Lifting functionals defined on maps to measure-valued maps via optimal transport

Hugo Lavenant

Bocconi University

2023 LMS Invited Lecture Series, Durham (United Kingdom),
July 19, 2023

online: `https://cvgmt.sns.it/paper/6151/`

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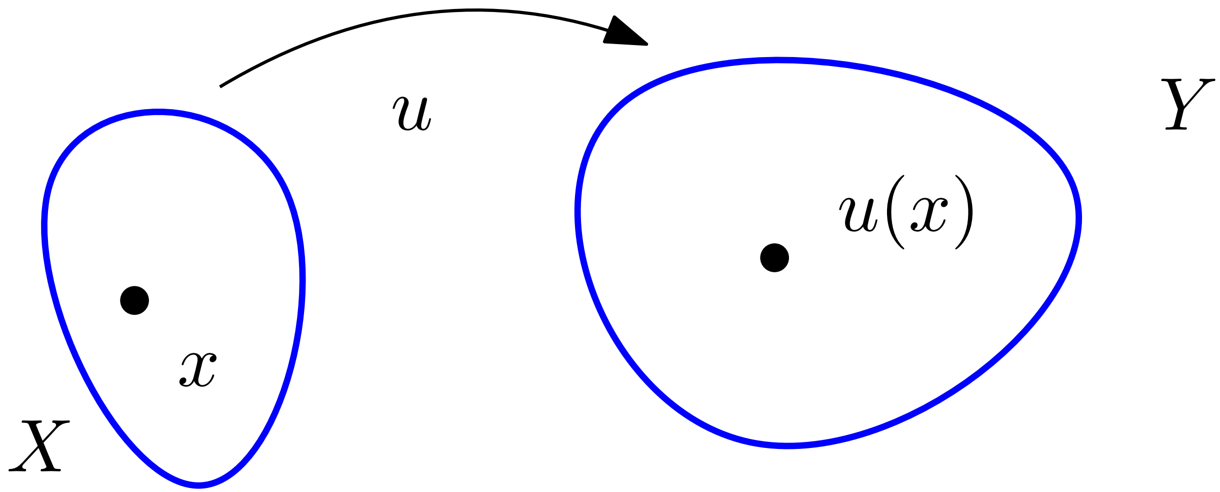
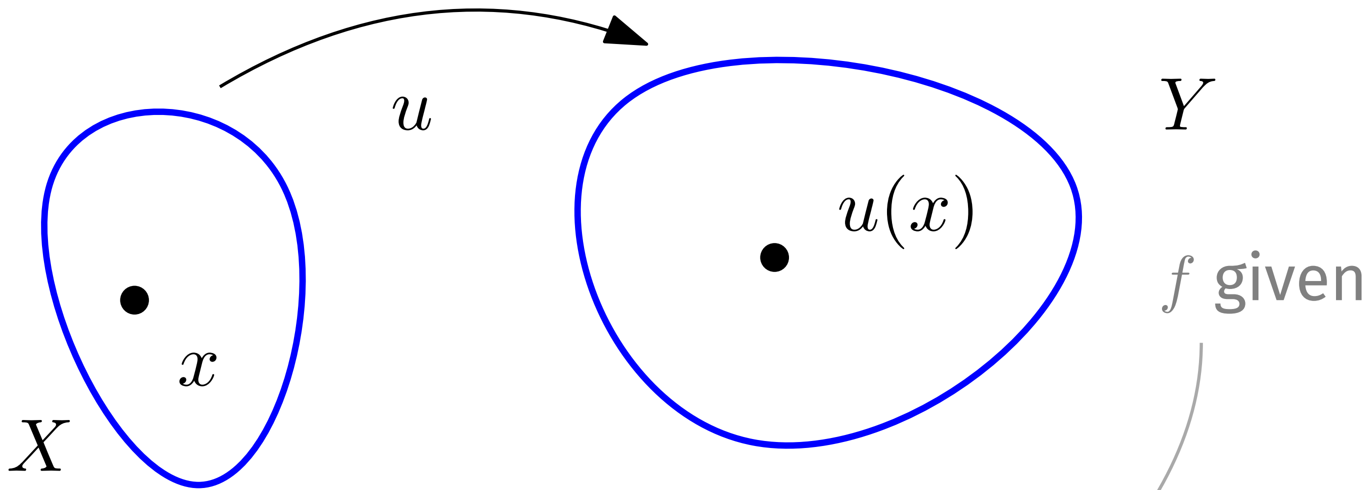
Lifting of functionals defined on maps



Given

$$\boxed{u \to E(u) \in [0, +\infty]}$$

Lifting of functionals defined on maps



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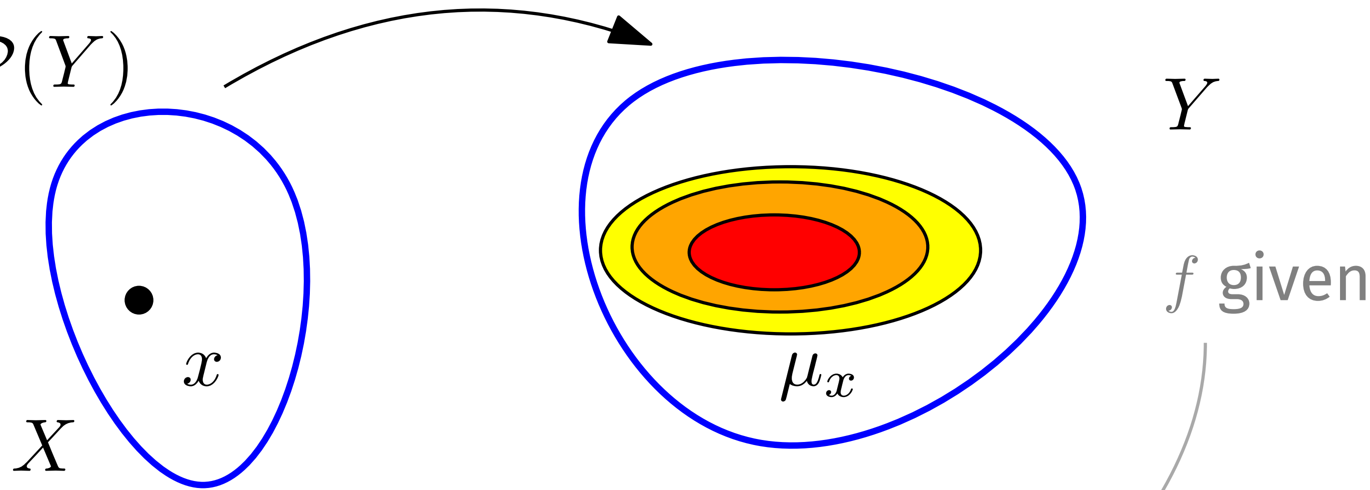
Example:

$$E(u) = \int_X f(x, u(x))\,\mathrm{d}x$$

Lifting of functionals defined on maps

$$\mu : X \to \mathcal{P}(Y)$$



Given

$$\boxed{u \to E(u) \in [0, +\infty]}$$

Looking for

$$\boxed{\mu \to \mathcal{T}_E(\mu) \in [0, +\infty]}$$

Example:

$$E(u) = \int_X f(x, u(x))\,\mathrm{d}x$$

$$\mathcal{T}_E(\mu) = \int_X \left(\int_Y f(x, y)\,\mathrm{d}\mu_x(y) \right) \mathrm{d}x$$

Lifting of functionals defined on maps

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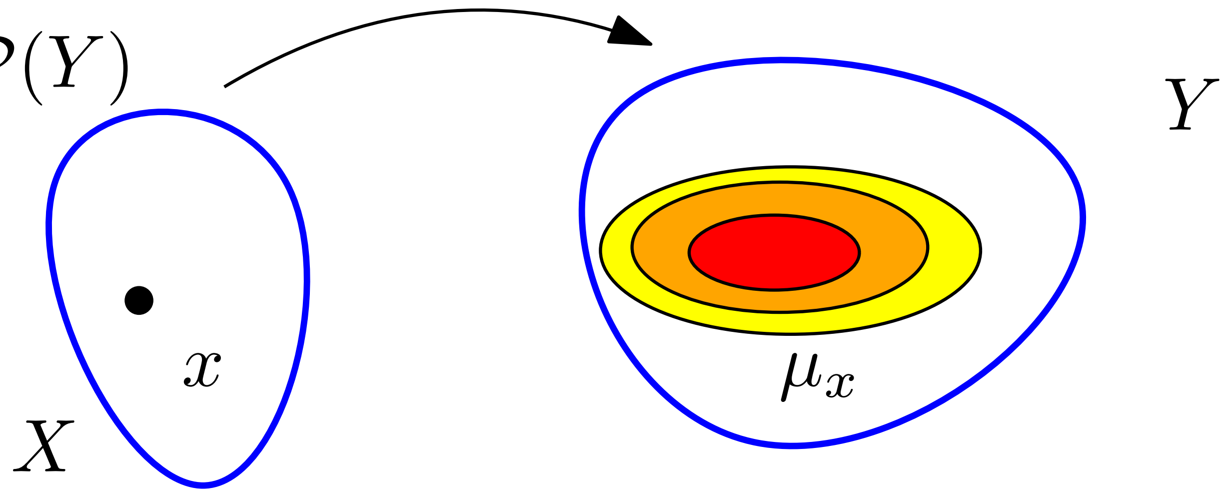

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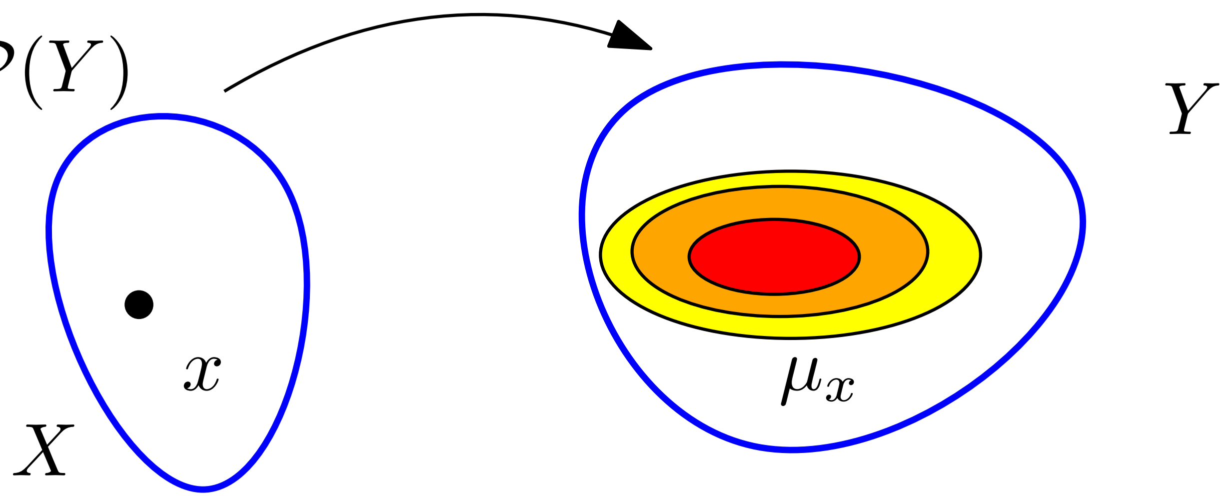
Example:

$$E(u) = \frac{1}{2} \int_X |\nabla u(x)|^2 \, \mathrm{d}x$$

???

Lifting of functionals defined on maps

$\mu : X \to \mathcal{P}(Y)$



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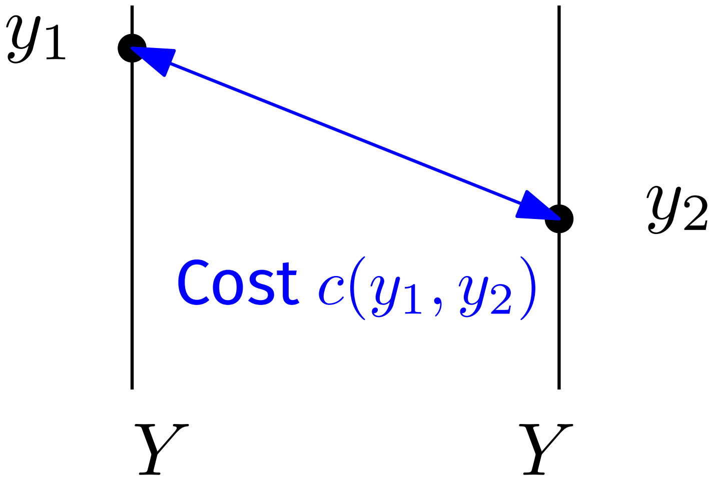
???

Today

- "Lagrangian" answer $\mathcal{T}_E$
- "Eulerian" answer $\mathcal{T}_{E,\mathrm{Eul}}$

The link with optimal transport

Simpler question: $c : Y \times Y \to [0, +\infty]$



Savaré and Sodini (2022). A simple relaxation approach to duality for Optimal Transport problems in completely regular spaces.

The link with optimal transport

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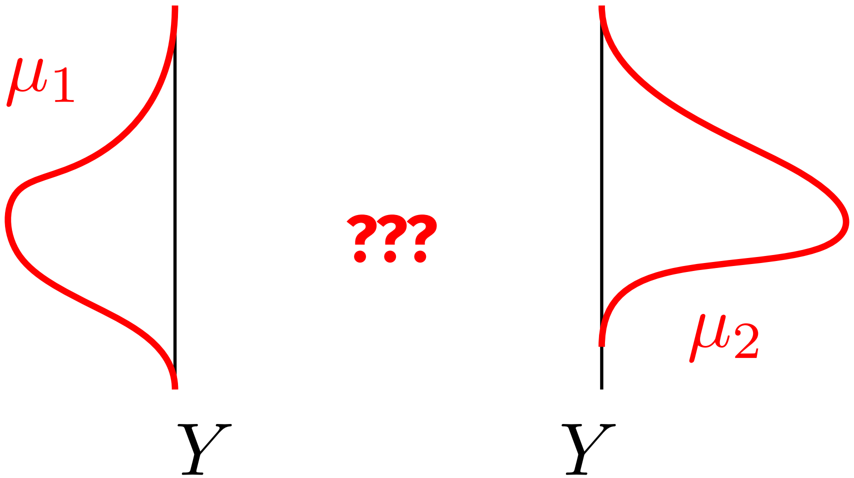

Question: how to extend c into $\mathcal{T}_c : \mathcal{P}(Y) \times \mathcal{P}(Y) \to [0, +\infty]$

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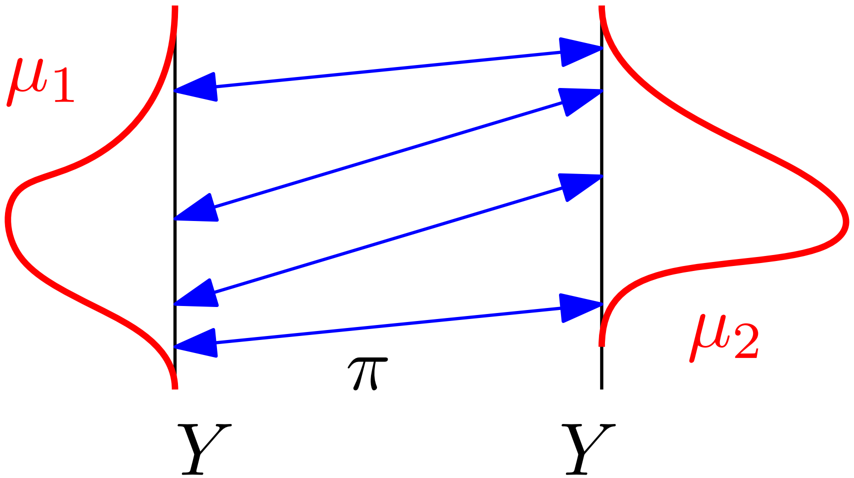

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Probabilities on $Y \times Y$ with marginals μ_1, μ_2

$$\mathcal{T}_c(\mu_1, \mu_2) = \min_{\pi} \left\{ \int_{Y \times Y} c(y_1, y_2)\, \pi(\mathrm{d}y_1, \mathrm{d}y_2) \ : \ \pi \in \Pi(\mu_1, \mu_2) \right\}$$

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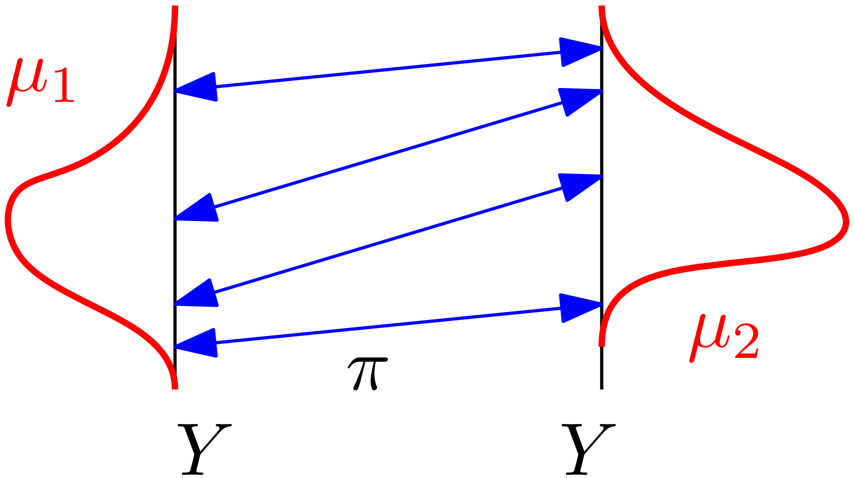

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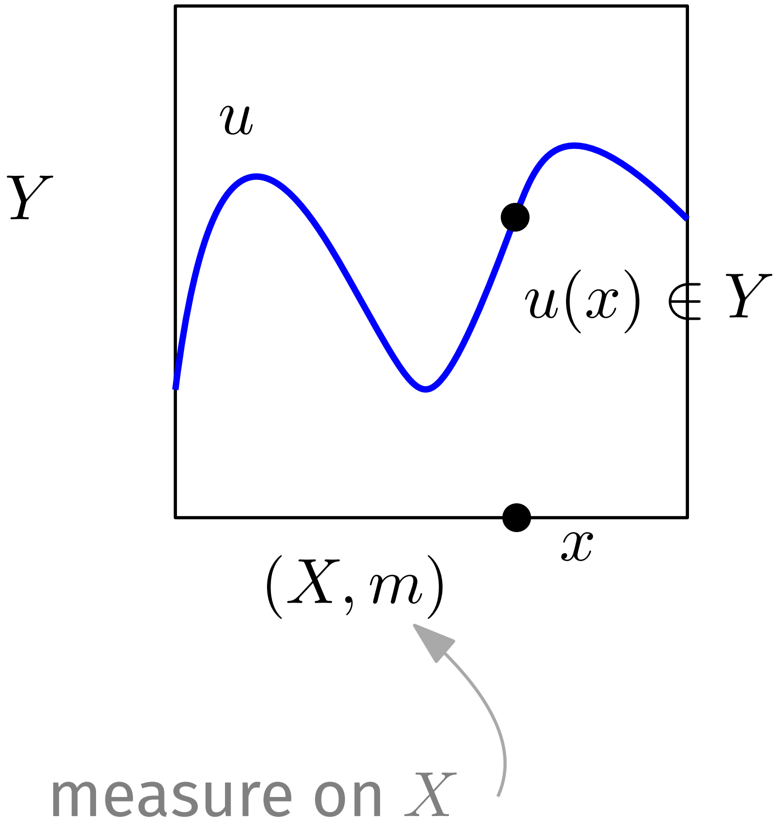
Theorem. $\mathcal{T}_c$ is the largest **convex** and **lower semi continuous** functional on $\mathcal{P}(Y) \times \mathcal{P}(Y)$ such that $\mathcal{T}_c(\delta_{y_1}, \delta_{y_2}) = c(y_1, y_2)$ for any y_1, y_2.

w.r.t. narrow convergence if c l.s.c. and, e.g. Y polish space

Savaré and Sodini (2022). A simple relaxation approach to duality for Optimal Transport problems in completely regular spaces.

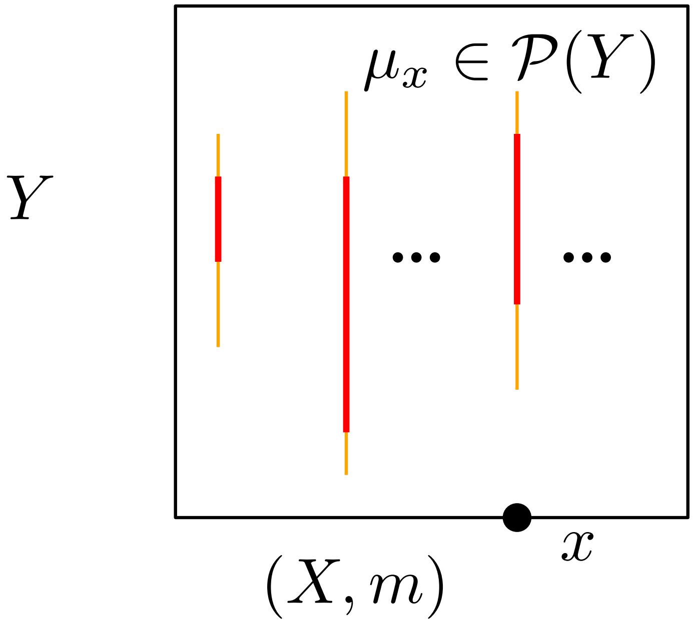
Today's question



Maps $u : X \to Y$, equivalent if equal m-a.e.

$$\boxed{E : L^0(X, Y, m) \to [0, +\infty]}$$

Today's question



measure on X

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Want to extend to $L^0(X, \mathcal{P}(Y), m)$

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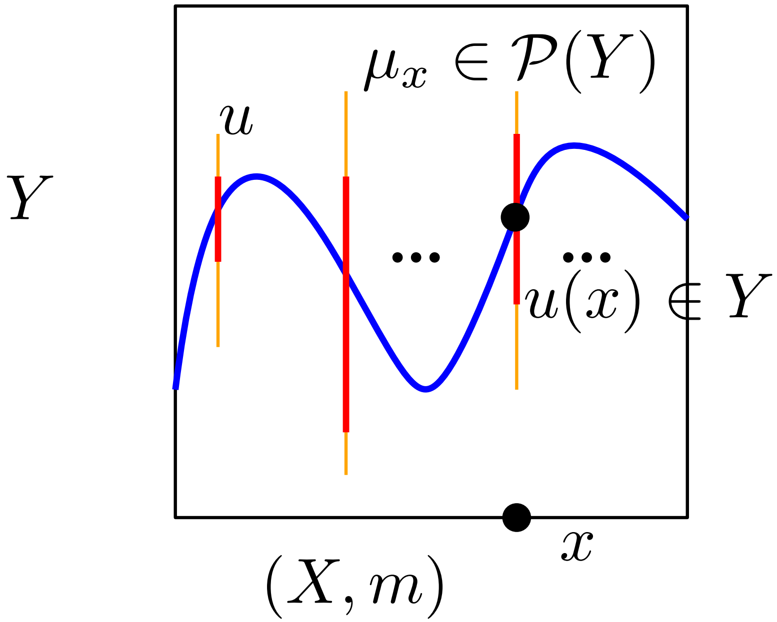

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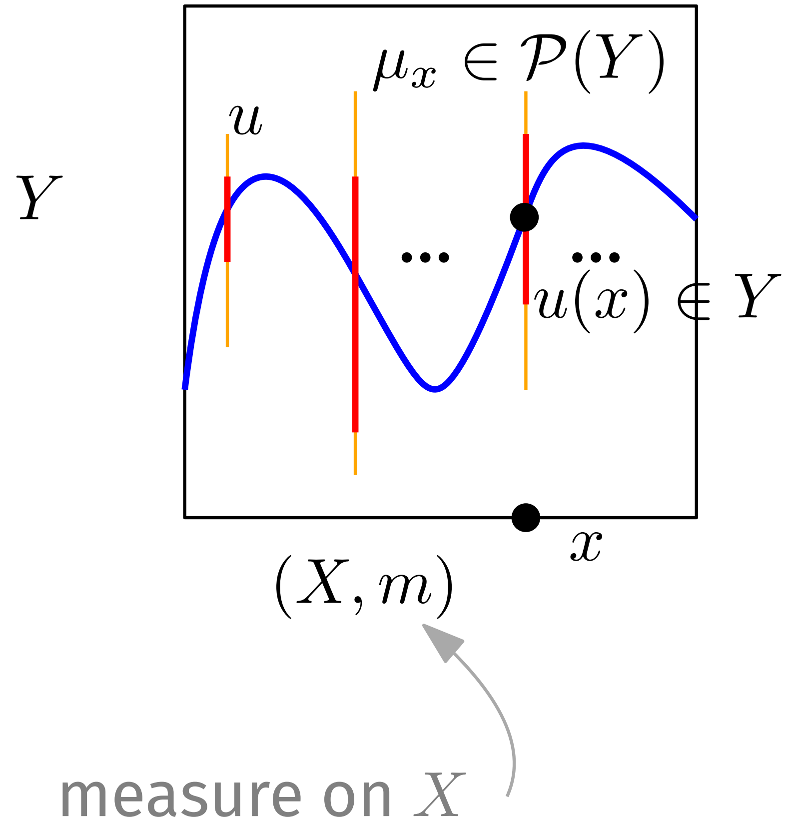
Define $\mu_u : x \mapsto \delta_{u(x)}$.

Today's question



Maps $u : X \to Y$, equivalent if equal m-a.e.

$$\boxed{E : L^0(X, Y, m) \to [0, +\infty]}$$

Want to extend to $L^0(X, \mathcal{P}(Y), m)$

Define $\mu_u : x \mapsto \delta_{u(x)}$.

Question. What is the **largest** convex and **lower semi continuous** (for which topology?) functional

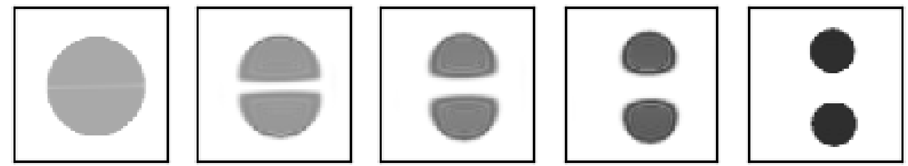
$$\mathcal{T} : L^0(X, \mathcal{P}(Y), m) \to [0, +\infty]$$

such that $\mathcal{T}(\mu_u) = E(u)$ for all u?

Why? Motivation coming from Optimal Transport

For curves ($X = [0,1]$), $E(u) = \displaystyle\int_0^1 |\dot u_t|^2 \, \mathrm{d}t$

Minimizers of the lifted action are **geodesics**.



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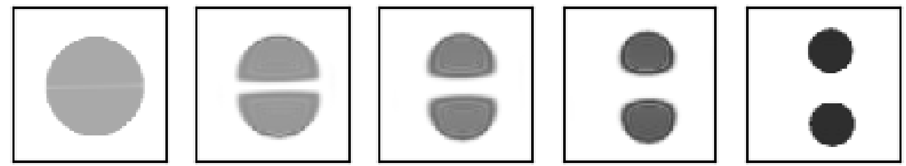

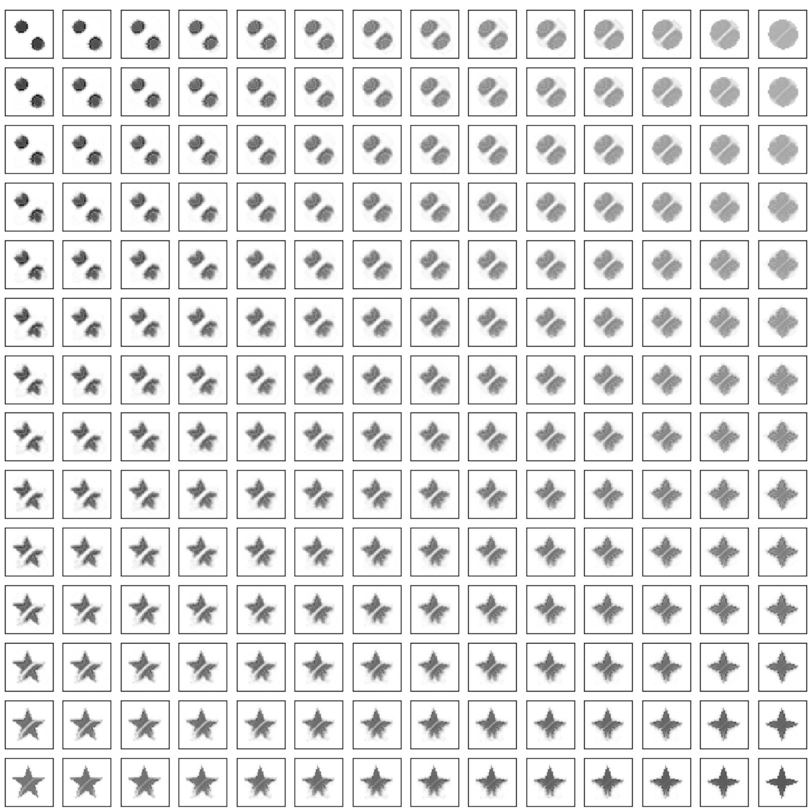

For maps, $E(u) = \int |\nabla u(x)|^2 \, \mathrm{d}x$

Minimizers of the **Eulerian** lifting of the Dirichlet energy are **harmonic maps**.

Brenier (2003). Extended Monge-Kantorovich theory.
Solomon, Guibas and Butscher (2013). Dirichlet energy for analysis and synthesisof soft maps.
Lavenant (2019). Harmonic mappings valued in the Wasserstein space.

Why? Motivation coming from imaging

Map denoising:

$$\text{minimize} \qquad E(u) = \int W(\nabla u(x))\,\mathrm{d}x + \int f(x, u(x))\,\mathrm{d}x$$

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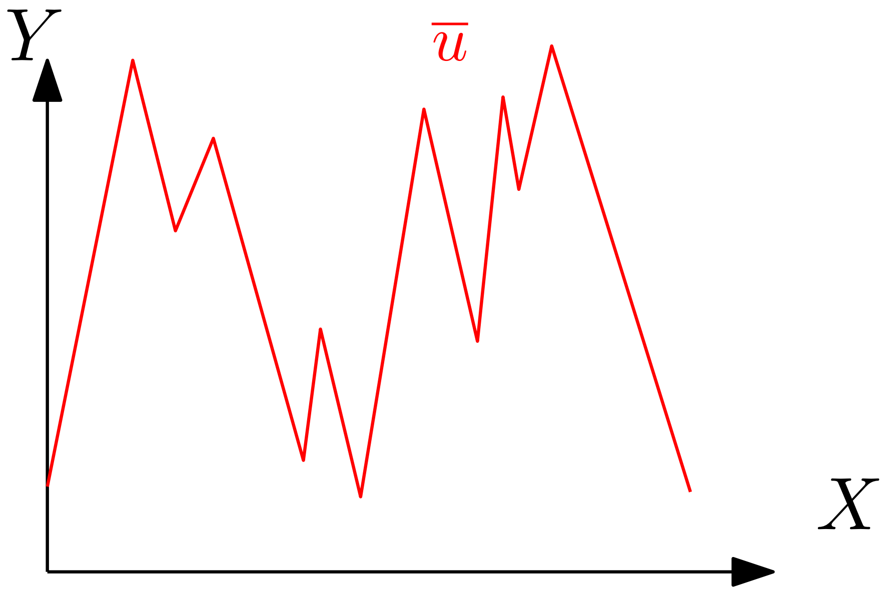
Regularization

data fitting, like
$f(x, u(x)) = |u(x) - \overline{u}(x)|^2$



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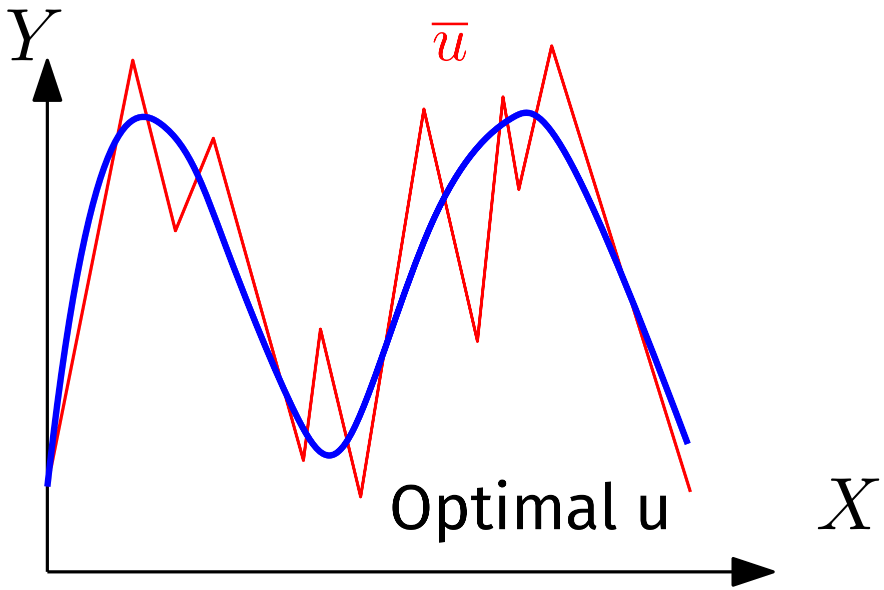

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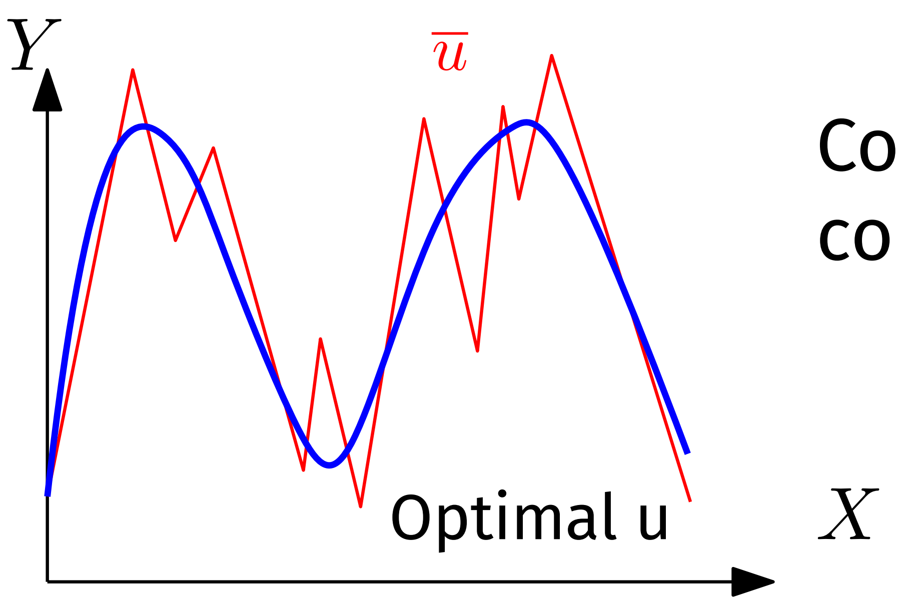

Codomain of u manifold, or f non convex $\to$ **convexification**.

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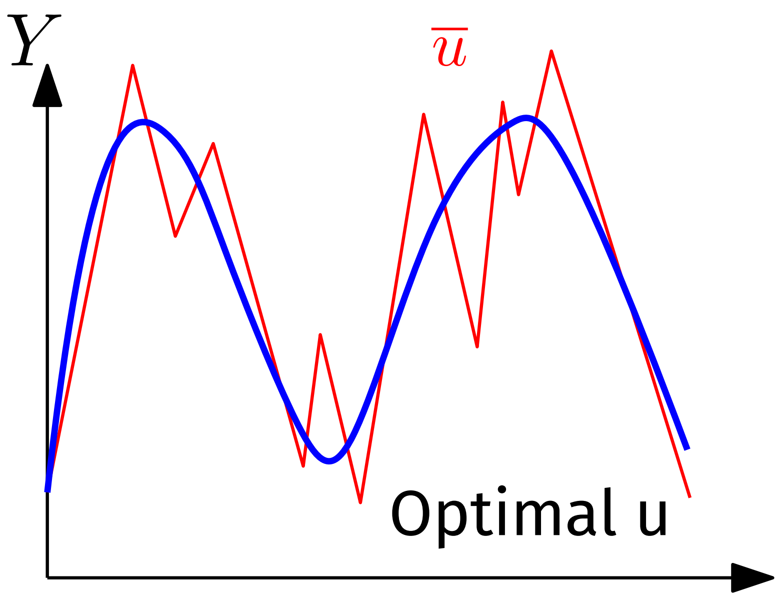
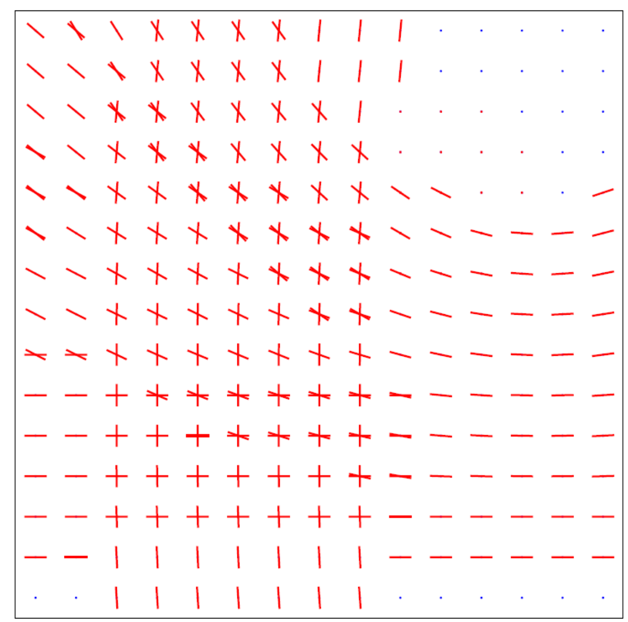


X

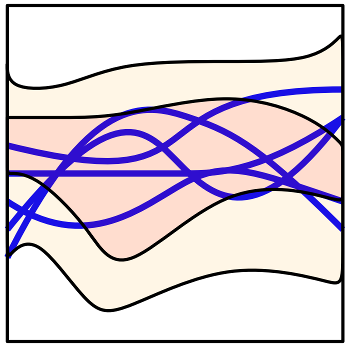
Codomain of u manifold, or f non convex $\rightarrow$ **convexification**.



Data really measure-valued:

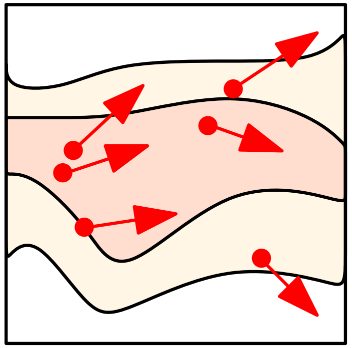
Magnetic Resonance Imaging:
distributions of directions, in $\mathcal{P}(\mathbb{S}^2)$

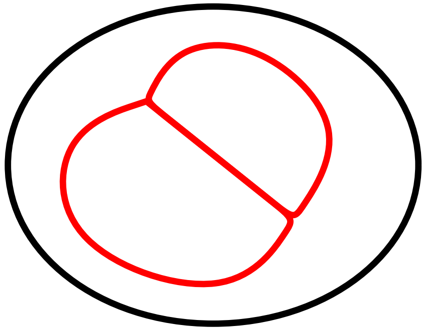
Vogt, Haase, Bednarski and Lellmann (2020). On the connection between dynamical optimal transport and functional lifting.
Vogt and Lellmann (2018). Measure-valued variational models with applications todiffusion-weighted imaging.



1 - The Lagrangian lifting or optimal transport with an infinity of marginals

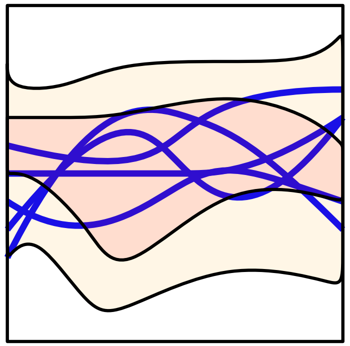
2 - The Eulerian lifting





3 - Understanding the difference: localization of functionals

X, Y polish (metric, complete, separable) spaces.



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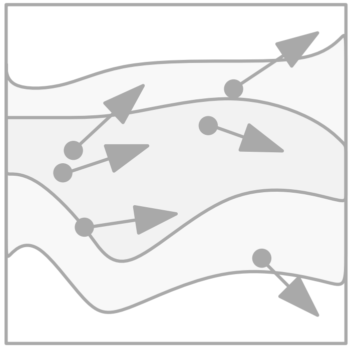
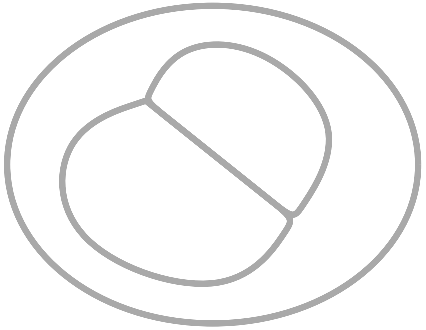
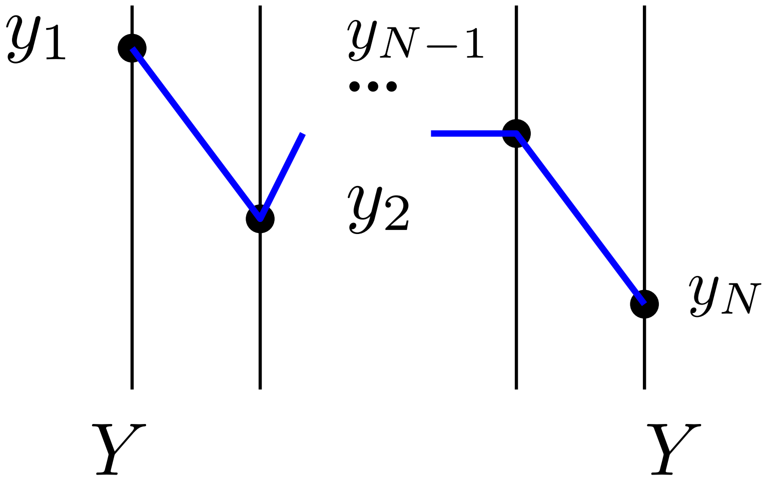




3 - Understanding the difference: localization of functionals

Optimal transport with several marginals

Transport problem with N marginals: $c : Y^N \to [0, +\infty]$



cost $c(y_1, \ldots, y_N)$

Optimal transport with several marginals

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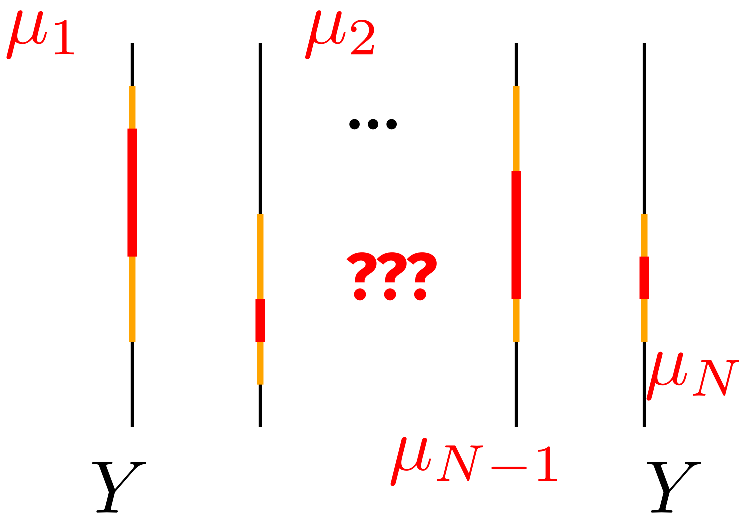

Question: how to extend c into $\mathcal{T}_c : \mathcal{P}(Y)^N \to [0, +\infty]$

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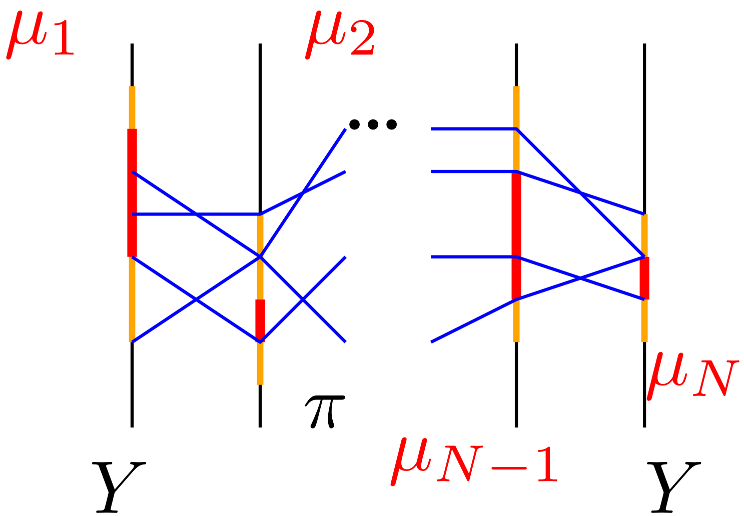

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Probabilities on Y^N with marginals $\mu_1, \dots, \mu_N$

$$\mathcal{T}_c(\mu_1, \dots, \mu_N) = \min_{\pi} \left\{ \int_{Y^N} c(y_1, \dots, y_N)\, \pi(\mathrm{d}y_1, \dots, \mathrm{d}y_2) \;:\; \pi \in \Pi(\mu_1, \dots, \mu_N) \right\}$$

Largest convex l.s.c. such that $\mathcal{T}_c(\delta_{y_1}, \dots, \delta_{y_N}) = c(y_1, \dots, y_N)$.

Optimal transport with several marginals

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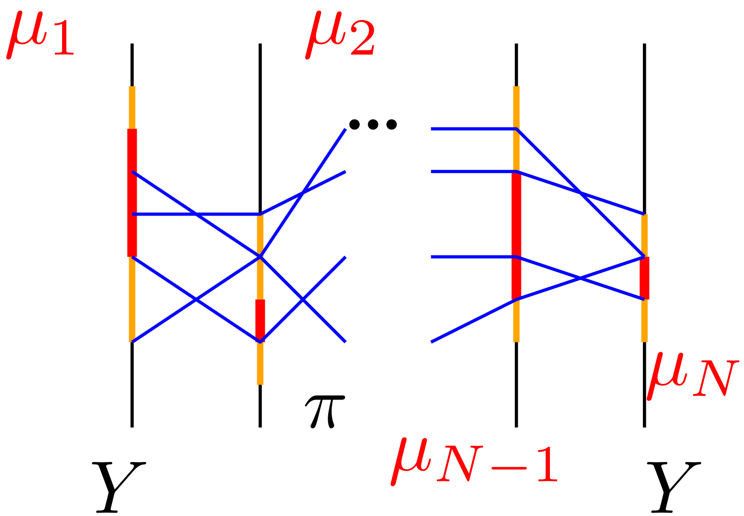

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Idea: take limit $N \to +\infty$: indexing set $\{1, \ldots, N\}$ becomes X

Optimal transport with several marginals

Transport problem with N marginals: $c : Y^N \to [0, +\infty]$

Y^N becomes $X \to Y$,
and c becomes E

$\mu_1, \ldots, \mu_N \in \mathcal{P}(Y)^N$
becomes
$\mu : X \to \mathcal{P}(Y)$

π becomes Q probability
on maps $X \to Y$.

$$\mathcal{T}_c(\mu_1, \ldots, \mu_N) = \min_\pi \left\{ \int_{Y^N} c(y_1, \ldots, y_N)\, \pi(\mathrm{d}y_1, \ldots, \mathrm{d}y_2) \;:\; \pi \in \Pi(\mu_1, \ldots, \mu_N) \right\}$$

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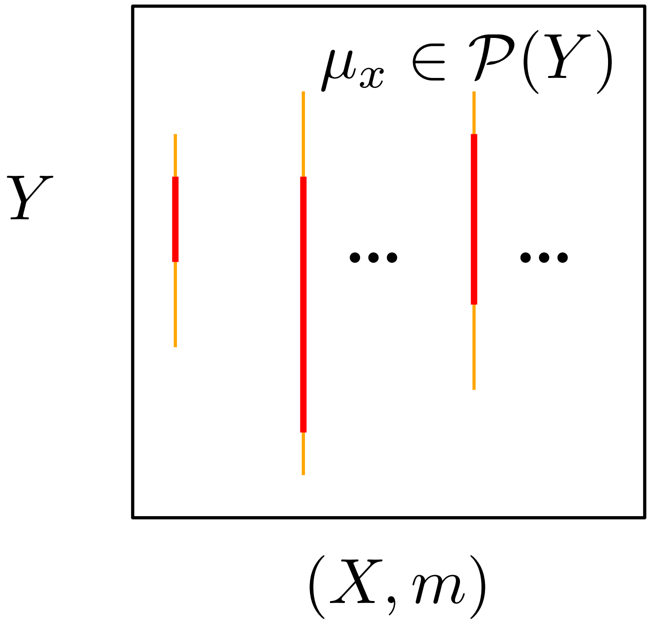
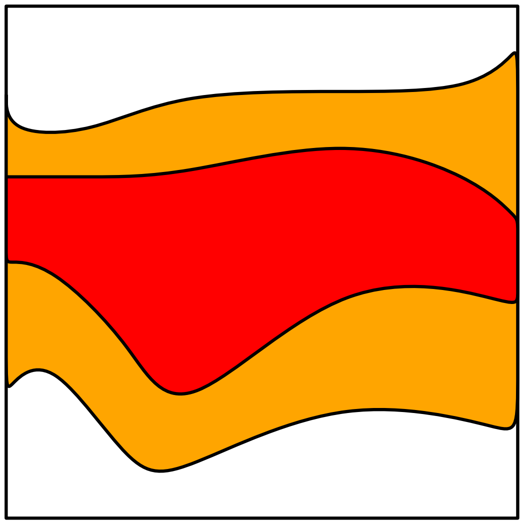
Maps of measures and measures on maps



Maps of measures and measures on maps

$\mathcal{M}_+(X \times Y) \leftrightarrow \mathcal{P}(Y)^N$



Y

(X, m)

View μ as measure on $X \times Y$ by

$$\int_{X \times Y} \varphi \, \mathrm{d}\mu = \int_X \left(\int_Y \varphi(x, y) \, \mathrm{d}\mu_x(y) \right) \mathrm{d}m(x)$$

Maps of measures and measures on maps

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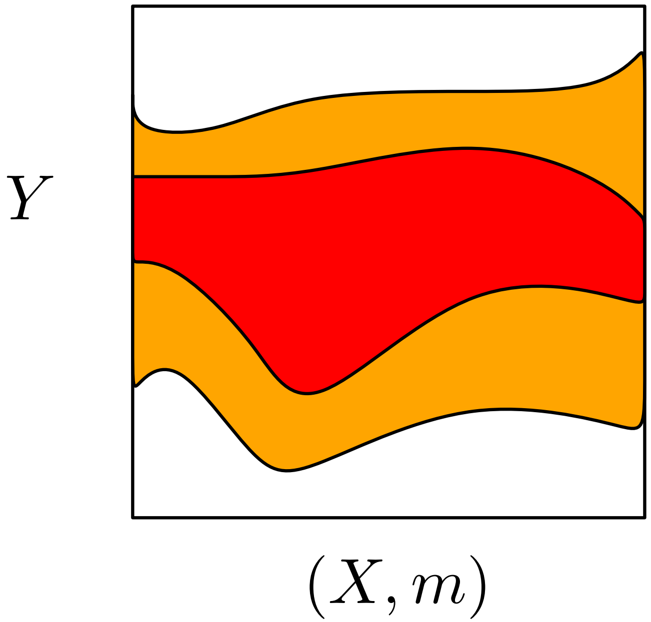

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Theorem. (disintegration). As sets, $L^0(X, \mathcal{P}(Y), m)$ coincides with measures on $X \times Y$ whose first marginal is m.

Maps of measures and measures on maps

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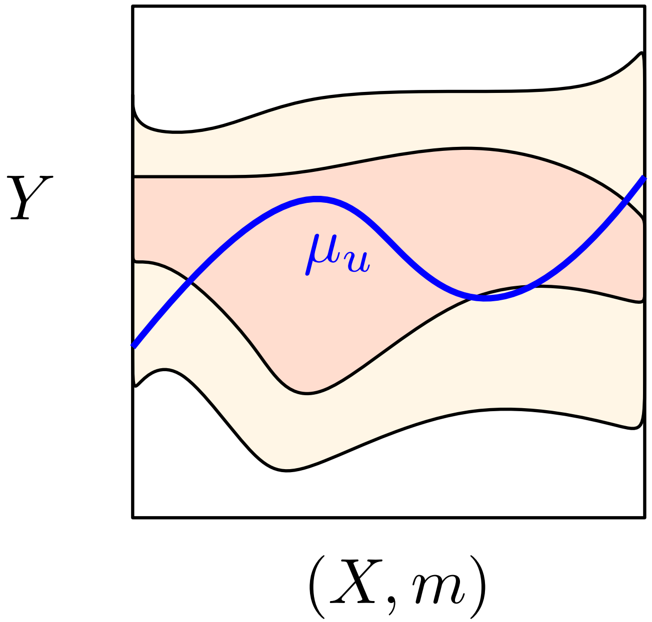

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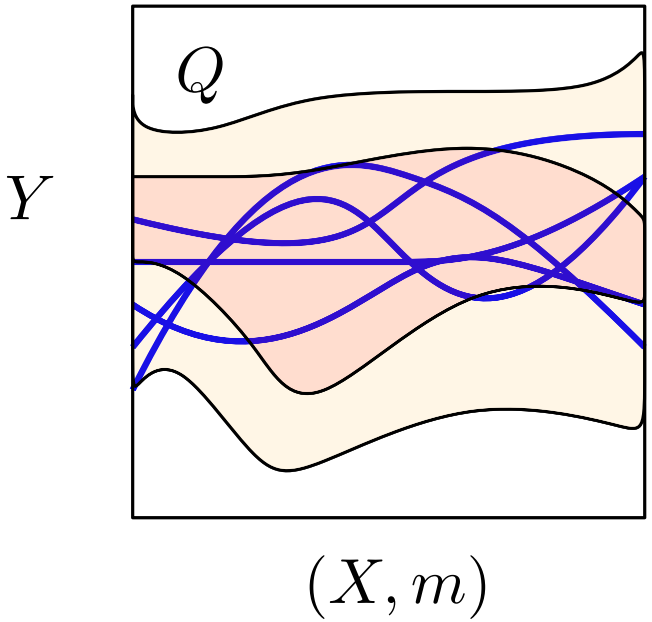
Recall: $\mu_u : x \mapsto \delta_{u(x)}$.

Maps of measures and measures on maps

$\mathcal{M}_+(X \times Y) \leftrightarrow \mathcal{P}(Y)^N$



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$Q \leftrightarrow \pi$

$L^0(X, Y, m) \leftrightarrow Y^N$

Definition. $Q \in \mathcal{P}(L^0(X, Y, m))$ belongs to $\Pi(\mu)$ if

$$\mu = \int_{L^0(X,Y,m)} \mu_u \,\mathrm{d}Q(u).$$

Proposition. $\Pi(\mu)$ if never empty (if X, Y polish spaces).

Multimarginal OT with an infinity of marginals

- $E : L^0(X, Y, m) \to [0, +\infty]$,
- μ measure on $X \times Y$ with first marginal m.

Definition.

$$\mathcal{T}_E(\mu) = \inf_Q \left\{ \int_{L^0(X,Y,m)} E(u) \, \mathrm{d}Q(u) \ : \ Q \in \Pi(\mu) \right\}.$$

Multimarginal OT with an infinity of marginals

- $E : L^0(X, Y, m) \to [0, +\infty]$,
- μ measure on $X \times Y$ with first marginal m.

Definition.
$$\mathcal{T}_E(\mu) = \inf_Q \left\{ \int_{L^0(X,Y,m)} E(u) \, \mathrm{d}Q(u) \ : \ Q \in \Pi(\mu) \right\} .$$

Theorem.
- $\mathcal{T}_E$ is always convex.
- Under additional assumption, it is the largest convex and l.s.c. functional such that
$$\forall u, \quad \mathcal{T}_E(\mu_u) = E(u)$$

See next slides

Narrow convergence on $\mathcal{M}_+(X \times Y)$

Idea of the proof

Any $\mathcal{T}$ convex l.s.c. such that $\mathcal{T}(\mu_u) = E(u)$

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$$\leq \int_{L^0} \mathcal{T}(\mu_u)\,\mathrm{d}Q(u) \qquad \text{Jensen}$$

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$$\rightsquigarrow \mathcal{T}_E(\mu)$$ Minimizing in $Q \in \Pi(\mu)$

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$$\rightsquigarrow \mathcal{T}_E(\mu) \qquad \text{Minimizing in } Q \in \Pi(\mu)$$

Left to do: prove that $\mathcal{T}_E$ is lower semi continuous.

Assumption on E

To guarantee existence of optimal $Q \in \Pi(\mu)$ and l.s.c. of $\mathcal{T}_E$.

Assumption. E is l.s.c. and for any $\psi : Y \to [0, +\infty)$ with compact sublevel sets, the following functional has compact sublevel sets in $L^0(X, Y, m)$:

$$u \mapsto \; E(u) + \int_X \psi(u(x)) \, \mathrm{d}m(x).$$

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Think $E(u) = \int |\nabla u|^2$ ✓

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"Coercivity of E + Tightness of μ"

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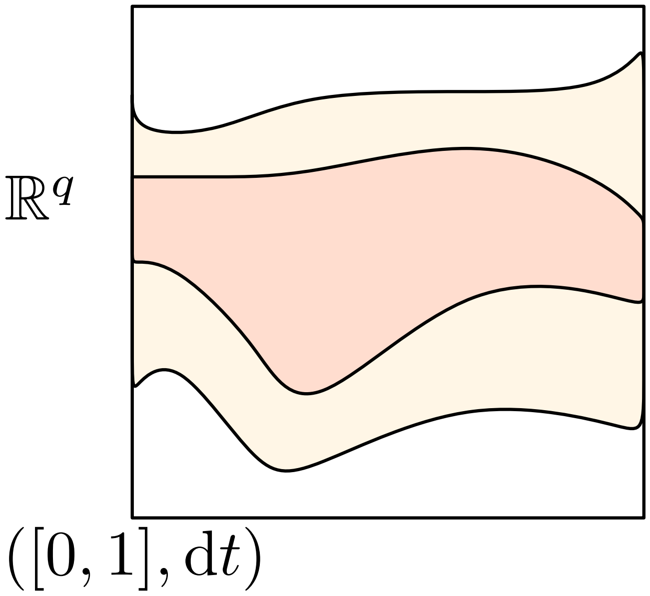
Think $E(u) = \int |\nabla u|^2$ ✓

Remark. If X finite, no assumption needed on E besides l.s.c.

Curves of measures and measures of curves



$$E(u) = \int_0^1 |\dot{u}_t|^p \, \mathrm{d}t$$

Lisini (2007). Characterization of absolutely continuous curves in Wasserstein spaces.
Ambrosio, Gigli and Savaré (2008). Gradient flows in metric spaces and in the space of probability measures.

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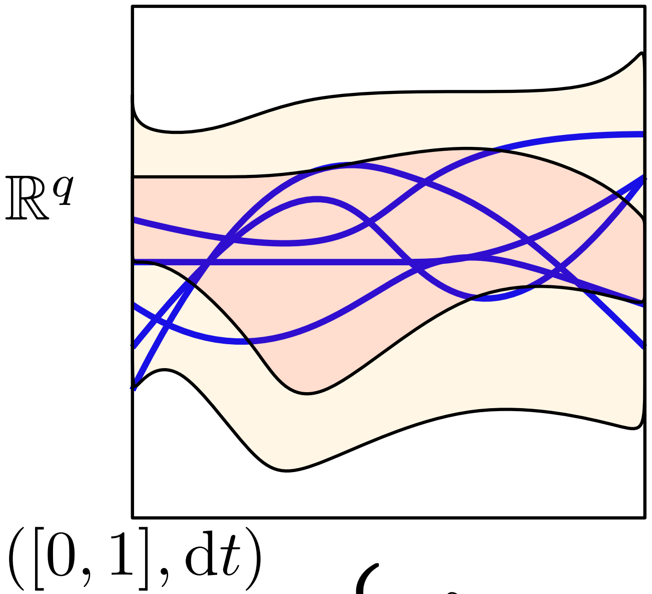

$$E(u) = \int_0^1 |\dot{u}_t|^p \, \mathrm{d}t$$

Curves are continuous, define $e_t : u \mapsto u_t$

$$\mathcal{T}_E(\mu) = \inf_Q \left\{ \int_{C(X,Y)} \int_0^1 |\dot{u}_t|^p \, \mathrm{d}t \, Q(\mathrm{d}u) : \ \forall t \in [0,1], \ e_t \# Q = \mu_t \right\}$$

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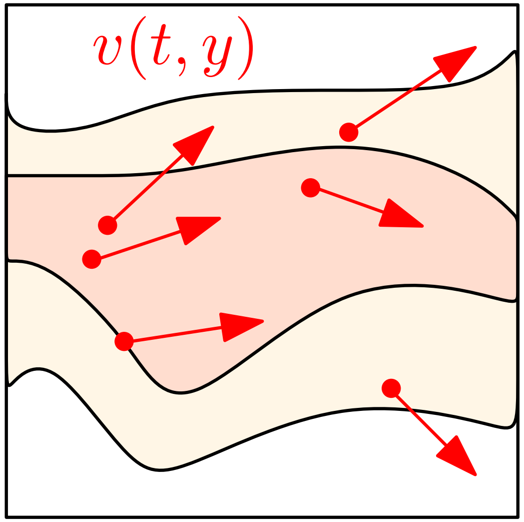

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Theorem. We can write $\mathcal{T}_E(\mu)$ as

$$\min_v \left\{ \int_0^1 \int_{\mathbb{R}^q} |v(t,y)|^p \,\mathrm{d}\mu_t(y)\mathrm{d}t \;:\; \partial_t \mu + \mathrm{div}_y(v\mu) = 0 \right\}.$$

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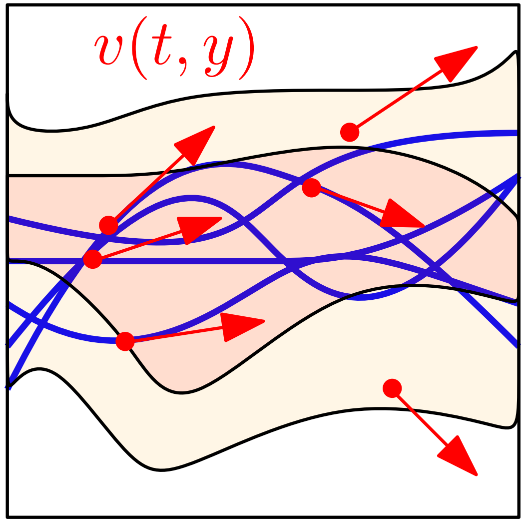

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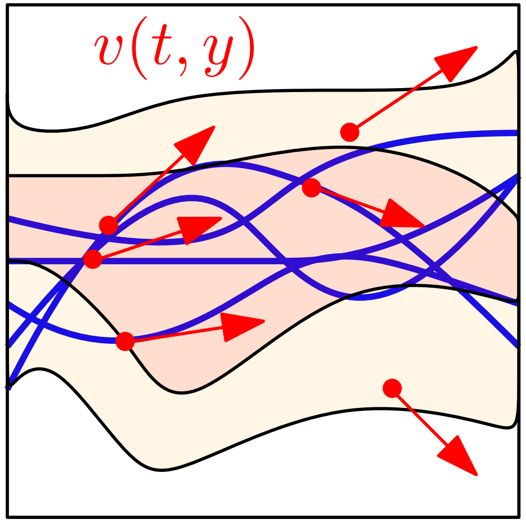

$\mathbb{R}^q$

$([0,1], \mathrm{d}t)$

$$E(u) = \int_0^1 |\dot{u}_t|^p \, \mathrm{d}t$$

$E(u) = \int_0^1 |\ddot{u}|^2$ for **splines**.

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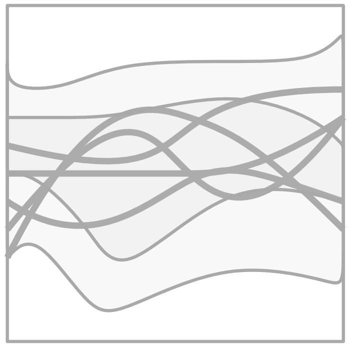
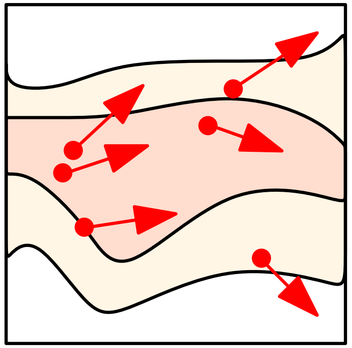
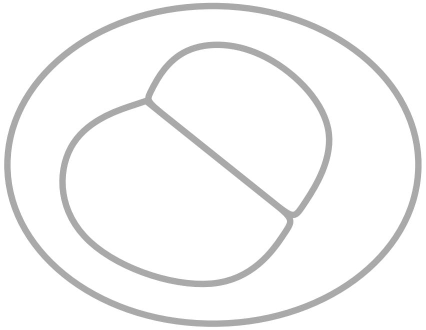
Benamou, Gallouët, Vialard (2019). Second-order models for optimal transport and cubic splines on the Wasserstein space.
Chen, Conforti, Georgiou (2018). Measure-valued spline curves: An optimal transport viewpoint.



1 - The Lagrangian lifting or optimal transport with an infinity of marginals

2 - The Eulerian lifting

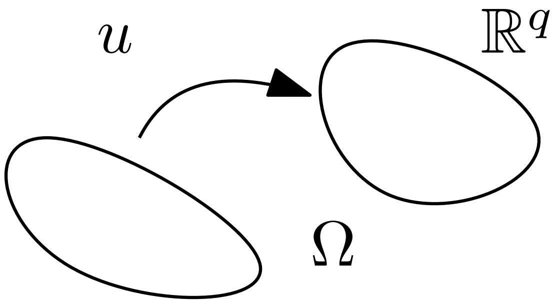




3 - Understanding the difference: localization of functionals

The Eulerian lifting

$X = \Omega \subset \mathbb{R}^d$ with Lebesgue measure, $Y = \mathbb{R}^q$, and W **convex**



$$E(u) = \int_\Omega W(\nabla u(x))\mathrm{d}x$$

Vogt, Haase, Bednarski and Lellmann (2020). On the connection between dynamical optimal transport and functional lifting.

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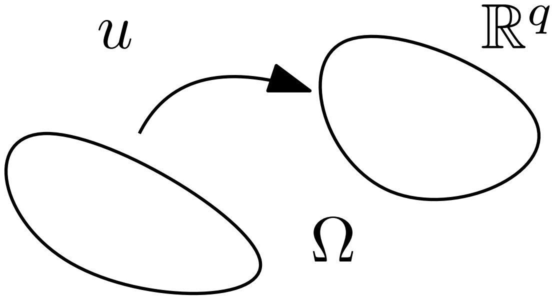

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Definition. We define $\mathcal{T}_{E,\mathrm{Eul}}(\mu)$ as

$$\min_v \left\{ \int_\Omega \int_{\mathbb{R}^q} W(v(x,y))\, \mathrm{d}\mu_x(y)\mathrm{d}x \quad \text{s.t. } \nabla_x \mu + \mathrm{div}_y(v\mu) = 0 \right\}$$

$v : \Omega \times \mathbb{R}^q \to \mathbb{R}^{q \times d}$ "density of Jacobian matrix".

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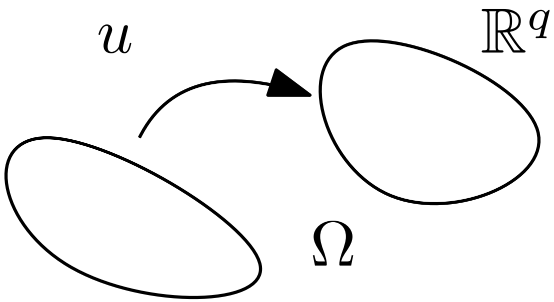

$$E(u) = \int_\Omega W(\nabla u(x))\mathrm{d}x + \int_\Omega f(x, u(x))\,\mathrm{d}x$$

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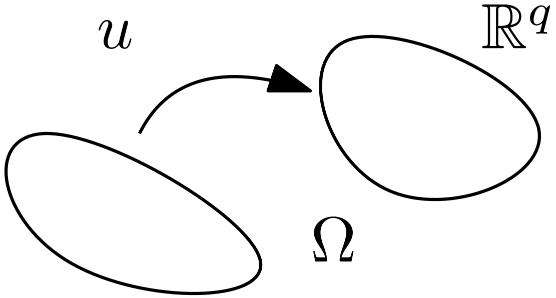

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$v : \Omega \times \mathbb{R}^q \to \mathbb{R}^{q\times d}$ "density of Jacobian matrix".

Remark. To have a convex formulation: $(\mu, v) \leftrightarrow (\mu, v\mu)$.

Vogt, Haase, Bednarski and Lellmann (2020). On the connection between dynamical optimal transport and functional lifting.

Example: harmonic mapps valued in the Wasserstein space

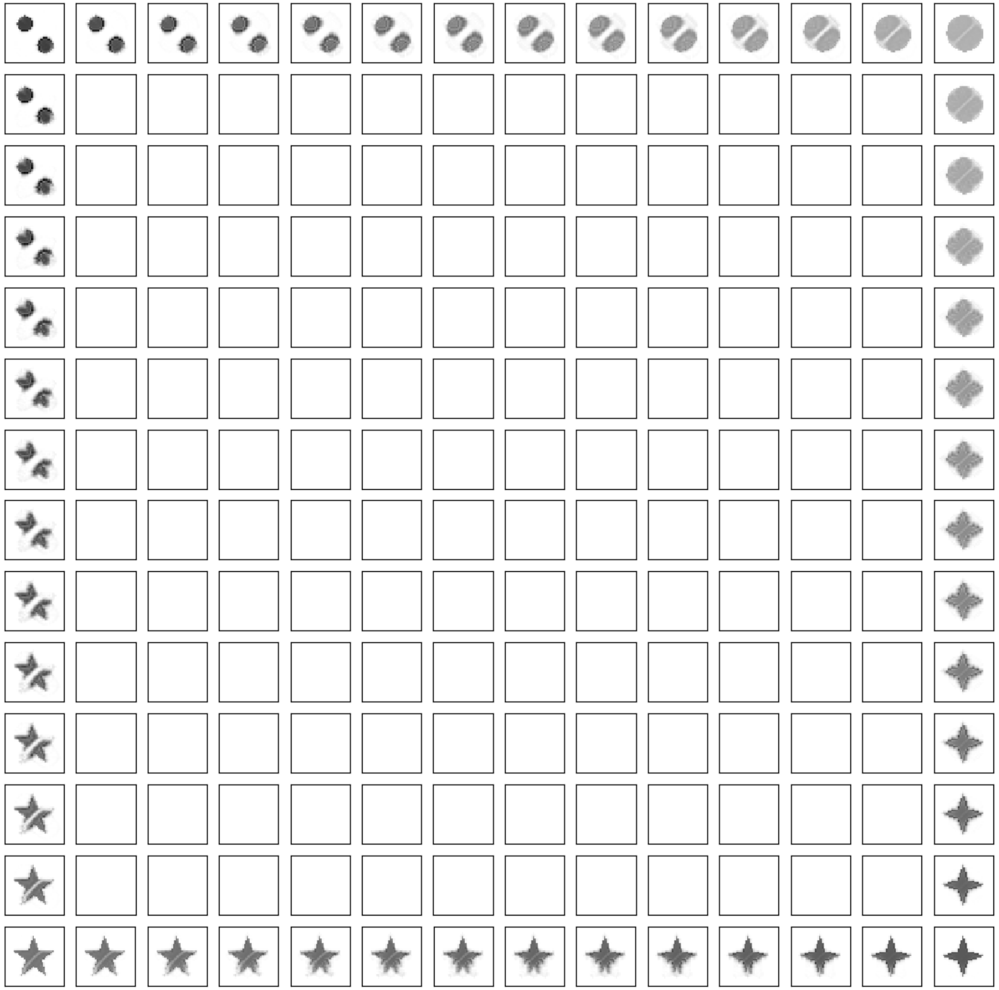
$$E(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, \mathrm{d}x$$

Dirichlet problem.

$$\min_\mu \{ \mathcal{T}_{E,\mathrm{Eul}}(\mu)$$

$$\mu_x \text{ given for } x \in \partial\Omega \}$$

Solutions are **harmonic** maps.



Lavenant (2019). Harmonic mappings valued in the Wasserstein space.

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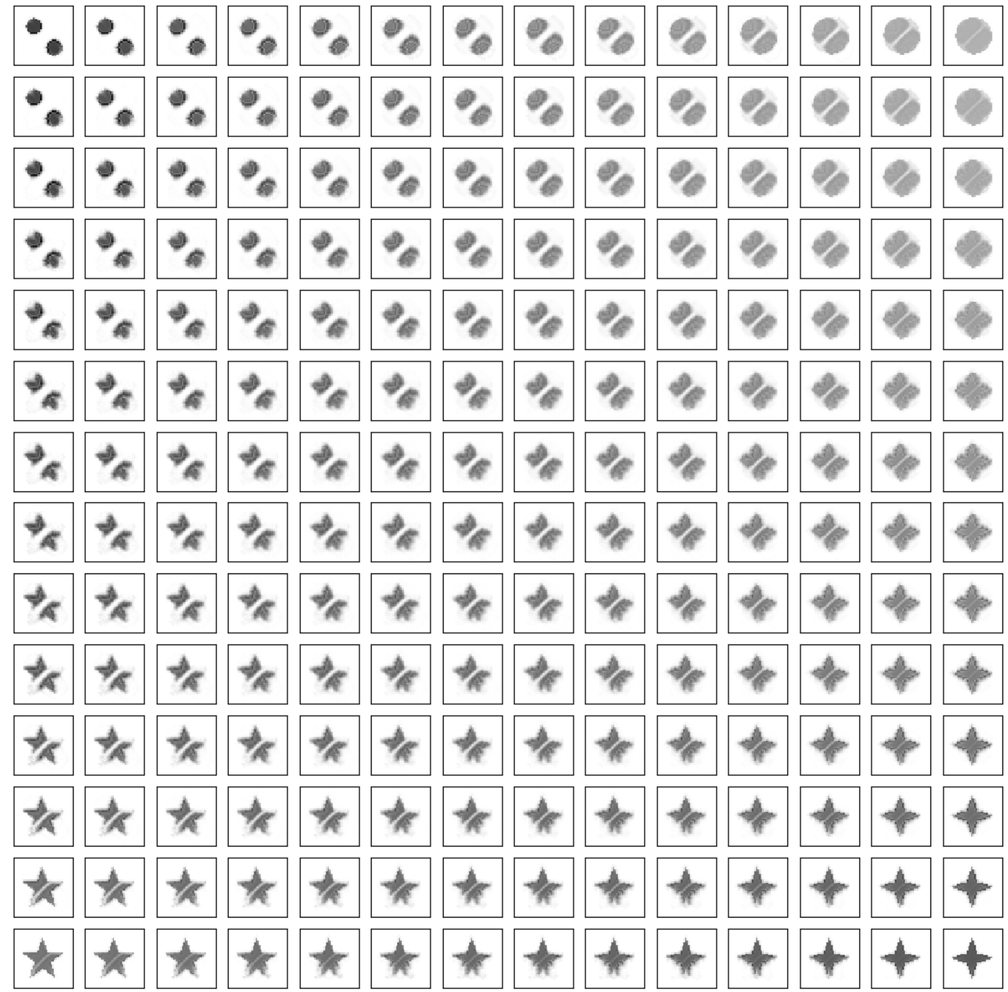

Solutions are **harmonic** maps.

Theorem. If $x \in \partial\Omega \to \mu_x$ is Lipschitz for $(\mathcal{P}(Y), W_2)$ then there exists a minimizer.

Lavenant (2019). Harmonic mappings valued in the Wasserstein space.

Some properties

Restrict to the case $E(u) = \int_\Omega W(\nabla u)$.

Proposition. The functional $\mu \to \mathcal{T}_{E,\mathrm{Eul}}(\mu)$ is convex and l.s.c.

under **assumption** that W grows at least like $|v|^p$ for some $p \geq 1$.

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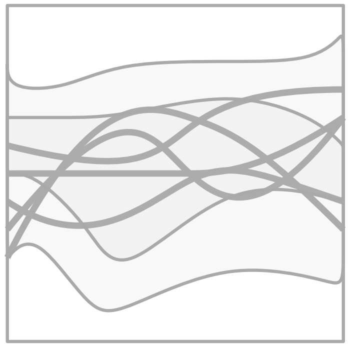
Proposition. $\mathcal{T}_{E,\mathrm{Eul}}(\mu_u) = E(u)$ for any u.

Consequence: $\mathcal{T}_{E,\mathrm{Eul}} \leq \mathcal{T}_E$. $\longrightarrow$ Equal if $\Omega = [0,1]$ is a segment!



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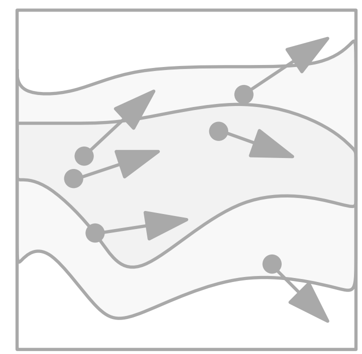
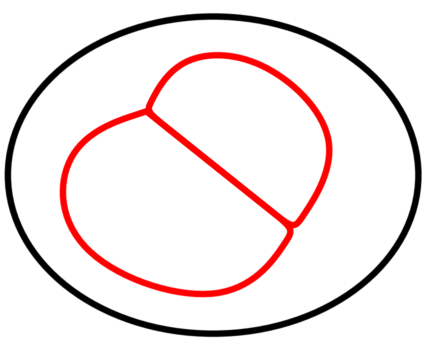
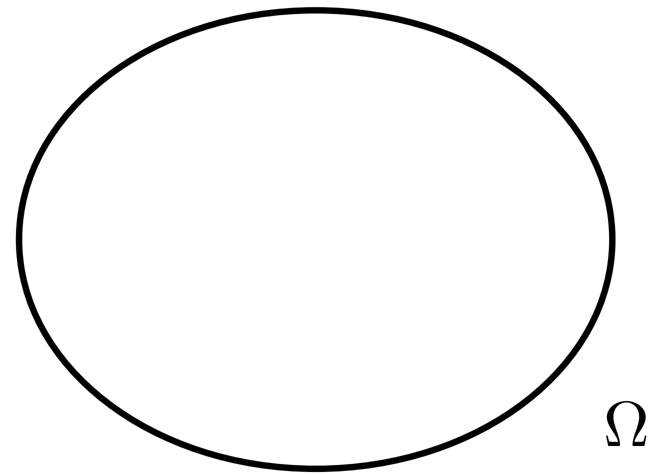




3 - Understanding the difference: localization of functionals

Localization of functionals

Previously: E depends on u: $\quad E(u) = \int_\Omega W(\nabla u).$



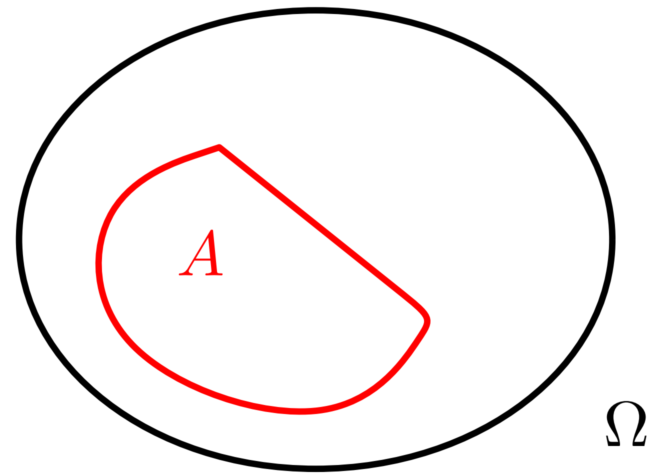
Localization of functionals

Previously: E depends on u: $\quad E(u) = \int_\Omega W(\nabla u).$

Localized version:

$$E(u, A) = \int_A W(\nabla u)$$

Function $\qquad$ Open set $A \subseteq \Omega$



Dal Maso (2012). An introduction to Γ-convergence.

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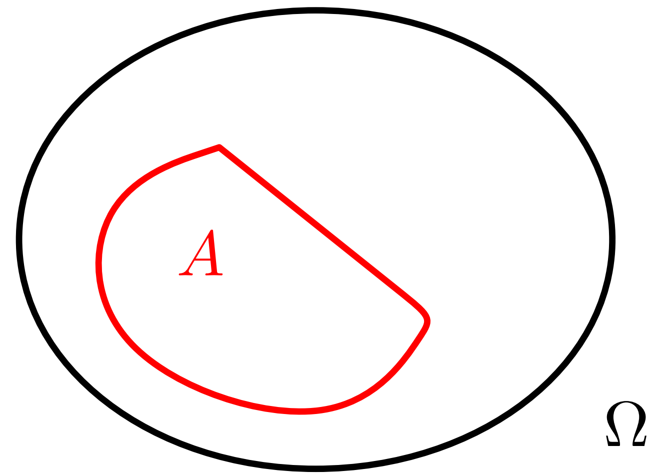

Definition. A localized functional E is

- **convex and l.s.c.** if $E(\cdot, A)$ is convex and l.s.c for any A.

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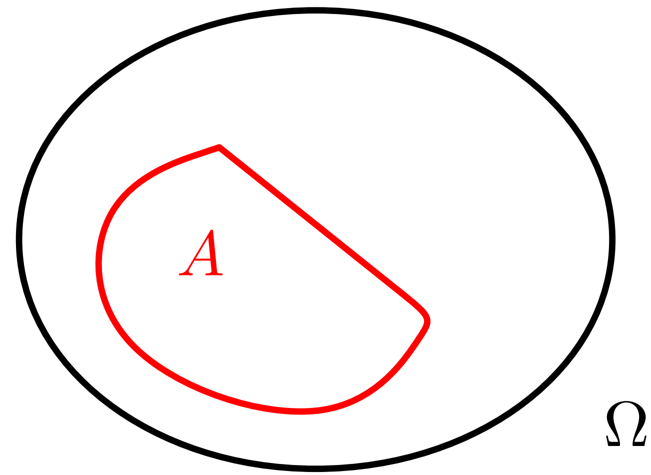

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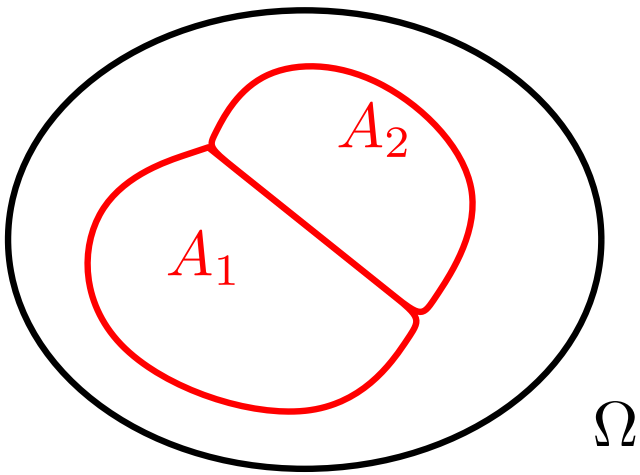

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- **local** if $E(u, A)$ only depends on the restriction of u to A.
- **additive** if for any u, A_1, A_2 with A_1, A_2 disjoint:

$$E(u, A_1 \cup A_2) = E(u, A_1) + E(u, A_2).$$

Dal Maso (2012). An introduction to Γ-convergence.

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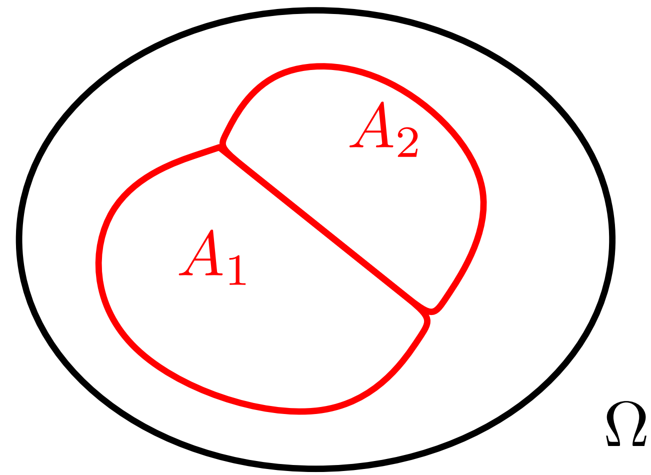

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- **a measure** if $A \mapsto E(u, A)$ is a measure.

Dal Maso (2012). An introduction to Γ-convergence.

For the Eulerian lifting

Under **assumption** that W grows at least like $|v|^p$ for some $p \geq 1$.

The functional $$E(u, A) = \int_A W(\nabla u)$$

is **convex**, **l.s.c.**, **local** and a **measure**.

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Localized Eulerian lifting:

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for $v : A \times \mathbb{R}^q \to \mathbb{R}^{q \times d}$

For the Eulerian lifting

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$$\mathcal{T}_{E,\mathrm{Eul}}(\mu, A) = \min_v \left\{ \int_A \int_{\mathbb{R}^q} W(v(x,y))\, \mathrm{d}\mu_x(y) \mathrm{d}x \text{ s.t. } \nabla_x \mu + \mathrm{div}_y(v\mu) = 0 \right\}$$

for $v : A \times \mathbb{R}^q \to \mathbb{R}^{q \times d}$

Proposition. This lifting is **convex, l.s.c., local** and a **measure**.

For the Lagrangian lifting

Localized version:

$$\mathcal{T}_E(\mu, A) = \inf_Q \left\{ \int_{L^0(X,Y,m)} E(u, A) \, \mathrm{d}Q(u) \ : \ Q \in \Pi(\mu) \right\}.$$

$\mathcal{T}_E$ is **local** if E is local.

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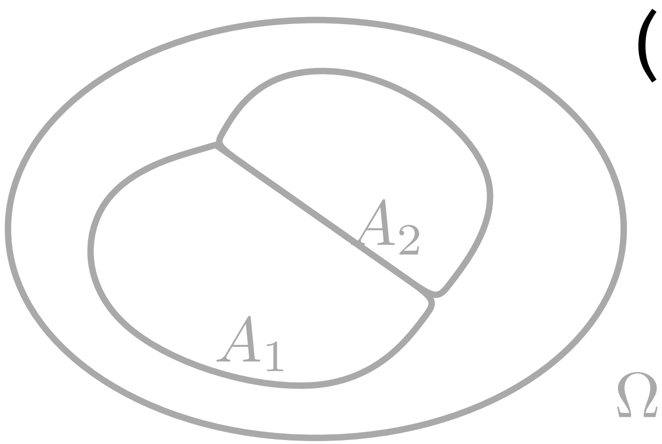
> **Proposition.** If E is a measure, then $\mathcal{T}_E$ is **superadditive**.

(But not a additive: see next slide)



$$\mathcal{T}_E(\mu, A_1 \cup A_2) \geq \mathcal{T}_E(\mu, A_1) + \mathcal{T}_E(\mu, A_2)$$

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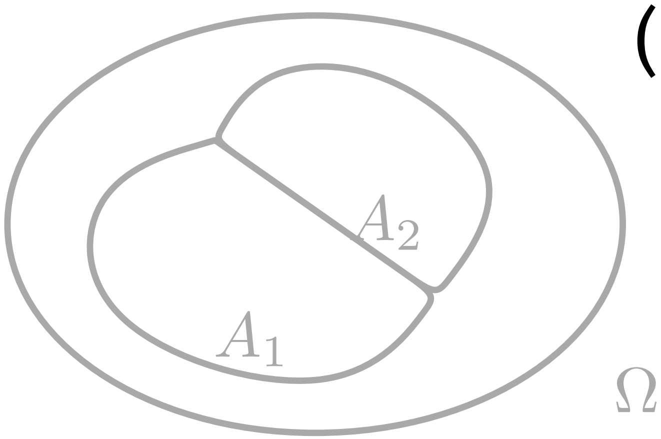

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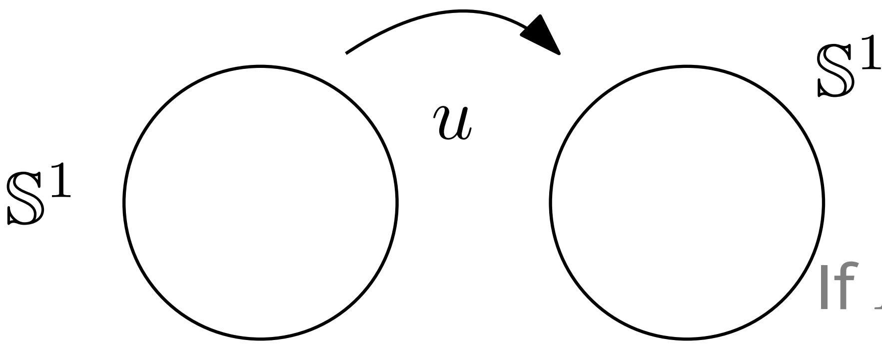
Counterexample for the Lagrangian lifting



$$E(u, A) = \frac{1}{2} \int_A |\dot{u}_t|^2 \, \mathrm{d}t$$

If $E(u, A) < +\infty$ then u continuous over A.

De Lellis and Spadaro (2011). Q-valued functions revisited.

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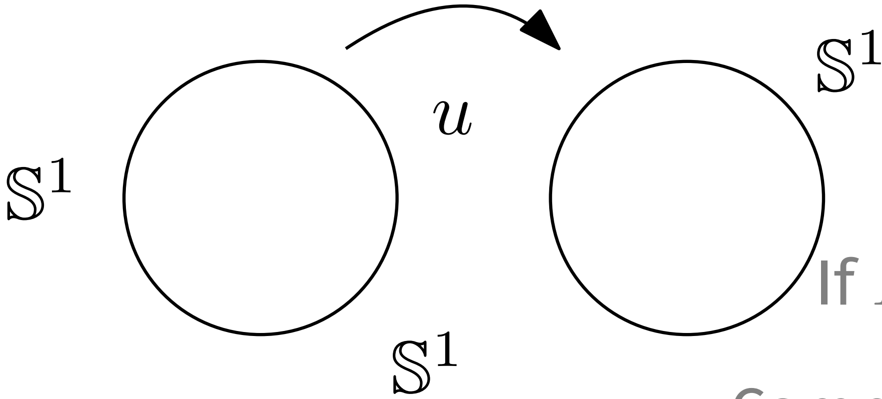

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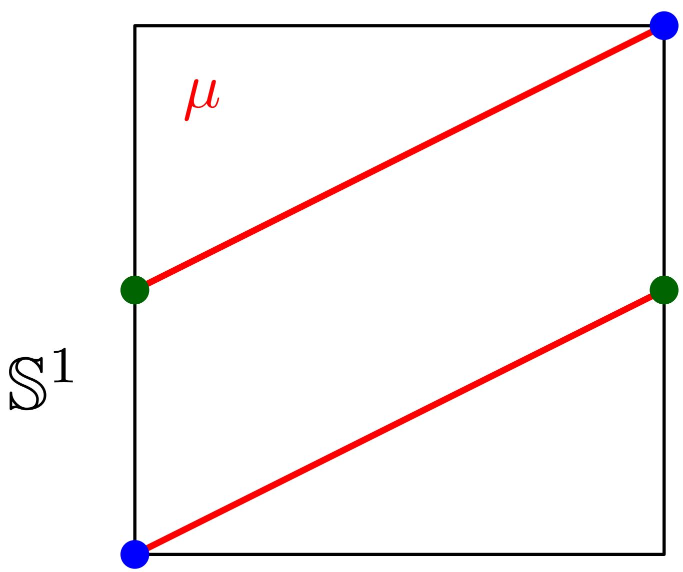
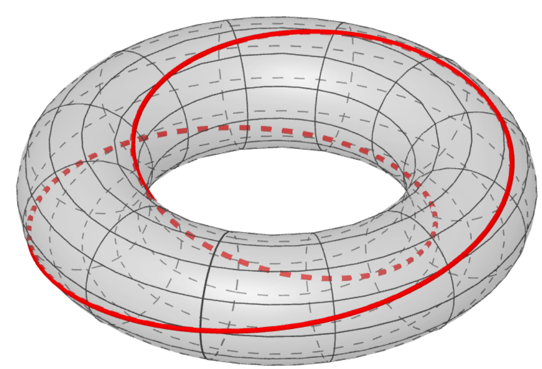


Complex square root

Define

$$\mu_x \to \frac{\delta_{\sqrt{x}} + \delta_{-\sqrt{x}}}{2}$$

$3d$ visualization



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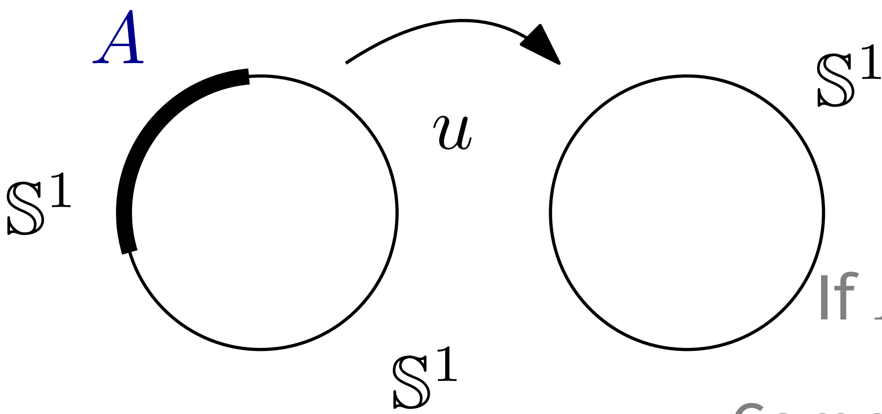

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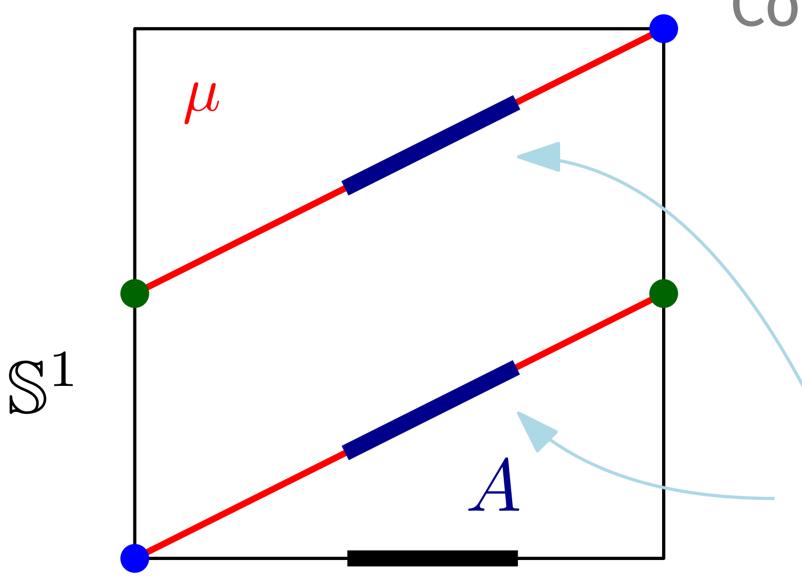

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Slope $1/2$

If A not dense in $\mathbb{S}^1$

$$\mathcal{T}_E(\mu, A) \leq \frac{1}{8}|A|$$

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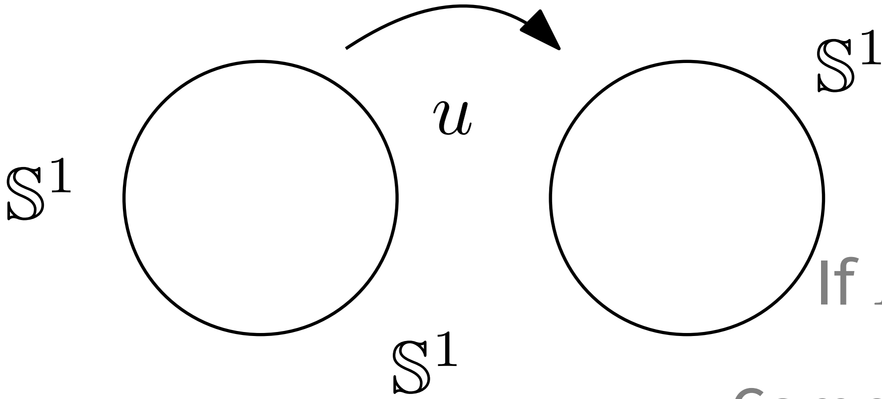

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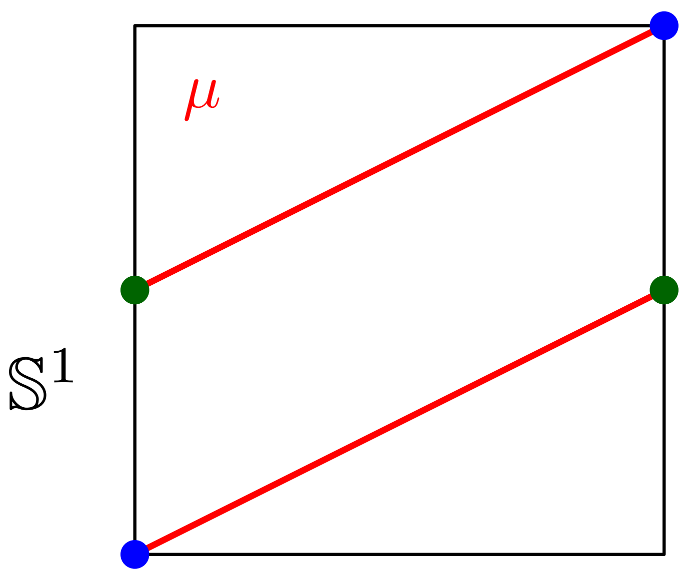

Complex square root

Define $\mu_x \to \dfrac{\delta_{\sqrt{x}} + \delta_{-\sqrt{x}}}{2}$

But $\mathcal{T}_E(\mu, \mathbb{S}^1) = +\infty$.

No **continuous** selection of the complex square root exists.

Thus $\mathcal{T}_E(\mu, \cdot)$ is **not** additive.

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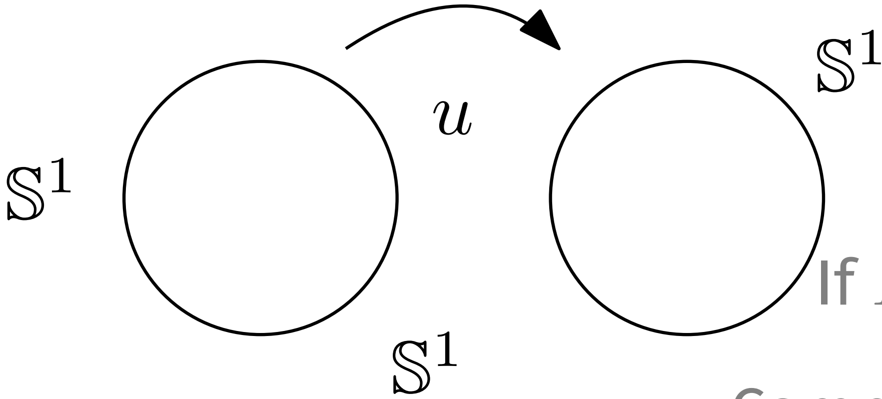

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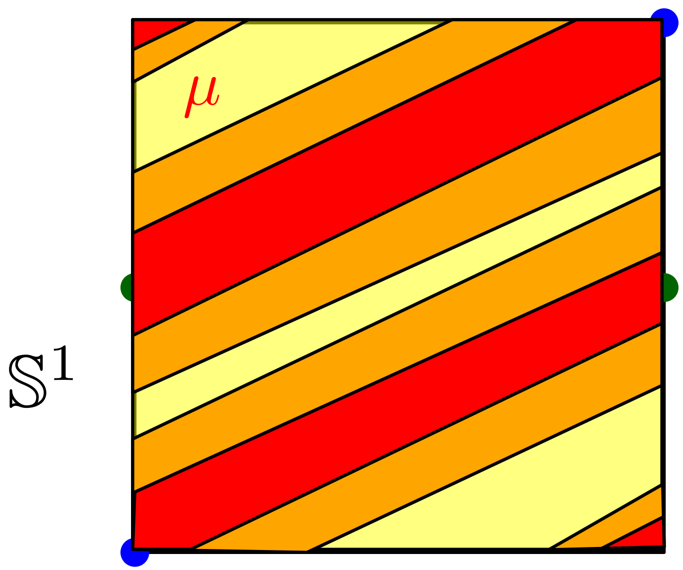

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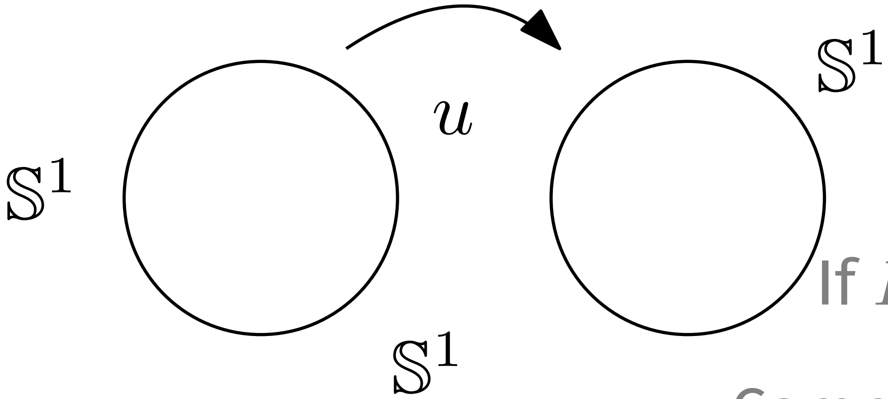

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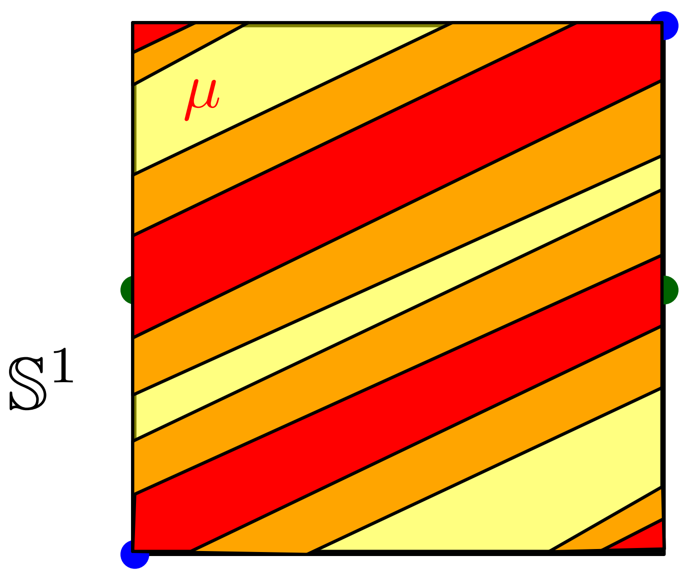

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- Extension to smoothed version.
- Extension to maps $\mathbb{R}^2 \to \mathbb{R}^2$.

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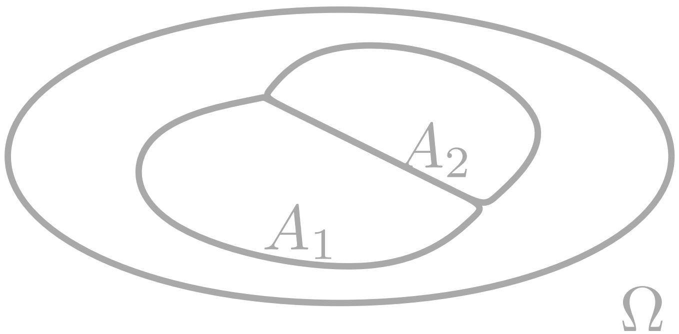
Optimality of the Eulerian lifting

We define $\overline{\mathcal{T}}_E$ the largest $\mathcal{T}$ **convex, l.s.c., subadditive**, increasing and inner regular such that $\mathcal{T}(\mu_u, A) = E(u, A)$.



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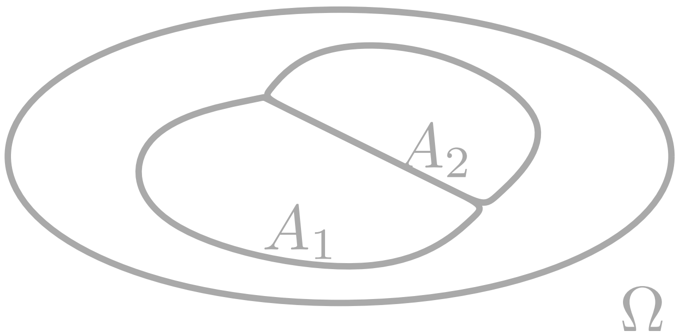

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Theorem. For $W : \mathbb{R}^{qd} \to [0, +\infty]$ convex, approximatively radial define

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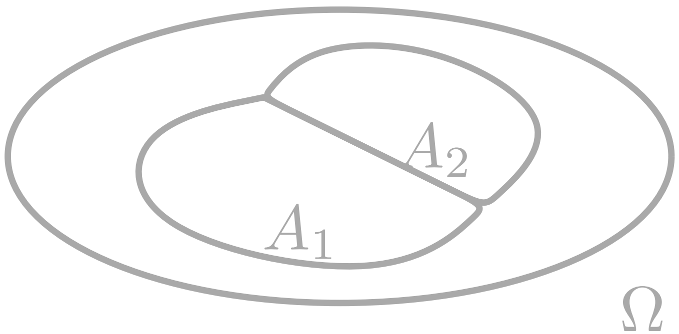

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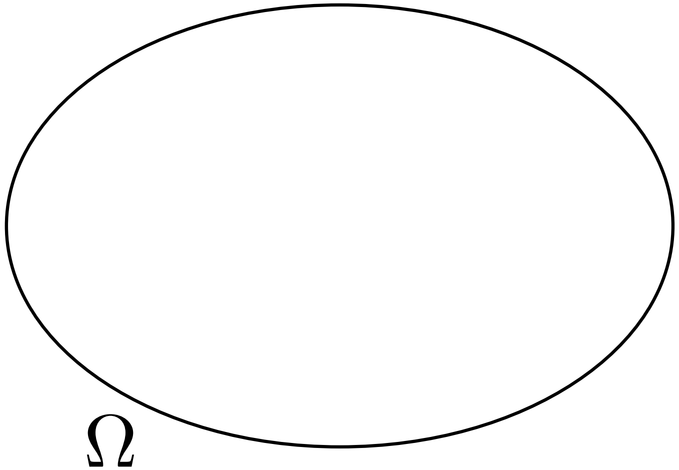
Idea of the proof

Q optimal for $\mathcal{T}_E$

If $\mathcal{T}_E(\mu) = \mathcal{T}_{E,\mathrm{Eul}}(\mu)$ then for Q-a.e. map $u : \Omega \to \mathbb{R}^q$

$$\boxed{\nabla u(x) = v(x, u(x))}$$

v optimal in $\mathcal{T}_{E,\mathrm{Eul}}$



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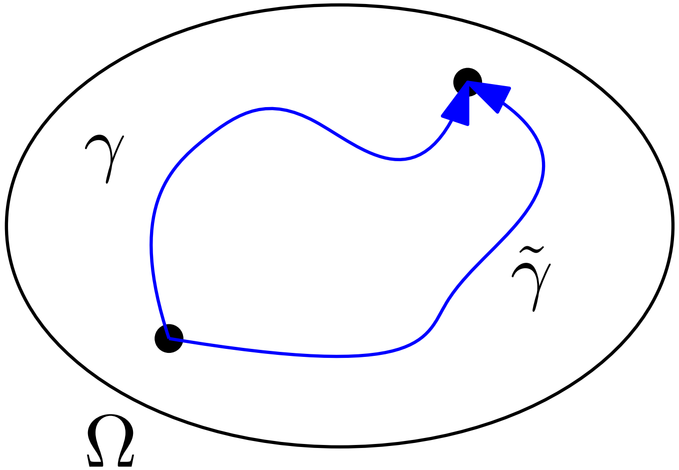

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For any $\gamma : I \to \Omega$, $y(t) = u(\gamma_t)$ solution of ODE $\dot{y}_t = v(t, y_t)\dot{\gamma}_t$.

$\rightsquigarrow$ incompatibility for different γ, $\tilde{\gamma}$ joining same points.

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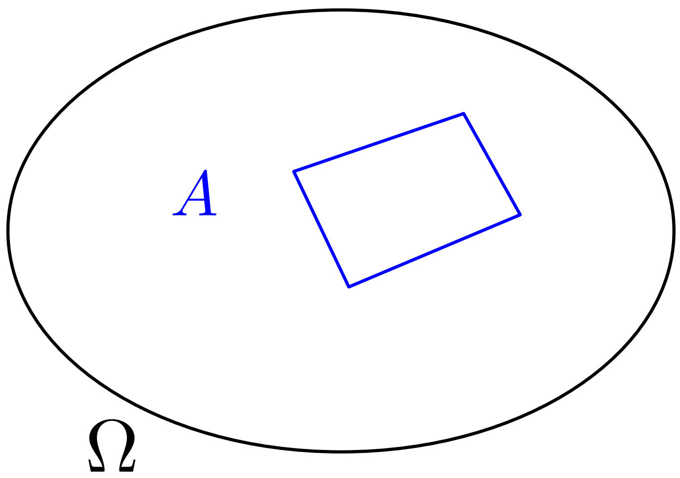

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Lemma. If v is smooth, for $A \subseteq \Omega$ starshaped,

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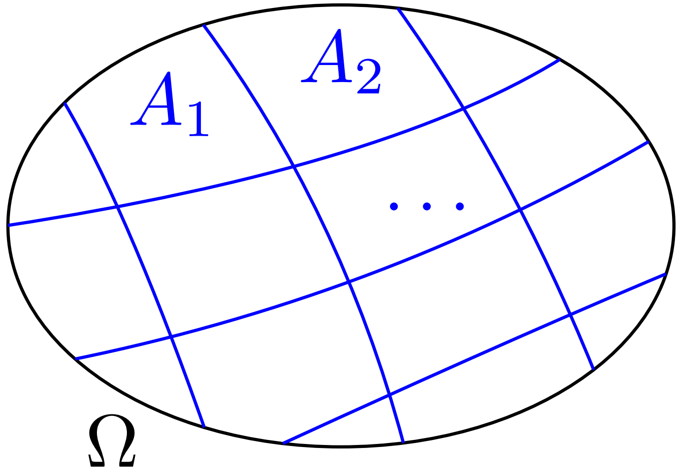
Small if A is small

Idea of the proof



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Regularize (μ, v), cut Ω in pieces $A_1, \dots, A_n$ of diameters ε,

$$\overline{\mathcal{T}}_E(\mu, \Omega) \leq \sum_i \mathcal{T}_E(\mu, A_i) \leq \sum_i \mathcal{T}_{E,\mathrm{Eul}}(\mu, A_i) + C\varepsilon m(A_i)$$
$$\leq \mathcal{T}_{E,\mathrm{Eul}}(\mu, \Omega) + C\varepsilon.$$

Conclusion

Question. What is the **largest** convex and **l.s.c.** (for narrow convergence on $\mathcal{M}_+(X \times Y)$) functional

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Eulerian formulation, subadditive envelope

Multimarginal OT

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Open question. Define $E(u) = \displaystyle\int W(\nabla^2 u)$ for W convex.

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Thank you for your attention

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