Math 340 Assignment #2 Due Friday March 6, 2020 at beginning of class. Richard Anstee

(two pages)

1. Give an example of a Linear Program for which an optimal dictionary has a z row with zero coefficients for at least one non basic variable and yet the optimal solution is unique.

2. Consider the LP

$$\begin{array}{lrrcr} \text{maximize} & -7x_1 & +8x_2 & & \\ \text{subject to} & 2x_1 & +x_2 & \leq & 5 \\ & x_1 & +2x_2 & \leq & 4 \\ & 3x_1 & +3x_2 & \leq & 27 \end{array} \qquad x_1, x_2 \geq 0$$

Explain, without solving the LP, that any optimal solution $(y_1^*, y_2^*, y_3^*)^T$ of the dual problem must satisfy $y_3^* = 0$. Deduce what happens to the optimal value of the primal problem if we replace 27 in the third constraint by 29.

3. Theorem 5.5 is taken from page 65-66 of V. Chvátal's book on Linear Programming. Consider the LP:

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1, 2, \ldots m) \quad (5.24) \\ & x_j \geq 0 & (j = 1, 2, \ldots, n) \end{array}$$

Theorem 5.5. If (5.24) has at least one non-degenerate basic feasible optimal solution, then there is a positive ϵ with the property: If $|t_i| \leq \epsilon$ for all $i = 1, 2, \ldots, m$, then the problem

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i + t_i & (i = 1, 2, \ldots m) \quad (5.25) \\ & x_j \geq 0 & (j = 1, 2, \ldots, n) \end{array}$$

has an optimal solution and its optimal value equals

$z^* + \sum_{i=1}^{m} y_i^* t_i$

with z^* standing for the optimal value of (5.24) and with $y_1^*, y_2^*, \ldots, y_m^*$ standing for the optimal solution of its dual.

Now consider the following LP

$$\begin{array}{lrrrl} \max & 12x_1 & +20x_2 & +18x_3 & \\ & 4x_1 & +6x_2 & +8x_3 & \leq 600 \\ & x_1 & +(7/2)x_2 & +2x_3 & \leq 300 \\ & 2x_1 & +4x_2 & +3x_3 & \leq 550 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

The final dictionary is:

$$\begin{array}{rclllll} x_1 & = & 75/2 & -2x_3 & -(7/16)x_4 & +(3/4)x_5 \\ x_2 & = & 75 & & +(1/8)x_4 & -(1/2)x_5 \\ x_6 & = & 175 & +x_3 & +(3/8)x_4 & +(1/2)x_5 \\ z & = & 1950 & -6x_3 & -(11/4)x_4 & -x_5 \end{array}$$

optimal basis $\{x_1, x_2, x_6\}$

$$B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 7/16 & -3/4 & 0 \\ -1/8 & 1/2 & 0 \\ -3/8 & -1/2 & 1 \end{pmatrix} \end{array}$$

Does Theorem 5.5 apply here?

For what value of ϵ (choose the largest possible for this particular data!) is the theorem true? In this case I will offer you an hint about computing ϵ. Hint: If you have been given an inequality $3t_1 - 5t_2 \leq 5$ which must hold for any choices t_1, t_2 satisfying $-\epsilon \leq t_1 \leq \epsilon$ and $-\epsilon \leq t_2 \leq \epsilon$, then we deduce that $\epsilon \leq 5/8$ (corresponds to choosing $t_1 = 5/8$ and $t_2 = -5/8$). Tricky inequalities.

4. Consider our standard LP: $\max \mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Assume every entry of A is strictly positive and $\mathbf{b} \geq \mathbf{0}$. Deduce that the LP has an optimal solution.

5. Consider our two phase method in the case that the LP is infeasible. We begin with the primal LP

$$\begin{array}{ll} \max & z \\ & A\mathbf{x} \leq \mathbf{b}\,. \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

We introduce an artificial variable x_0 and obtain the new LP (in Phase 1)

$$\begin{array}{ll} \max & -x_0 \\ & [-\mathbf{1}\,|\,A]\begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} \leq \mathbf{b}\,, \\ & x_0 \geq 0, \mathbf{x} \geq \mathbf{0}, \end{array}$$

where $(-\mathbf{1})$ is the vector of -1's and we use $[-\mathbf{1}\,|\,A]$ to denote the matrix obtained from A by adding one more column (of all -1's) on the left. We assume the maximum value of the objective function in the new LP is strictly negative. (In general this would depend on whether the original LP is infeasible; we assume the original LP is infeasible)

a) Derive the dual of the new LP. Verify that an optimal dual solution will verify (certify?) that the inequalities above do not have any feasible solution. Namely explain why

$$\mathbf{y} \geq \mathbf{0}\,, \quad \mathbf{y}^T A \geq \mathbf{0}^T\,, \quad \mathbf{y}^T\mathbf{b} < 0$$

will show that there is no feasible solution to the original LP.

b) Verify that for an optimal dual solution (the magic coefficients!) $\mathbf{y} = (y_1, y_2, \ldots, y_m)^T$ we have $y_1 + y_2 + \cdots + y_m = 1$. (You may note that in assignment 1, the sum was $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ which you now see was no coincidence).

6. Extend the standard theorem of the alternative as follows. Let A be an $m \times n$ matrix and $\mathbf{b}$ be an $m \times 1$ vector. Let $\mathbf{u}$ be an $n \times 1$ vector (of upper bounds). Prove that either:

i) there exists an $\mathbf{x}$ with $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \leq \mathbf{u}$

or

ii) there exists vectors $\mathbf{y}, \mathbf{z}$ with $A^T\mathbf{y} + \mathbf{z} = \mathbf{0}, \quad \mathbf{y}, \mathbf{z} \geq \mathbf{0}, \quad \mathbf{b} \cdot \mathbf{y} + \mathbf{u} \cdot \mathbf{z} < 0$

but not both.

(You should use without proving them again the rules for forming a dual as introduced in Assignment 1, Question 2 to handle free variables and equalities).