Chapter 1: Markov processes (MP)

- State: $S_t \in \mathcal{S}$ $\quad t = 0, 1, 2, \ldots$

  state space

- Markov evolution:

  past

$$P(S_{t+1} = s' \mid S_t = s, S_{t-1}, S_{t-2}, \ldots, S_0)$$

  future $\quad$ present

$$= P(S_{t+1} = s' \mid S_t = s)$$

  future $\quad$ present

  - Future independent of past given present
  - Markov chain: $S_0 \to S_1 \to S_2 \to \cdots$

- Transition probability:

$$p(s' \mid s) = P(S_{t+1} = s' \mid S_t = s)$$

  - Assumed homogeneous (stationary, time-independent)
  - Normalization: $\sum_{s'} p(s' \mid s) = 1 \quad \forall s$

- Transition matrix: $P_{ij} = p(j \mid i) = P(i \to j)$

  row $\quad$ column

  - $|\mathcal{S}| \times |\mathcal{S}|$ matrix
  - $\sum_j P_{ij} = 1 \quad \forall i \quad$ (row sums $= 1$)
  - $0 \le P_{ij} \le 1 \quad$ (stochastic matrix)

- Probability distribution: $p_t(s) = P(S_t = s)$

- Probability vector: p_t, $\quad p_{t,i} = p_t(i)$

  - $|\mathcal{S}|$-dim vector
  - Row vector

· Evolution equation:

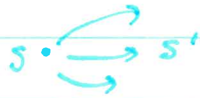
$$p_{t+1}(s') = \sum_s p(s'|s)\, p_t(s)$$

· Matrix notation: $p_{t+1} = p_t P$ right product

$$(-\ p_{t+1}\ -) = (-\ p_t\ -)\begin{pmatrix} & & \\ & P & \\ & & \end{pmatrix}$$

$$(p_{t+1})_j = \sum_i (p_t)_i\, P_{ij}$$

· Expectations:



$$E[f(S_{t+1}) \,|\, S_t = s] = \sum_{s'} f(s')\, p(s'|s)$$

· Matrix notation:

$$F_i = E[f(S_{t+1}) \,|\, S_t = i] = \sum_j f(j)\, p(j|i)$$

$$\begin{pmatrix} \vdots \\ F \\ \vdots \end{pmatrix} = \begin{pmatrix} & & \\ & P & \\ & & \end{pmatrix}\begin{pmatrix} \vdots \\ f \\ \vdots \end{pmatrix} \qquad = \sum_j P_{ij}\, f(j)$$

$$\Rightarrow \quad F = Pf$$ left product

· Marginalization: $p(x) = \sum_y p(x,y)$

$$p(x|z) = \sum_y p(x,y|z)$$

· Chain rule: $p(x,y) = p(x|y)\,p(y) = p(y|x)\,p(x)$

$$p(x,y|z) = p(x|y,z)\,p(y|z)$$

· Deterministic transition: $\underset{s \to s'}{p(s'|s)} = \begin{cases} 1 & \text{for same } s^* \\ 0 & \text{otherwise} \end{cases} = \delta_{s', s^*(s)}$