Linear Programming in Game Theory – Part II

MinMax theorem and the value of a two player zero sum game

Vocabulary

For this lecture, *game* means *two player zero sum game*.

For this lecture, *strategy* may mean *mixed strategy*.

1. The best strategy as a Linear Program
2. Duality
3. Consequences of duality
4. Symmetric games

1. The best strategy as a Linear Program

Goal

We look only at a *two player zero sum* game. The goal is to describe the best mixed strategy for each player.

So we look at game given described by a payoff matrix A. Player 1 chooses a row i, Player 2 chooses a column j, and the payoff of Player 1 is a_{ij} while the payoff of Player 2 is $-a_{ij}$.

As an example in this lecture we take

$$A = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 5 & -2 \end{pmatrix}$$

Mixed strategies for Player 1

Let $\mathbf{x} \in \mathbb{R}^m$ be a mixed strategy for Player 1, that is $\mathbf{x} \geqslant \mathbf{0}$ and $x_1 + \ldots x_m = 1$. Each x_i is the probability for Player 1 to play strategy i.

If Player 2 plays the first column, then the expected payoff of Player 1 is

$$x_1 a_{11} + x_2 a_{21} + \ldots + x_m a_{m1}.$$

More generally, if Player 2 plays column j, its expected payoff is

$$x_1 a_{1j} + x_2 a_{2j} + \ldots + x_m a_{mj}.$$

Worst case situation

Player 1 looks at the worst case situation, that is at its minimal expected payoff. This is the minimal payoff among all the strategies of Player 2. It reads

$$\min \begin{pmatrix} x_1 a_{11} & +x_2 a_{21} & + \dots & +x_m a_{m1}, \\ x_1 a_{12} & +x_2 a_{22} & + \dots & +x_m a_{m2}, \\ \vdots & \vdots & \vdots & \vdots \\ x_1 a_{1n} & +x_2 a_{2n} & + \dots & +x_m a_{mn} \end{pmatrix}$$

Player 1 computes the minimum of n numbers (the expected payoff for each strategy of Player 2) to compute the minimum payoff it can secure.

An example

In the case of the matrix A of this lecture which has 2 rows and 3 columns, $\mathbf{x} \in \mathbb{R}^2$ and the minimum payoff of Player 1 if it plays $\begin{pmatrix} x_1 & x_2 \end{pmatrix}^\top$ is

$$\begin{aligned} \min(\quad & 2x_1 && -x_2, \\ & -x_1 && +5x_2, \\ & 3x_1 && -2x_2,) \end{aligned}$$

This is the minimum of three real numbers. If for instance $\mathbf{x}^\top = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}$ then this value is

$$\min\left(\frac{1}{2}, 2, \frac{1}{2}\right) = \frac{1}{2}.$$

Optimization problem for Player 1

With this conservative approach, Player 1 wants to choose a strategy which maximizes its worst case situation payoff. That Player 1 will solve

$$\max_{\mathbf{x}} \Big(\min \big(\begin{array}{llll} x_1 a_{11} & +x_2 a_{21} & + \dots & +x_m a_{m1}, \\ x_1 a_{12} & +x_2 a_{22} & + \dots & +x_m a_{m2}, \\ \vdots & \vdots & \vdots & \vdots \\ x_1 a_{1n} & +x_2 a_{2n} & + \dots & +x_m a_{mn} \big) \Big) \end{array}$$

where the maximum is taken among all $\mathbf{x} \in \mathbb{R}^m$ such that $\mathbf{x} \geqslant \mathbf{0}$ and $x_1 + x_2 + \dots + x_m = 1$.

And this can be rewritten as linear program!

Linear program

Let's introduce z our objective function. Rather than trying to compute the minimum, we will impose only the inequalities

$$\begin{array}{lllll} x_1 a_{11} & +x_2 a_{21} & +\dots & +x_m a_{m1} & \geqslant z \\ x_1 a_{12} & +x_2 a_{22} & +\dots & +x_m a_{m2} & \geqslant z \\ \vdots & \vdots & \vdots & \vdots & \\ x_1 a_{1n} & +x_2 a_{2n} & +\dots & +x_m a_{mn} & \geqslant z \end{array}$$

Indeed if these inequalities are valid then z is smaller than the minimum of all the values in the left hand side. And later if we maximize in z then we will increase z until one of these inequalities is an equality, that is z is the minimum of these values.

The LP for the first player

Changing the signs of the inequalities to get standard form, the conclusion is that the Player 1 solves the LP

$$
\begin{array}{lrrcrrcl}
\text{maximize} & & & & & +z & & \\
\text{s.t.} & -x_1 a_{11} & -x_2 a_{21} & \dots & -x_m a_{m1} & +z & \leqslant & 0 \\
 & -x_1 a_{12} & -x_2 a_{22} & \dots & -x_m a_{n2} & +z & \leqslant & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & -x_1 a_{1m} & -x_2 a_{2m} & \dots & -x_m a_{mn} & +z & \leqslant & 0 \\
 & x_1 & +x_2 & \dots & +x_m & & = & 1
\end{array}
\qquad \mathbf{x} \geqslant \mathbf{0}, z \text{ free}
$$

This is a LP with decision variables $(\mathbf{x}, z) \in \mathbb{R}^{m+1}$ and $n+1$ constraints. It is not in standard inequality form because z is free and the last constraint is an equality.

An example

With the example of this section we end up with

$$\begin{array}{lrrrcl}
\text{maximize} & & & +z & & \\
\text{s.t.} & -2x_1 & +x_2 & +z & \leqslant & 0 \\
 & x_1 & -5x_2 & +z & \leqslant & 0 \\
 & -3x_1 & +2x_2 & +z & \leqslant & 0 \\
 & x_1 & +x_2 & & = & 1
\end{array} \qquad x_1, x_2 \geqslant 0,\ z \text{ free}.$$

LINDO gives the solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

with an optimal value of 1. This the best mixed strategy for Player 1.

Matrix notations

If we try to write this with matrix notations, recalling that A is the $m \times n$ payoff matrix of the game, the LP to determine the best strategy for Player 1 is

$$\begin{array}{lrrcl} \text{maximize} & & +z & & \\ \text{s.t.} & -A^\top \mathbf{x} & +z\mathbf{1} & \leqslant & \mathbf{0} \qquad \mathbf{x} \geqslant \mathbf{0},\ z \text{ free} \\ & \mathbf{1}^\top \mathbf{x} & & = & 1 \end{array}$$

where $\mathbf{1}$ is the vector with only 1. There is a slight abuse of notations because $z\mathbf{1}$ is the vector of size n where all component equal to z; while in $\mathbf{1}^\top \mathbf{x}$ the vector $\mathbf{1}$ is of size m, in such a way that

$$\mathbf{1}^\top \mathbf{x} = x_1 + x_2 + \ldots + x_m.$$

The LP above is nothing else than the matrix notation of the LP which appears in Frame 10.

2. Duality

Insight

We have a linear program, we can write its dual. Here the dual has also an interpretation in terms of game theory.

Main result

The dual of the LP computing the best strategy of Player 1 is the LP computing the best strategy of Player 2.

This result is a deep connection. It will help us to show in which sense the two strategies computed by the two players are the best ones.

Proof 1

The proof is not very involved, it is just a tedious rewriting of the dual of the LP which is written in Frame 10 or in Frame 12.

The LP is not in standard form because of the equality in the last constraint and z is a free variable. The LP has $n+1$ constraints, the dual variables will be denoted by $(\mathbf{y}, w)$ with $\mathbf{y} \in \mathbb{R}^n$ and $w \in \mathbb{R}$.

- The first n constraints of the primal are inequalities which impose $\mathbf{y} \geqslant \mathbf{0}$.
- The last constraint of the primal is an equality so w is a free variable.
- As the first m decision variables (that is the vector $\mathbf{x}$) are non negative, the first m constraints of the dual are inequalities.
- The last variable of the primal (that is z) is free so the last constraint in the dual is an equality.

Proof 2

With the notations of Frame 10, the dual LP is written

$$
\begin{array}{lccccccc}
\text{minimize} & & & & & +w & & \\
\text{s.t.} & -y_1 a_{11} & -y_2 a_{12} & \dots & -y_n a_{1n} & +w & \geqslant & 0 \\
 & -y_1 a_{21} & -y_2 a_{22} & \dots & -y_n a_{2n} & +w & \geqslant & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & -y_1 a_{m1} & -y_2 a_{m2} & \dots & -y_n a_{mn} & +w & \geqslant & 0 \\
 & y_1 & +y_2 & \dots & +y_n & & = & 1
\end{array}
\qquad \mathbf{y} \geqslant \mathbf{0}, w \text{ free}
$$

Proof 3

If you prefer matrix notations, you get that the dual of the LP for the first player is

$$\begin{array}{lrrcll} \text{minimize} & & +w & & & \\ \text{s.t.} & -A\mathbf{y} & +w\mathbf{1} & \geqslant & \mathbf{0} & \qquad \mathbf{y} \geqslant \mathbf{0},\ w \text{ free} \\ & \mathbf{1}^\top \mathbf{y} & & = & 1 & \end{array}$$

Proof 4

Let's go back to the dual of the LP written in Frame 15. The first m inequalities can be rewritten

$$w \geqslant \max\begin{pmatrix} y_1 a_{11} & +y_2 a_{12} & +\ldots & +y_n a_{1n}, \\ y_1 a_{21} & +y_2 a_{22} & +\ldots & +y_n a_{2n}, \\ \vdots & \vdots & \vdots & \vdots \\ y_1 a_{m1} & +y_2 a_{m2} & +\ldots & +y_n a_{mn} \end{pmatrix}$$

But for instance the first line, that is

$$y_1 a_{11} + y_2 a_{12} + \ldots + y_m a_{1m}$$

is what Player 1 wins in average (hence Player 2 loses) if Player 2 plays its mixed strategy $\mathbf{y}$ while Player 1 plays the first strategy.

Next, as we want to minimize w, we will take w equal to the right hand side.

Proof 5

In the end, the dual of the LP can be rewritten, by eliminating w as

$$\min_{\mathbf{y}}\Bigg(\max\big(\begin{array}{cccc} y_1a_{11} & +y_2a_{12} & +\ldots & +y_na_{1n}, \\ y_1a_{21} & +y_2a_{22} & +\ldots & +y_na_{2n}, \\ \vdots & \vdots & \vdots & \vdots \\ y_1a_{m1} & +y_2a_{m2} & +\ldots & +y_na_{mn}\big)\Bigg) \end{array}$$

where the minimum is taken among all $\mathbf{y}$ such that $\mathbf{y} \geqslant \mathbf{0}$ and $y_1 + y_2 + \ldots + y_n = 1$.

And this is nothing else than the best strategy for Player 2 with a conservative view: that is Player 2 tries to find the mixed strategy which minimizes the payoff of Player 1 in the worst possible situation.

You may go back to Frame 8 and the ones before to see that this is exactly the analogous of what happens for Player 1. This concludes the proof.

Example

If I go back to the example the dual problem is

$$\begin{array}{lrrrrcl} \text{minimize} & & & & +w & & \\ \quad\text{s.t.} & -2y_1 & +y_2 & -3y_3 & +w & \geqslant & 0 \\ & y_1 & -5y_2 & +2y_3 & +w & \geqslant & 0 \\ & y_1 & +y_2 & +y_3 & & = & 1 \end{array} \qquad y_1, y_2, y_3 \geqslant 0,\ w \text{ free}.$$

LINDO gives the solution

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}$$

with an optimal value of 1 (same as the primal, as expected). This the best mixed strategy for Player 2.

3. Consequences of duality

Strong duality

We are looking at these two LPs in duality, the first one being about the best strategy for Player 1 while the second is the best strategy for Player 2. "Best" is defined here with the conservative approach of the previous section.

Both the primal and the dual are feasible. For instance for the primal, take any $\mathbf{x} \geqslant \mathbf{0}$ which sums up to 1, and then take z very negative. Similarly for the dual. By strong duality we conclude the following.

Strong duality

There exists $(\mathbf{x}^*, z^*)$ an optimal solution to the primal and $(\mathbf{y}^*, w^*)$ an optimal solution to the dual and $z^* = w^*$.

Vocabulary

The value $z^* = w^*$ is called the *value* of the game and denoted by $v(A)$, where we recall that A is the payoff matrix.

Weak duality 1

Weak duality will explain us in which sense the LPs will yield “best” strategies.

Let’s take $(\mathbf{x}^*, z^*)$ and $(\mathbf{y}^*, w^*)$ to be optimal solutions to the primal and dual LPs respectively.

Optimal strategies

- If Player 1 plays an optimal strategy $\mathbf{x}^*$ then its expected payoff is at least $v(A)$, and it is $v(A)$ if Player 2 plays an optimal strategy $\mathbf{y}^*$.
- If Player 1 does not play an optimal strategy $\mathbf{x}^*$ but if Player 2 plays an optimal strategy $\mathbf{y}^*$, then the payoff of Player 1 is less than $v(A)$.

In other words: if Player 1 thinks that the other player will not make any mistake (that is choosing an optimal strategy $\mathbf{y}^*$) then its best interest is to play strategy $\mathbf{x}^*$.

Of course there is a completely analogous statement for Player 2.

Weak duality 2

To prove such a result we will use weak duality. We recall the two problems in duality with matrix notations

$$\begin{array}{lrrcl} \text{maximize} & & +z & & \\ \text{s.t.} & -A^\top \mathbf{x} & +z\mathbf{1} & \leqslant & \mathbf{0} & \qquad \mathbf{x} \geqslant \mathbf{0},\ z \text{ free} \\ & \mathbf{1}^\top \mathbf{x} & & = & 1 \end{array}$$

and

$$\begin{array}{lrrcl} \text{minimize} & & +w & & \\ \text{s.t.} & -A\mathbf{y} & +w\mathbf{1} & \geqslant & \mathbf{0} & \qquad \mathbf{y} \geqslant \mathbf{0},\ w \text{ free} \\ & \mathbf{1}^\top \mathbf{y} & & = & 1 \end{array}$$

Let's take $(\mathbf{x}, z)$ primal feasible and $(\mathbf{y}, w)$ dual feasible.

In the dual let me take the first constraint and multiply it by $\mathbf{x}^\top$: I can do that because $\mathbf{x} \geqslant \mathbf{0}$. We get

$$0 \leqslant -\mathbf{x}^\top A\mathbf{y} + w\mathbf{x}^\top \mathbf{1} = -\mathbf{x}^\top A\mathbf{y} + w\mathbf{1}^\top \mathbf{x} = -\mathbf{x}^\top A\mathbf{y} + w.$$

Weak duality 3

Let's do the same in the dual: we take the first constraint of the primal and we multiply it by $\mathbf{y}^\top \geqslant \mathbf{0}^\top$. We get

$$0 \geqslant -\mathbf{y}^\top A^\top \mathbf{x} + z\mathbf{y}^\top \mathbf{1} = -\mathbf{y}^\top A^\top \mathbf{x} + z\mathbf{1}^\top \mathbf{y} = -\mathbf{y}^\top A^\top \mathbf{x} + z.$$

I recall that $\mathbf{x}^\top A\mathbf{y}$ is a real number (it is the expected payoff of Player 1 if Player 1 plays its mixed strategy $\mathbf{x}$ while Player 2 plays its mixed strategy $\mathbf{y}$) and

$$\mathbf{x}^\top A\mathbf{y} = (\mathbf{x}^\top A\mathbf{y})^\top = \mathbf{y}^\top A^\top \mathbf{x}.$$

Hence we have are basically reproving weak duality, that is if $(\mathbf{x}, z)$ is feasible in the primal and $(\mathbf{y}, w)$ is feasible in the dual,

$$z \leqslant \mathbf{x}^\top A\mathbf{y} \leqslant w.$$

Weak duality 4

Now we can prove the statement in Frame 21 about the meaning of optimal strategies.

Assume that Player 1 plays strategy $\mathbf{x}^*$. Then if Player 2 plays strategy $\mathbf{y}$ by the last equation of the previous slide,

$$v(A) = z^* \leqslant {\mathbf{x}^*}^\top A\mathbf{y}$$

and there is equality if $\mathbf{y} = \mathbf{y}^*$. This is exactly the first part of the statement.

On the other hand if Player 2 plays $\mathbf{y}^*$ and Player 1 plays any strategy $\mathbf{x}$, then still by the same estimate its expected payoff is

$$\mathbf{x}^\top A\mathbf{y}^* \leqslant w^* = v(A)$$

which provided the second part of the statement.

MinMax Formulation

Actually, the result is sometimes phrased in an other way, and this is the how Von Neumann originally formulated it in 1929.

MinMax Theorem

Let A be a $m \times n$ matrix. Then

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \mathbf{x}^\top A\mathbf{y} = \min_{\mathbf{y}} \max_{\mathbf{x}} \mathbf{x}^\top A\mathbf{y} = v(A),$$

where the maxima in $\mathbf{x} \in \mathbb{R}^m$ are taken with respect to mixed strategies for Player 1 ($\mathbf{x} \geqslant \mathbf{0}$ and $\mathbf{1}^\top \mathbf{x} = 1$) while the minima in $\mathbf{y} \in \mathbb{R}^n$ are taken with respect to mixed strategies for Player 2 ($\mathbf{y} \geqslant \mathbf{0}$ and $\mathbf{1}^\top \mathbf{y} = 1$)

Informally, you can swap the $\max$ and the $\min$. The value of the left hand side is the LP solved by Player 1, while the value of the right hand side is the LP solved by Player 2 (and the two values coincide by duality).

4. Symmetric games

Definition

A symmetric game is one where the role of the two players can be interchanged. Without too much surprise a symmetric game is fair, that is the value is 0.

Definition

A game is symmetric if it is the same for Player 1 to play i and Player 2 to play j than for Player 1 to play j and Player 2 to play i.

By same, it means that the payoff of Player 1 in the first case is the payoff of Player 2 in the second case. As the payoff of Player 2 is given by the matrix $-A$, in the end the condition to impose is that for all i, j

$$a_{ij} = -a_{ji}.$$

Definition with matrices

A game with payoff matrix A is symmetric if and only if $A^\top = -A$.

Result

Theorem

Let's take a symmetric game with payoff matrix A. Then $v(A)$ the value of the game is 0.

Moreover, an optimal strategy for Player 1 is also an optimal strategy for Player 2.

Rock Paper Scissors is a symmetric game hence its value is 0. In this case by solving the LP (but also by symmetry argument) the optimal strategy is probability $1/3$ on each pure strategy.

Proof

The dual and primal LP are exactly the same. Indeed, if I start from the primal, that I use $A^\top = -A$ and that I multiply the first line of the constraint by -1 I end up with

$$\begin{array}{llcccl} \text{maximize} & & +z & & & \\ \text{s.t.} & -A^\top \mathbf{x} & +z\mathbf{1} & \leqslant & \mathbf{0} & \mathbf{x} \geqslant \mathbf{0},\ z \text{ free} \\ & \mathbf{1}^\top \mathbf{x} & & = & 1 & \end{array}$$

$$\text{is the same as} \quad \begin{array}{llcccl} \text{maximize} & & +z & & & \\ \text{s.t.} & -A\mathbf{x} & (-z)\mathbf{1} & \geqslant & \mathbf{0} & \mathbf{x} \geqslant \mathbf{0},\ z \text{ free} \\ & \mathbf{1}^\top \mathbf{x} & & = & 1 & \end{array}$$

Now if I call $w = -z$ and that I transform the maximization to a minimization by multiplying by -1, I end up with (up to a negative sign) the dual. Taking then just the values of the problem I have $v(A) = -v(A)$ which yields $v(A) = 0$. As the primal and the dual are the same (up to the negative sign), they have the same optimal solutions, so optimal strategies are the same.