# Métodos Numéricos Computacionais Implementações e Análise

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## 1 Introdução

#### 1.1 Entendendo o Problema

Nem sempre é possível resolver equações diferenciais analiticamente. Os métodos cujas implementações são apresentadas no relatório têm como objetivo resolver equações diferenciais numericamente. Trata-se de um problema de valor inicial, no qual são dados uma equação diferencial do tipo y'(x) = f(x, y(x)) e um ponto da função solução y na forma  $y(x_0) = y_0$ .

Já que estamos tratando de métodos numéricos, não teremos como saída uma função y(x) que seja uma resposta exata satisfazendo as condições iniciais dadas. Teremos, no entanto, uma lista de pontos  $y(x_i) = y_i$  estimados. O espaço entre  $x_i$  e  $x_{i+1}$  pode ser o quão pequeno se queira, desde que seja maior que zero. Torna-se claro que, quanto menor for esse  $\Delta x$ , melhor será a nossa aproximação em relação à suposta solução exata. Em compensação, quanto menor tal valor maior será o custo computacional, pois mais pontos

serão calculados. Dessa forma, o  $\Delta x$  deverá ser definido para cada execução de método.

Também torna-se necessário definir um  $x_f$  (x final) para cada execução de método numérico de forma que a execução tenha fim. Nas implementações desse exercício foi escolhido substituir a entrada do  $\Delta x$  pelo número de pontos n (e implicitamente calcular o  $\Delta x$ ), já que é mais usual pensarmos em termos do número de pontos do que do intervalo entre eles. Tendo isso em vista, cada método implementado nesse exercício recebe como entrada os valores de:  $x_0, y_0, x_f, n$  e y'(x, y(x)). Respectivamente x e y iniciais, x final, número de pontos a serem calculados e a função y'.

# 2 Os Métodos: Fórmulas, Gráficos e Análise de Erros

Na parte final desse documento estão as implementações dos Métodos Numéricos das famílias Adams-Bashforth (com ordem de precisão de 1 até 8), Adams-Moulton (com ordem de precisão de 1 até 8) e Runge-Kutta (com ordem de precisão de 1 até 6). Nessa seção, para cada família, traz-se imagens comparativas de seus respectivos gráficos, análise estatística de seus erros e tabelas com seus pontos obtidos.

É importante lembrar que os métodos de Euler, Euler Inverso e Euler Aprimorado são casos particulares dessas famílias listadas. O método de Euler é o caso de ordem 1 tanto da família Adams-Bashforth como Runge-Kutta. O Euler Inverso e o aprimorado são, respectivamente, os casos de ordem 1 e 2 de Adams-Moulton. E uma versão de implementação do método de Euler Aprimorado também pode ser entendida como uma caso de ordem 2 da família Runge-Kutta.

A seguir estão apresentadas as fórmulas dos métodos e uma análise comparativa entre os erros, os gráficos e os pontos (numericamente falando) de cada família de métodos. É importante perceber que estamos comparando métodos com diferentes ordens de precisão. A ordem de precisão n influencia diretamente no erro, de forma que o erro da solução numérica do método é definido como  $E(\Delta x) = c\Delta x^n$ , sendo c uma constante. Em notação O-grande,

um método de ordem n pode ser dito como  $O(\Delta x^n)$ . Essa informação será verificada quando analisarmos os erros.

#### 2.1 Adams-Bashforth

Fórmula do Método Adams-Bashforth

$$\begin{split} y_{n+1} &= y_n + hf(t_n, y_n), \qquad \text{(This is the Euler method)} \\ y_{n+2} &= y_{n+1} + h\left(\frac{3}{2}f(t_{n+1}, y_{n+1}) - \frac{1}{2}f(t_n, y_n)\right), \\ y_{n+3} &= y_{n+2} + h\left(\frac{23}{12}f(t_{n+2}, y_{n+2}) - \frac{4}{3}f(t_{n+1}, y_{n+1}) + \frac{5}{12}f(t_n, y_n)\right), \\ y_{n+4} &= y_{n+3} + h\left(\frac{55}{24}f(t_{n+3}, y_{n+3}) - \frac{59}{24}f(t_{n+2}, y_{n+2}) + \frac{37}{24}f(t_{n+1}, y_{n+1}) - \frac{3}{8}f(t_n, y_n)\right), \\ y_{n+5} &= y_{n+4} + h\left(\frac{1901}{720}f(t_{n+4}, y_{n+4}) - \frac{1387}{360}f(t_{n+3}, y_{n+3}) + \frac{109}{30}f(t_{n+2}, y_{n+2}) - \frac{637}{360}f(t_{n+1}, y_{n+1}) + \frac{251}{720}f(t_n, y_n)\right). \end{split}$$

Os coeficientes multiplicativos da fórmula de ordem k podem ser encontrados na seguinte tabela:

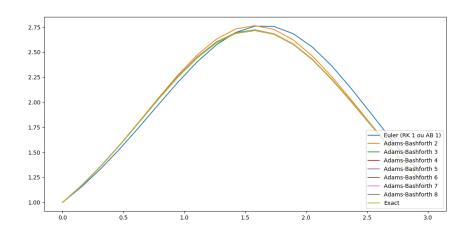
k	$\beta_1$	$eta_2$	$\beta_3$	$eta_4$	$eta_5$	$eta_6$	$eta_7$	$eta_8$
1	1							
2	$\frac{3}{2}$	$-\frac{1}{2}$						
3	$\frac{23}{12}$	$-\frac{4}{3}$	$\frac{5}{12}$					
4	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{3}{8}$				
5	$\frac{1901}{720}$	$-\frac{1387}{360}$	$\frac{109}{30}$	$-\frac{637}{360}$	$\frac{251}{720}$			
6	$\frac{4277}{1440}$	$-\frac{2641}{480}$	$\frac{4991}{720}$	$-\frac{3649}{720}$	$\frac{959}{480}$	$-\frac{95}{288}$		
7	$\frac{198721}{60480}$	$-\frac{18637}{2520}$	$\frac{235183}{20160}$	$-\frac{10754}{945}$	$\frac{135713}{20160}$	$-\frac{5603}{2520}$	$\frac{19087}{60480}$	
8	$\frac{16083}{4480}$ -	$-\frac{1152169}{120960}$	$\frac{242653}{13440}$	$-\frac{296053}{13440}$	$\frac{2102243}{120960}$	$-\frac{115747}{13440}$	$\frac{32863}{13440}$	$-\frac{5257}{17280}$

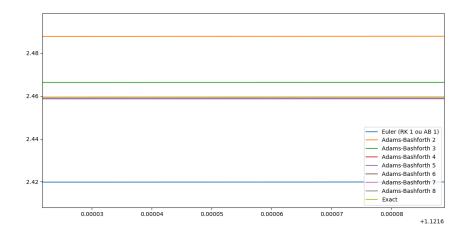
Percebe-se que os primeiros k pontos, sendo k a ordem de precisão do método dessa família, devem ser estimados para executar o algoritmo. Para isso foi escolhido usar o método Runge-Kutta

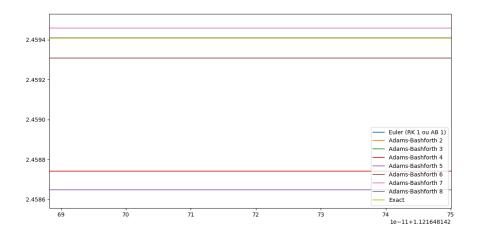
de ordem também k. Assim é preservada a ordem de precisão do método.

Análise comparativa da família Adams-Bashforth para o problema de valor inicial com  $x_0 = 0$ ,  $y_0 = 1$ ,  $x_f = 3$ , n = 20 e y' = cos(x) \* y utilizando as implementações contidas na parte final desse relatório:

Método				Média d	lo Erro	Des	svio Pad	lrão do	Erro	
Eul	er (RK 1	ou AB	1)   6	0.0656949116859			0.0392463786245			
Ad	ams-Bash	forth 2	6	0.0192732424196			0.0166155612885			
Ad	ams-Bash	forth 3	į e	.002922	9745720	9	0.00222625128307			
j Ad	ams-Bash	forth 4	į e	.001106	2920439	4	0.00117	3908234	34	
j Ad	ams-Bash	forth 5	į o.	0003789	2571181	4   0	.000354	8109626	62	
j Ad	ams-Bash	forth 6	į o.	0001501	.3122973	1   0	.000157	2955191	.81	
l Adams-Bashforth 7			i 7	7.69408736258e-05 7.93104830			303170-	.05 İ		
l Au	ams basii	TOT CIT 1	' •	0340013	02386 0	J   1	• 221040	202T16	0.5	
!	ams-Bash			6628917			.260792		!	
!									!	
!			3.	6628917	/5732e-0		260792		!	
Ad +	ams-Bash	forth 8	y Adans-Bashforth 3	9 Adams-Bashforth 4	75732e-0	5 4 +	y Adams-Bashforth 7	y Adams-Bashforth 8	-05	
Ad +	ams-Bash	forth 8	y Adans-Bashforth 3	9 Adams-Bashforth 4	75732e-0	y Adams-Bashforth 6	y Adans-Bashforth 7	y Adams-Bashforth 8	y exato  1.0 1.1702755401819866	
Ad +	y Euler (RK 1 ou AB 1)   1.0   1.15789473684   1.1384596438	forth 8	y Adans-Bashforth 3  1.0 1.17924764886 1.3641369429	y Adams-Bashforth 4	75732e-0	5 4 y Adams-Bashforth 6 1.17027583667 1.3641983999	y Adams-Bashforth 7	y Adans-Bashforth 8  1.0 1.17027593667 1.3641983999	y exato  1.0 1.1702758491819866 1.3641983999060567	
Ad +   x   0.0   0.157894736842   0.473684210526	ams-Bash    y Euler (RK 1 ou AB 1)    1.0    1.15789473684    1.33844596438  1.5392949753	forth 8  y Adans-Bashforth 2  1.0  1.16922298219 1.3637520853 1.57961560879	y Adams-Bashforth 3  1.9 1.17924764886 1.3641369429 1.57940422779	1.0 1.17027498799 1.36419652659 1.57801233268	y Adams-Bashforth 5 1.16 1.17027586123 1.364198451337 1.57691540768	5 4 y Adams-Bashforth 6 1.0 1.17027583667 1.3641983999 1.57801530191	y Adams-Bashforth 7  1.0 1.17027583667 1.3641983999 1.57801530191	y Adams-Bashforth 8  1.0 1.17027583667 1.3641983909 1.57801530191	y exato 1.10 1.1702758401819866 1.3641983999669567 1.5780153182286395	
+   x   0.0   0.157894736842   0.315789473684   0.473684210526   0.631578947368	y Euler (RK 1 ou AB 1)  1.0 1.15780473684 1.3384296428 1.53932949753 1.75561086463	forth 8	y Adams-Bashforth 3  1.0 Adams-Bashforth 3  1.17924764886 1.3643369429 1.57994022729 1.8793794665	y Adams-Bashforth 4   1.0   1.17027498799   1.36419620559   1.57881233268   1.80499081095	y Adams-Bashforth 5   1.0   1.17027586123   1.36419845137   1.57881540768   1.80474612111	y Adams-Bashforth 6  1.0 1.17027583667 1.3641983909 1.57801530191 1.8844795566	y Adams-Bashforth 7  1.0 1.7027583667 1.3641983909 1.57801530191 1.8847495656	y Adams-Bashforth 8  1.0 1.17027583667 1.3641983909 1.57801530191 1.8847456566	y exato  1.102754401819866 1.364198399966967 1.5780153180286395 1.88047599805531385	
Ad +   x   0.9   0.15789473684   0.315789473684   0.473684210526   0.631578947368   0.789473684211	y Euler (RK 1 ou AB 1)   1.0   1.15780473684   1.33844596438   1.5392949753   1.75561986463   1.0793494295	y Adams-Bashforth 2  1.0 1.6922298219 1.3637520853 1.57961506879 1.81820140591 2.0452536681	y Adams-Bashforth 3 1.6 1.7024764886 1.3643469429 1.65794022779 2.88733794605 2.03826644682	y Adams-Bashforth 4  1.0 1.170727498799 1.36419652059 1.57881233268 1.80490981005 2.03416531593	y Adams-Bashforth 5   1.0 1.17627586123 1.36419845137 1.5760156766 1.86474612111 2.03379419308	y Adams-Bashforth 6  1.0 1.17027583667 1.3641983909 1.57801530191 1.80474596566 2.0333609662	y Adams-Bashforth 7  1.0 1.17027583667 1.3641983999 1.57801530191 1.88474596566 2.0339669962	y Adams-Bashforth 8  1.0 1.17027583667 1.3641983909 1.57801530191 1.880474596566 2.0313669862	y exato 1.0 1.10 1.792754401819866 1.3641983999669567 1.5780153180286395 1.8847459895551385 2.03395613913173	
, and , e.e., e.e.	y fuler (8/ 1 or A8 1)  1.0  1.1780471564 1.13844596438 1.17956196643 1.17957196643 1.17957196643	y Adams-Bashforth 2  1.692298219 1.16922298219 1.156752668379 1.181020146991 2.045253846881 2.247103645284	y Adams-Bashforth 3  1.0 1.17024764886 1.3641369429 1.57594022729 1.8873374665 2.03826644682 2.25881176372	y Adams-Bashforth 4  1.0 1.17927498799 1.36419652659 1.57891233268 1.80490981095 2.03416531593 2.2521646883	y Adams-Bashforth 5  1.0  1.1702756122  1.36419845137  1.57801546768  1.86474612111  2.03379419308  2.2517029978	y Adams-Bashforth 6  1.0 1.17027583667 1.3641983099 1.57801530191 1.8847450566 2.03395609862 2.25207687544	y Adams-Bashforth 7  1.0 1.17027583667 1.3641983999 1.57801530191 1.88474505566 2.03395609862 2.25214237135	y Adans-Bashforth 8  1.0 1.17027533667 1.364193999 1.57801530191 1.8847459556 2.03195609862 2.2521427135	y exato  1.102758401819866 1.364198399966967 1.5768153180286395 2.033996138181713 2.25214240269976	
Ad +   x   0.0   0.157894736842   0.47368421652   0.6315789473684   0.47368421652   0.947368421653   1.18526312653	ams-Bash    y fuler (8K 1 our AB 1)   1.0   1.15789473664   1.13844596438   1.79541984635   1.19943827584   2.49924756633	y Adams-Bashforth 2  1.0 1.6922298219 1.3637520853 1.57961560879 1.81020140591 2.0452538681 2.27103645204 2.47133228676	y Adams-Bashforth 3 1.6 1.7624764886 1.3643469429 1.657946922779 2.88733794695 2.0382664682 2.25801170372 2.45874394826	y Adams-Bashforth 4  1.0  1.17027498799 1.360149632959 1.860496812965 2.03416531593 2.2521064883 2.43328441832	y Adams-Bashforth 5   1.0   1.17027596123   1.16021965233   1.160219652137   1.160474612111   2.03374419308   2.25170209978   2.431395989   2.43139589   2.4313989   2.4313989   2.4313989   2.4313989   2.4313989   2.4313989   2.4313989   2.4313989   2.4313989   2.4313989   2.4318999   2.4318999   2.4318999   2.431899   2.431899   2.431899   2.431899   2	5 4 y Adams-Bashforth 6 1.0 1.17027583667 1.3641983909 1.57801530191 1.80474596566 2.033960962 2.25207687544 2.4437625403	y Adams-Bashforth 7  1.0 1.17027583667 1.3641983999 1.57801530191 1.88474596566 2.0335609662 2.25214237135 2.4430138756	y Adams-Bashforth 8  1.0 1.7027543667 1.36419433909 1.57801330191 1.880474596566 2.0339669962 2.25214237135 2.4438674812	y exato  1.0 1.102175401819866 1.3641983999669567 1.578015318226395 1.8847459895551385 2.033956128181713 2.252142409269976 2.443868790475302	
Ad + 0.0 0.0 0.157894736842 0.3157894736842 0.315789473684211 1.052735789 1.10527315789 1.126315789474	y fater (8K 1 ou AB 1)  1.0 1.15789475644 1.135844596438 1.75932946733 1.75951586463 2.1094327564 2.4021479693 2.57243196411	y Adams-Bashforth 2  1.09 1.16922298219 1.16922298219 1.157961569879 1.15102416991 2.04525380681 2.271103645924 2.47135228676 2.62942739275	y Adams-Bashforth 3  1.0  1.17024764886 1.3641369429 1.57904022729 1.8793794695 2.03826644682 2.25801170372 2.45874394826 2.608983170476	y Adams-Bashforth 4  1.0 1.17027498799 1.36419652059 1.57681233268 1.86490981005 2.03416531593 2.2521664883 2.44328441822 2.59223374733	y dams-Bashforth 5  1.0  1.10  1.1702756122  1.36419845137  1.57801549768  1.80474612111  2.03379419308  2.2517029978  2.4431359580	y Adams-Bashforth 6  1.0 1.17027583667 1.3641983999 1.57801530191 1.8847450566 2.03395609862 2.15227678744 2.44376285403 2.5033636673	y Adams-Bashforth 7  1.0  1.7027583667  1.1641983999 1.57801530191 1.88474506566 2.03395609862 2.02514237135 2.44301038756	y Adams-Bashforth 8  1.0 1.17027583667 1.3641943999 1.57801530191 1.88474505566 2.03195609862 2.2521427135 2.44386874812 2.59363645946	y exato 1.10 1.10 1.1702754401819866 1.364198399669567 1.5784153186286395 2.033956130181713 2.25214240269976 2.443868796475392 2.4538151529904	
0.0 0.157894736842 0.3157894736842 0.315789473684 0.4736842169 0.631578947368 1.1263157894736 1.1263157894736 1.126315789473	ams-Bash  1.0 1.1.7 1.1.5932949733 1.7593196413 1.795919643 1.195919643 1.195919643 1.2.1959391754 1.2.57745196411 2.57745196411	forth 8  1.0 1.692298219 1.3637528653 1.57961508279 1.8169214991 2.27183645284 2.4713528676 2.62942739275 2.73942139566	y Adams-Bashforth 3  1.0  1.7024764886 1.3641369429 1.57964022772 2.1587646622 2.25801176372 2.45974394826 2.60958170476 2.60958170476	9 Adams-Bashforth 4 1.0 1.17027498799 1.186419652659 1.15769123326 2.024146531593 2.2521064883 2.45328441802 2.59223374733 2.66577229701	y Adams-Bashforth 5   1.0   1.17027566123   1.36418985137   1.37021546708   2.26374913908   2.25170209978   2.	5 4  y Adams-Bashforth 6  1.0  1.17027583667  1.3641983909  1.57801530191  2.0339669662  2.25207687544  2.4437625493  2.5033639673  2.68810953493	y Adams-Bashforth 7  1.0  1.702753567  1.36419439999  1.57801538191  2.282142371375  2.4831923769  2.59373829269	y Adams-Bashforth 8  1.0 1.17027753667 1.36410435909 1.57801530191 2.20336669662 2.25214237135 2.44318674812 2.59363645946	y exato  1.0  1.1027784401819866 1.3641983999969567 1.5780153189286395 2.031964139813173 2.252142499269976 2.443868799475392 2.5936115631529994	
Ad + 0.0 0.0 0.157894736842 0.3157894736842 0.315789473684211 1.052735789 1.10527315789 1.126315789474	y fater (8K 1 ou AB 1)  1.0 1.15789475644 1.135844596438 1.75932946733 1.75951586463 2.1094327564 2.4021479693 2.57243196411	y Adams-Bashforth 2  1.09 1.16922298219 1.16922298219 1.157961569879 1.15102416991 2.04525380681 2.271103645924 2.47135228676 2.62942739275	y Adams-Bashforth 3  1.0  1.17024764886 1.3641369429 1.57904022729 1.8793794695 2.03826644682 2.25801170372 2.45874394826 2.608983170476	y Adams-Bashforth 4  1.0 1.17927498799 1.36419652059 1.57891233268 1.86490991095 2.03416531593 2.2521664883 2.44328441822 2.59223374733	y dams-Bashforth 5  1.0  1.10  1.1702756122  1.36419845137  1.57801549768  1.80474612111  2.03379419308  2.2517029978  2.4431359580	y Adams-Bashforth 6  1.0 1.17027583667 1.3641983999 1.57801530191 1.8847450566 2.03395609862 2.15227678744 2.44376285403 2.5033636673	y Adams-Bashforth 7  1.0  1.7027583667  1.1641983999 1.57801530191 1.88474506566 2.03395609862 2.02514237135 2.44301038756	y Adams-Bashforth 8  1.0 1.17027583667 1.3641943999 1.57801530191 1.88474505566 2.03195609862 2.2521427135 2.44386874812 2.59363645946	y exato 1.10 1.10 1.1702754401819866 1.364198399669567 1.5784153186286395 2.033956130181713 2.25214240269976 2.443868796475392 2.4538151529904	
Ad  +  0.0  0.157894736842  0.137894736842  0.3157894736842  0.3157894736842  1.16256315789  1.16256315789  1.16215769474  1.4621052631589	ams-Bash  1.0 1.1578475644 1.13844596438 1.7934942554 2.19543827594 2.19543827594 2.25742196411 2.6954581569 2.758383332	y Adams-Bashforth 2  1.0  1.16922298219  1.3637528653  1.57961566879  1.81020140501  2.271035645204  2.271035645204  2.27103565232	y Adams-Bashforth 3 1.10 1.170.2476486 1.18613169429 1.57094022729 2.088256644689 2.08825644689 2.0880170476 2.60098170476 2.60098170476 2.60098170476 2.722240670818	y Adans-Bashforth 4  1.0  1.17027498799 1.36419652059 1.36419652059 2.03416511593 2.252106488 2.5923374733 2.68577292701 2.75139922993	y Adams-Bashforth 5  1.0  1.17027586123 1.16419845137 1.57801540766 1.80474612111 2.03179413980 2.2517020978 2.2517020978 2.2502676742 2.66693005661 2.717183274477	y Adams-Bashforth 6  1.0  1.17027583667  1.3641983999  1.57801539191  1.80474596566  2.23227687544  2.5032369672  2.6810053493  2.7842749495	y Adams-Bashforth 7  1.0  1.17027833667  1.3641983999  1.57801539191  1.88474596566  2.22214227131  2.2521422712059  2.6822676639  2.7843744442	y Adams-Bashforth 8  1.9 1.17027933667 1.36419433999 1.57801530191 1.880474596566 2.25214237135 2.25214237135 2.2531645946 2.68806664611 2.75819335816	y exato  1.10 1.170.7544018.1866 1.170.7544018.1866 1.170.7544018.1866 1.1804754908.0512 1.27814018.1866 2.25214.2492.9512 2.4338679047532 2.4338679047532 2.638032.953262558 2.638032.95326258	
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### 2.2 Adams-Moulton

Fórmula dos Métodos Adams-Moulton

$$\begin{split} y_n &= y_{n-1} + hf(t_n,y_n), \qquad \text{(This is the backward Euler method)} \\ y_{n+1} &= y_n + \frac{1}{2}h\left(f(t_{n+1},y_{n+1}) + f(t_n,y_n)\right), \qquad \text{(This is the trapezoidal rule)} \\ y_{n+2} &= y_{n+1} + h\left(\frac{5}{12}f(t_{n+2},y_{n+2}) + \frac{2}{3}f(t_{n+1},y_{n+1}) - \frac{1}{12}f(t_n,y_n)\right), \\ y_{n+3} &= y_{n+2} + h\left(\frac{3}{8}f(t_{n+3},y_{n+3}) + \frac{19}{24}f(t_{n+2},y_{n+2}) - \frac{5}{24}f(t_{n+1},y_{n+1}) + \frac{1}{24}f(t_n,y_n)\right), \\ y_{n+4} &= y_{n+3} + h\left(\frac{251}{720}f(t_{n+4},y_{n+4}) + \frac{646}{720}f(t_{n+3},y_{n+3}) - \frac{264}{720}f(t_{n+2},y_{n+2}) + \frac{106}{720}f(t_{n+1},y_{n+1}) - \frac{19}{720}f(t_n,y_n)\right). \end{split}$$

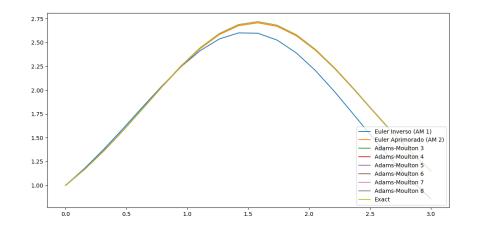
Os coeficientes multiplicativos da fórmula de ordem k+1 podem ser encontrados na seguinte tabela:

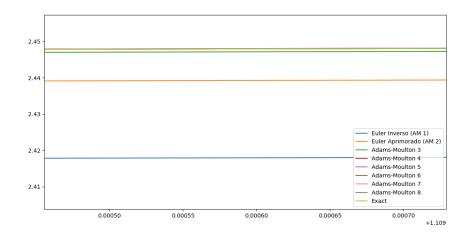
É importante notar que o  $y_{i+1}$  nos métodos de Adams-Moulton são definidos, também, em função dos próprios  $y_{i+1}$ . Diante dessa situação, para calcular algoritmicamente os pontos torna-se necessário realizar uma estimativa do próprio ponto com antecedência. Para manter a ordem de precisão do método compatível com o esperado, foi escolhido aproximar os pontos através do método Runge-Kutta de respectiva ordem.

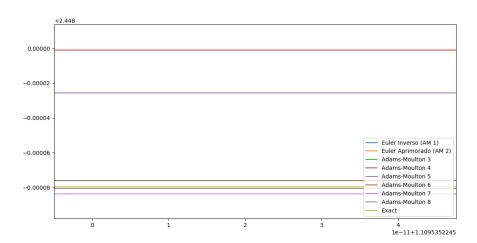
Análise comparativa da família Adams-Moulton para o problema de valor inicial com  $x_0 = 0$ ,  $y_0 = 1$ ,  $x_f = 3$ , n = 20 e y' = cos(x) \* y utilizando as implementações contidas na parte final desse relatório:

+   Método	Média do Erro	Desvio Padrão do Erro
Euler Inverso (AM 1) Euler Aprimorado (AM 2) Adams-Moulton 3 Adams-Moulton 4 Adams-Moulton 5 Adams-Moulton 6 Adams-Moulton 7 Adams-Moulton 8	0.138340007444   0.00543755696108   0.000357389705101   8.53927995198e-05   2.32434825008e-05   7.02920263779e-06   3.21871749741e-06   1.1968682765e-06	0.114763566621 0.00423500226462 0.00025002600381 8.79534953476e-05 1.99939367257e-05 7.42493052414e-06 3.1188850292e-06 1.27529333634e-06

х		y Euler Aprimorado (AM 2)							y exato
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
.157894736842	1.18055122753	1.17010619227	1.17024764886	1.17027498799	1.17027586123	1.17027583667	1.17027583667	1.17027583667	1.170275840181
.315789473684	1.38143467869	1.363528632	1.36404942344	1.36419652059	1.36419845137	1.3641983969	1.3641983909	1.3641983989	1.364198399906
.473684210526	1.59772512778	1.576436014	1.57770744395	1.57799799557	1.57801540768	1.57801530191	1.57801530191	1.57801530191	1.578015318028
.631578947368	1.8214547461	1.80181731356	1.80425821514	1.80472290303	1.88475442988	1.80474596566	1.80474596566	1.88474596566	1.804745989655
789473684211	2.04154353899	2.02928578572	2.03329709132	2.03394480294	2.03397814906	2.03395931736	2.03395609862	2.03395609862	2.03395613018
947368421053	2.24429295919	2.24548165038	2.25135935868	2.25216488157	2.25218094823	2.25214745252	2.25214098065	2.25214237135	2.25214248928
10526315789	2.41455722727	2.43520366431	2.44304687902	2.44394576009	2.44392263598	2.44387255546	2.44386477453	2.44386789112	2.44386879847
26315789474	2.53755107885	2.58322004806	2.59286000839	2.59375465183	2.59367418868	2.59361035856	2.59360451679	2.59361029863	2.59361156315
42105263158	2.60104359636	2.67648117411	2.68745870697	2.68823732979	2.68809264213	2.68802302456	2.68802272193	2.68803142995	2.68803205326
57894736842	2.59749286391	2.70626350693	2.71787317403	2.71843771586	2.71823835	2.71817431264	2.71818189637	2.71819243739	2.7181915298
73684210526	2.52558428854	2.66970977734	2.6811098666	2.68140381021	2.68117540273	2.68112803575	2.68114316012	2.68115342486	2.68115063668
89473684211	2.39071633928	2.57034736545	2.58071189918	2.58073728545	2.58051497403	2.58049085843	2.58051016578	2.58051778756	2.58051372479
.05263157895	2.20423597336	2.41744019975	2.42611204509	2.42592847534	2.42574624965	2.42574480987	2.42576310251	2.42576663125	2.42576259777
.21052631579	1.98158196342	2.22435852007	2.23096013596	2.23066596543	2.23054639066	2.23056022882	2.2305726301	2.23057227963	2.23056959803
36842105263	1.73981010635	2.00640724109	2.0108799044	2.01058329041	2.01053186584	2.01054990116	2.01055386047	2.01055137873	2.01055069015
52631578947	1.49512193863	1.77864612919	1.78121359025	1.78100456694	1.781010554	1.78102231612	1.7810184085	1.78101615505	1.78101715310
68421052632	1.26094734207	1.55415078679	1.55522782662	1.55516862481	1.55520304825	1.55520206071	1.55519330196	1.55519311462	1.55519484029
84210526316	1.04689159982	1.34296338861	1.34384339612	1.34313256316	1.34318810069	1.34317307822	1.34316333059	1.34316583376	1.34316726184
3.0	0.858565899824	1.15175928181	1.15131393936	1.15154191407	1.15159138933	1.15156527954	1.15155768841	1.15156231921	1.15156283651







# 2.3 Runge-Kutta

Fórmula do Método Runge-Kutta  $4\,$ 

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot k_1)$$

$$k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} \cdot k_2)$$

$$k_4 = f(t_n + h, y_n + h \cdot k_3)$$

$$y_{n+1} = y_n + \frac{h}{6} \cdot (k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4)$$

A generalização da família de métodos Runge-Kutta é dada pela seguinte imagem:

$$y_{n+1}=y_n+h\sum_{i=1}^s b_i k_i,$$

onde

$$egin{aligned} k_1 &= f(t_n, y_n), \ k_2 &= f(t_n + c_2 h, y_n + a_{21} h k_1), \ k_3 &= f(t_n + c_3 h, y_n + a_{31} h k_1 + a_{32} h k_2), \ &dots \ k_s &= f(t_n + c_s h, y_n + a_{s1} h k_1 + a_{s2} h k_2 + \cdots + a_{s,s-1} h k_{s-1}). \end{aligned}$$

Análise comparativa da família Runge-Kutta para o problema de valor inicial com  $x_0 = 0$ ,  $y_0 = 1$ ,  $x_f = 3$ , n = 20 e y' = cos(x) \* y utilizando as implementações contidas na parte final desse relatório:

+					+				
i	Método	· · · · · · · · · · · · · · · · · · ·	M 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4	F	I Baarra	. D. d ~ -	d		
1	метоао		Média do	Erro	pesvio	Padrao	do Erro		
+					+				
1 =	l / D// 1	. AD 1\	0.0050040	116050		222246276	00045		
Eu	ler (RK 1 ou	IABI)	0.0656949	116829	0.0	0.0392463786245			
Fuler	Aprimorado	2 (RK 2)	0.0079657	1285262	1 0.6	004193919	82569		
	•								
	Runge-Kutta	a 3	0.00013723	5021532	7.46	551779890	06e-05		
i	D	- a i	4 01053370	225-06	i 200	364949523	25-06		
I	Runge-Kutta	14	4.81853370	2336-06	2.00	004343523	55e-06		
1	Runge-Kutta	a 5	1.31486408	077e-07	1 5.50	24171900	02e-08		
!	J				!				
	Runge-Kutta	a 6   1	2.77150835	147e-08	1.56641320517e-08				
i	<b>_</b>	i			i				
<b>+</b>					T				
×	y Euler (RK 1 ou AB 1)	y Euler Aprimorado 2 (RK	2)   y Runge-Kutta 3	y Runge-Kutta 4	y Runge-Kutta 5	y Runge-Kutta 6	y exato		
0.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0		
0.157894736842	1.15789473684	1.16922298219	1.17024764886	1.17027498799	1.17027586123	1.17027583667	1.170275840181986		
0.315789473684	1.33844596438	1.36180619765	1.3641369429	1.36419652059	1.36419845137	1.3641983909	1.364198399906056		
0.473684210526	1.53932940753	1.57403366816	1.57792017433	1.57801233268	1.57801540768	1.57801530191	1.578015318028630		
0.631578947368	1.75561986463	1.79900384347	1.80462296474	1.80474192848	1.80474612111	1.80474596566	1.804745989655138		
0.789473684211	1.97934948295	2.0263967557	2.03381759932	2.03395111218	2.03395630109	2.03395609862	2.033956130181713		
0.947368421053	2.19943827584	2.24284109526	2.25200607829	2.25213660179	2.25214261086	2.25214237135	2.252142409209976		
1.10526315789	2.40218769603	2.43302838952	2.44375507863	2.44386237383	2.44386900914	2.44386874812	2.443868790475302		
1.26315789474	2.57245196411	2.58153995916	2.59354029566	2.59360471575	2.59361178276	2.59361151817	2.593611563152090		
1.42105263158	2.69544581569	2.67511255456	2.68801927158	2.68802495348	2.68803225858	2.68803200725	2.688032053262558		
	1.57894736842   2.7589383332   2.70485735982		2.718246956	2.71818435461	2.71819170935	2.718191484	2.71819152984382		
	.73684210526   2.75538760075   2.6678672862		2.68127597406	2.68114354772	2.68115078471	2.68115059194	2.681150636681273		
1.89473684211		2.68347902538 2.56776298046		2.58050685596	2.58051384253 2.42576269282	2.58051368213	2.580513724792542		
2.05263157895	2.54861107613	2.41401735776 2.22025348883	2.42599811975	2.42575606537	2.42576269282	2.42576255844	2.425762597777492		
2.21052631579	2.3621307102	2.22025348883 2.00198801625	2.23083065018	2.23056353555	2.23056968225	2.23056956365	2.230569598036628		
2.36842105263	1 1.89770484319	1.77438863648	1.78125715713	1.78101261288	1.78101725089	1.78101713336	1.781017153108046		
2.52631578947	1.89770484319	1.77438863648	1.78125715713	1.78101261288	1.78101725089	1.78101713336	1.781017153108046		
2.8421052632	1.41884207891	1.35051592771	1.35539583922	1.35519137231	1.35519495425	1.35519482878	1.343167261846345		
3.0	1.41884207891	1.1502445616	1.15166812727	1.34316497266	1.15156297397	1.15156283748	1.151562836514534		
3.0	1.204/8633666	1.1502445616	1.15166812727	1.15150168826	1.1515629/39/	1.15156283748	1.15156283651453		

