## CS 450Sample Problem Set

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## 0.1Theoretical Problems

Question-1: Consider using the fixed-point iteration to find the root of the following equation

 $x^3+x-1=0.$ When the can construct different forms of fixed-point relationships, e.g.,  $1) \ x=g_1(x), \text{ with } g_1(x)=1-x^3; \quad \text{not convergent.}$   $2) \ x=g_2(x), \text{ with } g_2(x)=\sqrt[3]{1-x}; \quad \text{convergent.}$ (1)

1) 
$$x = g_1(x)$$
, with  $g_1(x) = 1 - x^3$ ; not convergent

2) 
$$x = g_2(x)$$
, with  $g_2(x) = \sqrt[3]{1-x}$ ;

3) 
$$x = g_3(x)$$
, with  $g_3(x) = \frac{1+2x^3}{1+3x^2}$ .

Suppose the algorithm starts from the same initial point, do all the above fixedpoint iterations converge? Which one converges the fastest? Can you construct a fixed-point iteration that has the fastest convergence rate amongst all the possible candidates?

Question-2: Which of the following fixed-point iterations converge to  $\sqrt{5}$ ?

1)  $x = h_1(x)$ , with  $h_1(x) = \frac{4}{5}x + \frac{1}{x}$ ; converge iter 28

2)  $x = h_2(x)$ , with  $g_2(x) = \frac{x}{2} + \frac{5}{2x}$ ; converge iter 3

3)  $x = h_3(x)$ , with  $g_3(x) = \frac{x+5}{x+1}$ .

1) 
$$(x) = h_1(x)$$
, with  $h_1(x) = \frac{4}{5}x + \frac{1}{x}$ ; converge ; ter-28

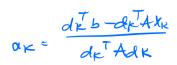
2) 
$$x = h_2(x)$$
, with  $g_2(x) = \frac{x}{2} + \frac{5}{2x}$ ;

3) 
$$x = h_3(x)$$
, with  $g_3(x) = \frac{x+5}{x+1}$ .

Question-3: Consider using the Conjugate Gradient Method to solve the following optimization problem

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \qquad f(\boldsymbol{x}) = \frac{1}{2} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} - \boldsymbol{b}^T \boldsymbol{x}$$
 (2)

where A is a symmetric and positive definite matrix. Let us stay with the notations adopted in the lecture notes, and answer the following questions.



- (a) Derive the expression for the step size  $\alpha_k$ .
- (b) Show that the residual error  $r_k$  at iteration k is orthogonal to the previous ones, i.e.,  $r_k^T r_j = 0$  for j < k.
- (c) Show that the update direction  $d_k$  at iteration k is conjugate to the previous ones, i.e.,  $\langle d_k, d_j \rangle_A = 0$  for j < k.
  - (d) What is the convergence rate of the conjugate gradient method?

**Question-4:** For Lagrange polynomial interpolation of m data points  $(x_i, y_i), i = 1, \dots, m$ .

- (a) What is the degree of each polynomial function  $L_j(x)$  in the Lagrange basis?
  - (b) How many polynomials of degree m+1 interpolates these data points?
  - (c) How many polynomials of degree m interpolates these data points?
  - (d) How many polynomials of degree m-1 interpolates these data points?
  - (e) How many polynomials of degree m-2 interpolates these data points?