Spectral Methods Notes

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1 Introduction

Spectral methods have been extensively developed mainly in order to solve the highly non-linear Einstein equations within the Numerical Relativity framework. Indeed, spectral methods can be used to solve any kind of nonlinear system of PDE, *e.g.* the EoM for a skyrmion or the equations for a rotating configuration of a scalar field, the socalled Q-balls.

The main idea of spectral methods is the expansion of the functions that we want to solve in series of a set of orthogonal polynomials (Chebyshev and Legendre are the most common). Then, the system of differential equations is converted into a system of algebraic equations in which the unknowns are the coefficients of the expansion. This is due to the fact that we know how derivatives act on the polynomials, as we will see.

1.1 Interpolation

In order to start the spectral methods technique we have to know the most basic idea in which it is based: any function $f(x)$ can be approximately interpolated by truncated series of a set of orthogonal polynomials.

$$f(x) \approx \sum_{i=0}^{\mathrm{N_c}-1} f_i T_i(x), \tag{1.1}$$

where $x \in [-1, 1]$, $T_i(x)$ is the set of polynomials (we will mostly work with the Chebyshev polynomials) and f_i are the coefficients of the expansion. These can be determined calculating the following integrals:

$$f_i = \frac{\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} f(x) T_i(x)}{\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} T_i(x) T_i(x)} \approx \frac{\displaystyle\sum_{j=0}^{\mathrm{N}-1} f(x_j) T_i(x_j) w_j}{\displaystyle\sum_{j=0}^{\mathrm{N}-1} T_i(x_j) T_i(x_j) w_j}, \tag{1.2}$$

where x_j and w_j are, respectively, the collocation points and the weights of the Chebyshev-Gauss-Lobatto (CGL) integral. The last step of the equation above shows how we calculate the integrals numerically in a fast and accurate way using weighted Gaussian quadratures. The collocation points and weights for the Chebyshev polynomials can be calculated as follows,

$$x_i = -\cos\left(\frac{i\pi}{\mathrm{N}-1}\right), \quad w_0 = w_{\mathrm{N}-1} = \frac{\pi}{2\mathrm{N}}, \quad w_i = \frac{\pi}{\mathrm{N}}. \tag{1.3}$$

Chebyshev polynomials can also be classified by its parity on the interval $[-1, 1]$: $T_{2i}(x) = T_{2i}(-x)$, $T_{2i+1}(x) = -T_{2i+1}(-x)$. Then if we want to interpolate a function $f(x)$ with a definite parity, even or odd, on that interval we may use only the polynomials with the same parity of $f(x)$. Besides we can reduce the interpolation range to the subinterval $[0, 1]$, this can be very useful in some problems with spherical symmetry.

1.2 Chebyshev and Legendre polynomials

Here we show how to construct both sets of polynomials easily using their respective recurrence relations. For the Chebyshev polynomials we have:

$$T_0(x) = 1, \;\; T_1(x) = x, \;\; T_i(x) = 2xT_{i-1}(x) - T_{i-2}, \tag{1.4}$$

while the Legendre polynomials recurrence relation is:

$$P_0(x) = 1, \;\; P_1(x) = x, \;\; P_l(x) = \frac{2n-1}{n} x P_{l-1}(x) - \frac{n-1}{n} P_{l-2}. \tag{1.5}$$

As we have said they are both sets of orthogonal polynomials:

$$\int_{-1}^{1} d\mu \, T_i(x) T_j(x) = \delta_{ij} \frac{\pi}{1+\delta_{i0}}, \tag{1.6}$$

$$\int_{-1}^{1} dx \, P_l(x) P_m(x) = \delta_{lm} \frac{2}{2l+1}, \tag{1.7}$$

where $d\mu = \frac{dx}{\sqrt{1-x^2}}$.

We will learn the basics of spectral methods, focusing each section on specific cases and using some differential equations as a guide, increasing the complexity to end with a real interesting problem.

2 Linear ODE in one domain

The simplest case that we can solve is a linear ODE in some domain $\mathcal{D}$. Whatever the range of our domain, we can always perform a change of coordinate in order to work on the interval $[-1, 1]$. We will consider a differential equation that contains the most general linear operators that we can solve using spectral methods,

$$\frac{df}{dx} + xf + \frac{1}{x} f = s(x). \tag{2.1}$$

This is not the most generic differential equation, since we could have higher derivative terms, polynomials of x or powers of $1/x$. However we will represent the different terms of the equation (2.1) by linear operators so that we can multiply and combine them in order to construct a more general differential equation, *e.g.* $x^2 = xx$, $\frac{d^2}{dx^2} = \frac{d}{dx}\frac{d}{dx}$. The function $s(x)$ is the source of the equation, it is a known function and it only depends on the coordinate x. The idea of the spectral methods is obtaining an approximation to the function $f(x)$ that satisfies the equation (2.1) by solving the coefficients f_i of its interpolation:

$$f(x) \approx \sum_{i=0}^{\mathrm{N_c}-1} f_i T_i(x). \tag{2.2}$$

Inserting the last expansion in the equation we have

$$\sum_i \left(f_i \frac{dT_i}{dx} + xT_i + \frac{1}{x} T_i \right) = \sum_i s_i T_i(x). \tag{2.3}$$

Since we know $s(x)$ it can be directly interpolated, so that the coefficients s_i are also known. The keypoint now is that we know how these operators act on the Chebyshev polynomials, indeed they act linearly on them.

$$\frac{dT_i}{dx} = 2iT_{i-1} + \frac{i}{i-2} \frac{dT_{i-2}}{dx} = \sum_j L^d_{ij} T_j, \tag{2.4}$$

$$xT_i = \frac{1}{2} (T_{i+1} + T_{i-1}) = \sum_j L^x_{ij} T_j, \tag{2.5}$$

$$\frac{1}{x} T_i = 2T_{i-1} - \frac{1}{x} T_{i-2}. = \sum_j L^{1/x}_{ij} T_j. \tag{2.6}$$

The differential equation is reformulated into a system of algebraic equations:

$$\sum_{i,j} \left(L^d_{ij} + L^x_{ij} + L^{1/x}_{ij} \right) f_i T_j \equiv \sum_{i,j} f_i L_{ij} T_j = \sum_i s_i T_i. \tag{2.7}$$

The entries of the matrices $L^d, L^x, L^{1/x}$ are shown in Appendix A. Now it is more clear that if the differential equation contains, *e.g.* second derivatives, we can define the second derivative operator $L^{dd}_{ij} = L^d_{ik} L^d_{kj}$, and the same for any polynomial of x or $1/x$. We multiply the last equation by T_k and integrate on $\int d\mu$ in order to obtain the system,

$$\sum_j L_{ij} f_j = s_i. \tag{2.8}$$

Nevertheless, we cannot solve the problem because the equation (2.1) is not complete since it requires a boundary condition (B.C.). Consider the following B.C. $f(x = -1) = A$, $f(x = 1) = B$,

$$f(-1) = \sum_i f_i T_i(-1) = \sum_i f_i (-1)^i = A \tag{2.9}$$

$$f(1) = \sum_i f_i T_i(1) = \sum_i f_i = B. \tag{2.10}$$

These conditions can be easily imposed in the linear system filling the last 2 rows of the linear operator and the source coefficients with the corresponding condition:

$$L_{-2j} = (-1)^j, \quad s_{-2} = A, \tag{2.11}$$

$$L_{-1j} = 1, \quad s_{-1} = B. \tag{2.12}$$

We choose the last two rows to impose the B.C. since these are the least important in the expansion (2.2). This choice is sometimes called the Tau-method. After imposing the B.C. the system (2.8) is solved inverting the linear operator and multiplying it by the source coefficients vector:

$$\boxed{f_i = \sum_j L_{ij}^{-1} s_j.} \tag{2.13}$$

3 Multidomain Linear ODE

It is frequent to find problems where we have to split our domain $\mathcal{D}$ in different regions. This often occurs in spherically symmetric problems in which the radial coordinate ranges the whole space, $r \in [0, \infty]$. In these cases the points 0 and ∞ might be problematic. For this reason we may consider as an example the radial part of the two-dimensional Poisson equation with a spherically symmetric source:

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} - \frac{l(l+1)}{r^2} f = s(r). \tag{3.1}$$

If the source has no compact support (it extends to infinity) we have to split $\mathcal{D}$ in two regions: $r_1 \in \mathcal{D}_1 = [0, C]$, $r_2 \in \mathcal{D}_2 = [C, \infty]$. The value of C is arbitrary, however we may choose it in order to exploit some properties from the Chebyshev polynomials. We know that f must respect regularity around the origin, then if we take C sufficiently small we can expand f in terms of a definite parity (even or odd) Chebyshev polynomials. Besides, as we explained in the Introduction, if we fix the parity of the polynomials, we can expand the function within the subdomain $x \in [0, 1]$.

Then, in $\mathcal{D}_1$ we may define the new coordinate $r_1 = C\,x_1$, so the equation (3.1) reads:

$$\frac{d^2 f}{dx_1^2} + \frac{2}{x_1}\frac{df}{dx_1} - \frac{l(l+1)}{x_1^2} f = C^2 s(x_1) \quad \text{in } \mathcal{D}_1. \tag{3.2}$$

We take the even parity Chebyshev polynomials to expand $f(x_1)$ and $s(x_1)$ and, as in the last section we express our differential equation as a system of algebraic linear equations.

$$f(x_1) = \sum_i f_i^{\mathcal{D}_1} T_{2i}(x_1), \quad s(x_1) = \sum_i s_i^{\mathcal{D}_1} T_{2i}(x_1), \tag{3.3}$$

$$\sum_j \left(L_{ij}^{dd} + 2L_{ij}^{1/x} - l(l+1) \sum_k L_{ik}^{1/x} L_{kj}^{1/x} \right) f_j^{\mathcal{D}_1} = \sum_j L_{ij}^{\mathcal{D}_1} f_j = s_i^{\mathcal{D}_1}. \tag{3.4}$$

Again, in our problem we must impose some B.C. for instance: $f(r = 0) = A$, $f(r = \infty) = B$. Then, in $\mathcal{D}_1$ we use the penultimate row of the linear operator L to impose the first B.C.

$$f(x_1 = 0) = \sum_i f_i^{\mathcal{D}_1} T_{2i}(0) = \sum_i f_i^{\mathcal{D}_1} (-1)^i \longrightarrow L_{-2j}^{\mathcal{D}_1} = (-1)^j, \quad s_{-2}^{\mathcal{D}_1} = A. \tag{3.5}$$

In $\mathcal{D}_2$ the radial coordinate extends to infinity, hence it is useful to express the equation first in terms of $u_2 = \frac{1}{r_2}$ and then define the coordinate $u_2 = \frac{1-x_2}{2C}$. After these changes the equation (3.1) is:

$$(1 - x_2)^4 \frac{d^2 f}{dx_2^2} = 4C^2 s(x_2) \quad \text{in } \mathcal{D}_2. \tag{3.6}$$

Once more, the equation is translated into an algebraic system and the B.C. at infinity is imposed, again, in the penultimate rows of $L^{\mathcal{D}_2}$ and $s^{\mathcal{D}_2}$.

To complete the problem we have to concatenate both domains in a single linear operator. If we use ${\mathrm{N_c}}^{\mathcal{D}_1}$ and ${\mathrm{N_c}}^{\mathcal{D}_2}$ coefficients for each domain respectively, the total operator will have size $({\mathrm{N_c}}^{\mathcal{D}_1}+{\mathrm{N_c}}^{\mathcal{D}_2})\times({\mathrm{N_c}}^{\mathcal{D}_1}+{\mathrm{N_c}}^{\mathcal{D}_2})$, such that in the diagonals we have the operators $L^{\mathcal{D}_1}$ and $L^{\mathcal{D}_2}$ and the rest is filled with 0. Furthermore we have to impose continuity conditions for f and its derivative in the boundary where the two domains join:

$$f(x_1=1)=\sum_{i=0}^{{\mathrm{N_c}}^{\mathcal{D}_1}} f_i^{\mathcal{D}_1}T_{2i}(1)=\sum_{i=0}^{{\mathrm{N_c}}^{\mathcal{D}_1}} f_i^{\mathcal{D}_1}=f(x_2=-1)=\sum_{i=0}^{{\mathrm{N_c}}^{\mathcal{D}_2}} f_i^{\mathcal{D}_2}(-1)^i \tag{3.7}$$

$$\frac{df}{dx_1}(x_1=1)=\sum_{i,j=0}^{{\mathrm{N_c}}^{\mathcal{D}_1}} f_i^{\mathcal{D}_1}L^d_{i2j}T_{2j}(1)=\sum_{i,j=0}^{{\mathrm{N_c}}^{\mathcal{D}_1}} f_i^{\mathcal{D}_1}=\frac{df}{dx_2}(x_2=-1)=\sum_{i,j=0}^{{\mathrm{N_c}}^{\mathcal{D}_2}} f_i^{\mathcal{D}_2}L^d_{ij}(-1)^j. \tag{3.8}$$

These conditions are imposed in the ${\mathrm{N_c}}^{\mathcal{D}_1}-1$ and the $({\mathrm{N_c}}^{\mathcal{D}_1}+{\mathrm{N_c}}^{\mathcal{D}_2})-1$ rows of the total linear operator L, which is schematically represented below:

$$\left(\begin{array}{c|c} L^{\mathcal{D}_1} & 0 \\ \hdashline \hline 0 & L^{\mathcal{D}_2} \\ \hdashline \end{array}\right)$$

The dashed line represents the continuity condition on f (above) and its derivative (below).

Finally, this operator is inverted and the coefficients of the function f are obtained as in the last section.

4 Non-linear ODE in one domain

Until now we can solve some simple differential equations, however interesting problems, as in General Relativity, typically involve nonlinearities. The keypoint of solving nonlinear differential equations using spectral methods is to introduce all the nonlinear terms in the source. This implies that we will need an initial configuration for our function. Lets focus on a simple nonlinear differential equation:

$$\frac{d^2\phi}{dx^2}+\phi^2=0. \tag{4.1}$$

Defining the source as $s\equiv-\phi^2$ we recover a problem more similar to those that we have solved before. Thus we have to find the field such that the equation (4.1) is satisfied, or, in spectral methods language, we want to obtain the coefficients ϕ_i such that the residuals $R_i=L_{ij}\phi_j-s_i$ are nearly 0. This is achieved iteratively using a Newton-Rapshon algorithm.

The first step in order to solve a nonlinear ODE is to separate the linear part of the equation and the rest is introduced in the source. Then, an initial configuration for the field is required, so that we can interpolate the source to obtain the coefficients s_i and, together with the linear operator (L^{dd} in our case), the residuals R_i can be calculated. The Newton-Raphson method proceeds as follows:

$$J_{ij}(\phi^n) = \frac{\partial R_i(\phi^n)}{\partial \phi_j^n} = L_{ij} - \frac{\partial s_i}{\partial \phi_j^n}, \quad X_i^n = \sum_j J_{ij}^{-1}(\phi^n) R_j(\phi^n) \tag{4.2}$$

$$\phi_i^{n+1} = \phi_i^n - X_i^n. \tag{4.3}$$

It states that the coefficients ϕ_i^n calculated after n iterations make R_i tend to 0 so that we will get closer to the true solution in each iteration. Indeed this can also be applied to linear ODEs, the true solution is obtained in the first iteration.

The calculation of the Jacobian matrix J_{ij} can be done analytically, however, if the source is sufficiently complicated the numerical calculation of J is a more efficient choice. In our case the source only depend on the field quadratically, however it might depend on a nontrivial combination of the field and its derivatives. Simply using the discrete definition of the derivative we have:

$$J_{ij} \approx \frac{R_i(\phi^n + \epsilon\phi_j^n) - R_i(\phi^n - \epsilon\phi_j^n)}{2\epsilon}. \tag{4.4}$$

In the last equation ϕ^n denotes the vector of coefficients in the current iteration, and ϕ_j^n is a vector of zeros except in the position j which is the value of the corresponding coefficient. We take the value $\epsilon \sim 1\%$. If the Jacobian matrix is calculated numerically we need more than one iteration to solve even a linear problem, however this method converges quite well.

There is one last important comment concerning the B.C. As in the previous cases, these are imposed on the linear operator as well as in the source. Besides, in the rows where we imposed the B.C. the Jacobian matrix is:

$$J_{-2j} = L_{-2j}, \quad J_{-1j} = L_{-1j}. \tag{4.5}$$

5 Multidomain non-linear ODE

The resolution of a nonlinear ODE in more than one domain combines almost exactly the steps of the last two section that we have seen. The idea is to obtain the linear operator and source of each domain individually following section 4, then the concatenation of both domains is explained in section 3. Finally we apply the Newton-Raphson method calculating the residuals and the Jacobian matrix from the total linear operator and the total vector of source coefficients. Again, the rows where we impose the continuity conditions we also impose:

$$J_{\mathrm{N_c}^{\mathcal{D}_1}-1\mathrm{j}} = L^{\mathcal{D}_1}_{-1j}, \quad J_{\mathrm{N_c}^{\mathcal{D}_1}+\mathrm{N_c}^{\mathcal{D}_2}-1\mathrm{j}} = L^{\mathcal{D}_2}_{-1j}. \tag{5.1}$$

6 Linear PDE in one domain

The problems that we already know to solve appear frequently in physics in spherically symmetric problem. However we may face problems with axial symmetry in which we cannot erase the angular coordinate from the equations. These problems involve PDEs, so we will start solving a linear PDE using spectral methods.

The main idea is to separate the dependence on the different coordinates expanding on different sets of polynomials. Depending on the coordinates of the problem there are better choices for the polynomials, in cartesian coordinates the expansion on two sets of Chebyshev polynomials is a good choice, while in cylindrical coordinates we may expand the radial coordinate on Chebyshev and the angular part on Legendre polynomials,

$$\phi(x,y) = \sum_{i,j} \phi_{ij} T_i(x) T_j(y), \quad \phi(r,\theta) = \sum_{i,l} \phi_i^l T_i(r) P_l(\cos\theta),$$
$$\phi(r,\theta,\varphi) = \sum_{i,l,m} \phi_{ilm} T_i(r) Y_m^l(\theta,\varphi).$$

In this section we will work with the Chebyshev and Legendre polynomials since we want to focus again on the Poisson equation in two dimensions. This section generalizes section 3 to the case in which the source depends on both coordinates.

The most important trick in this section is that we will always interpolate our functions using Chebyshev polynomials, but with a change of basis we will know the expansion on Legendre polynomials. Defining the matrix M as the relation between both sets of polynomials in the following way:

$$P_l(x) = \sum_m M_{lm} T_m(x), \tag{6.1}$$

we can obtain the Legendre coefficients of any function $\phi(x)$ easily if we know its Chebyshev coefficients,

$$\phi(x) = \sum_l \phi_l^L P_l(x) = \sum_l \phi_l^C T_l(x) \longrightarrow \phi_l^L = \sum_m M_{lm}^{-1} \phi_m^C. \tag{6.2}$$

The entries of the matrix M_{lm} can also be easily obtained since they are the m-th Chebyshev coefficient of the l-th Legendre polynomial interpolation,

$$M_{lm} = \frac{\int d\mu\, P_l(x) T_m(x)}{\int d\mu\, T_m(x) T_m(x)}. \tag{6.3}$$

Hence, now the coefficients that we want to obtain as the solution is a matrix ϕ_i^l of size $\mathrm{N_c}^{\mathrm{C}} \times \mathrm{N_c}^{\mathrm{L}}$.

Starting from the two-dimensional Poisson equation $\nabla^2\phi = s(r,\theta)$, we will insert the expansions step by step:

$$\phi(x,\theta) = \sum_l \phi_l(x) P_l(\cos\theta), \tag{6.4}$$

$$\sum_l \left(\frac{d^2\phi_l}{dx^2} + \frac{2}{x}\frac{d\phi_l}{dx} - \frac{l(l+1)}{x^2}\phi_l \right) P_l(\cos\theta) = \sum_l s_l(x) P_l(\cos\theta). \tag{6.5}$$

We have already converted our radial coordinate into the coordinate x defined on the interval $[-1,1]$. Since the source is a known function, we can interpolate it in the way that we explained above in order to obtain the functions $s_l(x)$. We may see that in this concrete case different values of l do not couple. Hence, we can solve the problem as a system of $\mathrm{N_c}^{\mathrm{L}}$ decoupled ODEs, where each equation is solved as explained in section 3.

We multiply now the equation (6.5) by $P_m(\cos\theta)$ and integrate to eliminate the dependence on θ.

Then expand each function $\phi_l(x)$ and $s_l(x)$ in Chebyshev polynomials.

$$\phi_l(x) = \sum_i \phi_i^l T_i(x), \tag{6.6}$$

$$\sum_{i,j} \phi_i^l L_{ij}^l T_j(x) = \sum_i s_i^l T_i(x). \tag{6.7}$$

We introduced a Legendre index in the linear operator since it depends on the value of l. As we have said before, different values of l do not couple in this case, but a generic case in which the indices l and m are coupled would require a new index m on L. This will be the case of nonlinear PDEs, hence we can represent the residuals of a generic PDE in the following way:

$$R_i^l = L_{ij}^{lm} \phi_j^m - s_i^l. \tag{6.8}$$

Obviously, if now ϕ_i^l and s_i^l are matrices, then L_{ij}^{lm} is a tensor of 4 indices.

The B.C. in a linear PDE are easily imposed. Usual B.C. in the current problem are,

$$\phi(r = R, \theta) = g(\theta), \quad \phi(r = \infty, \theta) = 0. \tag{6.9}$$

We proceed interpolating the function $g(\theta)$ to obtain the coefficients g_l, then we construct each matrix L^{lm} and the B.C. are imposed in the last rows as always. Also in the last rows of the source coefficients we have to set the values: $s_{-2}^l = g_l$, $s_{-1}^l = 0$.

Finally we have to solve the linear problem:

$$L_{ij}^{lm} \phi_j^m = s_i^l. \tag{6.10}$$

Only for our problem (6.5), where we can split in $\mathrm{N_c}^{\mathrm{L}}$ decoupled ODEs we could obtain the coefficients ϕ_i^l inverting the each linear operator $\mathrm{N_c}^{\mathrm{L}}$ times and multiplying it by the source coefficients. However in a more general problem what we can do is a conversion of the linear tensor L_{ij}^{lm} into a matrix $L_{\alpha\beta}$ mixing the indices:

$$\alpha = i + l\mathrm{N_c}^{\mathrm{C}}, \quad \beta = \mathrm{j} + \mathrm{m}\mathrm{N_c}^{\mathrm{C}}. \tag{6.11}$$

The new matrix $L_{\alpha\beta}$ has size $\mathrm{N_c}^{\mathrm{L}}\mathrm{N_c}^{\mathrm{C}} \times \mathrm{N_c}^{\mathrm{L}}\mathrm{N_c}^{\mathrm{C}}$ so it can be inverted and the problem is solved.

7 Non-linear PDE in one domain

In this section, again a Newton-Raphson method will be required. This is the most general case on this notes, but it is not much more complicated than the last case or the nonlinear ODE.

We can still focus again in the two-dimensional Poisson equation, but now the source depends on the field nonlinearly. We proceed as before, take an initial configuration for the field ϕ, calculate the corresponding source and obtain its coefficients. Then, compute the linear tensor L_{ij}^{lm} and impose the B.C. as always. Next, calculate the Jacobian tensor,

$$J_{ij}^{lm} = \frac{\partial R_i^l}{\partial \phi_j^m}. \tag{7.1}$$

It is highly recommendable to compute J_{ij}^{lm} numerically in this case. The Newton-Raphson algorithm requires to invert the Jacobian tensor, for which we have to compactify its indices as we did in the last section. Then, we can iterate the algorithm to solve the problem.

$$X_\alpha^n = \sum_\beta J_{\alpha\beta}^{-1} R_\beta^n, \;\; \phi_\alpha^n = \phi_\alpha^{n-1} - X_\alpha^n. \tag{7.2}$$

A Linear Operators

Here the entries of the linear operators for $\mathrm{N_c} = 8$ are shown in order to guide the reader.

$$L^d = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 5 & 0 & 7 \\ 0 & 0 & 4 & 0 & 8 & 0 & 12 & 0 \\ 0 & 0 & 0 & 6 & 0 & 10 & 0 & 14 \\ 0 & 0 & 0 & 0 & 8 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \;\; L^x = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$L^{1/x} = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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