & 1/2 & +1/2\,x_6 & +x_1 & +1/2\,x_2 \\ z & = & 21/2 & -3/2\,x_6 & -2x_1 & -1/2\,x_2 \end{array}$$

All the variables in the z-row have negative coefficients hence optimality is reached. We set the non-basic variables to 0 to get the optimal solution :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 5/2 \\ 3/2 \\ 0 \end{pmatrix}$$

with the optimal value of the objective function being $z = 21/2$.

Third example

In this example the dictionary we start with is

$$\begin{array}{rcllll} x_5 & = & 5 & -x_1 & -2x_2 & -3x_3 & -x_4 \\ x_6 & = & 3 & -x_1 & -x_2 & -2x_3 & -3x_4 \\ z & = & & 5x_1 & +6x_2 & +9x_3 & +8x_4 \end{array}$$

If you apply Anstee's rule, 4 pivots are needed to reach optimality.

Following Anstee's rule, the entering variable is x_3 and one can see that the leaving one is x_6. The computations then lead to the dictionary

$$\begin{array}{rclllll} x_5 & = & 1/2 & +1/2\,x_1 & -1/2\,x_2 & +3/2\,x_6 & +7/2\,x_4 \\ x_3 & = & 3/2 & -1/2\,x_1 & -1/2\,x_2 & -1/2\,x_6 & -3/2\,x_4 \\ z & = & 27/2 & +1/2\,x_1 & +3/2\,x_2 & -9/2\,x_6 & -11/2\,x_4 \end{array}$$

Again choosing the variable with the largest coefficient in the z row, x_2 becomes the entering variable, and x_5 is the leaving one. After computations we end up with

$$\begin{array}{rclllll} x_2 & = & 1 & +x_1 & -2x_5 & +3x_6 & +7x_4 \\ x_3 & = & 1 & -x_1 & +x_5 & -2x_6 & -5x_4 \\ z & = & 15 & +2x_1 & -3x_5 & & +5x_4 \end{array}$$

Next step we choose x_4 as the entering variable while x_3 is leaving. This reads

$$\begin{array}{rclllll} x_2 & = & 12/5 & -2/5\,x_1 & -3/5\,x_5 & +1/5\,x_6 & -7/5\,x_3 \\ x_4 & = & 1/5 & -1/5\,x_1 & +1/5\,x_5 & -2/5\,x_6 & -1/5\,x_3 \\ z & = & 16 & +x_1 & -2x_5 & -2x_6 & -x_3 \end{array}$$

We are not done yet as x_1 must enter, hence x_4 must leave. It gives

$$\begin{array}{rclllll} x_2 & = & 2 & -2x_4 & -x_5 & +x_6 & -x_3 \\ x_1 & = & 1 & -5x_4 & +x_5 & -2x_6 & -x_3 \\ z & = & 17 & -5x_4 & -x_5 & -4x_6 & -2x_3 \end{array}$$

And this is the end as all the coefficients in the z-row are negative. We set the non-basic variables to 0 to get an optimal solution which is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with the optimal value of the objective function being $z = 17$.

As a side remark, in the third pivot if we choose x_1 as the entering variable and x_3 as the leaving one we reach optimality after this pivot. It illustrates the fact that Anstee's rule does not always minimize the number of pivot operations.