1. Give an example of a primal LP which is infeasible while simultaneously its dual LP is infeasible.

   Hint. Either work with the simplest possible LP. Or you can also take for the A matrix

   $$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

   and adjust $\mathbf{b}$ and $\mathbf{c}$ to get both primal and dual infeasibility.

   The simplest example (but which may look weird) would be to take one variable, one constraint, and a matrix A which is zero, that is

   $$\begin{array}{lrcl} \max & x_1 & & \\ & 0 \cdot x_1 & \leq & -1 \end{array} \qquad x_1 \geq 0.$$

   Indeed, you can check that both primal and dual are infeasible.

   To obtain a less degenerate primal/dual pair that are both infeasible, we can try making a primal whose dual has the same infeasible constraints. A 2×2 example suffices relying on the matrix A given in the hint is enough.

   $$\text{primal:} \quad \begin{array}{lrrcr} \max & -x_1 & +2x_2 & & \\ & x_1 & -x_2 & \leq & 1 \\ & -x_1 & +x_2 & \leq & -2 \\ & & \mathbf{x} \geq \mathbf{0} & & \end{array} \qquad \text{dual:} \quad \begin{array}{lrrcr} \min & y_1 & -2y_2 & & \\ & y_1 & -y_2 & \geq & -1 \\ & -y_1 & +y_2 & \geq & 2 \\ & & \mathbf{y} \geq \mathbf{0} & & \end{array}$$

   Indeed, summing the two constraints of the primal yields infeasibility, and similarly for the dual.

2. Let us consider our usual LP in standard inequality form $\max \mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ with n decision variables and m constraints. We introduce slack variables, that is we rewrite it

   $$\text{(Primal)} \qquad \begin{array}{lrcl} \max & \mathbf{c} \cdot \mathbf{x}_D & & \\ & (A|\mathrm{I}_m) \begin{pmatrix} \mathbf{x}_D \\ \mathbf{x}_S \end{pmatrix} & = & \mathbf{b} \end{array} \qquad \mathbf{x}_D, \mathbf{x}_S \geq \mathbf{0},$$

   with $\mathbf{x}_D$ the decision variables and $\mathbf{x}_S$ the slack variables, while I_m is the identity matrix. We assume that this LP has an optimal solution, we denote by $\mathbf{c}_B, \mathbf{c}_N, B, A_N$ the usual objects of the revised simplex formulas for an optimal dictionary of the LP.

   (1) Write the dual of the LP (Primal). What is the number of variables in the dual?

   If I write with matrix notations, I have

   $$\begin{array}{lrcl} \min & \mathbf{b} \cdot \mathbf{y} & & \\ & \begin{pmatrix} A^\top \\ \mathrm{I}_m \end{pmatrix} \mathbf{y} & \geq & \begin{pmatrix} \mathbf{c} \\ \mathbf{0} \end{pmatrix} \end{array} \qquad \mathbf{y} \text{ free},$$

   The dual variable $\mathbf{y}$ has dimension m, same as the number of constraints in (Primal). Actually, expanding the LP above we retrieve the usual dual of a LP in standard inequality form.

(2) Write the system that you obtain if in the dual you transform the constraints associated to the basic variables of the primal into equalities and you drop the other ones. Justify that this system is well posed.

Keeping only some constraints in the dual means choosing the rows of the matrix

$$\begin{pmatrix} A^\top \\ \mathrm{I}_m \end{pmatrix}$$

indexed by the basic variables of (Primal). That is we have to keep the columns of the matrix $(A|\mathrm{I}_m)$ indexed by the basic variables: it is nothing else the matrix B. The system we end up with is

$$B^\top \mathbf{y} = \mathbf{c}_B.$$

As B is invertible, so is $B^\top$ and the system is well posed (actually we get $\mathbf{y} = B^{-\top}\mathbf{c}_B$ which is an expression that we used for the proof of strong duality).

(3) Prove that the method that we learned to determine an optimal dual solution from an optimal primal one yields a well posed system if and only if the primal optimal solution is non degenerate (that is all the basic variables are strictly positive when non basic variables are set to 0).

If the primal optimal solution is non degenerate, complementary slackness will make us transform into equalities exactly the constraints indexed by basic variables in the dual. By (2), we know that this leads to a well posed system. On the other hand, if the primal optimal solution is degenerate, then the number of non zero variables is strictly less than m (the number of basic variables). Hence in the dual the number of constraints which are transformed into equalities is strictly less than m. We end up with a system with m unknowns and strictly less than m equations: this is an ill-posed system.

3. a) I used $d_1 d_2 d_3 d_4 d_5 = 54321$ and obtained

$$\begin{array}{llllll}
\max & c_1x_1 & +7x_2 & +7x_3 & +11.1x_4 & \\
 & 15x_1 & +4.5x_2 & +1.2x_3 & +9x_4 & \le 100 \\
 & 14x_1 & +4x_2 & +1x_3 & +8.2x_4 & \le 100 \\
 & 13x_1 & +3x_2 & +3x_3 & +3x_4 & \le 100 \\
 & 12x_1 & +2x_2 & +4x_3 & +x_4 & \le 100 \\
 & 11x_1 & +x_2 & +5x_3 & +x_4 & \le 100
\end{array} \qquad x_1, x_2, x_3, x_4 \ge 0$$

There were three intervals:

$$c_1 \in (-\infty, 31.244] \text{ gives solution } (0, 15.55, 16.66, 1.11) \text{ with } z = 237.88$$

$$c_1 \in [31.244, 87.5] \text{ gives solution } (6.149, 0, 6.472, 0) \text{ with } z = 6.149c_1 + 45.304$$

$$c_1 \in [87.5, +\infty) \text{ gives solution } (6.666, 0, 0, 0) \text{ with } z = 6.666c_1$$

You may wish to note that since we are only changing c_1 and we look at intervals in which the optimal basis is unchanged, then the optimal solution $B^{-1}\mathbf{b}$ stays unchanged in that interval. Thus the slope of the line in an interval is the value of x_1. The extent of the interval is found by checking the objective function ranging for the coefficient of x_1 (c_1) which gives the interval (around the current value of c_1) for which the current basis remains an optimal basis and so for which the solution remains unchanged.

One lucky (?) student had a degeneracy arise ($d_1d_2d_3d_4d_5 = 41411$) and so there were two different bases and two associated intervals for which the value of x_1 and the other variables were the same in both and hence in the graph, the intervals could be combined into one interval.

b) To show that the (piecewise linear) curve is concave upwards it suffices to show that for each interval where we have computed $z = x_1c_1 + \text{constant}$ for $a \le c_1 \le b$ then $z \ge x_1c_1 + \text{constant}$ for all values of c_1. This is easily seen to be true since the feasible solution which achieves $z = x_1c_1 + \text{constant}$ is a feasible solution regardless of c_1 and so we have a feasible solution to our LP of value $x_1c_1 + \text{constant}$ for all values of c_1 and hence the optimal value of the objective function is at least this big.

4.

a) Show there is an $\mathbf{x} \ge \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$ if and only if there is an $\mathbf{x} \ge \mathbf{0}$ with $A\mathbf{x} \le -\mathbf{1}$.

Note: we use the definition $(x_1, x_2, \ldots, x_n) < (y_1, y_2, \ldots, y_n)$ if and only if $x_1 < y_1$, $x_2 < y_2$, ... **and** $x_n < y_n$. This is the standard notation in matrix theory for matrix or vector inequalities. This may be contrary to your expectations. Mathematically speaking, the symbol $>$ would generally mean $\ge$ and $\ne$ but this is not true for matrices or vectors. A vector $\mathbf{x}$ might satisfy $x \ge 0$ and also $\mathbf{x} \ne \mathbf{0}$. If $x \ge 0$ and yet $\mathbf{x}$ has still has some 0 entries then $\mathbf{x} \not> \mathbf{0}$.

If there is an $\mathbf{x} \ge \mathbf{0}$ with $A\mathbf{x} \le -\mathbf{1}$, then that $\mathbf{x}$ satisfies with $A\mathbf{x} \le -\mathbf{1} < \mathbf{0}$.

If there is an $\mathbf{x} \ge \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$ then assume such an $\mathbf{x}$ exists with $A\mathbf{x} = (-a_1, -a_2, \ldots, -a_m)^T$. Let $a = \min\{a_1, a_2, a_3, \ldots, a_m\}$. Thus $a > 0$. Then $A(\frac{1}{a}\mathbf{x}) = (-a_1/a, -a_2/a, \ldots, -a_m/a)^T \le -\mathbf{1}$ and $\frac{1}{a}\mathbf{x} \ge \mathbf{0}$.

This result is particularly interesting to see a strict inequality appear in an LP.

b) We set up a primal dual pair.

$$\text{primal P:} \quad \begin{array}{cc} \max & \mathbf{0}\cdot\mathbf{x} \\ A\mathbf{x} & \le -\mathbf{1} \\ \mathbf{x} & \ge \mathbf{0} \end{array} \qquad \text{dual D:} \quad \begin{array}{cc} \min & -\mathbf{1}\cdot\mathbf{y} \\ A^T\mathbf{y} & \ge \mathbf{0} \\ \mathbf{y} & \ge \mathbf{0} \\ \mathbf{z} & \ge \mathbf{0} \end{array}$$

We have two statements:

i) there exists an $\mathbf{x} \ge \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$,

ii) there exists $\mathbf{y} \ge \mathbf{0}$, $\mathbf{z} \ge \mathbf{0}$ with $A^T\mathbf{y} \ge 0$ and $\mathbf{y} \ne \mathbf{0}$

Our primal P is bounded (value of objective function at most 0) and so there are two cases by Fundamental Theorem of LP.

Case 1. Assume that the primal is infeasible.

The dual is feasible ($\mathbf{y} = \mathbf{0}$ works) so by the Fundamental Theorem of Linear Programming, the dual is either unbounded or has an optimal solution. But if the dual has an optimal solution, then by Strong Duality, we deduce that the primal has an optimal solution $\mathbf{x}$ which is feasible which is a contradiction. Thus the dual is unbounded and so we can find a feasible $\mathbf{y}$ (i.e. $A^T\mathbf{y} \ge \mathbf{0}$) with $-\mathbf{1}\cdot\mathbf{y} < 0$ and so $y \ne \mathbf{0}$ and hence ii) holds.

The primal is infeasible and so by a) we have that there can be no $\mathbf{x} \ge \mathbf{0}$ with $A\mathbf{x} \le -\mathbf{1}$ which means by our argument in part a) that there is no $\mathbf{x} \ge \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$ and we conclude i) doesn't hold.

Case 2. Assume the primal has an optimal solution $\mathbf{x}^*$.

We apply a) again to note that if there is a feasible solution to the primal then there is an $\mathbf{x} \geq \mathbf{0}$ with $A\mathbf{x} < \mathbf{0}$. Thus i) holds

Also, by Weak Duality, any feasible solution to the dual (i.e. any $\mathbf{y}$ with $A^T\mathbf{y} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}$) has $-\mathbf{1} \cdot \mathbf{y} \geq \mathbf{0} \cdot \mathbf{x}^* = 0$. But $-\mathbf{1} \cdot \mathbf{y} \geq 0$ and $\mathbf{y} \geq \mathbf{0}$ implies that $\mathbf{y} = \mathbf{0}$ and so ii) doesn't hold.

Since Case 1 and 2 exhaust all possibilities we know that either i) or ii) holds but not both.

5. (from an old exam) We seek a minimum cost diet selected from the following three foods.

	food 1	food 2	food 3
vitamins/100gms	13.23	18.4	36
calories/100gms	100	125	139
minimum (100gms)	10	10	8
cost $/100gm	3.00	5.00	8.00

We require a diet that has at least 760 units of vitamins and at least 3500 calories. The minimums are stated in units of 100gms. We let the variable food$i$ refer to the amount of food i purchased in units of 100gms.

a) There is a special on food 2 reducing the price to \$4.10/100gms. This would not change the your purchase strategy because the 'current basis remains optimal' and so the current basic feasible solution $B^{-1}\mathbf{b}$ remains fixed even with a drop of \$.91/100 gms. A price reduction to \$3.10 falls outside the range and one imagines the purchase strategy would change

b) The marginal cost of 10 units of vitamins is $10 \times .2222$ which is \$2.22. The chosen diet as a whole has 760 units of vitamins and costs \$178 and so the dollar cost of the whole diet per 10 units of vitamins obtained is $\$178/76 = \$2.34$. The marginal cost is cheaper (which is hardly surprising since the diet also has other constraints).

c) Integrality can be important in diet problems such as this if the foods come in integer amounts (e.g. apples) but many foods are available in continuous amounts (bulk food bins) and anyway can be divided at home into appropriate amounts. I expected some discussion of the possibility that the variables for the foods were integral.

d) A linear inequality that expresses the requirement that at least 20% of the weight of the purchased diet comes from food 2 is

$$food2 \geq .2(food1 + food2 + food3)$$

which in LINDO input would become $.2food1 - .8food2 + .2food3 \leq 0$.