大偶数表为一个素数及一个不超过二个素数的乘积之和

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摘 要

本文的目的在于用筛法证明了：每一充分大的偶数是一个素数及一个不超过两个素数乘积之和.

关于孪生素数问题亦得到类似的结果.

一、引 言

把命题"每一个充分大的偶数都能表示为一个素数及一个不超过 a 个素数的乘积之和"简记为 $(1, a)$.

不少数学工作者改进了筛法及素数分布的某些结果，并用以改善 $(1, a)$. 现在我们将 $(1, a)$ 发展历史简述如下：

$(1, c)$——Renyi[1]，

$(1, 5)$——潘承洞[2]、Барбан[3]，

$(1, 4)$——王元[4]、潘承洞[5]、Барбан[6]，

$(1, 3)$——Бухштаб[7]、Виноградов[8]、Bombieri[9]，

在文献 [10] 中我们给出了 $(1, 2)$ 的证明提要.

命 $P_x(1, 2)$ 为适合下列条件的素数 p 的个数：

$$x - p = p_1 \quad 或 \quad x - p = p_2p_3,$$

其中 p_1, p_2, p_3 都是素数.

用 x 表一充分大的偶数. 命 $C_x = \prod\limits_{\substack{p|x \\ p>2}} \frac{p-1}{p-2} \prod\limits_{p>2} \left(1 - \frac{1}{(p-1)^2}\right)$.

对于任意给定的偶数 h 及充分大的 x，用 $x_h(1, 2)$ 表示满足下面条件的素数 p 的个数：

$$p \leqslant x, \quad p + h = p_1 \quad 或 \quad p + h = p_2p_3,$$

其中 p_1, p_2, p_3 都是素数.

本文目的在于证明并改进作者在文献 [10] 内所提及的全部结果，现在详述如下.

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定理 1. (1, 2) 及 $P_x(1,2) \geqslant \frac{0.67xC_x}{(\log x)^2}$.

定理 2. 对于任意偶数 h, 都存在无限多个素数 p, 使得 $p+h$ 的素因子的个数不超过 2 个及 $x_h(1,2) \geqslant \frac{0.67xC_x}{(\log x)^2}$.

在证明定理 1 时,主要用到了本文中的引理 8 和引理 9. 在证明引理 8 时, 我们使用较为简单的数字计算方法;而证明引理 9 时,我们使用了 Bombieri 定理[9]及 Richert[11] 中的一个结果.

二、几 个 引 理

引理 1. 假设 $y \geqslant 0$, 而 $[\log x]$ 表示 $\log x$ 的整数部分, $x>1$,

$$\Phi(y)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{y^{\omega}d\omega}{\omega\left(1+\frac{\omega}{(\log x)^{1.1}}\right)^{[\log x]+1}}.$$

显见,当 $0\leqslant y\leqslant 1$ 时,有 $\Phi(y)=0$. 对于所有 $y\geqslant 0$,则 $\Phi(y)$ 是一个非减函数. 当 $\log x\geqslant 10^4$ 及 $y\geqslant e^{2(\log x)^{-0.1}}$ 时,则有

$$1-x^{-0.1}\leqslant \Phi(y)\leqslant 1.$$

证. 我们先来证明

$$\frac{\partial^r}{\partial\omega^r}\left(\frac{y^{\omega}}{\omega}\right)=\left(\frac{y^{\omega}}{\omega}\right)\left\{(\log y)^r+\sum_{i=1}^{r}\frac{(-1)^i r\cdots(r-i+1)(\log y)^{r-i}}{\omega^i}\right\} \tag{1}$$

成立. 显见, (1) 式当 $r=1$ 和 $r=2$ 时都成立. 现假定 (1) 式对于 $r=2,\cdots,S$ 时都成立,而证明对于 $S+1$ 也成立. 由于

$$\begin{aligned}
\frac{\partial^{S+1}}{\partial\omega^{S+1}}\left(\frac{y^{\omega}}{\omega}\right)&=\frac{\partial}{\partial\omega}\left\{y^{\omega}\left(\frac{(\log y)^S}{\omega}+\sum_{i=1}^{S}\frac{(-1)^iS\cdots(S-i+1)(\log y)^{S-i}}{\omega^{i+1}}\right)\right\}\\
&=y^{\omega}\left\{\frac{(\log y)^{S+1}}{\omega}+\sum_{i=1}^{S}\frac{(-1)^iS\cdots(S-i+1)(\log y)^{S+1-i}}{\omega^{i+1}}-\frac{(\log y)^S}{\omega^2}\right.\\
&\quad\left.+\sum_{i=1}^{S}\frac{(-1)^{i+1}S\cdots(S-i+1)(i+1)(\log y)^{S-i}}{\omega^{i+2}}\right\}=\left(\frac{y^{\omega}}{\omega}\right)\left\{(\log y)^{S+1}\right.\\
&\quad-\frac{(S+1)(\log y)^S}{\omega}+\frac{(-1)^{S+1}(S+1)!}{\omega^{S+1}}+\sum_{i=2}^{S}\left(\frac{(-1)^iS\cdots(S-i+1)(\log y)^{S+1-i}}{\omega^i}\right.\\
&\quad\left.\left.+\frac{(-1)^iS\cdots(S+2-i)i(\log y)^{S+1-i}}{\omega^i}\right)\right\}=\left(\frac{y^{\omega}}{\omega}\right)\left\{(\log y)^{S+1}\right.\\
&\quad\left.+\sum_{i=1}^{S+1}\frac{(-1)^i(S+1)\cdots(S+1-i+1)(\log y)^{S+1-i}}{\omega^i}\right\}.
\end{aligned}$$

故 (1) 式得证.

又当 $y\geqslant 1$ 时,我们有

$$\Phi(y)=1+\left\{\frac{(\log x)^{1.1+1.1[\log x]}}{[\log x]!}\right\}\left\{\frac{\partial^{[\log x]}}{\partial\omega^{[\log x]}}\left(\frac{y^{\omega}}{\omega}\right)\right\}_{\omega=-(\log x)^{1.1}}$$

$$= 1 - e^{-(\log x)^{1.1}(\log y)} \sum_{\nu=0}^{[\log x]} \frac{\{(\log x)^{1.1}(\log y)\}^{\nu}}{\nu!}$$

$$= \left\{\frac{1}{[\log x]!}\right\} \int_0^{(\log x)^{1.1}(\log y)} e^{-\lambda} \lambda^{[\log x]} d\lambda.$$

因为 $0 \leqslant y \leqslant 1$ 时, $\Phi(y) = 0$. 故由上式得到: 当 $y \geqslant 0$ 时,则 $\Phi(y)$ 是一个非减函数. 又当 $y \geqslant e^{2(\log x)^{-1.0}}$ 时,有

$$0 < 1 - \Phi(y) = \left\{\frac{1}{[\log x]!}\right\} \int_{(\log x)^{1.1}(\log y)}^{\infty} e^{-\lambda} \lambda^{[\log x]} d\lambda$$

$$\leqslant \left\{\frac{1}{[\log x]!}\right\} \int_{2[\log x]}^{\infty} e^{-\lambda} \lambda^{[\log x]} d\lambda = \left\{\frac{([\log x])^{1+[\log x]}}{[\log x]!}\right\}$$

$$\cdot \int_2^{\infty} e^{-\lambda[\log x]} \lambda^{[\log x]} d\lambda = \left\{\frac{e^{-[\log x]}([\log x])^{1+[\log x]}}{[\log x]!}\right\}$$

$$\cdot \int_1^{\infty} e^{-\lambda[\log x]} (1+\lambda)^{[\log x]} d\lambda \leqslant x^{-0.1}.$$

其中用到 $\log x \geqslant 10^4$ 及当 $\lambda \geqslant 1$ 时,有 $e^{\log(1+\lambda)} \leqslant e^{\lambda \log 2}$.

引理 2. 令 $e(\alpha) = e^{2\pi i\alpha}$, $S(\alpha) = \sum_{n=M+1}^{M+N} a_n e(n\alpha)$, $Z = \sum_{n=M+1}^{M+N} |a_n|^2$, 其中 a_n 是任意的实数. 我们用令 $\sum_{\chi_q}^{*}$ 来表示和式之中经过且只经过模 q 的所有原特征,则有

$$\sum_{q \leqslant X} \frac{q}{\varphi(q)} \sum_{\chi_q}^{*} \left| \sum_{n=M+1}^{M+N} a_n \chi_q(n) \right|^2 \leqslant (X^2 + \pi N) \sum_{n=M+1}^{M+N} |a_n|^2; \tag{2}$$

$$\sum_{D < q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi_q}^{*} \left| \sum_{n=M+1}^{M+N} a_n \chi_q(n) \right|^2 \ll \left(Q + \frac{N}{D}\right) \sum_{n=M+1}^{M+N} |a_n|^2. \tag{3}$$

证. 令 F 是一个周期为 1 的复数值可微函数,则有 $\left|F\left(\frac{a}{q}\right)\right| = \left|F(\alpha) - \int_{\frac{a}{q}}^{\alpha} dF(\beta)\right| \leqslant |F(\alpha)| + \int_{\frac{a}{q}}^{\alpha} |F'(\beta)||d\beta|$, 我们用 $I(a, q)$ 来表示以 $\frac{a}{q}$ 为中心,而长度 $\frac{1}{Q^2}$ 的区间, 显见, 当 $1 \leqslant a < q$, $(a, q) = 1$, $q \leqslant Q$ 时,所有的区间 $I(a, q)$ 都没有共同部分,故得

$$\sum_{q \leqslant Q} \sum_{\substack{(a,q)=1 \\ 1 \leqslant a < q}} \left|F\left(\frac{a}{q}\right)\right| \leqslant \sum_{q \leqslant Q} \sum_{\substack{(a,q)=1 \\ 1 \leqslant a < q}} \left\{Q^2 \int_{I(a,q)} |F(\alpha)| d\alpha + \frac{1}{2} \int_{I(a,q)} |F'(\beta)| d\beta\right\}$$

$$\leqslant Q^2 \int_0^1 |F(\alpha)| d\alpha + \frac{1}{2} \int_0^1 |F'(\beta)| d\beta.$$

我们取 $F(\alpha) = \{S(\alpha)\}^2$, 则得

$$\int_0^1 |F(\alpha)| d\alpha = Z \text{ 及 } \frac{1}{2} \int_0^1 |F'(\beta)| d\beta = \int_0^1 |S(\alpha)||S'(\alpha)| d\alpha$$

$$\leqslant \left\{\left(\int_0^1 |S(\alpha)|^2 d\alpha\right)\left(\int_0^1 |S'(\alpha)|^2 d\alpha\right)\right\}^{\frac{1}{2}} = Z^{\frac{1}{2}} \left(\int_0^1 |S'(\alpha)|^2 d\alpha\right)^{\frac{1}{2}}.$$

故有

$$\sum_{q \leqslant Q} \sum_{\substack{(a,q)=1 \\ 1 \leqslant a < q}} \left|S\left(\frac{a}{q}\right)\right|^2 = \sum_{q \leqslant Q} \sum_{\substack{(a,q)=1 \\ 1 \leqslant a < q}} \left\{\left|S\left(\frac{a}{q}\right)\right| \left|e\left(-\frac{a\left(M + \left[\frac{N}{2}\right]\right)}{q}\right)\right|\right\}^2$$

$$= \sum_{q \leqslant Q} \sum_{\substack{(a,q)=1 \\ 1 \leqslant a < q}} \left| \sum_{n=M+1}^{M+N} a_n e\left(\left\{n - \left(M + \left[\frac{N}{2}\right]\right)\right\} \frac{a}{q}\right) \right|^2$$

$$= \sum_{q \leqslant Q} \sum_{\substack{(a,q)=1 \\ 1 \leqslant a < q}} \left| \sum_{-\left[\frac{N}{2}\right]+1 \leqslant n \leqslant N-\left[\frac{N}{2}\right]} a_{n+M+\left[\frac{N}{2}\right]} e\left(\frac{na}{q}\right) \right|^2$$

$$\leqslant ZQ^2 + Z^{\frac{1}{2}} \left\{ \sum_{n=-\left[\frac{N}{2}\right]+1}^{N-\left[\frac{N}{2}\right]} \left((2\pi n) a_{n+M+\left[\frac{N}{2}\right]}\right)^2 \right\}^{\frac{1}{2}} \leqslant ZQ^2$$

$$+ \pi N Z^{\frac{1}{2}} \left(\sum_{n=-\left[\frac{N}{2}\right]+1}^{N-\left[\frac{N}{2}\right]} \left| a_{n+M+\left[\frac{N}{2}\right]} \right|^2 \right)^{\frac{1}{2}} \leqslant (Q^2 + \pi N) Z. \tag{4}$$

令 χ^* 表示原特征，$\tau(\chi_q^*) = \sum\limits_{1 \leqslant a < q} \chi_q^*(a) e\left(\frac{a}{q}\right)$，$\tau(\overline{\chi_q^*}) \chi_q^*(n) = \sum\limits_{a=1}^{q} \overline{\chi_q^*}(a) e\left(\frac{na}{q}\right)$. 由于 $|\tau(\overline{\chi_q^*})|^2 = q$, 故得到

$$\left(\frac{1}{\varphi(q)}\right) \sum_{\chi_q}{}^* \left| \sum_{n=M+1}^{M+N} a_n \chi_q(n) \right|^2 \leqslant \left(\frac{1}{q\varphi(q)}\right) \sum_{\chi_q}{}^* \left| \tau(\overline{\chi_q}) \sum_{n=M+1}^{M+N} a_n \chi_q(n) \right|^2$$

$$= \left(\frac{1}{q\varphi(q)}\right) \sum_{\chi_q}{}^* \left| \sum_{a=1}^{q} \overline{\chi_q}(a) \sum_{n=M+1}^{M+N} a_n e\left(\frac{na}{q}\right) \right|^2$$

$$\leqslant \left(\frac{1}{q\varphi(q)}\right) \sum_{\chi_q} \left| \sum_{a=1}^{q} \overline{\chi_q}(a) \sum_{n=M+1}^{M+N} a_n e\left(\frac{na}{q}\right) \right|^2$$

$$\leqslant \frac{1}{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{n=M+1}^{M+N} a_n e\left(\frac{na}{q}\right) \right|^2.$$

由上式及 (4) 式，即得到 (2) 式. 我们定义 h 是一个正整数，它使得 $2^h D < Q \leqslant 2^{h+1} D$, 则我们有

$$\sum_{D < q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi_q}{}^* \left| \sum_{n=M+1}^{M+N} a_n \chi_q(n) \right|^2 \leqslant \sum_{i=0}^{h} \left(\sum_{2^i D < q \leqslant 2^{i+1} D} \frac{1}{\varphi(q)} \sum_{\chi_q}{}^* \left| \sum_{n=M+1}^{M+N} a_n \chi_q(n) \right|^2 \right)$$

$$\leqslant \sum_{i=0}^{h} \left(\frac{1}{2^i D}\right) \left(\sum_{2^i D < q \leqslant 2^{i+1} D} \frac{q}{\varphi(q)} \sum_{\chi_q}{}^* \left| \sum_{n=M+1}^{M+N} a_n \chi_q(n) \right|^2 \right)$$

$$\leqslant \sum_{i=0}^{h} \left(2^{i+2} D + \frac{\pi N}{2^i D}\right) \sum_{n=M+1}^{M+N} |a_n|^2 \ll \left(Q + \frac{N}{D}\right) \sum_{n=M+1}^{M+N} |a_n|^2.$$

故引理 2 得证.

引理 3. 当 $S = \sigma + it$ 和 $\sigma \geqslant \frac{1}{2}$ 时，则有

$$\sum_{q \leqslant Q} \sum_{\chi_q}{}^* |L(S, \chi_q)|^4 \ll Q^2 |S|^2 (\log Q)^4.$$

证. 我们有

$$L(S,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^S}=\sum_{n=1}^{N}\frac{\chi(n)}{n^S}+\sum_{n=N+1}^{\infty}\frac{\sum_{i\leqslant n}\chi(i)-\sum_{i\leqslant n-1}\chi(i)}{n^S}$$

$$=\sum_{n=1}^{N}\frac{\chi(n)}{n^S}+\sum_{n=N+1}^{\infty}\left(\sum_{i\leqslant n}\chi(i)\right)\left(\frac{1}{n^S}-\frac{1}{(n+1)^S}\right)-\frac{\sum_{i\leqslant N}\chi(i)}{(N+1)^S}$$

$$=\sum_{n=1}^{N}\frac{\chi(n)}{n^S}+O\left(\frac{|S|q^{\frac{1}{2}}\log q}{N^{\sigma}}\right).$$

故由引理 2 及 $\sigma\geqslant\frac{1}{2}$, 我们有

$$\sum_{q\leqslant Q}\sum_{\chi_q}{}^{*}|L(S,\chi_q)|^4\ll\sum_{q\leqslant Q}\sum_{\chi_q}{}^{*}\left(\left|\sum_{n=1}^{[Q|S|]}\frac{\chi_q(n)}{n^S}\right|^4+Q^{-2}|S|^2q^2(\log q)^4\right)$$

$$\ll|S|^2Q^2(\log Q)^4+(Q^2+Q^2|S|^2)\sum_{n=1}^{[Q|S|]^2}\frac{d^2(n)}{n}\ll Q^2|S|^2(\log Q)^4.$$

故本引理得证.

引理 4. 当 k 是无平方因子的奇数,而 $m\neq 1$ 时,则我们有

$$\left|\sum_{\chi_k}{}^{*}\chi_k(m)\right|\leqslant|(m-1,k)|.$$

证. 令 $k=p_1\cdots p_l$, 而 $p_1<\cdots<p_l$. 令 g_i 是 mod p_i 的原根, 则有 $m\equiv g_j^{\xi_j}(\bmod p_j)$, $0\leqslant\xi_j\leqslant p_j-2$, $j=1,\cdots,l$, 则关于模 k 的所有原特征可表示为

$$\chi_k^*(m)=e^{2\pi i\left(\frac{\nu_1\xi_1}{p_1-1}+\cdots+\frac{\nu_l\xi_l}{p_l-1}\right)},$$

其中 $1\leqslant\nu_j\leqslant p_j-2$, 而 $j=1,\cdots,l$.

令 $Z(m,k)=\left|\sum_{\chi_k}{}^{*}\chi_k(m)\right|$, 则有

$$Z(m,k)=\prod_{j=1}^{l}Z(m,p_j)=\prod_{j=1}^{l}\left|\sum_{\nu_j=1}^{p_j-2}e^{2\pi i\frac{\nu_j\xi_j}{p_j-1}}\right|=\prod_{\substack{j=1\\ \xi_j=0}}^{l}(p_j-2)<\prod_{p_j|(m-1)}p_j=|(m-1,k)|.$$

故本引理得证.

设 x 是偶数,令 $\lambda_1=1$; 当 $d>x^{\frac{1}{4}-\frac{\epsilon}{2}}$ 时,令 $\lambda_d=0$; 而当 $1<d\leqslant x^{\frac{1}{4}-\frac{\epsilon}{2}}$ 时,令

$$\lambda_d=\frac{\mu(d)}{f(d)g(d)}\left\{\sum_{\substack{1\leqslant k\leqslant(x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}}/d\\(k,xd)=1}}\frac{\mu^2(k)}{f(k)}\right\}\left\{\sum_{\substack{1\leqslant k\leqslant(x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}}\\(k,x)=1}}\frac{\mu^2(k)}{f(k)}\right\}^{-1}.$$

其中 $g(k)=\frac{1}{\varphi(k)}$, $f(k)=\varphi(k)\prod_{p|k}\frac{p-2}{p-1}$. 又当 d 为奇数, $\mu(d)\neq 0$ 时,有

$$\sum_{\substack{1\leqslant k\leqslant(x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}}\\(k,x)=1}}\frac{\mu^2(k)}{f(k)}=\sum_{t|d}\sum_{\substack{1\leqslant k\leqslant(x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}}\\(k,x)=1,\ (k,d)=t}}\frac{\mu^2(k)}{f(k)}=\sum_{t|d}\left\{\frac{1}{\prod_{p|t}(p-2)}\right\}\sum_{\substack{1\leqslant k\leqslant(x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}}/t\\(k,xd)=1}}\frac{\mu^2(k)}{f(k)}$$

$$\geqslant\left\{\prod_{p|d}\left(1+\frac{1}{p-2}\right)\right\}\left\{\sum_{\substack{1\leqslant k\leqslant(x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}}/d\\(k,xd)=1}}\frac{\mu^2(k)}{f(k)}\right\}.$$

故对于所有正整数 d，都有 $|\lambda_d| \leqslant 1$. 设 x 是偶数，$\log x > 10^4$，又令 $Q = \prod_{2 \leqslant p \leqslant x^{\frac{1}{4}}} p$，

$$\Omega = \sum_{\substack{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}} \\ p_3 \leqslant \frac{x}{p_1 p_2} \\ (x - p_1 p_2 p_3, Q) = 1}} 1, \quad M = \sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}}} \left(\frac{1}{\log \frac{x}{p_1 p_2}}\right) \left(\sum_{\substack{n \leqslant \frac{x}{p_1 p_2} \\ (x - p_1 p_2 n, Q) = 1}} \Lambda(n)\right),$$

则有 $\Omega \leqslant \frac{M}{1-\epsilon} + N$，其中 $N \ll \sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}}} \left(\frac{x}{p_1 p_2}\right)^{1-\epsilon} \ll x^{1-\epsilon} \int_{x^{\frac{1}{10}}}^{x^{\frac{1}{3}}} \frac{dS}{S^{1-\epsilon}} \int_{x^{\frac{1}{3}}}^{\left(\frac{x}{S}\right)^{\frac{1}{2}}} \frac{dt}{t^{1-\epsilon}}$

$$\ll x^{1-\frac{\epsilon}{2}} \int_{x^{\frac{1}{10}}}^{x^{\frac{1}{3}}} \frac{dS}{S^{1-\frac{\epsilon}{2}}} \ll x^{1-\frac{\epsilon}{3}}.$$

由引理 1，我们有

$$M \leqslant \sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}}} \left(\frac{1}{\log \frac{x}{p_1 p_2}}\right) \sum_{\substack{n \leqslant \frac{x}{p_1 p_2} \\ (x - p_1 p_2 n, Q) = 1}} \Lambda(n) \Phi\left(\frac{x}{p_1 p_2 n}\right) + O\left(\frac{x}{(\log x)^{2.01}}\right)$$

$$\leqslant \sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}}} \left(\frac{1}{\log \frac{x}{p_1 p_2}}\right) \sum_{n \leqslant \frac{x}{p_1 p_2}} \Lambda(n) \Phi\left(\frac{x}{p_1 p_2 n}\right) \left(\sum_{\substack{d \mid (x - p_1 p_2 n, Q) \\ (d, x) = 1}} \lambda_d\right)^2$$

$$+ O\left(\frac{x}{(\log x)^{2.01}}\right) = \sum_{\substack{(d_1, x) = 1 \\ d_1 \mid Q}} \sum_{\substack{(d_2, x) = 1 \\ d_2 \mid Q}} \lambda_{d_1} \lambda_{d_2} N_{\frac{d_1 d_2}{(d_1, d_2)}} + O\left(\frac{x}{(\log x)^{2.01}}\right). \tag{5}$$

其中

$$N_{\frac{d_1 d_2}{(d_1, d_2)}} = \sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}}} \left(\frac{1}{\log \frac{x}{p_1 p_2}}\right) \sum_{\substack{n \leqslant \frac{x}{p_1 p_2} \\ x - p_1 p_2 n \equiv 0 \left(\bmod \frac{d_1 d_2}{(d_1, d_2)}\right)}} \Lambda(n) \Phi\left(\frac{x}{p_1 p_2 n}\right)$$

$$= \left\{\frac{1}{\varphi\left(\frac{d_1 d_2}{(d_1, d_2)}\right)}\right\} \left\{ \sum_{\substack{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}} \\ n \leqslant \frac{x}{p_1 p_2} \\ (p_1 p_2 n, d_1 d_2) = 1}} \left(\frac{1}{\log \frac{x}{p_1 p_2}}\right) \Lambda(n) \Phi\left(\frac{x}{p_1 p_2 n}\right) \right.$$

$$\left. + \sum_{\chi_{\frac{d_1 d_2}{(d_1, d_2)}} \neq \chi_0} \overline{\chi_{\frac{d_1 d_2}{(d_1, d_2)}}}(x) \sum_{\substack{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}} \\ n \leqslant \frac{x}{p_1 p_2}}} \left(\frac{\Lambda(n)}{\log \frac{x}{p_1 p_2}}\right) \Phi\left(\frac{x}{p_1 p_2 n}\right) \chi_{\frac{d_1 d_2}{(d_1, d_2)}}(p_1 p_2 n) \right\}$$

$$= \left\{\frac{1}{\varphi\left(\frac{d_1 d_2}{(d_1, d_2)}\right)}\right\} \left\{ \sum_{\substack{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}} \\ n \leqslant \frac{x}{p_1 p_2} \\ (p_1 p_2 n, d_1 d_2) = 1}} \left(\frac{1}{\log \frac{x}{p_1 p_2}}\right) \Lambda(n) \Phi\left(\frac{x}{p_1 p_2 n}\right) \right\} - \left\{\frac{1}{2\pi i\, \varphi\left(\frac{d_1 d_2}{(d_1, d_2)}\right)}\right\}$$

$$\cdot \left\{ \int_{2 - i\infty}^{2 + i\infty} \left(1 + \frac{\omega}{(\log x)^{1.1}}\right)^{-[\log x] - 1} \left(\frac{x^\omega}{\omega}\right) \sum_{\chi_{\frac{d_1 d_2}{(d_1, d_2)}} \neq \chi_0} \overline{\chi_{\frac{d_1 d_2}{(d_1, d_2)}}}(x) \frac{L'}{L}\left(\omega, \chi_{\frac{d_1 d_2}{(d_1, d_2)}}\right) \right.$$

$$\left. \cdot \sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant \left(\frac{x}{p_1}\right)^{\frac{1}{2}}} \chi_{\frac{d_1 d_2}{(d_1, d_2)}}(p_1 p_2) \cdot \left(\frac{1}{\log \frac{x}{p_1 p_2}}\right) \left(\frac{d\omega}{(p_1 p_2)^\omega}\right) \right\}. \tag{6}$$

令

$$M_1=\sum_{(d_1,x)=1}\sum_{(d_2,x)=1}\frac{\lambda_{d_1}\lambda_{d_2}}{\varphi\left(\frac{d_1d_2}{(d_1,d_2)}\right)}\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}\\ n\leqslant\frac{x}{p_1p_2}}}\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\Lambda(n)\Phi\left(\frac{x}{p_1p_2n}\right),$$

$$M_2=\sum_{\substack{d\leqslant x^{\frac{1}{2}-\epsilon}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{\varphi(d)}\Bigg|\sum_{\chi_d\neq\chi_0}\overline{\chi^*_{d^*}}(x)\int_{2-i\infty}^{2+i\infty}\left(\frac{x^\omega}{\omega}\right)\left(1+\frac{\omega}{(\log x)^{1.1}}\right)^{-[\log x]-1}$$

$$\cdot\frac{L'}{L}(\omega,\chi^*_{d^*})\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}\\(p_1p_2,d)=1}}\chi^*_{d^*}(p_1p_2)\left((p_1p_2)^\omega\log\frac{x}{p_1p_2}\right)^{-1}d\omega\Bigg|.$$

其中 d^* 是 χ_d 的 conductor，而 $\chi^*_{d^*}$ 是等价于 χ_d 的 mod d^* 的原特征. $\nu(d)$ 是 d 的素数因子的个数.

引理 5. 设 x 是偶数，则有

$$\Omega\leqslant\frac{M_1+M_2}{1-\epsilon}+O\left(\frac{x}{(\log x)^{2.01}}\right).$$

证. 由(5)式和(6)式，我们有

$$M\leqslant M_1+|M_3|+M_4+O\left(\frac{x}{(\log x)^{2.01}}\right),\tag{7}$$

其中

$$M_3=\sum_{(d_1,x)=1}\sum_{(d_2,x)=1}\frac{\lambda_{d_1}\lambda_{d_2}}{\varphi\left(\frac{d_1d_2}{(d_1,d_2)}\right)}\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}\\ n\leqslant\frac{x}{p_1p_2}\\(d_1d_2,p_1p_2n)>1}}\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\Lambda(n)\Phi\left(\frac{x}{p_1p_2n}\right).$$

$$M_4=\sum_{(d_1,x)=1}\sum_{(d_2,x)=1}\left(-\frac{\lambda_{d_1}\lambda_{d_2}}{2\pi i\varphi\left(\frac{d_1d_2}{(d_1,d_2)}\right)}\right)\int_{2-i\infty}^{2+i\infty}\left(\frac{x^\omega}{\omega}\right)\left(1+\frac{\omega}{(\log x)^{1.1}}\right)^{-[\log x]-1}$$

$$\cdot\sum_{\chi_{\frac{d_1d_2}{(d_1,d_2)}}\neq\chi_0}\overline{\chi_{\frac{d_1d_2}{(d_1,d_2)}}}(x)\frac{L'}{L}\left(\omega,\chi_{\frac{d_1d_2}{(d_1,d_2)}}\right)\sum_{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}}\frac{\chi_{\frac{d_1d_2}{(d_1,d_2)}}(p_1p_2)}{(p_1p_2)^\omega\log\frac{x}{p_1p_2}}d\omega.$$

首先估计 M_3，

$$M_3\ll x^\epsilon\sum_{d\leqslant x^{\frac{1}{2}-\epsilon}}\frac{1}{d}\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}\\ n\leqslant\frac{x}{p_1p_2}\\(d,p_1p_2n)>1}}\Lambda(n)\ll\sum_{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}}\left(\frac{x^{1+\epsilon}}{p_1p_2}\right)\Bigg(\sum_{\substack{d\leqslant x^{\frac{1}{2}-\epsilon}\\p_1|d}}\frac{1}{d}$$

$$+\sum_{\substack{d\leqslant x^{\frac{1}{2}-\epsilon}\\p_2|d}}\frac{1}{d}\Bigg)+\sum_{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}}\sum_{p\leqslant\frac{x}{p_1p_2}}(\log p)\sum_{\substack{d\leqslant x^{\frac{1}{2}-\epsilon}\\p|d}}\frac{x^\epsilon}{d}+x^{1-\epsilon}\ll x^{1-\epsilon}.\tag{8}$$

再估计 M_4，设 $\mu(d)\neq0$，$d=p_1\cdots p_k$，则正整数 d_1 和 d_2 满足 $\frac{d_1d_2}{(d_1,\ d_2)}=d$ 的充分和必要的

条件是 $d_1=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, $d_2=p_1^{\beta_1}\cdots p_k^{\beta_k}$, 其中 $0\leqslant\alpha_i\leqslant1$, $0\leqslant\beta_i\leqslant1$, $\alpha_i+\beta_i\geqslant1$ $(1\leqslant i\leqslant k)$. 故当 $d>0$, $\mu(d)\neq0$ 时,则满足 $\frac{d_1d_2}{(d_1,\ d_2)}=d$ 的正整数 d_1, d_2 的组数为 $3^{\nu(d)}$. 由于 $|\lambda_d|\leqslant1$, 故有

$$M_4\leqslant\sum_{\substack{d\leqslant x^{\frac12-\epsilon}\\(d,x)=1}}\frac{3^{\nu(d)}|\mu(d)|}{\varphi(d)}\Bigg|\sum_{\chi_d\neq\chi_0}\int_{2-i\infty}^{2+i\infty}\left(\frac{x^\omega}{\omega}\right)\left(1+\frac{\omega}{(\log x)^{1.1}}\right)^{-[\log x]-1}$$

$$\cdot\ \overline{\chi_d}(x)\,\frac{L'}{L}(\omega,\ \chi_d)\sum_{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}}\left(\frac{\chi_d(p_1p_2)}{(p_1p_2)^\omega}\right)\left(\frac{d\omega}{\log\frac{x}{p_1p_2}}\right)\Bigg|.$$

由于 $\frac{L'}{L}(\omega,\ \chi_d)=\frac{L'}{L}(\omega,\ \chi^*_{d^*})+\sum_{p\mid\frac{d}{d^*}}\frac{\chi^*_{d^*}(p)\log p}{p^\omega-\chi^*_{d^*}(p)}$, 故有

$$M_4\leqslant M_2+M_5,\tag{9}$$

其中

$$M_5=\sum_{\substack{d\leqslant x^{\frac12-\epsilon}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{\varphi(d)}\Bigg|\sum_{\chi_d\neq\chi_0}\overline{\chi^*_{d^*}}(x)\int_{2-i\infty}^{2+i\infty}\left(\frac{x^\omega}{\omega}\right)\left(1+\frac{\omega}{(\log x)^{1.1}}\right)^{-[\log x]-1}$$

$$\cdot\left(\sum_{p\mid\frac{d}{d^*}}\frac{\chi^*_{d^*}(p)\log p}{p^\omega-\chi^*_{d^*}(p)}\right)\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}\\(p_1p_2,d)=1}}\frac{\chi^*_{d^*}(p_1p_2)}{(p_1p_2)^\omega\log\frac{x}{p_1p_2}}d\omega\Bigg|.$$

又当 $\mathrm{Re}\,\omega=2$ 时,有 $\frac{\chi^*_{d^*}(p)}{p^\omega-\chi^*_{d^*}(p)}=\sum_{\lambda=1}^{\infty}\left(\frac{\chi^*_{d^*}(p)}{p^\omega}\right)^\lambda$. 又当 $\lambda\geqslant1$, $\mu(d^*)\neq0$, $(d^*,\ xp_1p_2p^\lambda)=1$ 时,则使用引理 4, 我们有

$$\left|\sum_{\chi_{d^*}}{}^{*}\,\overline{\chi_{d^*}}(x)\chi_{d^*}(p_1p_2p^\lambda)\right|=\left|\sum_{\chi_{d^*}}{}^{*}\,\chi_{d^*}(p_1p_2p^\lambda y)\right|$$

$$\leqslant|(p_1p_2p^\lambda y-1,\ d^*)|=|(x-p_1p_2p^\lambda,\ d^*)|,\tag{10}$$

其中 y 满足 $xy\equiv1\ (\mathrm{mod}\,d^*)$ 的解. 又由 (10) 式及引理 1 得到

$$M_5\ll\sum_{\substack{d\leqslant x^{\frac12-\epsilon}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{\varphi(d)}\Bigg|\sum_{\substack{d^*\mid d\\d^*>1}}\sum_{p\mid\frac{d}{d^*}}(\log p)\sum_{\lambda=1}^{\infty}\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}\\(p_1p_2,d)=1}}\sum_{\chi_{d^*}}{}^{*}\,\overline{\chi_{d^*}}(x)\chi_{d^*}(p_1p_2p^\lambda)$$

$$\cdot\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\Phi\left(\frac{x}{p_1p_2p^\lambda}\right)\Bigg|\ll\sum_{\substack{d\leqslant x^{\frac12-\epsilon}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{\varphi(d)}\sum_{\substack{d^*\mid d\\d^*>1}}\sum_{p\mid\frac{d}{d^*}}\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}\\(p_1p_2,d)=1}}$$

$$\cdot\sum_{1\leqslant\lambda\leqslant\left(\log\frac{x}{p_1p_2}\right)(\log p)^{-1}}\left(\frac{\log p}{\log\frac{x}{p_1p_2}}\right)(x-p_1p_2p^\lambda,\ d^*)\ll\sum_{\substack{k_1k_2\leqslant x^{\frac12-\epsilon}\\(k_1k_2,x)=1}}\frac{|\mu(k_1)||\mu(k_2)|x^{\frac{\epsilon}{4}}}{\varphi(k_1)\varphi(k_2)}$$

$$\cdot\sum_{p\mid k_2}\sum_{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}}\sum_{1\leqslant\lambda\leqslant\left(\log\frac{x}{p_1p_2}\right)(\log p)^{-1}}(x-p_1p_2p^\lambda,\ k_1)\ll x^{\frac{\epsilon}{3}}\sum_{\substack{k_1\leqslant x^{\frac12-\epsilon}\\(k_1,x)=1}}$$

$$\cdot\frac{1}{k_1}\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}\\p^\lambda\leqslant\frac{x}{p_1p_2}}}(x-p_1p_2p^\lambda,\ k_1)\sum_{\substack{k_2\leqslant x^{\frac12-\epsilon}\\k_2\equiv0(\mathrm{mod}\,p)}}\frac{1}{k_2}\ll x^{\frac{\epsilon}{2}}\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}\\p^\lambda\leqslant\frac{x}{p_1p_2}}}$$

$$\cdot \frac{1}{p}\sum_{d\mid(x-p_1p_2p^{\lambda})} d \sum_{\substack{k_1\leqslant x^{\frac{1}{2}-\epsilon}\\ d\mid k_1}}\frac{1}{k_1}\ll x^{1-\epsilon}. \tag{11}$$

由(7)式，(8)式，(9)式及(11)式，本引理得证。

引理 6. 我们有

$$M_2\ll\frac{x}{(\log x)^{2.01}}.$$

证. 令

$$\begin{aligned}\Phi(y,\chi)&=\int_{2-i\infty}^{2+i\infty}\left(\frac{y^{\omega}}{\omega}\right)\left(1+\frac{\omega}{(\log x)^{1.1}}\right)^{-[\log x]-1}\frac{L'}{L}(\omega,\chi)d\omega\\&=\int_{1+\frac{1}{\log x}-i\infty}^{1+\frac{1}{\log x}+i\infty}\left(\frac{y^{\omega}}{\omega}\right)\left(1+\frac{\omega}{(\log x)^{1.1}}\right)^{-[\log x]-1}\frac{L'}{L}(\omega,\chi)d\omega.\end{aligned}$$

则有

$$\begin{aligned}M_2\leqslant&\sum_{\substack{1<l\leqslant x^{\frac{1}{2}-\epsilon}\\(l,x)=1}}\left\{\sum_{\substack{1<d\leqslant x^{\frac{1}{2}-\epsilon}\\ l\mid d,(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{\varphi(d)}\right\}\left|\sum_{\chi_l}{}^{*}\,\overline{\chi_l}(x)\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}\\(p_1p_2,d)=1}}\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\right.\\&\left.\cdot\,\Phi\left(\frac{x}{p_1p_2},\chi_l\right)\chi_l(p_1p_2)\right|\leqslant\sum_{\substack{1<d\leqslant x^{\frac{1}{2}-\epsilon}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{\varphi(d)}\\&\cdot\left\{\sum_{\substack{1<l\leqslant x^{\frac{1}{2}-\epsilon}\\(l,xd)=1}}\frac{|\mu(l)|3^{\nu(l)}}{\varphi(l)}\left|\sum_{\chi_l}{}^{*}\,\overline{\chi_l}(x)\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}\\(p_1p_2,d)=1}}\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\right.\right.\\&\left.\left.\cdot\,\Phi\left(\frac{x}{p_1p_2},\chi_l\right)\chi_l(p_1p_2)\right|\right\}.\end{aligned}$$

令 $\tau(l)=\sum_{d\mid l}1$，则有

$$\sum_{1<d\leqslant x^{\frac{1}{2}-\epsilon}}\frac{3^{\nu(d)}|\mu(d)|}{\varphi(d)}\ll(\log x)\sum_{d\leqslant x^{\frac{1}{2}-\epsilon}}\frac{(\tau(d))^2}{d}\ll(\log x)^5.$$

故有

$$M_2\ll(\log x)^6\max_{1<m\leqslant x^{\frac{1}{2}}}N_m. \tag{12}$$

其中

$$\begin{aligned}N_m=&\sum_{\substack{1<l\leqslant x^{\frac{1}{2}-\epsilon}\\(l,x)=1}}\frac{|\mu(l)|3^{\nu(l)}}{l}\left|\sum_{\chi_l}{}^{*}\,\overline{\chi_l}(x)\sum_{\substack{x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant(\frac{x}{p_1})^{\frac{1}{2}}\\(p_1p_2,m)=1}}\right.\\&\left.\cdot\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\Phi\left(\frac{x}{p_1p_2},\chi_l\right)\chi_l(p_1p_2)\right|.\end{aligned}$$

我们用 $\sum_{(k,m)}$ 来表示一个和式，其中的 p_1 和 p_2 经过且只经过 $x^{\frac{1}{10}}<p_1\leqslant x^{\frac{1}{3}}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac{1}{2}}$，$x^{\frac{13}{30}}2^k<p_1p_2\leqslant x^{\frac{13}{30}}2^{k+1}$，$(p_1p_2,m)=1$。令 l_1 是一个正整数，满足 $2^{l_1-1}(\log x)^{100}<x^{\frac{1}{2}-\epsilon}<$

$2^{l_1}(\log x)^{100}$, $l_2=\left[\frac{7\log x}{30\log 2}\right]$, 则有

$$N_m \leqslant \sum_{l=0}^{l_1}\sum_{k=0}^{l_2} N_m^{(l,k)}. \tag{13}$$

其中

$$N_m^{(0,k)}=\sum_{\substack{1<d\leqslant(\log x)^{100}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{d}\left|\sum_{\chi_d}{}^{*}\overline{\chi}_d(x)\sum_{(k,m)}\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\Phi\left(\frac{x}{p_1p_2},\chi_d\right)\chi_d(p_1p_2)\right|.$$

而当 $l\geqslant 1$ 时,

$$N_m^{(l,k)}=\sum_{\substack{2^{l-1}(\log x)^{100}<d\leqslant 2^l(\log x)^{100}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{d}$$

$$\cdot\left|\sum_{\chi_d}{}^{*}\overline{\chi}_d(x)\sum_{(k,m)}\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\Phi\left(\frac{x}{p_1p_2},\chi_d\right)\chi_d(p_1p_2)\right|.$$

令 $S(H,\omega,\chi_d)=\sum_{n=1}^{H}\frac{\mu(n)\chi_d(n)}{n^{\omega}}$, 其中 $H\ll x$. 我们知道当 $\operatorname{Re}\omega\geqslant 1$ 时,有

$$S(H,\omega,\chi_d)\ll\log x;\quad L(\omega,\chi_d)=\sum_{n=1}^{H}\frac{\chi_d(n)}{n^{\omega}}+O\left(\frac{|\omega|d^{\frac{1}{2}}\log d}{H}\right).$$

故得到当 $\operatorname{Re}\omega\geqslant 1$ 时,有

$$1-L(\omega,\chi_d)S(H,\omega,\chi_d)=\sum_{n=1}^{\infty}\frac{C_H(n)\chi_d(n)}{n^{\omega}}+O\left(\frac{|\omega|d^{\frac{1}{2}}(\log x)^2}{H}\right),$$

其中 $C_H(1)=0$, 当 $n>H^2$ 时, $C_H(n)=0$; 而当 $n>1$ 时, $C_H(n)=-\sum_{d}\mu(d)$, 其中 d 经过 n 的因子,它使得 $1\leqslant d\leqslant H$ 及 $\frac{n}{d}\leqslant H$; 当 $1\leqslant n\leqslant H$ 时, 有 $C_H(n)=0$; 而当 $n>H$ 时, $C_H(n)\leqslant\tau(n)$. 故 $H\ll x$ 时,由 Schwarz 不等式得到

$$\left|\sum_{n=1}^{\infty}\frac{C_H(n)\chi_d(n)}{n^{\omega}}\right|^2\ll(\log x)\sum_{l=0}^{3l_1}\left|\sum_{n=2^lH+1}^{2^{l+1}H}\frac{C_H(n)\chi_d(n)}{n^{\omega}}\right|^2.$$

令 $\alpha=1+\frac{1}{\log x}$, 由上式、$\sum_{n\leqslant x}\tau^2(n)\ll x(\log x)^3$ 及(3)式我们得到: 当 $Q\ll x$ 时,有

$$\sum_{D<d\leqslant Q}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\left|\sum_{n=2^lH+1}^{2^{l+1}H}\frac{C_H(n)\chi_d(n)}{n^{\alpha+i\nu}}\right|^2\ll\left(Q+\frac{2^lH}{D}\right)\sum_{n=2^lH+1}^{2^{l+1}H}\frac{(\tau(n))^2}{n^2}$$

$$\ll\left(\frac{Q}{2^lH}+\frac{1}{D}\right)(\log x)^3,$$

及

$$\sum_{D<d\leqslant Q}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}|1-L(\alpha+i\nu,\chi_d)S(H,\alpha+i\nu,\chi_d)|^2$$

$$\ll\sum_{D<d\leqslant Q}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\left|\sum_{n=1}^{\infty}\frac{C_H(n)\chi_d(n)}{n^{\alpha+i\nu}}\right|^2+\frac{|\alpha+i\nu|^2Q^2(\log x)^4}{H^2}$$

$$\ll \left(\frac{Q}{H}+\frac{1}{D}+\frac{|\alpha+i\nu|^2Q^2}{H^2}\right)(\log x)^5. \tag{14}$$

令 $\beta=\frac{1}{2}+\frac{1}{\log x}$，由于 $\{S(H,\beta+i\nu,\chi_d)\}^2=\sum_{n=1}^{H^2}\frac{j(n)\chi_d(n)}{n^{\beta+i\nu}}$，其中 $|j(n)|\leqslant\tau(n)$，故由(3)式可知，当 $l\geqslant 1$，$H\ll x$ 时，有

$$\sum_{2^{l-1}(\log x)^{100}<d\leqslant 2^l(\log x)^{100}}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}|S(H,\beta+i\nu,\chi_d)|^4$$

$$\ll\left(2^l(\log x)^{100}+\frac{H^2}{2^l(\log x)^{100}}\right)\sum_{n=1}^{H^2}\frac{(\tau(n))^2}{n}\ll 2^l(\log x)^{104}+\frac{H^2}{2^l(\log x)^{96}}. \tag{15}$$

由于 $L'(\omega,\chi_d)=\frac{1}{2\pi i}\int_r\frac{L(\xi,\chi_d)}{(\xi-\omega)^2}d\xi$，其中 r 是以 ω 为中心，$(\log x)^{-1}$ 为半径的圆，故有

$$|L'(\omega,\chi_d)|\ll(\log x)^2\int_r|L(\xi,\chi_d)|\,d\xi.$$

利用 Holder 不等式，得到

$$|L'(\omega,\chi_d)|^4\ll(\log x)^5\int_r|L(\xi,\chi_d)|^4|d\xi|.$$

又由引理 3，我们有

$$\sum_{2^{l-1}(\log x)^{100}<d\leqslant 2^l(\log x)^{100}}\left(\frac{1}{\varphi(d)}\right)\sum_{\chi_d}{}^{*}|L'(\beta+i\nu,\chi_d)|^4\ll 2^l(\log x)^{109}(|\beta+i\nu|)^2.$$

当 $\mathrm{Re}\,\omega\geqslant\alpha=1+\frac{1}{\log x}$ 时，我们得到

$$\frac{L'}{L}(\omega,\chi_d)=\left\{\frac{L'}{L}(\omega,\chi_d)\right\}\{1-L(\omega,\chi_d)S(H,\omega,\chi_d)\}+L'(\omega,\chi_d)S(H,\omega,\chi_d). \tag{16}$$

令

$$A(l,k,\omega,m,H)=\sum_{\substack{2^{l-1}(\log x)^{100}<d\leqslant 2^l(\log x)^{100}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{d}$$

$$\cdot\sum_{\chi_d}{}^{*}\left|\sum_{(k,m)}\frac{\chi_d(p_1p_2)}{(p_1p_2)^{\omega}\log\frac{x}{p_1p_2}}\right||1-L(\omega,\chi_d)S(H,\omega,\chi_d)|.$$

$$B(l,k,\omega,m,H)=\sum_{\substack{2^{l-1}(\log x)^{100}<d\leqslant 2^l(\log x)^{100}\\(d,x)=1}}\frac{|\mu(d)|3^{\nu(d)}}{d}$$

$$\cdot\sum_{\chi_d}{}^{*}\left|\sum_{(k,m)}\frac{\chi_d(p_1p_2)}{(p_1p_2)^{\omega}\log\frac{x}{p_1p_2}}\right||L'(\omega,\chi_d)S(H,\omega,\chi_d)|.$$

若 $l\geqslant 1$ 时，由(16)式我们有

$$N_m^{(l,k)}\ll x(\log x)^2\int_0^{\infty}\frac{A(l,k,\alpha+i\nu,m,H)}{|\alpha+i\nu|\left(1+\frac{|\alpha+i\nu|}{(\log x)^{1.1}}\right)^{[\log x]+1}}d\nu$$

$$+x^{\frac{1}{2}}\int_0^{\infty}\frac{B(l,k,\beta+i\nu,m,H)}{|\beta+i\nu|\left(1+\frac{|\beta+i\nu|}{(\log x)^{1.1}}\right)^{[\log x]+1}}d\nu. \tag{17}$$

显见，当 $|\mu(d)| \neq 0$ 及 d 很大时，有

$$3^{\nu(d)} \leqslant e^{\frac{3\log d}{\log\log d}}. \tag{18}$$

现在我们首先对 $l \geqslant 1$ 时，$2^k x^{\frac{13}{30}} > x^{\frac{1}{2}-\epsilon}$ 及 $x^{\frac{1}{2}-\epsilon} \geqslant 2^k x^{\frac{13}{30}} > 2^l(\log x)^{100}$ 这二种情形的 $N_m^{(l,k)}$ 进行估计，此时我们取 $H = 2^l(\log x)^{200} I_{l,x}$，其中 $I_{l,x} = e^{\frac{6\log\{2^l(\log x)^{100}\}}{\log\log\{2^l(\log x)^{100}\}}}$。则根据(14)—(18)式，我们有

$$\begin{aligned}
N_m^{(l,k)} \ll\ & x(\log x)^4 \int_0^\infty \Bigg[\Bigg\{ \sum_{\substack{2^{l-1}(\log x)^{100} < d \leqslant 2^l(\log x)^{100} \\ (d,x)=1}} \frac{|\mu(d)|}{d} \sum_{\chi_d}{}^* \Bigg| \sum_{(k,m)} \frac{\chi_d(p_1p_2)}{(p_1p_2)^{\alpha+i\nu}\log\frac{x}{p_1p_2}} \Bigg|^2 \Bigg\} \\
& \cdot \Bigg\{ \sum_{\substack{2^{l-1}(\log x)^{100} < d \leqslant 2^l(\log x)^{100} \\ (d,x)=1}} \frac{|\mu(d)|}{d} \sum_{\chi_d}{}^* \left| 1 - L(\alpha+i\nu,\chi_d) S(H,\alpha+i\nu,\chi_d) \right|^2 \Bigg\} I_{l,x} \Bigg]^{\frac{1}{2}} \\
& \cdot \left(\frac{d\nu}{1+\nu^{2.1}}\right) + x^{\frac{1}{2}}(\log x)^4 \int_0^\infty \Bigg\{ (I_{l,x}) \sum_{\substack{2^{l-1}(\log x)^{100} < d \leqslant 2^l(\log x)^{100} \\ (d,x)=1}} \frac{|\mu(d)|}{d} \sum_{\chi_d}{}^* \\
& \cdot \Bigg| \sum_{(k,m)} \frac{\chi_d(p_1p_2)}{(p_1p_2)^{\beta+i\nu}\log\frac{x}{p_1p_2}} \Bigg|^2 \Bigg\}^{\frac{1}{2}} \Bigg\{ \sum_{\substack{2^{l-1}(\log x)^{100} < d \leqslant 2^l(\log x)^{100} \\ (d,x)=1}} \frac{|\mu(d)|}{d} \sum_{\chi_d}{}^* \left| S(H,\beta+i\nu,\chi_d) \right|^4 \Bigg\}^{\frac{1}{4}} \\
& \cdot \left(\frac{d\nu}{1+\nu^4}\right) \Bigg\{ \sum_{\substack{2^{l-1}(\log x)^{100} < d \leqslant 2^l(\log x)^{100} \\ (d,x)=1}} \frac{|\mu(d)|}{d} \sum_{\chi_d}{}^* |L'(\beta+i\nu,\chi_d)|^4 \Bigg\}^{\frac{1}{4}} \\
\ll\ & x(\log x)^8 \int_0^\infty \Bigg\{ \left(2^l(\log x)^{100} + \frac{2^k x^{\frac{13}{30}}}{2^l(\log x)^{100}} \right) \Bigg(\sum_{2^k x^{\frac{13}{30}} < n \leqslant 2^{k+1} x^{\frac{13}{30}}} \frac{1}{n^2} \Bigg) \Bigg(\frac{2^l(\log x)^{100}}{H} \\
& + \frac{1}{2^l(\log x)^{100}} + \frac{(1+\nu^2)2^{2l}(\log x)^{200}}{H^2} \Bigg) (I_{l,x}) \Bigg\}^{\frac{1}{2}} \left(\frac{d\nu}{1+\nu^{2.1}}\right) + x^{\frac{1}{2}}(\log x)^8 \int_0^\infty \Bigg\{ \Bigg(2^l(\log x)^{100} \\
& + \frac{2^k x^{\frac{13}{30}}}{2^l(\log x)^{100}} \Bigg) (I_{l,x}) \Bigg\}^{\frac{1}{2}} \Big\{ 2^{2l}(\log x)^{213} + H^2(\log x)^{13} \Big\}^{\frac{1}{4}} (1+\nu^2)^{\frac{1}{4}} \left(\frac{d\nu}{1+\nu^4}\right) \ll \frac{x}{(\log x)^{20}}.
\end{aligned} \tag{19}$$

现在我们再对 $2^k x^{\frac{13}{30}} \leqslant 2^l(\log x)^{100} \leqslant 2x^{\frac{1}{2}-\epsilon}$ 时的 $N_m^{(l,k)}$ 进行估计，此时我们取

$$H = \max(2^{2l-k} x^{-\frac{13}{30}} (\log x)^{400} I_{l,x},\ x^{\frac{1}{2}-\epsilon}),$$

则有

$$\begin{aligned}
N_m^{(l,k)} \ll\ & x(\log x)^8 \int_0^\infty \Bigg\{ \left(2^l(\log x)^{100} + \frac{2^k x^{\frac{13}{30}}}{2^l(\log x)^{100}} \right) \Bigg(\sum_{2^k x^{\frac{13}{30}} < n \leqslant 2^{k+1} x^{\frac{13}{30}}} \frac{1}{n^2} \Bigg) \\
& \cdot \left(\frac{2^l(\log x)^{100}}{H} + \frac{1}{2^l(\log x)^{100}} + \frac{(1+\nu^2)2^{2l}(\log x)^{200}}{H^2} \right) (I_{l,x}) \Bigg\}^{\frac{1}{2}} \left(\frac{d\nu}{1+\nu^{2.1}}\right) \\
& + x^{\frac{1}{2}}(\log x)^4 \int_0^\infty \Bigg\{ \sum_{2^{l-1}(\log x)^{100} < d \leqslant 2^l(\log x)^{100}} \frac{|\mu(d)|}{d} \sum_{\chi_d}{}^* |S(H,\beta+i\nu,\chi_d)|^2 \Bigg\}^{\frac{1}{2}} (I_{l,x})^{\frac{1}{2}} \\
& \cdot \Bigg\{ \sum_{2^{l-1}(\log x)^{100} < d \leqslant 2^l(\log x)^{100}} \frac{|\mu(d)|}{d} \sum_{\chi_d}{}^* |L'(\beta+i\nu,\chi_d)|^4 \Bigg\}^{\frac{1}{4}}
\end{aligned}$$

$$\cdot\left\{\sum_{2^{l-1}(\log x)^{100}<d\leqslant 2^{l}(\log x)^{100}}\frac{|\mu(d)|}{d}\sum_{\chi_d}{}^{*}\left|\left(\sum_{(k,m)}\frac{\chi_d(p_1p_2)}{(p_1p_2)^{\beta+i\nu}\log\frac{x}{p_1p_2}}\right)^2\right|^2\right\}^{\frac14}\left(\frac{d\nu}{1+\nu^4}\right)$$

$$\ll\frac{x}{(\log x)^{20}}+x^{\frac12}(\log x)^{20}\left\{2^l(\log x)^{100}+\frac{H}{2^l(\log x)^{100}}\right\}^{\frac12}(I_{l,x})^{\frac12}(2^l(\log x)^{100})^{\frac14}$$

$$\cdot\left(2^l(\log x)^{100}+\frac{2^{2k}x^{\frac{13}{15}}}{2^l(\log x)^{100}}\right)^{\frac14}\int_0^{\infty}\frac{(1+\nu^2)^{\frac14}}{1+\nu^4}d\nu\ll\frac{x}{(\log x)^{20}}.\tag{20}$$

现在来估计 $N_m^{(0,k)}$，其中 $0\leqslant k\leqslant I_2$。当 χ_d 是原特征及 $\operatorname{Re}S\geqslant 1-\frac{c}{d^{\frac{1}{300}}}$时，$L(S,\chi_d)\neq 0$。其中 c 是一个常数，故有

$$N_m^{(0,k)}\ll\sum_{1<d\leqslant(\log x)^{100}}\frac{3^{\nu(d)}|\mu(d)|}{d}\sum_{\chi_d}{}^{*}\left|\int_{1-\frac{1}{(\log x)^{1/2}}-i\infty}^{1-\frac{1}{(\log x)^{1/2}}+i\infty}\sum_{(k,m)}\left(\frac{1}{\log\frac{x}{p_1p_2}}\right)\right.$$

$$\left.\cdot\ \chi_d(p_1p_2)\left(\frac{x}{p_1p_2}\right)^{\omega}\left(1+\frac{\omega}{(\log x)^{1.1}}\right)^{-[\log x]-1}\frac{L'}{L}(\omega,\chi_d)\frac{d\omega}{\omega}\right|$$

$$\ll(\log x)^{200}\sum_{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}}\left(\frac{x}{p_1p_2}\right)^{1-\frac{1}{(\log x)^{1/2}}}\ll\frac{x}{(\log x)^{20}}.\tag{21}$$

由 (12)，(13) 式及 (19)—(21) 式，本引理得证。

引理 7. 对于大偶数 x，我们有

$$M_1\leqslant\left\{\frac{(8+24\varepsilon)xC_x}{\log x}\right\}\left\{\sum_{x^{\frac{1}{10}}<p_1\leqslant x^{\frac13}<p_2\leqslant\left(\frac{x}{p_1}\right)^{\frac12}}\frac{1}{p_1p_2\log\frac{x}{p_1p_2}}\right\},$$

其中 $C_x=\prod_{\substack{p\mid x\\ p>2}}\frac{p-1}{p-2}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$。

证. 令 $S=\sum_{\substack{1\leqslant k\leqslant(x^{\frac12-\epsilon})^{\frac12}\\ (k,x)=1}}\frac{\mu^2(k)}{f(k)}$，则有

$$\lambda_d g(d)=\left(\frac{1}{S}\right)\sum_{\substack{1\leqslant k\leqslant(x^{\frac12-\epsilon})^{\frac12}/d\\ (k,xd)=1}}\frac{\mu(kd)\mu(k)}{f(kd)}.$$

当 $(m,x)=1$ 时，我们有

$$\sum_{\substack{d\leqslant(x^{\frac12-\epsilon})^{\frac12}\\ (d,x)=1,\ m\mid d}}\lambda_d g(d)=\left(\frac{1}{S}\right)\left(\sum_{\substack{d\leqslant(x^{\frac12-\epsilon})^{\frac12}\\ (d,x)=1,\ m\mid d}}\ \sum_{\substack{1\leqslant k\leqslant(x^{\frac12-\epsilon})^{\frac12}/d\\ (k,xd)=1}}\frac{\mu(kd)\mu(k)}{f(kd)}\right)$$

$$=\left(\frac{1}{S}\right)\sum_{\substack{1\leqslant r\leqslant(x^{\frac12-\epsilon})^{\frac12}\\ (r,x)=1}}\frac{\mu(r)}{f(r)}\sum_{m\mid d\mid r}\mu\left(\frac{r}{d}\right)=\frac{\mu(m)}{Sf(m)}.$$

由于 $\frac{1}{\varphi\left(\frac{d_1d_2}{(d_1,d_2)}\right)}=g(d_1)g(d_2)\sum_{d\mid(d_1,d_2)}f(d)$。故有

$$\sum_{\substack{d_1\leqslant(x^{\frac12-\epsilon})^{\frac12}\\ (d_1d_2,x)=1}}\ \sum_{d_2\leqslant(x^{\frac12-\epsilon})^{\frac12}}\frac{\lambda_{d_1}\lambda_{d_2}}{\varphi\left(\frac{d_1d_2}{(d_1,d_2)}\right)}=\sum_{\substack{d_1\leqslant(x^{\frac12-\epsilon})^{\frac12}\\ (d_1d_2,x)=1}}\ \sum_{d_2\leqslant(x^{\frac12-\epsilon})^{\frac12}}\lambda_{d_1}\lambda_{d_2}g(d_1)g(d_2)\sum_{k\mid(d_1,d_2)}f(k)$$

$$= \sum_{\substack{k \leqslant (x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}} \\ (k,x)=1}} f(k)\Bigl(\sum_{\substack{d \leqslant (x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}} \\ k \mid d,(d,x)=1}} \lambda_d g(d)\Bigr)^2 = \frac{1}{S}. \tag{22}$$

令 $V_k(x) = \sum\limits_{\substack{1 \leqslant n \leqslant x \\ (n,k)=1}} \dfrac{\mu^2(n)}{\varphi(n)}$，则有

$$\log x \leqslant \sum_{n=1}^{x} \frac{1}{n} \leqslant \sum_{1 \leqslant n \leqslant x} \frac{\mu^2(n)}{n} \prod_{p \mid n}\Bigl(\sum_{l=0}^{\infty} \frac{1}{p^l}\Bigr) = \sum_{1 \leqslant n \leqslant x} \frac{\mu^2(n)}{n} \prod_{p \mid n}\Bigl(1-\frac{1}{p}\Bigr)^{-1}$$

$$= V_1(x) = \sum_{d \mid k} \sum_{\substack{1 \leqslant n \leqslant x \\ (n,k)=d}} \frac{\mu^2(n)}{\varphi(n)} = \sum_{d \mid k} \frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{1 \leqslant m \leqslant x/d \\ (m,k)=1}} \frac{\mu^2(m)}{\varphi(m)} \leqslant \sum_{d \mid k} \frac{\mu^2(d)}{\varphi(d)} V_k(x)$$

$$= \frac{k V_k(x)}{\varphi(k)},$$

故有 $V_k(x) \geqslant \dfrac{\varphi(k)\log x}{k}$. 令 $\phi(1)=1$，而当 $q>2$ 时，令 $\phi(q) = \prod\limits_{p \mid q}(p-2)$，则有

$$S = \sum_{\substack{1 \leqslant k \leqslant (x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}} \\ (k,x)=1}} \frac{\mu^2(k)}{\varphi(k)} \prod_{p \mid k}\Bigl(1+\frac{1}{p-2}\Bigr) = \sum_{\substack{1 \leqslant k \leqslant (x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}} \\ (k,x)=1}} \frac{\mu^2(k)}{\varphi(k)} \sum_{q \mid k} \frac{1}{\phi(q)}$$

$$= \sum_{\substack{q \leqslant (x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}} \\ (q,x)=1}} \frac{\mu^2(q)}{\phi(q)\varphi(q)} \sum_{\substack{r \leqslant (x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}}/q \\ (r,qx)=1}} \frac{\mu^2(r)}{\varphi(r)} \geqslant \sum_{\substack{q \leqslant (x^{\frac{1}{2}-\epsilon})^{\frac{1}{2}} \\ (q,x)=1}} \frac{\mu^2(q)}{\phi(q)\varphi(q)} \Bigl\{\frac{\varphi(qx)}{qx} \log \frac{x^{\frac{1}{4}-\frac{\epsilon}{2}}}{q}\Bigr\}$$

$$= \Bigl(\frac{\varphi(x)}{x}\Bigr)(\log x^{\frac{1}{4}-\frac{\epsilon}{2}}) \prod_{p \nmid x}\Bigl(1+\frac{1}{p(p-2)}\Bigr) + O(1) = \frac{\Bigl(\frac{1}{8}-\frac{\epsilon}{4}\Bigr)(\log x)}{C_x} + O(1).$$

由(22)式及上式，当 x 很大时，有

$$M_1 \leqslant (8+24\epsilon)C_x(\log x)^{-1} \sum_{\substack{x^{\frac{1}{10}} < p_1 < x^{\frac{1}{3}} < p_2 \leqslant (\frac{x}{p_1})^{\frac{1}{2}} \\ n \leqslant \frac{x}{p_1 p_2}}} \Bigl(\frac{\Lambda(n)}{\log \frac{x}{p_1 p_2}}\Bigr) \Phi\Bigl(\frac{x}{p_1 p_2 n}\Bigr).$$

由引理 1，本引理得证.

引理 8. 设 x 是大偶数，则有

$$\Omega \leqslant \frac{3.9404 x C_x}{(\log x)^2}.$$

证. 当 x 很大时，由引理 5 到引理 7，我们有

$$\Omega \leqslant \Bigl\{\frac{8(1+5\epsilon)xC_x}{\log x}\Bigr\} \Bigl\{\sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant (\frac{x}{p_1})^{\frac{1}{2}}} \frac{1}{p_1 p_2 \log \frac{x}{p_1 p_2}}\Bigr\}, \tag{23}$$

又有

$$\sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}} < p_2 \leqslant (\frac{x}{p_1})^{\frac{1}{2}}} \frac{1}{p_1 p_2 \log \frac{x}{p_1 p_2}} \leqslant (1+\epsilon) \sum_{x^{\frac{1}{10}} < p_1 \leqslant x^{\frac{1}{3}}} \int_{x^{\frac{1}{3}}}^{(\frac{x}{p_1})^{\frac{1}{2}}} \frac{dt}{p_1 t (\log t) \log \frac{x}{p_1 t}}$$

$$\leqslant (1+2\epsilon) \int_{x^{\frac{1}{10}}}^{x^{\frac{1}{3}}} \frac{dS}{S \log S} \int_{x^{\frac{1}{3}}}^{(\frac{x}{S})^{\frac{1}{2}}} \frac{dt}{t(\log t)\Bigl(\log \frac{x}{St}\Bigr)} = (1+2\epsilon) \int_{\frac{1}{10}}^{\frac{1}{3}} \frac{d\alpha}{\alpha} \int_{\frac{1}{3}}^{\frac{1-\alpha}{2}} \frac{d\beta}{\beta(1-\alpha-\beta)\log x},$$

及

$$\int_{\frac{1}{10}}^{\frac{1}{3}}\frac{d\alpha}{\alpha}\int_{\frac{1}{3}}^{\frac{1-\alpha}{2}}\left(\frac{1}{1-\alpha}\right)\left(\frac{1}{\beta}+\frac{1}{1-\alpha-\beta}\right)d\beta=\int_{\frac{1}{10}}^{\frac{1}{3}}\frac{\log\frac{1-\alpha}{2}-\log\frac{1}{3}-\log\frac{1-\alpha}{2}+\log\left(\frac{2}{3}-\alpha\right)}{\alpha(1-\alpha)}d\alpha$$

$$=\int_{\frac{1}{10}}^{\frac{1}{3}}\frac{\log(2-3\alpha)}{\alpha(1-\alpha)}d\alpha=\sum_{i=0}^{6}\int_{\frac{1}{10}+\frac{i}{30}}^{\frac{1}{10}+\frac{i+1}{30}}\frac{\log\left(1.6-\frac{i}{10}\right)}{\alpha(1-\alpha)}d\alpha+\sum_{i=0}^{6}\int_{\frac{1}{10}+\frac{i}{30}}^{\frac{1}{10}+\frac{i+1}{30}}\frac{\log\frac{2-3\alpha}{1.6-0.i}}{\alpha(1-\alpha)}d\alpha$$

$$\leqslant\sum_{i=0}^{6}\left\{\log\left(1.6-\frac{i}{10}\right)\right\}\left\{\log\frac{\frac{9}{10}-\frac{i}{30}}{\frac{1}{10}+\frac{i}{30}}-\log\frac{\frac{9}{10}-\frac{1}{30}-\frac{i}{30}}{\frac{1}{10}+\frac{i+1}{30}}\right\}$$

$$+\sum_{i=0}^{6}\int_{\frac{1}{10}+\frac{i}{30}}^{\frac{1}{10}+\frac{i+1}{30}}\frac{(0.4+0.i-3\alpha)}{(1.6-0.i)\alpha(1-\alpha)}d\alpha\leqslant\sum_{i=0}^{6}\left\{\log(1.6-0.i)+\frac{4+i}{16-i}\right\}\left\{\log\frac{27-i}{3+i}\right.$$

$$\left.-\log\frac{26-i}{4+i}\right\}-3\sum_{i=0}^{6}\int_{\frac{1}{10}+\frac{i}{30}}^{\frac{1}{10}+\frac{i+1}{30}}\frac{d\alpha}{(1.6-0.i)(1-\alpha)}=\sum_{i=0}^{6}\left\{\log(1.6-0.i)+\frac{4+i}{16-i}\right\}$$

$$\cdot\left\{\log\frac{108+23i-i^2}{78+23i-i^2}\right\}-3\sum_{i=0}^{6}\left(\frac{1}{1.6-0.i}\right)\left(\log\frac{27-i}{26-i}\right)$$

$$\leqslant(0.47+0.25)(0.32542)+(0.40547+0.33334)(0.26236)+(0.33647$$
$$+0.42858)(0.22315)+(0.26236+0.53847)(0.19671)+(0.18232$$
$$+0.66667)(0.17799)+(0.09531+0.81819)(0.16431)$$
$$+0.15415-3\left(\frac{0.03774}{1.6}+\frac{0.03922}{1.5}+\frac{0.04082}{1.4}+\frac{0.04256}{1.3}+\frac{0.04445}{1.2}\right.$$
$$\left.+\frac{0.04652}{1.1}+0.04879\right)\leqslant0.234303+0.193837+0.17073+0.15754+0.151115$$
$$+0.1501+0.15415-3(0.023587+0.026146+0.029157+0.032738$$
$$+0.037041+0.04229+0.04879)\leqslant1.21178-0.71924=0.49254. \tag{24}$$

由(23)和(24)式,引理8得证.

设 x 是一大偶数,令 $P_x(x, x^{\frac{1}{10}})$ 表示满足下面条件的素数 p 的个数: $p\leqslant x$, $p\not\equiv x(\operatorname{mod} p_i)$ $(1\leqslant i\leqslant j)$, 其中 $3=p_1<p_2<\cdots<p_j\leqslant x^{\frac{1}{10}}$. 对于一个素数 p', 则令 $P_x(x, p', x^{\frac{1}{10}})$ 表示满足下面条件的素数 p 的个数: $p\leqslant x$, $p\equiv x\ (\operatorname{mod} p')$, $p\not\equiv x\ (\operatorname{mod} p_i)$ $(1\leqslant i\leqslant j)$. 其中 $3=p_1<p_2<\cdots<p_j\leqslant x^{\frac{1}{10}}$.

引理 9. 设 x 是大偶数,则有

$$P_x(x, x^{\frac{1}{10}})-\left(\frac{1}{2}\right)\sum_{x^{\frac{1}{10}}<p\leqslant x^{\frac{1}{3}}}P_x(x, p, x^{\frac{1}{10}})\geqslant\frac{2.6408xC_x}{(\log x)^2},$$

其中 $C_x=\prod_{\substack{p\mid x\\ p>2}}\frac{p-1}{p-2}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$.

证. 在文献[11]中取 $r(p)=\frac{p}{p-1}$, $K=x$, $Z=x^{\frac{1}{10}}$, 则显见文献[11]中的条件 (A_1) 和

(A_2) 都满足，由文献 [11] 中的 (2.11) 式，我们有

$$\begin{aligned}\Gamma_x(x^{\frac{1}{10}}) &= \frac{x}{\varphi(x)}\prod_{p\nmid x}\frac{1-\dfrac{1}{p-1}}{1-\dfrac{1}{p}}\frac{e^{-r}}{\log x^{\frac{1}{10}}}\left\{1+O\left(\frac{1}{\log x}\right)\right\}\\ &= \frac{x}{\varphi(x)}\prod_{\substack{p\mid x\\ p>2}}\frac{(p-1)^2}{p(p-2)}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\frac{e^{-r}}{\log x^{\frac{1}{10}}}\left\{1+O\left(\frac{1}{\log x}\right)\right\}\\ &= \frac{20e^{-r}C_x}{\log x}\left\{1+O\left(\frac{1}{\log x}\right)\right\}. \qquad (25)\end{aligned}$$

其中 r 是 Eular 常数. 又当 $0<u\leqslant 2$ 时，令 $F(u)=\dfrac{2e^r}{u}$，$f(u)=0$. 而当 $u\geqslant 2$ 时，令 $(uF(u))'=f(u-1)$，$(uf(u))'=F(u-1)$，当 $2<u\leqslant 3$ 时，有 $uF(u)=2F(2)$，$F(u)=\dfrac{2e^r}{u}$. 又当 $2<u\leqslant 4$ 时，则有

$$uf(u)=\int_2^u F(t-1)dt=2e^r\log(u-1),\quad f(u)=\frac{2e^r\log(u-1)}{u}.$$

当 $3\leqslant u\leqslant 4$ 时，我们有

$$uF(u)=2e^r+\int_3^u f(t-1)dt=2e^r\left(1+\int_2^{u-1}\frac{\log(t-1)}{t}dt\right),$$

又有

$$5f(5)=2e^r\log 3+\int_4^5 F(u-1)du=2e^r\left(\log 4+\int_3^4\frac{du}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt\right).$$

在文献[11]的定理A中，取 $\xi^2=x^{\frac{1}{2}-\epsilon}$，$q=1$，$z=x^{\frac{1}{10}}$，则由 (25) 式及文献 [11] 中的 (2.19)，(4.18) 及 (3.24) 式，我们知道当 x 很大时，有

$$\begin{aligned}P_x(x,x^{\frac{1}{10}}) &\geqslant \frac{2(1-\sqrt{\epsilon})e^{-r}xC_xf(5)}{(\log x)(\log x^{\frac{1}{10}})}\geqslant\left\{\frac{8(1-\sqrt{\epsilon})xC_x}{(\log x)^2}\right\}\\ &\quad\cdot\left\{\log 4+\int_3^4\frac{du}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt\right\}. \qquad (26)\end{aligned}$$

又在文献[11]的定理A中取 $\xi^2=\dfrac{x^{\frac{1}{2}-\epsilon}}{p}$，$q=p$，$z=x^{\frac{1}{10}}$，则由 (25) 式及文献 [11] 中的 (2.18)，(3.24) 及 (4.18)，我们有

$$\begin{aligned}\sum_{x^{\frac{1}{10}}<p\leqslant x^{\frac{1}{3}}}P_x(x,p,x^{\frac{1}{10}}) &\leqslant\left\{\frac{20(1+\sqrt{\epsilon})e^{-r}xC_x}{(\log x)^2}\right\}\left\{\sum_{x^{\frac{1}{10}}<p\leqslant x^{\frac{1}{5}}}\left(\frac{2e^r}{p}\right)\right.\\ &\quad\cdot\left(1+\int_2^{4-\frac{10\log p}{\log x}}\frac{\log(t-1)}{t}dt\right)\left(\frac{\log x^{\frac{1}{10}}}{\log\frac{x^{\frac{1}{2}}}{p}}\right)+\left.\sum_{x^{\frac{1}{5}}<p\leqslant x^{\frac{1}{3}}}\frac{2e^r\log x^{\frac{1}{10}}}{p\log\frac{x^{\frac{1}{2}}}{p}}\right\}\\ &\leqslant\left\{\frac{(4+5\sqrt{\epsilon})xC_x}{\log x}\right\}\left\{\int_{x^{\frac{1}{10}}}^{x^{\frac{1}{5}}}\frac{dS}{S(\log S)\left(\log\frac{x^{\frac{1}{2}}}{S}\right)}\int_2^{4-\frac{10\log S}{\log x}}\frac{\log(t-1)}{t}dt\right.\end{aligned}$$

$$+\int_{x^{\frac{1}{10}}}^{x^{\frac{1}{5}}}\frac{dS}{S(\log S)\left(\log\frac{x^{\frac{1}{2}}}{S}\right)}\Bigg\}=\left\{\frac{(4+5\sqrt{\epsilon})xC_x}{(\log x)^2}\right\}\left\{\int_{\frac{1}{10}}^{\frac{1}{5}}\frac{d\alpha}{\alpha\left(\frac{1}{2}-\alpha\right)}\int_2^{4-10\alpha}\frac{\log(t-1)}{t}dt\right.$$

$$+\int_{\frac{1}{10}}^{\frac{1}{5}}\frac{d\alpha}{\alpha\left(\frac{1}{2}-\alpha\right)}\Bigg\}=\left\{\frac{(8+10\sqrt{\epsilon})xC_x}{(\log x)^2}\right\}$$

$$\cdot\left\{\log 8+\int_{\frac{1}{10}}^{\frac{1}{5}}\frac{d\alpha}{2\alpha\left(\frac{1}{2}-\alpha\right)}\int_2^{4-10\alpha}\frac{\log(t-1)}{t}dt\right\}.$$

令 $4-10\alpha=u-1$, $\alpha=\frac{5-u}{10}$, $\frac{d\alpha}{\alpha\left(\frac{1}{2}-\alpha\right)}=-\frac{10du}{u(5-u)}$, 又当 $\alpha=\frac{1}{10}$ 时, 有 $u=4$, 而当 $\alpha=\frac{1}{5}$ 时, $u=3$, 故有

$$\int_{\frac{1}{10}}^{\frac{1}{5}}\frac{d\alpha}{\alpha\left(\frac{1}{2}-\alpha\right)}\int_2^{4-10\alpha}\frac{\log(t-1)}{t}dt=\int_3^4\frac{10du}{u(5-u)}\int_2^{u-1}\frac{\log(t-1)}{t}dt.$$

显见,当 $1\leqslant x\leqslant 2$ 时,有 $\log x\leqslant\frac{x-1}{2}+\frac{x-1}{1+x}$, 故有

$$\int_3^4\frac{du}{u}\int_2^{u-1}\frac{\log(t-1)}{t}dt-\left(\frac{1}{4}\right)\int_{\frac{1}{10}}^{\frac{1}{5}}\frac{d\alpha}{\alpha\left(\frac{1}{2}-\alpha\right)}\int_2^{4-10\alpha}\frac{\log(t-1)}{t}dt$$

$$=\int_3^4\left(\frac{1}{u}-\frac{2.5}{u(5-u)}\right)du\int_2^{u-1}\frac{\log(t-1)}{t}dt\geqslant\int_3^4\left\{\frac{2.5-u}{u(5-u)}\right\}du$$

$$\cdot\int_2^{u-1}\left(\frac{t-2}{2}+\frac{t-2}{t}\right)\left(\frac{dt}{t}\right)=\int_3^4\left\{\frac{2.5-u}{2u(5-u)}\right\}\left(u-3+\frac{4}{u-1}-2\right)du$$

$$=\int_3^4\left(\frac{1}{2}-\frac{2.25}{u}-\frac{1}{4(5-u)}+\frac{0.75}{u-1}\right)du=\frac{1}{2}-2.25\log\frac{4}{3}-\frac{\log 2}{4}$$

$$+0.75\log\frac{3}{2}=\frac{1}{2}+0.75\log\frac{9}{8}-1.5\log\frac{4}{3}-\frac{\log 2}{4}$$

$$\geqslant 0.588335-0.6048075=-0.0164725. \tag{27}$$

由(26)和(27)式,我们有

$$P_x(x,x^{\frac{1}{10}})-\left(\frac{1}{2}\right)\sum_{x^{\frac{1}{10}}<p\leqslant x^{\frac{1}{3}}}P_x(x,p,x^{\frac{1}{10}})\geqslant\left(\frac{(8-50\sqrt{\epsilon})xC_x}{(\log x)^2}\right)$$

$$\cdot\left(\log 4-\frac{\log 8}{2}-0.0164725\right)\geqslant\frac{(8xC_x)(0.3301)}{(\log x)^2}.$$

故引理 9 得证。

三、结　　果

显见，我们有

$$P_x(1,\ 2) \geqslant P_x(x,\ x^{\frac{1}{10}}) - \left(\frac{1}{2}\right) \sum_{x^{\frac{1}{10}} < p \leqslant x^{\frac{1}{3}}} P_x(x,\ p,\ x^{\frac{1}{10}}) - \frac{\Omega}{2} - x^{0.91}. \tag{28}$$

由(28)式、引理 8 和引理 9，即得到定理 1

$$(1,\ 2)\ \text{及}\ P_x(1,\ 2) \geqslant \frac{0.67 x C_x}{(\log x)^2}$$

的证明.

完全类似的方法可得到定理 2 的证明.

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