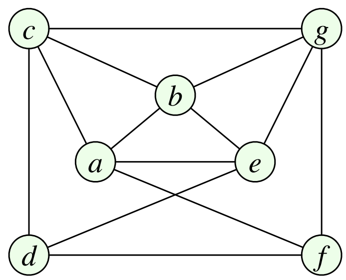
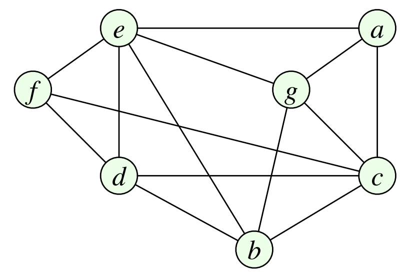
CS/MATH111 ASSIGNMENT 5

Problem 1. Determine whether the two graphs below are planar or not. To show planarity, give a planar embedding. To show that a graph is not planar, use Kuratowski's theorem.

Graph G:

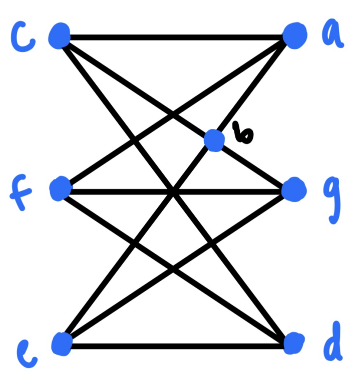


Graph H:



Solution:

(i) **Graph G:**

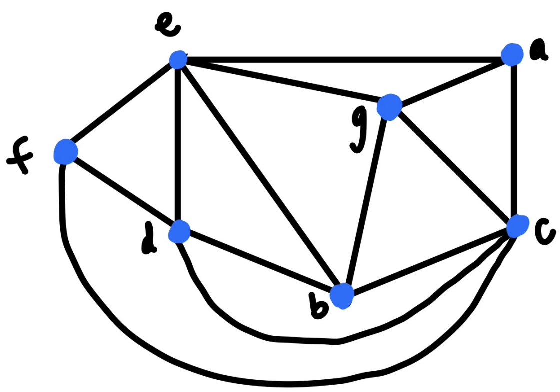


None of the vertices in graph G have a degree of 5, so therefore we can rule out that K_5 is a possible subdivision of G. The vertices in G have the following degrees:

$$deg(a) = deg(b) = deg(c) = deg(e) = deg(g) = 4$$
$$deg(d) = deg(f) = 3$$

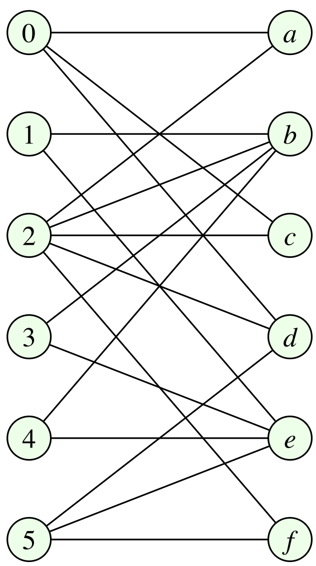
All vertices have at least a degree of 3, so we can now try to create the same graph in the form of $K_{3,3}$. G's vertex set can be partitioned into two disjoint subsets L = {c,f,e} and R = {a,g,d}, such that every edge has one endpoint in L and the other endpoint in R. We can see that G has a subgraph that is homeomorphic to $K_{3,3}$ so according to Kuratowski's Theorem, **G is nonplanar**.

(ii) **Graph H:**
Graph H is **planar** because it has the following planar embedding:

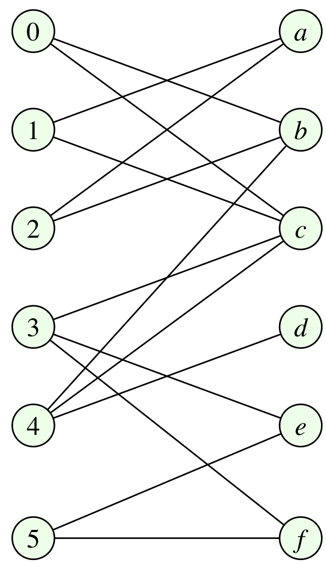


Problem 2. You are given two bipartite graphs G and H below. For each graph determine whether it has a perfect matching. Justify your answer, either by listing the edges that are in the matching or using Hall's Theorem to show that the graph does not have a perfect matching.

graph G



graph H



Solution:

(i) **graph G:**
Using Hall's Theorem: G is a graph with $|L| = |R|$ that has a perfect matching if and only if each set $X \subseteq L$ satisfies $|N(X) \geq |X|$

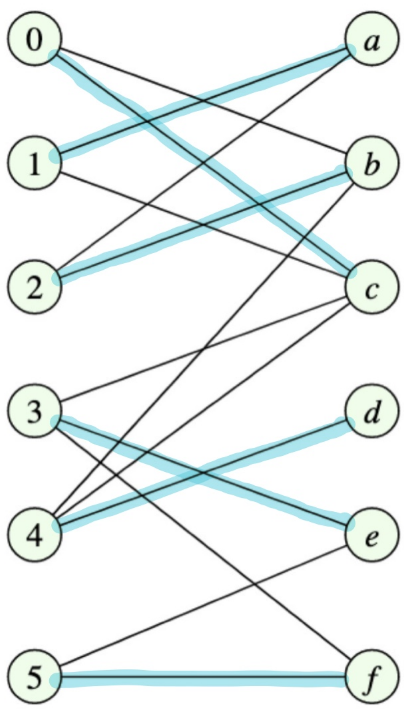
The endpoints for each vertex in G are as follows:

$$0 = \{a, c, d\}$$
$$1 = \{b, e\}$$
$$2 = \{a, b, c, d, f\}$$
$$3 = \{b, e\}$$
$$4 = \{b, e\}$$
$$5 = \{d, e, f\}$$

Let X = {1,3,4}
Let N(X) = {b,e}
$|N(X)| = 2$ and $|X| = 3$
$|N(X)| \leq |X|$ which does not satisfy Hall's Theorem.

Therefore, G is not a perfect matching.

(ii) **graph H:**
This is a perfect matching because $M = \{(0, c), (1, a), (2, b), (3, e), (4, d), (5, f)\}$



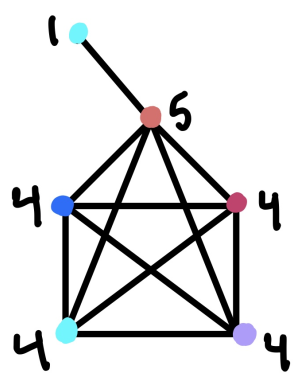
Problem 3. (a) For each degree sequence below, determine whether there is a graph with 6 vertices where vertices have these degrees. If a graph exists, (i) draw it, (ii) find the chromatic number and justify. If it doesn't, justify that it doesn't exist.

(a1) $4, 4, 4, 3, 3, 1$.
(a2) $5, 4, 4, 4, 4, 1$.
(a3) $5, 5, 3, 3, 3, 1$.

Solution:

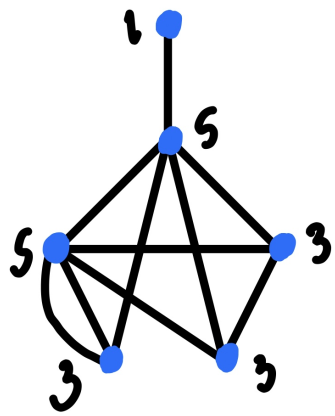
(a1) $\sum$deg(v) $= 4 + 4 + 4 + 3 + 3 + 1 = 19$, 19 is odd.
A graph is connected if there is a path between every pair of vertices $\implies$ $\sum$deg(v) have to be even. Therefore, a graph cannot exist if $\sum$deg(v) is odd.

(a2) $\sum$deg(v) $= 5 + 4 + 4 + 4 + 4 + 1 = 22$, 22 is even. Also, according to Handshaking Lemma, $\sum$deg(v) $= 2|E| \implies 22 = 2|E| \implies E(edges) = 11$. Therefore, it can form a graph with 11 edges like the one below



As seen in the graph, the chromatic number is 5. There are 5 vertices in the graph where every vertex neighbors every other vertex. There is also an extra vertex sticking of degree 1 that only neighbors 1 other vertex. So the least number of colors needed for this graph is 5 where each adjacent vertex does not have the same color

(a3) $\sum$deg(v) $= 5 + 5 + 3 + 3 + 3 + 1 = 20$, 20 is even. Although we get an even number, this cannot form a simple graph because there are 2 vertices with a degree of 5, which means each of them have to be connected to 5 other vertices. However, since there are only a total of 6 vertices, only one vertex can be connected to 5 others, whereas the others can only connect with a maximum of 4 other vertices. Although it cannot form a simple graph, a graph with multiple edges does exist for this degree sequence:

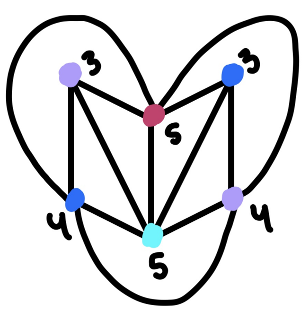


(b) For each degree sequence below, determine whether there is a planar graph with 6 vertices where vertices have these degrees. If a graph exists, (i) draw it, (ii) find the chromatic number and justify. If it doesn't, justify that it doesn't exist.

(b1) $5, 5, 4, 4, 3, 3$.
(b2) $5, 5, 4, 4, 4, 4$.

Solution:

(b1) $\sum$deg(v) $= 5+5+4+4+3+3 = 24$, 24 is even. Also, according to Handshaking Lemma, $\sum$deg(v) $= 2|E| \implies 24 = 2|E| \implies E(edges) = 12$. Therefore, it can form a planar graph with 12 non-intersecting edges such as the one below



As seen in the graph, the least number of colors needed in the graph so that no adjacent vertices can have the same color, is 4. Therefore, the chromatic number of the graph is 4.

(b2) $\sum$deg(v) $= 5+5+4+4+4+4 = 26$, 26 is even. Also, according to Handshaking Lemma, $\sum$deg(v) $= 2|E| \implies 26 = 2|E| \implies E(edges) = 13$.
Let n = number of vertices = 6 and m = number of edges = 13.
According to Corollary 1 of Euler's Formula, if graph G is a connected planar graph with $n \geq 3$, then $m \leq 3n - 6$.

$$m \leq 3n - 6$$
$$13 \leq 3 \cdot 6 - 6$$
$$13 \nleq 12$$

So therefore, there is not a planar graph for this degree sequence.

Academic integrity declaration. We, Binh Le and Hugo Wan, did the assignment together.