Gears based on general smooth and sinusodial rack profile

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Abstract

The general geometrical and mechanical properties of gears based on general smooth rack profiles are discussed.

1 Geometry

1.1 Motion of gear and rack

We are considering a gear rotating with angular velocity ω around the coordinate origin $(0, 0)$ in the two-dimensional (complex) plane. On the other hand we are considering a rack moving along the x-axis (real axis) with velocity v. Thus we have two two-dimensional moving rigid objects. We consider body-space coordinates z_g and z_r for the gear and the rack respectively. The time-dependent transform to world coordinates z are as follows:

$$\Phi_g(z_g, t) = z_g e^{i\omega t} \tag{1.1}$$

$$\Phi_r(z_r, t) = z_r + vt \tag{1.2}$$

Of course we have the inverse functions (inverse with respect to coordinates z):

$$\Phi_g^{-1}(z, t) = ze^{-i\omega t} \tag{1.3}$$

$$\Phi_r^{-1}(z, t) = z - vt \tag{1.4}$$

Thus, we get the mapping from one body space into the other as a function of time as follows:

$$\Xi_g(z_r, t) = \Phi_g^{-1}(\Phi_r(z_r, t), t) = (z_r + vt)e^{-i\omega t} \tag{1.5}$$

$$\Xi_r(z_g, t) = \Phi_r^{-1}(\Phi_g(z_g, t), t) = z_g e^{i\omega t} - vt \tag{1.6}$$

with these time derivatives (velocity fields):

$$\dot{\Xi}_g(z_r, t) = (v - i(z_r + vt)\omega)e^{-i\omega t} \tag{1.7}$$

$$\dot{\Xi}_r(z_g, t) = i\omega z_g e^{i\omega t} - v \tag{1.8}$$

For any t, there is exactly one z_r respectively z_g, such that the respective velocity field vanishes. We denote these with $z_{r0}(t)$ $z_{g0}(t)$:

$$z_{r0}(t) = -i\frac{v}{\omega} - vt \tag{1.9}$$

$$z_{g0}(t) = -i\frac{v}{\omega}e^{-i\omega t} \tag{1.10}$$

The respective world space point does no longer depend on time t:

$$z_0 = \Phi_r(z_{r0}(t), t) = \Phi_g(z_{g0}(t), t) = -i\frac{v}{\omega} \tag{1.11}$$

We denote the quantity $\frac{v}{\omega}$ with r (pitch radius), obtaining the typical relation

$$v = \omega r \tag{1.12}$$

1.2 Shape of the rack and gear module

In body coordinates z_r, the surface of the rack is given by a (smooth and derivable) real function $f(x)$ such that

$$z_r(x) = x + i(f(x) - r) \tag{1.13}$$

We further assume that $f(x)$ is periodic with periodicity p:

$$f(x + p) = f(x) \tag{1.14}$$

We consider all points $z = x + iy$ (x, y real) with $y < f(x) - r$ to be actual rack points (means, that these points really belong to the rack in the sense of a rigid body). We denote this set by

$$\Omega_r = \{x + yi | y < f(x) - r\} \tag{1.15}$$

After mapping this to the gear body space for a given time t gives

$$\Omega_g(t) = \Xi_g(\Omega_r, t) = \{(x + yi + vt)e^{-i\omega t} | y < f(x) - r\} \tag{1.16}$$

After a full rotation of the gear at $\tau = \frac{2\pi}{\omega}$, we assume that

$$\Omega_g(0) = \Omega_g(\tau) = \{(x + yi + \frac{2\pi v}{\omega}) | y < f(x) - r\} = \{(x + yi + 2\pi r) | y < f(x) - r\} \tag{1.17}$$

We introduce a magnitude

$$\mathbf{z} = \frac{2\pi r}{p} \tag{1.18}$$

If $\mathbf{z}$ is an integer number (the number of teeth of the gear), then we get indeed

$$\begin{aligned} \Omega_g(\tau) &= \{(x + yi + \mathbf{z}p) | y < f(x) - r\} = \{(x + yi + \mathbf{z}p) | y < f(x + \mathbf{z}p) - r\} = \{(x + yi) | y < f(x) - r\} \\ &= \Omega(0) \end{aligned} \tag{1.19}$$

In order to get rational numbers in the context of construction, the module number m is introduced to express the linear relation between the number of teeth and the diameter of the gear

$$2r = \mathbf{z}m \tag{1.20}$$

1.3 The shape of the gear

We define the swapped area of the rack in gear-body-coordinates:

$$\tilde{\Omega}_g = \bigcup_t \Omega_g(t) \tag{1.21}$$

Then the set

$$\bar{\Delta} = \mathbb{C} \backslash \tilde{\Omega}_g \tag{1.22}$$

is the largest set of gear body points that never coincide with the rack. Notice, from a topological point of view, the sets $\Omega_g(t)$ are open sets as well as $\tilde{\Omega}_g$. Thus $\bar{\Delta}$ is a closed set. We define Δ to be the interior of $\bar{\Delta}$:

$$\Delta = \mathtt{int}(\bar{\Delta}) \tag{1.23}$$

and Γ to be the border of $\bar{\Delta}$

$$\Gamma = \partial \bar{\Delta} \tag{1.24}$$

We now prove

Theorem 1.1 *For any $z_g \in \Gamma$ there exists a t and an x such that*

$$z_g = (x + (f(x) - r)i + vt)e^{-i\omega t} \tag{1.25}$$

Proof 1.25 is equivalent to

$$\begin{aligned}\texttt{Re}(z_g e^{i\omega t}) &= x + vt \\ \texttt{Im}(z_g e^{i\omega t}) &= f(x) - r\end{aligned} \tag{1.26}$$

Assuming that for a given z_g, t, and x we have

$$\begin{aligned}\texttt{Re}(z_g e^{i\omega t}) &= x + vt \\ \texttt{Im}(z_g e^{i\omega t}) &< f(x) - r\end{aligned} \tag{1.27}$$

Then we must have $z_g \in \Omega_g(t)$. And therefore $z_g \in \tilde{\Omega}$ and $g_z \notin \bar{\Delta}$ and finally $z_g \notin \Gamma$ which contradicts the proposition. Thus for all t we must have

$$\texttt{Im}(z_g e^{i\omega t}) \geq f(\texttt{Re}(z_g e^{i\omega t}) - vt) - r \tag{1.28}$$

We now define the function

$$g(z,t) = \texttt{Im}(z e^{i\omega t}) - f(\texttt{Re}(z e^{i\omega t}) - vt) + r \tag{1.29}$$

which is now known to satisfy

$$g(z_g, t) \geq 0 \tag{1.30}$$

for all t. Assuming that $g(z_g, t) > 0$ for all t leads to another contradiction. Whenever $g(z_g, t) > 0$ for a t, then there is a maximum radius $\rho(t)$ such that $g(z,t)$ is non-negative for all z in the open disc $u_{\rho(t)}(z_g)$. $\rho(t)$ is a continuous function of t. Since $g(z,t)$ is periodic with respect to t we can take the minimum radius ρ_{min} over a compact set. This means that $u_{\rho_{min}}(z_g) \subset \bar{\Delta}$ and thus $z_b \in \Delta$ and finally $z_b \notin \Gamma$. ∎

1.3.1 Another necessary condition for $z_b \in \Gamma$

For any $z_g \in \Gamma$, from theorem 1.1 we know about the existence of a t and an x such that

$$z_g = (x + (f(x) - r)i + vt)e^{-i\omega t} \tag{1.31}$$

This corresponds to a point in rack coordinates

$$z_r = \Xi_r(z_g, t) = z_g e^{i\omega t} - vt = x + (f(x) - r)i \tag{1.32}$$

On the other hand the velocity of the gear at that point in rack coordinates is

$$\begin{aligned}\dot{\Xi}_r(z_g, t) &= i\omega z_g e^{i\omega t} - v = i\omega(x + (f(x) - r)i + vt) - v \\ &= -\omega(f(x) - r) - v + i\omega(x + vt) \\ &= -\omega f(x) + i\omega(x + vt) \\ &= i\omega(x + vt + if(x))\end{aligned} \tag{1.33}$$

Now intuition tells us that the velocity must be tangential to the rack boundary. Thus for a real λ we must have:

$$i\omega(x + vt + if(x)) = \lambda(1 + f'(x)i) \tag{1.34}$$

or equivalently

$$i(x + vt + if(x)) = \lambda(1 + f'(x)i) \tag{1.35}$$

Equating real and imaginary parts separately yields 2 real equations:

$$\begin{aligned}-f(x) &= \lambda \\ x + vt &= \lambda f'(x)\end{aligned} \tag{1.36}$$

After eliminating λ we get

$$x + vt + f(x)f'(x) = 0 \tag{1.37}$$

Theorem 1.2 *Given a* $z_b \in \Gamma$*, then for a* t *and* x *satisfying*

$$z_g = (x + (f(x) - r)i + vt)e^{-i\omega t} \tag{1.38}$$

this second condition is also satisfied:

$$x + vt + f(x)f'(x) = 0 \tag{1.39}$$

Proof The velocity of that contact point on the gear is (1.33)

$$\dot{\Xi}_r(z_g, t) = i\omega(x + vt + if(x)) \tag{1.40}$$

This means, after Δt the point z_g is found at

$$z = z_r + i\omega(x + vt + if(x))\Delta t \tag{1.41}$$

in rack coordinates. The change of the rack-x-coordinate (the real part) is thus

$$\Delta x = -\omega f(x)\Delta t \tag{1.42}$$

while the change of profile elevation for this Δx is

$$\Delta y = f'(x)\Delta x = -\omega f'(x)f(x)\Delta t \tag{1.43}$$

On the other hand, the change of elevation of the moving gear point is (the imaginary part):

$$\Delta y' = \omega(x + vt)\Delta t \tag{1.44}$$

The difference

$$\Delta y' - \Delta y = \omega(x + vt + f'(x)f(x)) \tag{1.45}$$

only vanishes if the expression 1.39 is also vanishing. But a nonzero value of the above expression immediately implies a collision of the gear point z_g shortly before or after time t. ∎

1.4 A curve that contains Γ

We now construct a smooth curve $\gamma(x)$ such that, due to the two above theorems, contains all points of Γ. First, for any x there is the only one

$$t(x) = -\frac{f(x)f'(x) + x}{v} \tag{1.46}$$

which satisfies 1.37. Based on this, we have a point $z(x)$ on the rack

$$z(x) = x + (f(x) - r)i \tag{1.47}$$

and finally the point on the gear.

$$\begin{aligned} \gamma(x) &= \Xi_g(z(x), t(x)) \\ &= (x + (f(x) - r)i + vt(x))e^{-i\omega t(x)} \\ &= (-f(x)f'(x) + (f(x) - r)i)e^{-i\omega t(x)} \end{aligned} \tag{1.48}$$