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CS111 Homework 1
due Wednesday, Jan 20th

Problem 1. Give an asymptotic estimate for the number h(n) of "Hello"s printed by Algorithm Print-Hellos below. Your solution *must* consist of the following steps:

(a) First express $h(n)$ using the summation notation $\sum$

(b) Next, give the closed-form expression[1] for $h(n)$

(c) Finally, give the asymptotic value of $h(n)$ using the Θ-notation

Show your work and include justification for each step
Solution:

$$\begin{aligned}
h(n) &= \sum_{i=1}^{n}(i^2+i) + \sum_{i=1}^{n^2}(i) \\
&= \sum_{i=1}^{n}(i^2) + \sum_{i=1}^{n}(i) + \sum_{i=1}^{n^2}(i) \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + \frac{n^2(n^2+1)}{2} \\
&= \frac{(n^2+n)(2n+1)}{6} + \frac{3n(n+1)}{6} + \frac{3n^2(n^2+1)}{6} \\
&= \frac{2n^3+n^2+2n^2+n}{6} + \frac{3n^2+3n}{6} + \frac{3n^4+3n^2}{6} \\
&= \frac{2n^3+3n^2+n}{6} + \frac{3n^4+6n^2+3n}{6} \\
&= \frac{3n^4+2n^3+9n^2+4n}{6} \\
h(n) &= \frac{1}{2}n^4 + \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{2}{3}n = \Theta(n^4)
\end{aligned}$$

Problem 2.

(a) Use properties of quadratic functions to prove that $3x^2 \geq (x+1)^2$ for all real $x \geq 4$

(b) Use mathematical induction and the inequality from part (a) to prove that $3^n \geq 2^n + 3n^2$ for all integers $n \geq 4$

(c) Let $g(n) = 2^n + 3n^2$ and $h(n) = 3^n$. Using the inequality from part (b), prove that $g(n) = O(h(n))$. You need to give a rigorous proof derived directly from the definition of the O-notation, without using any theorems from class. (First, give a complete statement of the definition. Next, show how $g(n) = O(h(n))$ follows from this definition.)

Solutions

(a) $3x^2 - (x+1)^2 \geq 0$
$3x^2 - (x+1)^2 \geq 0$
$3x^2 - x^2 - 2x - 1 \geq 0$
$2x^2 - 2x - 1 \geq 0$
Use the quadratic formula to find the roots
$= \dfrac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$
$= \dfrac{2 \pm \sqrt{12}}{4}$

One of the roots evaluates to 1.366, making the quadratic expression true. The parabola for this quadratic is positive and keeps increasing on the right hand side of the graph. So the x-values increase, and as the x-values get bigger than 4, the function continues to increase, so this proves $3x^2 - (x+1)^2 \geq 0$

(b) Let $3^n \geq (2^n + 3n^2)$ ... (A).
Base case: $n = 4$. For n = 4 the LHS of the inequality of (A) becomes $3^4 = 81$. For n = 4 the RHS of the inequality of (A) becomes $2^4 + 3(4)^2 = 64$. So, $81 \geq 64$. Therefore, $3^4 \geq 2^4 + 3(4)^2$. Hence, (A) is true for $n = 4$.

Hypothesis (assume it's true for n = m):
Let (A) is true for some n = m, m ¿ 4. Then, $3^m \geq 2^m + 3m^2$ ... (B).

Inductive step: n = m+1:
$3^(m+1) = 3.3^m \geq 3.[2^m + 3m^2]$ by (B) $= 3.2^m + 9m^2 \geq 2.2^m + 9m^2 = 2^(m+1) + 9m^2 \geq [2^(m+1) + 3m^2]$ by (A). So, (A) is true for n=m+1.

Hence, by Mathematical Induction, we have (A) is true for all $n \geq 4$.
i.e. $3^n \geq 2^n + 3n^2$ for all $n \geq 4$.

(c) The definition of O-notation describes the upper bound of a function. And in this case, let g(n) and h(n) be two functions defined on the set of natural numbers. Then g(n) = O(h(n)) if and only if there exists a positive constants c and n-zero such that $|g(n)| \leq c|h(n)|$ for all $n \leq n$-zero.

To prove that g(n) = O(h(n)), we need to find positive constants c and n-zero such that $|g(n)| \leq c|h(n)|$ for all $n \leq n$-zero.

Given that g(n) = $2^n + 3n^2$ and h(n) = 3^n and using the inequality $3^n \geq 2^n + 3n^2$ from question 2b, we can use c=1 and n-zero = 4.

Therefore, for all n $\geq$ 4:
$|g(n)| = |2^n + 3n^2| \geq |2^n| + |3n^2| \geq 2^n + 3n^2 = g(n)$. And, $|h(n)| = |3^n| = 3^n$.
So, g(n) $\leq$ h(n) for all n $\geq$ 4. Therefore, we can say that g(n) = O(h(n)).

Problem 3. Give asymptotic estimates, using Θ-notation for the following functions:

(a) $3n^3 + 5n^2 - 2n + 4$

(b) $3n^2logn + 2\sqrt{n} + n^3$

(c) $3n + \dfrac{n^2}{logn} + nlog^3n$

(d) $7 \cdot n^4 + n^3log^2n + 3^n$

(e) $log^9n + n^34^n + 5^n$

Solutions

(a) $\Theta(n^3)$
$f(n) = 3n^3 + 5n^2 - 2n + 4$
$3n^3 + 5n^2 - 2n + 4 \geq 3n^3 + 0 - 2n^3 + 0$
$3n^3 + 0 - 2n^3 + 0 = 1 \cdot n^3$ Needed to put this in the form of $c \cdot n^3$
$f(n) = \Omega(n^3)$

$3n^3 + 5n^2 - 2n + 4 \leq 3n^3 + 5n^3 + 0 + 4n^3$
$= 12n^3$
$f(n) = O(n^3)$
Therefore, $\Theta(n^3)$

(b) $\Theta(n^3)$
The order of increasing time complexity is $n < n^2logn < n^3$

(c) $\Theta(\dfrac{n^2}{logn})$
Compare $\dfrac{n^2}{logn}$ and $nlog^3n$. Factor out the n from both expressions and then compare $\dfrac{n}{logn}$ and log^3n. Multiply both expressions by $logn$.
$logn * \dfrac{n}{logn} = n$ and $logn * log^3n = log^4n$
$n > log^4n$ or $\dfrac{n^2}{logn} > nlog^3n$
$3n < \dfrac{n^2}{logn}$
So the order of increasing time complexity is $n < nlog^3n < \dfrac{n^2}{logn}$

(d) $\Theta(3^n)$ The order of increasing time complexity is $n^3log^2n < n^4 < 3^n$

(e) $\Theta(5^n)$
$log^9n < n^34^n$ and $log^9n < 5^n$
So now we just compare n^34^n and 5^n
We want to get 5^n to have something similar to n^34^n so can rewrite 5^n as $(\frac{5}{4})^n \cdot 4^n$.
Now we can compare n^3 and $(\frac{5}{4})^n$.
$n^3 < (\frac{5}{4})^n$
The order of increasing time complexity is therefore $log^9n < n^34^n < 5^n$ so the asymptotic estimate is $\Theta(5^n)$

Academic integrity declaration. We, Binh Le and Hugo Wan, did the assignment together.