Degeneracy and its consequences

In short, degeneracy happens when some variables which are supposed to be non zero are equal to zero. Most of the time if you apply the rules word for word everything goes smoothly, but they can be minor catches.

Degenerate basic solutions

When in a dictionary, setting the non basic variables to 0 gives the value 0 to a basic variable, then by definition the solution you get is a **degenerate basic solution**. As an example, let's consider the dictionary

$$\begin{array}{rcrrrr} x_3 & = & 2 & -x_2 & +x_4 & -x_1 \\ x_5 & = & & +3x_2 & -x_4 & +2x_1 \\ z & = & -3 & +2x_2 & -x_4 & -3x_1 \end{array}$$

Then by setting the non basic variables to 0 you get the solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

In the solution above $x_5 = 0$ even though x_5 is a basic variable: this is a degenerate basic solution.

Note that in this particular example there is no problem to perform a pivot to optimality. Indeed one sees that x_2 enters while x_3 leaves and you get as a new dictionary

$$\begin{array}{rcrrrr} x_2 & = & 2 & -x_3 & +x_4 & -x_1 \\ x_5 & = & 6 & -3x_3 & +2x_4 & -x_1 \\ z & = & 1 & -2x_3 & +x_4 & -x_1 \end{array}$$

The next pivot with x_4 entering will tell you that the LP is unbounded.

Remark. How can you end up with a degenerate basic solution when pivoting? If it isn't the case in the initial dictionary, it means that at some point in a pivot you have a tie for the leaving variable[1]. Then the variable that could leave but doesn't will still be basic but set to 0 when setting non basic variables to 0 in the next dictionary.

Degenerate pivots

A **degenerate pivot** is a pivot in which the solution (obtained by setting the non basic variables to 0) does not change, in particular the value of z does not change either[2]. A degenerate pivot can only happen if you pivot

[1] In this case by convention choose the one with the smallest subscript

[2] This is equivalent: if in a pivot to optimality the value of z doesn't change then it is a degenerate pivot meaning that the whole solution doesnt' change.

from a dictionary with a degenerate basic solution. As an example[3], consider the dictionary

$$\begin{array}{rcrrrr} x_4 & = & 2 & +2x_5 & +2x_2 & -x_3 \\ x_1 & = & 1 & -x_5 & -x_2 & \\ x_6 & = & & +x_5 & +x_2 & -x_3 \\ z & = & 2 & -2x_5 & & +x_3 \end{array}$$

This dictionary leads to a degenerate solution as x_6 is set to 0 when the non basic variables are set to 0.

The only candidate for the entering variable is x_3. Now which variable is leaving? This is the one which is first driven to 0 when x_3 in increased from 0 to $+\infty$. We can write, setting x_5 and x_2 to 0,

$$\left\{ \begin{array}{rcrr} x_4 & = & 2 & -x_3 \\ x_1 & = & 1 & \\ x_6 & = & 0 & -x_3 \end{array} \right.$$

So x_6 is the first variable driven to 0! It is actually already 0 when x_3 is equal to 0. So x_3 increases from 0 to 0 but becomes basic, while x_6 leaves and becomes non basic. The pivot is still performed exactly in the same way then: we use the x_6-row to eliminate x_3 in the right hand side, and we end up with

$$\begin{array}{rcrrrr} x_4 & = & 2 & +x_5 & x_2 & +x_6 \\ x_1 & = & 1 & -x_5 & -x_2 & \\ x_3 & = & & x_5 & +x_2 & -x_6 \\ z & = & 2 & -x_5 & +x_2 & -x_6 \end{array}$$

The new dictionary still leads to a degenerate basic solution. Notice that in the two dictionaries, setting the non basic variables to 0 yields the same basic (degenerate) solution, that is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix},$$

and the value of z, that is 2 in this case, hasn't changed.

In summary, **in a degenerate pivot, you go from a degenerate basic solution to the same basic solution, not changing the value of z, changing only the dictionary** (that is which variables are labeled as basic and non basic).

After a degenerate pivot, you can still do a new pivot if optimality isn't reached! In this case, x_2 enters and x_1 leaves. This is no longer a degenerate pivot and we get

$$\begin{array}{rcrrrr} x_4 & = & 3 & & -x_1 & +x_6 \\ x_2 & = & 1 & -x_5 & -x_1 & \\ x_3 & = & 1 & & -x_1 & -x_6 \\ z & = & 3 & -2x_5 & -x_1 & -x_6 \end{array}$$

It's all good as optimality is reached and you can report the optimal solution.

Remark. From a theoretical point of view, the potential problem with degenerate pivots would be cycling. It means that by pivoting you visit a periodic sequence of dictionaries. For this to happen, as the value of z must be the same in all dictionaries, you have to visit only dictionaries associated to a single degenerate basic solution and perform only degenerate pivots. Some rules for choosing the entering and leaving variables in case of ties prevent that from happening.

[3]See also this pdf from where the example is taken: `https://www.math.ubc.ca/~anstee/math340/340degeneracy.pdf`

Degenerate dual solutions and non-uniqueness in the primal

Another way in which degeneracy can show up is in the z-row. As coefficients in front of slack variables are linked to the dual problem, it corresponds to a degeneracy for the solution of the dual problem. **What can happen is that in the final dictionary some coefficients in the z-row are equal to 0. This leads to non-uniqueness of optimal solutions to the primal problem**.

As an example let's take the dictionary

$$\begin{array}{rclrrr} x_4 & = & 2 & & +x_1 & -3x_6 \\ x_2 & = & 1 & -2x_5 & +3x_1 & \\ x_3 & = & 5 & -x_5 & +x_1 & +x_6 \\ z & = & 7 & & -3x_1 & -2x_6 \end{array}$$

As all the coefficients in the z-row are negative (or 0), **optimality is reached**. However the coefficient in front of x_5 in the z-row is equal to 0. Setting the non basic variables to 0 we get an optimal solution, that is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 5 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

We have taken $x_5 = 0$ but actually we can change x_5 (while keeping x_1 and x_6 to 0) and z won't change! Indeed, the reasoning is that

$$z = 7 - 3x_1 - 2x_6 \leqslant 7$$

as $x_1, x_6 \geqslant 0$ and there is equality if $x_1 = x_6 = 0$, but no constraint is imposed on x_5 with the z-row. So we can create more optimal solutions. What constraints do we have to impose on x_5? It has to be non-negative and the basic variables have to be non-negative: that is,

$$\left\{ \begin{array}{rclrcl} x_4 & = & 2 & & \geqslant & 0 \\ x_2 & = & 1 & -2x_5 & \geqslant & 0 \\ x_3 & = & 5 & -x_5 & \geqslant & 0 \end{array} \right. \qquad \text{and} \qquad x_5 \geqslant 0.$$

The most constraining inequality is the second one, added to the non-negativity we get $0 \leqslant x_5 \leqslant 1/2$. Then we can use the dictionary to build a solution once we choose the value of x_5. Namely, calling $x_5 = t$, we get that all the optimal solutions of the LP are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1-2t \\ 5-t \\ 2 \\ t \\ 0 \end{pmatrix} \qquad \text{for} \qquad t \in \left[0, \frac{1}{2}\right].$$

So in general having a coefficient equal to zero in the z-row of the final dictionary will lead to non-uniqueness of optimal solutions.

Remark. As an exercise, you can try to build a dictionary which is optimal, where at least one coefficient in the z-row is 0, but which is also coming from a degenerate basic solution, and such that the LP has a unique solution.