A Geometric standpoint on Linear Programming – Part II

Simplex algorithm, duality, sensitivity analysis

Disclaimers

The goal is to give an alternative (geometric) standpoint on Linear Programming. You will *not* be tested on that in the final, but I hope it will give you more insights about LP.

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We will talk about geometric insights. Although we will do geometry in dimension n with sometimes $n \geqslant 4$, all the intuition comes from $n = 2$ or $n = 3$. Don't try to imagine what is a space of dimension 4!

1. Geometric representation of a Linear Program
2. Simplex algorithm
3. Duality
4. Sensitivity analysis

1. Geometric representation of a Linear Program

An example

Let's consider the following LP

$$\begin{array}{llllll} \text{maximize} & x_1 & +x_2 & & & \\ \text{subject to} & 3x_1 & +2x_2 & \leqslant & 3 & \qquad x_1, x_2 \geqslant 0. \\ & x_1 & +2x_2 & \leqslant & 2 & \end{array}$$

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The feasible region of this LP was already plotted in the previous lecture. New data here:

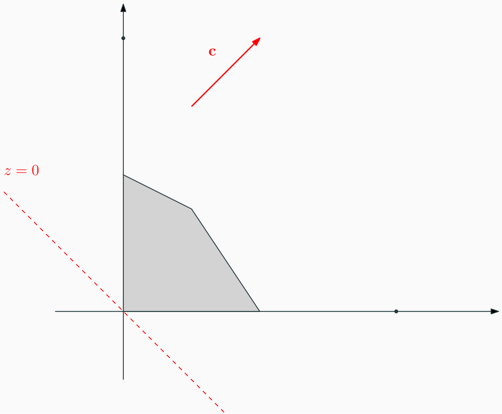
$$\mathbf{c} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Geometric representation

The feasible region is the gray area.

$\mathbf{c}$ gives the direction of the objective function.

Geometric representation



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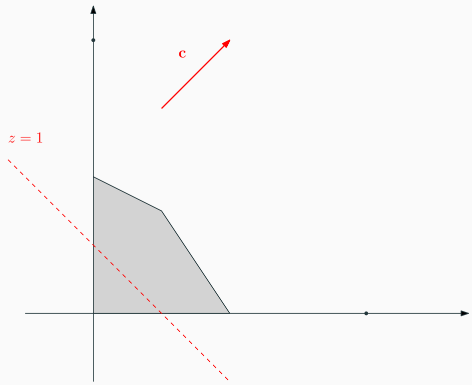
Different level sets of the objective function (hyper-planes normal to $\mathbf{c}$).

Geometric representation



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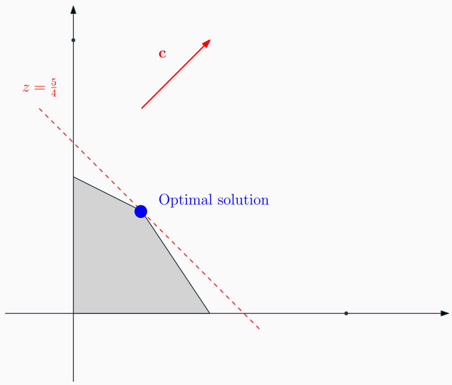

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Different level sets of the objective function (hyperplanes normal to $\mathbf{c}$).

Optimal solution: point of the feasible region “the farthest” in the direction defined by $\mathbf{c}$.

Main insight

Geometric description of Linear Programming

The constraints define a convex polytope.

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(Almost) all the course could be read with this geometric insight.

Fundamental theorem of Linear Programming

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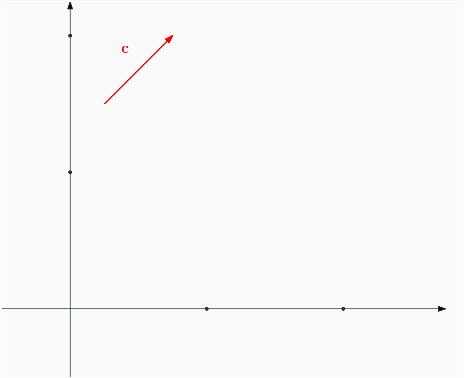
Each Linear Program is one out of the three following outcome:



Infeasible

The polytope is empty.

Fundamental theorem of Linear Programming

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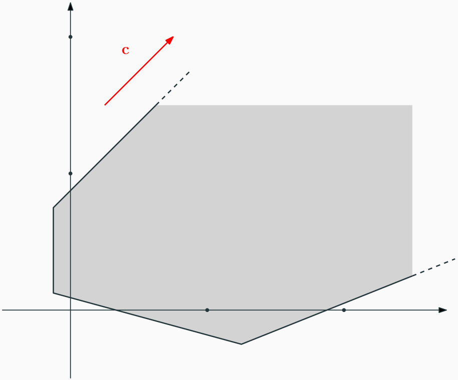
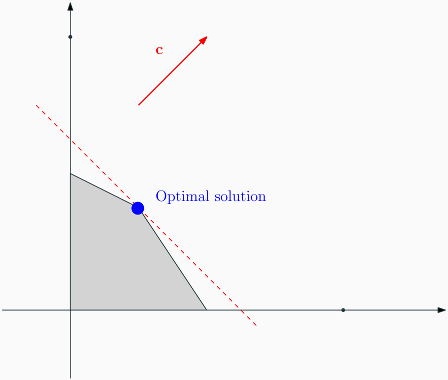


Unbounded

The polytope is not bounded in the direction $\mathbf{c}$.

Fundamental theorem of Linear Programming

Each Linear Program is one out of the three following outcome:

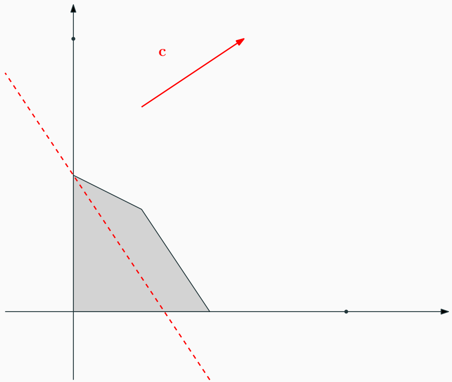


Has an optimal solution

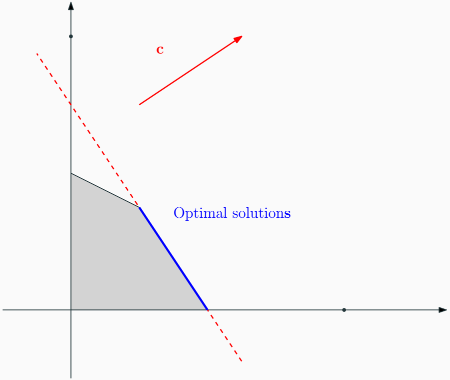
This is Slide **??**.

Non uniqueness

Non uniqueness

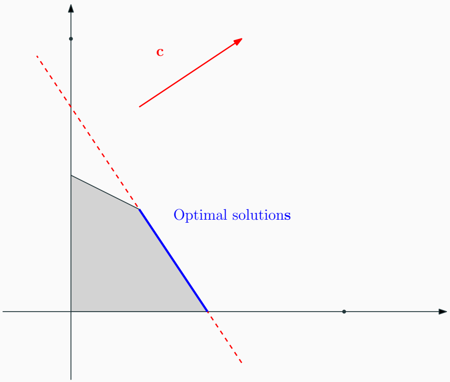


Non uniqueness



Here the set of optimal solutions is an edge of the polytope.

Non uniqueness



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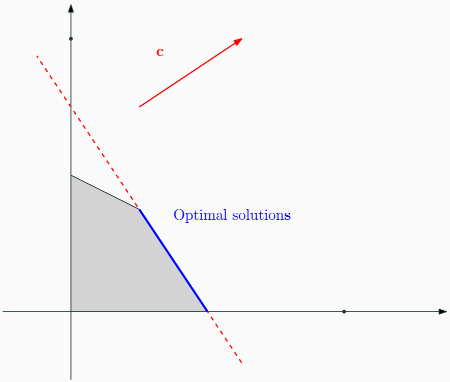
In 3 dimensions, the set of optimal solutions can be:

- a vertex (unique solution),
- an edge,
- or a face.

Non uniqueness



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In 3 dimensions, the set of optimal solutions can be:

- a vertex (unique solution),
- an edge,
- or a face.

The set of optimal solutions can even be unbounded.

2. Simplex algorithm

Main idea

Last time: feasible dictionaries are linked to the vertices of the polytope representing the feasible region.

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The simplex algorithm is about moving from one vertex to another of the feasible region in the direction given by $\mathbf{c}$.

Main idea

Last time: feasible dictionaries are linked to the vertices of the polytope representing the feasible region.

Simplex algorithm: going from one feasible dictionary to the next.

Insight

The simplex algorithm is about moving from one vertex to another of the feasible region in the direction given by $\mathbf{c}$.

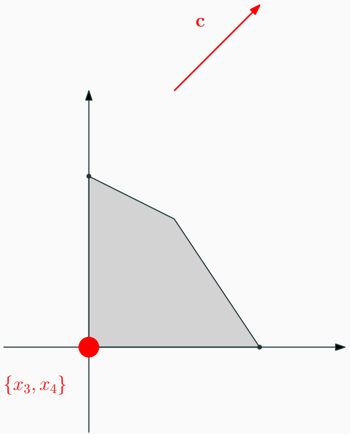
Choosing a entering and a leaving variable means choosing an *edge* of the polytope and moving along this edge to reach the new vertex (that is a new dictionary).

Example

We consider the same LP:

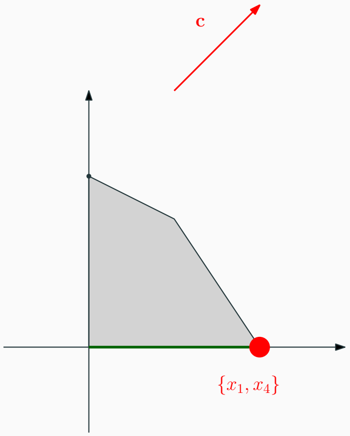
$$\begin{array}{rlllll} \text{maximize} & x_1 & +x_2 & & & \\ \text{subject to} & 3x_1 & +2x_2 & \leqslant & 3 & \qquad x_1, x_2 \geqslant 0. \\ & x_1 & +2x_2 & \leqslant & 2 & \end{array}$$

Example



$$
\begin{array}{lcccc}
x_3 & = & 3 & -3x_1 & -2x_2 \\
x_4 & = & 2 & -x_1 & -2x_2 \\
z & = & & x_1 & +x_2
\end{array}
$$

Example



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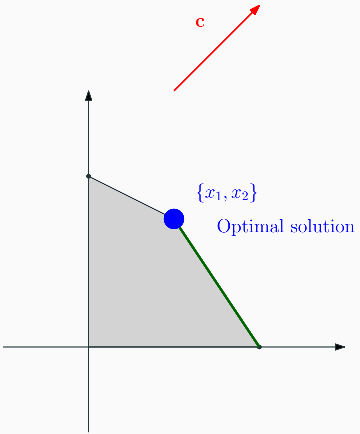
z & = & & x_1 & +x_2
\end{array}
$$

x_1 enters, x_3 leaves.

$$
\begin{array}{rcrrr}
x_1 & = & 1 & -1/3\, x_3 & -2/3\, x_2 \\
x_4 & = & 1 & +1/3\, x_3 & -4/3\, x_2 \\
z & = & 1 & -1/3\, x_3 & +1/3\, x_2
\end{array}
$$

Example



$$\begin{array}{rcrrr} x_3 & = & 3 & -3x_1 & -2x_2 \\ x_4 & = & 2 & -x_1 & -2x_2 \\ z & = & & x_1 & +x_2 \end{array}$$

x_1 enters, x_3 leaves.

$$\begin{array}{rcrrr} x_1 & = & 1 & -1/3\,x_3 & -2/3\,x_2 \\ x_4 & = & 1 & +1/3\,x_3 & -4/3\,x_2 \\ z & = & 1 & -1/3\,x_3 & +1/3\,x_2 \end{array}$$

x_2 enters, x_4 leaves.

$$\begin{array}{rcrrr} x_1 & = & 1/2 & -1/2\,x_3 & +1/2\,x_4 \\ x_2 & = & 3/4 & +1/4\,x_3 & -3/4\,x_4 \\ z & = & 5/4 & -1/4\,x_3 & -1/4\,x_4 \end{array}$$

A remark on cycling

Cycling in the simplex is when we visit a periodic sequence of dictionaries.

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Cycling in the simplex is when we visit a periodic sequence of dictionaries.

From a geometric point of view, in cycling you always stay at the same vertex, but this vertex is degenerate. So more than one dictionary is associated to this vertex, and you cycle between dictionaries associated to the same vertex.

3. Duality

Disclaimer

Duality is maybe the hardest to fit in this geometric standpoint.

I’m not aware of a clean and nice interpretation of the dual problem which explains it much better than algebraic manipulations.

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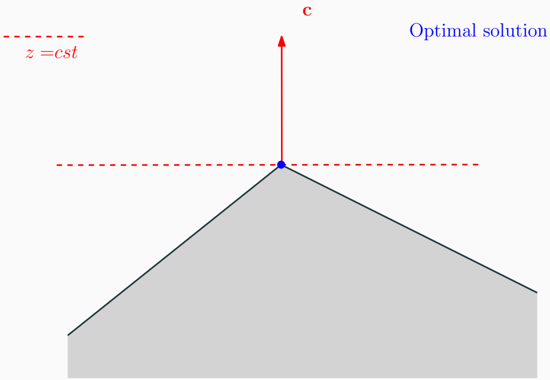
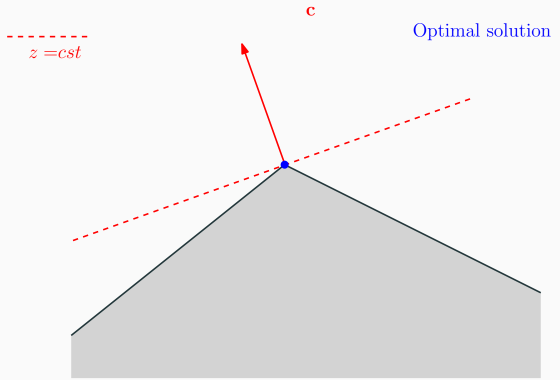
One approach to duality: certificate to check the optimality of a solution.

Is a vertex the optimal solution?



In this situation, we know that the vertex is the optimal solution.

Is a vertex the optimal solution?



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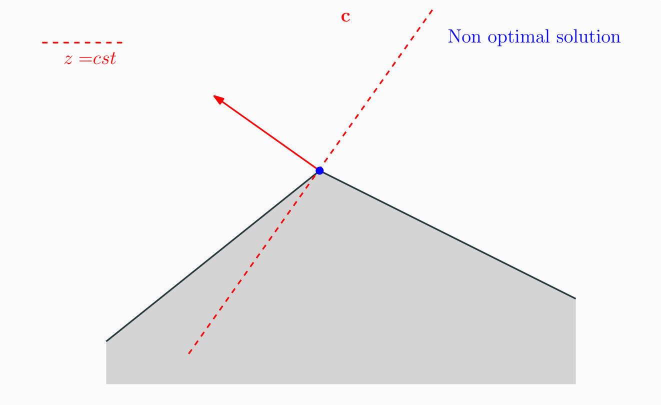
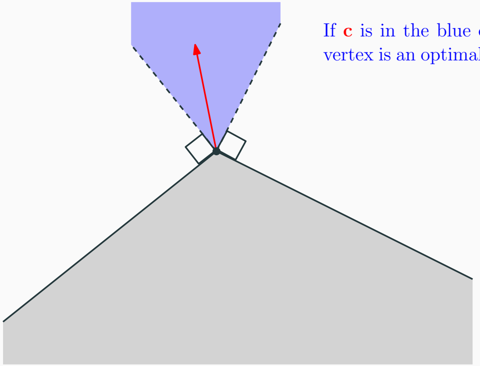
Is a vertex the optimal solution?



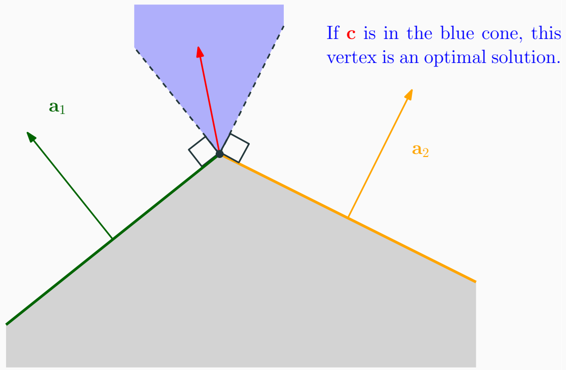
In this situation, we know that the vertex is not the optimal solution.

Optimal cone



The vertex is the optimal solution if $\mathbf{c}$ is in the blue cone,

Optimal cone



The vertex is the optimal solution if $\mathbf{c}$ is in the blue cone, that is

$$\mathbf{c} = y_1 \, \mathbf{a}_1 + y_2 \, \mathbf{a}_2 \qquad \text{with } y_1, y_2 \geqslant 0$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are normal to the two hyperplanes defining the “active” constraints.

A justification via the duality theorem 1

Now let's show analytically what was understood geometrically in the previous slide.

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Let's consider the primal dual pair with n decision variables and m constraints in the primal

$$\begin{array}{lrcl} \text{maximize} & \mathbf{c}^\top \mathbf{x} & & \\ \text{such that} & A\mathbf{x} & \leqslant & \mathbf{b} \\ & \mathbf{x} & & \text{free} \end{array} \qquad \begin{array}{lrcl} \text{minimize} & \mathbf{b}^\top \mathbf{y} & & \\ \text{such that} & A^\top \mathbf{y} & = & \mathbf{c} \\ & \mathbf{y} & \geqslant & \mathbf{0} \end{array}$$

We assume that both LPs have optimal solutions $\mathbf{x}^*, \mathbf{y}^*$.

A justification via the duality theorem 2

Let $\mathbf{a}_j^\top \in \mathbb{R}^n$ be the j-th row of A, that is

$$\mathbf{a}_j = \begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{pmatrix}.$$

The vector $\mathbf{a}_j$ is normal to the hyperplane delimited by the j-th constraint of the primal.

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The equation $A^\top \mathbf{y}^* = \mathbf{c}$ can be rewritten

$$\sum_{j=1}^{m} y_j^* \, \mathbf{a}_j = \mathbf{c}$$

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The equation $A^\top \mathbf{y}^* = \mathbf{c}$ can be rewritten

$$\sum_{j=1}^{m} y_j^* \, \mathbf{a}_j = \mathbf{c}$$

But $\mathbf{y}^* \geqslant \mathbf{0}$ and the y_j^* are zero if the constraints in the primal are not active, that is if they are strict inequalities, thanks to complementary slackness.

This is exactly the equation on Slide **??**.

4. Sensitivity analysis

Goal

Some sensitivity analysis questions can be interpreted geometrically. We will only consider:

- changing the objective function vector $\mathbf{c}$,
- changing the right hand side of the constraints $\mathbf{b}$.

Adding a decision variable is less easy to picture because the dimension of the feasible region changes.

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We look at the same example:

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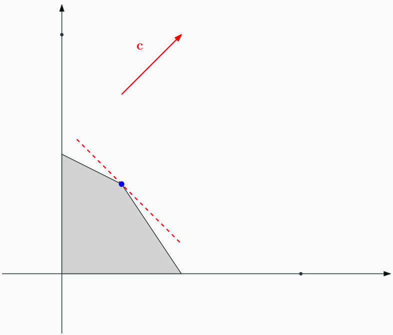
Changing the objective function



$$
\begin{array}{lrrcr}
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\text{subject to} & 3x_1 & +2x_2 & \leqslant & 3 \\
 & x_1 & +2x_2 & \leqslant & 2
\end{array}
$$

with $x_1, x_2 \geqslant 0$.

Optimal solution

$\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1} A_N \leqslant \mathbf{0}$

Changing the objective function

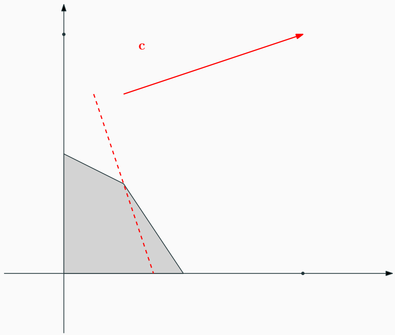
$$
\begin{array}{llllll}
\text{maximize} & \color{red}{5/4}\, x_1 & +x_2 & & \\
\text{subject to} & 3x_1 & +2x_2 & \leqslant & 3 \\
 & x_1 & +2x_2 & \leqslant & 2
\end{array}
$$

with $x_1, x_2 \geqslant 0$.

“The basis stays optimal”

$$\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1} A_N \leqslant \mathbf{0}$$

Changing the objective function



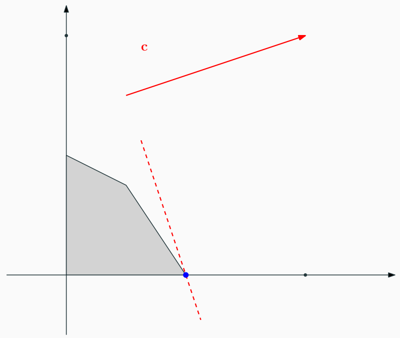
$$\begin{array}{lrrcr} \text{maximize} & {\color{red}3}x_1 & +x_2 & & \\ \text{subject to} & 3x_1 & +2x_2 & \leqslant & 3 \\ & x_1 & +2x_2 & \leqslant & 2 \end{array}$$

with $x_1, x_2 \geqslant 0$.

"The basis is no longer optimal"

$$\mathbf{c}_N^\top - \mathbf{c}_B^\top B^{-1} A_N \nleqslant \mathbf{0}$$

Changing the objective function

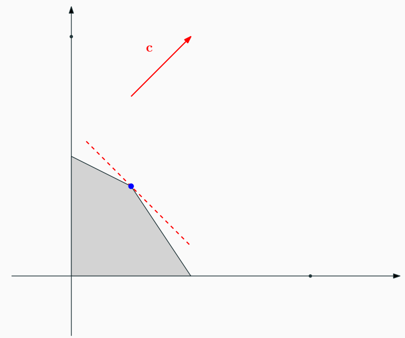


New optimal solution
via primal simplex

$$\begin{array}{llllll} \text{maximize} & 3x_1 & +x_2 & & \\ \text{subject to} & 3x_1 & +2x_2 & \leqslant & 3 \\ & x_1 & +2x_2 & \leqslant & 2 \end{array}$$

with $x_1, x_2 \geqslant 0$.

Changing the right hand side of the constraints



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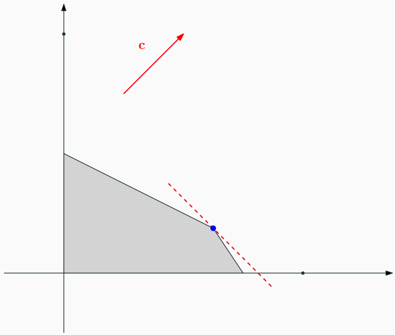
Optimal solution

$B^{-1}\mathbf{b} \geqslant \mathbf{0}$

Changing the right hand side of the constraints



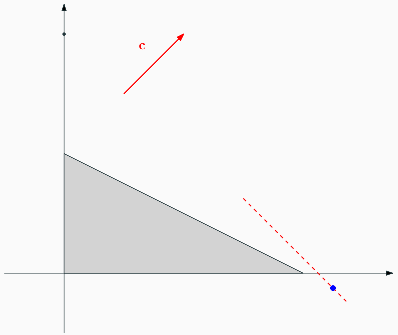
$$
\begin{array}{lrrcl}
\text{maximize} & x_1 & +x_2 & & \\
\text{subject to} & 3x_1 & +2x_2 & \leqslant & 9/2 \\
& x_1 & +2x_2 & \leqslant & 2
\end{array}
$$

with $x_1, x_2 \geqslant 0$.

"The basis stays optimal"

$$B^{-1}\mathbf{b} \geqslant \mathbf{0}$$

Changing the right hand side of the constraints



$$\begin{array}{lcccc} \text{maximize} & x_1 & +x_2 & & \\ \text{subject to} & 3x_1 & +2x_2 & \leqslant & {}^{25}\!/_{4} \\ & x_1 & +2x_2 & \leqslant & 2 \end{array}$$

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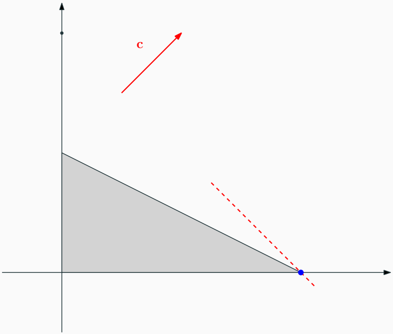

New optimal solution
via dual simplex

$$\begin{array}{lrrcl} \text{maximize} & x_1 & +x_2 & & \\ \text{subject to} & 3x_1 & +2x_2 & \leqslant & {}^{25}\!/_{4} \\ & x_1 & +2x_2 & \leqslant & 2 \end{array}$$

with $x_1, x_2 \geqslant 0$.