$$N \le \frac{R}{2f_{\max}} \tag{3-87}$$

The number of included harmonics must generally be smaller for high-frequency tones than for low-frequency tones. However, if $N$ is set according to the highest fundamental frequency that occurs over a considerable duration, such as an entire piece of music, low-frequency tones may be "harmonic-poor" in the sense that their spectra will be unnecessarily limited to low frequencies, making them sound dull or muted.

A possible solution to this problem is to use additive synthesis to add each sinusoidal component that is both desired and will "fit" underneath the Nyquist "ceiling." This solution is, however, computationally expensive (it would have replaced one of the oscillators in the previous example by eight oscillators, for example). Another solution is to define a few different versions of the required complex waveform, each containing a different number of included harmonics. This solution requires considerable memory space rather than computation time. We must take some care to normalize the amplitudes of the components in such a way that tones with more or fewer harmonics seem to have the same loudness (if all tones are normalized to the same peak value, for example, the relative loudness of the fundamental to the total sound of the waveform would be different).

A third alternative exists—one that is neither computationally expensive nor memory-intensive. To understand this alternative, we must take a brief detour into the realm of nonlinear synthesis techniques.

### 3.4.3 Closed-Form Summation Formulas

Closed-form summation formulas exist for many trigonometric functions (Jolley, 1961; Moorer, 1977; Stirling, 1749). Some of these mathematical relationships have a form that is useful for the efficient synthesis of band-limited complex waveforms. One of the simplest of these trigonometric relationships is given by the formula

$$\sum_{k=1}^{n} \sin k\theta = \sin\left[\frac{(n+1)\theta}{2}\right]\frac{\sin(n\theta/2)}{\sin(\theta/2)} \tag{3-88}$$

With $\theta$ set to $2\pi ft$ or $2\pi fn/R$, it is easy to see that equation (3-88) describes a waveform consisting of the sum of $n$ equal-strength harmonics of frequency $f$. The left-hand side of the equation describes the additive synthesis approaches discussed previously—these may be implemented directly. The right-hand side describes a new way of computing this waveform, however, which involves two multiplications and one division of the sine functions of three quantities easily computed from $\theta$ and $n$. The major attraction of the right-hand side of equation (3-88) as a synthesis technique is its efficiency, because the amount of computation involved is independent of the values for $\theta$ and $n$. In other words, the "closed form" of the summation given on the right-hand side of equation (3-88) may be used to compute a waveform consisting of the sum of 100 harmonics with no more work than a waveform consisting of the sum of only two harmonics.

As might be expected, the implementation of such formulas is not without its subtleties, because the division of sin $(n\theta/2)$ by sin $(\theta/2)$ can be somewhat tricky when $\theta$ is equal to zero or, in fact, any integer multiple of $\pi$. One way of defining

$$B(n, \theta) = \frac{\sin(n\theta/2)}{\sin(\theta/2)} \tag{3-89}$$

to resolve this difficulty (Moore, 1977) is to note that

$$\lim_{\theta \to m\pi} B(n, \theta) = \begin{cases} +n & n \text{ odd} \\ +n & n \text{ even and } m \text{ even} \\ -n & n \text{ even and } m \text{ odd} \end{cases} \tag{3-90}$$

where $m$ and $n$ are any integers. Whenever $\theta$ approaches an integer multiple of $\pi$, we may simply replace the value of $B(n, \theta)$ by $+n$ or $-n$, depending on the circumstances.

$B(n, \theta)$ is such an interesting and useful function that it deserves further comment. For $\theta = 2\pi f_1 t$, it has a spectrum consisting of exactly $n$ sinusoidal components—spaced at intervals of $f_1$ Hz—symmetrically arranged around 0 Hz. It also has a peak amplitude of precisely $\pm n$. Multiplying $B(n, \theta)$ by a sinusoidal waveform sin $\phi$, where $\phi = 2\pi f_2 t$, has the effect of shifting the spectrum of $B(n, \theta)$ (by convolution in the frequency domain) so that it is centered around $f_2$ Hz. We are therefore not restricted to using the sinusoidal multiplier of $B(n, \theta)$ specified in equation (3-88), which sets $f_2$ to $f_1(n + 1)/2$, thereby placing the lowest of the $n$ frequency components at $f_2$ Hz, but may choose any appropriate frequency for $f_2$. For example, equation (3-88) may be rewritten in the following form:

$$\sum_{k=0}^{n-1} \sin(\theta + k\beta) = \sin\left[\theta + \frac{(n-1)\beta}{2}\right]\frac{\sin(n\beta/2)}{\sin(\beta/2)} \tag{3-91}$$

This equation describes a waveform with exactly $n$ components that start at a frequency determined by $\theta$ and are spaced upward from that frequency at intervals determined by $\beta$. If $\theta$ is set equal to $\beta$, equation (3-91) is the same as equation (3-88).

A version of equation (3-91) is implemented in cmusic as the `blp` (for *band-limited pulse*) unit generator, which has the general statement

```
blp output[b] amp[bvpn] incr1[bvpn] incr2[bvpn] n[bvpn]
      sum1[dpv] sum2[dpv] ;
```

`blp` generates a band-limited pulse wave that contains `n` equal-amplitude sinusoidal components starting at frequency f1 (specified by `inc1`) and spaced upward by frequency f2 (specified by `inc2`). The only difference between the cmusic implementation of `blp` and equation (3-91) is that `blp` also automatically divides the output waveform by `n` so that the basic waveform is normalized to have a peak amplitude of unity (this peak amplitude is then further modified by multiplication by the `amp` input to `blp`).

Many other closed-form summation formulas exist, some of which are useful in the creation of band-limited waveforms such as those needed for excitation functions in the subtractive synthesis of musical tones. A few of the more useful ones are listed in Table 3-6.

**TABLE 3-6** Useful summation formulas for computer music. The closed-form expressions on the right may be used for efficient computation of waveforms with a specifiable number of frequency components, such as band-limited pulse waveforms. Note that by definition, the cosecant function is the inverse of the sine function (that is, csc θ = 1 / sin θ).

| Summation | Closed Form |
|---|---|
| $\sum_{k=1}^{n} \sin k\theta$ | $\sin\left[\frac{1}{2}(n+1)\theta\right] \sin\left(\frac{n\theta}{2}\right) \csc\left(\frac{\theta}{2}\right)$ |
| $\sum_{k=1}^{n} \cos k\theta$ | $\cos\left[\frac{1}{2}(n+1)\theta\right] \sin\left(\frac{n\theta}{2}\right) \csc\left(\frac{\theta}{2}\right)$ |
| $\sum_{k=0}^{n-1} \sin(\theta + k\beta)$ | $\sin\left[\theta + \frac{1}{2}(n-1)\beta\right] \sin\left(\frac{n\beta}{2}\right) \csc\left(\frac{\beta}{2}\right)$ |
| $\sum_{k=0}^{n-1} \cos(\theta + k\beta)$ | $\cos\left[\theta + \frac{1}{2}(n-1)\beta\right] \sin\left(\frac{n\beta}{2}\right) \csc\left(\frac{\beta}{2}\right)$ |
| $\sum_{k=0}^{n-1} a^k \cos k\theta$ | $\frac{(1 - a\cos\theta)(1 - a^n \cos n\theta) + a^{n+1}\sin\theta \sin n\theta}{1 - 2a\cos\theta + a^2}$ |
| $\sum_{k=0}^{n-1} a^k \sin(\theta + k\beta)$ | $\frac{\sin\theta - a\sin(\theta - \beta) - a^n \sin(\theta + n\beta) + a^{n+1}\sin[\theta + (n-1)\beta]}{1 - 2a\cos\beta + a^2}$ |

### 3.4.4 Hornlike Tones

We can combine band-limited excitation sources with time-varying filters to create many musically interesting and useful effects. For example, in his pioneering work on the timbral qualities of brass tones done at AT&T Bell Laboratories in the late 1960s, Jean-Claude Risset determined that a subjectively important characteristic of brass tones is that their spectrum bandwidth varies proportionally to their amplitude (Risset and Mathews, 1969). That is, as the amplitude of a brass instrument builds up from zero at the beginning of a note, the number and strength of the higher harmonics builds up also, perceived as an increase in the "brightness" of the tone during the attack. Similarly, as the tone dies away at the end of a note, the number and strength of the upper harmonics decreases also, making the tail end of the note nearly a sinusoid.

We can simulate this behavior with the `blp` and `nres` unit generators in cmusic by placing the center frequency of the filter on the fundamental frequency of a band-limited pulse waveform and causing the filter bandwidth to track the amplitude envelope of the note (this general type of brasslike tone synthesis has been used extensively in the analog synthesis domain). By appropriate choice of amplitude envelopes and scale factors that determine the bandwidths associated with minimum and maximum amplitude values, we can create a simple instrument capable of a wide variety of brasslike tone effects.