Mecánica de Fluidos

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Apellidos, Nombre:

Grupo:

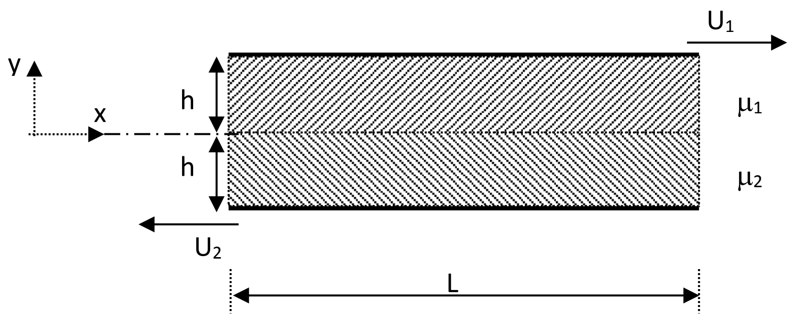
Problema 2

Dos fluidos de densidades diferentes ρ_1 y ρ_2 ($\mu_1 = 2\mu_2$) fluyen entre dos placas paralelas planas, horizontales, e infinitamente anchas en régimen laminar y de forma estacionaria. Ambas placas se mueven con velocidades constantes U_1 y U_2. Si se asume la misma variación de presión (dp/dx) para ambos fluidos, entonces:

a) Calcular el perfil de velocidad.

b) Determinar dp/dx para que el caudal sea nulo.

c) Si $U_1 = U_2 = U$, calcular los esfuerzos cortantes en ambas paredes, en función de dp/dx.



Solución

a) Se consideran las siguientes hipótesis:

1. Régimen estacionario: $\frac{\partial}{\partial t} = 0$
2. Flujo bidimensional: $u = u(x, y);\ v = w = 0;\ \frac{\partial}{\partial z} = 0$
3. Placas muy largas y anchas, flujo completamente desarrollado
4. $\left.\frac{\partial p}{\partial x}\right|_{\rho_1} = \left.\frac{\partial p}{\partial x}\right|_{\rho_2} = \frac{\partial p}{\partial x}$

Por ser dos placas paralelas horizontales, la única componente de la velocidad debe ser u.

$$\vec{v} = u\,\vec{\imath}$$

Se parte de la ecuación de la conservación de la masa para fluido incompresible.

$$\nabla \cdot \vec{v} = 0 \rightarrow \frac{\partial u}{\partial x} + \cancel{\frac{\partial v}{\partial y}} + \cancel{\frac{\partial w}{\partial z}} = 0 \rightarrow \frac{\partial u}{\partial x} = 0$$

Además, se asume por la hipótesis 3, que no existe variación del perfil de velocidad en dirección x. Así:

$$u = u(y)$$

De las ecuaciones de Navier-Stokes:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w\right)$$

En el eje z, al no existir w, se llega a:

$$-\frac{\partial p}{\partial z} = 0 \rightarrow \text{la presión no depende de z}$$

En el eje y, al no existir v, se llega a:

$$-\rho g - \frac{\partial p}{\partial y} = 0 \rightarrow\ p(x, y, z) = -\rho g y + f(x) \quad (1)$$

En el eje x:

$$\underset{1}{\cancel{\rho g_x}} - \frac{\partial p}{\partial x} + \mu\left(\underset{2}{\cancel{\frac{\partial^2 u}{\partial x^2}}} + \frac{\partial^2 u}{\partial y^2} + \underset{3}{\cancel{\frac{\partial^2 u}{\partial z^2}}}\right) = \rho\left(\underset{4}{\cancel{\frac{\partial u}{\partial t}}} + \underset{2}{\cancel{\frac{\partial u}{\partial x}}}u + \underset{5}{\frac{\partial u}{\partial y}\cancel{v}} + \underset{6}{\frac{\partial u}{\partial z}\cancel{w}}\right)$$

1. No hay g en esta dirección.
2. La componente $u(y)$, no depende de x.
3. La componente $u(y)$, no depende de z.
4. No depende del tiempo.
5. No hay v.
6. No hay w.

Por lo que queda:

$$-\frac{\partial p}{\partial x}+\left(\frac{\partial^2 u}{\partial y^2}\right)=0\rightarrow\frac{1}{\mu}\frac{\partial p}{\partial x}=\left(\frac{\partial^2 u}{\partial y^2}\right)\quad (2)$$

La componente u solamente depende de y, por lo que $\partial p/\partial x$ sólo podría depender de y según la ecuación anterior (2). Por otro lado, según (1) $\partial p/\partial x=\partial f(x)/\partial x$, y por tanto solo podría depender de x. De esta forma, para que la función $\partial p/\partial x$ cumpla las ecuaciones (1) y (2), $\partial p/\partial x$ ha de ser igual a una constante K. Por tanto:

$$\frac{1}{\mu}\frac{\partial p}{\partial x}=\left(\frac{\partial^2 u}{\partial y^2}\right)=K$$

Resolviendo la EDO en derivadas parciales:

$$\int\partial^2 u=\int K\partial y^2\Rightarrow\int\partial u=\int(Ky+A)\partial y\Rightarrow u=K\frac{y^2}{2}+Ay+B=\frac{1}{\mu}\frac{\partial p}{\partial x}\frac{y^2}{2}+Ay+B$$

Según las condiciones de contorno:

Fluido 1: $y=h\rightarrow u=U_1$

$$U_1=\frac{1}{\mu_1}\frac{\partial p}{\partial x}\frac{h^2}{2}+A_1h+B_1\quad (1)$$

Fluido 2: $y=-h\rightarrow u=-U_2$

$$-U_2=\frac{1}{\mu_2}\frac{\partial p}{\partial x}\frac{h^2}{2}-A_2h+B_2\quad (2)$$

Para $y=0\rightarrow\ u_1=u_2\rightarrow B_1=B_2$

Para $y=0\rightarrow\ \tau_1=\tau_2$

$$\tau_1=\tau_2\rightarrow\mu_1\frac{\partial u_1}{\partial y}=\mu_2\frac{\partial u_2}{\partial y}$$

Para la igualdad en esfuerzos cortantes, de la resolución de la EDO:

$$\tau=\mu\frac{\partial u}{\partial y}=\mu\left(\frac{1}{\mu}\frac{\partial p}{\partial x}h+A\right)$$

Así,

$$\mu_1\left(\frac{1}{\mu_1}\frac{\partial p}{\partial x}0+A_1\right)=\mu_2\left(\frac{1}{\mu_2}\frac{\partial p}{\partial x}0+A_2\right)\Rightarrow\mu_1A_1=\mu_2A_2$$

Sustituyendo el dato de $\mu_1=2\mu_2$

$$2\mu_2A_1=\mu_2A_2\Rightarrow 2A_1=A_2$$

Con todo esto, sustituyendo en las expresiones anteriores (1) y (2):

$$U_1=\frac{1}{\mu_1}\frac{\partial p}{\partial x}\frac{h^2}{2}+\frac{A_2}{2}h+B_2$$

$$-U_2 = \frac{1}{\mu_2}\frac{\partial p}{\partial x}\frac{h^2}{2} - A_2 h + B_2$$

Y restando ambas ecuaciones:

$$U_1 + U_2 = \frac{\partial p}{\partial x}\frac{h^2}{2}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) + \frac{A_2}{2}h + A_2 h = \frac{\partial p}{\partial x}\frac{h^2}{2}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right) + \frac{3A_2}{2}h$$

$$A_2 = \left[U_1 + U_2 - \frac{\partial p}{\partial x}\frac{h^2}{2}\left(\frac{1}{2\mu_2} - \frac{1}{\mu_2}\right)\right]\frac{2}{3h} = \frac{2(U_1 + U_2)}{3h} - \frac{\partial p}{\partial x}\frac{h^2}{2}\left(-\frac{1}{2\mu_2}\right)\frac{2}{3h} =$$

$$= \frac{2(U_1 + U_2)}{3h} + \frac{\partial p}{\partial x}h\frac{1}{6\mu_2}$$

$$A_1 = \frac{A_2}{2} = \frac{U_1 + U_2}{3h} + \frac{\partial p}{\partial x}h\frac{1}{12\mu_2}$$

Sustituyendo el valor anterior en (1):

$$U_1 = \frac{1}{\mu_1}\frac{\partial p}{\partial x}\frac{h^2}{2} + \left(\frac{U_1 + U_2}{3h} + \frac{\partial p}{\partial x}h\frac{1}{12\mu_2}\right)h + B_1$$

se obtiene,

$$B_1 = U_1 - \frac{U_1 + U_2}{3} - \frac{\partial p}{\partial x}h^2\frac{1}{3\mu_2} = \frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x}h^2\frac{1}{3\mu_2} = B_2$$

Finalmente, los perfiles de velocidad de cada fluido son:

$$u_1 = \frac{1}{\mu_1}\frac{\partial p}{\partial x}\frac{y^2}{2} + \left(\frac{U_1 + U_2}{3h} + \frac{\partial p}{\partial x}h\frac{1}{12\mu_2}\right)y + \frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x}h^2\frac{1}{3\mu_2}$$

$$u_1 = \frac{1}{4\mu_2}\frac{\partial p}{\partial x}y^2 + \left(\frac{U_1 + U_2}{3h} + \frac{\partial p}{\partial x}h\frac{1}{12\mu_2}\right)y + \frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x}h^2\frac{1}{3\mu_2}$$

$$u_2 = \frac{1}{\mu_2}\frac{\partial p}{\partial x}\frac{y^2}{2} + \left(\frac{2(U_1 + U_2)}{3h} + \frac{\partial p}{\partial x}h\frac{1}{6\mu_2}\right)y + \frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x}h^2\frac{1}{3\mu_2}$$

b) Q = 0

$$Q = \int \vec{v} \circ \vec{n} dA = \int_{-h}^{h} u dy = \int_{-h}^{0} u_2 dy + \int_{0}^{h} u_1 dy = 0$$

$$\int_{-h}^{0} u_2 dy = \int_{-h}^{0}\left[\frac{1}{\mu_2}\frac{\partial p}{\partial x}\frac{y^2}{2} + \left(\frac{2(U_1 + U_2)}{3h} + \frac{\partial p}{\partial x}h\frac{1}{6\mu_2}\right)y + \frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x}h^2\frac{1}{3\mu_2}\right]dy =$$

$$= \frac{1}{2\mu_2}\frac{\partial p}{\partial x}\frac{y^3}{3}\bigg|_{-h}^{0} + \left(\frac{2(U_1 + U_2)}{3h} + \frac{\partial p}{\partial x}h\frac{1}{6\mu_2}\right)\frac{y^2}{2}\bigg|_{-h}^{0} + \left(\frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x}h^2\frac{1}{3\mu_2}\right)y\bigg|_{-h}^{0} =$$

$$= \frac{h^3}{6\mu_2}\frac{\partial p}{\partial x} - \frac{2(U_1 + U_2)}{3h}\frac{h^2}{2} - \frac{\partial p}{\partial x}\frac{h}{6\mu_2}\frac{h^2}{2} + \left(\frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x}h^2\frac{1}{3\mu_2}\right)h =$$

$$= \left(\frac{2U_1 - U_2}{3}\right)h - \left(\frac{U_1 + U_2}{3}\right)h + \frac{h^3}{6\mu_2}\frac{\partial p}{\partial x} - \frac{h^3}{12\mu_2}\frac{\partial p}{\partial x} - \frac{h^3}{3\mu_2}\frac{\partial p}{\partial x} =$$

$$\frac{U_1 - 2U_2}{3} h + \frac{2 - 1 - 4}{12\mu_2} h^3 \frac{\partial p}{\partial x} = \frac{U_1 - 2U_2}{3} h - \frac{h^3}{4\mu_2} \frac{\partial p}{\partial x}$$

$$\int_0^h u_1 dy = \int_0^h \left[\frac{1}{4\mu_2} \frac{\partial p}{\partial x} y^2 + \left(\frac{U_1 + U_2}{3h} + \frac{\partial p}{\partial x} h \frac{1}{12\mu_2} \right) y + \frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x} h^2 \frac{1}{3\mu_2} \right] dy =$$

$$= \frac{1}{4\mu_2} \frac{\partial p}{\partial x} \frac{y^3}{3} \bigg|_0^h + \left(\frac{(U_1 + U_2)}{3h} + \frac{\partial p}{\partial x} h \frac{1}{12\mu_2} \right) \frac{y^2}{2} \bigg|_0^h + \left(\frac{2U_1}{3} - \frac{U_2}{3} - \frac{\partial p}{\partial x} h^2 \frac{1}{3\mu_2} \right) y \bigg|_0^h =$$

$$= \frac{h^3}{12\mu_2} \frac{\partial p}{\partial x} + \frac{U_1 + U_2}{3h} \frac{h^2}{2} + \frac{\partial p}{\partial x} \frac{h}{12\mu_2} \frac{h^2}{2} + \frac{2U_1 - U_2}{3} h - \frac{h^3}{3\mu_2} \frac{\partial p}{\partial x} =$$

$$= \frac{U_1 + U_2 + 4U_1 - 2U_2}{6} h + \frac{\partial p}{\partial x} h^3 \left(\frac{1}{12\mu_2} + \frac{1}{24\mu_2} - \frac{1}{3\mu_2} \right) =$$

$$= \frac{5U_1 - U_2}{6} h + \frac{\partial p}{\partial x} h^3 \frac{2 + 1 - 8}{24\mu_2} = \frac{5U_1 - U_2}{6} h - \frac{\partial p}{\partial x} h^3 \frac{5}{24\mu_2}$$

$$0 = \frac{U_1 - 2U_2}{3} h - \frac{h^3}{4\mu_2} \frac{\partial p}{\partial x} + \frac{5U_1 - U_2}{6} h - \frac{\partial p}{\partial x} h^3 \frac{5}{24\mu_2}$$

$$\frac{U_1 - 2U_2}{3} h + \frac{5U_1 - U_2}{6} h = \frac{\partial p}{\partial x} \left(\frac{5h^3}{24\mu_2} + \frac{h^3}{4\mu_2} \right)$$

$$\frac{2U_1 - 4U_2 + 5U_1 - U_2}{6} h = \frac{\partial p}{\partial x} \frac{5h^3 + 6h^3}{24\mu_2} = \frac{\partial p}{\partial x} \frac{11h^3}{24\mu_2}$$

$$\frac{\partial p}{\partial x} = \frac{7U_1 - 5U_2}{6} \frac{24\mu_2}{11h^2} = \frac{\mu_2}{11h^2} (28U_1 - 20U_2)$$

c) Para $y = h$:

$$\tau_1 = \mu_1 \frac{\partial u_1}{\partial y} \bigg|_{y=h} = \mu_1 \left(\frac{2}{\mu_1} \frac{\partial p}{\partial x} \frac{h}{2} + \frac{U_1 + U_2}{3h} + \frac{\partial p}{\partial x} \frac{h}{6\mu_1} \right) = \mu_1 \left(\frac{U_1 + U_2}{3h} \right) + \frac{7h}{6} \frac{\partial p}{\partial x}$$

Si $U_1 = U_2 = U$

$$\tau_1 = \frac{2\mu_1 U}{3h} + \frac{7h}{6} \frac{\partial p}{\partial x} = \frac{4\mu_2 U}{3h} + \frac{7h}{6} \frac{\partial p}{\partial x}$$

Para $y = -h$

$$\tau_2 = \mu_2 \frac{\partial u_2}{\partial y} \bigg|_{y=-h} = \mu_2 \left(\frac{2}{\mu_2} \frac{\partial p}{\partial x} \frac{(-h)}{2} + \frac{2(U_1 + U_2)}{3h} + \frac{\partial p}{\partial x} \frac{h}{6\mu_2} \right) = \mu_2 2 \left(\frac{U_1 + U_2}{3h} \right) - \frac{5h}{6} \frac{\partial p}{\partial x}$$

Si $U_1 = U_2 = U$

$$\tau_2 = \frac{4\mu_2 U}{3h} - \frac{5h}{6} \frac{\partial p}{\partial x}$$