Title:	Association Rules
Course:	Machine Learning
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BOLOGNA BUSINESS SCHOOL

Alma Mater Studiorum Università di Bologna

Association Rules – Discovering co–occurrencies in a market basket

- Given a set of commercial transactions, find rules that will predict the occurrence of an item based on the occurrences of other items in the transaction
- Example of Association Rules
  - $\{\text{Diaper}\} \rightarrow \{\text{Beer}\}$,
  - $\{\text{Bread, Milk}\} \rightarrow \{\text{Coke, Eggs}\}$,
  - $\{\text{Beer, Bread}\} \rightarrow \{\text{Milk}\}$
    - Implication means co-occurrence, not causality!
    - The implication of Association Rules is different from that of logic (boolean): it can be true *with some level of truth*

TID	*Items*
1	Bread, Milk
2	Beer, Bread, Diaper, Eggs
3	Beer, Coke, Diaper, Milk
4	Beer, Bread, Diaper, Milk
5	Bread, Coke, Diaper, Milk

Market Basket Transactions

Definition: Frequent Itemset

- Itemset
  - A collection of one or more items
  - Example: $\{$Bread, Diaper, Milk$\}$
- k-itemset
  - An itemset that contains k items
- Support count (σ)
  - Frequency of occurrence of an itemset
  - E.g. $\sigma(\{\text{Bread, Diaper, Milk}\}) = 2$
- Support
  - Fraction of transactions that contain an itemset
  - E.g. $\sigma(\{\text{Bread, Diaper, Milk}\}) = 2/5$
- Frequent Itemset
  - An itemset whose support is greater than or equal to a *minsup* threshold

TID	*Items*
1	Bread, Milk
2	Beer, Bread, Diaper, Eggs
3	Beer, Coke, Diaper, Milk
4	Beer, Bread, Diaper, Milk
5	Bread, Coke, Diaper, Milk

Market Basket Transactions

Definition: Association Rule

- Association Rule
  - An expression of the form $A \Rightarrow C$, where A and C are itemsets
    - A =Antecedent and C =Consequent
  - Example: $\{$Diaper, Milk$\} \rightarrow \{$Beer$\}$
- Rule Evaluation Metrics
  - Support (sup)
    - Fraction of the N transactions that contain both A and C
  - Confidence (conf)
    - Measures how often all the items in C appear in transactions that contain A

TID	*Items*
1	Bread, Milk
1	Beer, Bread, Diaper, Eggs
3	Beer, Coke, Diaper, Milk
4	Beer, Bread, Diaper, Milk
5	Bread, Coke, Diaper, Milk

Market Basket Transactions

$$sup = \frac{\sigma(Beer, Diaper, Milk)}{N} = \frac{2}{5} = 0.4$$

$$conf = \frac{\sigma(Beer, Diaper, Milk)}{\sigma(Milk, Diaper)}$$

Why support and confidence?

- Rules with low support can be generated by random associations
- Rules with low confidence are not really reliable
- Nevertheless a rule with relatively low support but high confidence can represent an uncommon but interesting phenomenon

Association Rule Mining Task

- Given a set of transactions N, the goal of association rule mining is to find all rules having
  - support $\geqslant$ *minsup* threshold
  - confidence $\geqslant$ *minconf* threshold
- Brute-force approach:
  - List all possible association rules
  - Compute the support and confidence for each rule
  - Prune rules that fail the *minsup* and *minconf* thresholds

    $\Rightarrow$ Computationally prohibitive!

Mining Association Rules

Example of Rules:

$$
\begin{aligned}
\{Diaper, Milk\} &\Rightarrow \{Beer\} && (s = 0.4, c = 0.67) \\
\{Beer, Milk\} &\Rightarrow \{Diaper\} && (s = 0.4, c = 1.0) \\
\{Beer, Diaper\} &\Rightarrow \{Milk\} && (s = 0.4, c = 0.67) \\
\{Beer\} &\Rightarrow \{Diaper, Milk\} && (s = 0.4, c = 0.67) \\
\{Diaper\} &\Rightarrow \{Beer, Milk\} && (s = 0.4, c = 0.5) \\
\{Milk\} &\Rightarrow \{Beer, Diaper\} && (s = 0.4, c = 0.5)
\end{aligned}
$$

- All the rules above are binary partitions of the same itemset: {Beer,Diaper,Milk}
- Rules originating from the same itemset have identical support but can have different confidence

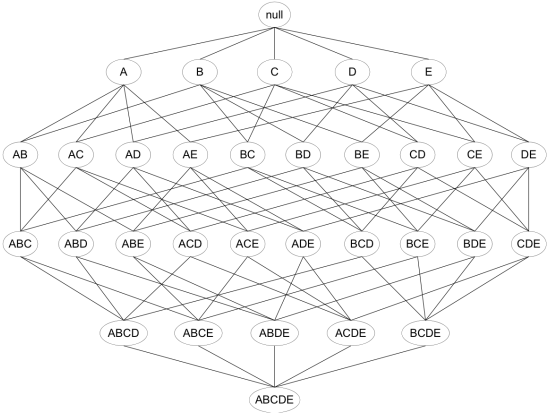
  $\Rightarrow$ we may decouple the support and confidence requirements

Mining Association Rules

- Two-step approach:
  - 1. Frequent Itemset Generation
    - Generate all itemsets whose support is greater than minsup
  - 2. Rule Generation
    - Generate high confidence rules from each frequent itemset, where each rule is a binary partitioning of a frequent itemset
- Frequent itemset generation is still computationally expensive

Frequent Itemset Generation

Given D items, there are $M = 2^D$ possible candidate itemsets



Frequent Itemset Generation

Brute-force approach:

- Each itemset in the lattice is a candidate frequent itemset
- Count the support of each candidate by scanning the database
- Match each transaction against every candidate
- Complexity: $\mathcal{O}(NWM) \Rightarrow$ Expensive

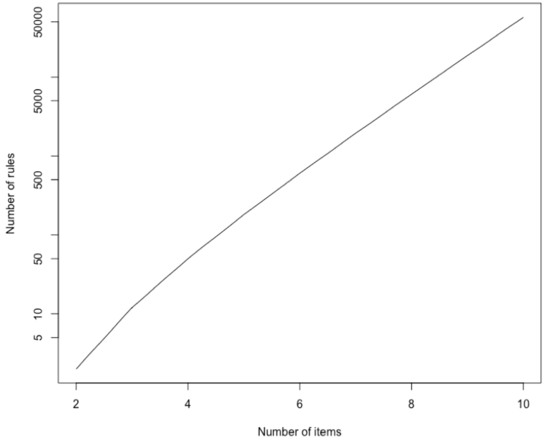
TID	*Items*
1	Bread, Milk
1	Beer, Bread, Diaper, Eggs
3	Beer, Coke, Diaper, Milk
4	Beer, Bread, Diaper, Milk
5	Bread, Coke, Diaper, Milk

$\leftarrow W \rightarrow$

Brute Force - Computational Complexity OPTIONAL

- Given D unique items:
  - Total number of itemsets $= 2^D$
  - Total number of possible association rules:

$$R = \sum_{k=1}^{D-1} \left(\binom{D}{k} \times \sum_{j=1}^{D-k} \binom{D-k}{j} \right)$$
$$= 3^D - 2^{D+1} + 1$$



Explanation of the formula

OPTIONAL

- count the number of ways to create an itemset that forms the left hand side of the rule
- for each size k itemset selected for the left-hand side, count the number of ways to choose the remaining $D - k$ items to form the right-hand side of the rule

Going deeper

- choose k of the D items for the left hand side of the rule, there are $\binom{D}{k}$ ways to do this
- there are $\binom{D-k}{i}$ ways to choose the right hand side of the rule, $1 \leqslant i \leqslant D - k$
- the double summation derives from the two points above
- the *binomial theorem* states that $\sum_{i=0}^{n} \binom{n}{i} x^i = (1 + x)^n$
- using the theorem for $x = 1$ and $x = 2$ leads to the final result (pay attention to the starting value of the summation)

Frequent Itemset Generation Strategies

- Reduce the number of candidates M
  - Complete search: $M = 2^D$
  - Use pruning techniques to reduce M
- Reduce the number of comparisons NM
  - Use efficient data structures to store the candidates or transactions
  - No need to match every candidate against every transaction

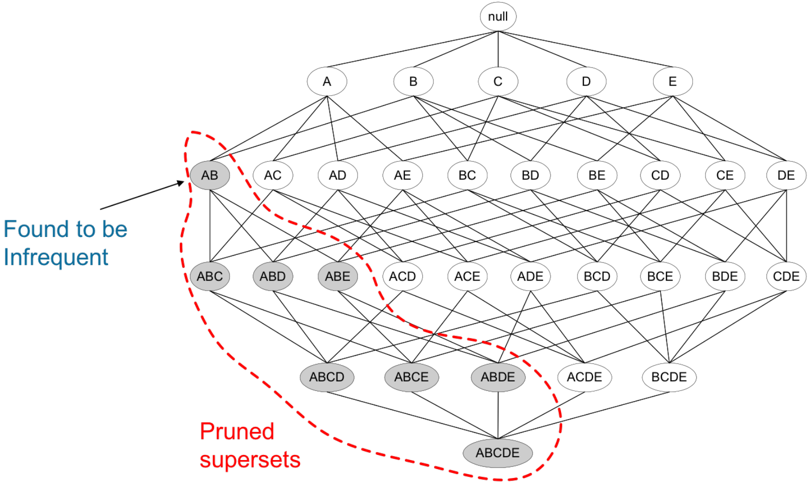
Reducing Number of Candidates

- Apriori principle
  - If an itemset is frequent, then all of its subsets must also be frequent
- It holds due to the following property of the support measure:

$$\forall X, Y : (X \subseteq Y) \Rightarrow sup(X) \geqslant sup(Y)$$

  - The Support of an itemset never exceeds the support of its subsets
  - This is known as the anti-monotone property of support

Pruning strategy



Apriori algorithm - Candidate generation OPTIONAL

Definitions

C_k: candidate itemsets of size k

L_k: frequent itemsets of size k

$subset_k(c)$: set of the subsets of c with k elements

Candidate generation – Join Step — Optional

- Let L_k be represented as a table with k columns where each row is a frequent itemset
- Let the items in each row of L_k be in lexicographic order
- C_{k+1} is generated by a self join of L_k

$$
\begin{array}{l}
\texttt{insert into } C_{k+1} \\
\texttt{select p.item}_1\texttt{, p.item}_2\texttt{, ... , p.item}_k\texttt{, q.item}_k \\
\texttt{from L}_k\texttt{ p, L}_k\texttt{ q} \\
\texttt{where p.item}_1\texttt{=q.item}_1\texttt{ and ... and p.item}_{k-1}\texttt{=q.item}_{k-1} \\
\quad \texttt{and p.item}_k\texttt{ < q.item}_k\texttt{;}
\end{array}
$$

Candidate generation – Prune Step

OPTIONAL

Each $(k+1)$-itemset which includes a k-itemset which is not in L_k is deleted from C_{k+1}

for all $c \in C_k$ **do**
    for all $s \in subset_{k-1}(c)$ **do**
        if $s \notin L_{k-1}$ **then**
            delete c from C_k
return C_k

Frequent itemset generation

OPTIONAL

$L_1 \leftarrow$ frequent 1–itemsets
$k \leftarrow 1$
while $L_k \neq \emptyset$ **do**
  C_{k+1} = candidates generated from L_k
  for all t transaction in database **do**
    increment candidate count in C_{k+1} for candidates found in t
  $L_{k+1} \leftarrow \{c\} \in C_{k+1} : \ sup(c) \geqslant minsup$
  $k \leftarrow k + 1$
return k, L_k

Pruning example – minsup=3

C_1

Item	*Count*
Beer	3
Bread	4
Coke	2
Diaper	4
Eggs	1
Milk	4

The support of $\{Coke\}$ *and* $\{Eggs\}$ *is below minsupp, therefore they do not generate* C_2 *candidates* $\longrightarrow$

C_2

Item	*Count*
Beer,Bread	2
Beer,Diaper	3
Beer,Milk	2
Bread,Diaper	3
Bread,Milk	3
Diaper,Milk	3

No C_3 *candidate will include* $\{Beer, Bread\}$ *or* $\{Beer, Milk\}$ $\longrightarrow$

C_3

Item	*Count*
Bread,Diaper,Milk	2

Number of itemsets to evaluate:

No pruning $= \binom{6}{1} + \binom{6}{2} + \binom{6}{3} = 41$

Support based pruning $= 13$

Origin of the name *Apriori*

- Level–wise computation
  - the level is the cardinality of the itemsets under evaluation
- The evaluations at level k use the *prior knowledge* acquired for the previous levels to reduce the search space

Factors Affecting Complexity I

- Choice of minimum support threshold
  - lowering support threshold results in a greater number of frequent itemsets
  - this may reduce pruning and increase the maximum length of frequent itemsets
  - the number of complete reads of the dataset is given by the maximum length of frequent itemsets plus one
- Dimensionality (number of items) of the data set
  - more space is needed to store support count of each item
  - if number of frequent items also increases, both computation and I/O costs may also increase

Factors Affecting Complexity II

- Size of database
  - since Apriori makes multiple passes, run time of algorithm may increase with number of transactions
- Average transaction width
  - transaction width increases with denser data sets
  - This may increase max length of frequent itemsets and traversals of data structures (number of subsets in a transaction increases with its width)

Confidence

From [Agrawal et al.(1993)Agrawal, Imieliński, and Swami]

- The confidence of a rule can be computed from the supports

$\Rightarrow$ for conficence based pruning of rules it is sufficient to know the supports of frequent itemsets

$$conf(A \Rightarrow C) = \frac{sup(A \Rightarrow C)}{sup(A)}$$

Rule Generation

Optional I

Give a frequent itemset L

- find all the non-empty subsets $f \in L$ such that the confidence of rule $f \Rightarrow (L - f)$ is not less than the minimum confidence (set by the experiment designer)
  - from $\{Beer, Diaper, Milk\}$ the possible rules are
    $Beer, Diaper \Rightarrow Milk, \quad Beer \Rightarrow Diaper, Milk,$
    $Beer, Milk \Rightarrow Diaper, \quad Milk \Rightarrow Beer, Diaper,$
    $Diaper, Milk \Rightarrow Beer, \quad Diaper \Rightarrow Beer, Milk$
- if $|L| = k$ then there are $2^k - 2$ candidate rules
  - $L \Rightarrow \emptyset$ and $\emptyset \Rightarrow L$ can be ignored

Rule Generation

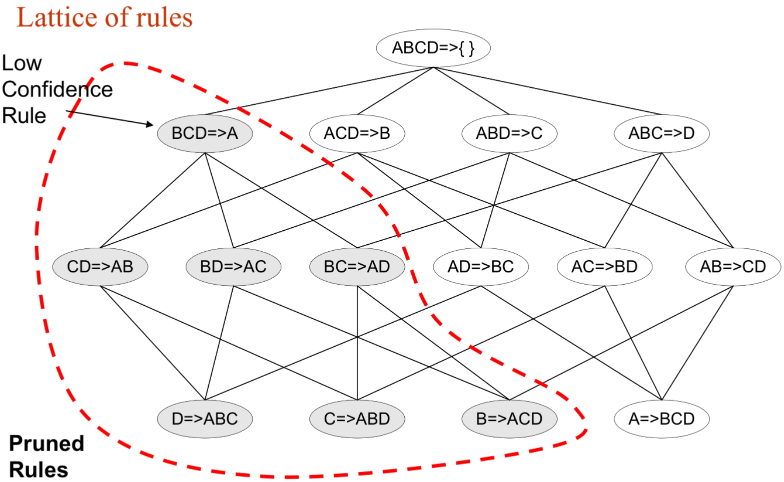
OPTIONAL II

- How to efficiently generate rules from frequent itemsets?
  - In general, confidence does not have an anti-monotone property
    - $conf(ABC \rightarrow D)$ can be larger or smaller than $conf(AB \rightarrow D)$
  - But let us consider rules generated from the same itemset
    - e.g., $i = \{A, B, C, D\} \in L$:

$$conf(ABC \rightarrow D) \geqslant conf(AB \rightarrow CD) \geqslant conf(A \rightarrow BCD)$$

- Confidence of rules generated from the same itemset is anti-monotone w.r.t. the number of items on the RHS of the rule
  - i.e. it decreases when we move an item from the left hand to the right hand

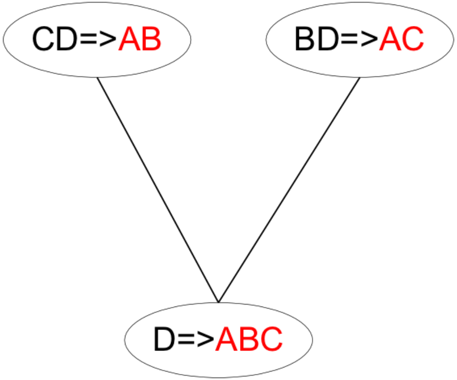
Rule Pruning



Rule Generation in Apriori

OPTIONAL

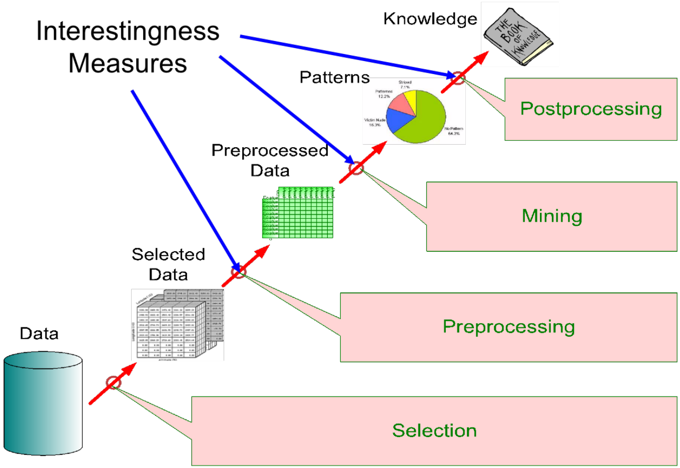
- Candidate rule is generated by merging two rules that share the same prefix in the rule consequent
- join(CD⇒AB,BD⇒AC) would produce the candidate rule $D \Rightarrow ABC$
- Prune rule D⇒ABC if its subset AD⇒BC does not have high confidence



Pattern Evaluation

- Association rule algorithms tend to produce too many rules
  - many of them are uninteresting or redundant
  - Redundant if $\{A, B, C\} \Rightarrow \{D\}$ and $\{A, B\} \Rightarrow \{D\}$ have same support and confidence
- Interestingness measures can be used to prune/rank the derived patterns
- In the original formulation of association rules, support and confidence are the only measures used

Application of Interestingness Measure



Computing Interestingness Measures

- Given a rule $A \Rightarrow C$, the information needed to compute rule interestingness can be obtained from a contingency table
- The elements of the contingency table are the basis for most of the interestiness measures

	C	$\overline{C}$	
A	f_{11}	f_{10}	f_{1+}
$\overline{A}$	f_{01}	f_{00}	f_{0+}
	f_{+1}	f_{+0}	

Drawback of Confidence

- $conf(Tea \Rightarrow Coffee) = \frac{sup(Tea, Coffee)}{sup(Tea)} = \frac{15}{20} = 0.75$
  - fairly high
- $\mathbf{Pr}(Coffee) = 0.9$ and
  $\mathbf{Pr}\left(Coffee \,\middle|\, \overline{Tea}\right) = \frac{75}{80} = 0.9375$
  - despite the high confidence of $Tea \Rightarrow Coffee$, the absence of Tea increases the probability of $Coffee$
  - for this rule the confidence is misleading

	$Coffee$	$\overline{Coffee}$	
Tea	15	5	20
$\overline{Tea}$	75	5	80
	90	10	100

$Tea \Rightarrow Coffee$

Statistical Independence

- Population of 1000 students
  - 600 students know how to swim (S)
  - 700 students know how to bike (B)
  - 420 students know how to swim and bike (S,B)

- $\mathbf{Pr}(S \wedge B) = \frac{420}{1000} = 0.42$
- $\mathbf{Pr}(S) * P(B) = 0.6 * 0.7 = 0.42$

- $\mathbf{Pr}(S \wedge B) = P(S) * P(B) \Rightarrow$ Statistical independence
- $\mathbf{Pr}(S \wedge B) > P(S) * P(B) \Rightarrow$ Positively correlated
- $\mathbf{Pr}(S \wedge B) < P(S) * P(B) \Rightarrow$ Negatively correlated

Statistical-based Measures I

Measures that take into account the deviation from statistical independence

$$lift(A \Rightarrow C) = \frac{conf(A \Rightarrow C)}{sup(C)} = \frac{\mathbf{Pr}(A, C)}{\mathbf{Pr}(A)\,\mathbf{Pr}(C)}$$

- lift evaluates to 1 for independence
- insensitive to rule direction
- it is the ratio of true cases w.r.t. independence

Statistical-based Measures II

Measures that take into account the deviation from statistical independence

$$
\begin{aligned}
leve(A \Rightarrow C) &= \mathbf{Pr}(A, C) - \mathbf{Pr}(A) * \mathbf{Pr}(C) \\
&= sup(A \cup C) - sup(A)sup(C)
\end{aligned}
$$

- leverage evaluates to 0 for independence
- insensitive to rule direction
- it is the number of additional cases w.r.t. independence

Statistical-based Measures III

Measures that take into account the deviation from statistical independence

$$conv(A \Rightarrow C) = \frac{1 - sup(C)}{1 - conf(A \Rightarrow C)} = \frac{\mathbf{Pr}(A)(1 - \mathbf{Pr}(C))}{\mathbf{Pr}(A) - \mathbf{Pr}(A, C)}$$

- conviction is infinite if the rule is always true
- sensitive to rule direction
- it is the ratio of the expected frequency that A occurs without C (that is to say, the frequency that the rule makes an incorrect prediction) if A and C were independent divided by the observed frequency of incorrect predictions
- also called novelty

Intuition about Measures

higher support	$\Rightarrow$	rule applies to more records
higher confidence	$\Rightarrow$	chance that the rule is true for some record is higher
higher lift	$\Rightarrow$	chance that the rule is just a coincidence is lower
higher conviction	$\Rightarrow$	the rule is violated less often than it would be if the antecedent and the consequent were independent

Example of page 35 – Interestingness measures

$Tea \Rightarrow Coffee$

$$conf = \frac{0.15}{0.20} = 0.75$$

in a 0 to 1 scale it is apparently high

$$lift = \frac{0.15}{0.90*0.20} = 0.83$$

is less than 1, therefore not interesting

$$leve = 0.15 - 0.90 * 0.20 = -0.03$$

is less than 0, therefore not interesting

$$conv = \frac{1-0.9}{1-0.75} = 0.4$$

is low, remembering that absolute truth gives *infinite*

Comparison of measures

	$C1$	$\overline{C1}$	
$A1$	88	5	93
$\overline{A1}$	5	2	7
	93	7	100

Rule $(A1 \Rightarrow C1)$

$$\begin{aligned} conf &= 0.88/0.93 &&= 0.946 \\ lift &= 0.88/(0.93 * 0.93) &&= 1.017 \\ leve &= 0.88 - 0.93 * 0.93 &&= 0.015 \\ conv &= (1 - 0.93)/(1 - 0.946) &&= 1.302 \end{aligned}$$

A high confidence rule can have small lift if both sides are very frequent

	$C2$	$\overline{C2}$	
$A2$	2	5	7
$\overline{A2}$	5	88	93
	7	93	100

Rule $(A2 \Rightarrow C2)$

$$\begin{aligned} conf &= 0.02/0.07 &&= 0.286 \\ lift &= 0.02/(0.07 * 0.07) &&= 4.082 \\ leve &= 0.02 - 0.07 * 0.07 &&= 0.015 \\ conv &= (1 - 0.07)/(1 - 0.286) &&= 1.302 \end{aligned}$$

A low confidence rule can have high lift if both sides are very infrequent

Conclusion on measures

- There are lots of measures proposed in the literature, beyond the four presented here
- Confidence is usually the base tool
- Other measures can be used to test the results given by confidence and for additional filtering

Multidimensional association rules

Let's consider a dataset deriving from sensors measuring the concentration of air pollutants

TID	*CO*	*Tin_Oxide*	*Titanium*
1	high	medium	high
2	medium	low	medium
3	medium	high	low
4	low	medium	medium

- Look for rules such as $CO = high\ and\ Tin\ Oxide = high\ then\ Titanium = high$ (support 0.25 and confidence 1)
- This can be used for example, if one of the sensor is not available, to guess its qualitative value given the others
- Useful for a qualitative analysis, in substitution of regression

Comparison mono– vs multi–dimensional

- Mono–dimensional (intra-attribute)
    - event: transaction
    - event description:
        - items A, B, and C are together in a transaction
- Multi–dimensional (inter–atrribute)
    - event: tuple
    - event description:
        - attribute A has value a, attribute B has value b and attribute C has value c in a tuple

Equivalence mono/multi–dimensional

Multi–dimensional		*Mono–dimensional*
Schema: (TID,CO,Tin_Oxide,Titanium)		
1, high, medium, high	$\longrightarrow$	1, {CO/high, Tin_Oxide/medium, Titanium/high}
2, medium, low, medium		2, {CO/medium, Tin_Oxide/low, Titanium/medium}
Schema: (TID,a?,b?,c?,d?)		
1, yes, yes, no, no	$\longleftarrow$	1, {a, b}
2, yes, no, yes, no		2, {a, c}

Quantitative attributes

TID	*CO*	*Tin Oxide*	*Titanium*
1	2.6	1360	1046
2	2.0	1292	955
3	2.2	1402	939
4	1.6	1376	948

- Too many distinct values for the multi/mono transformation
- Most software packages for association rules discovery do not deal with quantitative attributes

$\Rightarrow$ discretization

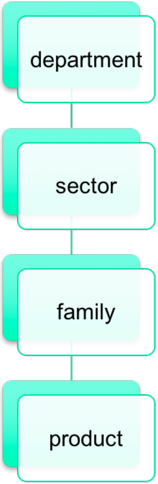
- possibly *equifrequency* or with *mono–dimensional clustering*, for optimal covering of the original value domains
- discretisation leads to a dataset like that of page 45

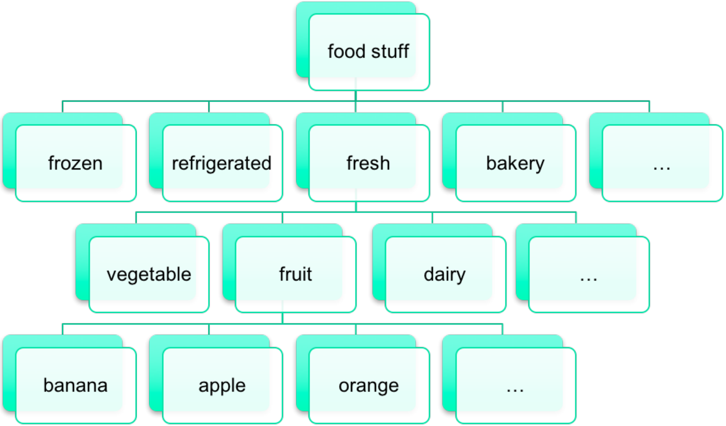
- Association rules can involve items at different qualitative levels

Multilevel Association Rules

- A real MBA database can include tens of thousands of distinct items
- Frequently it is necessary to find a tradeoff between general and detailed reasoning
  - choose the right level of abstraction
- A common background knowledge is the organization of the items into a hierarchy of concepts
  - it can be easily coded in the transactions
  - it can help the choice of the right level of abstraction

Concept Hierarchy





Support in Multilevel AR

- From specialized to general

  (apple $\Rightarrow$ milk) $\rightarrow$ (fruit $\Rightarrow$ dairy)

  - the support of rules increases, in general
  - new rules can become interesting
- From general to specialized

  (fruit $\Rightarrow$ dairy) $\rightarrow$ (apple $\Rightarrow$ milk)

  - the support of rules decreases, in general
  - the support of rules can go under the threshold

Confidence in Multilevel AR

- A level change can influence the confidence in any direction
- If the specialized rule has (approximately) the same confidence as the general one, then it is redundant

Example

Low–fat milk is a subclass of milk

- 1000 transactions, 80 with milk and bread, 114 with milk, 20 with low–fat milk and bread, 28 with low–fat milk
  a) milk $\Rightarrow$ bread (support $=$ 8%, confidence $=$ 70%)
  b) low–fat milk $\Rightarrow$ bread (support $=$ 2%, confidence $=$ 71%)
- rule b) has almost the same confidence as rule a)
- rule b) is a descendant of rule a)

$\Rightarrow$ rule b) is redundant

Mining Multilevel Association Rules

- Look for frequent itemsets at each level of abstraction, top down
  - Each level requires a new run of the rule discovery algorithm
- Decrease the support threshold in lower levels

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