Be sure this exam has 5 pages.

THE UNIVERSITY OF BRITISH COLUMBIA

Sessional Examination - April 15, 2020

MATH 340: Linear Programming

Instructor: Dr. Richard Anstee, Section 201, Dr. Hugo Lavenant, Section 202

Special Instructions: Open book exam but no software aids (such as LINDO?LINGO). You must show your work and **explain** your answers. Answers are to be uploaded to Crowdmark, each question separately. Quote names of our main LP theorems used, as appropriate.

Time: 4 hours (extra time provided for uploading) Total marks: 100

1. Sign and upload Integrity Pledge. It is required.

2. [12 marks]

a) [10 marks] Solve the following linear programming problem, using our standard two phase method and using Anstee's rule. It will take three pivots in phase 1 (including special pivot to feasibility) and one pivot in phase 2.

$$\begin{array}{lrrrcr} \text{Maximize} & -2x_1 & & -x_3 & & \\ & -x_1 & -x_2 & +x_3 & \leq & 1 \\ & & -x_2 & +x_3 & \leq & -2 \\ & -x_1 & & -x_3 & \leq & -1 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

b) [2 marks] Either give two optimal solutions or show the optimal solution is unique.

3. [10 marks] Given $A, \mathbf{b}, \mathbf{c}$, current basis (and B^{-1} for your computational ease), use our Revised Simplex method and Anstee's rule to determine the next entering variable (if there is one), the next leaving variable (if there is one), and (if there is both an entering and leaving variable) the new basic feasible solution after the pivot as well as the eta matrix E that has (new B) = (old B)E. The current basis is $\{x_3, x_2, x_6\}$.

$$A = \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 5 & 2 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 & 0 \\ 15 & 2 & 3 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \\ x_4 \\ x_5 \\ x_6 \end{array} \begin{array}{c} \mathbf{b} \\ \begin{pmatrix} 4 \\ 3 \\ 11 \end{pmatrix} \end{array} \quad B^{-1} = \begin{array}{c} \\ x_3 \\ x_2 \\ x_6 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} -1 & 2 & 0 \\ 1 & -1 & 0 \\ 1 & -4 & 1 \end{pmatrix} \end{array}$$

$$c^T = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 8 & -1 & 2 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

4. [6 marks] Consider a primal in standard inequaliity form. There are six independent questions below. What can you say about the dual if you already know: (answer for each of the following considered individually)

   a) a feasible solution to the primal exists?

   b) an optimal solution to the primal exists?

   c) an optimal solution to the primal exists with $x_1 > 0$?

   d) an optimal solution to the primal exists with $x_1 = 0$?

   e) there is no feasible solution to the primal?

   f) the primal is unbounded?

5. [25 marks] A student wants to make healthy smoothies. The student will use a food processor, kale and carrots. The food processor has only a limited number of hours available to the student. The supplies of kale and carrots are limited (the students doesn't want to go to the store again soon). Each unit of these smoothies brings some health boost. We have listed the requirements for a unit of each smoothie in terms of food processor hours, bunches of Kale, Kgs Carrots as well as the resulting health boosts. The student would like to maximize the total health boost.

	light green shake	just green shake	dark green shake	availability
food processor	2	3	3	13
kale	2	3	4	14
carrots	1	2	1	7
health boost	4	7	5	

Let x_1 denote the units of light green shake, x_2 denote the units of just green shake and x_3 denote the units of dark green shakes and let x_{3+i} denote the ith slack for $i = 1, 2, 3$. We allow fractional values for our variables. The final dictionary is:

$$\begin{array}{rcrrrr} x_1 & = & 2 & -5x_4 & +3x_5 & +3x_6 \\ x_2 & = & 2 & +2x_4 & -x_5 & -2x_6 \\ x_3 & = & 1 & +x_4 & -x_5 & \\ z & = & 35 & -x_4 & & -2x_6 \end{array} \qquad B^{-1} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 5 & -3 & -3 \\ -2 & 1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

NOTE: All questions are independent of one another. They are not cumulative.

a) [2 marks] Give the marginal values for each of the resources: food processor, kale, and carrots. What are the units for the marginal values?

b) [3 marks] Given a new product with requirements $\begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$, give the range on the health boost associated with this new product for which the solution above remains optimal.

c) [5 marks] Give the range on c_2 (health boost from green shakes) so that the current solution remains optimal.

d) [5 marks] Consider changing the availability of the food processor time to $13 + p$ hours, the bunches kale to $14 + p$ and the kilograms of carrots to $7 + p$. Determined the range on p so that the current basis $\{x_1, x_2, x_3\}$ remains optimal and report the health boost as a function of p in that interval.

Hint for e),f): You need not complete all of the very final dictionary, merely the variables in the basis and their values and **all** the entries in the z row (a partial dictionary). Show your work.

e) [5 marks] Change the availability of food processor to 8, kale to 9 and carrots to 3. Determine the new optimal solution using the Dual Simplex method. Report the new solution as well as the new marginal values.

f) [5 marks] Consider adding a new constraint $x_2 \geq 2\frac{1}{2}$. Solve using the Dual Simplex method. Report the new solution as well as the new marginal values.

6. [12 marks] We are choosing a diet of Indian food (Canadian style) to meet nutrional requirements at minimum cost. Each food has a $ cost for the ingredients and in addition a cost associated with time: we value time at $15 per hour. Thus the cost of an item is the cost of the ingredients plus the cost of the time. We have limited availability of time (25 hours), spices (18) and onions (25). We also have a minimum constraint on bread (chapatis and naan total at least 20) and maximum constraints on chapatis (maximum 15) and naan (maximum 15). We seek a diet of total nutrition of 50.

	time(hours)	cost ingredients	spices	onions	nutrition
lentils	1	5	2	3	4
butter chicken	1	7	3	2	4
black beans	1	3	2	3	4
chapatis	.5	0	0	0	1
naan	0	8	0	0	1

Our LINDO input is:

```
max -20lentil-22chicken-18blckbean-7.5chapatis-8naan
subject to
spice)2lentil+2chicken+3blckbean<18
onion)3lentil+2chicken+2blckbean<25
nutrtion)4lentil+4chicken+4blckbean+chapatis+naan>50
time)lentil+chicken+blckbean+0.5*chapatis<25
bread)chapatis+naan>20
chapmax)chapatis<15
naanmax)naan<15
end
```

The LINDO output is on the next page. Each of the following questions are independent of each other.

a) [3 marks] Chapatis are made with whole wheat flour which is essentially free. What happens to the cost of the diet if whole wheat flour is hard to get and you are only able to make 10 chapatis?

b) [3 marks] You have found an old recipe for curry that costs 1.5 for ingredients plus .5 hour time, 1 unit of spices, 2 units of onions with a nutritional value of 2. Should you consider making this dish?

c) [3 marks] Thinking of the diet as a whole, what is ratio of the cost to the nutrition contribution. Compare it with the dual price of just the nutrition constraint.

d) [3 marks] What would you predict about the cost if the spice supply rises by 2.25 and simultaneously we allow 2.5 more chapatis. Explain

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LP OPTIMUM FOUND AT STEP      0

        OBJECTIVE FUNCTION VALUE

        1)      -296.5000
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Variable	Value	Reduced Cost
LENTIL	4.500000	0.000000
CHICKEN	0.000000	2.000000
BLCKBEAN	3.000000	0.000000
CHAPATIS	15.00000	0.000000
NAAN	5.000000	0.000000

Row	Slack or Surplus	Dual Price
SPICE	0.000000	2.000000
ONION	5.500000	0.000000
NUTRTION	0.000000	-6.000000
TIME	10.00000	0.000000
BREAD	0.000000	-2.000000
CHAPMAX	0.000000	0.5000000
NAANMAX	10.00000	0.000000

Ranges in which the basis is unchanged:

Objective Coefficient Ranges:

Variable	Current Coefficient	Allowable Increase	Allowable Decrease
LENTIL	-20.00000	2.000000	2.000000
CHICKEN	-22.00000	2.000000	INFINITY
BLCKBEAN	-18.00000	4.000000	2.000000
CHAPATIS	-7.500000	INFINITY	0.5000000
NAAN	-8.000000	0.5000000	INFINITY

Righthand Side Ranges:

Row	Current RHS	Allowable Increase	Allowable Decrease
SPICE	18.00000	4.500000	3.000000
ONION	25.00000	INFINITY	5.500000
NUTRTION	50.00000	4.400000	6.000000
TIME	25.00000	INFINITY	10.00000
BREAD	20.00000	6.000000	4.400000
CHAPMAX	15.00000	5.000000	10.00000
NAANMAX	15.00000	INFINITY	10.00000

7. [10 marks] Let A be a given $m \times n$ matrix and let B be a given $p \times n$ matrix. Let $\mathbf{b}$ be a $m \times 1$ vector. Show that either

   i) There exist an $\mathbf{x}$ with $A\mathbf{x} = \mathbf{b},\ B\mathbf{x} \leq \mathbf{0}$,
   or
   ii) There exists $\mathbf{y}, \mathbf{z}$ with $A^T\mathbf{y} + B^T\mathbf{z} = \mathbf{0}, \quad \mathbf{z} \geq \mathbf{0}, \quad \mathbf{b} \cdot \mathbf{y} < 0$,
   but not both.

8. [5 marks] Consider an LP in standard inequality form whose optimal solution has an objective function value of 10. What happens to the optimal objective function value if we delete a constraint from the LP. Explain.

9. [5 marks] Consider the game given by payoff matrix A below (the payoff to the row player).

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 5 & -1 & 3 \end{bmatrix}$$

   If the row player plays a strategy $\mathbf{x} = (1/4, 3/4)^T$ and the column player knows that, what should the column player play? If the column player plays a strategy $\mathbf{y} = (1/3, 1/3, 1/3)^T$ and the row player knows that, what should the row player play? What can you say about the value of the game?

10. [5 marks] Consider an LP in standard inequality form whose coefficient in the objective function of x_1 is c_1. If we change c_1 from 7 to 10, the optimal value of the objective function increases from -5 to +10. What can you say about the value of x_1 at optimality before and after the change? Explain.

11. [5 marks] You are given the optimal solution $\mathbf{x}^*$ to the LP:

$$\begin{aligned} \max \quad & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

   Assume $A > 0$ and $\mathbf{b} > \mathbf{0}$ (all entries strictly positive). Consider adding a new column A_{new} to A and a new variable x_{new} with profit coeficient c_{new}. If $A_{new} = \mathbf{b}$, under what conditions on c_{new} would it be an optimal solution (to the new LP) to have $x_{new} = 1$. Would such a solution be degenerate?

12. [5 marks] Let A, $\mathbf{b}$ and $\mathbf{c}$ be given. Assume that our standard LP: max $\mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has an optimal solution. Assume the LP: max $(-\mathbf{c}) \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ has an optimal solution. Show that for any $\mathbf{u}$ satisfying $A\mathbf{u} \leq \mathbf{0}$ and $\mathbf{u} \geq \mathbf{0}$ that we may deduce that $\mathbf{u}$ also satisfies $\mathbf{c} \cdot \mathbf{u} = 0$.

100 total marks

End of Exam