Complement on Game theory

Quick notes about what happens in Game theory when we know the strategy of one of the player.

Payoff and LPs

Let's consider a game with payoff matrix

$$\begin{pmatrix} 3 & -1 \\ 17 & -3 \end{pmatrix}.$$

I recall that there is a primal dual pair associated to this game. The primal LP, giving the optimal strategy of the first player is

$$\begin{array}{lrrrcl} \max & & & +z & & \\ & -3x_1 & -17x_2 & +z & \leqslant & 0 \\ & x_1 & +3x_2 & +z & \leqslant & 0 \\ & x_1 & +x_2 & & = & 1 \end{array} \qquad x_1, x_2 \geqslant 0.$$

On the other hand, the LP solved by the second player is the dual LP:

$$\begin{array}{lrrrcl} \min & & & +w & & \\ & -3y_1 & +y_2 & +w & \geqslant & 0 \\ & -17y_1 & +3y_2 & +w & \geqslant & 0 \\ & y_1 & +y_2 & & = & 1 \end{array} \qquad y_1, y_2 \geqslant 0.$$

A strategy of the second player

Suppose the column player plays

$$\mathbf{y} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

What does it means for the row player?

I will present three reasonings, one based on Game theory while the other two is rely on LP. The three reasonings are valid, though I think that the first one is the simplest.

Game theory approach

If the column player plays its $\mathbf{y}$, then we can compute the payoff of the row players in each of the alternative. If the row player plays the first row, s/he gets an expected payoff of

$$\frac{1}{2} \times 3 + \frac{1}{2} \times (-1) = 1.$$

For the second strategy, it becomes

$$\frac{1}{2} \times 17 + \frac{1}{2} \times (-3) = 7.$$

The row player wants to maximize its expected payoff, so **s/he should play the second row** and would then get a payoff of 7.

Primal LP approach

In this case, we rather reason on the LP solved by the first player. As the **y** can be interpreted as dual variables, we use them to do a linear combination of the constraints of the LP for the first player. If we do 1/2 times the first constraint + 1/2 times the second constraint, the value of the LP for the first player is less than the value of the LP

$$\begin{array}{lrrrcl}
\max & & & +z & & \\
& -x_1 & -7x_2 & +z & \leqslant & 0 \\
& x_1 & +x_2 & & = & 1
\end{array} \qquad x_1, x_2 \geqslant 0.$$

Now the first constraint can be written $z \leqslant x_1 + 7x_2$ and we want to maximize z. So we will take $z = x_1 + 7x_2$, hence we end up with the LP

$$\begin{array}{lrrcl}
\max & x_1 & +7x_2 & & \\
& x_1 & +x_2 & = & 1
\end{array} \qquad x_1, x_2 \geqslant 0.$$

This LP can be solved "by hand": we just take x_2 as large as possible (because of $7 > 1$), and we can only take $x_2 = 1$. So the solution for the last LP is $(x_1, x_2) = (0, 1)$, and the value of the LP is 7.

So we can say that **the value of the original LP solved by the first player is less than** 7.

Dual LP approach

Maybe even simpler than the previous one: we use **y** as a feasible solution in the dual LP. Indeed, once we replace **y** by $\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}^\top$ to choose w we have the two constraints,

$$\left\{ \begin{array}{rrcl}
-1 & +w & \geqslant & 0 \\
-7 & +w & \geqslant & 0
\end{array} \right. .$$

As we want to minimize in w, we choose $w = 7$, and we deduce that the value of the dual LP is less than 7. **Hence the value of the primal LP is less than** 7 **by weak duality**.