Machine Learning I

Review on Probability and Statistics

Souhaib Ben Taieb

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University of Mons

Overview

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Random variables

Discrete random variables

Continuous random variables

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References

- **Introduction to Probability for Data Science**, Stanley H. Chan. [Link] (Book, slides and videos)
- Probability Theory Review for Machine Learning, Samuel Ieong. [Link]
- *All of Statistics*, Larry Wasserman. [Link]

Probability

Sample space and events

- When we speak about probability, we often refer to the probability of **an event of uncertain nature** taking place.
- We first need to clarify what the **possible events** are to which we want to attach probability.
- We often conduct an experiment, i.e. take some measurements of a **random (stochastic) process**.
- Our measurements take values in some set Ω, the **sample space** (or the outcome space)., which defines *all possbile outcomes* of our measurements.

Sample space and events

- We toss one coin heads (H) or tails (T)
  - $\Omega = \{H, T\}$
- We toss two coins
  - $\Omega = \{HH, HT, TH, TT\}$
- We measure the reaction time to some stimulus
  - $\Omega = (0, \infty)$

Sample space and events

An **event** A is a subset of Ω ($A \subseteq \Omega$), i.e., it is a subset of possible outcomes of our experiment. We say that an event A **occurs** if the outcome of our experiment belongs to the set A.

- Let $\Omega = \{HH, HT, TH, TT\}$, and consider the following events: $A_1 = \{HH, TH, TT\}$ and $A_2 = \{TH, TT\}$. We observe $\omega = HT$. Which events did occur?
- Let $\Omega = (0, \infty)$, and consider the following events $A_1 = (3, 6)$, $A_2 = (1, 2)$ and $A_3 = (2, 8)$. We observe $\omega = 4$. Which events did occur?

Probability space

A **probability space** is defined by the triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω is the **sample space**
- $\mathcal{F} = 2^\Omega$ is the **space of events** (or event space)[1]
- $\mathbb{P}$ is the **probability measure/distribution** that maps an event $A \in \mathcal{F}$ to a real value between zero and one

[1] 2^S is the set of all subsets of S including S and the empty set $\varnothing$. Note that $\mathcal{F} = 2^\Omega$ is not fully general, but it is often sufficient for practical purposes.

Probability axioms

A **probability distribution** is a mapping from events to real numbers that satisfy certain **axioms**:

1. *Non-negativity*: $\mathbb{P}(A) \geq 0, \forall A \subseteq \Omega$
2. *Unity of* Ω: $\mathbb{P}(\Omega) = 1$
3. *Additivity*: For all disjoint events $A, B \in \mathcal{F}$ (i.e. $A \cap B = \varnothing$), we have that, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

Using set theory and the probability axioms, we can show several useful and intuitive properties of probability distributions.

- $\mathbb{P}(\varnothing) = 0$
- $A \subset B \implies \mathbb{P}(A) \leq \mathbb{P}(B)$
- $0 \leq \mathbb{P}(A) \leq 1$
- $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

All of these properties can be understood via a Venn diagram.

Probability properties

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

$$\begin{aligned}
&\mathbb{P}(\Omega) = 1 \quad \text{(Axiom 2)} \\
&\iff \mathbb{P}(A \cup A^c) = 1, \quad \forall A \subseteq \Omega \\
&\iff \mathbb{P}(A^c) + \mathbb{P}(A) = 1 \quad \text{(Axiom 3)} \\
&\iff \mathbb{P}(A^c) = 1 - \mathbb{P}(A)
\end{aligned}$$

$A \subseteq B \implies \mathbb{P}(A) \leq \mathbb{P}(B)$.

$$\begin{aligned}
&A \subseteq B \\
&\implies B = A \cup (B \backslash A) \quad (A \cap (B \backslash A) = \varnothing) \\
&\implies \mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \backslash A) \quad \text{(Axiom 3)} \\
&\implies \mathbb{P}(B) \geq \mathbb{P}(A) \quad \text{(Axiom 1)}
\end{aligned}$$

Probability of an event (discrete case)

- The probability of any event $A = \{\omega_1, \omega_2, \ldots, \omega_k\}$ ($\omega \in \Omega$) is the sum of the probabilities of its elements:

$$\mathbb{P}(A) = \mathbb{P}(\{w_1, w_2, \ldots, w_k\}) = \sum_{i=1}^{k} \mathbb{P}(\{w_i\})$$

- If Ω consists of n equally likely outcomes (i.e. a uniform distribution), then the probability of any event A is

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{n}$$

- Suppose we toss a fair dice twice. The sample space is $\Omega = \{(t_1, t_2) : t_1, t_2 = 1, 2, \ldots, 6\}$. Let A be the event that the sum of two tosses being less than five. What is $\mathbb{P}(A)$?

Conditional probability

If $\mathbb{P}(B) > 0$, the **conditional probability** of A *given* B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Note: $\mathbb{P}(A|B) \neq \mathbb{P}(B|A) \quad$ (in general)

The **chain rule** can be obtained by rewriting the above expression as follows:

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A|B) = \mathbb{P}(A)\mathbb{P}(B|A)$$

More generally, we have

$$\mathbb{P}(A_1 \cap A_2 \cap A_3 \ldots) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 A_2)\ldots$$

Independence of events

Two events A and B are called **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

A set of events $A_j (j \in J)$ are called **mutually independent** if

$$\mathbb{P}\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} \mathbb{P}(A_j).$$

Conditional probability gives another interpretation of independence: A and B are independent if the *unconditional probability* is the same as the conditional probability.

When combined with other properties of probability, independence can sometimes simplify the calculation of the probability of certain events.

Example

Consider a fair coin. What is the probability of at least one head in the first 10 tosses?

Let A be the event "at least one head in 10 tosses". Then, A^c is the event "No heads in 10 tosses" (all 10 tosses being tails).

We have

$$\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}(A^c) &(1)\\
&= 1 - \mathbb{P}(T \cap T \cap T \cap \cdots \cap T) &(2)\\
&= 1 - \prod_{i=1}^{10} \mathbb{P}(T) &(3)\\
&= 1 - (1/2)^{10} &(4)
\end{aligned}$$

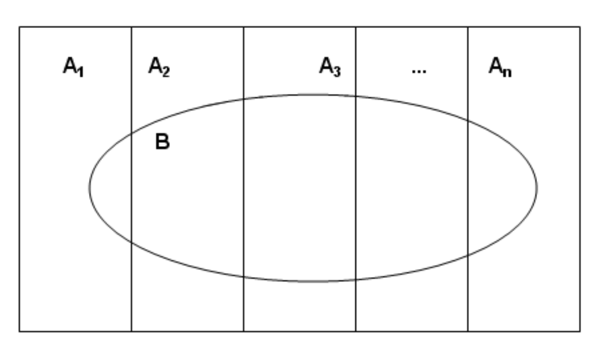
Exercise

Consider tossing a fair dice. Let A be the event that the result is an odd number, and $B = \{1, 2, 3\}$.

- Compute $\mathbb{P}(A|B)$
- Compute $P(A)$
- Are A and B independent?

Law of total probability

Let $A_1, A_2, \ldots, A_n$ be a partition of Ω. What is the probability of B?



Law of total probability

Let $A_1, A_2, \ldots, A_n$ be a partition of Ω. Then, for any $B \subseteq \Omega$, we have that

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$

The **law of total probability** is a combination of **additivity** and **conditional probability**. In fact, we have

$$\begin{aligned}
\mathbb{P}(B) &= \mathbb{P}((B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k)) \\
&= \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) \\
&= \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)
\end{aligned}$$

Bayes' Rule

(**Bayes' Rule**) Let $A_1, A_2, \ldots, A_n$ be a partition of Ω. Then, we have that

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)}$$

Roughly, Bayes' rule allows us to calculate $\mathbb{P}(A_i|B)$ from $\mathbb{P}(B|A_i)$. This is useful when $\mathbb{P}(A_i|B)$ is not obvious to calculate but $\mathbb{P}(B|A_i)$ and $\mathbb{P}(A_i)$ are easy to obtain.

Bayes' Rule is a combination of **conditional probability** and the **law of total probability**:

$$\mathbb{P}(A_i|B) = \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(B|A_i)\mathbb{P}(A_i)}{\mathbb{P}(B)}$$

Example

Suppose there are three types of emails: $A_1 =$ SPAM, $A_2 =$ Low Priority and $A_3 =$ High Priority. Based on previous experience, we have $\mathbb{P}(A_1) = 0.85, \mathbb{P}(A_2) = 0.1, \mathbb{P}(A_3) = 0.05$.

Let B the event that an email contains the word "free", then $\mathbb{P}(B|A_1) = 0.9, \mathbb{P}(B|A_2) = 0.1, \mathbb{P}(B|A_3) = 0.1$. When we receive an email containing the word "free", what is the probability that it is a spam?

Random variables

Random variables

Often we are interested in dealing with *summaries of experiments* rather than the actual *outcome*. For instance, suppose we toss a coin three times. But we may only be interested in a summary such as the number of heads. We have

$$\Omega = \{\underbrace{HHH}_{\substack{\downarrow \\ 3}}, \underbrace{HHT}_{\substack{\downarrow \\ 2}}, \underbrace{HTH}_{\substack{\downarrow \\ 2}}, \underbrace{THH}_{\substack{\downarrow \\ 2}}, \underbrace{TTH}_{\substack{\downarrow \\ 1}}, \underbrace{THT}_{\substack{\downarrow \\ 1}}, \underbrace{HTT}_{\substack{\downarrow \\ 1}}, \underbrace{TTT}_{\substack{\downarrow \\ 0}}\}$$

These summary statistics are called **random variables**. Specifically, a random variable is a function from the sample space Ω to the reals.

Random variables

A random variable can be seen as a **mapping** between a distribution on Ω to a distribution on the reals (or the range of the random variable, $\mathcal{X} \subseteq \mathbb{R}$). Formally, we have that for some subset $S \subseteq \mathcal{X}$,

$$\mathbb{P}_X(X \in S) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in S\})$$

For the previous example, we have

$$\Omega = \{\underbrace{HHH}_{\substack{\downarrow \\ 3}}, \underbrace{HHT}_{\substack{\downarrow \\ 2}}, \underbrace{HTH}_{\substack{\downarrow \\ 2}}, \underbrace{THH}_{\substack{\downarrow \\ 2}}, \underbrace{TTH}_{\substack{\downarrow \\ 1}}, \underbrace{THT}_{\substack{\downarrow \\ 1}}, \underbrace{HTT}_{\substack{\downarrow \\ 1}}, \underbrace{TTT}_{\substack{\downarrow \\ 0}}\}$$

and

$$\mathbb{P}_X(X = 0) = 1/8, \quad \mathbb{P}_X(X = 1) = 3/8,$$

$$\mathbb{P}_X(X = 2) = 3/8, \quad \mathbb{P}_X(X = 3) = 1/8.$$

In the following, we will use $\mathbb{P}$ to denote probability.

Discrete random variables

Probability mass function

The **probability mass function** (PMF) of a random variable X is a function which specifies the probability of obtaining a number x. We denote the PMF as

$$p_X(x) = \mathbb{P}(X = x).$$

What is the PMF of the previous coin flip example?

A function p_X is a PMF if and only if

1. $p_X(x) \geq 0, \forall x \in \mathcal{X}$
2. $\sum_{x \in \mathcal{X}} p_X(x) = 1$

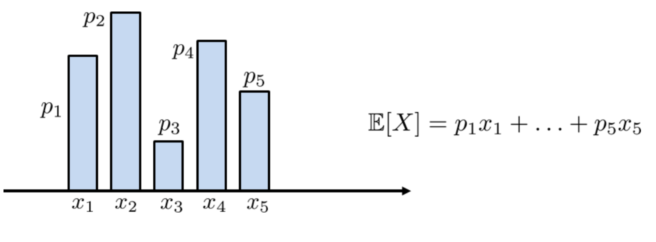
Some important discrete distributions

- Discrete **uniform** distribution on K categories ($X \in \{C_1, C_2, \ldots, C_K\}$). The PMF is given by
$$p_X(x) = \frac{1}{k}, \qquad \forall x \in \{C_1, C_2, \ldots, C_K\}$$
- The **Bernouilli** distribution with parameter $p \in [0, 1]$ ($X \in \{0, 1\}$). The PMF is given by
$$p_X(x) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases} = p^x (1-p)^{1-x}$$
It can represent a coin toss when the coin has bias p where 1 denotes heads and 0 denotes tails.
- Other important distributions: Binomial, Geometric, Poisson, etc.
- The symbol "$\sim$" denotes "distributed as", i.e. $X \sim \text{Ber}(p)$ means that X has a Bernoulli distribution with parameter p.

Expectation

The **expectation** of a random variable X is

$$\mathbb{E}[X] = \sum_{x \in \mathcal{X}} x \; p_X(x).$$



Expectation and its properties

For any function g, we have

$$\mathbb{E}[g(X)] = \sum_{x \in \mathcal{X}} g(x)\ p_X(x).$$

For any function g and h,

$$\mathbb{E}[g(X) + h(X)] = \mathbb{E}[g(X)] + \mathbb{E}[h(X)].$$

For any constant c,

$$\mathbb{E}[cX] = c\ \mathbb{E}[X].$$

For any constant c,

$$\mathbb{E}[X + c] = \mathbb{E}[X] + c.$$

Moments and variance

The k-th **moment** of a random variable X is

$$\mathbb{E}[X^k] = \sum_{x \in \mathcal{X}} x^k \; p_X(x).$$

The **variance** of a random variable X is

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mu_X)^2],$$

where $\mu_X = \mathbb{E}[X]$. The **standard deviation** of X is $\sqrt{\mathrm{Var}(X)}$.

Useful properties of the variance include:

- $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- $\mathrm{Var}(cX) = c^2 \mathrm{Var}(X)$
- $\mathrm{Var}(X + c) = \mathrm{Var}(X)$

Continuous random variables

Probability density function

The **probability density function** (PDF) of a continuous random variable X is a function f_X, when integrated over an interval $[a, b]$, yields the probability of obtaining $a \leq X \leq b$:

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

A PDF has the following properties:

1. $f_X(x) \geq 0, \forall x \in \mathcal{X}$
2. $\int_{\mathcal{X}} f_X(x)\ dx = 1$

Note that $f_X(x)$ is not the probability of having $X = x$. In fact, we can have $f_X(x) > 1$.

Some important continuous distributions

- Continuous **uniform** distribution on interval $[a, b]$. The PDF is given by
$$f_X(x) = \frac{1}{b-a} \quad (x \in [a, b]).$$
We write $X \sim \mathcal{U}[a, b]$.
- **Gaussian** distribution. With a location (mean) μ and scale (standard deviation) σ, the PDF is given by
$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (x \in \mathbb{R}).$$
We write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Expectation and its properties

The **expectation** of a continuous random variable X is given by

$$\mathbb{E}[X] = \int_{\mathcal{X}} x \ f_X(x) \ dx.$$

For any function g, we have

$$\mathbb{E}[g(X)] = \int_{\mathcal{X}} g(x) f_X(x) dx.$$

Let $I_A(X) = \begin{cases} 1, & X \in A \\ 0, & X \notin A \end{cases}$. Then, we have

$$\mathbb{E}[I_A(X)] = \int_{\mathcal{X}} I_A(x) \ f_X(x) \ dx = \int_{\mathcal{A}} f_X(x) \ dx = \mathbb{P}(X \in A).$$

Moments and variance

The k-th **moment** of a continuous random variable X is

$$\mathbb{E}[X^k] = \int_{\mathcal{X}} x^k \ f_X(x) dx$$

The **variance** of a continuous random variable X is

$$\text{Var}(X) = \mathbb{E}[(X - \mu_X)^2] = \int_{\mathcal{X}} (x - \mu_X)^2 \ f_X(x) dx,$$

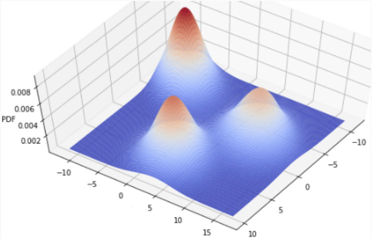
where $\mu_X = \mathbb{E}[X]$. The **standard deviation** of X is $\sqrt{\text{Var}(X)}$.

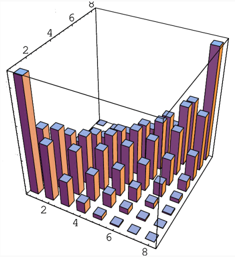
See the useful properties of the variance introduced previously.

Multivariate random variables

More than one random variable?

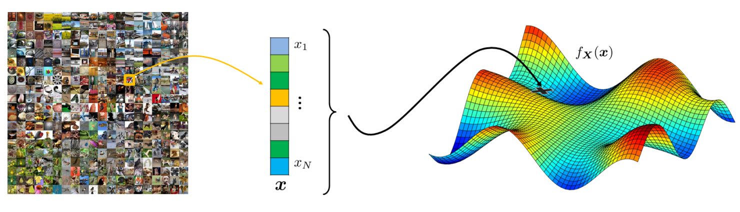
- Multivariate random variables or random vectors are ubiquitous in modern data analysis.
- The uncertainty in the random vector is characterized by a **joint** PDF or PMF.





More than one random variable?

An image from a dataset can be represented by a high-dimensional vector.



Joint distributions

- $f_X(x)$
- $f_{X_1,X_2}(x_1, x_2)$
- $f_{X_1,X_2,X_3}(x_1, x_2, x_3)$
- $\ldots$
- $f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$
- We often just write $f_X(x)$ when the dimensionality is clear from context.

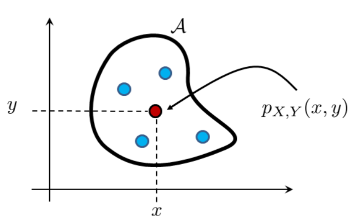
Joint PMF

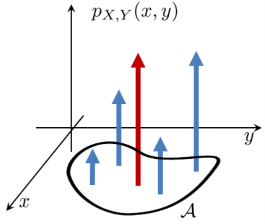
Let X and Y be two discrete random variables. The **joint PMF** of X and Y is defined as

$$p_{X,Y}(x,y) = \mathbb{P}(X = x \text{ and } Y = y).$$

For any $A \subseteq \mathcal{X} \times \mathcal{Y}$, we have

$$\mathbb{P}((X,Y) \in A) = \sum_{(x,y)\in A} p_{X,Y}(x,y).$$





Example

Let X be a coin flip, Y be a dice. Find the joint PMF.

The sample space of X is $\{0, 1\}$. The sample space of Y is $\{1, 2, 3, 4, 5, 6\}$. The joint PMF is

	Y					
	1	2	3	4	5	6
X = 0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
X = 1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

Equivalently, we have

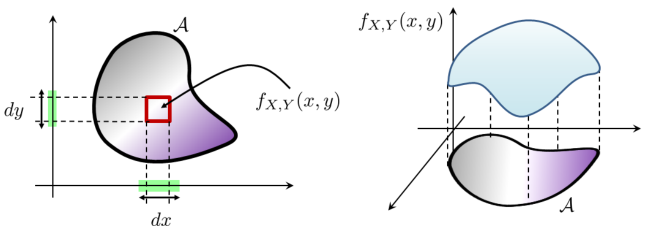
$$p_{X,Y}(x, y) = \frac{1}{12}, \quad x = 0, 1, \quad y = 1, 2, 3, 4, 5, 6.$$

Joint PDF

Let X and Y be two continuous random variables. The **joint PDF** of X and Y is a function $f_{X,Y}(x, y)$ that can be integrated to yield a probability:

$$\mathbb{P}((X, Y) \in A) = \int_A f_{X,Y}(x, y) dx\, dy,$$

for any $A \subseteq \mathcal{X} \times \mathcal{Y}$.



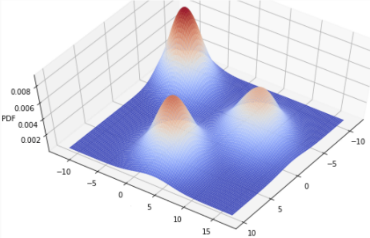
Marginal distribution

The **marginal PMF** is defined as

$$p_X(x) = \sum_{y \in \mathcal{Y}} p_{X,Y}(x, y) \text{ and } p_Y(y) = \sum_{x \in \mathcal{X}} p_{X,Y}(x, y),$$

and the **marginal PDF** is defined as

$$f_X(x) = \int_{\mathcal{Y}} f_{X,Y}(x, y) dy \text{ and } f_Y(y) = \int_{\mathcal{X}} f_{X,Y}(x, y) dx.$$



Independence

If two random variables X and Y are **independent**, then

$$p_{X,Y}(x,y) = p_X(x)p_Y(y), \qquad \text{and } f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

If a sequence of random variables $X_1, \ldots, X_N$ are independent, then their joint PDF (or joint PMF) can be factorized:

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_N) = \prod_{j=1}^{n} f_{X_j}(x_j)$$

Independent and Identically Distributed (i.i.d.)

A collection of random variables $X_1, \dots, X_N$ are called independent and identically distributed (i.i.d.) if

1. All $X_1, \dots, X_N$ are independent.
2. All $X_1, \dots, X_N$ have the same distribution.