Sequences

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1 Introduction

There is not much theory one can cover on the topic of sequences. Your intuition and some rudimentary knowledge of inequalities, like the AMGM inequality or the Cauchy-Schwartz inequality will often prove most helpful.

Advice : General advice

- Define an auxiliary sequence - a related sequence making the condition nicer.
- Conjecture about the properties of the sequence.
- Observe extremal elements (if they exist).

Lemma 1.1

A monotonic, periodic sequence is constant.

Another fact that you should keep in mind is that integer sequences are much nicer to work with than real or rational sequences. One of the reasons for this is that a non-constant integer sequence must have "large" differences between consecutive terms - namely the difference is at least 1. This fact clearly not true for rational or real sequences.

Problem 1.2

Find all sets of real numbers $\{a_1, \ldots, a_{2015}\}$ such that

$$2 \cdot \sqrt{a_n - n + 1} \geq a_{n+1} - n + 1$$

for all $1 \leq n \leq 2014$ and additionally $2 \cdot \sqrt{a_{2015} - 2014} \geq a_1 + 1$.

Proof. Use the equality condition of the AMGM inequality and observe the cyclic structure of the problem.

Problem 1.3

Define the sequence $\{a_i\}_{i=0}^{\infty}$ by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{2} \cdot \left(3a_n + \sqrt{5a_n^2 + 20}\right).$$

Prove all terms of the sequence are integers.

Proof outline. The condition is equivalent to $a_{n+1}^2 - 3a_n a_{n+1} + a_n^2 - 5 = 0$, which means a_{n+1} is a zero of $X^2 - 3a_n X + a_n^2 - 5$. The two conditions are symmetric in a_{n+1} and a_{n-1} □

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Problem 1.4: IMO 2014 P1

Let $a_0, a_1, \ldots$ be a strictly increasing sequence of positive integers. Prove that there exists a unique $n \in \mathbb{N}$, such that

$$a_n < \frac{a_0 + \ldots + a_n}{n} \leq a_{n+1}$$

Proof. Transform the inequality into the equivalent form:

$$n \cdot a_n < \sum_{i=0}^{n} a_i \leq n \cdot a_{n+1}.$$

First we may try and subtract the term $n \cdot a_n$ from both sides, but unfortunately we do not get anything nice. However, if we subtract $a_1 + \ldots + a_n$ from the middle then we get something nice, namely a sequence

$$b_n = \sum_{i=1}^{n-1} (a_n - a_i),$$

and the condition transformed into $b_n < a_0 \leq b_{n+1}$, which is clearly fulfilled as $\{a_n\}$ are integers. This problem is proof of the fact that choosing a good auxiliary sequence can make the problem much easier. □

Problem 1.5: ISL 2019 A2

Let $u_1, \ldots, u_{2019}$ be real numbers such that

$$\sum_{i=1}^{2019} u_i = 0 \quad \text{and} \quad \sum_{i=1}^{2019} u_i^2 = 1.$$

Prove that

$$\min\{u_i\} \cdot \max\{u_i\} \leq \frac{-1}{2019}.$$

Proof. We split the numbers into non-negatives and negatives, $\{a_i\}_{i=1}^{k}$ and $\{b_i\}_{i=1}^{2019-k}$. The conditions transform as follows:

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{2019-k} b_i \quad \text{and} \quad \sum_{i=1}^{k} a_i^2 + \sum_{i=1}^{2019-k} b_i^2 = 1,$$

and we wish to prove that $\max\{a_i\} = a$ and $\max\{b_i\} = b$ satisfy $ab \geq \frac{1}{2019}$.

Using the Cauchy-Schwartz inequality doesn't seem possible, as the two sets do not have the same number of elements. Note that the constant 2019 in the denominator is equal to the number of terms, so we may try:

$$1 = \sum_{i=1}^{k} a_i^2 + \sum_{i=1}^{2019-k} b_i^2 \leq a \cdot \sum_{i=1}^{k} a_i + b \cdot \sum_{i=1}^{2019-k} b_i =$$

$$b \cdot \sum_{i=1}^{k} a_i + a \cdot \sum_{i=1}^{2019-k} b_i \leq k \cdot (ab) + (2019 - k) \cdot ab = 2019 \cdot (ab),$$

which we wanted to prove. The decision to split the set into non-negatives and negatives is quite natural, as inequalities are usually nicer to work with when all terms are non-negative. The last step of the solution is motivated by the presence of the constant 2019 and the fact that we need a mixed term - something of the form $a \cdot b$; whereas the condition in the problem statement only tells us information about what happens when we multiply two terms from family $\{a_i\}_{i=1}^{k}$ or from family $\{b_i\}_{i=1}^{2019-k}$. □

Problem 1.6: ISL 2020 A1

Let $\{a_n\}$ be a sequence of positive real numbers, such that

$$(a_{n+1})^2 + a_n a_{n+2} \leq a_n + a_{n+2}.$$

Show that $a_{2025} \leq 1$.

Proof. The problem statement seems true, as if the sequence grows larger than 1, then the product of two elements of the sequence of the left side of the inequality should be larger than the sum of two such numbers. As there are no initial terms of the sequence and since no actual recursive relation is given, there is very little experimentation we can do. We might try to either transform the inequality into something nicer to work with, or we could try defining an auxiliary sequence. By adding the inequalities in the cases $n = k + 1$ and $n = k$ we can see:

$$(a_{k+1})^2 + a_k a_{k+2} + (a_k)^2 + a_{k-1} a_{k+1} \leq a_k + a_{k+2} + a_{k-1} + a_{k+1}.$$

Motivated by the desired inequality $a_k \leq 1$ we factorize the expression as follows.

$$(a_{k+1} - 1)(a_{k+1} + a_{k-1}) + (a_k - 1)(a_k + a_{k+2}) \leq 0,$$

from which it follows that for every $k > 1$, one of a_{k+1} and a_k must be smaller or equal to 1.

Since we want to prove that $a_{2025} \leq 1$, we define the auxiliary sequence $b_k = 1 - a_k$ and factorize the expression in the following way:

$$(1 - b_{k+1})^2 + (1 - b_k)(1 - b_{k+2}) \leq 2 - b_k - b_{k+2}$$
$$-2b_{k+1} + (b_{k+1})^2 - b_k - b_{k+2} + b_k b_{k+2} \leq -b_k - b_{k+2}$$
$$b_{k+1}^2 + b_k b_{k+2} \leq 2b_{k+1}$$

Since if $b_k < 0$, then $b_{k+1} \geq 0$ and $b_{k-1} \geq 0$, we would reach contradiction by the equation above, hence $b_k \geq 0$ $\forall k > 1$, meaning $a_{2025} \leq 1$. □

Problem 1.7: IMO 2023 P4

Let $\{x_i\}_{i=1}^{2023}$ be pairwise distinct positive real numbers such that

$$a_i = \sqrt{(x_1 + \ldots + x_i)(\frac{1}{x_1} + \ldots + \frac{1}{x_i})},$$

are integers for all $1 \leq i \leq 2023$. Prove that $a_{2023} \geq 3034$.

Proof. Some preliminary observations are that $3034 = \frac{3}{2} \cdot 2024 - \frac{1}{2}$, so we may want to prove that $a_i \geq \frac{3}{2} \cdot i$, but since $\{a_i\}$ are integers, we will actually prove that $a_{2i+1} \geq 3i + 1$ by induction.

We prove this by using the Cauchy-Schwartz inequality:

$$a_{k+2}^2 = (x_1 + \ldots + x_{k+2}) \left(\frac{1}{x_1} + \ldots + \frac{1}{x_{k+2}} \right) =$$
$$((x_1 + \ldots + x_k) + x_{k+1} + x_{k+2}) \left(\left(\frac{1}{x_1} + \ldots + \frac{1}{x_k} \right) + \frac{1}{x_{k+2}} + \frac{1}{x_{k+1}} \right) \geq$$
$$\left(\sqrt{(x_1 + \ldots + x_k) \cdot \left(\frac{1}{x_1} + \ldots + \frac{1}{x_k} \right)} + \sqrt{\frac{x_{k+1}}{x_{k+2}}} + \sqrt{\frac{x_{k+2}}{x_{k+1}}} \right)^2 >$$
$$(a_k + 2)^2$$

The inequality between row 2 and 3 follows from the Cauchy-Schwartz inequality, while the strict inequality from row 3 to row 4 follows from the AMGM inequality, as x_{k+1} and x_{k+2} are distinct, hence equality can't be reached. As a_{k+2} is an integer strictly larger than $a_k + 2$, it follows that $a_{k+2} \geq a_k + 3$, which gives the desired conclusion.

□

Problem 1.8: ISL 2015 A1

Let $\{a_n\}$ be a sequence of positive real numbers, such that

$$a_{k+1} \geq \frac{k a_k}{(a_k)^2 + (k-1)}$$

for every positive integer k. Show that $a_n + a_{n-1} + \ldots + a_1 \geq n$ for every $n \geq 2$.

Proof. We define the auxiliary sequence $s_k = a_k + \ldots + a_1$, where $s_0 = 0$ and transform the given inequality in the following way:

$$\frac{k}{a_{k+1}} \leq a_k + \frac{k-1}{a_k} \implies \frac{k}{a_{k+1}} - \frac{k-1}{a_k} \leq a_k.$$

By summing n such inequalities we get:

$$\frac{n}{a_{n+1}} \leq s_n.$$

Now we can use induction to see:

$$s_{n+1} = s_n + a_{k+1} \geq s_n + \frac{n}{s_n} \geq n+1.$$

Where the last inequality is achieved by noting that $s_n = 1$ and $s_n = n$ produce equality, hence $s_n \geq n$ must produce the desired inequality. The main part of the problem was transforming the inequality into a nicer form. This specific manipulation can be motivated by the fact that taking inverses of both sides seems good, as we get rid of the sum in the denominator on the right and replace it with a product in the denominator, which behaves much more nicely. $\square$

Problem 1.9

Define the following sequence of real numbers: $a_1 = 1$ and

$$a_{n+1} = a_n + \frac{1}{a_n}.$$

Determine the integer part of a_{100}.

Proof. Observe:

$$(a_{n+1})^2 = (a_n)^2 + 2 + (\frac{1}{a_n})^2 \implies (a_n)^2 = 2n + \sum_{i=2}^{n-1} \left(\frac{1}{a_i}\right)^2 \implies a_n > \sqrt{2n}.$$

By using the inequality from above we get:

$$(a_n)^2 = 2n + \sum_{i=2}^{n-1} \left(\frac{1}{a_i}\right)^2 \leq 2n + \sum_{i=2}^{n-1} \frac{1}{2i} \leq 2n + \frac{n-3}{4} = \frac{9n-3}{4},$$

where the last inequality was reached by observing that $a_n \geq 2$ for $n \geq 2$. In fact, a much shaper bound of

$$(a_n)^2 \leq 2n + \frac{1}{2} \ln{(n-2)}$$

can be reached with some rudimentary knowledge of analysis. Anyway, the integer part of a_{100} is 14, given by some simple calculation. In fact, we can also get that $\lfloor a_{2025} \rfloor = 63$. $\square$

Problem 1.10: EGMO 2020 P6

Let $m > 1$ be a positive integer. Define the sequence $\{a_n\}$ by $a_1 = a_2 = 1$, $a_3 = 4$ and the recursive relation:

$$a_n = m \cdot (a_{n-1} + a_{n-2}) - a_{n-3}.$$

Determine all m for which all terms of the defined sequence are perfect squares.

Proof. We start by some preliminary calculation:

$$a_4 = 5m - 1$$
$$a_5 = 5m^2 + 3m - 1$$
$$a_6 = 5m^3 + 8m^2 - 2m - 4$$
$$a_7 = 5m^4 + 13m + m^2 - 10m - 1$$

Recall one of the standard arguments for proving that an expression is not a perfect square: we bound it strictly between two consecutive perfect squares. The expressions we have found are not very promising, as the leading coefficients are not perfect squares, meaning that the bounds we produce by blindly following the above advice will not be very sharp. We use the same approach, but more carefully: if all of the terms of the sequence are perfect squares, so are their products. By multiplying two expressions that are both odd degree, or both even degree we will find a polynomial of odd degree with leading coefficient that is a square, which seems much more promising. We chose a_4 and a_6 as the two candidates, as they have smallest degrees.

$$a_4 \cdot a_6 = (5m - 1) \cdot (5m^3 + 8m^2 - 2m - 4) = 25m^4 + 35m^3 - 18m^2 - 18m + 16.$$

The candidate for one of the bounds is

$$(5m^2 + bm + c)^2 = 25m^4 + (10b)m^3 + (10c + b^2)m^2 + (2bc)m + c^2$$

But since the coefficient of the cubic term in the expression above is a multiple of 10, we make the following technical shift:

$$f(m) = 4 \cdot a_4 \cdot a_6 = 100m^4 + 140m^3 - 72m^2 - 72m + 16,$$

which is still a square as it is a product of three squares, and

$$P_{b,c}(m) = (10m + b + c)^2 = 100m^4 + (20b)m^3 + (20c + b^2)m^2 + (2bc)m + c^2.$$

Specifically, we look at

$$P_{7,-6}(m) = 100m^4 + 140m^3 - 71m^2 - 84m + 36.$$

We see that the following inequality

$$f(m) < P_{7,-6}(m)$$
$$100m^4 + 140m^3 - 72m^2 - 72m + 16 < 100m^4 + 140m^3 - 71m^2 - 84m + 36$$
$$12m < m^2 + 20$$

holds for all positive integers larger than 10. Analogously:

$$P_{7,-7}(m) \leq f(m)$$
$$100m^4 + 140m^3 - 91m^2 - 98m + 48 \leq 100m^4 + 140m^3 - 72m^2 - 72m + 16$$
$$32 \leq 19m^2 + 26m$$

holds for all positive integers m. Now we must check which of the $m \in \{1, \dots, 10\}$ give solutions. By simple calculation, we eliminate all values except 2 and 10, as at least one of a_4, a_5 and a_6 are not squares in those cases. By some not-so-trivial guessing and checking we can see that $m = 2$ gives $a_n = (F_n)^2$, where F_n are the Fibonacci numbers, and $m = 10$ gives $a_n = (G_n)^2$, where $G_{n+2} = 3 \cdot G_{n+1} + G_n$ with initial conditions $G_1 = G_2 = 1$ $\square$

References

[1] Alexander Remorov. *Sequences*. 2012. URL: https://alexanderrem.weebly.com/uploads/7/2/5/6/72566533/sequences.pdf.