Universal constructions, limits, and colimits[edit]

Main articles: Universal property and Limit (category theory)

Using the language of category theory, many areas of mathematical study can be categorized. Categories include sets, groups and topologies.

Each category is distinguished by properties that all its objects have in common, such as the empty set or the product of two topologies, yet in the definition of a category, objects are considered atomic, i.e., we *do not know* whether an object *A* is a set, a topology, or any other abstract concept. Hence, the challenge is to define special objects without referring to the internal structure of those objects. To define the empty set without referring to elements, or the product topology without referring to open sets, one can characterize these objects in terms of their relations to other objects, as given by the morphisms of the respective categories. Thus, the task is to find *universal properties* that uniquely determine the objects of interest.

Numerous important constructions can be described in a purely categorical way if the *category limit* can be developed and dualized to yield the notion of a *colimit*.

Equivalent categories[edit]

Main articles: Equivalence of categories and Isomorphism of categories

It is a natural question to ask: under which conditions can two categories be considered *essentially the same*, in the sense that theorems about one category can readily be transformed into theorems about the other category? The major tool one employs to describe such a situation is called *equivalence of categories*, which is given by appropriate functors between two categories. Categorical equivalence has found numerous applications in mathematics.

Further concepts and results[edit]

The definitions of categories and functors provide only the very basics of categorical algebra; additional important topics are listed below. Although there are strong interrelations between all of these topics, the given order can be considered as a guideline for further reading.

- The functor category D^C has as objects the functors from *C* to *D* and as morphisms the natural transformations of such functors. The Yoneda lemma is one of the most famous basic results of category theory; it describes representable functors in functor categories.
- Duality: Every statement, theorem, or definition in category theory has a *dual* which is essentially obtained by "reversing all the arrows". If one statement is true in a category *C*then its dual is true in the dual category C^{op}. This duality, which is transparent at the level of category theory, is often obscured in applications and can lead to surprising relationships.
- Adjoint functors: A functor can be left (or right) adjoint to another functor that maps in the opposite direction. Such a pair of adjoint functors typically arises from a construction defined by a universal property; this can be seen as a more abstract and powerful view on universal properties.

Higher-dimensional categories[edit]

Main article: Higher category theory

Many of the above concepts, especially equivalence of categories, adjoint functor pairs, and functor categories, can be situated into the context of *higher-dimensional categories*. Briefly, if we consider a morphism between two objects as a "process taking us from one object to another", then higher-dimensional categories allow us to profitably generalize this by considering "higher-dimensional processes".

For example, a (strict) 2-category is a category together with "morphisms between morphisms", i.e., processes which allow us to transform one morphism into another. We can then "compose" these "bimorphisms" both horizontally and vertically, and we require a 2-dimensional "exchange law" to hold, relating the two composition laws. In this context, the standard example is **Cat**, the 2-category of all (small) categories, and in this example, bimorphisms of morphisms are simply natural transformations of morphisms in the usual sense. Another basic example is to consider a 2-category with a single object; these are essentially monoidal categories. Bicategories are a weaker notion of 2-dimensional categories in which the composition of morphisms is not strictly associative, but only associative "up to" an isomorphism.

This process can be extended for all natural numbers *n*, and these are called *n*-categories. There is even a notion of *ω-category* corresponding to the ordinal number ω.

Higher-dimensional categories are part of the broader mathematical field of higher-dimensional algebra, a concept introduced by Ronald Brown. For a conversational introduction to these ideas, see John Baez, 'A Tale of *n*-categories' (1996).

Historical notes[edit]

This section **needs additional citations for verification**. Please help improve this article by adding citations to reliable sources. Unsourced material may be challenged and removed.
Find sources: "Category theory" – news · newspapers · books · scholar · JSTOR *(November 2015) (Learn how and when to remove this template message)*

Main article: Timeline of category theory and related mathematics

> “It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation [...]”
>
> — *Samuel Eilenberg and Saunders Mac Lane, General theory of natural equivalences*[6]

In 1942–45, Samuel Eilenberg and Saunders Mac Lane introduced categories, functors, and natural transformations as part of their work in topology, especially algebraic topology. Their work was an important part of the transition from intuitive and geometric homology to axiomatic homology theory. Eilenberg and Mac Lane later wrote that their goal was to understand natural transformations. That required defining functors, which required categories.

Stanislaw Ulam, and some writing on his behalf, have claimed that related ideas were current in the late 1930s in Poland. Eilenberg was Polish, and studied mathematics in Poland in the 1930s. Category theory is also, in some sense, a continuation of the work of Emmy Noether (one of Mac Lane's teachers) in formalizing abstract processes; Noether realized that understanding a type of mathematical structure requires understanding the processes that preserve that structure. To achieve this understanding, Eilenberg and Mac Lane proposed an axiomatic formalization of the relation between structures and the processes that preserve them.

The subsequent development of category theory was powered first by the computational needs of homological algebra, and later by the axiomatic needs of algebraic geometry. General category theory, an extension of universal algebra having many new features allowing for semantic flexibility and higher-order logic, came later; it is now applied throughout mathematics.

Certain categories called topoi (singular *topos*) can even serve as an alternative to axiomatic set theory as a foundation of mathematics. A topos can also be considered as a specific type of category with two additional topos axioms. These foundational applications of category theory have been worked out in fair detail as a basis for, and justification of, constructive mathematics. Topos theory is a form of abstract sheaf theory, with geometric origins, and leads to ideas such as pointless topology.

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with applications in functional programming and domain theory, where a cartesian closed category is taken as a non-syntactic description of a lambda calculus. At the very least, category theoretic language clarifies what exactly these related areas have in common (in some abstract sense).

Category theory has been applied in other fields as well. For example, John Baez has shown a link between Feynman diagrams in Physics and monoidal categories.[7] Another application of category theory, more specifically: topos theory, has been made in mathematical music theory, see for example the book *The Topos of Music, Geometric Logic of Concepts, Theory, and Performance* by Guerino Mazzola.

More recent efforts to introduce undergraduates to categories as a foundation for mathematics include those of William Lawvere and Rosebrugh (2003) and Lawvere and Stephen Schanuel (1997) and Mirroslav Yotov (2012).

See also[edit]

- ***Category theory portal***
- ***Mathematics portal***

- Domain theory
- Enriched category theory
- Glossary of category theory
- Group theory
- Higher category theory
- Higher-dimensional algebra
- Important publications in category theory
- Lambda calculus
- Outline of category theory
- Timeline of category theory and related mathematics

Notes[edit]

1. ^ Some authors compose in the opposite order, writing fg or $f \circ g$ for $g \circ f$. Computer scientists using category theory very commonly write $f\,;\,g$ for $g \circ f$
2. ^ Note that a morphism that is both epic and monic is not necessarily an isomorphism! An elementary counterexample: in the category consisting of two objects A and B, the identity morphisms, and a single morphism f from A to B, f is both epic and monic but is not an isomorphism.