Solution

$$\hat{p} = \frac{\langle 3\times10^{29}, -6\times10^{28}, 0\rangle \text{ kg}\cdot\text{m/s}}{\sqrt{(3\times10^{29})^2 + (-6\times10^{28})^2} \text{ kg}\cdot\text{m/s}}$$
$$= \langle 0.98, -0.20, 0\rangle$$
$$\vec{F}\bullet\hat{p} = (-4\times10^{22})(0.98) + (1\times10^{23})(-0.20) + (0)(0) \text{ N}$$
$$= -5.9\times10^{22} \text{ N}$$
$$\vec{F}_{\parallel} = (-5.9\times10^{22} \text{ N})\langle 0.98, -0.20, 0\rangle$$
$$= \langle -5.8\times10^{22}, 1.2\times10^{22}, 0\rangle \text{ N}$$
$$\vec{F}_{\perp} = \langle -4\times10^{22}, 1\times10^{23}, 0\rangle \text{ N} - \langle -5.8\times10^{22}, 1.2\times10^{22}, 0\rangle \text{ N}$$
$$= \langle 1.8\times10^{22}, 8.8\times10^{23}, 0\rangle \text{ N}$$

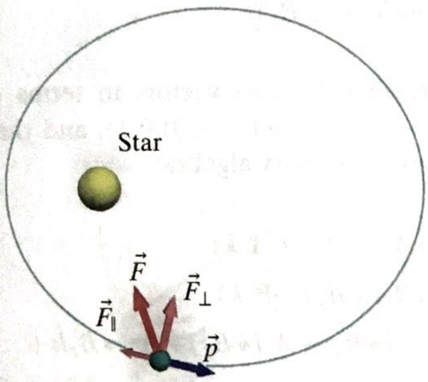

Figure 5.34 $\vec{F}$, $\vec{F}_{\parallel}$, and $\vec{F}_{\perp}$ at a particular instant for a planet orbiting a star.

Because the direction of $\vec{F}_{\parallel}$ is opposite to the direction of $\hat{p}$, the momentum of the planet will decrease during the next small time interval. Since $\vec{F}_{\perp}$ is not zero, the direction of the planet's momentum will also change.

In a computational model of curving motion, it can often be quite informative to display the parallel and perpendicular parts of the net force, since one determines changes in speed and the other changes in direction. The dot function in VPython provides a convenient way to calculate a dot product, and the norm function calculates a unit vector in the direction of a given vector. Assuming the current momentum and net force are the vectors p and F, one could use these VPython statements to calculate the force parts:

```
phat = norm(p)    # create unit vector from p
Fpara = dot(F, phat)*phat
Fperp = F - Fpara
```

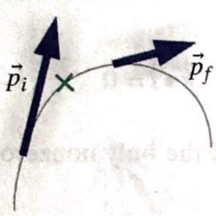

Figure 5.35 A system moves along the path shown. The arrows shown represent the momentum of the system at equal times before and after the time of interest, when the object is at the green x.

Figure 5.36 Graphically determine $\Delta\vec{p}$.

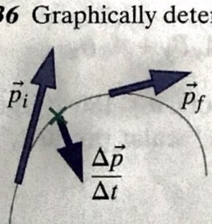

Figure 5.37 Divide $\Delta\vec{p}$ by Δt to get $\Delta\vec{p}/\Delta t$. Draw an arrow representing $\Delta\vec{p}/\Delta t$ at the location of interest. The smaller Δt is, the closer the result is to $d\vec{p}/dt$.

5.7 $d\vec{p}/dt$ FOR CURVING MOTION

It is not difficult to move from thinking about finite time steps to thinking about the instantaneous rate of change of momentum, $d\vec{p}/dt$, for curving motion. In Chapter 4 we used a three-step procedure to find $d\vec{p}/dt$ for a mass oscillating on a spring. We can use the same procedure, which is based on the definition of a derivative, to find $d\vec{p}/dt$ for an object whose path curves. Because we need to know the direction of $d\vec{p}/dt$ we often do this graphically, as in Figures 5.35–5.37.

FINDING $d\vec{p}/dt$ GRAPHICALLY

1. Draw two arrows representing $\vec{p}$ at equal times before and after the time of interest (Figure 5.35).
2. Find $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$ (Figure 5.36).
3. Divide $\Delta\vec{p}$ by Δt to get $\Delta\vec{p}/\Delta t$ (Figure 5.37). The smaller Δt is, the closer the result is to $d\vec{p}/dt$.

QUESTION For curving motion, does $d\vec{p}/dt$ always point toward the inside of the curve?

Yes. Refer to Figures 5.27–5.30, in which we saw how a force perpendicular to motion caused an object's path to curve in the direction of that force. $d\vec{p}/dt$ is in the same direction as $\vec{F}_{\text{net}}$, and must point at least partly toward the inside of the curve, although it may also have a part parallel to $\vec{p}$ if the speed is changing as well.

Parallel and Perpendicular Parts of $d\vec{p}/dt$

We have seen that at any instant the net force on an object can be expressed as the sum of two parts: one that is parallel to the object's momentum and one that is perpendicular to the system's momentum. The same is true for the vector $d\vec{p}/dt$:

$$\frac{d\vec{p}}{dt} = \frac{d\vec{p}}{dt}_{\parallel} + \frac{d\vec{p}}{dt}_{\perp}$$

We can see this mathematically by "factoring" momentum into the product of the magnitude of the momentum (a scalar) and a unit vector in the direction of the momentum, and then taking the derivative, by applying the product rule. The resulting expression has two pieces, which correspond to the two ways in which momentum can change, in magnitude and in direction. (We use the notation convention that $p = |\vec{p}|$ to make the following discussion easier to follow.)

$$\vec{p} = p\hat{p}$$

$$\frac{d\vec{p}}{dt} = \frac{dp}{dt}\hat{p} + p\frac{d\hat{p}}{dt}$$

This may look unfamiliar, but we can make sense of it by considering each term separately.

> **QUESTION** Which term in the expression above involves a change in magnitude of momentum (change in speed)?

$\frac{dp}{dt}\hat{p}$ is nonzero if speed is changing, and zero otherwise.

> **QUESTION** Which term involves a change in direction?

$p\frac{d\hat{p}}{dt}$ is nonzero if direction is changing, and zero otherwise.

We can relate the parallel and perpendicular parts of $d\vec{p}/dt$ to the parallel and perpendicular parts of the net force responsible for changing the momentum of a system:

PARALLEL AND PERPENDICULAR

$$\frac{d\vec{p}}{dt}_{\parallel} = \frac{dp}{dt}\hat{p} = \vec{F}_{\text{net}\,\parallel}$$

$$\frac{d\vec{p}}{dt}_{\perp} = p\frac{d\hat{p}}{dt} = \vec{F}_{\text{net}\,\perp}$$

Note that this division is useful only for smooth, continuous curving motion. At turning points or other times when $\hat{p}$ changes abruptly, $d\hat{p}/dt$ is undefined.

This way of splitting the Momentum Principle is useful in finding the values of unknown forces that cause curving motion. If we can find the value of either part of $d\vec{p}/dt$, this will allow us to deduce the value of the corresponding part of the net force. We can show that, if we know an object's momentum and the instantaneous radius of curvature of its curving path, we can calculate $d\vec{p}/dt_{\perp}$.

The Kissing Circle

Motion along a circular path is easily described by giving the radius of the circle. However, we need a way to describe the instantaneous curvature of an object's path when that path is not a circle.

$R = L\sin\theta$.

Special case: If $\theta = 0$ the string hangs vertically and there is no motion, so the tension in the string ought to be mg, which is consistent with the equation for F_T that we found above.

Note that $\vec{F}_{net}$ and $d\vec{p}/dt$ are not the same thing. In the example above, we knew the x and y components of $d\vec{p}/dt$ (the effect), and from this deduced the x and y components of $\vec{F}_{net}$ (the cause). In the z direction, it was the other way around: we knew that there were no forces in this direction, so we deduced that the z component of momentum was not changing.

There could in principle be many possible motions of this system. The ball could swing back and forth in a plane; this is the motion of a "simple pendulum." The ball could go around in a path something like an ellipse, in which case the ball's height would vary, just as with simple pendulum motion. The ball could move so violently that the string goes slack for part of the time, with abrupt changes of momentum every time the string suddenly goes taut. If the string can stretch noticeably, there can be a sizable oscillation superimposed on the swinging motion.

All of these motions are difficult to analyze. In principle, we could use a computer to predict the general motion of the ball hanging from the string if we had an equation for the tension force in the string, as a function of its length. Essentially, we would need the effective spring stiffness for this stiff "spring." However, the string may be so stiff that even a tiny stretch implies a huge tension (a nearly inextensible string). The observable length of the string is nearly constant, but the tension force that the string applies to the ball can vary a great deal. This makes it challenging to do a computer numerical integration, because a tiny error in position makes a huge change in the force. Special techniques have been developed to handle such situations.

SUMMARY

Determining unknown forces

Make a clear choice of the system; everything else is the surroundings.

Identify objects in the surroundings that can exert either a distant or contact force on the chosen system.

Draw the forces on a free-body diagram and include the name of the object responsible.

Apply the Momentum Principle to the chosen system.

Use what is known about $d\vec{p}/dt$, such as $d\vec{p}/dt = \vec{0}$ in the case of a system at rest or in uniform motion, or what can be obtained from the radius of the kissing circle.

Solve for the unknown forces. For a system containing several objects, it may be necessary to make a different choice of system in order to be able to determine forces between these objects.

Finding the rate of change of momentum graphically

1. Draw two arrows representing $\vec{p}$ at equal time intervals before and after the time of interest (Figure 5.35).
2. Find $\Delta\vec{p} = \vec{p}_f - \vec{p}_i$ (Figure 5.36).

3. Divide $\Delta\vec{p}$ by Δt to get $\Delta\vec{p}/\Delta t$ (Figure 5.37). The smaller Δt, the closer the result is to $d\vec{p}/dt$.

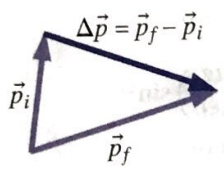

Figure 5.62

Momentarily at rest vs. uniform motion and equilibrium

For a system in equilibrium $d\vec{p}/dt = \vec{0}$.

For a system in uniform motion $d\vec{p}/dt = \vec{0}$.

For a system that is momentarily at rest, $d\vec{p}/dt \neq \vec{0}$.

The Momentum Principle: Parallel and perpendicular components

$$\frac{dp}{dt}\hat{p} = \vec{F}_{\parallel} = (\vec{F}_{\text{net}} \bullet \hat{p})\hat{p} \quad \text{and} \quad p\frac{d\hat{p}}{dt} = \vec{F}_{\perp} = \vec{F}_{\text{net}} - \vec{F}_{\parallel}$$

Valid only for a moving object, because $\hat{p}$ and $d\hat{p}/dt$ are undefined when $\vec{p} = \vec{0}$.

Component of net force parallel to momentum changes magnitude of momentum.

Component of net force perpendicular to momentum changes direction of momentum.

Effect of perpendicular component of net force on a particle moving along a curving path

$$p\left|\frac{d\hat{p}}{dt}\right| = p\frac{v}{R} = |\vec{F}_{\text{net}\perp}|$$

R is the radius of the kissing circle. The component of net force perpendicular to the particle's momentum changes its direction but not the magnitude of its momentum (speed). Must be a smoothly curving motion; not valid if there is an abrupt change of direction.

The dot product

$$\vec{A} \bullet \vec{B} = |\vec{A}||\vec{B}|\cos\theta = A_x B_x + A_y B_y + A_z B_z$$

QUESTIONS

Q1 You are riding in the passenger seat of an American car, sitting on the right side of the front seat. The car makes a sharp left turn. You feel yourself thrown to the right and your right side hits the right door. Is there a force that pushes you to the right? What object exerts that force? What really happens? Draw a diagram to illustrate and clarify your analysis.

Q2 A student said, "When the Moon goes around the Earth, there is an inward force due to the Moon and an outward force due to centrifugal force, so the net force on the Moon is zero." Give two or more physics reasons why this is wrong.

Q3 A space shuttle is in a circular orbit near the Earth. An astronaut floats in the middle of the shuttle, not touching the walls. On a diagram, draw and label **(a)** the momentum $\vec{p}_1$ of the astronaut at this instant, **(b)** all of the forces (if any) acting on the astronaut at this instant, **(c)** the momentum $\vec{p}_2$ of the astronaut a short time Δt later, and **(d)** the momentum change (if any) $\Delta\vec{p}$ in this time interval. **(e)** Why does the astronaut seem to "float" in the shuttle?

It is ironic that we say the astronaut is "weightless" despite the fact that the only force acting on the astronaut is the astronaut's weight (that is, the gravitational force of the Earth on the astronaut).

Q4 Tarzan swings back and forth on a vine. At the microscopic level, why is the tension force on Tarzan by the vine greater than it would be if he were hanging motionless?

Q5 Tarzan swings from a vine. When he is at the bottom of his swing, as shown in Figure 5.63, which is larger in magnitude: the force by the Earth on Tarzan, the force by the vine (a tension force) on Tarzan, or neither (same magnitude)? Explain how you know this.

Figure 5.63

PROBLEMS

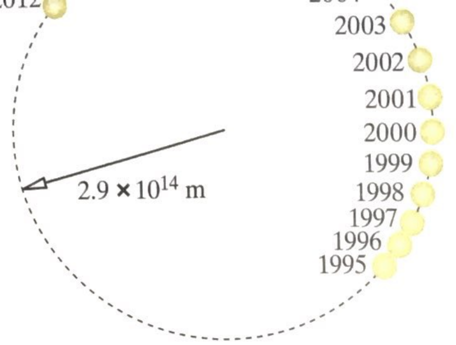

Figure 5.84 Positions on the sky of the star S0-20 near the center of our galaxy.

(a) Using the positions and times shown in Figure 5.84, what is the approximate speed of this star in m/s? Also express the speed as a fraction of the speed of light. **(b)** This is an extraordinarily high speed for a macroscopic object. Is it reasonable to approximate the star's momentum as mv? (Some other stars near the galactic center with highly elliptical orbits move even faster when they are closest to the center.) **(c)** Based on these data, estimate the mass of the massive black hole about which this star is orbiting. (You can see in Figure 5.84 that the speed appears greater in 2007–2008 than in 1995–1996. You're looking at a projection of an elliptical orbit, which only happens to look circular as viewed from Earth, and the black hole is not at the center of this circle. Therefore

It is now thought that most galaxies have such a black hole at their centers, as a result of long periods of mass accumulation. When many bodies orbit each other, sometimes in an interaction an object happens to acquire enough speed to escape from the group, and the remaining objects are left closer together. Some simulations show that over time, as much as half the mass may be ejected, with agglomeration of the remaining mass. This could be part of the mechanism for forming massive black holes.

For more information, search the web for Andrea Ghez. You may see the term "arc seconds," which is an angular measure of how far apart objects appear in the sky, and "parsecs," which is a distance equal to 3.3 light-years (a light-year is the distance light goes in one year).

•••**P56** You put a 10 kg object on a bathroom scale at the North Pole, and the scale reads exactly 10 kg (actually, it measures the force F_N that the scale exerts on the object, but displays a reading in kg). At the North Pole you are 6357 km from the center of the Earth. At the equator, the scale reads a different value due to two effects: (1) The Earth bulges out at the equator (due to its rotation), and you are 6378 km from the center of the Earth. (2) You are moving in a circular path due to the rotation of the Earth (one rotation every 24 hours). Taking into account ***both*** of these effects, what does the scale read at the equator?

COMPUTATIONAL PROBLEMS

More detailed and extended versions of some computational modeling problems may be found in the lab activities included in the *Matter & Interactions, 4th Edition*, resources for instructors.

••**P57** Start with the program you wrote to model the motion of a spacecraft near the Earth (Chapter 3). Choose initial conditions that produce an elliptical orbit. **(a)** Create an arrow to represent the net force on the spacecraft, and place the tail of the arrow at the position of the spacecraft. Inside the calculation loop, update the arrow's position every time you update the craft's position. **(b)** Update the axis of the arrow inside the loop, so it always points in the direction of the net force. Find a scale factor that gives the arrow an appropriate length when the craft is near the Earth, and allows the arrow to be visible when the craft is far from the Earth. **(c)** Now create two additional arrows to represent $\vec{F}_{\parallel}$ and $\vec{F}_{\perp}$, the parallel and perpendicular parts of the net force on the spacecraft. In VPython you can take the dot product of two vectors, $\vec{A}$ and $\vec{B}$, like this: `C = dot(A,B)`. Make your program display $\vec{F}_{\text{net}}$, $\vec{F}_{\parallel}$, and $\vec{F}_{\perp}$ as the craft orbits the Earth. **(d)** In what part of the orbit does $\vec{F}_{\parallel}$ point in the same direction as $\vec{p}$? What effect does it have on the craft's momentum? **(e)** In what part of the orbit does $\vec{F}_{\parallel}$ point in the opposite direction to $\vec{p}$? What effect does it have on the craft's momentum? **(f)** Are there any locations in the orbit where $\vec{F}_{\parallel} = 0$? If so, what are they?

••**P58** Start with the program you wrote to model the 3D motion of a mass hanging from a spring (Chapter 4). **(a)** Create an arrow to represent the net force on the mass, and place the tail of the arrow at the position of the mass. Update both the position and axis of the arrow inside the calculation loop, using an appropriate scale factor, so the arrow always shows the net force on the mass. **(b)** Now create two additional arrows to represent $\vec{F}_{\parallel}$ and $\vec{F}_{\perp}$, the parallel and perpendicular parts of the net force on the mass. Make your program display $\vec{F}_{\text{net}}$, $\vec{F}_{\parallel}$, and $\vec{F}_{\perp}$ as the mass oscillates. **(c)** Find initial conditions for which $\vec{F}_{\parallel}$ is zero and remains zero. Explain. **(d)** Find initial conditions for which $\vec{F}_{\perp}$ is zero and remains zero. Explain. **(e)** Find initial conditions for which both $\vec{F}_{\parallel}$ and $\vec{F}_{\perp}$ are nonzero most of the time. Explain.

•••**P59** Start with the program you wrote to model the motion of a spacecraft near the Earth. Use initial conditions that produce an elliptical orbit. Modify the program so the kissing circle is continuously displayed as the craft orbits the Earth. You may want to use a `ring` object in VPython to display the circle.

ANSWERS TO CHECKPOINTS

1 **(a)** Baseball: left diagram in Figure 5.85, **(b)** Yo-yo: right diagram in Figure 5.85

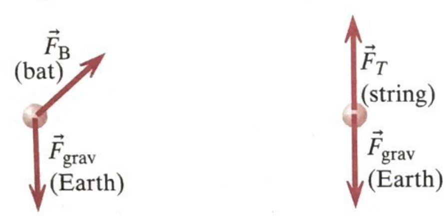

Figure 5.85 Left diagram: baseball. Right diagram: yo-yo.

2 $\vec{0}$; $\vec{0}$; $\langle 30, -90, 130\rangle$ N
3 **(a)** Both blocks. **(b)** Either one of the blocks; note that the magnitude of the force that block m_2 exerts on m_1 is the same as the magnitude of the force that block m_1 exerts on block m_2.
4 **(a)** Figure 5.86 shows the forces:

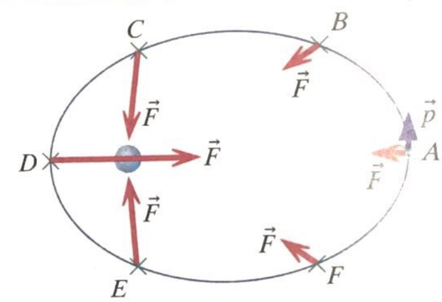

Figure 5.86

A: **(b)** zero, **(c)** nonzero, **(d)** zero, **(e)** changing; *B*: **(b)** nonzero, **(c)** nonzero, **(d)** positive, **(e)** changing; *C*: **(b)** nonzero, **(c)** nonzero, **(d)** positive, **(e)** changing; *D*: **(b)** zero, **(c)** nonzero, **(d)** zero, **(e)** changing; *E*: **(b)** nonzero, **(c)** nonzero, **(d)** negative, **(e)** changing; *F*: **(b)** nonzero, **(c)** nonzero, **(d)** negative, **(e)** changing