A Geometric standpoint on Linear Programming – Part I

Polytopes as feasible regions of Linear Programs

Disclaimers

The goal is to give an alternative (geometric) standpoint on Linear Programming. You will *not* be tested on that in the final, but I hope it will give you more insights about LP.

Disclaimers

The goal is to give an alternative (geometric) standpoint on Linear Programming. You will *not* be tested on that in the final, but I hope it will give you more insights about LP.

We will talk about geometric insights. Although we will do geometry in dimension n with sometimes $n \geqslant 4$, all the intuition comes from $n = 2$ or $n = 3$. Don't try to imagine what is a space of dimension 4!

1. Convex polytopes

2. Dictionaries and vertices

3. A geometric application of the theorem of the alternative

1. Convex polytopes

Question

In this lecture we will look only at the feasible region of a LP. Let's consider the set

$$\begin{array}{rrcl} 3x_1 & +2x_2 & \leqslant & 3 \\ x_1 & +2x_2 & \leqslant & 2 \end{array} \qquad x_1, x_2 \geqslant 0.$$

Question

In this lecture we will look only at the feasible region of a LP. Let's consider the set

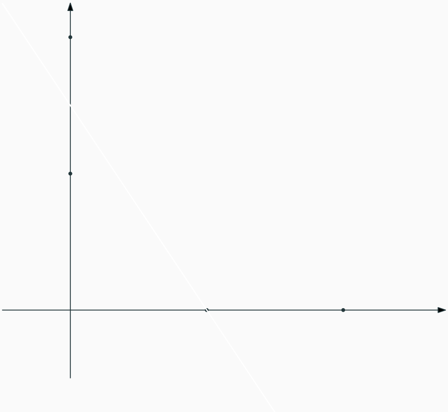
$$\begin{array}{rrcl} 3x_1 & +2x_2 & \leqslant & 3 \\ x_1 & +2x_2 & \leqslant & 2 \end{array} \qquad x_1, x_2 \geqslant 0.$$

Question: Plot in the plane the set of x_1, x_2 which satisfy the constraints.

Half space

Let's look only at

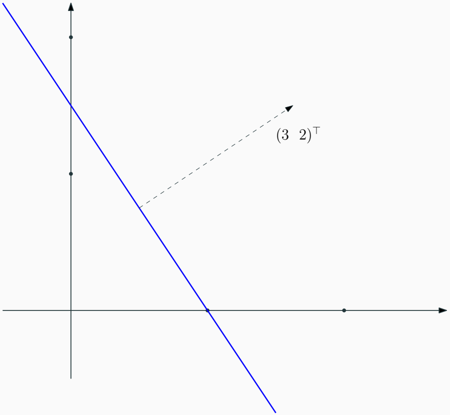
$$3x_1 + 2x_2 \leqslant 3.$$



Half space

Let's look only at

$$3x_1 + 2x_2 \leqslant 3.$$

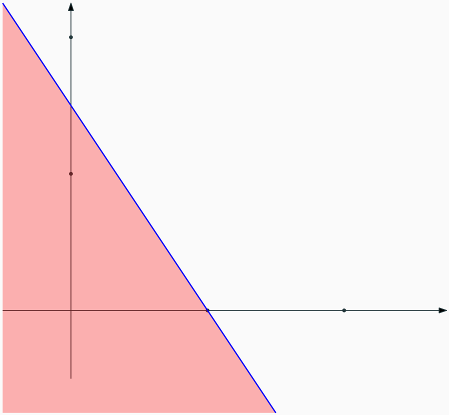


$3x_1 + 2x_2 = 3$

Half space

Let's look only at

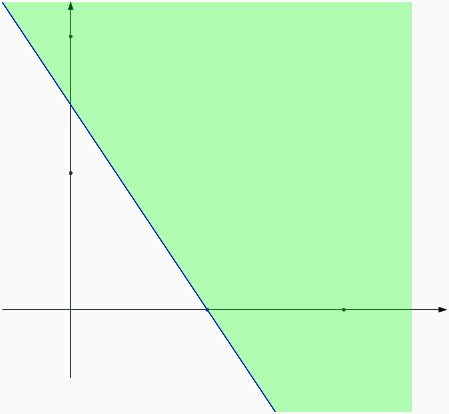
$$3x_1 + 2x_2 \leqslant 3.$$



Half space

Let's look only at

$$3x_1 + 2x_2 \leqslant 3.$$



$3x_1 + 2x_2 \geqslant 3$

General case

Vocabulary

In $\mathbb{R}^n$, if $a_1, a_2, \ldots, a_n$ are numbers not all equal to 0, then the set of $\mathbf{x} \in \mathbb{R}^n$ such that

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

is a *hyperplane* (a line if $n = 2$, a plane if $n = 3$).

General case

Vocabulary

In $\mathbb{R}^n$, if $a_1, a_2, \ldots, a_n$ are numbers not all equal to 0, then the set of $\mathbf{x} \in \mathbb{R}^n$ such that

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

is a *hyperplane* (a line if $n = 2$, a plane if $n = 3$). The vector

$$\mathbf{a} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}^\top$$

is normal to the hyperplane.

General case

Vocabulary

In $\mathbb{R}^n$, if $a_1, a_2, \ldots, a_n$ are numbers not all equal to 0, then the set of $\mathbf{x} \in \mathbb{R}^n$ such that

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$$

is a *hyperplane* (a line if $n = 2$, a plane if $n = 3$). The vector

$$\mathbf{a} = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \end{pmatrix}^\top$$

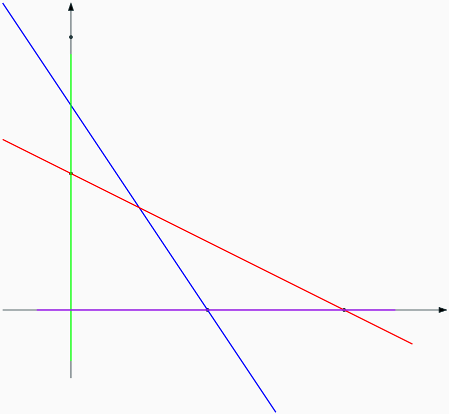
is normal to the hyperplane.

The set of $\mathbf{x} \in \mathbb{R}^n$ such that

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \leqslant b$$

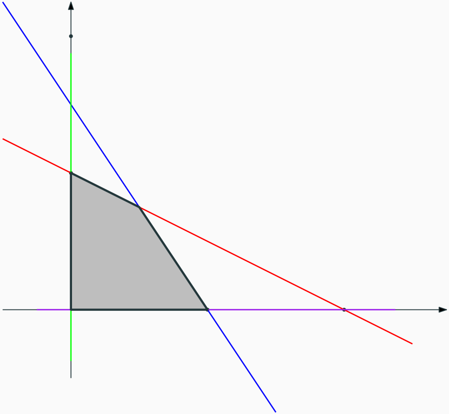
is a *half-space*.

Back to the example



$$\begin{array}{cccc} 3x_1 & +2x_2 & \leqslant & 3 \\ x_1 & +2x_2 & \leqslant & 2 \\ x_1 & & \geqslant & 0 \\ & x_2 & \geqslant & 0 \end{array}$$

Back to the example



$$\begin{array}{rrrr} 3x_1 & +2x_2 & \leqslant & 3 \\ x_1 & +2x_2 & \leqslant & 2 \\ x_1 & & \geqslant & 0 \\ & x_2 & \geqslant & 0 \end{array}$$

The feasible region is the gray area.

Feasible region in general

Vocabulary

The feasible region of a Linear Program is an intersection of half-spaces. It is called a *convex polytope* (convex polygon in 2 dimensions, convex polyhedron in 3 dimensions).

Feasible region in general

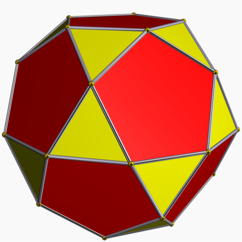
Vocabulary

The feasible region of a Linear Program is an intersection of half-spaces. It is called a *convex polytope* (convex polygon in 2 dimensions, convex polyhedron in 3 dimensions). Actually, every convex polytope can be written as the feasible region of a Linear Program (not necessarily in standard inequality form).

Feasible region in general

Vocabulary

The feasible region of a Linear Program is an intersection of half-spaces. It is called a *convex polytope* (convex polygon in 2 dimensions, convex polyhedron in 3 dimensions). Actually, every convex polytope can be written as the feasible region of a Linear Program (not necessarily in standard inequality form).

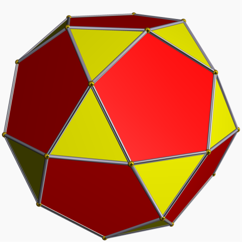


An example of convex polyhedron in 3 dimensions
Credit given to Robert Webb's Stella software
`http://www.software3d.com/Stella.php`

Feasible region in general

Vocabulary

The feasible region of a Linear Program is an intersection of half-spaces. It is called a *convex polytope* (convex polygon in 2 dimensions, convex polyhedron in 3 dimensions). Actually, every convex polytope can be written as the feasible region of a Linear Program (not necessarily in standard inequality form).



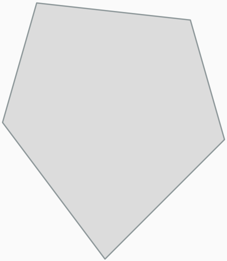
An example of convex polyhedron in 3 dimensions
Credit given to Robert Webb's Stella software
`http://www.software3d.com/Stella.php`

If there are some equality constraints, the polytope can have zero volume, for instance a flat square floating in a 3-dimensional space.

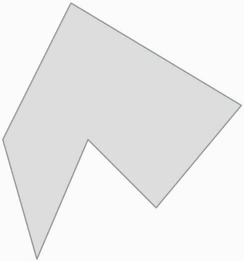
Convexity: geometric definition

Definition

A set S is convex if, for any two points $\mathbf{x}, \mathbf{y}$ in the set S, the segment joining $\mathbf{x}$ to $\mathbf{y}$ stays in the set.



Convex set

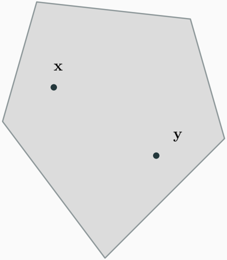


Non convex set

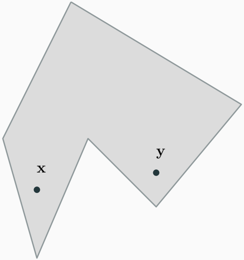
Convexity: geometric definition

Definition

A set S is convex if, for any two points $\mathbf{x}, \mathbf{y}$ in the set S, the segment joining $\mathbf{x}$ to $\mathbf{y}$ stays in the set.



Convex set

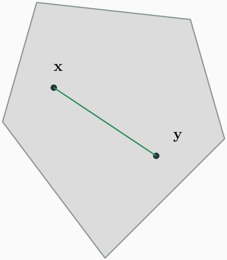


Non convex set

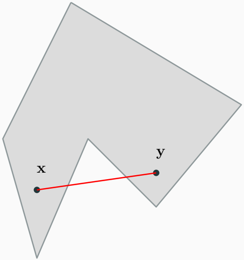
Convexity: geometric definition

Definition

A set S is convex if, for any two points $\mathbf{x}, \mathbf{y}$ in the set S, the segment joining $\mathbf{x}$ to $\mathbf{y}$ stays in the set.



Convex set



Non convex set

Convexity: analytic definition

Definition

A set S is convex if, for any two points $\mathbf{x}, \mathbf{y}$ in the set S, and for any $t \in [0,1]$ then

$$(1-t)\mathbf{x} + t\mathbf{y} \in S$$

The map $t \in [0,1] \mapsto (1-t)\mathbf{x} + t\mathbf{y}$ is nothing else than the parametrization of the segment joining $\mathbf{x}$ to $\mathbf{y}$.

Convexity: analytic definition

Definition

A set S is convex if, for any two points $\mathbf{x}, \mathbf{y}$ in the set S, and for any $t \in [0, 1]$ then

$$(1-t)\mathbf{x} + t\mathbf{y} \in S$$

The map $t \in [0,1] \mapsto (1-t)\mathbf{x} + t\mathbf{y}$ is nothing else than the parametrization of the segment joining $\mathbf{x}$ to $\mathbf{y}$.

Exercise. If the set S is the feasible region of a LP in standard inequality form, that is defined by

$$S = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{x} \geqslant \mathbf{0} \text{ and } A\mathbf{x} \leqslant \mathbf{b}\},$$

show analytically that this set is convex.

2. Dictionaries and vertices

Link between algebra and geometry

Insight

Each feasible dictionary of a LP corresponds to a vertex of the feasible region. Conversely, to each vertex of the feasible region one can associate at least one feasible dictionary.

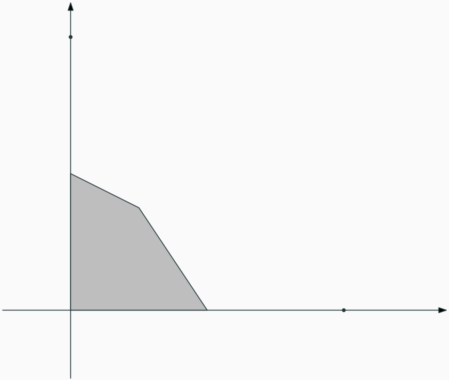
Link between algebra and geometry

Insight

Each feasible dictionary of a LP corresponds to a vertex of the feasible region. Conversely, to each vertex of the feasible region one can associate at least one feasible dictionary.

To use a different vocabulary, the basic feasible solutions (the solutions obtained by setting the non basic variables to 0 in a dictionary) are the vertices of the feasible region.

Example

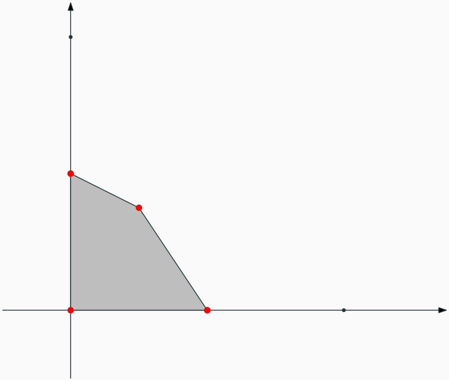


We take our example

$$\begin{array}{cccc} 3x_1 & +2x_2 & \leqslant & 3 \\ x_1 & +2x_2 & \leqslant & 2 \end{array}$$

with $x_1, x_2 \geqslant 0$. The slack variables are x_3, x_4.

Example

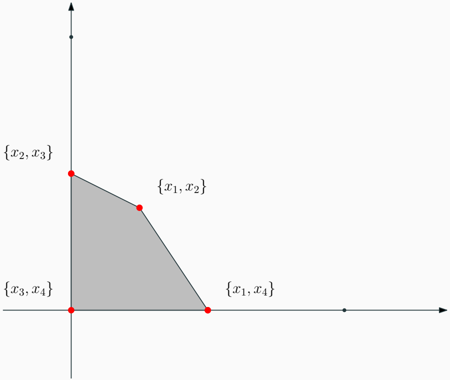


We take our example

$$\begin{array}{rrcr} 3x_1 & +2x_2 & \leqslant & 3 \\ x_1 & +2x_2 & \leqslant & 2 \end{array}$$

with $x_1, x_2 \geqslant 0$. The slack variables are x_3, x_4.

Example



We take our example

$$\begin{array}{cccc} 3x_1 & +2x_2 & \leqslant & 3 \\ x_1 & +2x_2 & \leqslant & 2 \end{array}$$

with $x_1, x_2 \geqslant 0$. The slack variables are x_3, x_4.

Next to each vertex, set of the basic variable yielding the vertex when setting the non basic variables to 0.

Why is that?

I won't do a full proof:

- it would require a precise definition of a vertex,
- then the proof would use the revised simplex formulas.

Why is that?

I won't do a full proof:

- it would require a precise definition of a vertex,
- then the proof would use the revised simplex formulas.

Rough intuition:

- Setting non basic variables to 0 means choosing which inequality constraints are actually equality constraints.

Why is that?

I won't do a full proof:

- it would require a precise definition of a vertex,
- then the proof would use the revised simplex formulas.

Rough intuition:

- Setting non basic variables to 0 means choosing which inequality constraints are actually equality constraints.
- Having an inequality constraint which is an equality tells you that, instead of being on a half-space, you are on the hyperplane, that is on a *face* of the polytope.

Why is that?

I won't do a full proof:

- it would require a precise definition of a vertex,
- then the proof would use the revised simplex formulas.

Rough intuition:

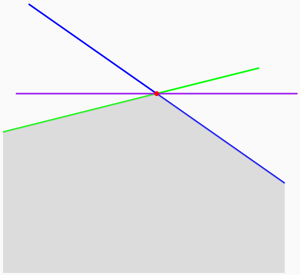
- Setting non basic variables to 0 means choosing which inequality constraints are actually equality constraints.
- Having an inequality constraint which is an equality tells you that, instead of being on a half-space, you are on the hyperplane, that is on a *face* of the polytope.
- By specifying enough hyperplanes on which you are supposed to be, you define uniquely a point, that is a vertex. In other words, each vertex can be defined as the intersection of enough faces of the polytope.

Degenerate primal dictionaries

It can happen when too many hyperplanes intersect at the same vertex.

Degenerate primal dictionaries

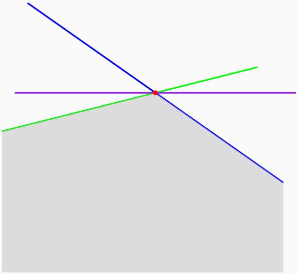
It can happen when too many hyperplanes intersect at the same vertex.



Three halfspaces with boundaries being hyperplanes (in blue, purple and green) intersect at a vertex. The gray area is the feasible region.

Degenerate primal dictionaries

It can happen when too many hyperplanes intersect at the same vertex.

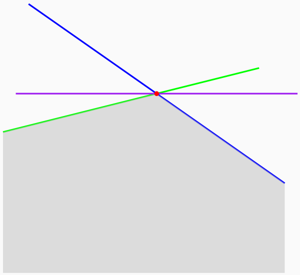


Three halfspaces with boundaries being hyperplanes (in blue, purple and green) intersect at a vertex. The gray area is the feasible region.

More than one dictionary correspond to this vertex: constraining to be both on the blue hyperplane and the purple hyperplane will also enforce that we are on the green hyperplane.

Degenerate primal dictionaries

It can happen when too many hyperplanes intersect at the same vertex.



Three halfspaces with boundaries being hyperplanes (in blue, purple and green) intersect at a vertex. The gray area is the feasible region.
More than one dictionary correspond to this vertex: constraining to be both on the blue hyperplane and the purple hyperplane will also enforce that we are on the green hyperplane.

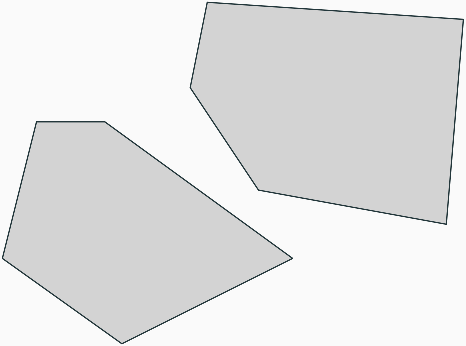
One could believe that degeneracy is very unlikely: in 2 dimensions, if you take 3 lines (that is 3 hyperplanes) at random they will *not* intersect at a single point. However, LPs are not drawn "at random" and degeneracy is common in LP.

3. A geometric application of the theorem of the alternative

Statement of the result

Theorem

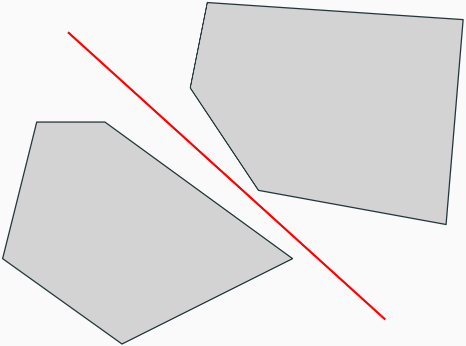
Take two non empty convex polytopes and assume that they do not intersect.



Statement of the result

Theorem

Take two non empty convex polytopes and assume that they do not intersect. Then there exists a hyperplane which separates them, that is one polytope is in one of the half-space and the other one is on the other half space.



Comments

The statement in itself can seem pretty intuitive and not require a proof, and it is a fair point. Actually, the theorem of the alternative could be seen as an algebraic statement of this “obvious” geometric result.

Comments

The statement in itself can seem pretty intuitive and not require a proof, and it is a fair point. Actually, the theorem of the alternative could be seen as an algebraic statement of this "obvious" geometric result.

To make the link with Linear Programming, we will use the following definition of "convex polytope".

Definition

In $\mathbb{R}^n$, a set S is called a convex polytope if and only if there exists a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^m$ such that

$$S = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } A\mathbf{x} \leqslant \mathbf{b}\}.$$

In other words, a convex polytope is the feasible region of a LP where all decision variables are free.

Proof 1

We take S_1 and S_2 two convex polytopes in $\mathbb{R}^n$. By definition, there exists A_1, A_2 matrices and $\mathbf{b}_1, \mathbf{b}_2$ vectors (not necessarily of the same dimension) such that

$$S_1 = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } A_1\mathbf{x} \leqslant \mathbf{b}_1\} \text{ and } S_2 = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } A_2\mathbf{x} \leqslant \mathbf{b}_2\}$$

Proof 1

We take S_1 and S_2 two convex polytopes in $\mathbb{R}^n$. By definition, there exists A_1, A_2 matrices and $\mathbf{b}_1, \mathbf{b}_2$ vectors (not necessarily of the same dimension) such that

$$S_1 = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } A_1\mathbf{x} \leqslant \mathbf{b}_1\} \text{ and } S_2 = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } A_2\mathbf{x} \leqslant \mathbf{b}_2\}$$

As the two polytopes do not intersect by assumption, we cannot find $\mathbf{x} \in \mathbb{R}^n$ such that $A_1\mathbf{x} \leqslant \mathbf{b}_1$ and $A_2\mathbf{x} \leqslant \mathbf{b}_2$. In other words, the following set of equations does not have a solution:

$$\begin{aligned} A_1\mathbf{x} &\leqslant \mathbf{b}_1, \\ A_2\mathbf{x} &\leqslant \mathbf{b}_2. \end{aligned}$$

Proof 1

We take S_1 and S_2 two convex polytopes in $\mathbb{R}^n$. By definition, there exists A_1, A_2 matrices and $\mathbf{b}_1, \mathbf{b}_2$ vectors (not necessarily of the same dimension) such that

$$S_1 = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } A_1\mathbf{x} \leqslant \mathbf{b}_1\} \text{ and } S_2 = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } A_2\mathbf{x} \leqslant \mathbf{b}_2\}$$

As the two polytopes do not intersect by assumption, we cannot find $\mathbf{x} \in \mathbb{R}^n$ such that $A_1\mathbf{x} \leqslant \mathbf{b}_1$ and $A_2\mathbf{x} \leqslant \mathbf{b}_2$. In other words, the following set of equations does not have a solution:

$$\begin{aligned} A_1\mathbf{x} &\leqslant \mathbf{b}_1, \\ A_2\mathbf{x} &\leqslant \mathbf{b}_2. \end{aligned}$$

The system above can be written $A\mathbf{x} \leqslant \mathbf{b}$ with

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}.$$

Proof 2

A variant of the theorem of the alternative

Assume that there does not exist $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leqslant \mathbf{b}$. Then there exists $\mathbf{y} \geqslant \mathbf{0}$ such that $A^\top \mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Exercise. Prove it.

Proof 2

A variant of the theorem of the alternative

Assume that there does not exist $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leqslant \mathbf{b}$. Then there exists $\mathbf{y} \geqslant \mathbf{0}$ such that $A^\top \mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Exercise. Prove it.

In our case, given what A and $\mathbf{b}$ are, we know that there exists $\mathbf{y} \geqslant \mathbf{0}$ such that

$$A^\top \mathbf{y} = A_1^\top \mathbf{y} + A_2^\top \mathbf{y} = \mathbf{0} \quad \text{and} \quad \mathbf{b}^\top \mathbf{y} = \mathbf{b}_1^\top \mathbf{y} + \mathbf{b}_2^\top \mathbf{y} < 0.$$

Proof 2

A variant of the theorem of the alternative

Assume that there does not exist $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leqslant \mathbf{b}$. Then there exists $\mathbf{y} \geqslant \mathbf{0}$ such that $A^\top \mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Exercise. Prove it.

In our case, given what A and $\mathbf{b}$ are, we know that there exists $\mathbf{y} \geqslant \mathbf{0}$ such that

$$A^\top \mathbf{y} = A_1^\top \mathbf{y} + A_2^\top \mathbf{y} = \mathbf{0} \quad \text{and} \quad \mathbf{b}^\top \mathbf{y} = \mathbf{b}_1^\top \mathbf{y} + \mathbf{b}_2^\top \mathbf{y} < 0.$$

The vector $A_1^\top \mathbf{y}$ will give the normal to an hyperplane separating the two polytopes. (One should justify that $A_1^\top \mathbf{y} = -A_2^\top \mathbf{y}$ is not $\mathbf{0}$. If this where the case, using that either $\mathbf{b}_1^\top \mathbf{y} < 0$ or $\mathbf{b}_2^\top \mathbf{y} < 0$, one of the two polytopes would be empty.)

Proof 2

A variant of the theorem of the alternative

Assume that there does not exist $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leqslant \mathbf{b}$. Then there exists $\mathbf{y} \geqslant \mathbf{0}$ such that $A^\top \mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Exercise. Prove it.

In our case, given what A and $\mathbf{b}$ are, we know that there exists $\mathbf{y} \geqslant \mathbf{0}$ such that

$$A^\top \mathbf{y} = A_1^\top \mathbf{y} + A_2^\top \mathbf{y} = \mathbf{0} \quad \text{and} \quad \mathbf{b}^\top \mathbf{y} = \mathbf{b}_1^\top \mathbf{y} + \mathbf{b}_2^\top \mathbf{y} < 0.$$

The vector $A_1^\top \mathbf{y}$ will give the normal to an hyperplane separating the two polytopes. (One should justify that $A_1^\top \mathbf{y} = -A_2^\top \mathbf{y}$ is not $\mathbf{0}$. If this where the case, using that either $\mathbf{b}_1^\top \mathbf{y} < 0$ or $\mathbf{b}_2^\top \mathbf{y} < 0$, one of the two polytopes would be empty.)

Let $c \in \mathbb{R}$ such that $\mathbf{y}^\top \mathbf{b}_1 < c < -\mathbf{y}^\top \mathbf{b}_2$.

Proof 2

A variant of the theorem of the alternative

Assume that there does not exist $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} \leqslant \mathbf{b}$. Then there exists $\mathbf{y} \geqslant \mathbf{0}$ such that $A^\top \mathbf{y} = \mathbf{0}$ and $\mathbf{b}^\top \mathbf{y} < 0$.

Exercise. Prove it.

In our case, given what A and $\mathbf{b}$ are, we know that there exists $\mathbf{y} \geqslant \mathbf{0}$ such that

$$A^\top \mathbf{y} = A_1^\top \mathbf{y} + A_2^\top \mathbf{y} = \mathbf{0} \quad \text{and} \quad \mathbf{b}^\top \mathbf{y} = \mathbf{b}_1^\top \mathbf{y} + \mathbf{b}_2^\top \mathbf{y} < 0.$$

The vector $A_1^\top \mathbf{y}$ will give the normal to an hyperplane separating the two polytopes. (One should justify that $A_1^\top \mathbf{y} = -A_2^\top \mathbf{y}$ is not $\mathbf{0}$. If this where the case, using that either $\mathbf{b}_1^\top \mathbf{y} < 0$ or $\mathbf{b}_2^\top \mathbf{y} < 0$, one of the two polytopes would be empty.)

Let $c \in \mathbb{R}$ such that $\mathbf{y}^\top \mathbf{b}_1 < c < -\mathbf{y}^\top \mathbf{b}_2$.

(As $\mathbf{b}_1^\top \mathbf{y} + \mathbf{b}_2^\top \mathbf{y} < 0$, this is always possible to find.)

Proof 3

With the objects from the previous slide we define the hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} = c\}$$

Proof 3

With the objects from the previous slide we define the hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} = c\}$$

and we define the two half spaces delimited by H:

$$H_< = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} < c\},$$

$$H_> = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} > c\}.$$

Proof 3

With the objects from the previous slide we define the hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} = c\}$$

and we define the two half spaces delimited by H:

$$H_< = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} < c\},$$

$$H_> = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} > c\}.$$

You can check that $\mathbf{y}^\top A_1$ is a row vector, so that $\mathbf{y}^\top A_1 \mathbf{x}$ is a scalar, it can be written

$$\mathbf{y}^\top A_1 \mathbf{x} = a_1 x_1 + a_2 x_2 + \ldots a_n x_n$$

where the a_i are the components of the row vector $\mathbf{y}^\top A_1$.

Proof 3

With the objects from the previous slide we define the hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} = c\}$$

and we define the two half spaces delimited by H:

$$H_< = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} < c\},$$

$$H_> = \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \mathbf{y}^\top A_1 \mathbf{x} > c\}.$$

You can check that $\mathbf{y}^\top A_1$ is a row vector, so that $\mathbf{y}^\top A_1 \mathbf{x}$ is a scalar, it can be written

$$\mathbf{y}^\top A_1 \mathbf{x} = a_1 x_1 + a_2 x_2 + \dots a_n x_n$$

where the a_i are the components of the row vector $\mathbf{y}^\top A_1$.

Claim

The polytope S_1 is included in the half space $H_<$ while the polytope S_2 is included in the half space $H_>$.

Notice that the claim implies the theorem.

Proof 4

Now we just need to prove the claim.

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$.

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1\mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_1$$

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1 \mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_1 = \mathbf{b}_1^\top \mathbf{y} < c.$$

This proves $\mathbf{x} \in H_<$.

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1\mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_1 = \mathbf{b}_1^\top \mathbf{y} < c.$$

This proves $\mathbf{x} \in H_<$.

Then let's take $\mathbf{x} \in S_2$, that is $A_2\mathbf{x} \leqslant \mathbf{b}_2$.

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1\mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_1 = \mathbf{b}_1^\top \mathbf{y} < c.$$

This proves $\mathbf{x} \in H_<$.

Then let's take $\mathbf{x} \in S_2$, that is $A_2\mathbf{x} \leqslant \mathbf{b}_2$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$

$$\mathbf{y}^\top A_2\mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_2$$

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1 \mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_1 = \mathbf{b}_1^\top \mathbf{y} < c.$$

This proves $\mathbf{x} \in H_<$.

Then let's take $\mathbf{x} \in S_2$, that is $A_2\mathbf{x} \leqslant \mathbf{b}_2$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$

$$\mathbf{y}^\top A_2 \mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_2 = \mathbf{b}_2^\top \mathbf{y} < -c.$$

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1 \mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_1 = \mathbf{b}_1^\top \mathbf{y} < c.$$

This proves $\mathbf{x} \in H_<$.

Then let's take $\mathbf{x} \in S_2$, that is $A_2\mathbf{x} \leqslant \mathbf{b}_2$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$

$$\mathbf{y}^\top A_2 \mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_2 = \mathbf{b}_2^\top \mathbf{y} < -c.$$

But taking the transpose of $A_1^\top \mathbf{y} + A_2^\top \mathbf{y} = \mathbf{0}$ we have $\mathbf{y}^\top A_2 = -\mathbf{y}^\top A_1$

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1\mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_1 = \mathbf{b}_1^\top \mathbf{y} < c.$$

This proves $\mathbf{x} \in H_<$.

Then let's take $\mathbf{x} \in S_2$, that is $A_2\mathbf{x} \leqslant \mathbf{b}_2$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$

$$\mathbf{y}^\top A_2\mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_2 = \mathbf{b}_2^\top \mathbf{y} < -c.$$

But taking the transpose of $A_1^\top \mathbf{y} + A_2^\top \mathbf{y} = \mathbf{0}$ we have $\mathbf{y}^\top A_2 = -\mathbf{y}^\top A_1$ which gives us

$$-\mathbf{y}^\top A_1\mathbf{x} < -c.$$

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1\mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_1 = \mathbf{b}_1^\top \mathbf{y} < c.$$

This proves $\mathbf{x} \in H_<$.

Then let's take $\mathbf{x} \in S_2$, that is $A_2\mathbf{x} \leqslant \mathbf{b}_2$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$

$$\mathbf{y}^\top A_2\mathbf{x} \leqslant \mathbf{y}^\top \mathbf{b}_2 = \mathbf{b}_2^\top \mathbf{y} < -c.$$

But taking the transpose of $A_1^\top \mathbf{y} + A_2^\top \mathbf{y} = \mathbf{0}$ we have $\mathbf{y}^\top A_2 = -\mathbf{y}^\top A_1$ which gives us

$$-\mathbf{y}^\top A_1\mathbf{x} < -c.$$

Multiplying by -1 changes the inequality, hence $\mathbf{y}^\top A_1\mathbf{x} > c$, that is $\mathbf{x} \in H_>$.

Proof 4

Now we just need to prove the claim.

Let's take $\mathbf{x} \in S_1$, that is $A_1\mathbf{x} \leqslant \mathbf{b}_1$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$ to get

$$\mathbf{y}^\top A_1\mathbf{x} \leqslant \mathbf{y}^\top\mathbf{b}_1 = \mathbf{b}_1^\top\mathbf{y} < c.$$

This proves $\mathbf{x} \in H_<$.

Then let's take $\mathbf{x} \in S_2$, that is $A_2\mathbf{x} \leqslant \mathbf{b}_2$. We take the scalar product of this inequality with $\mathbf{y} \geqslant \mathbf{0}$

$$\mathbf{y}^\top A_2\mathbf{x} \leqslant \mathbf{y}^\top\mathbf{b}_2 = \mathbf{b}_2^\top\mathbf{y} < -c.$$

But taking the transpose of $A_1^\top\mathbf{y} + A_2^\top\mathbf{y} = \mathbf{0}$ we have $\mathbf{y}^\top A_2 = -\mathbf{y}^\top A_1$ which gives us

$$-\mathbf{y}^\top A_1\mathbf{x} < -c.$$

Multiplying by -1 changes the inequality, hence $\mathbf{y}^\top A_1\mathbf{x} > c$, that is $\mathbf{x} \in H_>$.

This concludes the proof.