Binh Le SID

Hugo Wan SID

CS/MATH111 ASSIGNMENT 4

Problem 1: Give an asymptotic estimate, using the Θ-notation, of the number of letters printed by the algorithms given below. Give a complete justification for your answer, by providing an appropriate recurrence equation and its solution.

(a) **algorithm** PrintAs(n)
    if $n \le 1$ **then**
        print("AAA")
    else
        for $j \leftarrow 1$ **to** n^3
            do print("A")
        for $i \leftarrow 1$ **to** 5 **do**
            PrintAs($\lfloor n/2 \rfloor$)

(b) **algorithm** PrintBs(n)
    if $n \ge 4$ **then**
        for $j \leftarrow 1$ **to** n^2
            do print("B")
        for $i \leftarrow 1$ **to** 6 **do**
            PrintBs($\lfloor n/4 \rfloor$)
        for $i \leftarrow 1$ **to** 10 **do**
            PrintBs($\lceil n/4 \rceil$)

(c) **algorithm** PrintCs(n)
    if $n \le 2$ **then**
        print("C")
    else
        for $j \leftarrow 1$ **to** n
            do print("C")
        PrintCs($\lfloor n/3 \rfloor$)
        PrintCs($\lfloor n/3 \rfloor$)
        PrintCs($\lfloor n/3 \rfloor$)
        PrintCs($\lfloor n/3 \rfloor$)

(d) **algorithm** PrintDs(n)
    if $n \ge 5$ **then**
        print("D")
        print("D")
        if ($x \equiv 0 \pmod 2$) **then**
            PrintDs($\lfloor n/5 \rfloor$)
            PrintDs($\lceil n/5 \rceil$)
            $x \leftarrow x + 3$
        else
            PrintDs($\lceil n/5 \rceil$)
            PrintDs($\lfloor n/5 \rfloor$)
            $x \leftarrow 5x + 3$

In part (d), variable x is a global variable initialized to 1.

Solution:

(a) Recurrence equation: $A_n = 5A(\frac{n}{2}) + n^3$
Apply Master Theorem with: a = 5, b = 2, c = 1, d = 3.
Case $a < b^d$ or $5 < 2^3 \implies A(n) = \theta(n^3)$.

(b) Recurrence equation: $B_n = (6B(\frac{n}{4}) + 10B(\frac{n}{4})) + n^2 = 16B(\frac{n}{4}) + n^2$
Apply Master Theorem with: a = 16, b = 4, c = 1, d = 2.
Case $a = b^d$ or $16 = 4^2 \implies B(n) = \theta(n^2 logn)$.

(c) Recurrence equation: $C_n = 4C(\frac{n}{3}) + n$
Apply Master Theorem with: a = 4, b = 3, c = 1, d = 1.
Case $a > b^d$ or $4 > 3^1 \implies C(n) = \theta(n^{log_3 4})$.

(d) Recurrence equation: $D_n = 2D(\frac{n}{5}) + 2$
Apply Master Theorem with: a = 2, b = 5, c = 2, d = 0.
Case $a > b^d$ or $2 > 5^0 \implies D(n) = \theta(n^{log_5 2})$.

Problem 2: We have three sets A, B, C with the following properties:

(a) $|B| = 2|A|$, $|C| = 3|A|$,

(b) $|A \cap B| = 18$, $|A \cap C| = 20$, $|B \cap C| = 24$,

(c) $|A \cap B \cap C| = 11$,

(d) $|A \cup B \cup C| = 129$.

Use the inclusion-exclusion principle to determine the number of elements in A. Show your work.

Solution:

Inclusion-exclusion principle with 3 sets, A, B, and C
$\implies |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$$\text{Let } x = |A|$$
$$x + 2x + 3x - 18 - 20 - 24 + 11 = 129$$
$$6x - 51 = 129$$
$$6x = 180$$
$$x = 30$$

So the number of elements in A, $|A| = 30$ elements.

Problem 3: A company, Nice Inc., will award 40 fellowships to high-achieving UCR students from four different majors: computer science, biology, political science and history. They decided to give fellowship awards to at least 7 students majoring in computer science and at most 8 biology majors. The number of political science and history majors should be between 5 and 10 students each. How many possible lists of awardees are there? You need to give a complete derivation for the final answer, using the method developed in class. (Brute force listing of all lists will not be accepted.)

Solution:

$$a = \text{computer science major}$$
$$b = \text{biology major}$$
$$c = \text{political science major}$$
$$d = \text{history major}$$

We have 40 in total students ($a + b + c + d = 40$) in 4 different majors. (k = 4). The boundaries of each major are as follows:

$$\begin{aligned}
&a \geq 7 \\
&b \leq 8 \\
&5 \leq c, d \leq 10
\end{aligned}$$

We need to manipulate the expressions so that the lower bounds of each major are all 0. Let's give a an upper bound of 40 and let $a' = a - 7, c' = c - 5$, and $d' = d - 5$. We can rewrite the equation as $a' + b + c' + d' = 23$ with the following constraints:

$$\begin{aligned}
&0 \leq a' \leq 33 \\
&0 \leq b \leq 8 \\
&0 \leq c' \leq 5 \\
&0 \leq d' \leq 5
\end{aligned}$$

The value of m is now $m = 23$ because $a' + b + c' + d' = 40 - 7 - 5 - 5 = 23$

Since a' has no upper bound as $a' \geq 0$, a' is not needed in computation. So,

$$S(0 \leq b \leq 8 \wedge 0 \leq c' \leq 5 \wedge 0 \leq d' \leq 5) = S_{total} - S(b \geq 9 \vee c' \geq 6 \vee d' \geq 6)$$
$$S_{total} = \binom{23 - 0 + 4 - 1}{4 - 1} = \binom{26}{3} = 2600$$

Now we use the Inclusion-Exclusion Principle to find $S(b \geq 9 \vee c' \geq 6 \vee d' \geq 6)$, when m = 23 and k = 4.

$$\begin{gathered}
S(b \geq 9 \vee c' \geq 6 \vee d' \geq 6) = S(b \geq 9) + S(c' \geq 6) + S(d' \geq 6) - S(b \geq 9 \wedge c' \geq 6) \\
-S(b \geq 9 \wedge d' \geq 6) - S(c' \geq 6 \wedge d' \geq 6) + S(b \geq 9 \wedge c' \geq 6 \wedge d' \geq 6) \\
= \binom{17}{3} + 2 * \binom{20}{3} - 2 * \binom{11}{3} - \binom{14}{3} + \binom{5}{3} = 680 + (2 * 1140) - (2 * 165) - 364 + 10 = 2276.
\end{gathered}$$

Since $S(0 \leq b \leq 8 \wedge 0 \leq c' \leq 5 \wedge 0 \leq d' \leq 5) = S_{total} - S(b \geq 9 \vee c' \geq 6 \vee d' \geq 6) \implies 2600 - 2276 = 324.$

Thus, there are 324 possible lists of awardees.

Academic integrity declaration. We, Binh Le and Hugo Wan, did the assignment together.