

My Notes and Viewpoints on The Complex and Real

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I. RELATIONSHIP BETWEEN COMPLEX AND REAL DERIVATIONS

To begin with, in this manuscript, derivation of a real function with respect to real variable is named as *real derivation*, and derivation of a complex function with respect to complex variable is named as *complex derivation*.

A. Scalar case

Define a complex variable $z = x + i \cdot y \in \mathbb{C}$, where i stands for the imaginary unit, and $x, y \in \mathbb{R}$. Then the conjugate of variable z is a new variable as $z^* = x - i \cdot y$. Here, z and z^* can be viewed as functions with respect to x and y , and then we have the following complex derivative operators:

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2} \cdot 2 \cdot \frac{\partial}{\partial z} \\ &= \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \\ &= \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial z} \right) \\ &= \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} \cdot \frac{1}{\frac{\partial z}{\partial x}} + \frac{\partial}{\partial y} \cdot \frac{1}{\frac{\partial z}{\partial y}} \right) \\ &= \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} \cdot 1 + \frac{\partial}{\partial y} \cdot \frac{1}{i} \right) \\ &= \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} - i \cdot \frac{\partial}{\partial y} \right) \end{aligned} \tag{1a}$$

$$\frac{\partial}{\partial z^*} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i \cdot \frac{\partial}{\partial y} \right). \tag{1b}$$

B. Vector case

In what follows, we use Eq. (1) and the definition of vector derivation to deduce the (complex) vector derivative operators.

Define a column vector variable $\mathbf{z} = \mathbf{x} + i \cdot \mathbf{y}$, where $\mathbf{z} = [z_1, z_2, \dots, z_p]^T \in \mathbb{C}^p$, $\mathbf{x} = [x_1, x_2, \dots, x_p]^T \in \mathbb{R}^p$ and $\mathbf{y} = [y_1, y_2, \dots, y_p]^T \in \mathbb{R}^p$. According to the definition of a vector variable, we have

$$\frac{\partial}{\partial \mathbf{z}} \triangleq \left[\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_p} \right]^T. \tag{2}$$

Based on (1a), (2) can be reformed as

$$\begin{aligned} \frac{\partial}{\partial \mathbf{z}} &= \left[\frac{1}{2} \cdot \left(\frac{\partial}{\partial x_1} - i \cdot \frac{\partial}{\partial y_1} \right), \frac{1}{2} \cdot \left(\frac{\partial}{\partial x_2} - i \cdot \frac{\partial}{\partial y_2} \right), \dots, \frac{1}{2} \cdot \left(\frac{\partial}{\partial x_p} - i \cdot \frac{\partial}{\partial y_p} \right) \right]^T \\ &= \frac{1}{2} \cdot \left(\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_p} \right]^T - i \cdot \left[\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_p} \right]^T \right) \\ &= \frac{1}{2} \cdot \left(\frac{\partial}{\partial \mathbf{x}} - i \cdot \frac{\partial}{\partial \mathbf{y}} \right). \end{aligned} \tag{3a}$$

Similarly, we have

$$\frac{\partial}{\partial \mathbf{z}^*} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial \mathbf{x}} + i \cdot \frac{\partial}{\partial \mathbf{y}} \right). \tag{3b}$$

C. An example

In this section, we deduce the derivation of $\mathbf{w}^H \mathbf{R} \mathbf{w}$ with respect to \mathbf{w} . Here, \mathbf{R} is a Hermitian matrix. $\mathbf{w}^H \mathbf{R} \mathbf{w}$ can be written as

$$\begin{aligned} \mathbf{w}^H \mathbf{R} \mathbf{w} &= (\mathbf{w}_R + i \cdot \mathbf{w}_I)^H (\mathbf{R}_R + i \cdot \mathbf{R}_I) (\mathbf{w}_R + i \cdot \mathbf{w}_I) \\ &= \mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_R - \mathbf{w}_R^T \mathbf{R}_I \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_I \mathbf{w}_R \\ &\quad + i \cdot (\mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_I + \mathbf{w}_R^T \mathbf{R}_I \mathbf{w}_R - \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_R + \mathbf{w}_I^T \mathbf{R}_I \mathbf{w}_I) \\ &= \mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_R - \mathbf{w}_R^T \mathbf{R}_I \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_I \mathbf{w}_R \end{aligned} \quad (4)$$

where \mathbf{w}_R and \mathbf{R}_R are the real part of \mathbf{w} and \mathbf{R} , respectively, while \mathbf{w}_I and \mathbf{R}_I are the imaginary part of \mathbf{w} and \mathbf{R} , respectively, i.e., $\mathbf{w} \triangleq \mathbf{w}_R + i \cdot \mathbf{w}_I$ and $\mathbf{R} \triangleq \mathbf{R}_R + i \cdot \mathbf{R}_I$. The last equation in (4) holds because the imaginary part of $\mathbf{w}^H \mathbf{R} \mathbf{w}$ is 0 since \mathbf{R} is a positive semi-definite matrix (\mathbf{R} is Hermitian).

According to (3a), we can obtain

$$\begin{aligned} \frac{\partial \mathbf{w}^H \mathbf{R} \mathbf{w}}{\partial \mathbf{w}} &= \frac{1}{2} \cdot \left(\frac{\partial \mathbf{w}^H \mathbf{R} \mathbf{w}}{\partial \mathbf{w}_R} - i \cdot \frac{\partial \mathbf{w}^H \mathbf{R} \mathbf{w}}{\partial \mathbf{w}_I} \right) \\ &= \frac{1}{2} \cdot [2\mathbf{R}_R \mathbf{w}_R - \mathbf{R}_I \mathbf{w}_I + \mathbf{R}_I^T \mathbf{w}_I - i \cdot (-\mathbf{R}_I^T \mathbf{w}_R + 2\mathbf{R}_R \mathbf{w}_I + \mathbf{R}_I \mathbf{w}_R)] \\ &= \frac{1}{2} \cdot [2\mathbf{R}_R \mathbf{w}_R - 2\mathbf{R}_I \mathbf{w}_I - i \cdot (2\mathbf{R}_I \mathbf{w}_R + 2\mathbf{R}_R \mathbf{w}_I)] \\ &= \mathbf{R}_R \mathbf{w}_R - \mathbf{R}_I \mathbf{w}_I - i \mathbf{R}_I \mathbf{w}_R - i \mathbf{R}_R \mathbf{w}_I \\ &= (\mathbf{R}_R - i \mathbf{R}_I)(\mathbf{w}_R - i \mathbf{w}_I) \\ &= (\mathbf{R}_R^T + i \mathbf{R}_I^T)(\mathbf{w}_R - i \mathbf{w}_I) \\ &= \mathbf{R}^T \mathbf{w}^* \end{aligned} \quad (5)$$

where $\mathbf{R}_R^T = \mathbf{R}_R$ and $\mathbf{R}_I^T = -\mathbf{R}_I$ (since \mathbf{R} is a Hermitian matrix, i.e., $\mathbf{R}^H = \mathbf{R}$) are adopted in the third and sixth equations.

II. TRANSFER FROM COMPLEX OPERATION TO REAL OPERATION

In many references, some complex operations are almost always translated into the real ones, such as $\mathbf{w}^H \mathbf{R} \mathbf{w}$. Specifically speaking, given $\mathbf{w} \in \mathbb{C}^{M \times 1}$, $\mathbf{R} \in \mathbb{C}^{M \times M}$ and \mathbf{R} is a Hermitian matrix, we can calculate $\mathbf{w}^H \mathbf{R} \mathbf{w}$ using

$$\mathbf{w}^H \mathbf{R} \mathbf{w} = \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix}^T \begin{bmatrix} \mathbf{R}_R & -\mathbf{R}_I \\ \mathbf{R}_I & \mathbf{R}_R \end{bmatrix} \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix} \quad (6)$$

where $\mathbf{w} \triangleq \mathbf{w}_R + i \cdot \mathbf{w}_I$ and $\mathbf{R} \triangleq \mathbf{R}_R + i \cdot \mathbf{R}_I$.

The proof of (6) is provided as follows. Firstly, the left side of (6) can be written as

$$\begin{aligned} \mathbf{w}^H \mathbf{R} \mathbf{w} &= [\mathbf{w}_R + i \cdot \mathbf{w}_I]^H [\mathbf{R}_R + i \cdot \mathbf{R}_I] [\mathbf{w}_R + i \cdot \mathbf{w}_I] \\ &= [\mathbf{w}_R^T - i \cdot \mathbf{w}_I^T] [\mathbf{R}_R + i \cdot \mathbf{R}_I] [\mathbf{w}_R + i \cdot \mathbf{w}_I] \\ &= \mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_R + \mathbf{w}_R^T \mathbf{R}_R i \mathbf{w}_I + \mathbf{w}_R^T i \mathbf{R}_I \mathbf{w}_R + \mathbf{w}_R^T i \mathbf{R}_I i \mathbf{w}_I \\ &\quad - i \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_R - i \mathbf{w}_I^T \mathbf{R}_R i \mathbf{w}_I - i \mathbf{w}_I^T i \mathbf{R}_I \mathbf{w}_R - i \mathbf{w}_I^T i \mathbf{R}_I i \mathbf{w}_I \end{aligned} \quad (7)$$

in which, the sum of the second and the fifth terms equals to zero, i.e., $\mathbf{w}_R^T \mathbf{R}_R i \mathbf{w}_I - i \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_R = 0$. Since \mathbf{R} is a Hermitian matrix, \mathbf{R}_I has the form as

$$\mathbf{R}_I = \begin{bmatrix} 0 & r_{12} & \cdots & r_{1M} \\ -r_{12} & 0 & \cdots & r_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ -r_{1M} & -r_{2M} & \cdots & 0 \end{bmatrix} \quad (8)$$

and it is easy to find that the third and the eighth terms in (7) are both zero, i.e., $\mathbf{w}_R^T i \mathbf{R}_I \mathbf{w}_R = 0$ and $i \mathbf{w}_I^T i \mathbf{R}_I i \mathbf{w}_I = 0$. Therefore, (7) can be finally revised as

$$\begin{aligned} \mathbf{w}^H \mathbf{R} \mathbf{w} &= \mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_R + \mathbf{w}_R^T i \mathbf{R}_I i \mathbf{w}_I - i \mathbf{w}_I^T \mathbf{R}_R i \mathbf{w}_I - i \mathbf{w}_I^T i \mathbf{R}_I \mathbf{w}_R \\ &= \mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_R - \mathbf{w}_R^T \mathbf{R}_I \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_I \mathbf{w}_R. \end{aligned} \quad (9)$$

On the other hand, the right side of (6) can be written as

$$\begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix}^T \begin{bmatrix} \mathbf{R}_R & -\mathbf{R}_I \\ \mathbf{R}_I & \mathbf{R}_R \end{bmatrix} \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix} = \mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_R + \mathbf{w}_I^T \mathbf{R}_I \mathbf{w}_R - \mathbf{w}_R^T \mathbf{R}_I \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_I. \quad (10)$$

By comparing (9) with (10), we obtain (6).