the ring $X(M)$

Chapter 1. Tensors

Let $v_1, \cdots, v_s$ modules over a ring.

ring $\times G \to G$

transfer G_1 to G_2

information of coordinate

(~ vector field over $F(M)$ ~~geometries~~ functions on M

or T_pM over $\mathbb{R}$)

$V_1 \otimes \cdots \otimes V_s = \{ (v_1, \cdots, v_s) \quad v_i \in V_{is} \}$ two index ?

A quemetion

$A: V_1 \otimes \cdots \otimes V_s \longrightarrow \mathbb{R}$ is komulimann

If each componient is linear,

$v \longrightarrow A(v_1, \cdots, v_s)$ is keimen

linearly $\{v_1, \cdots, v_s\} \to \mathbb{R}$ ↑ any way

If v is a module over k, the V^* is the set of kolimen quetions from v to k.

Def. A tensor of type (r, s). $(r, s \in \mathbb{N}^+)$ ...

is a k-multilinear koliman function.

A dual $(V^*)^s \otimes V^s \longrightarrow$ function

(Metrix_1 ~~θ~~ $\times \text{Metrix}_2 \to |\ |$)

$T^r_s(V)$ is the set of tensors of type (r, s).

A tensor of type $(0, 0)$ is an element of k.

A tensor field is a tensor on the ring $X(M)$.

set of vector fields on M

Over the ring, $F(M) = \begin{cases} X: M \longrightarrow TM \\ p \longmapsto X(p) \in T_pM \end{cases}$

geometries on M

A tensor of type (r, s)

$A: (X^*(M))^r \otimes (X(M))^s \longrightarrow F(M)$

If $\theta^1, \cdots, \theta^r$ 1-forms on M

$\in X^*(M)$ vector field

and $X_1, \cdots, X_s$

$A(\theta^1, \cdots, \theta^r; X_1, \cdots, X_s)$ is a function on M.

A tensor field of $(0, 0)$ is a function.

$F(M)$ multilinear

↑ check it before ~~being~~ tensor

Important questions

f $\downarrow$ $\qquad\qquad$ f(1-forms) = f apply to vectors ?

$$A(\theta^1, \cdots, \theta^r, X_1, \cdots fX_i, \cdots, X_s) = f\, A(\theta^1, \cdots, X_1, \cdots)$$

$f \in F(M)$ $\qquad$ if f is non-linear, ~~the amplitude of 1-form change variously~~.

1) Examples

$$E: \mathcal{X}^*(M) \times \mathcal{X}(M) \longrightarrow F(M)$$

$$(\theta, X) \longrightarrow \underbrace{\theta(X)}_{\text{Scalar}}$$

R linear respect to X

$$E(\theta, X) \longrightarrow \theta(X)$$

$$E(f\theta, X) \longrightarrow f(\theta)(X) = f(\theta X)$$

$$E(\theta, fX) \longrightarrow \theta f(X) = f\theta(X)$$

g is function. If $g, f \in F(M)$, then define $\underline{\theta(gX) = gX(\overset{\theta}{f})}$

2) Counterexamples.

For $\omega \in \mathcal{X}^*(M)$

$$F: \mathcal{X}(M)^2 \longrightarrow F(M)$$

$$(X, Y) \longmapsto X(\omega(Y))$$

This is not $F(M)$-linear in Y

$$X(\omega(gY)) = X(g(\omega Y))$$

we can sum up tensors of the same type and <u>multiply tensors of different types.</u>

$$\left.\begin{array}{l} A \in T^{r,s} \\ B \in T^{r',s'} \end{array}\right. \Rightarrow A \otimes B \in T^{r+s', s+r'}$$

$$(A \otimes B)(\theta^1, \cdots, \theta^{r+s'}, X_1, \cdots, X_{s+r'})$$
$$= A(\theta^1, \cdots, \theta^r, X_1, \cdots X_s)\, B(\theta^{r+1}, \cdots, \theta^{r+s'}, X_{s+1}, \cdots, X_{s+r'})$$

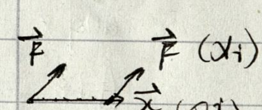
Proposition. Let $p \in M$, $A \in T^{r,s}(M)$

Let $\overline{\theta^1}, \cdots, \overline{\theta^r}$ and $\theta^1, \cdots, \theta^r$ 1-forms

such that $\overline{\theta^i}(p) = \theta^i(p)$, $\forall\, i = 1, \cdots, r$

Let $\overline{X_1}, \cdots, \overline{X_s}, X_1, \cdots, X_s$ vector fields.

Such that $\overline{X_i}(p) = X_i(p)$, $\forall\, i = 1, \cdots, s$



$W = \vec{F} \cdot \vec{x}$

Then. $A(\overline{\theta^1}, \cdots, \overline{\theta^r}, \overline{X_1}, \cdots, \underset{\text{vector}}{\overline{X_s}})(p) = A(\theta^1, \cdots, \theta^r, \underset{\uparrow}{X_1}, \cdots, X_s)$

if one = 0, all = 0

Tensor doesn't depends on vector field, only depends on $(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s)$ on point p.

We can define $A_p \in (T_p^*M)^r \otimes (T_pM)^s \longrightarrow \mathbb{R}$

$A \quad \underset{\text{associate to}}{p \in M \longrightarrow A_p}$ depends on the basis we choose

To proof

Then, we also show that if $\theta^{s_0}(p) = 0$ or $X_{j_0}(p) = 0$

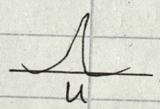
then $A(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s)(p) = 0$.

Chart $(u, \eta = (x^1, \cdots, x^n))$

$p \in u$. $X_{j_0} = \sum_{i,j} \underbrace{X^i}_{\text{functions}} \partial_j$ (locally defined) $X_j(p) = 0 \Leftrightarrow X^i(p) = 0, \forall i$

/. amplitude along direction of j

f a bump function on u



$f(p) = 1$

$\underbrace{f^2}_{\text{global function}} X_{j_0}$ is a vector field

$= \sum_i \underbrace{g x^i}_{\text{function}} (f \partial_i)$

$f^2 A(\theta^1, \cdots, \theta^r, X_1, \cdots, X_{j_0}, \cdots, X_s)$

$= A(\theta^1, \cdots, \theta^r, X_1, \cdots, f^2 X_{j_0}, \cdots, X_s)$

$= \sum_i f X^i A(\theta^1, \cdots, \theta^r, X_1, \cdots f^2 X_{j_0}, \cdots, X_s)$

$\Rightarrow A(\_\_\_ \quad) = 0$ evalue at $p \rightarrow$ global

Chart $(u, \xi = (x^1, \cdots, x^m))$ $p \in u \in M$

$A = (dx^{i_1}, \cdots, dx^{i_r}, \partial_{j_1}, \cdots, \partial_{j_s})$ $\partial_i, i = 1, \cdots, n$ vector field

(on u) $dx^i, i = 1, \cdots, n$ formal way

$A^{i_1, \cdots, i_r}_{j_1, \cdots, j_s}$ (functions on u)

and $A = \sum A^{i_1, \cdots, i_r}_{j_1, \cdots, j_s} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \cdots \otimes dx^{j_s}$

The contraction is $C : \mathcal{X}^*(M) \otimes \mathcal{X}(M) \longrightarrow F(M)$

$(\underset{\text{1-forms}}{\theta}, \underset{\text{field vectors associate}}{X}) \longmapsto \underset{\text{scalar of point } p}{\theta(X)}$

The constraction of A (on the index i of j)

is the tensor field of type $(r-1, s-1)$

$(C^i_j)(\theta^1, \cdots, \theta^{r-1}, X_1, \cdots, X_{s-1})$

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a compaction of ϵ and $(\theta, X) \longmapsto A(\theta', \cdots, \underset{i}{\theta}, \cdots, \theta^r, X_1, \cdots \underset{j}{X}, \cdots X_s)$

In local coordinates,

the component of $(C^i_j A)$ are $\sum_{m=1}^{m} A^{i_1, \cdots, i_m, \cdots i_r}_{j_1, \cdots j_m, \cdots j_s}$

Def. Let $\phi : M \longrightarrow N$ be a differentiable map

If $A \subset T^r_s(N)$, then $(\phi^* A)(x_1, \cdots, x_s) = A\phi(d\phi(x_1), \cdots, d\phi(x_s))$

$x_1, \cdots, x_s \in \mathcal{X}(\underline{M})$ vector fields on M

$\phi^* A$ = product of A by ϕ

$\phi^* A$ is a $(0, s)$ type of tensors.

$(\theta', \cdots, \theta^{r-1}, X_1, \cdots, X_{s-1}) \longrightarrow \sum_{s=1}^{m} A(dx_i \theta', dx_i \alpha, \theta^{r-1}, \partial x_i X_1, \partial x_i X, \partial x_{s-1} X_{s-1})$

i position　　　　j position

Def. A deviation of tensor is a $\mathbb{R}$ linear map.

$D : T^{r,s}(M) \longrightarrow T^{r,s}(M)$

such that:

- $D(A \otimes B) = DA \otimes B + A \otimes DB$
- $D(CA) = C D(A)$　　　(constraction)

For a function of f　$(0,0)$ tensor

$f \otimes A \cdots$　　$\underline{fA}$　　D^r_s

$D(fA) = fDA + (D^r_s)A$

Def. A derivation of tensor is a $\mathbb{R}$ linear map
$D : T^{r,s}(M) \longrightarrow T^{r,s}(M)$
such that

- $D(A \otimes B) = DA \otimes B + A \otimes DB$
- $D(CA) = C(DA)$ (construction)

For a function of f $(0,0)$ tensor
$f \otimes A \cdots fA \quad D^r_s$
So $D(fA) = fDA + (\underbrace{D^r_s}_{\text{a tensor}})A$

And D is a derivation of functions,
So, there exists a $V \in$ ____ , $Df = V(f)$
The chain rule becomes $D < A(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s)$

$$= DA(\theta^1, \cdots, \theta^r, X_1, \cdots, X_s) \quad \text{product}$$
$$+ \sum_1^n A(\theta^1, \ldots, D\theta^j, \theta^r, X_1 \cdots X_s)$$
$$+ \sum_1^n A(\theta^1, \cdots, \theta^r, X_1, DX_j, \cdots X_s)$$

become $A(\theta, X) = \Sigma(A \otimes \theta \otimes X)$. double construction
Theorem. Given a vector field V and an $\mathbb{R}$ linear function δ :
$\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ such that $\underbrace{\delta}_{\text{derivator}}(fX) = VfX + f(\delta X)$
$df \cdot X$

$\Rightarrow$ there exists a vector derivation of tensors.

$$\left| \begin{array}{l} = \delta \text{ on } \mathfrak{X}(M) \text{ and} \\ = V \text{ on } \mathcal{F}(M) \end{array} \right|$$

Derivation of field $\xrightarrow{\text{know}}$ tensors

Def. Let $\underbrace{V}_{\text{vector field.}} \in \mathfrak{X}(M)$

then L_V. $L_V(f) = V(f) \quad \forall f \in \mathcal{F}(M)$
$L_V(X) = [V, X], \forall X \in \mathfrak{X}(M)$
$\underbrace{[V, X]}_{\text{vector field}}(f) = V(\underbrace{X(f)}_{\text{not vector field}}) - X(V(f))$

is a derivation of tensors. called the Lie derivative Linear Algebra.
V is a linear space.

Def. A bilinear form on V, is a function on $V \times V$, synthetics. It is positive (negative) defined if $\forall u \neq 0$,

$b(u,u) > 0$ ($b(u,u) < 0$)

2) positive if $b(u,u) \geq 0$.
negtive $b(u,u) \leq 0$

3) nondegenerate if $b(u,w) = 0$, $\forall w \in V \Rightarrow$ ~~the ux~~ $u = 0$

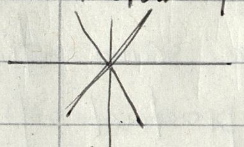
The index of b is the dimention of the largest subspace $W \subset V$, such that $b|_W$ is negtive define.

Then $q(u) = b(u,u)$ is the quadratic form.

A scalar product $U \cdot V$ is a symmetric bilinear form that is nondegenerate.

Ex. $g : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

index $= 1$ $g(u,w) = v_1 w_1 - v_2 w_2$



g is symetric bilinear non degenerate.

A vector is called null (isotrophic) if $q(u) = 0$, but $u \neq 0$

(eg $(1,1)$ for g)

The constuction of A (on the index i of j) is the tensor field of type $(r-1, s-1)$

$(C^i_j A)(\theta^1, \cdots, \theta^{r-1}, X_1, \cdots, X_{s-1})$

If b is undefined $\Rightarrow$ orthorgnal changes: $w = w' = (1,1)$ are orthorgnal

$(b, 1)$ and $(1, b)$ are orthorgnal

$W \subset V$

If $W^\perp = \{ u \in W, u \perp W \}$

$\Rightarrow W + W^\perp \neq V$

A subspace W is called nondegenerate, if $g|_W$ is non-degenerate (not automatic)

restrict to W

The norm of a vector is $|u| = \sqrt{|g(u,u)|}$

$u = (0,1)$, $g(u,u) = -1$

A unit vector is a vector such that
$g(u, u) = \pm 1$
we can always construct a (pseudo) orthogonal basis
$\{e_i\}_{i=1,\dots,n}$
$g(e_i, e_j) = \delta_{ij} \varepsilon_j \qquad \delta_{ij} = 0 \text{ or } 1$
$\varepsilon_j = \pm 1 = g(e_j, e_j)$

The signature of $g = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_j)$
with the convention of putting the (-1) ~~first~~ final.
The number of (-1)'s is the index of g.
If $\{e_1, \dots, e_n\}$ is an orthogonal basis.
$\forall u \in V, \quad u = \sum_{i=1}^{n} \varepsilon_i \, g(u, u) \, e_i$

Lema. If $V \subset dW$ are the space of the same dimention
The V and W are equiped with scalar product of
Some indexes, iff there exists a linear isometry $T: V \to W$

Chap Semi-Reimannian manifold
Def. A metric g on M is a type $(0, 2)$ tensor field
which is symmetric, nondegenerate with constant index.

two vector field $\longrightarrow$ function

$\uparrow$

tensor space
$g \in \infty$
$p \in M \longrightarrow g_p$

product
index product
on $T_p M$
index constant

Regularity is
Def. A Semi-Riemannian manifold (M, g) is a $\underset{\text{Smooth}}{C^\infty}$ manifold
equiped with a metric g.

⚠ $\neq$ metrics $\Rightarrow \neq$ Semi-Riemanian manifold on Domain M

If the index $= 0$ (g is positive defined)
(M, g) is called Riemannian ~~manifold~~.
If the index $= 1 \Rightarrow (M, g)$ is called Lorentzian.
$\forall$ x local coordinates $\{u, \xi = (x^1, \cdots, x^n)\}$
$g = \sum g_{ij}\, dx^i \otimes dx^j \quad (1 \leq i, j \leq n)$

$g_{ij} = g(\partial_i, \partial_j)$
$V = \sum X^i \partial_i$
$W = \sum W^j \partial_j$
$g(V, W) = \sum_{i,j} g_{ij} V^i W^j$

The matrix $(g_{ij})_{1 \leq i, j \leq n}$ is inversable and its inverse ~~g~~ $(g^{ij})_{1 \leq i, j \leq n}$
$g_{ij} = g_{ji}$
commute
On the Euclidean space $\mathbb{R}^n$
we have the Riemannian metric
$\langle u, w \rangle = \sum_{i=1}^{n} u^i w^i$
and if $0 \leq \nu \leq m$, there is a Semi-Riemannian metric
$\mathbb{R}^n_\nu$ $\langle v, w \rangle = -\sum_{i=1}^{\nu} u^i w^i + \sum_{i=\nu+1}^{n} u^i w^i$

if $\varepsilon_i = \begin{cases} -1 & 1 \leq i \leq \nu \\ +1 & \nu+1 \leq i \leq m \end{cases}$

Def. A tangent vector to M at $p \in M$, $u \in T_pM$
is space-like if $g(u,u) > 0$ or $u = 0$.
causal type
null if $g(u,u) = 0$ and $u \neq 0$
time-like if $g(u,u) < 0$
None vectors, from the null-zone is then concution metrics,
null vector are called light like if P is a submanifold
of M and M is equiped with a Riemannian metric g.
then (P, g) is a Riemannian manifold.
P equiped with g

for example sphere S^2, admits a Riemannian metric.
But this is not always the case. for Semi-Riemannian metrics

Lemma. If (M, g) and (N, h) are Semi-Riemannian manifolds
and $\pi: M \times N \longrightarrow M$
$\sigma: M \times N \longrightarrow N$
then $g \longrightarrow \pi^* g_M + \sigma^* g_N$ are Semi-Riemannian metric
$M \times N$

Def. An Isometry of (M, g) into (N, h) is a diffeomorphism which persevues the metrics. $\phi^* g_N = g_M$

$\forall, v, w \in T_p M$, $p \in M$
~~$h_{\phi(p)}(d\phi_p(u)$~~ ~~$(d\phi_p(w) = g_p(w, w))$~~
$h_{\phi(p)}(d\phi_p(u), d\phi_p(w)) = g_p(u, w)$

Ex: $(u^1, \dots, u^n)$ natural coordinates on $\mathbb{R}^n_s \longrightarrow$ Semi-Rieman metric
Let V and W two vector fields on $\mathbb{R}^n$
The $W = \Sigma\, W^i \partial_i \cdots \partial_i = \frac{\partial}{\partial w^i}$

Definition. $D_V W = dW(V) = \Sigma\, V(w^i)\partial_i$
This is the covariant derivative of W with respect to V.

Def. (Axioms)
A connection on a manifold M is a map $D: \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$
Such that.

- $D_V W$ is $\mathcal{F}(M)$ linear in V $\longleftarrow$ ~~tensorial~~ in V
- $D_V W$ is $\mathbb{R}$ linear in W

$D_V(fW) = V(f)W + f D_V W \quad \forall f \in \mathcal{F}(M)$
$D_V W$ is the corvariant dericative of W with respect to V.
If (M, g) is a Semi-Riemannian manifold.
If $V \in \mathfrak{X}(M)$. Let V^* is a one-form $V^*(X)$
$V^*(X) = g(V, X) \quad X \in \mathfrak{X}(M)$
associate manifold through metric

$V \longmapsto V^*$ is $\mathcal{F}(N)$ linear and an Isomorphism

(Semi) Riemannian manifold (M, g) symmetric

$g.\ p \in M \longmapsto g_p$

Given non-degenerate } scalar product

signature $(-1, -1, \cdots, +1, +1)$ is the same for all $p \in M$

main signature: $(+1, +1, -1, +1)$ Riemannian

$(-1, +1, +1, +1)$ Lorentz

Examples.

- $S^2 = \{p \in \mathbb{R}^3.\ |p| = 1\}$

$\forall p \in S^2$

inner product

$T_p S^2 = \{x \in \mathbb{R}^3,\ \langle p, x \rangle = 0\}$

So we can take $\forall x, y \in T_p S^2$

$g_p^{S^2}(x, y) = \langle x, y \rangle_{\mathbb{R}^3}$ is a Riemannian metrix on S^2

If you replace $\langle\ ,\ \rangle_{\mathbb{R}^3}$ by another Semi-Riemanian metrix on $\mathbb{R}^3$ (Ex. $\langle v, w \rangle = 0$, $-v_1 w_1 + v_2 w_2 + v_3 w_3$)

Reimannian and Semi-Reimannian follow different rule.

then $\langle x, y \rangle_{\mathbb{R}^3}$ is no longer a Semi-Reimannian matrix on S^2.

Another Riemanian metrix can be contructed by:

$\tilde{g}_p(x, y) = e^f g_p(x, y)$, $\forall f \in C^\infty(M, \mathbb{R})$

$\tilde{g}$ is different from g_p

Conformal metrix $\tilde{g}$ is respect to metrix g.

Ex. $S^3 = \{p \in \mathbb{R}^4,\ |p|^2 = 1\}$

$\forall p \in S^3$, $T_p S^3 = \{x \in \mathbb{R}^4,\ \langle p, x \rangle = 0\}$

$\exists \{E_1, E_2, E_3\}$ vector fields on S^3

$|E_1|^2 = |E_2|^2 = |E_3|^2$ $\langle E_i, E_j \rangle = 0$, $i \neq j$

where we use the Riemanian metrix. $\forall p \in S^3$, $x, y \in T_p S^3$

$g_p(x, y) = \langle x, y \rangle_{\mathbb{R}^4}$

Then we can define a new Semi-Reimannian metric.

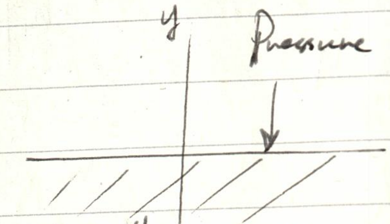
$\langle E_i, E_j \rangle = 0$, $i \neq j$

$|E_1|^2 = 1$ $|E_2|^2 = |E_3|^2 = 1$ globally defined

3) $\mathbb{H}^2 = \{p \in \mathbb{R}^2, p = (x, y), y > 0\}$

$T_p\mathbb{H}^2 \simeq \mathbb{R}^2$

$g_{\mathbb{H}^2} = \frac{dx^2 + dy^2}{y^2} = \frac{\langle\ ,\ \rangle_{\mathbb{R}^2}}{y^2}$



Describe pressure when $y \to 0$

Question. when can you equipe M with a Semi-Riemannian metric?

a) not always fun

Semi-Riemannian metric with index > 0

b) Always from Riemannian metrics.

2 ways:

1) - Witney's theorem:

i) an immersion of M in $\mathbb{R}^N$, for N large enough.

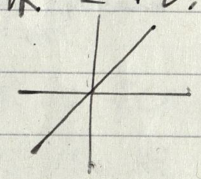
$M \overset{i}{\hookrightarrow} \mathbb{R}^N$

ii) Pull back the euclidean metric on $\mathbb{R}^N$ by i

$\forall p \in M$ $\quad g_p(x, y) = \langle di_p(x), di(y)\rangle_{\mathbb{R}^N} = i^*\langle\ ,\ \rangle$

$y \cdot Y \in T_pM$ $\quad$ generate $\quad$ positive defined

$\mathbb{R}^2 = \langle v, w\rangle = -v_1w_1 + v_2w_2$



degenerate

2) (U, x^i) atlas of M

(partition of unity α_i adapted to $\{U_i\}$)

Then define $g = \sum_i \alpha_i x_i^* \langle\ ,\ \rangle_{\mathbb{R}^n}$

g is Riemannien

Let (M, g) a Semi-Riemannian manifold

Def. A connection D on a manifold M is a map

$\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ $\quad$ $\mathfrak{X}(M)$ {vector field on M}

$(X, Y) \longmapsto D_XY$

i) $v \longmapsto D_v\phi$ is $\mathcal{F}(M)$ linear $\quad$ space of f

ii) $w \longmapsto D_v\phi$ is $\mathbb{R}$ linear

iii) $D_v(fw) = v(f)w + fD_vw$ $\quad (f \in \mathcal{F}(M))$

Theorem. Let (M, g) be a Semi-Riemannian manifold
Then there exists a unique connection D such that:
$V, W \in \mathfrak{X}(M)$

a) $[V, W] = D_V W - D_W V$ (torsion free)

b) $X(\underbrace{g(V, W)}_{\text{inner product}}) = g(D_X V, W) + g(V, D_X W)$ (metric connection)

D is characterized by the koszul formula:

c) $2g(D_V W, X) = Vg(X, W) + W(g(X, V)) - Xg(V, W)$
$\cancel{+g(V}$ $\mp g(W, [X, V]) - g(V, [W, X]) + g(X, [V, W])$
symmetric part

$D_V(\nabla)$ is the Levi-Civita connection

Proof. Let D be a connection satisfying a) and b)
1. the right hand side of c) use b) and a) to abtain
~~2.~~ $2g(D_V W, X)$
So D (is f c) $\Rightarrow$ conseques
1-1 correspondence between vector fields and 1-forms.

Existance at $F(V, W, X)$ = right hand side of c)
$\mathfrak{X}(M) \longmapsto \mathbb{R}$
$x \longmapsto F(V, W, X)$ is $\mathfrak{F}(M)$-linear.
So it is a 1-form $\Rightarrow \exists$ a unique vector field
$D_V W$. such that $g(D_V W, X) = F(V, W, X)$ $\forall X$ ($\Rightarrow$ c)
and D is a connection.

Eg. $2g(D_V(fW), X) \Rightarrow g(V(f)W + f D_V W, X)$
similarity.
$2\langle D_V W - D_W V, X) \cancel{\Rightarrow g(V(f)W + f D_V W, X)}$
$2g\langle [V, W], X)$

Def. The Christoffel symbols for the local chart (u, x) are the functions on u.

$$D_{\partial_i}(\partial_j) = \sum_{k=1}^{n} \Gamma_{ij}^{k} \underbrace{\partial_k}_{\text{Local basis}} \quad (\text{on } u)$$

$\partial_i = \frac{\partial}{\partial j}$

basis of tangent space

3. Recall that $[\partial_i, \partial_j] = 0 = D_{\partial_i}(\partial_j) - D_{\partial_j}(\partial_i) \Rightarrow \Gamma^k_{ij} = \Gamma^k_{ji} \quad \forall i, j, k$
(on u)

If $W = \sum_j \underbrace{W^j}_{\text{function}} \underbrace{\partial_j}_{\text{basis vector}}$ then

$$D_{\partial_i}(W) = \sum_j \partial_i(w^j)\partial_i + \sum_j W_j \Gamma^k_{ij}\partial_k$$

$$= \sum_{k=1}^n (\partial_i(w^k) + \sum_j W^j \Gamma^k_{ij}) \partial_k$$

$$g(\partial_i, \partial_j) = g_{ij}$$

By Koszul

$$\Gamma^k_{ij} \overset{\Delta}{=} \sum_{l=1}^n g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Christoffel symbol

Ex. on $\mathbb{R}^n$, $\Gamma^k_{ij} = 0 \quad \forall i, j, k$

$$(g^{ij}) = (g_{ij})^{-1}$$

· Curvature

Def/Lemma

(M, g) Semi-Riemannian manifold at ∇ (the Levi-Civita connection)

then $R: \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$

$$(X, Y, Z) \longmapsto R(X, Y)Z$$

defined by

$$R(X, Y).Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y \nabla_X Z$$

is a $(1, 3)$ tensor field called the Riemannian Curvature (tensor)

1) Be careful with the sign convention.

2) There is $(0, 4)$ -version

$$R(X, Y, Z, W) = g(R(X,Y)Z, W) \text{ (scalar)}$$

3) We need to check

$$R(fX, Y)Z = fR(X, Y)Z$$

and $R(X, Y)(fZ) = fR(X, Y)Z$

$R(fX,Y) = \nabla_{[fX,Y]}Z + \nabla_Y \nabla_{fX} Z + \nabla_{fX}\nabla_Y Z$

$fXY - XfX) = fXY - Y(f)X - fXX$

$= \nabla_{f[X,Y]} - Y(f)XZ + \nabla_Y(f\nabla_Y Z) - f\nabla_X\nabla_Y Z$

$= \nabla_{f[X,Y]} - Y(f)XZ + Y(f)\underset{\nabla_X Z}{\cancel{\nabla_Y Z}} + f\nabla_Y\nabla_X Z - f\nabla_X\nabla_Y Z$

Prop.

a) $R(X,Y)Z = -R(Y,X)Z$

b) $g(R(X,Y)_x Z, W) = -(R(X,Y)W, Z)$ (1st ~~Bianchi~~ Bianchi identity)

c) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

d) $g(R(X,Y)Z, W) = g(R(Z,W)X, Y)$

$g(\ ,\ ) = \langle\ ,\ \rangle$

Pf

b) $g\langle R(X,Y),Z, Z\rangle = \langle \nabla_{[X,Y]} Z - \nabla_X\nabla_Y + \nabla_Y\nabla_X Z, Z\rangle$

$= \langle \nabla_{[X,Y]} Z, Z\rangle - \langle \nabla_X\nabla_Y Z, Z\rangle + \langle \nabla_Y\nabla_X Z, Z\rangle$

$= [X,Y]\frac{|Z|^2}{2} - X\langle\nabla_Y Z, Z\rangle + \langle\nabla_Y Z, \nabla_X Z\rangle + Y\langle\nabla_X Z, Z\rangle - \langle\nabla_X Z, \nabla_Y Z\rangle$

$\frac{\langle Z, Z\rangle}{2}$

$\begin{cases} X\langle Y, Z\rangle = \langle\nabla_X Z, Y\rangle + \langle\nabla_X Y, Z\rangle \\ [X,Y] = \nabla_X Y - \nabla_Y Y \end{cases}$

$= [X,Y]\frac{\langle Z,Z\rangle}{2} - XY\frac{\langle Z,Z\rangle}{2} + Y\left(X\frac{\langle Z,Z\rangle}{2}\right) = 0$

So $g(R(X,Y)(Z+W), (Z+W)) = 0$

$g(R(X,Y)Z, W + g(R(X,Y)W, Z) = 0$ b)

c) $R(X,Y)Z + R(Y,Z)X + R(Z,X)X$

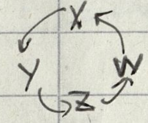
$= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z + \cdots$

$\nabla_Z\nabla_Y X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]}X + \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[Z,X]}X$

$= \nabla_Y[X,Z] + \nabla_X[Z,Y] + \nabla_Z[Y,X]$

$= [Y,[X,Z]]$

d) $\langle R(X,Y)Z, W\rangle + \langle R(Z,X)Y, W\rangle + \langle R(Y,Z)X, W\rangle$



$+ \langle R(X,Z)W, X\rangle + \langle R(W,Y)Z, X\rangle + \langle R(Z,W)Y, X\rangle$

$+ \langle R(Z,W)X, Y\rangle + \langle R(X,Z)W, Y\rangle + \langle R(W,X)Z, Y\rangle$

$+ \langle R(W,X)Y, Z\rangle + \langle R(Y,W)X, Z\rangle + \langle R(X,Y)W, Z\rangle$

$= 2\{\langle R(Z,X)Y, W\rangle + \langle R(Y,W)X, Z\rangle\}$

Remark. R is a tensor.

$X, Y, Z \in \mathcal{X}(M)$, $p \in M$

$(R(X,Y),Z)$ p only depends on $X(p)$, $Y(p)$, $Z(p)$ evaluate its at p

So we can made some of $R_p(u, v)w$ f on $u, v, w \in T_pM$

Proposition. $\left.\begin{array}{l} V(f) \\ D_V W \end{array}\right\} \longrightarrow D_V R$

$X, Y, Z \in \mathcal{X}(M)$

$(\nabla_Z R)(X,Y) + (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) = 0$

R is a $(1,3)$ ~~tensor~~ $\left.\begin{array}{l} X(f) \\ \nabla_X Y \end{array}\right) \longrightarrow \nabla_X R$

$\nabla_X R$ is a $(1,3)$ tensor

$(\nabla_X R)(Y,Z)W$

$= \nabla_X(R(Y,Z)W) - R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W - R(Y,Z)\nabla_X W$

Rk. $(\nabla_X g)(Y,Z)$ $(0,2)$ ~~tensor~~

$= Xg(Y,Z) - g(\nabla_X Y) - g(X\nabla_X) = 0$

$\underline{\nabla g = 0}$

Pf. Put $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$

then $[X,Y] = [X,Z] = [Y,Z] = 0$

$(\nabla_Z R)(X,Y)W = [\nabla_Z, R(X,Y)]W - R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W$

(*) becomes ⓐ

$= [\nabla_Z, R(X,Y)]W + [\nabla_X, R(Y,Z)]W + [\nabla_Y, R(Z,X)]W$

$- R(\nabla_Y Z, X)W - R(Z, \nabla_Y X)W$

$= [\nabla, R] + [\nabla, R] + [\nabla, R]$ ⓑ

$+ R([X,Z], Y)W + R([Z,Y], X)W + R([Y,X], Z)W$

but $[\nabla_Z, R(X,Y)] = [\nabla_Z, [\nabla_X, \nabla_Y]]$

ⓐ + ⓑ $= [\nabla_Z, [\nabla_X, \nabla_Y]] - \nabla[~~X~~[Y,X], Z] = 0$

$(X) = [\nabla_Z, [\nabla_X, \nabla_Y]] + [\nabla_X, [\nabla_Y, \nabla_Z]] + [\nabla_Y, [\nabla_Z, \nabla_X]] = 0$

Jacobin

Sectional Curvature

Let π be a plane in T_pM $(p \in M)$

If $u, w \in T_pM$

a and $w \neq 0$, not collection

Define $Q(u,w) = \langle u, u\rangle\langle w, w\rangle - \langle u, w\rangle^2$

Semi-Reimannian metric

π is non-degenerate if $Q(u,w) \neq 0$

$|Q(u,w)|$ = volume of the parallelgram defined by u and w

If π is nondegenerate then we define the semilinal curvature of π by $K(\pi) = \dfrac{g(R(u,w)u,w)}{Q(u,w)} = K(u,w)$

$K(\pi)$ does not depends on the choice of u and w.

Proof. If at the point $p \in M$,

$K = 0$. ($K = 0$ for all the planes)

then $R = 0$ at p

Pf. 1) If u and w define a non-degenerate plane.

then u and w can be approximated by vector which define a non-degenerate plane.

If u is null $\left(\begin{array}{l} u \neq 0 \\ \langle u, u\rangle = 0 \end{array}\right)$

Let x such that $\langle u, x\rangle \neq 0$

If not let x be of the opposite ~~causal~~ ~~type~~ of u.

$\rightarrow Q(u,x) = \langle u,u\rangle\langle x,x\rangle - \langle u, x\rangle^2$

< 0 negative

Let $\delta \neq 0$, small then

$u, w+\delta x$ define a nondegenerate plane

$Q(u, w+\delta x) = Q(u,w) + 2\delta(\langle u,u\rangle\langle w,x\rangle - \langle u,w\rangle\langle u,x\rangle) + \delta^2 Q(u,x)$

non-degenerate

if $b = 0 \Rightarrow$ ok

if $b \neq 0 \Rightarrow \delta \gg \delta^2$

BRUNNEN