Applications of Linear Programming

1. Historical remarks

2. An example of a scheduling problem

3. Branch and bound for integer linear programming

1. Historical remarks

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- And now? Still relying on simplex or interior point method. Huge progress in the scale of the problems solved, better theoretical understanding.

On the importance of solving LP

Being able to solve LP was a major breakthrough. For a while, LP were the state of the art of “artificial intelligence”.

‘I don’t want to bore you,’ Harvey said, ‘but you should understand that these heaps of wire can practically think—linear programming—which means that instead of going through all the alternatives they have a hunch which is the right one.’

From Billion-Dollar Brain *by Len Deighton.*
Copyright © VICO Patentverwertungs-und Vermögensverwaltungs-G-m.b.H., 1966.

Taken from the textbook by Chvátal. *Billion-Dollar Brain* is a movie from the 60s.

Report of a progress in LP solvers

The New York Times

AMERICANS IN POLL VIEW GOVERNMENT MORE CONFIDENTLY

TREASURY MAY ASK INCREASES IN TAXES OF SOME CONCERNS

Albany Leaders Predicting a Cut In Income Taxes

Cocaine Traffickers Kill 17 in Peru Raid On Antidrug Team

Vote Comes to a 'Homeland,' But African Problems Linger

Breakthrough in Problem Solving

Homeless Spend Nights in City Welfare Office

High Cuban Defector Speaks Out, Denouncing Castro as 'Impulsive'

Front page of the *New York Times* of November 19th 1984, "Breakthrough in problem solving". Report the use of interior point method in LP.

Applications

From the practical point of view[1]:

- planning, “efficient” allocation of resources;
- scheduling;
- design;
- etc.

in lots of areas (transportation, finance, manufacturing, etc.).

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From a theoretical point of view:

- convex geometry;
- game theory;
- complexity theory;
- etc.

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Before	After
Experience, personal judgment	Statistics, formal techniques like Linear Programming

2. An example of a scheduling problem

Bus schedule

A bus company must decide of the schedule of its workers. Its goal is to minimize the number of drivers to hire[2].

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Constraint for the demand: the number of drivers required depends on the time of the day.

12am	1am	2am	3am	4 am	5 am	6am	7am	8am	9am	10am	11am
2	2	2	2	2	2	8	8	8	8	4	4

12pm	1pm	2pm	3pm	4 pm	5 pm	6pm	7pm	8pm	9pm	10pm	11pm
3	3	3	3	6	6	5	5	5	5	3	3

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Formulation as a LP

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The x_is should be constrained to be integers, but one can solve the LP and round up the optimal solution.

LINDO gives instantaneously an optimal solution. Rounding up the solution to the nearest larger integer gives 17 drivers.

Additional features

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- Try to avoid that drivers start too early: increase the coefficient in front of x_i if i corresponds to a nocturnal hour.
- Allow some drivers to do a 4 hours shift, 4 hours of break and again a 4 hours shift will decrease the cost[3]. Just call x_i' the number of drivers with these conditions starting at hour i, it amounts to the introduction of 24 additional variables without changing the number of constraints.

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More realistic case: airplane scheduling

Airline companies have to schedule their planes and crews:

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And all of this with integer constraints! But the amount of money at stake is gigantic.

3. Branch and bound for integer linear programming

(Mixed) integer linear programming

A Mixed Integer Linear Program is is a LP where some variables are constrained to be integers.

$$
\begin{array}{lrrrrrrcr}
\text{maximize} & 10x_1 & +7x_2 & +4x_3 & +3x_4 & +x_5 & & & \\
& 2x_1 & +6x_2 & +x_3 & & & +x_6 & \leqslant & 7 \\
& 2x_1 & -3x_2 & +4x_3 & +x_4 & +x_5 & & \leqslant & 3 \\
& x_1 & & +2x_3 & -3x_4 & +x_5 & -x_6 & \leqslant & -1 \\
& x_1 & +x_2 & +x_3 & +x_4 & -x_5 & & \leqslant & 3 \\
& & \multicolumn{7}{c}{x_1, x_2, x_3, x_4, x_5 \in \{0,1\} \text{ and } x_6 \geqslant 0}
\end{array}
$$

[4]LP are solvable in polynomial time. The simplex is not polynomial in the worst case situation, but the ellipsoid method is for instance.

[5]Integer Linear programming is a NP complete.

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Mixed integer LP and Integer LP are much harder than LP.

- Optimal solution of large LPs can be found in reasonable time[4].
- Some small scale mixed integer LPs are not solvable in reasonable time[5].

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Branch and bound

Start by solving the *convex relaxation of the LP*

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If the solution is integer, great. If not, *branch* on a variable, that is set this variable to 0 or 1 and explore again the two new resulting LPs.

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One stops branching if:

- The LP obtained has an integer optimal solution.
- The LP is infeasible.
- The value of the LP is worse than an already found integer feasible solution.