Solutions for Chapter 2 problems

Theoretical

Q1. (Birth-death process) Let $P(i \to i+1) = \alpha$ and $P(i \to i-1) = \beta$, where $i \in \{0, 1, 2, \ldots\}$ and $\alpha + \beta \leq 1$.

(a) With the natural ordering of states, the transition matrix is tri-diagonal:

$$\Pi = \begin{pmatrix} 1-\alpha & \alpha & 0 & 0 & \cdots \\ \beta & 1-\alpha-\beta & \alpha & 0 & \\ 0 & \beta & 1-\alpha-\beta & \alpha & \\ \vdots & & & & \ddots \end{pmatrix} \tag{1}$$

(b) For $i > 0$:

$$p_{n+1}(i) = \alpha p_n(i-1) + (1-\alpha-\beta)p_n(i) + \beta p_n(i+1) \tag{2}$$

For $i = 0$:

$$p_{n+1}(0) = (1-\alpha)p_n(0) + \beta p_n(1). \tag{3}$$

(c) The solution truncated to 3 states that you are likely to get from Mathematica, without normalisation, is

$$p^* = \begin{bmatrix} \left(\frac{\beta}{\alpha}\right)^2 & \left(\frac{\beta}{\alpha}\right) & 1 \end{bmatrix} \tag{4}$$

This is a good solution, but not in the right form because considering more states in the truncation will only increase the power of the first term. It is better to divide this solution by that first term to get instead

$$p^* = \begin{bmatrix} 1 & \left(\frac{\alpha}{\beta}\right) & \left(\frac{\alpha}{\beta}\right)^2 \end{bmatrix}. \tag{5}$$

The full solution is then guessed to be

$$p^* = \begin{bmatrix} 1 & \left(\frac{\alpha}{\beta}\right) & \left(\frac{\alpha}{\beta}\right)^2 & \left(\frac{\alpha}{\beta}\right)^3 & \cdots \end{bmatrix}, \tag{6}$$

giving

$$p^*(i) = \left(1 - \frac{\alpha}{\beta}\right)\left(\frac{\alpha}{\beta}\right)^i \tag{7}$$

after normalisation. It can be checked that this distribution satisfies the stationary Chapman–Kolmogorov equation obtained by setting $p_{n+1} = p_n = p^*$ in (2)-(3).

(d) The expectation is

$$\sum_{i=0}^{\infty} i\ p^*(i) = \frac{\alpha}{\beta - \alpha}. \tag{8}$$

(e) If $\alpha > \beta$, there are more births on average than deaths, so the population explodes in the long-time limit and there is no stationary distribution. For $\beta > \alpha$ we have more deaths than births, so we could think that the population will go extinct. Note however that the 0 population has probability α to get to 1 again, so the population never goes extinct for good. This is like having spontaneous births or allowing immigration only when no one is left.

Q2. (Bi-stochastic Markov chain) The result can be proved directly in components, so I'll give a proof in matrix notation instead. Note first that the normalization of Π can be expressed as $\Pi \mathbf{1}^\mathsf{T} = \mathbf{1}^\mathsf{T}$, where $\mathbf{1} = (1, 1, \ldots, 1)$ is the row vector of 1's. Hence, $\mathbf{1}^\mathsf{T}$ is a column (right) eigenvector of Π with eigenvalue 1. The bi-stochastic property, on the other hand, means that $\mathbf{1}\Pi = \mathbf{1}$, so $\mathbf{1}$ is a row (left) eigenvector of Π. Since eigenvectors can be multiplied by any constant, we therefore have $p^*\Pi = p^*$ for

$$p^* = \frac{\mathbf{1}}{\sum_{i=1}^{|\mathcal{X}|}(\mathbf{1})_i} = \frac{\mathbf{1}}{|\mathcal{X}|}, \tag{9}$$

which implies that p^* is a stationary distribution of Π.

Q3. **(Random walk on graphs)** By direct substitution in the stationary equation:

$$\sum_i p_i^* \Pi_{ij} = \sum_i a k_i \frac{A_{ij}}{k_i} = a \sum_i A_{ij} = a k_j. \tag{10}$$

The normalisation constant a is found by $a \sum_{i\in V} k_i = 1$.

Q4. (a) By direct substitution and using the detailed balance condition:

$$\sum_i p_i \Pi_{ij} = \sum_i p_j \Pi_{ji} = p_j \sum_i \Pi_{ji} = p_j, \tag{11}$$

since $\sum_j \Pi_{ij} = 1$ for all i.

(b) Directly using the detailed balance condition:

$$(\hat{\Pi}^{\mathsf{T}})_{ij} = \hat{\Pi}_{ji} = (p_j)^{1/2}\Pi_{ji}(p_i)^{-1/2} = (p_j)^{1/2}\left(\frac{p_i \Pi_{ij}}{p_j}\right)(p_i)^{-1/2} = p_i^{1/2}\Pi_{ij}(p_j)^{-1/2}. \tag{12}$$

(c) The eigenvalues of Π are real, since Π is conjugated to a symmetric matrix.

Q5. **(Markov chain Monte Carlo)** There are three possible cases depending on the value of $\pi(x')/\pi(x)$:

Case 1: $\pi(x')/\pi(x) > 1$. Then $P(x \to x') = 1$, so

$$\pi(x)P(x \to x') = \pi(x), \tag{13}$$

while $P(x' \to x) = \pi(x)/\pi(x') < 1$, so

$$\pi(x')P(x' \to x) = \pi(x')\frac{\pi(x)}{\pi(x')} = \pi(x). \tag{14}$$

Hence we have detailed balance with respect to π.

Case 2: $\pi(x')/\pi(x) < 1$. Then $P(x \to x') = \pi(x')/\pi(x)$, so

$$\pi(x)P(x \to x') = \pi(x') \tag{15}$$

while $P(x' \to x) = 1$, so

$$\pi(x')P(x' \to x) = \pi(x'), \tag{16}$$

implying again detailed balance.

Case 3: $\pi(x')/\pi(x) = 1$. Then $P(x \to x') = P(x' \to x) = 1$, which is consistent with detailed balance being $\pi(x) = \pi(x')$.

Or, directly, as shown in some submissions I received:

$$\pi(x)P(x \to x') = \min\{\pi(x), \pi(x')\} = \pi(x')\min\left\{1, \frac{\pi(x)}{\pi(x')}\right\} = \pi(x')P(x' \to x). \tag{17}$$

From these calculations, it is clear that π is the stationary distribution.

Numerical

See the Jupyter notebook.