Semi-Riemannian geometry

Chap 0. Differential geometry Atlas. charts.

Chap 1. Tensors abstract / Differential manifold + practical / Linear algebra

Chap 2. Semi-Riemannian manifolds matrix curvature

Chap 3. Geodesics (auto-parallel

Chap 4. examples

Chap 5. Geodesics (Dual nature of Geodesics) – calculus of variations

Chap 6. Homonic maps (k – manifold)

0. Differential manifolds

(To ~~do~~ specifiey where to do calculus)

Let S to be a topological space / connected / Hausdorff ~~paraco-~~ pact

A chart is a homomorphism ξ from an open subset u of S.

In an open subset $\xi(u)$ of $\mathbb{R}^n$, $\xi(p) = (x^1(p), \cdots, x^n(p))$, $\forall p \in u$

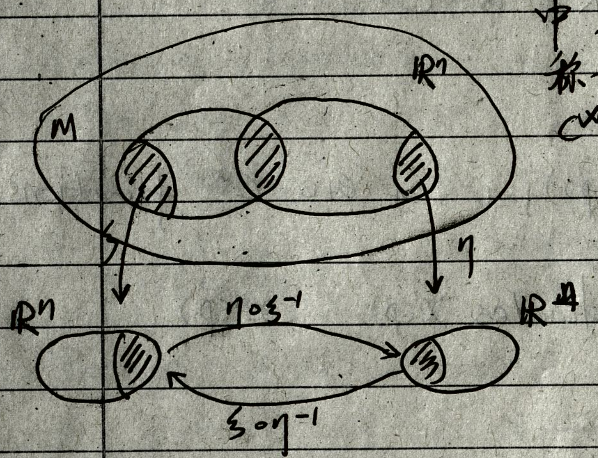
The x^i are the coordinate function of η.

$\eta = (x^1, \cdots, x^n)$ $u \longrightarrow \mathbb{R}^n$, n is the ~~direct~~ dimention of η

Two charts ξ and η of same ~~direction~~ intersect in a smooth mannan if $\xi \circ \eta^{-1}$ and $\eta \circ \xi^{-1}$ ~~one~~ are C^∞ maps from $\mathbb{R}^n$ to $\mathbb{R}^n$.

/ 设 $\mathbb{R}^n$ 中曲面 ~~(ξ, η,)~~ M 的局部坐标变换公式中所有函数都是 C^k 类的，则 ~~ξ, η~~ M 称为 C^k 类曲面，后续讨论的都是 C^∞ 类曲面。（k次、∞次可微函数）

/ transition functions



$\xi \circ \eta^{-1} = \eta^{-1} \circ \xi$?

Let M be a differential manifold.

A function of $M \to \mathbb{R}$ is C^∞

if for any chart (u, ξ). $f \circ \xi^{-1}$

$\xi^{-1}(u) \subset \mathbb{R}^n \to \mathbb{R}$

∞ differentiable

the sum product and inverse, C^∞

low regularity

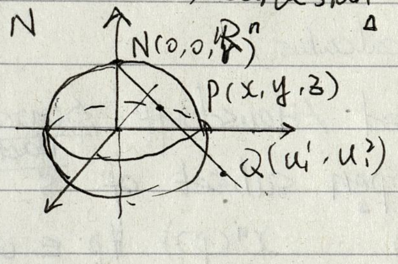
An atlas is a collection of charts of dimention m, such that:

1) $\forall p \in S$, $\exists u$ such that $p \in u$.
2) $\forall u, v \Rightarrow u$ and v intersect.

An atlas is said to be complete if it contains all the chats of S. which intersect in a smooth moman. Any atlas admits a complete atlas.

Defination. A differential manifold (C) is a topological space equiped with complete atlas.

examples.
1) Euclidean space
2) Sphere $S^1 = \{ x \in \mathbb{R}^{n+1} \ |x| = 1 \}$
3) Cartesian product



2 charts
Stedographic projection

Def. Let M be a differentiablble manifold. A function of $M \to \mathbb{R}$ is C^∞ if for any chat (u, ξ), $f \circ \xi^{-1}$ $\xi^{-1}(u) \subset \mathbb{R}^n \longrightarrow \mathbb{R}$

∞ differentrial

The sum product and inverse (where to semaum), C^∞

Def. Let M and N be two differential manifolds. A map $\phi : M \to N$ is C^∞ if for any charts (u, ξ) of M and (u, η) of N,

$$\eta \circ \phi \circ \xi^{-1} \quad \xi(u) \subset \mathbb{R}^n \longrightarrow \eta(u) \subset \mathbb{R}^n \quad \text{is } C^\infty$$

$\xi = (x^1, \cdots, x^n)$

$\eta = (y^1, \cdots, y^n)$

If $p \in \phi^{-1}(u)$, then ϕ is C^∞ if $y^j(\phi(p))_{j=1,\cdots,n}$ are C^∞ functions.

$\Updownarrow$

$y^j(\phi(p))$ depends smoothly on the coordinates $x^1(p), \cdots, x^n(p)$.

Def. Let $p \in M$,

Let $F(M)$ be the space of C^∞ functions on M.

A tangent vector v is a map $v : F(M) \to \mathbb{R}$, such that:

1) $\mathbb{R}$ linear: $v_p(af + bg) = a v_p(f) + b v_p(g)$ $\quad f, g \in F(M)$, $a, b \in \mathbb{R}$
2) Leibniznal: $(v(fg))_p = g_p v(\cancel{g}) + f_p v(g)$ On blackboard $g_p v(g) + f_p v(f)$

linear space

T_pM = the space of all tangent vectors at the point p.

(u, ξ), $p \in u$, $\xi = (x^1, \cdots, x^n)$

Let $\underset{\text{function}}{f} \in F(M)$

As to the Dimention, $n \to n$ or $m \to n$? $\mathbb{R}^-$ $(\mathbb{R}^n)$ $\partial u \to M$

$$\frac{\partial f}{\partial x^i}(p) = \frac{\partial (f \circ \xi^{-1})}{\partial u^i}(\xi(p))$$ at the point $\xi(p)$

$(u^1, \cdots, u^n)$ are the coordinates on $\mathbb{R}^{n/m}$

$\partial y_p = \frac{\partial}{\partial x^1}\Big|_p, \cdots, \frac{\partial}{\partial x^n}\Big|_p$ gonna a basis of T_pM

Def. Let $\phi: M \longrightarrow N$ a C^∞ map

$\forall p \in M$, the map: $d\phi_p: T_pM \longrightarrow T_{\phi(p)}N$ is a linear map.

defined by, if $v \in T_pM$, $d\phi_p(v_p) = v_\phi \in T_{\phi(p)}N$

$\phi(v_p)$ is better ?

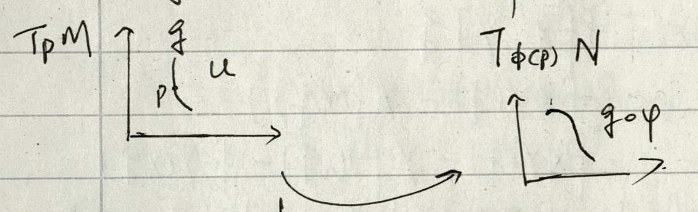
$v_\phi(g) = v(g \circ \phi)$ $\quad \cancel{\phi(v_g) = v(\phi(g))}$ not clean enough to present vector v

$\forall g \in F(N)$ $\quad g: N \to \mathbb{R}$ $\quad \forall g \in F(M)$ $\quad g: M \to \mathbb{R}$

$g \circ \phi^{-1}$ $\quad M \to \mathbb{R}$ $\quad g \circ \phi$, $\quad N \to \mathbb{R}$

How to compose

$\Rightarrow v(g \circ \phi)$ well defined



$d\phi_p \Rightarrow u(g \circ \varphi)$

to basis or to g ?

$d\phi_p$ is characterized by $d\phi_p(v(g)) = v(g \circ \phi)$,

for ~~all~~ $\forall v \in T_pM$, $g \in F(M)$, with coordinate system $(x^1, \cdots, x^n)$ on charts on M. any $(y^1, \cdots, y^n)$ on chats on N.

$$d\phi_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \sum_{i=1}^{n} \frac{\partial (y^1 \circ \varphi^1)}{\partial x^i} \frac{\partial}{\partial y^1}\Big|_{\phi(p)}$$

basis express any vectors

Remark: If M, N and P are differentiable manifold and $\phi: M \longrightarrow N$ and $\varphi: N \longrightarrow P$ are C^∞ maps then, $\varphi \circ \phi: M \to P$ is a smooth map.

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Def. A vector field is a map which sends to each point p of M, a tangent vector to M at p.

$$V: \quad M \longrightarrow \{T_pM,\ p\in M\}$$
$$p \longrightarrow v_p \in T_pM$$

Sample: $f \in F(M)$ then $V(f)$ is the map $V(f)(p) = v_p(f)$ $\forall p \in M$

Then $\dfrac{V \text{ is } C^\infty}{\text{map is } C^\infty}$, if f is C^∞, $\forall f \in F(M)$.

i) The sum of two vector fields is a vector field.
The multiplication of a vector field by a function is a vector field.

ii) The braket of V and W is $[V, W]_p(f) = V_p(W(f)) - W_p(V(f))$

$f \in F(M)$

geometrical meaning... $V_p(W(f)) \overset{?}{=} W_p(V(f))$ in which condition

$W(f) \neq f \to$ vector field $\downarrow$ $V_p(W(f))$

Commutator

(Poission braket ?)

It satisfies the Jacobi identity: • How to prove?

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

But $[gX, fY] = fg[X,Y] + g(X|f)Y - f(Y|g)X$ by Snaltz

and $[\partial_i, \partial_j] = 0$

$$\left(\frac{\partial^2 g}{\partial x^i \partial x^j} = \frac{\partial^2 g}{\partial x^j \partial x^i} \right)$$

$[fX, Y]g$
$= fXY(g) - Y(f(Xg))$
$= fXY(g) - Y(f)(Xg) - fY(Xg)$

$$[gX, fY]z = fg[X,Y]z + gX(f)Y(z) - fY(g)X(z)$$

Def. A differential manifold P is a submanifold of M, if P is topological subspace of M.

~~Def. $j: P \hookrightarrow M$~~

Def. $j: P \subset M$, inclusion is a C^∞ map.
and $dj_p: T_pP \longrightarrow \underline{T_{j(p)}M}$ is injective
image of P
local $\to$ global

(post a power size!) machen moment

Reference. De comma "Riemannien geometry" "Differential manifold"
P. Peterson "Riemannien Geometry"
Vol. 2 Spivak "A comprehensive introduction to differential geometry"
S O' Neil "Semi - Riemannian Geometry"

identity m by it's image

If M is a C^∞ differential manifold, then there exist an immersion
$f : M^n \longrightarrow \mathbb{R}^{2n}$ / f injective, df injective.
(Diffeomersion low)
and an $\underset{\text{immersion}}{\underline{\text{embedding}}}$ (diffeomorfism admits image)

$f : M^n \to \mathbb{R}^{2n+1}$

Self intersection

Tangent bundle tangent space M
A manifold TM.
$TM = \bigcup_{p \in M} T_pM$
$= \bigcup_{p\in M} \{p\} \times T_pM$
$= \bigcup_{x \in M} \{(p, v_p) \mid v_p \in T_pM\}$
$= \{(p, v_p) \mid p \in M, v_p \in T_pM\}$

Tangent bundle TM

$TM = \bigcup T_pM$
disjoint union of $p \in M$

$TM = \langle (p, v_p) : p \in M, \ v_p \in T_pM \rangle$

$\underline{\pi : TM \to M}$ canonical projection
Levengre transform?

Lagrangian formism
$\downarrow$
Hamiltonian formism

$(p, v_p) \longrightarrow p \qquad \pi^{-1}(p) = T_pM$

TM is a differential manifold of dimention $\underline{2n}$

(p, v_p)
$H(q, p)$

Let (u, ξ) be a local chat on M,
$\xi = (x^1, \cdots, x^n)$

Let $v \in T_pM$, then $v = \sum_i v^i \frac{\partial}{\partial x^i}\Big|_p = \sum_i v(x^i) \frac{\partial}{\partial x^i}$ (the vector along x^i)

$\underline{\tilde{\xi} : \pi^{-1}(u) \subset TM \longrightarrow \mathbb{R}^{2n}}$ (phase flow?) $\underline{\text{coordinates}}$ $(u^1, \cdots, u^n, u^{n+1}, \cdots, u^{n+n})$
$u \in M \quad \pi^{-1}(u) \quad (\dot{q}, p)$

$\tilde{\xi}(p, v) = x^1(\pi(p, v)), x^2(\pi(p, v)), \cdots, x^n(\pi(p, v))$.

$\mathbb{R}^{2n}$ ~~M~~ $\leftarrow TM$ (Nether's theorem..

$(v^1, \cdots, v^n)$ where $v^i = v(x^i) = \dot{x}^1(0)$

(Compatibility of differential charts)

The transition functions are

$v^i \tilde{\xi} \circ \tilde{\eta}^{-1}(a, b) \ (1 \le i \le n) = x^i \circ \pi \circ \tilde{\eta}^{-1}(a, b) = x^i \tilde{\eta}^{-1}(a)$

$v^{i+n} \tilde{\xi} \circ \tilde{\eta}^{-1}(a, b) = \dot{x}^i(u) \circ \underline{\tilde{\eta}^{-1}(a, b)} = \sum b^k \frac{\partial x^i}{\partial y^k} (\eta^{-1}(a))$
$\|$
$(\cdots, \sum b^k \frac{\partial}{\partial y^k}) \in T_pM$

$\xi = (x^1, \cdots, x^n)$
$\eta = (y^1, \cdots, y^n)$

$\Rightarrow \tilde{\xi} \circ \tilde{\eta}^{-1}$ is $C^\infty \in \mathbb{R}^{2n}, \mathbb{R}^{2n})$

A vector field X is a C^∞ map from M to TM such that $\pi \circ X = Id$

Remark. In general, $TM \not\cong M \times \mathbb{R}^n$

not trivial : it is a product

This is the same from ($S^3 \cong SU(3)$, Lie groups

Ex. TS^2 $\quad S^2 = (p \in \mathbb{R}^3, |p|^2 = 1)$

is a 2-dim manifold and a submanifold of $\mathbb{R}^3$

we want to ~~determine~~ T_pS^2 from $p \in S^2$

we take γ be a curve on $S^2 (C^\infty)$

$\gamma :]-\varepsilon, +\varepsilon[\longrightarrow S^2 \qquad \gamma(0) = p$

$t \longmapsto \gamma(t)$

γ acts on functions on S^2

$(f : S^2 \to \mathbb{R} \to f(\gamma(t))]-\varepsilon, +\varepsilon[\to \mathbb{R})$

$\left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)(t)$ is a direction

$\gamma(0) \cdot (f)$

$\left.\frac{d\gamma}{dt}\right|_{t=0} = \gamma(0) \in T_pS^2$

and all vectors tangent to S^2 at p can be obtained this way.

$\forall t \in]-\varepsilon, +\varepsilon[\qquad |\gamma(t)|^2 = 1$

$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} |\gamma(t)|^2 = 0 = 2 < \gamma'(0), \gamma(0)> \Rightarrow < p, \gamma(0)> = 0$

$T_pS^2 = \{x \in \mathbb{R}^3, <x, p> = 0\}$

$TS^2 = \{(p, x) \in \mathbb{R}^3 \times \mathbb{R}^3\}$

$|p|^2 = 1$ and $<p, x> = 0$

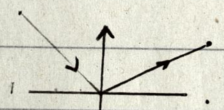
Examples Immersion and Embeddings

Defination: Let M^m and N^n be differentiable manifolds.
A differentiable mapping $\varphi: M \longrightarrow N$ is said to be immersion if $d\varphi_p: T_pM \longrightarrow T_{\varphi(p)}N$ is injective for all $p \in M$.
If, inaddition, φ is a homeomorphism onto $\varphi(M) \subset N$, we say that φ is an embedding. If $M \subset N$ and the inclusion $i: M \subset N$ is an embedding, we say that M is a submanifold of N.

Ex. Open subset of $\mathbb{R}^2$ are differential manifolds.
$p \in \mathbb{H}^2$, $\mathbb{H}^2 = \{p \in \mathbb{R}^2,\ p = (x, y),\ y > 0\}$
$T_p\mathbb{H}^2 = \mathbb{R}^2$
$T\mathbb{H}^2 = \underset{p}{\mathbb{H}^2} \times \underset{v}{\mathbb{R}^2}$ Cartesian product

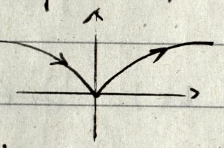
Counterexamples.

1) $\alpha:]-1, +1[\longrightarrow \mathbb{R}$
$t \longmapsto (t, |t|)$



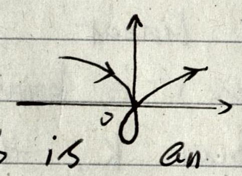
$M = \alpha(]-1, +1[)$ is not a differentiable manifold
image of α and not a submanifold of $\mathbb{R}^2$ (α is not differential at $t = 0$)

2) $\alpha: \mathbb{R} \longrightarrow \mathbb{R}^2$
$t \longmapsto (t^3, t^2)$



is a differentiable mapping
but not an immersion at $t=0$ ($d\alpha$ injective)

3) $\alpha: t \longrightarrow \mathbb{R}^2$
$t \longmapsto (t^3 - 4t, t^2 - 4)$

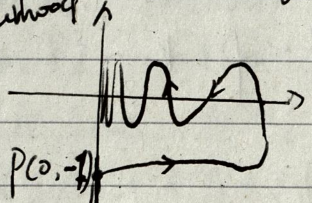


is differential and it is an immersion,
but there is a self-intersection f ($t=2$, $t=-2$)
α is not an embedding (not a homeomorphism)

4) $\mathbb{R}^2$

$$\alpha(t) = \begin{cases} (0, -t+2), & t \in (-3, -1) \\ \text{regular curve}, & t \in (-1, \frac{1}{\pi}) \\ (-t, -\sin\frac{1}{t}), & t \in (-\frac{2}{\pi}, 0) \end{cases}$$

open neibourhood



$P(0,-2)$

a set inter ∞ cohomony
This is an immersion, differential at a point, not embedding.
(Topologies are not competible)

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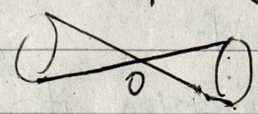
5) Curve $= \{x^2+y^2-z^2=0\}$



C

a neibourhood of 0 of C, is

If C such a manifold of $\mathbb{R}^3$, $\Rightarrow$ a neibourhood of 0 in $\mathbb{R}^2$

$\setminus\{0\}$ is disconnected

Submanifold detect whole structure, but this is not possible for a manifold bolns.hoed of $\mathbb{R}^2$.