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CS 111 ASSIGNMENT 3

Problem 1: a) Consider the following linear homogeneous recurrence relation: $R_n = 4R_{n-1} - 3R_{n-2}$. It is known that: $R_0 = 1$, $R_2 = 5$. Find R_3.
Solution:
If n = 2, then $R_2 = 4R_{2-1} - 3R_{2-2} \implies R_2 = 4R_1 - 3R_0$
$\implies 5 = 4R_1 - 3 * 1 \implies 5 = 4R_1 - 3 \implies 8 = 4R_1 \implies R_1 = 2$
To find R_3, n = 3, then $R_3 = 4R_{3-1} - 3R_{3-2} \implies R_3 = 4R_2 - 3R_1$
$\implies R_3 = 4 * 5 - 3 * 2 \implies R_3 = 20 - 6 \implies R_3 = 14$
Answer: $R_3 = 14$.

b) Determine the general solution of the recurrence equation if its characteristic equation has the following roots: 1, -2, -2, 2, 7, 7.
Solution:
The general solution of the recurrence equation:
$R_n = a_1 * (1)^n + a_2 * (-2)^n + a_3 * n(-2)^n + a_4 * (2)^n + a_5 * (7)^n + a_6 * n(7)^n$

c) Determine the general solution of the recurrence equation $A_n = 256A_{n-4}$.
Solution:
Characteristic equation is $x^4 = 256$. Its roots are 4 and -4.
General form of the solution: $A_n = a_1 * 4^n + a_2 * (-4)^n$

d) Find the general form of the particular solution of the recurrence $B_n = 3B_{n-2} - 2B_{n-3} + 2$.
Solution:
Characteristic equation and its roots: $x^3 - 3x - 2 = 0, x_1 = 1, x_2 = 1, x_3 = -2$
General solution of the homogeneous equation: $f'_n = a_1(1)^n + a_2n(1)^n + a_3(-2)^n$
Particular solution of the non-homogeneous equation: $f"_n = B$.
(Answer) The general form of the particular solution of the recurrence: $B = 3B(n-2) - 2B(n-3) + 2$
Plug in: $\implies n = 3, B = 3B - 0 + 2 \implies 2B = -2 \implies B = -1$
The general solution: $B_n = a_1(1)^n + a_2n(1)^n + a_3(-2)^n - 1$

Problem 2: Solve the following recurrence equations:
a)

$$\begin{aligned} f_n &= f_{n-1} + 4f_{n-2} + 2f_{n-3} \\ f_0 &= 0 \\ f_1 &= 1 \\ f_2 &= 4 \end{aligned}$$

Show your work (all steps: the characteristic polynomial and its roots, the general solution, using the initial conditions to compute the final solution.)
Solution:
The first step is to find the characteristic polynomial, which is $x^3 - x^2 - 4x - 2 = 0$, and from that, we can factor it into $(x+1)(x^2 - 2x - 2)$. One of the roots we get is -1. Then we use the quadratic formula to find

the roots of x^2-2x-2, and we get $1+\sqrt{3}, 1-\sqrt{3}$. So in sum,
Characteristic polynomial and its roots: $x^3-x^2-4x-2=0$, $x=-1, 1+\sqrt{3}, 1-\sqrt{3}$
General solution: $f_n = a_1(-1)^n + a_2(1+\sqrt{3})^n + a_3(1-\sqrt{3})^n$
Using the initial conditions:
$f_0 = 0 = a_1(-1)^0 + a_2(1+\sqrt{3})^0 + a_3(1-\sqrt{3})^0 = a_1+a_2+a_3 \implies a_1+a_2+a_3=0$
$f_1 = 1 = a_1(-1)^1 + a_2(1+\sqrt{3})^1 + a_3(1-\sqrt{3})^1 \implies -a_1 + a_2(1+\sqrt{3}) + a_3(1-\sqrt{3}) = 1$
$\implies -a_1 + a_2 + \sqrt{3}a_2 + a_3 - \sqrt{3}a_3 = 1$
$f_2 = 4 = a_1(-1)^2 + a_2(1+\sqrt{3})^2 + a_3(1-\sqrt{3})^2 \implies a_1 + a_2(4+2\sqrt{3}) + a_3(4-2\sqrt{3}) = 4$
$\implies a_1 + 4a_2 + 2\sqrt{3}a_2 + 4a_3 - 2\sqrt{3}a_3 = 4$
After solving these systems of equations, we find that:
$a_1 = 2, a_2 = \dfrac{5-6\sqrt{3}}{2\sqrt{3}} + 2, a_3 = \dfrac{5\sqrt{3}-6}{6}$
The final solution is: $2(-1)^n + (\dfrac{5-6\sqrt{3}}{2\sqrt{3}} + 2)(1+\sqrt{3})^n + (\dfrac{5\sqrt{3}-6}{6})(1-\sqrt{3})^n$.

b)

$$\begin{aligned} t_n &= t_{n-1} + 2t_{n-2} + 2^n \\ t_0 &= 0 \\ t_1 &= 2 \end{aligned}$$

Show your work (all steps: the associated homogeneous equation, the characteristic polynomial and its roots, the general solution of the homogeneous equation, computing a particular solution, the general solution of the non-homogeneous equation, using the initial conditions to compute the final solution.)
Solution:
Characteristic polynomial and its roots: $x^2 - x - 2 = 0, x_1 = -1, x_2 = 2$
General solution of the homogeneous equation: $f'_n = a_1(-1)^n + a_2(2)^n, q(n) = 2^n$.
A particular solution of the form: $f"_n = \beta n 2^n$,
$\beta n 2^n = \beta(n-1)2^{n-1} + 2\beta(n-2)2^{n-2} + 2^n$
$n = 2, 8\beta = 2\beta + 0 + 4 \implies 6\beta = 4 \implies \beta = \frac{2}{3}$
The general form: $t_n = a_1(-1)^n + a_2(2)^n + \frac{2}{3}n2^n$
Using the initial conditions:
$f_0 = a_1(-1)^0 + a_2(2)^0 + \frac{2}{3} * 0 * 2^0 \implies a_1 + a_2 = 0$
$f_1 = a_1(-1)^1 + a_2(2)^1 + \frac{2}{3} * 1 * 2^1 \implies -a_1 + 2a_2 = 2$
$a_1 = -\frac{2}{9}, a_2 = \frac{2}{9}$
The final solution: $t_n = -\frac{2}{9}(-1)^n + \frac{2}{9}(2)^n + \frac{2}{3}n2^n$

Problem 3: We want to tile an $n \times 1$ strip with 1×1 tiles that are green (G), blue (B), and red (R), 2×1 purple (P) and 2×1 orange (O) tiles. Green, blue and purple tiles cannot be next to each other, and there should be no two purple or three blue or green tiles in a row (for ex., GGOBR is allowed, but GGGOBR, GROPP and PBOBR are not). Give a formula for the number of such tilings. Your solution must include a recurrence equation (with initial conditions!), and a full justification. You do not need to solve it.

Solution:

$$\begin{aligned} t_n &= 3t_{n-1} + 2t_{n-2} \\ t_0 &= 1 \\ t_1 &= 3 \end{aligned}$$

For $n = 0$, $t_0 = 1$ because this is just one tiling that has no tiles. For $n = 1$, $t_1 = 3$ because we have three 1x1 tiles. Together, we get the above as the recurrence equation because we have two different types of tilings: 1x1 tiles and 2x1 tiles. We have three 1x1 tiles, representing $3t_{n-1}$, and two 2x1 tiles, representing $2t_{n-2}$. By adding those two expressions, we will get the number of tilings.

Academic integrity declaration. We, Hugo Wan and Binh Le, did the assignment together.