CS 450
Sample Problem Set

Howard Yang

Dec. 29, 2022

0.1 Theoretical Problems

Question-1: Consider using the fixed-point iteration to find the root of the following equation

Start: 0.68
tolerance: 1e-9
Maxstep: 1000

$$x^3 + x - 1 = 0. \tag{1}$$

One can construct different forms of fixed-point relationships, e.g.,

1) $x = g_1(x)$, with $g_1(x) = 1 - x^3$; not convergent.
2) $x = g_2(x)$, with $g_2(x) = \sqrt[3]{1 - x}$; convergent iter-28
3) $x = g_3(x)$, with $g_3(x) = \frac{1+2x^3}{1+3x^2}$. convergent iter-2

Suppose the algorithm starts from the same initial point, do all the above fixed-point iterations converge? 2) 3) Which one converges the fastest? 3 Can you construct a fixed-point iteration that has the fastest convergence rate amongst all the possible candidates?

$x^2 - 5 = 0$

Start: 2
tolerance: 1e-9
maxstep: 1000

Question-2: Which of the following fixed-point iterations converge to $\sqrt{5}$?

1) $x = h_1(x)$, with $h_1(x) = \frac{4}{5}x + \frac{1}{x}$; converge iter-28
2) $x = h_2(x)$, with $g_2(x) = \frac{x}{2} + \frac{5}{2x}$; converge iter-3
3) $x = h_3(x)$, with $g_3(x) = \frac{x+5}{x+1}$. converge iter-26

Rank the ones converge from fastest and slowest.
2) 1)

Question-3: Consider using the Conjugate Gradient Method to solve the following optimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \quad f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^T\boldsymbol{x} \tag{2}$$

where $\boldsymbol{A}$ is a symmetric and positive definite matrix. Let us stay with the notations adopted in the lecture notes, and answer the following questions.

$$\alpha_K = \frac{d_K^T b - d_K^T A x_K}{d_K^T A d_K}$$

(a) Derive the expression for the step size α_k.

(b) Show that the residual error $\boldsymbol{r}_k$ at iteration k is orthogonal to the previous ones, i.e., $\boldsymbol{r}_k^T \boldsymbol{r}_j = 0$ for $j < k$.

(c) Show that the update direction $\boldsymbol{d}_k$ at iteration k is conjugate to the previous ones, i.e., $\langle \boldsymbol{d}_k, \boldsymbol{d}_j \rangle_{\boldsymbol{A}} = 0$ for $j < k$.

(d) What is the convergence rate of the conjugate gradient method?

Question-4: For Lagrange polynomial interpolation of m data points $(x_i, y_i), i = 1, \cdots, m$.

(a) What is the degree of each polynomial function $L_j(x)$ in the Lagrange basis? $m-1$

(b) How many polynomials of degree $m+1$ interpolates these data points? inf

(c) How many polynomials of degree m interpolates these data points? inf

(d) How many polynomials of degree $m-1$ interpolates these data points? 1

(e) How many polynomials of degree $m-2$ interpolates these data points? 0