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Assignment 1 - Set Theory and Logic, Function**Group of 3, Due date: 17 November 2024**

1. Let $A = \{1, 3, 5, 7\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6, 7, 8\}$.

- a) Prove that $(A \cap B) \cup (B \cap C) \subseteq B$.
b) Prove or disprove: $(A \cup C) - B = (A - B) \cup (C - B)$.
c) Find $A \oplus (B \cap C)$ and determine whether this result is equal to $(A \oplus B) \oplus C$.

$$a) A \cap B = \{3, 5\}$$

$$B \cap C = \{5, 6\}$$

$$(A \cap B) \cup (B \cap C) = \{3, 5, 6\} \subseteq B = \{3, 4, 5, 6\}$$

$$(A \cap B) \cup (B \cap C) \subseteq B$$

$$b) A \cup C = \{1, 3, 5, 6, 7, 8\}$$

$$(A \cup C) - B = \{1, 7, 8\}$$

$$A - B = \{1, 7\}$$

$$C - B = \{7, 8\}$$

$$(A - B) \cup (C - B) = \{1, 7, 8\}$$

$$(A \cup C) - B = (A - B) \cup (C - B)$$

$$c) B \cap C = \{5, 6\}$$

$$A \oplus (B \cap C) = \{1, 3, 6, 7\}$$

$$A \oplus B = \{1, 4, 6, 7\}$$

$$(A \oplus B) \oplus C = \{1, 4, 5, 8\}$$

$$A \oplus (B \cap C) \neq (A \oplus B) \oplus C$$

2. Define the following sets:

- $P = \{x \in \mathbb{Z} \mid x \text{ is a prime number less than } 20\}$
- $E = \{x \in \mathbb{Z} \mid x \text{ is an even number less than } 20\}$
- $D = \{x \in \mathbb{Z} \mid x \text{ is a divisor of } 36 \text{ and less than } 20\}$

Using these sets:

- Translate the statement: "All elements of P that are not in D are also in E " into a formal logic expression using quantifiers.
- Prove or disprove the above statement in (a) by checking each element.
- Prove that $(P \cap E) \cup D = D$ using set theory laws.

$$P = \{2, 3, 5, 7, 11, 13, 17, 19\}$$

$$E = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\}$$

$$D = \{1, 2, 3, 4, 6, 9, 12, 18\}$$

$$a) \forall x ((P(x) \wedge \neg D(x)) \rightarrow E(x))$$

$$b) P(x) \wedge \neg D(x) = \{5, 7, 11, 13, 17, 19\}$$

$$E(x) = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$$

$$\forall x ((P(x) \wedge \neg D(x)) \rightarrow E(x)) = \emptyset, \text{ so this statement is FALSE.}$$

$$c) (P \cap E) \cup D = D$$

$$P \cap E = \{2\}$$

$$\{2\} \subseteq D$$

$$(P \cap E) \subseteq D$$

$$\text{So, } (P \cap E) \cup D = D$$

3. Let U be the universal set of all integers from 1 to 30. Define the following subsets of U:

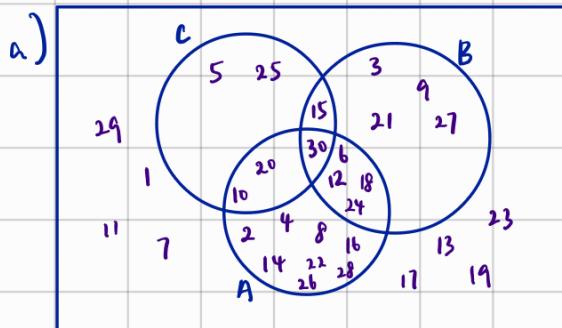
- $A = \{x \in U \mid x \text{ is a multiple of } 2\}$
- $B = \{x \in U \mid x \text{ is a multiple of } 3\}$
- $C = \{x \in U \mid x \text{ is a multiple of } 5\}$

- Construct a Venn diagram that represents U, A, B, and C.
- Identify the number of elements in each region of the Venn diagram (e.g., $A \cap B \cap C$, $A - B - C$, etc.).

$$A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30\}$$

$$B = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}$$

$$C = \{5, 10, 15, 20, 25, 30\}$$



b) $|A - B - C| = 8$

$$|B - A - C| = 4$$

$$|C - A - B| = 2$$

$$|(A \cap B) - C| = 4$$

$$|(A \cap C) - B| = 2$$

$$|(B \cap C) - A| = 1$$

$$|A \cap B \cap C| = 1$$

$$|U - (A \cup B \cup C)| = 8$$

4. Let $X = \{0, 1, 2\}$ and $Y = \{a, b\}$.
- Find the Cartesian product $X \times Y$ and list each ordered pair.
 - Define a relation $R \subseteq X \times X$ such that $(x, y) \in R$ if and only if $x + y$ is even. List all pairs in R .
 - Determine if R is reflexive, symmetric, or transitive, and provide proof for each property.

a) $X \times Y = \{(0, a), (0, b), (1, a), (1, b), (2, a), (2, b)\}$

b) $0 + 0 = 0$ $1 + 0 = 1$ $2 + 0 = 2$

$0 + 1 = 1$ $1 + 1 = 2$ $2 + 1 = 3$

$0 + 2 = 2$ $1 + 2 = 3$ $2 + 2 = 4$

$R = \{(0, 0), (0, 2), (1, 1), (2, 0), (2, 2)\}$

c) R is reflexive because $\{(0, 0), (1, 1), (2, 2)\} \in R$

R is symmetric because $\{(0, 2), (2, 0)\} \in R$

R is transitive because $\{(0, 2), (2, 0)\} \in R$ $(0, 0) \in R$

$\{(0, 0), (0, 2)\} \in R$ $(0, 2) \in R$

$\{(2, 0), (0, 2)\} \in R$ $(2, 2) \in R$

5. Consider the following statements:

- Prove the contrapositive of the following statement: "If $x \in \mathbb{Z}$ is even and x^2 is also even, then x is a multiple of 4."
- Determine whether the following statement is true or false. If true, prove it. If false, provide a counterexample:
 - "For all integers m and n , if mn is even, then m is even or n is even."
- Using a proof by contradiction, show that the square root of 3 is irrational.

Contrapositive:

a) If x is not a multiple of 4, then $x \in \mathbb{Z}$ is odd and x^2 is also odd.
 assume x not a multiple of 4,

$x = 2k + 1$ for some integer k

$x = 2k + 1$ x is an odd integer

$x^2 = (2k+1)^2$

$x^2 = (2k+1)(2k+1)$

$x^2 = 4k^2 + 4k + 1$

$x^2 = 2(2k^2 + 2k) + 1$

$x^2 = 2m + 1$ where $m = 2k^2 + 2k$ is an integer

x^2 is an odd integer

multiple of 4 = {4, 8, 12, 16, ...}

\Rightarrow multiple of 4 = even odd

Counter example = 2
 If $x = 2$, is even
 $x^2 = 4$, is also even
 but $2 \notin$ multiple of 4

b) when m / n is even

$$mn = (2k+1)(2k)$$

$$= 4k^2 + 2k$$

$$= 2(2k^2 + k) \quad 2k^2 + k \text{ for some integer}$$

$$= 2x$$

\downarrow
 mn is even

TRUE

Contradiction:

c) If $\sqrt{3}$ is irrational, then $\sqrt{3}$ cannot be form $\frac{p}{q}$

Suppose that the conclusion is false. Then $\sqrt{3}$ can be form $\frac{p}{q}$

$$\sqrt{3} = 1.732050808\dots$$

$\sqrt{3}$ cannot be form of $\frac{p}{q}$

The statement is true.

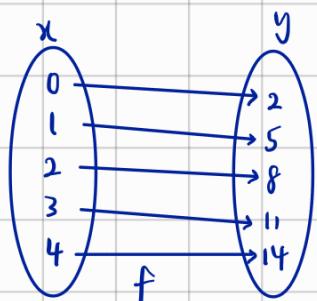
6. Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $f(x) = 3x + 2$.

a) Prove or disprove that f is one-to-one.

b) Prove or disprove that f is onto.

c) Find $f^{-1}(y)$, if it exists, for $y \in \mathbb{Z}$, and determine the conditions under which f has an inverse function.

a)



f is one to one

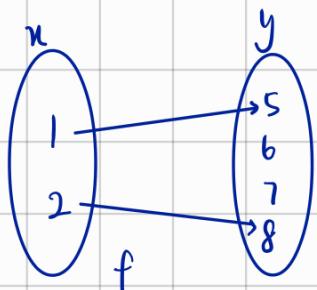
because each element in y has at most one arrow

$$\text{if } f(x_1) = f(x_2)$$

$$3x_1 + 2 = 3x_2 + 2$$

$$x_1 = x_2$$

b)



f is not onto

because each element in y don't have at least one arrow pointing to it

c)

$$y = 3x + 2$$

$$3x = y - 2$$

$$x = \frac{y-2}{3}$$

$$f^{-1}(y) = \frac{y-2}{3}$$

inverse function exists
when $3x - 1, x \in \mathbb{Z}$

7. Let $g: R \rightarrow R$ be defined by $g(x) = x^2$ and $h(x) = x + 1$.

- Compute $(g \circ h)(x)$ and $(h \circ g)(x)$.
- Determine whether $g \circ h$ and $h \circ g$ are one-to-one, onto, or neither.
- If possible, find the inverse of $(h \circ g)(x)$.

$$\begin{aligned} a) \quad (g \circ h)(x) &= g(h(x)) \\ &= g(x+1) \\ &= (x+1)^2 \\ &= x^2 + 2x + 1 \end{aligned}$$

$$\begin{aligned} (h \circ g)(x) &= h(g(x)) \\ &= h(x^2) \\ &= x^2 + 1 \end{aligned}$$

$$\begin{aligned} b) \quad g \circ hg &= g(h(g(x))) \\ &= g(h(x^2)) \\ &= g(x^2 + 1) \\ &= (x^2 + 1)^2 \\ &= x^4 + 2x^2 + 1 \end{aligned}$$

$$x = -1, (-1)^4 + 2(-1)^2 + 1 = 4$$

$$x = 1, 1^4 + 2(1)^2 + 1 = 4$$

not one to one,

not onto because no number can point at codomain which < 1

$$\begin{aligned} h \circ gh &= h(g(h(x))) \\ &= h(g(x+1)) \\ &= h(x+1)^2 \\ &= h(x^2 + 2x + 1) \\ &= x^2 + 2x + 1 + 1 \end{aligned}$$

$$x = 0, 0^2 + 2(0) + 1 + 1 = 2$$

$$x = -2, (-2)^2 + 2(-2) + 1 + 1 = 2$$

not one to one,

not onto because no number can point at codomain which < 1 .

$$\begin{aligned} c) \quad (h \circ g)(x) &= h(g(x)) \\ &= h(x^2) \\ &= x^2 + 1 \end{aligned}$$

$$\text{let } h(g(x)) = y$$

$$y = x^2 + 1$$

$$x^2 = y - 1$$

$$x = \pm\sqrt{y-1}, y \geq 1$$

$$f^{-1}(y) = \pm\sqrt{y-1}, y \geq 1$$