The Two Phase method

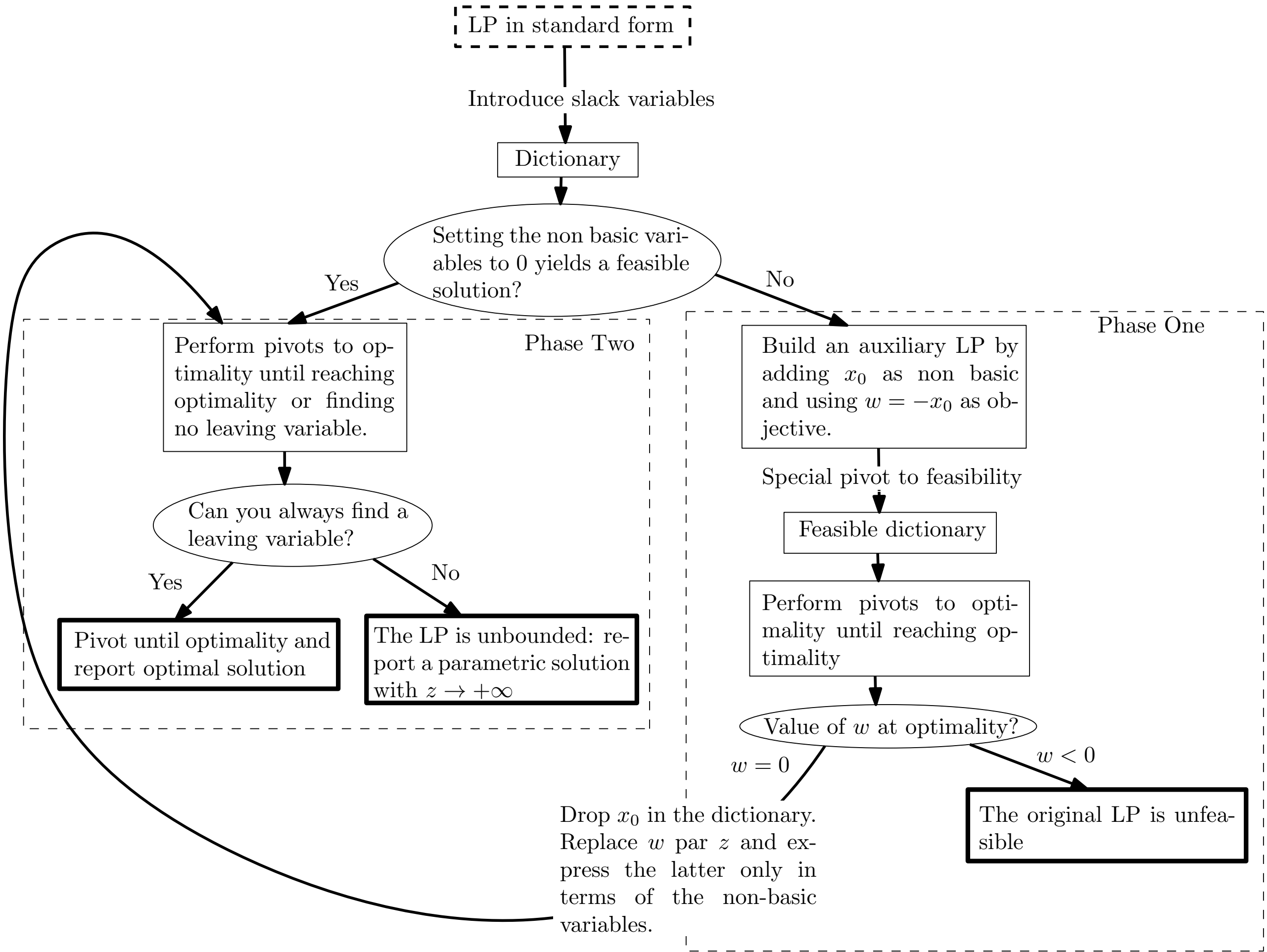
The **Two Phase method** is an algorithm which solves an LP in standard form. Its input is:

- A linear program in standard inequality form

Its output is **one out of the three** following options:

- The Linear Program has at least one optimal solution that you should report.
- The Linear Program is unbounded, which means the objective function can take values as large as you wanted. In this case, one must give a parametric solution which shows that the value of the objective function can tend to $+\infty$.
- The Linear Program is infeasible, which means that all the constraints cannot be satisfied simultaneously.

A flowchart of the Two Phase method can be found below: the boxes with a thick boundary correspond to the cases where the algorithm terminates.



Usually, Phase Two is what is called in the literature the **Simplex Method**.

Some details about the workflow

The basic brick is a pivot, either to optimality or to feasibility. A pivot is an operation which takes you from one dictionary to another: it consists in choosing a entering variable, a leaving variable, and to use the row of the leaving variable to eliminate the entering variable from the set of non-basic variables.

In a **pivot to optimality**:

- The entering variable is as the one in the z-row (or w-row) which has the largest positive coefficient (in case of ties choose the one with the smallest subscript). If no entering variable can be chosen, it means that optimality is reached: an optimal solution is obtained by setting all the non-basic variables to 0.
- The leaving variable is the first one to reach 0 when the entering variable is increased starting from 0. If there is no leaving variable it means that the problem is unbounded and you should report a parametric solution, see below.

In a **pivot to feasibility**:

- The entering variable is always x_0.
- The leaving variable is always the last one to become non-negative to 0 when x_0 increases. Actually, it is the one with the smallest "constant term".

Below is a (non exhaustive) list of things that you can check to detect potential mistakes.

- When you perform pivots to optimality, setting the non-basic variables to 0 must yield a feasible solution, that is all basic variables should be non-negative. If not, it means that you chose the wrong leaving variable.
- When you perform a pivot to optimality, the value of z (evaluated when the non basic variables are set to 0) must increase, or stay the same. If not, it means that you chose the wrong entering variable.
- When you perform the pivot to feasibility, setting the non-basic variables to 0 must yield a feasible solution, that is all basic variables should be non-negative. If not, it means that you chose the wrong leaving variable. This is precisely the point of the pivot to feasibility.
- When you perform a pivot to feasibility, the value of w (evaluated when the non basic variables are set to 0) decreases.
- When you are in Phase One and do pivots to optimality, the value of w (evaluated when the non basic variables are set to 0) can never be larger than 0. In particular, you can never be unbounded and you will always find a leaving variable.
- In Phase One, after you performed the pivot to feasibility, you must always choose x_0 as a leaving variable if possible. Indeed, it if you reach optimality with optimal value $w = 0$, it means that x_0 can be taken as a non basic variable, and in this case $w = -x_0$.

Examples: Phase Two

See the previous lecture notes on the topic: `https://hugolav.github.io/teaching/pivoting_process.pdf` and the practice for Quizz 1.

Example: Phase One, then Phase Two and Optimality is reached

See the example on Anstee's website, this is what we did in class: `https://www.math.ubc.ca/~anstee/math340/340anothertwophase.pdf`

You should also check the practice for Quizz 2.

Example: unbounded problem

Here is an example of an unbounded problem. Let us start directly with a dictionary:

$$\begin{array}{rclllll} x_4 & = & 5 & -x_1 & +x_2 & \\ x_5 & = & 3 & +2x_1 & -x_2 & \\ x_6 & = & 5 & & -x_2 & +2x_3 \\ z & = & & & +2x_2 & +x_3 \end{array}$$

Following Anstee's rule, x_2 enters while x_5 leaves. It gives

$$\begin{array}{rclllll} x_4 & = & 8 & +x_1 & -x_5 & \\ x_2 & = & 3 & +2x_1 & -x_5 & \\ x_6 & = & 2 & -2x_1 & +x_5 & +2x_3 \\ z & = & 6 & +4x_1 & -2x_5 & +x_3 \end{array}$$

Then x_1 enters and x_6 leaves:

$$\begin{array}{rclllll} x_4 & = & 9 & -1/2\, x_6 & -1/2\, x_5 & +x_3 \\ x_2 & = & 5 & -x_6 & & +2x_3 \\ x_1 & = & 1 & -1/2\, x_6 & +1/2\, x_5 & +x_3 \\ z & = & 10 & -2x_6 & & +5x_3 \end{array}$$

Now x_3 enters but... no variable is driven to 0 while we increase x_3. We can indefinitely increase x_3 to get larger and larger values of z. **The LP is unbounded**. More precisely, given a name to x_3, namely t, we get a feasible solution by setting all other non basic variables to 0:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 1+t \\ 5+2t \\ t \\ 9+t \\ 0 \\ 0 \end{pmatrix}$$

while $z = 10 + 5t$. As long as $t \geqslant 0$, this solution is feasible. In other words, we have a set of feasible solutions, parametrized by t, such that $z \to +\infty$ if $t \to +\infty$. This is what it means for the LP to be unbounded.

Example: Phase One, then in Phase Two the LP is unbounded

See the example on Anstee's website: `https://www.math.ubc.ca/~anstee/math340/340twophaselecture.pdf`

Example: infeasible problem

Let us consider the LP in standard form

$$\begin{array}{lllll} \text{maximize} & 3x_1 & -5x_2 & & \\ \text{subject to} & x_1 & +x_2 & \leqslant & 1 \\ & -x_1 & -2x_2 & \leqslant & -3 \end{array}$$

Notice that the property for a LP of being infeasible depends only on the constraints, so that the objective function is not relevant in this example. We form a dictionary by adding slack variables.

$$\begin{array}{rcrrr} x_3 & = & 1 & -x_1 & -x_1 \\ x_4 & = & -3 & +x_1 & +2x_2 \\ z & = & & 3x_1 & -5x_2 \end{array}$$

If we set the non-basic variables to 0, then $x_4 < 0$ so the dictionary is not valid. We must go through Phase One of the Two Phase Method. We add an additional variable x_0 and change the objective function:

$$\begin{array}{rcrrrr} x_3 & = & 1 & -x_1 & -x_2 & +x_0 \\ x_4 & = & -3 & +x_1 & +2x_2 & +x_0 \\ w & = & & & & -x_0 \end{array}$$

Then we do the pivot to feasibility: x_0 enters while the last variable to become non-negative when x_0 increases, namely x_4, leaves. We get

$$\begin{array}{rcrrrr} x_3 & = & 4 & -2x_1 & -3x_2 & +x_4 \\ x_0 & = & 3 & -x_1 & -2x_2 & +x_4 \\ w & = & -3 & +x_1 & +2x_2 & -x_4 \end{array}$$

Now we can do pivots to optimality. Following Anstee's rule, we choose x_2 as the entering variable while x_3 is leaving. We get

$$\begin{array}{rcrrrr} x_2 & = & 4/3 & -2/3\,x_1 & -1/3\,x_3 & +1/3\,x_4 \\ x_0 & = & 1/3 & +1/3\,x_1 & -2/3\,x_3 & +1/3\,x_4 \\ w & = & -1/3 & -1/3\,x_1 & -2/3\,x_3 & -1/3\,x_4 \end{array}$$

As all the coefficients in front of the non basic variables are negative, optimality is reached. However, as the optimal value of w, namely $-1/3$ is strictly negative, **the original LP is infeasible**.

To convince you of that (and we will understand why with duality later), notice that each of the slack variable (here x_3 and x_4) is naturally associated with a constraint (x_3 with the first one and x_4 with the second one). Now, at optimality the coefficient in front of x_3 is $-2/3$ while the one in front of x_4 is $-1/3$. Provided we forget about the minus sign, it tells us to do a clever combination of the equations in the constraints. Indeed, if we remember that the original constraint are

$$\begin{array}{rrcr} x_1 & +x_2 & \leqslant & 1 \\ -x_1 & -2x_2 & \leqslant & -3 \end{array}$$

then they imply that

$$\frac{2}{3}(x_1 + x_2) + \frac{1}{3}(-x_1 - 2x_2) \leqslant \frac{2}{3} \times 1 + \frac{1}{3} \times (-3)$$

which reads

$$\frac{1}{3}x_1 \leqslant -\frac{1}{3}.$$

This clearly contradicts the non-negativity assumption on x_1. In other words, there is no x_1, x_2 which are non-negative and which satisfy the constraints.