**Math 340** Assignment #2 Due Friday Mar 6, 2020 at beginning of class. Richard Anstee (two pages)

1. Give an example of a Linear Program for which an optimal dictionary has a $z$ row with zero coefficients for at least one non basic variable and yet the optimal solution is unique.

   When thinking of this problem you realize that the non basic variable with 0 coefficient in the $z$ row should not be allowed to increase. That it, a basic variable must be driven to 0 as the non basic variable increases. For instance we can take the dictionary

$$\begin{array}{rcrrr} x_3 & = & 0 & -x_1 & -2x_2 \\ x_4 & = & 3 & +2x_1 & +5x_2 \\ z & = & & -x_1 & \end{array} \qquad x_1, x_2, x_3, x_4 \geq 0$$

   We have a zero coefficient for $x_2$ in the $z$ row. We know that $(x_1\ x_2) = (0\ 0)$ is an optimal solution with optimal value $z = 0$. Moreover, for $z = -x_1$ to be 0 we need $x_1 = 0$. But once $x_1 = 0$, the $x_3$ row yields that $x_3 = -2x_2 \geq 0$, that is $x_2 \leq 0$. As we already know $x_2 \geq 0$, we conclude that $x_2 = 0$ at optimality. Hence $(x_1\ x_2) = (0\ 0)$ is the unique optimal solution.

   This dictionary comes for instance from the LP

$$\begin{array}{lrrl} \max & -x_1 & & \\ & x_1 & +2x_2 & \leq 0 \\ & -2x_1 & -5x_2 & \leq 3 \end{array} \qquad x_1, x_2 \geq 0$$

   as this is the one we obtain when introducing the slack variables. Note that the second constraint is superfluous and is here for decorative purposes.

2. Consider the LP

$$\begin{array}{lrrcr} \text{maximize} & -7x_1 & +8x_2 & & \\ \text{subject to} & 2x_1 & +x_2 & \leq & 5 \\ & x_1 & +2x_2 & \leq & 4 \\ & 3x_1 & +3x_2 & \leq & 27 \end{array} \qquad x_1, x_2 \geq 0$$

   Explain, without solving the LP, that any optimal solution $(y_1^*, y_2^*, y_3^*)^T$ of the dual problem must satisfy $y_3^* = 0$. Deduce what happens to the optimal value of the primal problem if we replace 27 in the third constraint by 29.

   We note that adding the first two inequalities results in the inequality $3x_1 + 3x_2 \leq 9$ and (note that $9 < 27$), we must have a primal slack of a at least 18 in the third constraint. Thus by Complementary slackness, any optimal solution to dual has $y_3 = 0$. Of course the same is true for 27 replaced by 29. In particular, there will be no change in the value of the objective function for the primal at optimality. This corresponds to the third resource as given by the third constraint having a marginal value of 0.

3. Theorem 5.5 is taken from page 65-66 of V. Chvátal's book on Linear Programming. Consider the LP:

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1, 2, \ldots m) \qquad (5.24) \\ & x_j \geq 0 & (j = 1, 2, \ldots, n) \end{array}$$

**Theorem 5.5**. If (5.24) has at least one non-degenerate basic feasible optimal solution, then there is a positive $\epsilon$ with the property: If $|t_i| \le \epsilon$ for all $i = 1, 2, \ldots, m$, then the problem

$$\begin{array}{lll} \text{maximize} & \sum_{j=1}^{n} c_j x_j & \\ \text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \le b_i + t_i & (i = 1, 2, \ldots m) \\ & x_j \ge 0 & (j = 1, 2, \ldots, n) \end{array} \qquad (5.25)$$

has an optimal solution and its optimal value equals

$z^* + \sum_{i=1}^{m} y_i^* t_i$

with $z^*$ standing for the optimal value of (5.24) and with $y_1^*, y_2^*, \ldots, y_m^*$ standing for the optimal solution of its dual.

Now consider the following LP

$$\begin{array}{lrrrl} \max & 12x_1 & +20x_2 & +18x_3 & \\ & 4x_1 & +6x_2 & +8x_3 & \le 600 \\ & x_1 & +(7/2)x_2 & +2x_3 & \le 300 \\ & 2x_1 & +4x_2 & +3x_3 & \le 550 \end{array} \qquad x_1, x_2, x_3 \ge 0$$

The final dictionary is:

$$\begin{array}{rcrrrr} x_1 & = & 75/2 & -2x_3 & -(7/16)x_4 & +(3/4)x_5 \\ x_2 & = & 75 & & +(1/8)x_4 & -(1/2)x_5 \\ x_6 & = & 175 & +x_3 & +(3/8)x_4 & +(1/2)x_5 \\ z & = & 1950 & -6x_3 & -(11/4)x_4 & -x_5 \end{array}$$

optimal basis $\{x_1, x_2, x_6\}$

$$B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 7/16 & -3/4 & 0 \\ -1/8 & 1/2 & 0 \\ -3/8 & -1/2 & 1 \end{pmatrix} \end{array}$$

Theorem 5.5 applies here because the current basic feasible solution is non degenerate. With $\Delta\mathbf{b} = (t_1, t_2, \ldots, t_3)^T$, the conclusions of Therem 5.5 are valid if $B^{-1}(\mathbf{b} + \Delta\mathbf{b}) \ge \mathbf{0}$. Thus we need

$$B^{-1}(\mathbf{b} + \Delta\mathbf{b}) = \begin{bmatrix} 7/16 & -3/4 & 0 \\ -1/8 & 1/2 & 0 \\ -3/8 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 600 + t_1 \\ 300 + t_2 \\ 550 + t_3 \end{bmatrix} = \begin{bmatrix} 75/2 + (7/16)t_1 - (3/4)t_2 \\ 75 - (1/8)t_1 + (1/2)t_2 \\ 175 - (3/8)t_1 - (1/2)t_2 + t_3 \end{bmatrix} \ge \mathbf{0}.$$

We must choose $\epsilon$ so that for all choices for each $t_i$ satisfying $-\epsilon \le t_i \le \epsilon$, the inequalities are true. From an inequality of the form $a + a_1t_1 + a_2t_2 + a_3t_3 \ge 0$ with $a \ge 0$, we have form $a_1t_1 + a_2t_2 + a_3t_3 \ge -a$. Now using $-\epsilon \le t_i \le \epsilon$, we have $a_1t_1 + a_2t_2 + a_3t_3 \ge -(|a_1| + |a_2| + |a_3|)\epsilon$ which can be achieved by appropriate choices of $t_1, t_2, t_3$ (e.g. if $a_1 < 0$ take $t_1 = \epsilon$ and if $a_1 \ge 0$ take $t_1 = -\epsilon$). Thus $a_1t_1 + a_2t_2 + a_3t_3 \ge -(|a_1| + |a_2| + |a_3|)\epsilon \ge -a$ and so $\epsilon \le \frac{a}{|a_1|+|a_2|+|a_3|}$. From the first inequality $75/2 + (7/16)t_1 - (3/4)t_2 \ge 0$ we deduce that the worst case would be to have $t_1 = -\epsilon$, $t_2 = \epsilon$ and then deduce that $-((7/16) + (3/4))\epsilon \ge -75/2$ and hence $\epsilon \le 600/19$. From the second inequality $75 - (1/8)t_1 + (1/2)t_2 \ge 0$ we deduce that $\epsilon \le \frac{600}{5}$. From the third inequality $175 - (3/8)t_1 - (1/2)t_2 + t_3 \ge 0$ we deduce that $\epsilon \le \frac{280}{3}$. Thus the largest possible $\epsilon$ for which if $|t_i| \le \epsilon$ for all $i = 1, 2, 3$, then the conclusions of Theorem 5.5 hold, is to take $\epsilon = \frac{600}{19}$.

4. Consider our standard LP: $\max \mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$. Assume every entry of $A$ is strictly positive and $\mathbf{b} \ge \mathbf{0}$. Deduce that the LP has an optimal solution.

The primal is feasible since $\mathbf{x} = \mathbf{0}$ works. There are two approaches to this problem either show the Primal is not unbounded or show the dual is feasible. Let me do the first approach. Consider just the first row of $A$ which results in the inequality $a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$. Given the positivity of the terms we have $a_{1,j}x_j \leq b_1$ for all $j = 1, 2, \ldots, n$. If $b_1 = 0$, then $x_j = 0$. If $b_1 > 0$, then $x_j \leq b_1/a_{1,j}$. But then the objective function is bounded by $\mathbf{c} \cdot \mathbf{x} = \sum_{i=1}^{n} c_j x_j \leq \sum_{i=1}^{n} |c_j| b_1/a_{1,j}$. So the primal is not unbounded. By the Fundamental Theorem of Linear Programming, the primal has an optimal solution.

Note we don't need all the hypotheses.

5. Consider our two phase method in the case that the LP is infeasible. We begin with the primal LP

$$\begin{array}{cc} \max & z \\ & A\mathbf{x} \leq \mathbf{b} \,. \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

We introduce an artificial variable $x_0$ and give the new LP (in Phase 1)

$$\begin{array}{cc} \max & -x_0 \\ & [-\mathbf{1} \,|\, A] \begin{bmatrix} x_0 \\ \mathbf{x} \end{bmatrix} \leq \mathbf{b} \,, \\ & \mathbf{x} \geq \mathbf{0}, x_0 \geq 0 \end{array}$$

a) Its dual is

$$\begin{array}{cc} \min & \mathbf{b} \cdot \mathbf{y} \\ & (-\mathbf{1})^T \mathbf{y} \geq -1 \\ & A^T \mathbf{y} \geq \mathbf{0} \\ & \mathbf{y} \geq \mathbf{0} \end{array} ,$$

We have assumed the maximum value of the objective function in the new LP is strictly negative and so by Strong Duality, the objective function in the dual is strictly negative. Hence for an optimal dual solution $\mathbf{y}^*$ we have $\mathbf{b} \cdot \mathbf{y}^* < 0$ and $\mathbf{y}^*$ feasible and hence $A^T\mathbf{y}^* \geq \mathbf{0}$ and $\mathbf{y}^* \geq \mathbf{0}$. These conditions yield that if we take the sum of $y_i^*$ times the $i$th inequality of the primal for $i = 1, 2, \ldots, m$, then we obtain a new inequality (the $y_i^* \geq 0$ ensure no inequalities flip) whose coefficients for each $x_j$ are all positive (because $A^T\mathbf{y}^* \geq \mathbf{0}$) and whose right hand side is strictly negative ($\mathbf{b} \cdot \mathbf{y}^* < 0$) which shows that the inequalities have no feasible solution.

b) An optimal primal solution for our new LP has $x_0 > 0$ and hence by Complementary Slackness we deduce that for any optimal dual solution $\mathbf{y}^*$ we have that the inequality $(-\mathbf{1})^T\mathbf{y} \geq -1$ is an equality $(-\mathbf{1})^T\mathbf{y}^* = -1$ which for $\mathbf{y}^* = (y_1^*, y_2^*, \ldots, y_m^*)^T$ becomes $y_1^* + y_2^* + \cdots + y_m^* = 1$.

6. Extend the standard theorem of the alternative as follows. Let $A$ be an $m \times n$ matrix and $\mathbf{b}$ be an $m \times 1$ vector. Let $\mathbf{u}$ be an $n \times 1$ vector (of upper bounds). Prove that either:

i) there exists an $\mathbf{x}$ with $A\mathbf{x} \leq \mathbf{b}, \mathbf{x} \leq \mathbf{u}$

or

ii) there exists vectors $\mathbf{y}, \mathbf{z}$ with $A^T\mathbf{y} + \mathbf{z} = \mathbf{0}, \quad \mathbf{y}, \mathbf{z} \geq \mathbf{0}, \quad \mathbf{b} \cdot \mathbf{y} + \mathbf{u} \cdot \mathbf{z} < 0$

but not both.

Let $A$ be an $m \times n$ matrix and let $\mathbf{u}$ be an $n \times 1$ vector (of upper bounds). We wish to show that either:

i) there exists an $\mathbf{x}$ with $A\mathbf{x} \le \mathbf{b}, \mathbf{x} \le \mathbf{u}$

or

ii) there exists vectors $\mathbf{y}, \mathbf{z}$ with $A^T\mathbf{y} + \mathbf{z} = 0, \quad \mathbf{y}, \mathbf{z} \ge 0, \quad \mathbf{b} \cdot \mathbf{y} + \mathbf{u} \cdot \mathbf{z} < 0$

but not both.

**Proof**: Consider the following primal dual pair of Linear Programs:

$$\text{primal:} \quad \begin{array}{cccc} \max & \mathbf{0} \cdot \mathbf{x} & & \\ & A\mathbf{x} & \le & \mathbf{b}\,, \\ & \mathbf{x} & \le & \mathbf{u} \end{array} \qquad \text{dual:} \quad \begin{array}{cccc} \min & & \mathbf{b} \cdot \mathbf{y} + \mathbf{u} \cdot \mathbf{z} & \\ & A^T\mathbf{y} + I\mathbf{z} & = & \mathbf{0}\,. \\ & \mathbf{y} \ge \mathbf{0} & \mathbf{z} \ge \mathbf{0} & \end{array}$$

We note that $\mathbf{y} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$ yields a feasible solution to the Dual (these vectors have $m$ and $n$ coordinates respectively). By the Fundamental Theorem of Linear Programming, we deduce that the dual either has an optimal solution or is unbounded.

CASE 1: Dual has an optimal solution $\mathbf{y}^*, \mathbf{z}^*$.

Thus, by Strong Duality, the primal has an optimal solution $\mathbf{x}^*$ which is a feasible solution and hence i) holds. Also $0 = \mathbf{0} \cdot \mathbf{x}^* = \mathbf{b} \cdot \mathbf{y}^* + \mathbf{u} \cdot \mathbf{z}^*$ which means by Weak Duality that every feasible solution $\mathbf{y}, \mathbf{z}$ to the dual has $0 \le \mathbf{b} \cdot \mathbf{y} + \mathbf{u} \cdot \mathbf{z}$ and hence ii) does not hold

CASE 2: Dual is unbounded.

Thus there is a feasible solution $\mathbf{y}, \mathbf{z}$ to the dual has $\mathbf{b} \cdot \mathbf{y} + \mathbf{u} \cdot \mathbf{z} \le -1 < 0$ and hence ii) holds. The dual being unbounded implies, using Weak Duality, that the primal has no feasible solution and hence i) does not hold.

Thus we have established the theorem in both cases.