Math 340 Assignment #4 Solutions Due Thursday April 2, 2020 R. Anstee

1. (for practice only; won't be graded but solution will be posted) Consider the LP:

$$\begin{array}{lrrrl} \text{Maximize} & -x_1 & -x_2 & +x_3 & \\ & -x_1 & & -x_3 & \leq -1 \\ & -x_1 & -x_2 & & \leq -2 \\ & & -x_2 & +x_3 & \leq -1 \end{array} \qquad x_1, x_2, x_3 \geq 0$$

The final dictionary is

$$\begin{array}{rcrrrr} x_4 & = & 0 & -x_6 & & +x_5 \\ x_2 & = & 2 & & -x_1 & +x_5 \\ x_3 & = & 1 & -x_6 & -x_1 & +x_5 \\ z & = & -1 & -x_6 & -x_1 & \end{array}$$

We now consider adding the constraint $x_1 + x_2 + x_3 \leq 1$. We introduce a slack variable $x_7 = 1 - x_1 - x_2 - x_3$ which we rewrite in terms of non basic variables as
$x_7 = 1 - x_1 - (2 - x_1 + x_5) - (1 - x_6 - x_1 + x_5) = -2 + x_6 + x_1 - 2x_5$. The dictionary is now

$$\begin{array}{rcrrrr} x_4 & = & 0 & -x_6 & & +x_5 \\ x_2 & = & 2 & & -x_1 & +x_5 \\ x_3 & = & 1 & -x_6 & -x_1 & +x_5 \\ x_7 & = & -2 & +x_6 & +x_1 & -2x_5 \\ z & = & -1 & -x_6 & -x_1 & \end{array}$$

We follow the rules of the dual simplex method first choosing x_7 to leave and then choosing x_1 to enter (there is a tie with x_6 when we choose the largest t so that $(-1,-1,0)+t(1,1,-2) \leq (0,0,0)$)

$$\begin{array}{rcrrrr} x_4 & = & 0 & -x_6 & & +x_5 \\ x_2 & = & 0 & +x_6 & -x_7 & -x_5 \\ x_3 & = & -1 & & -x_7 & -x_5 \\ x_1 & = & 2 & -x_6 & +x_7 & +2x_5 \\ z & = & -3 & & -x_7 & -2x_5 \end{array}$$

We choose x_3 to leave but now find that there is no candidate for an entering variable (there is no largest t such that $(0,-1,-2)+t(0,-1,-1) \leq (0,0,0)$) And so the dual is unbounded. In dual solution we seek is $(0,2,0,1)+t(0,1,0,1)$, i.e. $y_1 = 0$, $y_2 = 2+t$, $y_3 = 0$, $y_4 = 1+t$ with $z = -3-t$. You may check this in the dual as determined below:

primal:
$$\begin{array}{lrrrl} \text{Maximize} & -x_1 & -x_2 & +x_3 & \\ & -x_1 & & -x_3 & \leq -1 \\ & -x_1 & -x_2 & & \leq -2 \\ & & -x_2 & +x_3 & \leq -1 \\ & x_1 & +x_2 & +x_3 & \leq 1 \\ & \multicolumn{3}{l}{x_1, x_2, x_3 \geq 0} & \end{array}$$
, dual:
$$\begin{array}{lrrrrl} \text{Minimize} & -y_1 & -2y_2 & -y_3 & +y_4 & \\ & -y_1 & -y_2 & & +y_4 & \geq -1 \\ & & -y_2 & -y_3 & +y_4 & \geq -1 \\ & -y_1 & & +y_3 & +y_4 & \geq -1 \\ & \multicolumn{4}{l}{y_1, y_2, y_3, y_4 \geq 0} & \end{array}$$

2. Consider our standard LP: $\max \mathbf{c} \cdot \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. Assume $\mathbf{u}$ and $\mathbf{v}$ are both feasible solutions to the LP. Show that for any choice $\lambda \in [0,1]$, that $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ is also a feasible solution to the LP. You could start with $\lambda = 1/2 = (1-\lambda)$ to try on this problem.

Also show that if $\mathbf{u}$ and $\mathbf{v}$ are both optimal solutions to the LP , then $\lambda\mathbf{u}+(1-\lambda)\mathbf{v}$ are optimal solutions for any choice $\lambda\in[0,1]$.

Note: A set of points P in $\mathbf{R}^n$ is called convex *if for every pair $\mathbf{x},\mathbf{y}\in P$, that all the points on the* line segment *joining them is also in P. The set of points on the line segment joining $\mathbf{x}$ and $\mathbf{y}$ is $\{\lambda\mathbf{x}+(1-\lambda)\mathbf{y} : \lambda\in[0,1]\}=\{\mathbf{y}+\lambda(\mathbf{x}-\mathbf{y}) : \lambda\in[0,1]\}$.*

First assume $\mathbf{u}$ and $\mathbf{v}$ are feasible so that $A\mathbf{u}\leq\mathbf{b}$ and $A\mathbf{v}\leq\mathbf{b}$ and $\mathbf{u},\mathbf{v}\geq\mathbf{0}$. We compute

$$A(\lambda\cdot\mathbf{u}+(1-\lambda)\cdot\mathbf{v})=\lambda A\mathbf{u}+(1-\lambda)A\mathbf{v}\leq\lambda\mathbf{b}+(1-\lambda)\mathbf{b}=\mathbf{b}.$$

(linearity of Matrix multiplication is important). Similarly

$$\lambda\cdot\mathbf{u}+(1-\lambda)\cdot\mathbf{v}\geq\lambda\mathbf{0}+(1-\lambda)\mathbf{0}=\mathbf{0}.$$

Thus $\lambda\mathbf{u}+(1-\lambda)\mathbf{v}$ is a fesible solution to the LP.

Assume $\mathbf{u},\mathbf{v}$ are both optimal so that $\mathbf{c}\cdot\mathbf{u}=\mathbf{c}\cdot\mathbf{v}$. We check

$$\mathbf{c}\cdot(\lambda\cdot\mathbf{u}+(1-\lambda)\cdot\mathbf{v})=\lambda\mathbf{c}\cdot\mathbf{u}+(1-lambda)\mathbf{c}\cdot\mathbf{v}=\mathbf{c}\cdot\mathbf{u},$$

using $\mathbf{c}\cdot\mathbf{u}=\mathbf{c}\cdot\mathbf{v}$. But then $\lambda\cdot\mathbf{u}+(1-\lambda)\cdot\mathbf{v}$ is an optimal solution.

3. We know by our Marginal Value Theorem that the marginal values given by the dual variables predict the exact changes in the objective function for changes $\Delta\mathbf{b}$ to $\mathbf{b}$ for which $B^{-1}(\mathbf{b}+\Delta\mathbf{b})\geq\mathbf{0}$. Assume we know (perhaps from LINDO output) that $B^{-1}(\mathbf{b}+\Delta\mathbf{b})\geq\mathbf{0}$ for $\Delta\mathbf{b}=(6,0,0,0)^T$ and also for $\Delta\mathbf{b}=(0,0,8,0)^T$. Show that $B^{-1}(\mathbf{b}+\Delta\mathbf{b})\geq\mathbf{0}$ for $\Delta\mathbf{b}=(3,0,4,0)^T$. You might note that $\frac{1}{2}\times(6,0,0,0)^T+\frac{1}{2}\times(0,0,8,0)^T=(3,0,4,0)^T$.

   We use the ideas of convexity. Recall that $B^{-1}(\mathbf{b}+\Delta\mathbf{b})\geq\mathbf{0}$ if and only if $B^{-1}\mathbf{b}\geq-B^{-1}\Delta\mathbf{b}$ which is the same as $(-B^{-1})\Delta\mathbf{b}\leq B^{-1}\mathbf{b}$. We can now use the ideas of feasibility above directly thinking of $A=-B^{-1}$ and $\mathbf{b}$ being replaced by $B^{-1}\mathbf{b}$. Or do it again for the specific problem. We have $B^{-1}\mathbf{b}\geq-B^{-1}(6,0,0,0)^T$ and $B^{-1}\mathbf{b}\geq-B^{-1}(0,0,8,0)^T$. Then $(1/2)B^{-1}\mathbf{b}\geq-B^{-1}(3,0,0,0)^T$ and $(1/2)B^{-1}\mathbf{b}\geq-B^{-1}(0,0,4,0)^T$. Adding the two inequalities yields

$$(1/2)B^{-1}\mathbf{b}+(1/2)B^{-1}\mathbf{b}=B^{-1}\mathbf{b}\geq-B^{-1}(3,0,0,0)^T+-B^{-1}(0,0,4,0)^T=-B^{-1}(3,0,4,0)^T.$$

   Thus $B^{-1}\mathbf{b}\geq-B^{-1}(3,0,4,0)^T$ i.e. $B^{-1}(\mathbf{b}+(3,0,4,0)^T)\geq\mathbf{0}$.

4.

a) Consider the game given by payoff matrix A below (the payoff to the row player).

$$A=\begin{bmatrix}5&-3&-4\\-4&-2&5\end{bmatrix}$$

State explictly the LP for the row player. Deduce a bound on $v(A)$ if we use the mixed strategy for the column player $\mathbf{y}=(1/4,1/2,1/4)^T$.

The LP for the row player becomes

$$\begin{array}{lccccc}
\max & & & z & & \\
& -5x_1 & +4x_2 & +z & \leq 0 & \\
& 3x_1 & 2x_2 & +z & \leq 0 & \qquad \mathbf{x}\geq\mathbf{0}, z \text{ free }. \\
& 4x_1 & -5x_2 & +z & \leq 0 & \\
& x_1 & +x_2 & & =1 &
\end{array}$$

When the column player plays strategy $\mathbf{y} = (1/4, 1/2, 1/4)^T$, then the game, as far as the row player is concerned, has become

$$A\mathbf{y} = \begin{bmatrix} -5/4 \\ -5/4 \end{bmatrix}$$

From there it is clear that the row player can do no better against this column player strategy than $-5/4$ and hence $v(A) \leq -5/4$.

b) Find the optimal strategy for the row player and the column player for the game whose payoff matrix (for the row player) is as follows with $e, f > 0$. Hint: Try a column player strategy $(1/2, 1/2)^T$.

$$A = \begin{bmatrix} e & -e \\ -f & f \end{bmatrix}$$

With $\mathbf{y} = (1/2, 1/2)^T$, we find

$$A\mathbf{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which ensures, as in part a), that $v(A) \leq 0$. Is $v(A) = 0$? It is fairly straightforward to solve for the row player strategy if you assume $v(A) = 0$ (use complementary slackness such as used in the game of Morra) and you will obtain

$$\mathbf{x} = \begin{bmatrix} \frac{f}{e+f} \\ \frac{e}{e+f} \end{bmatrix}, \qquad \mathbf{x}^T A = [0\ 0],$$

from which we see that the row player has a strategy that guarantees winnings of 0 for the row player i.e. $v(A) \geq 0$. So $v(A) = 0$ and we have the optimal strategies.

5. (from an old exam) We have a gasoline blending problem where we may mix four gasoline products x_1, x_2, x_3, x_4 from 3 types of gas (gas1,gas2,gas3). The availability of the three gases is given as a function of a parameter p:

$$\begin{array}{lccccll} \text{Maximize} & 12.1x_1 & +8.3x_2 & +14.2x_3 & 18.1x_4 & (= z) & \\ (gas1) & 2x_1 & +x_2 & +3x_3 & +6x_4 & \leq 37 + 2p & \\ (gas2) & 3x_1 & & +2x_3 & +x_4 & \leq 42 + p & x_1, x_2, x_3, x_4 \quad \geq \quad 0 \\ (gas3) & x_1 & +4x_2 & +2x_3 & +x_4 & \leq 39 - 3p & \end{array}$$

We are interested in the value of the objective function z as a function of the parameter p near $p = 1$. We make p a variable (and move it to the other side of the inequalities), and make p free (allowing it to be negative). We also impose the somewhat arbitrary condition $p \leq 1$ and then use this constraint in our analysis. We send this off to LINDO. The LINDO output given in the assignment will be useful.

a) When $p = \frac{-37}{2}$, then the right hand side of the first inequality is 0 and we have $2x - 1 + x_2 + 3x_3 + 6x_4 = 0$ and yet $x_1, x_2, x_3, x_4 \geq 0$. We deduce that $x_1 = x_2 = x_3 = x_4 = 0$ and so $z = 0$ at optimality? For $p < \frac{-37}{2}$ we find that inequality 1 becomes $2x_1 + x_2 + 3x_3 + 6x_4 < 0$ and yet $x_1, x_2, x_3, x_4 \geq 0$ which is a contradiction. Thus the LP is now infeasible.

b) For for $p = 1$, $z = 230.8062$? The slope of the graph of z as a function of p is rate of increase of z as p increases. But $p = 1$ is imposed by the constraint PARAM $p \leq 1$ (p wishes to increase and so $p = 1$ at optimality) and the dual price for relaxing that constraint (and allowing p to increase) is 2.5875. The range on p for which the slope is valid is $(1 - 2.269231, 1 + 3.045455)$.

Note we are describing a way to determine the piecewise linear curve that gives the vaue of the objective function as a function of p. Dealing with the right hand side is much like dealing with parametric changes in $\mathbf{c}$ (e.g our previous assignment considered $\mathbf{c}$ with only c_1 varying).

c) Increasing p by Δ (which should increase the objective function by Δ times the dual price for the 4th inequality) is like increasing the righthand side of the first inequality by 2Δ, the righthand side of the second inequality by Δ and decreasing the righthand side of the third inequality by 3Δ resulting in a predicted change in the z value by 2Δ times dual price for constraint 1 and adding Δ times dual price for constraint 2 and subtracting 3 Δ times the dual price for constraint 3. This yields the equality $2(2.525000)+1(1.868750)-3(1.443750)=2.587500$ after dividing by Δ. Another solution is to notice that in the dual, the constraint associated with variable p in the primal, is $-2y_1 - y_2 + 3y_3 + y_4 \geq 0$ and so since $p > 0$ we have equality in this constraint by complementary slackness!

The input to LINDO is:

max 12.1x1 + 8.3x2 + 14.2x3 + 18.1x4

subject to

gas1) 2x1+x2+3x3+6x4-2p<37

gas2) 3x1+2x3+x4-p<42

gas3) x1+4x2+2x3+x4+3p<39

param)p<1

end

free p

The following is the output from LINDO:

OBJECTIVE FUNCTION VALUE

1) 230.8062

VARIABLE	VALUE	REDUCED COST
$X1$	11.875000	0.000000
$X2$	4.187500	0.000000
$X3$	3.687500	0.000000
$X4$	0.000000	0.362500
P	1.000000	0.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
$GAS1$)	0.000000	2.525000
$GAS2$)	0.000000	1.868750
$GAS3$)	0.000000	1.443750
$PARAM$)	0.000000	2.587500

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
$X1$	12.100000	0.161111	2.990000
$X2$	8.300000	0.322222	4.620000
$X3$	14.200000	4.271429	0.093548
$X4$	18.100000	0.362500	$INFINITY$
P	0.000000	$INFINITY$	2.587500

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
$GAS1$	37.000000	16.750000	4.916667
$GAS2$	42.000000	8.428572	19.000000
$GAS3$	39.000000	19.666666	13.400000
$PARAM$	1.000000	3.045455	2.269231

6. (from an old exam) We are running a factory and can produce products of three possible types from four types of parts as follows.

	product 1	product 2	product 3	available parts
part 1	3	5	2	286
part 2	4	6	2	396
part 3	5	8	3	440
part 4	4	7	4	396
profit $	21	35	15	

We wish to choose our product mix to obtain maximum profit subject both to the limitations on the inventory of available parts but also subject to the restriction that at most 50% of the number of produced products can be of one type

The LINDO input/output in the assignment will be useful.

a)[2 marks] The marginal values of the four parts are 0, 0, 3, 1.545455.

b)[4 marks] Mr. Edison visits the factory and offers to make a remarkable new part that substitutes for any of the four parts and will only charge $2 for each of these new parts. Would you buy some? Yes, because the new part costs less than the marginal value for Part 3 and so you could substitute the new part for some number of Part 3's. We can buy 24 new parts and in essence have 24 more part 3's and then make 3×24 more profit at an outlay of 2×24. We are using the right hand side ranges to see the range for which the marginal value of 3 for Part 3 is valid.

c)[4 marks] The market for product 1 crashes and the profit drops to that of product 3. Should you change your production? We see that the current basis remains the same and hence the current production remains optimal even if we decrease the profit for Product 1 from $21 to $15 (allowable decrease is 6). Thus we need not change the production.

d)[4 marks] The marginal cost for the total parts that make up product 2 are $0 \times 5 + 0 \times 6 + 3 \times 8 + \frac{17}{11} \times 7 = 34\frac{9}{11}$ which is less than the given profit of 35 per product 2. Normally for a product in the basis you have the sum of the marginal values equal to the profit but you are forgetting the 50% constraints. Increasing the production of Product 2 will violate the PROD2<50 constraint and so we get an additional contribution of $\frac{4}{11} \times 0.5 = \frac{2}{11}$ to the marginal cost of Product 2 which then yields equality of the marginal cost and the profit.

e) This question was not asked in the exam but was implicitly needed to explain the 'discrepancy' in d). The constraint PROD1<50

$$0.5 \text{ PROD1 - } 0.5 \text{ PROD2 - } 0.5 \text{ PROD3} < 0$$

can be rewritten as $PROD1 \leq (1/2)(PROD1 + PROD2 + PROD3)$ which was the constraint that the number of Product 1 being produced is at most 50% of the total production.