IMC 2025

Hugo Trebše (hugo.trebse@gmail.com)

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1 Day 1

Problem 1.1

Let $P \in \mathbb{R}[x]$ with $\deg(P) \geq 2$. For every $x \in \mathbb{R}$, let $\ell_x \subset \mathbb{R}^2$ be the line tangent to P at the point x. Prove or disprove the following assertions.

- If P is an odd degree polynomial, then
$$\bigcup_{x \in \mathbb{R}} \ell_x = \mathbb{R}^2.$$
- If P is an even degree polynomial, then
$$\bigcup_{x \in \mathbb{R}} \ell_x \neq \mathbb{R}^2.$$

Solution. In the answers to both questions we will make use of the following: the tangent line to P at the point $(\alpha, P(\alpha))$ has equation

$$y = P'(\alpha)(x - \alpha) + P(\alpha).$$

First we will prove that the first assertion is true. Assume the assertion to be false and let

$$(x_0, y_0) \in \mathbb{R}^2 \setminus \bigcup_{x \in \mathbb{R}} \ell_x \neq \emptyset.$$

Since the point (x_0, y_0) does not lie on any line tangent to P at some point, the equation

$$y_0 = P'(x)(x_0 - x) + P(x)$$

has no real solutions in x. This is clearly equivalent to the polynomial

$$G(x) = P'(x)(x_0 - x) + P(x) - y_0 = P(x) - xP'(x) + x_0P'(x) - y_0$$

having no real zeros. The leading term of $xP'(x)$ is na_nx^n, where a_n is the leading coefficient of P and $n = \deg(P) \geq 2$, which means that the leading term of G has coefficient $a_n(1 - n) \neq 0$. In particular, G is odd since P is odd. But all polynomials of odd degree have zeros, which is a contradiction. The assertion is hence true.

We will prove that the second assertion is also true. Let P be any non-constant even degree polynomial. We know that

$$(x_0, y_0) \in \bigcup_{x \in \mathbb{R}} \ell_x \iff y_0 = P'(x)(x_0 - x) + P(x) \text{ has a real solution in } x.$$

The latter is equivalent to the polynomial

$$G(x) = P'(x)(x_0 - x) + P(x) - y_0 = P(x) - xP'(x) + x_0P'(x) - y_0$$

having a zero for all $(x_0, y_0) \in \mathbb{R}^2$. A verbatim argument as above proves that G is of even degree. It is well known that even degree polynomials are either bounded from above or bounded from below (this can be proven by containing the zeros in a closed interval and noting that limits to $\pm\infty$ are equal, and then using the minimum or maximum on the compact to bound the polynomial). But it is then clear that the polynomial can't have zeros for all $(x_0, y_0) \in \mathbb{R}^2$; take $x_0 = 0$ and set y_0 to be either larger or smaller than the upper or lower bound of the polynomial $P(x) - xP'(x)$ $\square$

Problem 1.2

Let $f : \mathbb{R} \to \mathbb{R}$ be twice continously differentiable. Suppose

$$\int_{-1}^{1} f(x)\,dx = 0,$$

and $f(-1) = f(1) = 1$. Show that

$$\int_{-1}^{1} \left(f''(x)\right)^2\,dx \geq 15$$

and determine all f, for which equality holds.

Solution. We begin by some preliminary calculations that follow from the fundamental theorem of analysis.

$$\int_{-1}^{1} f'(x)\,dx = f(x)\Big|_{-1}^{1} = 0$$
$$\int_{-1}^{1} f''(x)\,dx = f'(x)\Big|_{-1}^{1} = f'(1) - f'(-1)$$

Furthermore, we use integration by parts to establish some equalities.

$$\int_{-1}^{1} f(x)\,dx = f(x)x\Big|_{-1}^{1} - \int_{-1}^{1} f'(x)x\,dx \implies \int_{-1}^{1} f'(x)x\,dx = 2$$
$$\int_{-1}^{1} f'(x)\,dx = f'(x)x\Big|_{-1}^{1} - \int_{-1}^{1} f''(x)x\,dx \implies \int_{-1}^{1} f''(x)x\,dx = f'(1) + f'(-1)$$
$$2\int_{-1}^{1} f'(x)x\,dx = f'(x)x^2\Big|_{-1}^{1} - \int_{-1}^{1} f''(x)x^2\,dx \implies \int_{-1}^{1} f''(x)x^2\,dx = f'(1) - f'(-1) - 4$$

Now we use the Cauchy-Schwarz inequality on the space $\mathcal{C}([-1,1])$. It follows that

$$\left(\int_{-1}^{1} f''(x)^2\,dx\right)\left(\int_{-1}^{1} (\alpha x^2 + \beta x + \gamma)^2\,dx\right) \geq \left(\int_{-1}^{1} \alpha f''(x)x^2 + \beta f''(x)x + \gamma f''(x)\,dx\right)^2$$

$$\left(\int_{-1}^{1} f''(x)^2\,dx\right)\left(\frac{2\alpha^2}{5} + \frac{2(2\alpha\gamma + \beta^2)}{3} + 2\gamma^2\right) \geq$$
$$\left(\alpha\int_{-1}^{1} f''(x)x^2\,dx + \beta\int_{-1}^{1} f''(x)x\,dx + \gamma\int_{-1}^{1} f''(x)\,dx\right)^2$$

By using the established identities we conclude

$$\left(\int_{-1}^{1} f''(x)^2\,dx\right)\left(\frac{2\alpha^2}{5} + \frac{2(2\alpha\gamma + \beta^2)}{3} + 2\gamma^2\right) \geq \left((\alpha+\beta+\gamma)f'(1) + (-\alpha+\beta-\gamma)f'(-1) - 4\alpha\right)^2.$$

Since we have no information about $f'(1)$ and $f'(-1)$ it would do us well to select the coefficients α, β and γ in such a way that the coefficients of $f'(1)$ and $f'(-1)$ on the right side of the inequality evaluate to zero. By adding the two equations arising from this observation it is clear that $\beta = 0$ and $\alpha = -\gamma \neq 0$ achieve this goal. We hence deduce

$$\int_{-1}^{1} f''(x)^2\, dx \geq \frac{16\alpha^2}{\alpha^2(\frac{2}{5} - \frac{4}{3} + 2)} = \frac{16}{\frac{6-20+30}{15}} = 15,$$

which we wanted to show.

Now we turn to the question of which functions achieve equality. Observe that the value of f outside of the interval $[-1, 1]$ are irrelevant both for the problem's conditions and the problem's conclusions. This means that if we add a function $h \in \mathcal{C}^2(\mathbb{R})$ with support disjoint from $[-1, 1]$ to any function meeting the problem criteria and achieving equality, the problem conditions are met and equality is still achieved by their sum. Hence we need to consider only $f \in \mathcal{C}^2([-1, 1])$. Since $\mathcal{C}([-1, 1])$ with the standard inner product forms an inner product space, equality is achieved if and only if the two vectors are linearly dependent. There hence exist constants $\psi, \theta \in \mathbb{R}$, not both zero, such that

$$\psi f''(x) + \theta(\alpha x^2 - \alpha) = 0.$$

It is clear that in fact neither of the constants is zero, meaning

$$f''(x) = \lambda(\alpha x^2 - \alpha) \quad \lambda \in \mathbb{R}.$$

By integrating this equation twice, then using the boundary condition and the fact that the integral of f on $[-1, 1]$ equals 0 to determine λ and α, we conclude that the only function in $\mathcal{C}^2([-1, 1])$ achieving equality is

$$f(x) = \frac{-5x^4 + 30x^2 - 9}{16}.$$

Hence any function achieving equality is of the form

$$f(x) = \frac{-5x^4 + 30x^2 - 9}{16} + h(x),$$

where $h \in \mathcal{C}^2(\mathbb{R})$ and $h(x) = 0$ for all $x \in [-1, 1]$. $\square$

Problem 1.3

Denote by S the set of all real symmetric 2025×2025 matrices of rank 1 whose entries take values -1 or $+1$. Let $A, B \in S$ be matrices chosen independently uniformly at random. Find the probability that A and B commute.

Solution. Firstly, we will explore the structure of the elements of S. The following claim is well-known.

Claim 1.4. *If $A \in \mathbb{R}^{n\times n}$ has rank 1, then there exist $u, v \in \mathbb{R}^n$ such that*

$$A = uv^T$$

For symmetric matrices of rank 1 we can prove the following stronger statement:

Claim 1.5. *If $A \in \mathbb{R}^{n\times n}$ is symmetric and has rank 1, then there exists a $v \in \mathbb{R}^n$ such that*

$$A = \lambda vv^T, \quad \text{where } \lambda \in \{-1, 0, 1\}.$$

Proof. We use the first claim and the symmetric property to deduce that

$$uv^T = vu^T \implies u_i v_j = v_i u_j \;\; \forall\; 1 \le i.j \le n.$$

If u_i and u_j are nonzero, then $\frac{x_i}{y_i} = \frac{x_j}{y_j}$. This means that all non-zero coordinates of u and v have the same ratio, hence $u = \alpha v$ for some $\alpha \in \mathbb{R}$. Now set $w = \sqrt{|\alpha|}v$ to deduce

$$A = \alpha vv^T = \pm(\sqrt{|\alpha|} \cdot v)(\sqrt{|\alpha|} \cdot v)^T = \pm ww^T,$$

which we wanted to show. □

For $A \in S$, clearly $w \neq 0$. Since the first column of A only has entries in $\{-1, 1\}$, w only has entries in $\{-1, 1\}$. From now on, we will denote the set of vectors of length n with entries in $\{-1, 1\}$ by $\{-1, 1\}^n$.

We now consider when two elements of S commute. Let $A, B \in S$ and $A = \pm vv^T$ and $B = \pm uu^T$. It follows that

$$AB = \pm vv^T uu^T = \pm v(v^T u)u^T = \pm\langle v, u\rangle vu^T$$
$$BA = \pm uu^T vv^T = \pm u(u^T v)v^T = \pm\langle u, v\rangle uv^T,$$

where we used the fact that the product of a column with a row is a scalar, which commutes. It is clear that the $\pm$ signs of AB and BA are equal, and that $v^T u = u^T v$, since both are just dot products. A and B hence commute if $\langle u, v\rangle = 0$ or if $vu^T = uv^T$.

Notice that for $u, v \in \{-1, 1\}^{2025}$ $\langle u, v\rangle \neq 0$, since $\langle u, v\rangle = 1 \pmod 2$. Assume $u, v \in \{-1, 1\}^{2025}$ and $vu^T = uv^T$. It follows that $u_1 \cdot v = v_1 \cdot u$, which means that u and v are scalar multiples of one another. Since they are both elements of $\{-1, 1\}^{2025}$ it follows that $u = \pm v$. In any case, we deduce that if $A = \pm vv^T$ and $B = \pm uu^T$ commute, then

$u = \pm v$ meaning $B = \pm A$. Since A commutes with A and $-A$, it follows that $A, B \in S$ commute if and only if $A = \pm B$.

The only step remaining is to determine the number of elements of S. Observe that any element of $\{-1,1\}^{2025}$ generates an element of S by Claim 1.5. One can check that vectors $u, v \in \{-1,1\}^{2025}$ generate the same element of S if and only if $u = \pm v$. Additionally, any matrix of the form vv^T has the left uppermost entry equal to 1, so in fact the $\pm$ sign in Claim 1.5 is necessary. Combining these observations leads one to conclude that there are 2025 binary choices to be made; one choice regarding the $\pm$ sign in $\pm vv^T$, 2025 choices regarding the entries of $v \in \{-1,1\}^{2025}$ and 1 correction since v and $-v$ generate the same element. It follows that $|S| = 2^{2025}$.

Since $A \in S$ commutes only with A and $-A$, the probability that two independently uniformly chosen matrices commute is

$$\frac{2}{2^{2025}} = 2^{-2024}$$

□

Interestingly enough, an analogous argument can solve a more general problem. Let D be the set of all not necessarily symmetric 2025×2025 matrices of rank 1 with entries either -1 or $+1$. An analogous argument in which one uses Claim 1.4 in place of Claim 1.5 can be used to prove that elements of D commute if and only if they are equal or opposite. Hence the symmetric condition only needs to be taken into account when calculating the size of S.

The fact that 2025 is odd made the problem somewhat easier, since the scalar product of two vectors in $\{-1,1\}^{2k+1}$ is always non-zero. The only place in our argument, where we used any properties of the integer 2025, is the calculation of the size of S and when we deduced that no two elements of $\{-1,1\}^{2025}$ have dot product zero. It hence follows, that two symmetric matrices of any dimension, that have rank 1 and entries in $\{-1,1\}$ commute if and only if they are the same or opposite, or if the dot product of the two vectors generating them is zero. Additionally, one now needs to count the size of the orthogonal complement of a vector $v \in \{-1,1\}^{2k}$ (clearly v isn't in its own orthogonal complement, which means we don't overcount commuting matrices) in the set $\{-1,1\}^{2k}$. A simple combinatorial argument proves that the size of the orthogonal complement of any $v \in \{-1,1\}^{2k}$ in the set $\{-1,1\}^{2k}$ is

$$\binom{2k}{k}.$$

In the case of $2k \times 2k$ matrices the probability of two independently uniformly at random chosen matrices commuting hence equals

$$\frac{2 + \binom{2k}{k}}{2^k}.$$

2 Day 2

Problem 2.1

Let $f : (0, \infty) \to \mathbb{R}$ be continuously differentiable. Assume there exist $b > a > 0$, such that $f(b) = f(a) = k$. Prove there exists a point $\xi \in (a, b)$ such that

$$f(\xi) - \xi f'(\xi) = k.$$

Solution. The expression is reminiscent of the numerator of the formula for the derivative of the quotient of two functions. Noting this, one can define a continuously differentiable function $g : (0, \infty) \to \mathbb{R}$ given by

$$g(x) = \frac{f(x) - k}{x}.$$

It is clear that $g(a) = g(b) = 0$. Using Rolle's theorem we conclude that there exists a $\xi \in (a, b)$ such that $g'(\xi) = 0$. Hence

$$0 = g'(\xi) = \frac{f'(\xi)\xi - f(\xi) + k}{\xi^2}.$$

Since $\xi \neq 0$, the desired point has been found. □

Problem 2.2

Let $\mathbb{N}$ be the set of positive integers. Find all nonempty subsets $M \subseteq \mathbb{N}$ satisfying the following properties:

- if $x \in M$, then $2x \in M$,
- if $x, y \in M$ and $2 \mid x+y$, then $\frac{x+y}{2} \in M$.

Solution. Combining the two conditions we find that the set is closed under addition.

Claim 2.3. *If $x, y \in M$, then $x + y \in M$.*

Proof.

$$x, y \in M \implies 2x, 2y \in M. \;\; 2 \mid (2x + 2y) \implies \frac{2x + 2y}{2} = x + y \in M.$$

□

Recall the following theorem

Theorem 2.4: Frobenius coin problem/ Chicken McNugget theorem

Let $m, n \in \mathbb{N}$ and $g = \gcd(m, n)$. Then the greatest integer divisible by g, that can't be written in the form $am + bn$ where a, b are nonnegative integers is

$$g \cdot \left(\frac{mn}{g^2} - \frac{m}{g} - \frac{n}{g} \right).$$

The explicit value in the theorem above is not particularly relevant, but it tells us M must contain all multiples of its gcd from some point onwards.

Observe that M must contain some odd number, as $2 \mid x \in M \implies \frac{x+2x}{2} = \frac{3}{2}x \in M$, which forces descent.

Motivated by these observations we conjecture that all sets meeting the required properties are of the form

$$\{d \cdot n | n \geq c\}$$

for some odd $d \in \mathbb{N}$ and some arbitrary $c \in \mathbb{N}$. These types of sets clearly meet the criteria.

Let M be any set meeting the required criteria, and let $c \cdot d$ be any one of its elements with $d = \gcd(M)$. It has been shown that $2 \neq d$. We prove that $(c+1) \cdot d \in M$, which clearly implies our claim.

By Theorem 2, there exists some $k \in \mathbb{N}$, such that $2^k \cdot d \in M$. We will prove that $\forall\, i \in \{0, 1, \ldots, k\}$

$$(2^i + c) \cdot d \in M,$$

which clearly implies the desired statement for $i = 0$. Since M is closed under addition, the claim is true for $i = k$. Assume the claim is false and let j be the largest integer in $\{0, 1, \ldots, k-1\}$ for which $(2^j + c) \cdot d \notin M$. However

$$(2^{j+1} + c) \cdot d \in M \text{ and } 2 \mid 2^{j+1} + 2c \implies \frac{2^{j+1} + 2c}{2} \cdot d = (2^j + c) \cdot d \in M,$$

which is a contradiction. $\square$

Problem 2.5

For a real $n \times n$ matrix $A \in M_n(\mathbb{R})$ denote by A^R its counter-clockwise $90°$ rotation. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}^R = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 5 & 8 \\ 1 & 4 & 7 \end{bmatrix}.$$

Prove that if $A = A^R$ then for any eigenvalue λ of A, we have either $\mathrm{Re}(\lambda) = 0$ or $\mathrm{Im}(\lambda) = 0$.

Solution. In this solution, we denote by $A^{(i)}$ and $A_{(i)}$ respectively the i-th column and i-th row of the matrix A.

Note that the map $^R : \mathbb{F}^{n\times n} \to \mathbb{F}^{n\times n}$ which maps any matrix to its counter-clockwise $90°$ rotation is linear. We may hope to find the matrix that this map defines, but such an approach would not be practical, since we can't for example multiply the matrix form of the map ^{R} and a matrix $A \in \mathbb{F}^{n\times n}$, as they belong to vector spaces of different dimensions. However, we may hope to express the map ^{R} with some well-known linear map from $\mathbb{F}^{n\times n}$ to $\mathbb{F}^{n\times n}$.

Claim 2.6. *Let $J \in M_n(\mathbb{R})$ be the square matrix with entries 1 on the main anti-diagonal and entries 0 elsewhere. Then for $A \in M_n(\mathbb{R})$*

$$A^R = JA^T$$

Proof. It is clear that $A^R_{(i)} = A^{(n+1-i)}$ and that $\left(A^T\right)^{(i)} = A_{(i)}$. Since the i-th row of JA^T consists of elements

$$(\langle e_{n+1-i}, A_{(j)}\rangle)_{j=1}^n = (a_{j,n+1-i})_{j=1}^n = A^{(n+1-i)},$$

which proves the desired identity as A^R and JA^T have the same columns. □

We are asked to prove that if $A = A^R$, then eigenvalues of A are either real or purely imaginary. By the spectral mapping theorem, this is equivalent to A^2 having only real eigenvalues. Recall the following theorem:

Theorem 2.7

A real symmetric matrix has only real eigenvalues.

It is hence sufficient to prove that A^2 is real and symmetric. The fact it is real is obvious since A is real. Note that

$$A^2 = AA^R = AJA^T \quad \text{and} \quad (AJA^T)^T = (A^T)^T J^T A^T = AJA^T.$$

It follows that A^2 is symmetric, which implies the desired conclusion. □

References

[1] *International Mathematics Competition for University Students 2025*. 2025. URL: https://www.imc-math.org.uk/?year=2025.