Stellenbosch UNIVERSITY IYUNIVESITHI UNIVERSITEIT

Department of Mathematical Sciences
Division of Applied Mathematics

AM783 | Applied Markov Processes

Problems for Chapter 2: Markov chains

Theoretical

Q1. (Birth-death process) Consider the simple population model seen in class for which $P(i \to i+1) = \alpha$ (birth) and $P(i \to i-1) = \beta$ (death), where $i \in \{0, 1, 2, \ldots\}$ and $\alpha + \beta \leq 1$.

(a) Write down the matrix Π of transition probabilities.

(b) Write down the propagation equation for $\bar{p}_n$ (also called the Chapman–Kolmogorov equation) in components. [Hint: There are only two relevant equations.]

(c) Find the stationary distribution $\bar{p}^*$, assuming $\beta > \alpha > 0$. [Hint: Truncate the system to a finite number of states (say 3) and find its stationary distribution by diagonalisation (on Mathematica). Generalise the solution for an arbitrary number of states.]

(d) Calculate the expected stationary population.

(e) Why do we need $\alpha < \beta$. What happens if $\alpha > \beta$?

Q2. A discrete Markov chain with $|\mathcal{X}|$ states is called **bi-stochastic** or **doubly stochastic** if $\sum_i \Pi_{ij} = 1$ in addition to the usual normalisation property $\sum_j \Pi_{ij} = 1$. Show in this case that the stationary distribution is the uniform distribution $p_i^* = 1/|\mathcal{X}|$.

Q3. (Random walk on graphs) Show that the stationary distribution of the unbiased random walk on an undirected, connected graph is, as seen in class,

$$p_i^* = \frac{k_i}{2M}, \tag{1}$$

where k_i is the degree of the node i, obtained from the adjacency matrix A_{ij} by $k_i = \sum_j A_{ij}$, and $M = \frac{1}{2}\sum_i k_i$ is the total number of edges in the graph.

Q4. A Markov chain with transition matrix Π_{ij} is said to be **reversible** with respect to a distribution p_i if it satisfies the following condition:

$$p_i \Pi_{ij} = p_j \Pi_{ji}, \qquad i, j \in \mathcal{X}, \tag{2}$$

known as the **detailed balance condition**.

(a) Show that p_i is a stationary distribution of the Markov chain.

(b) Show that the transformed matrix

$$\hat{\Pi}_{ij} = (p_i)^{1/2} \Pi_{ij} (p_j)^{-1/2} \tag{3}$$

is symmetric.

(c) What can be said about the eigenvalues of Π?

Q5. (Markov chain Monte Carlo) Let $\pi(x)$ be a probability distribution with $\pi(x) > 0$ for all x. Show that the Metropolis algorithm, based on the following transition probability:

$$P(x \to x') = \min\left\{1, \frac{\pi(x')}{\pi(x)}\right\} \tag{4}$$

for going from x to x', defines a reversible Markov chain. What is its stationary distribution?

Numerical

Q6. (Stationary distribution) We have seen in class that the stationary distribution p^* of an ergodic Markov chain can be computed using three methods:

1. Solve $p^* \Pi = p^*$, that is, find the left eigenvector of Π of eigenvalue 1;
2. Simulate *many* independent Markov chains in parallel or one after the other and compute the histogram of their final state X_n for n large enough;

3. Simulate a *single* Markov chain $\{X_i\}_{i=1}^n$ with n time steps and compute the time-averaged occupation

$$\hat{p}_n(x) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i,x} = \frac{\text{\# states with value } x}{n}. \tag{5}$$

Show for the two-state Markov chain seen in class (consider the symmetric or non-symmetric one) that the last two numerical methods agree with the analytical result of Method 1. Which method do you see as more effective and why?

Q7. **(Random walk on graphs)** Choose a large enough connected graph, say with 10 or more states, and write a program that simulates a trajectory of the unbiased random walk on that graph. The program should use the adjacency matrix of the graph as the internal representation of the graph and should output a trajectory $\{x_0, x_1, \ldots, x_n\}$ of length n starting at x_0. Use a long enough trajectory to estimate the stationary distribution of the random walk (use Method 3 of the previous question) and show that it agrees with the stationary distribution in Q3.

Q8. **(Markov chain Monte Carlo)** Let us revisit the estimation of π, as seen in CW1. Instead of dropping independent random points in the square $[-1, 1] \times [-1, 1]$, choose one point in the square – any point, e.g., $P_0 = (0, 0)$ – and iterate the following steps to construct a Markov chain $P_1 \to P_2 \to \cdots \to P_L$ of L points:

Step 1: Choose a random displacement $\delta P = (\delta P_x, \delta P_y)$ according to *any* symmetric distribution;

Step 2: Set $P_{i+1} = P_i + \delta P$ if P_{i+1} stays in the square (accept move); otherwise, set $P_{i+1} = P_i$ (reject move).

For this algorithm,

(a) Show that the set of points $\{P_i\}_{i=1}^L$ is uniform, no matter what distribution is used for generating δP (!). [Hint: This is a Metropolis algorithm.]

(b) Construct from the Markov chain an estimator of π and show that it converges to the correct value. [Hint: Use a “good” distribution for δP.]

(c) Can we construct error bars for this estimator the way we have seen in class? Explain your answer.