Methods: $\boxed{\text{① Model}}$

$$S_i^\alpha(t)=\sum_k \delta(t-t_{ik}^\alpha) \quad \cdots(21)$$

$$R^\alpha I^\alpha_{syn,i}(t)=\underset{\downarrow\, s}{\tau_m^\alpha}\sum_{\beta=1}^{M}\underset{\downarrow\, mV}{w^{\alpha\beta}}\sum_{j\in\Gamma_i^\beta}\underset{\downarrow\, s^{-1}}{\left(\epsilon^{\alpha\beta}*s_j^\beta\right)(t)} \quad \cdots(22)$$

$$\tau_m^\alpha\frac{du_i^\alpha}{dt}=-u_i^\alpha+\mu^\alpha(t)+R^\alpha I^\alpha_{syn,i}(t) \quad \cdots(23)$$

$$\vartheta_i^\alpha(t)=u_{th}^\alpha+\sum_{t_{i,j}^\alpha<t}\theta^\alpha(t-t_{ik}^\alpha) \quad \cdots(24)$$

$$=u_{th}^\alpha+\int_{-\infty}^{t}\theta^\alpha(t-t')s_i^\alpha(t')dt'$$

$$\lambda_i^\alpha(t)=f^\alpha(u_i^\alpha(t)-\vartheta_i^\alpha(t)) \quad \cdots(25)$$

$$f^\alpha(x)=c^\alpha\exp(x/\Delta_u^\alpha)$$

$$u_i^\alpha(t)=h_i^\alpha(t)+\sum_{\hat t_{i,j}^\alpha<t}\eta^\alpha(t-\hat t_{ij}^\alpha) \quad \cdots(26)$$

$$h_i^\alpha(t)=\left[\kappa^\alpha*(\mu^\alpha+R^\alpha I^\alpha_{syn,i})(t)\right] \quad \cdots(27)$$

$$\kappa^\alpha=\Theta(t)\,e^{t/\tau_m^\alpha}/\tau_m^\alpha$$

At low firing rates, $\eta(t)=(u_r-u_{th})e^{-(t-t_{ref}^\alpha)/\tau_m^\alpha}\Theta(t)$

$\boxed{\text{② Mean-field approximation}}$

$$\frac{1}{N^\alpha}\sum_{i=1}^{N^\alpha}\lambda_i^\alpha(t)\approx\frac{1}{N^\alpha}\sum_{i=1}^{N^\alpha}\lambda_A^\alpha(t|\hat t_i^\alpha) \quad \cdots(28)$$

$$R^\alpha I^\alpha_{syn,i}(t)=\tau_m^\alpha\sum_{\beta=1}^{M}p^{\alpha\beta}N^\beta w^{\alpha\beta}(\epsilon^{\alpha\beta}*A_N^\beta)(t) \quad \cdots(29)$$

$$\tau_m^\alpha\frac{\partial u_A^\alpha}{\partial t}=-u_A^\alpha+\mu^\alpha(t)+\tau_m^\alpha\sum_{\beta=1}^{M}J^{\alpha\beta}(\epsilon^{\alpha\beta}*A_N^\beta)(t) \quad \cdots(30)$$

$J^{\alpha\beta}=p^{\alpha\beta}N^\beta w^{\alpha\beta}$, initial condition $u_A^\alpha(\hat t,\hat t)=u_r$

$$p^{\alpha\beta}N^\beta\left(\frac{1}{p^{\alpha\beta}N^\beta}\sum_{j\in\Gamma_i^\beta}s_j^\beta(t)\right)\approx p^{\alpha\beta}N^\beta A_N^\beta(t) \quad \cdots(31)$$

$$\lambda_i^\alpha(t)\approx f^\alpha(u_A^\alpha(t,\hat t_i^\alpha)-\vartheta_A^\alpha(t,\hat t_i^\alpha))\equiv\lambda_A^\alpha(t|\hat t_i^\alpha) \quad \cdots(32)$$

$$\vartheta_A^\alpha(t,\hat t)=u_{th}+\theta^\alpha(t-\hat t)+\int_{-\infty}^{\hat t}\tilde\theta^\alpha(t-t')\underline{A_N^\alpha}(t')dt' \quad \cdots(33)$$

$\tilde\theta^\alpha(t)=\Delta_u^\alpha[1-e^{-\theta^\alpha(t)/\Delta_u^\alpha}]$ quasi-renewal kernel

$\boxed{\text{③ Discretized population density equations}}$

$$\sum_{k=-\infty}^{l-1}m(t_l,t_k)=N \quad \cdots(34) \qquad A_N(t_k)=\frac{\Delta n(t_k)}{N\Delta t}$$

$$\mathcal{X}(t_l)=\{(\Delta n(t_k),m(t_l,t_k))\}_{k\in Z,\,k<l} \quad \cdots(35)$$

$$m(t_l+\Delta t,t_k)=m(t_l,t_k)-X_{lk} \quad \cdots(36)$$

$$P_\lambda(t_l|t_k)=1-\exp\left(-\int_{t_l}^{t_l+\Delta t}\lambda(s|t_k)ds\right)\approx 1-e^{-\bar\lambda(t_l|t_k)\Delta t} \quad \cdots(37)$$

$$\bar\lambda(t_l|t_k)=[\lambda(t_l|t_k)+\lambda(t_{l+1}|t_k)]/2$$

$$X_{lk}\sim B(m(t_l,t_k),P_\lambda(t_l|t_k)) \quad \cdots(38)$$

$$\Delta n(t_l)=\sum_{k=-\infty}^{l-1}X_{lk} \quad \cdots(39)$$

$$\downarrow\; P_\lambda(t|\hat t)\approx\lambda(t|\hat t)\Delta t$$

$$m(t_l+\Delta t,t_k)-m(t_l,t_k)=-P_\lambda(t_l|t_k)m(t_l,t_k)+\sqrt{P_\lambda(t_l|t_k)[m(t_l,t_k)]_+}\,\mathcal{N}(t_l,t_k) \quad \cdots(40)$$

$$\frac{\partial Q_N(t,\hat t)}{\partial t}=-\lambda(t|\hat t)Q_N(t,\hat t)+\sqrt{\frac{\lambda(t|\hat t)[Q_N(t,\hat t)]_+}{N}}\,\xi(t,\hat t) \quad \cdots(41)$$

$$\xi(t,\hat t)\equiv\lim_{\Delta t\to 0}\mathcal{N}(t,\hat t)/\Delta t$$

$$A_N(t)=\int_{-\infty}^{t}\lambda(t|\hat t)Q_N(t,\hat t)d\hat t-\int_{-\infty}^{t}\sqrt{\frac{\lambda(t|\hat t)[Q_N(t,\hat t)]_+}{N}}\,\xi(t,\hat t)d\hat t \quad \cdots(42)$$

$$\mathcal{H}(t_l)=\{\Delta n(t_k)\}_{k\in Z,\,k<l} \quad \cdots(43)$$

$$X_{lk}\sim Pois(E[X_{lk}|m(t_l,t_k)]) \quad \cdots(44)$$

$$E[X_{lk}|m(t_l,t_k)]=P_\lambda(t_l|t_k)m(t_l,t_k) \quad \cdots(45)$$

$$\Delta n(t_l)\sim Pois(\Delta\bar n(t_l)) \quad \cdots(46)$$

$$\Delta\bar n(t_l)\equiv E[\Delta n(t_l)|\{m(t_l,t_k)\}_{k<l},\mathcal{H}(t_l)]=\sum_{k=-\infty}^{l-1}P_\lambda(t_l|t_k)\cdot m(t_l,t_k) \quad \cdots(47)$$

$$\langle\hat m_{l+1,k}\rangle=[1-P_\lambda(t_l|t_k)]\langle\hat m_{l,k}\rangle \quad \cdots(48)$$

$$\Delta\hat m_{l,k}=\hat m_{l,k}-\langle\hat m_{l,k}\rangle \quad \cdots(49)$$

$$\langle \hat{m}_{l,k}^2 \rangle = \mathrm{Var}[E[m_{l+1,k}|m_{l,k},\mathcal{H}_l]] + \langle \mathrm{Var}[m_{l+1,k}|m_{l,k},\mathcal{H}_l]\rangle \cdots (50)$$

$$\langle \Delta\hat{m}^2_{l+1,k}\rangle = [1-P_\lambda(t_l|t_k)]^2\langle\Delta\hat{m}^2_{l,k}\rangle + P_\lambda(t_l|t_k)\langle\hat{m}_{l,k}\rangle \cdots (51)$$

initial condition: $\langle \Delta\hat{m}^2_{k+1,k}\rangle = 0$

$$m_{l,k} = \langle\hat{m}_{l,k}\rangle + \delta m_{l,k} \cdots (52)$$

$$N = \sum_{k=-\infty}^{l-1}\langle \hat{m}_{l,k}\rangle + \sum_{k=-\infty}^{l-1}\delta m_{l,k} \cdots (53)$$

$$\Delta\bar{n}(t_l) = \sum_{k=-\infty}^{l-1} P_\lambda(t_l|t_k)\langle\hat{m}_{l,k}\rangle + \sum_{k=-\infty}^{l-1} P_\lambda(t_l|t_k)\delta m_{l,k} \cdots (54)$$

$$\sum_{k=-\infty}^{l-1} P_\lambda(t_l|t_k)\delta m_{l,k} \approx P_\Lambda(t_l)\sum_{k=-\infty}^{l-1} \delta m_{l,k} \cdots (55)$$

$$\sum_{k=-\infty}^{l-1} P_\lambda(t_l|t_k)\Delta\hat{m}_{l,k} = P_\Lambda(t_l)\sum_{k=-\infty}^{l-1}\Delta\hat{m}_{l,k} + \varepsilon_l \cdots (56)$$

$$\frac{d\mathcal{E}}{dP_\Lambda} = 2P_\Lambda\sum_{k,k'<l}\langle\Delta\hat{m}_{l,k}\Delta\hat{m}_{l,k'}\rangle - 2\sum_{k,k'<l}P_\lambda(t_l|t_k)\langle\Delta\hat{m}_{l,k},\Delta\hat{m}_{l,k'}\rangle \cdots (57)$$

$$\mathcal{E}(P_\Lambda) = \langle\varepsilon_l^2\rangle$$

$$P_\Lambda(t_l) = \frac{\sum_{k=-\infty}^{l-1} P_\lambda(t_l|t_k)\langle\Delta\hat{m}^2_{l,k}\rangle}{\sum_{k=-\infty}^{l-1}\langle\Delta\hat{m}^2_{l,k}\rangle} \cdots (58)$$

$$\Delta\bar{n}(t_l) = \sum_{k=-\infty}^{l-1} P_\lambda(t_l|t_k)\langle\hat{m}_{l,k}\rangle + P_\Lambda(t_l)\left(N - \sum_{k=-\infty}^{l-1}\langle\hat{m}_{l,k}\rangle\right) \cdots (59)$$

④ Mesoscopic population density equations in continuous time

$$A_N(t_l) = \frac{\Delta n(t_l)}{N\Delta t},\quad \bar{A}(t_l) = \frac{\Delta\bar{n}(t_l)}{N\Delta t} \cdots (60)$$

$$A_N(t) = \frac{dn(t)}{N\,dt},\quad dn(t) \sim \mathrm{Pois}(N\bar{A}(t)dt) \cdots (61)$$

$$A_N(t) = \frac{1}{N}\sum_k \delta(t - t_{pop,k}) \cdots (62)$$

$$S(t_l|t_k) = \frac{\langle\hat{m}_{l,k}\rangle}{\Delta n(t_k)},\quad v(t_l,t_k) = \frac{\langle\Delta\hat{m}^2_{l,k}\rangle}{N\Delta t} \cdots (63)$$

$$P_\lambda(t_l|t_k) = \lambda(t_l|t_k)\Delta t + O(\Delta t^2)$$

$$\bar{A}(t) = \lim_{\Delta t\to 0}\left\{\sum_{k=-\infty}^{\frac{t}{\Delta t}-1}\lambda(t|k\Delta t)S(t|k\Delta t)\frac{\Delta n(k\Delta t)}{N} + \frac{\sum_{k=-\infty}^{\frac{t}{\Delta t}-1}\lambda(t|k\Delta t)\,v(t,k\Delta t)\Delta t}{\sum_{k=-\infty}^{\frac{t}{\Delta t}-1} v(t,k\Delta t)\Delta t}\left(1 - \sum_{k=-\infty}^{\frac{t}{\Delta t}-1} S(t|k\Delta t)\frac{\Delta n(k\Delta t)}{N}\right)\right\} \cdots (64)$$

$$\bar{A}(t) = \int_{-\infty}^{t}\lambda(t|\hat{t})S(t|\hat{t})A_N(\hat{t})d\hat{t} + \Lambda(t)\left(1 - \int_{-\infty}^{t} S(t|\hat{t})A_N(\hat{t})d\hat{t}\right) \cdots (65)$$

$$\Lambda(t) = \frac{\int_{-\infty}^{t}\lambda(t|\hat{t})\,v(t,\hat{t})d\hat{t}}{\int_{-\infty}^{t} v(t,\hat{t})d\hat{t}} \cdots (66)$$

$$\frac{\partial S(t|\hat{t})}{\partial t} = -\lambda(t|\hat{t})S(t|\hat{t}),\quad S(\hat{t}|\hat{t}) = 1 \cdots (67)$$

$$S(t|\hat{t}) = \exp\left(-\int_{\hat{t}}^{t}\lambda(t'|\hat{t})dt'\right) \cdots (68)$$

$$\frac{\partial v}{\partial t} = -2\lambda(t|\hat{t})\underbrace{v}_{\tilde{v}(t|\hat{t})} + \lambda(t|\hat{t})S(t|\hat{t})A_N(\hat{t}),\quad v(\hat{t}|\hat{t}) = 0 \cdots (69)$$

$$\Delta n(t_k) \longrightarrow A_N(t_k) \xrightarrow{Eq.(33)} \vartheta_A^\alpha(t,\hat{t})$$

$$\left(A_N(t_k) = \frac{\Delta n(t_k)}{N \Delta t} \right)$$

$$A_N^\alpha, A_N^\beta \cdots \xrightarrow[Eq.(30)]{} u_A^\alpha(t,\hat{t})$$

$$\left. \begin{matrix} \vartheta_A^\alpha(t,\hat{t}) \\ u_A^\alpha(t,\hat{t}) \end{matrix} \right\} \xrightarrow[Eq.(32)]{} \lambda_A^\alpha(t|\hat{t}) \xrightarrow[Eq.(37)]{} P_\Lambda(t_l|t_k)$$

$$\left. \begin{matrix} \Delta n(t_k) \\ \text{initial condition of } m \text{ (or } \Delta m\text{)} \\ \langle \hat{m}(t_{k+1}, t_k) \rangle = \Delta n(t_k) \end{matrix} \right\} \xrightarrow[Eq.(48)]{} \langle \hat{m}_{l,k} \rangle$$

$$\left. \begin{matrix} \hat{m}_{l,k} \\ P_\Lambda(t_l|t_k) \\ \text{initial condition:} \\ \langle \Delta \hat{m}^2_{k+1,k} \rangle = 0 \end{matrix} \right\} \xrightarrow[Eq.(51)]{} \langle \Delta \hat{m}^2_{l+1,k} \rangle$$

The mesoscopic description (state) does not require the knowledge of the detail distribution of last spike time $m(t_l, t_k)$. But the mean $\langle \hat{m}_{l,k} \rangle$ of unconstrained process is a mesoscopic variable.