Linear Programming in Game Theory – Part III

Examples

Summary

A two player zero sum game is given by a payoff matrix A of size $m \times n$.

Player 1 chooses a row, Player 2 chooses a column. The payoff of Player 1 is the coefficient at the intersection the chosen row and the chosen column. The payoff of Player 2 is the opposite to the payoff of Player 1.

A mixed strategy for Player 1 is a vector $\mathbf{x} \in \mathbb{R}^m$ with $\mathbf{x} \geqslant \mathbf{0}$ and $x_1 + x_2 + \ldots + x_m = 1$. The value x_i is the probability for Player 1 to follow strategy i.

If Player 1 follows mixed strategy $\mathbf{x}$ and Player 2 follows mixed strategy $\mathbf{y}$, then the expected payoff of Player 1 is $\mathbf{x}^\top A\mathbf{y}$.

Summary (continued)

Player 1 solves the LP

$$
\begin{array}{llcccl}
\text{maximize} & & +z & & & \\
\text{s.t.} & -A^\top \mathbf{x} & +z\mathbf{1} & \leqslant & \mathbf{0} & \quad \mathbf{x} \geqslant \mathbf{0},\ z \text{ free} \\
 & \mathbf{1}^\top \mathbf{x} & & = & 1 &
\end{array}
$$

while Player 2 solves the dual LP

$$
\begin{array}{llcccl}
\text{minimize} & & +w & & & \\
\text{s.t.} & -A\mathbf{y} & +w\mathbf{1} & \geqslant & \mathbf{0} & \quad \mathbf{y} \geqslant \mathbf{0},\ w \text{ free} \\
 & \mathbf{1}^\top \mathbf{y} & & = & 1 &
\end{array}
$$

If $(\mathbf{x}^*, z^*)$ and $(\mathbf{y}^*, w^*)$ are optimal solutions of the primal and dual respectively, then it is in Player 1 best interest to play $\mathbf{x}^*$ and Player 2 best interest to play $\mathbf{y}^*$. If they both do that, the expected payoff of Player 1 is $z^* = w^* = {\mathbf{x}^*}^\top A\mathbf{y}^*$. It is called the value of the game, denoted by $v(A)$.

Questions one can ask

- Modeling questions: from the description of the game build the payoff matrix and then interpret the results.
- LP questions: any thing that we have seen on Linear Programming can be asked in the framework of Game Theory. Indeed, any question on the optimal strategy of a Player is a question about an optimal solution of a LP.

1. Game of Morra[1]

2. Political strategy[2]

[1] Chapter 15 of the textbook.

[2] Inspired from: `https://nordstromjf.github.io/IntroGameTheory/S_IntroZeroSum.html`

1. Game of Morra[a]

[a] Chapter 15 of the textbook.

Description

Two players called Roberta (the row player) and Charles (the column player). Each of them hides one or two coins and try to guess the number of coins of the other player. They announce simultaneously their guess.

If one and only one of them is correct, then the player with the wrong guess has to give the other player a sum equal to the total number of coins hidden. If not (if both are correct or both are wrong) then it is draw and the payoff for each is zero.

This is a symmetric game (the role of Roberta and Charles are interchangeable), we already know that the value is 0: if both play perfectly (that is they play their optimal strategies) the the expected payoff is 0.

Strategies

How to describe strategies for each of the player? Let's look at Roberta, she has four choices:

[1,1] Hide one, Guess one.
[2,1] Hide one, Guess two.
[1,2] Hide two, Guess one.
[2,2] Hide two, Guess two.

Similarly for Charles.

How to fill the payoff matrix? Just follow the rules of the game! For instance

- If Roberta plays [1,2] (hide one, guess two) while Charles plays [2,1] (hide two, guess one) they are both correct: it is a draw hence the payoff is 0.
- If Roberta plays [1,2] (hide one, guess two) while Charles plays [2,2] (hide two, guess two), only Roberta has made a correct guess. She wins the total number of coins hidden, that is $2 + 1 = 3$

The payoff matrix

It's tedious but the payoff matrix can be built. Its rows and columns are indexed by the strategies of the players.

$$
\begin{array}{c@{\;}c}
 & \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\
\begin{array}{c} [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \end{array} &
\begin{pmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{pmatrix}
\end{array}
$$

Optimal strategies

To compute optimal strategies, Roberta must solve a LP with 5 decisions variables and 5 constraints. It reads as follows (beware that the matrix $-A^\top$ appears):

$$
\begin{array}{lrrrrrcl}
\max & & & & & z & & \\
\text{s.t.} & & 2x_2 & -3x_3 & & +z & \leqslant & 0 \\
& -2x_1 & & & +3x_4 & +z & \leqslant & 0 \\
& 3x_1 & & & -4x_4 & +z & \leqslant & 0 \\
& & -3x_2 & +4x_3 & & +z & \leqslant & 0 \\
& x_1 & +x_2 & +x_3 & +x_4 & & = & 1.
\end{array}
\qquad x_1, x_2, x_3, x_4 \geqslant 0,\ z \text{ free}
$$

If I use LINDO, I find as a solution

$$\mathbf{x}^* = \begin{pmatrix} 0 & 3/5 & 2/5 & 0 \end{pmatrix}^\top$$

and $z^* = 0$ (as expected because the game is symmetric).

Optimal strategies (continued)

The optimal strategy is playing a little bit of [1,2] (hide one, guess two) and [2,1] (hide two, guess one).

Actually the LP has not a unique solution. See on Anstee's website for the use of complementary slackness to get all optimal solutions. In all optimal strategies $\mathbf{x}$, there holds $x_1 = 0$ that is Roberta never plays [1,1].

As the game is symmetric Charles solves exactly the same LP and its optimal strategies are the same.

A variant

Now let's look at the following variant: Charles announces first its guess, and then Roberta does. In particular, Roberta can choose her guess depending what Charles has just announced. She has 4 additional strategies:

[1,S] Hide one, guess the same as Charles.

[1,O] Hide two, guess the opposite to Charles.

[2,S] Hide two, guess the same as Charles.

[2,O] Hide two, guess the opposite to Charles.

She has now more strategies: she can get at least as much as before, maybe more. In other words, we know that the value of the game is non negative. The game is no longer symmetric.

New payoff matrix

Even more tedious to write: 8 rows, 4 columns.

$$
\begin{array}{c c}
 & \begin{array}{cccc} [1,1] & [1,2] & [2,1] & [2,2] \end{array} \\
\begin{array}{c} [1,1] \\ [1,2] \\ [2,1] \\ [2,2] \\ [1,S] \\ [1,O] \\ [2,S] \\ [2,O] \end{array} &
\begin{pmatrix}
0 & 2 & -3 & 0 \\
-2 & 0 & 0 & 3 \\
3 & 0 & 0 & -4 \\
0 & -3 & 4 & 0 \\
0 & 0 & -3 & 3 \\
-2 & 2 & 0 & 0 \\
3 & -3 & 0 & 0 \\
0 & 0 & 4 & -4
\end{pmatrix}
\end{array}
$$

New LP solved by Roberta

Now 9 decisions variables, still 5 constraints.

$$
\begin{array}{lrrrrrrrrrcl}
\max & & & & & & & & & z & & \\
\text{s.t.} & & 2x_2 & -3x_3 & & & +2x_6 & -3x_7 & & +z & \leqslant & 0 \\
& -2x_1 & & & +3x_4 & & -2x_6 & +3x_7 & & +z & \leqslant & 0 \\
& 3x_1 & & & -4x_4 & +3x_5 & & & -4x_8 & +z & \leqslant & 0 \\
& & -3x_2 & +4x_3 & & -3x_5 & & & +4x_8 & +z & \leqslant & 0 \\
& x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & +x_8 & & = & 1.
\end{array}
$$

with $x_1, x_2, \ldots, x_8 \geqslant 0$ and z free.

Again with LINDO, new optimal solution

$$\mathbf{x}^* = \begin{pmatrix} 0 & 56/99 & 40/99 & 0 & 0 & 2/99 & 0 & 1/99 \end{pmatrix}^\top$$

and $z^* = 4/99$.

Comments

The value of the game is now $4/99 > 0$: Roberta has an advantage! Her optimal strategy is almost as the previous one except playing from time to time [1,O] and [2,O] (that is guessing the opposite to what Charles just said). Now the solution of the LP is unique.

One can find the optimal strategy of Charles either y solving the dual problem or looking at complementary slackness. The answer is

$$\mathbf{y}^* = \begin{pmatrix} 28/99 & 30/99 & 21/99 & 20/99 \end{pmatrix}^\top$$

The strategy of Charles is involved but if he doesn't follow it then Roberta will make even more money.

I hope you appreciate the power of Linear Programming to solve such questions. As you can see, optimal strategies are not easy to guess.

A more complicated variant

Just some suggestions, way more complicated than what you will be asked on assessments.

Question

What happens if you allow for each Player to hide and guess up to p coins? We stick to the symmetric case (Roberta and Charles announce simultaneously).

- The game is still symmetric so the value is still 0.
- The number of strategies will be p^2.
- You could write some code to compute the optimal strategy.
- Is there an analytic expression for the optimal strategy? (I have no idea.)
- And if Charles announces before Roberta? Is there an expression of how much the advantage of Roberta is? (I have no idea.)

2. Political strategy[a]

[a] Inspired from: https://nordstromjf.github.io/IntroGameTheory/S_IntroZeroSum.html

Description

Two politicians are trying to win an election. They have to decide what to say about some political issue: Approve, Disapprove or stay Neutral.

The payoff of Player 1 is its share (in percent) of the voters she/he gets in the next election. The payoff matrix is (the values are completely made up)

$$\begin{array}{cc} & \begin{array}{ccc} A & D & N \end{array} \\ \begin{array}{c} A \\ D \\ N \end{array} & \begin{pmatrix} 55 & 45 & 60 \\ 40 & 60 & 50 \\ 40 & 10 & 30 \end{pmatrix} \end{array}$$

Optimal strategy

The LP for the best strategy for Player 1 is

$$\begin{array}{lrrrrcl} \max & & & & z & & \\ \text{s.t.} & -55x_1 & -40x_2 & -40x_3 & +z & \leqslant & 0 \\ & -45x_1 & -60x_2 & -10x_3 & +z & \leqslant & 0 \\ & -60x_1 & -50x_2 & -30x_3 & +z & \leqslant & 0 \\ & x_1 & +x_2 & +x_3 & & = & 1. \end{array}$$

If I use LINDO, I find a unique optimal solution

$$\mathbf{x}^* = \begin{pmatrix} 2/3 & 1/3 & 0 \end{pmatrix}^\top$$

and $z^* = 50$.

Comments

Here the payoff of Player 1 should rather be its percentage at the election minus 50 if we really want to keep the idea of "zero-sum". Here as the value of the game is $z^* = 50$, we could say that the "game" is fair.

What does a mixed strategy mean in this case? It's less clear if the election is played only once. Maybe Game Theory is not such a useful in this case.

Of course, I'm not sure that the stand one takes on a political issue should be driven by a Game Theory analysis.