CS 4495 Computer Vision

Linear Filtering 1: Filters, Convolution, Smoothing

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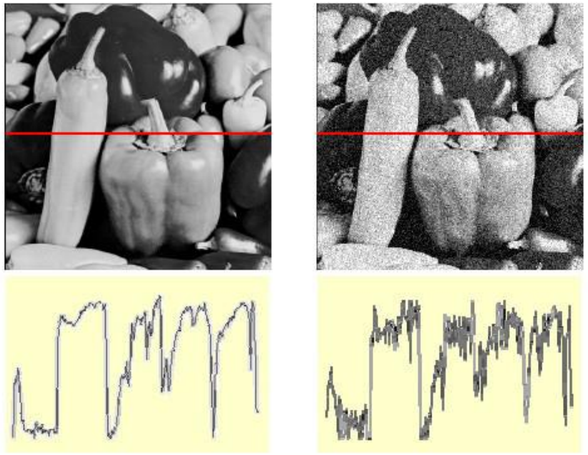
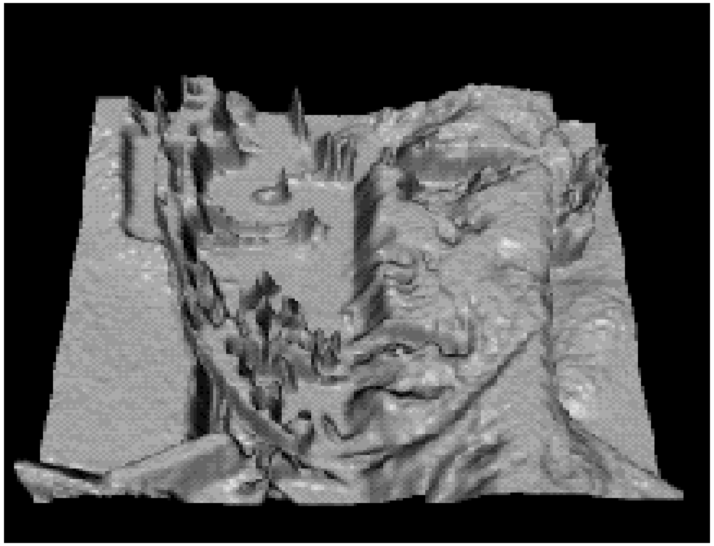
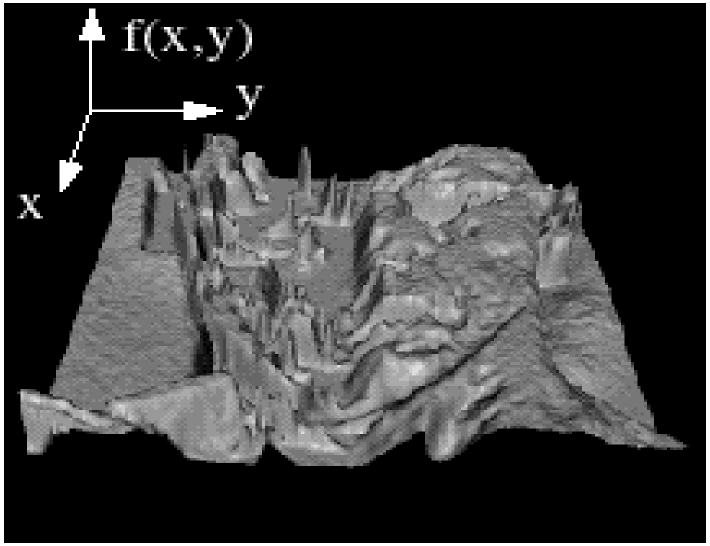


Linear outline

- Images are really *functions* $\vec{I}(x, y)$ where the vector can be any dimension but typical are 1, 3, and 4. (When 4?) Or thought of as a multi-dimensional *signal* as a function of spatial location.
- ***Image processing*** is (mostly) computing new functions of image functions. Many involve ***linear*** *operators.*
- Very useful linear operator is *convolution /correlation* - what most people call filtering – because the new value is determined by local values.
- With convolution can do things like noise reduction, smoothing, and edge finding (last one is next time).

Images as functions





Source: S. Seitz

Images as functions

- We can think of an image as a function, f or I, from R^2 to R:

  $f(x, y)$ gives the *intensity* or value at position (x, y)

  Realistically, we expect the image only to be defined over a rectangle, with a finite range:

  f: $[a,b] \times [c,d] \rightarrow [0, 1.0]$ *(why sometimes 255???)*

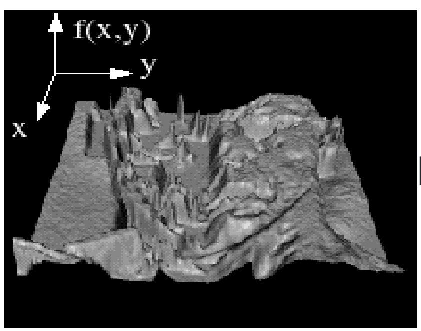
- A color image is just three functions "pasted" together. We can write this as a "vector-valued" function:

$$f(x, y) = \begin{bmatrix} r(x, y) \\ g(x, y) \\ b(x, y) \end{bmatrix}$$

Source: S. Seitz

Digital images

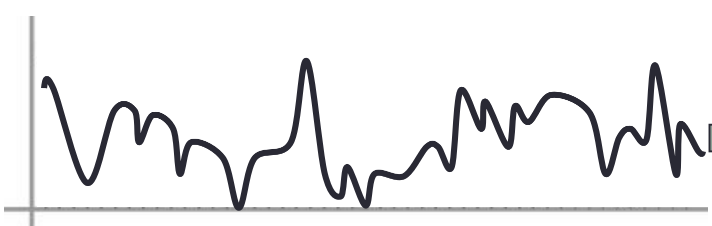
- In computer vision we typically operate on **digital** (**discrete**) images:
  - **Sample** the 2D space on a regular grid
  - **Quantize** each sample (round to "nearest integer")
- Image thus represented as a matrix of integer values.

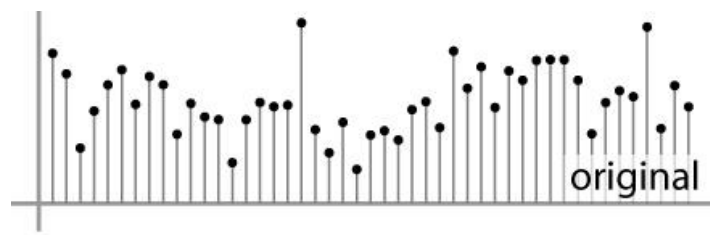


$j \rightarrow$, $i \downarrow$

62	79	23	119	120	105	4	0
10	10	9	62	12	78	34	0
10	58	197	46	46	0	0	48
176	135	5	188	191	68	0	49
2	1	1	29	26	37	0	77
0	89	144	147	187	102	62	208
255	252	0	166	123	62	0	31
166	63	127	17	1	0	99	30

2D

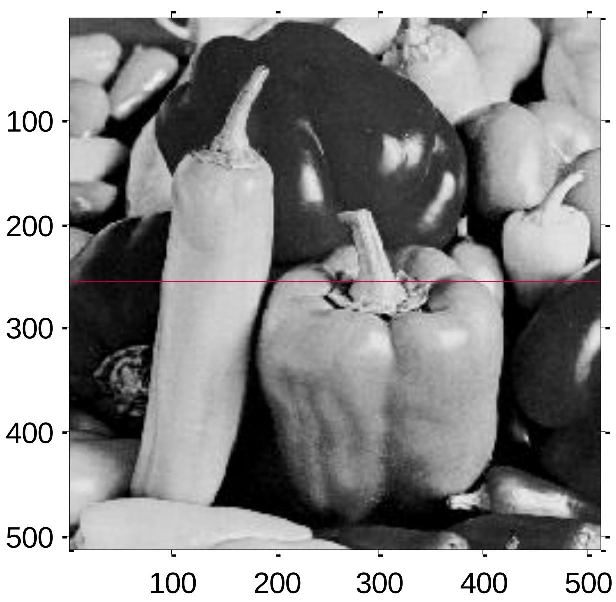


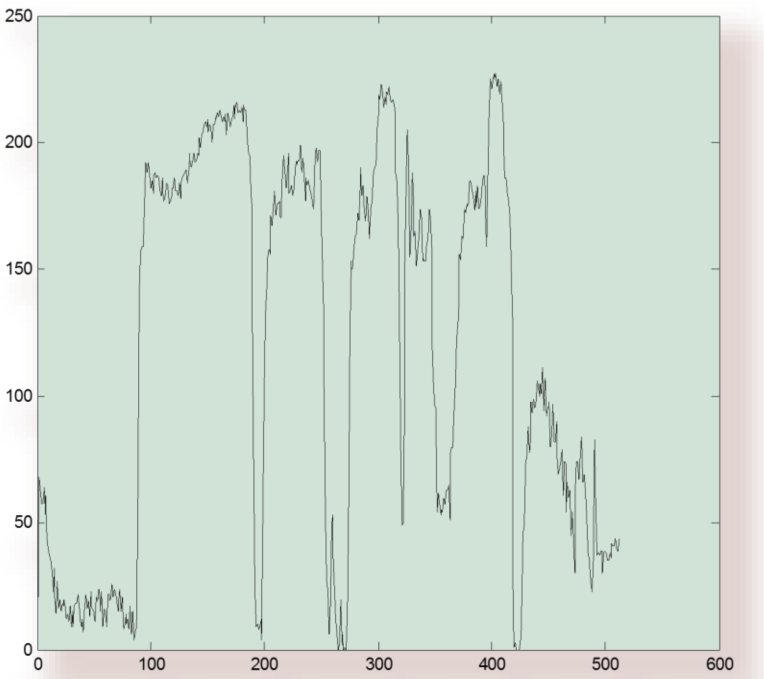


1D

Matlab – images are matrices

```
>> im = imread('peppers.png');  % semicolon or many numbers
>> imgreen = im(:,:,2);
>> imshow(imgreen)
>> line([1 512], [256 256],'color','r')
>> plot(imgreen(256,:));
```





Noise in images

- Noise as an example of images really being functions

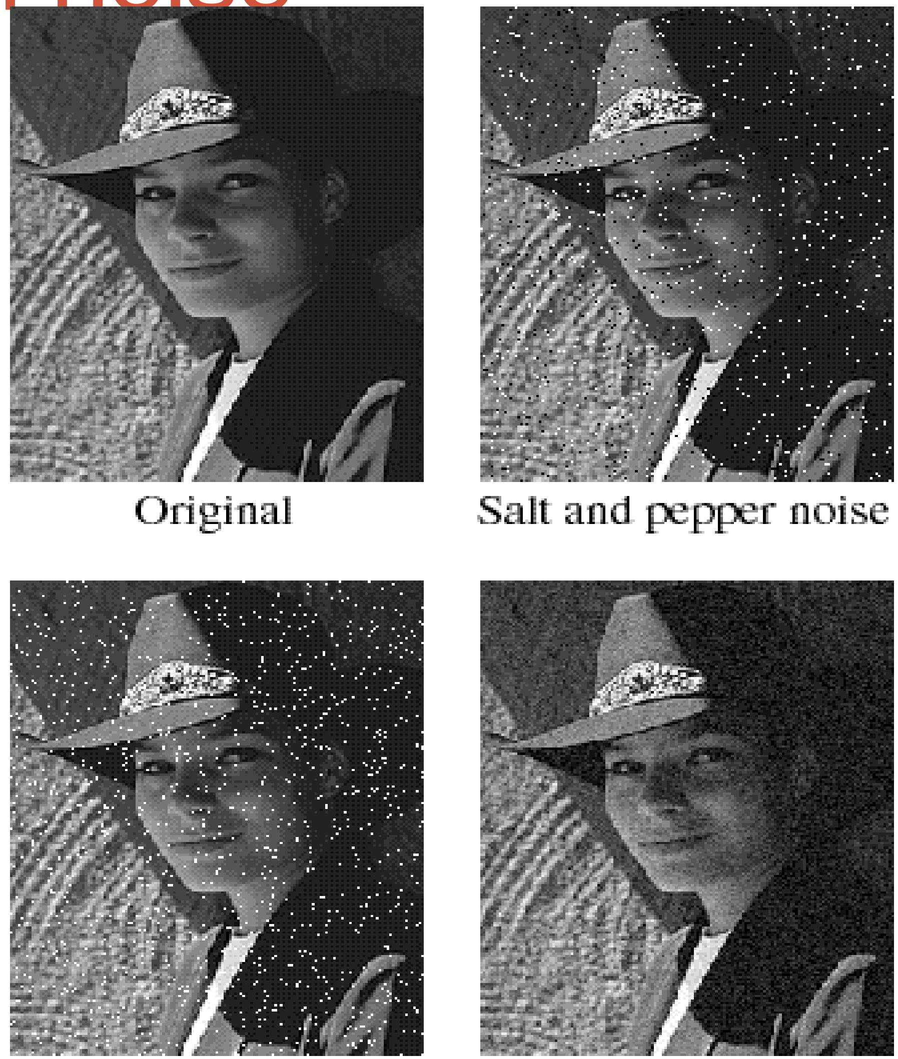
- Noise is just another function that is combined with the original function to get a new – guess what – function

$$\vec{I}'(x, y) = \vec{I}(x, y) + \vec{\eta}(x, y)$$

- In images noise looks, well, noisy.

Common types of noise

- **Salt and pepper noise**: random occurrences of black and white pixels
- **Impulse noise:** random occurrences of white pixels
- ***Gaussian noise:*** variations in intensity drawn from a Gaussian normal distribution



Original

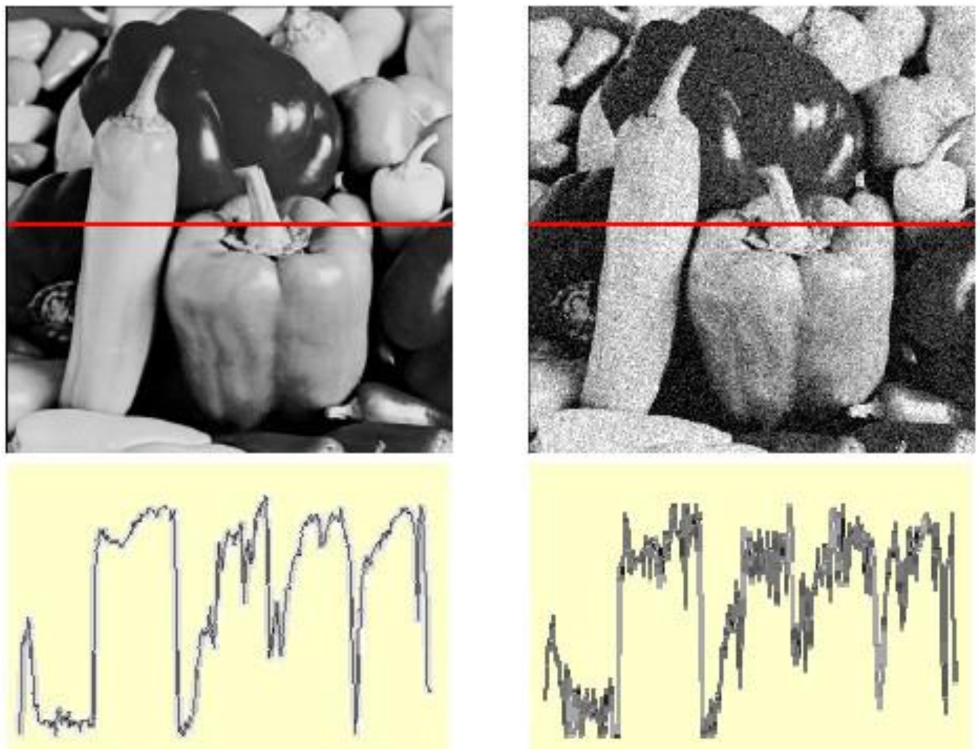
Salt and pepper noise

Impulse noise

Gaussian noise

Source: S. Seitz

Gaussian noise



$$f(x,y) = \overbrace{\bar{f}(x,y)}^{\text{Ideal Image}} + \overbrace{\eta(x,y)}^{\text{Noise process}}$$

Gaussian i.i.d. ("white") noise:
$\eta(x,y) \sim \mathcal{N}(\mu,\sigma)$

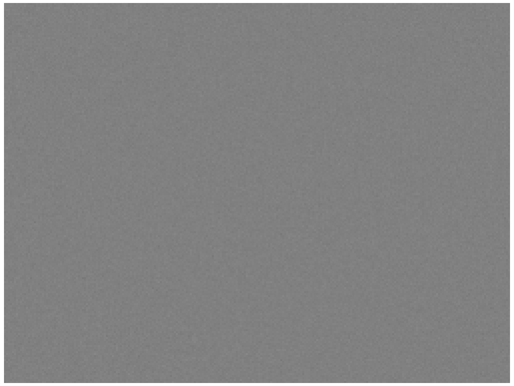
```
>> noise = randn(size(im)).*sigma;

>> output = im + noise;
```

Fig: M. Hebert

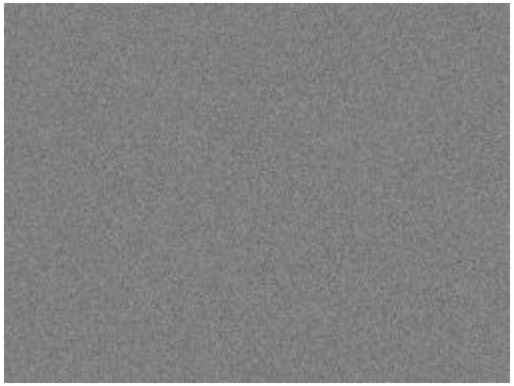
Effect of σ on Gaussian noise

Image shows the noise values themselves.

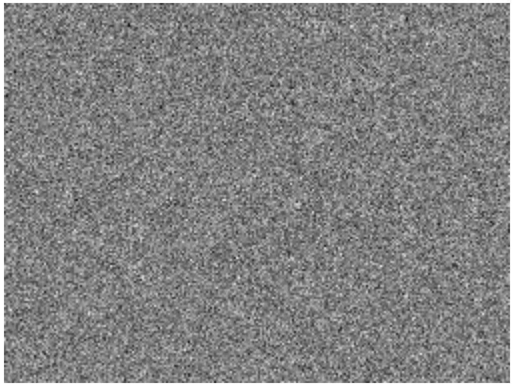
```
noise = randn(size(im)).*sigma;
```



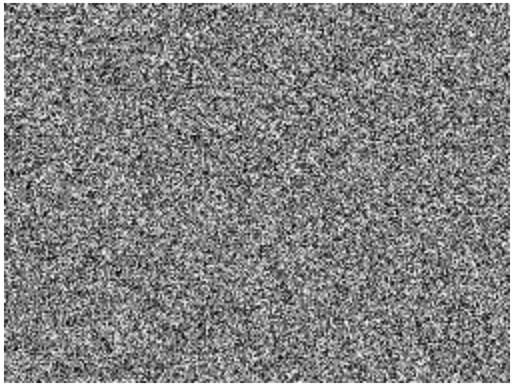
Sigma = 2



Sigma = 8



Sigma = 32



Sigma = 64

BE VERY CAREFUL!!!

- In previous slides, I did not say (at least wasn't supposed to say) what the range of the image was. A σ of 1.0 would be tiny if the range is [0 255] but huge if [0.0 1.0].

- Matlab can do either and you need to be very careful. If in doubt convert to double.

- Even more difficult can be displaying the image. Things like:
  - `imshow(I,[LOW HIGH])`

  display the image from [low high]

Don't worry – you'll get used to these hassles… see problem set PS0.

Back to our program…

Suppose want to remove the noise…

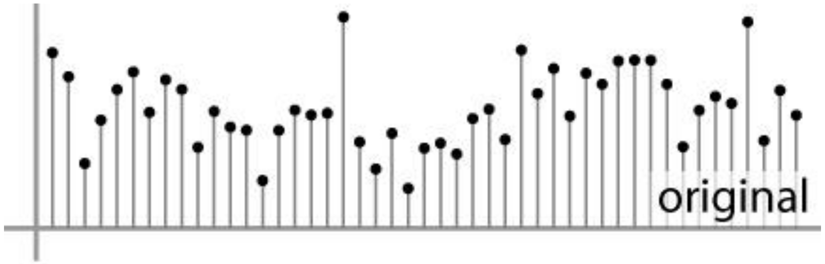
First attempt at a solution

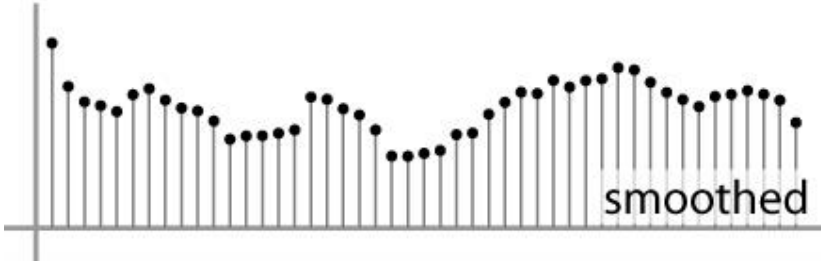
- Let’s replace each pixel with an average of all the values in its neighborhood
- Assumptions:
  - Expect pixels to be like their neighbors
  - Expect noise processes to be independent from pixel to pixel

K. Grauman

First attempt at a solution

- Let’s replace each pixel with an average of all the values in its neighborhood
- Moving average in 1D:

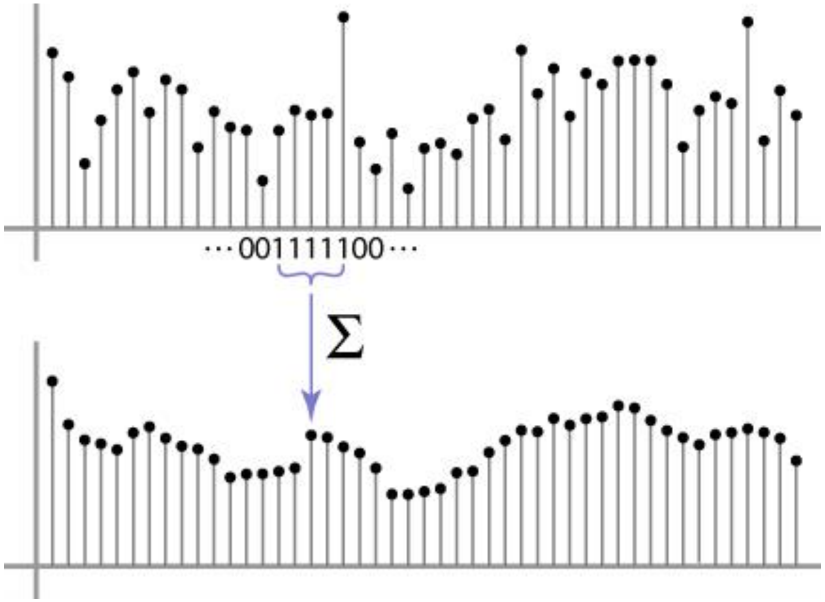




Source: S. Marschner

Weighted Moving Average

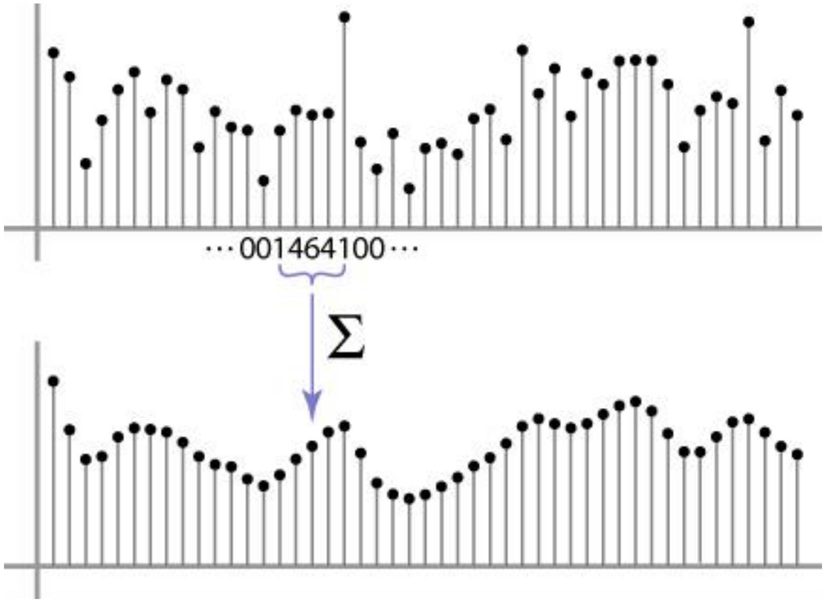
- Can add weights to our moving average
- *Weights* [1, 1, 1, 1, 1] / 5



Source: S. Marschner

Weighted Moving Average

- Non-uniform weights [1, 4, 6, 4, 1] / 16



Source: S. Marschner

Moving Average In 2D

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0								

Source: S. Seitz

Moving Average In 2D

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10							

Source: S. Seitz

Moving Average In 2D

$$F[x, y]$$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$G[x, y]$$

	0	10	20						

Source: S. Seitz

Moving Average In 2D

$F[x,y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x,y]$

	0	10	20	30					

Source: S. Seitz

Moving Average In 2D

$$F[x, y]$$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$G[x, y]$$

	0	10	20	30	30				

Source: S. Seitz

Moving Average In 2D

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Source: S. Seitz

Correlation filtering

Say the averaging window size is 2k+1 x 2k+1:

$$G[i,j] = \underbrace{\frac{1}{(2k+1)^2}}_{\text{Attribute uniform weight to each pixel}} \underbrace{\sum_{u=-k}^{k} \sum_{v=-k}^{k} F[i+u, j+v]}_{\text{Loop over all pixels in neighborhood around image pixel } F[i,j]}$$

Now generalize to allow **different weights** depending on neighboring pixel's relative position:

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} \underbrace{H[u,v]}_{\text{Non-uniform weights}} F[i+u, j+v]$$

Correlation filtering

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v]F[i+u,j+v]$$

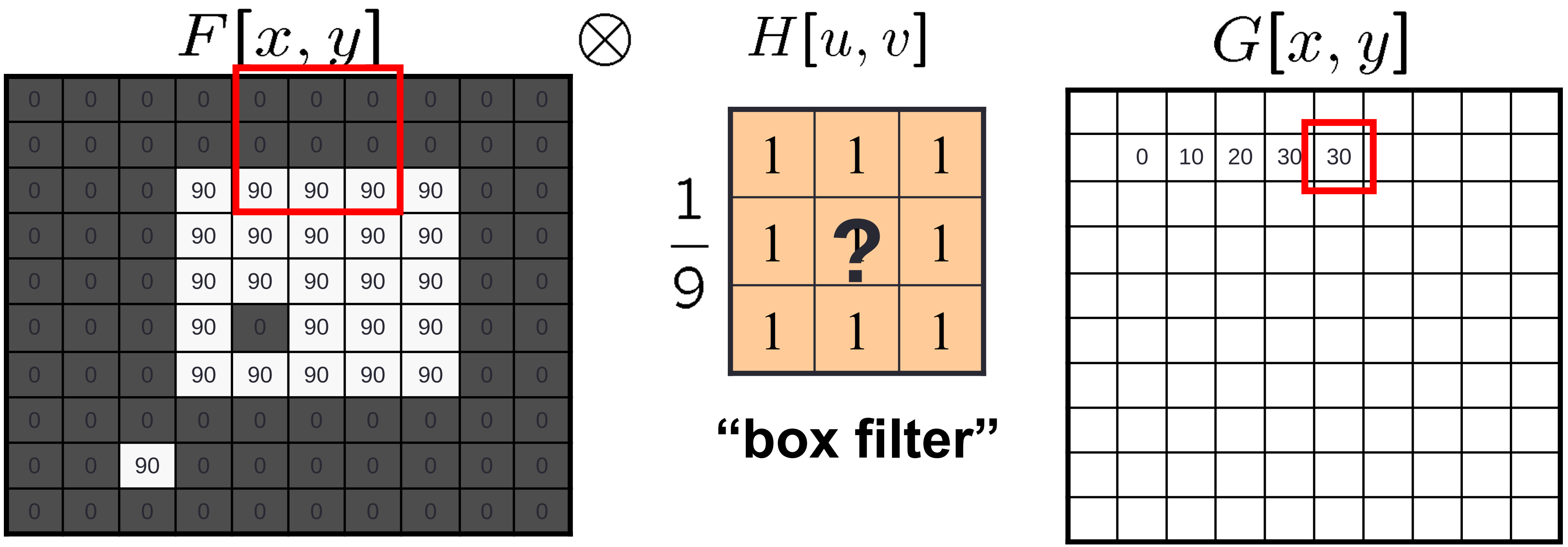
This is called **cross-correlation**, denoted $G = H \otimes F$

Filtering an image: replace each pixel with a linear combination of its neighbors.

The filter "**kernel**" or "**mask**" *H*[*u*,*v*] is the prescription for the weights in the linear combination.

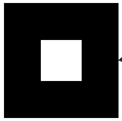
Averaging filter

- What values belong in the kernel *H* for the moving average example?



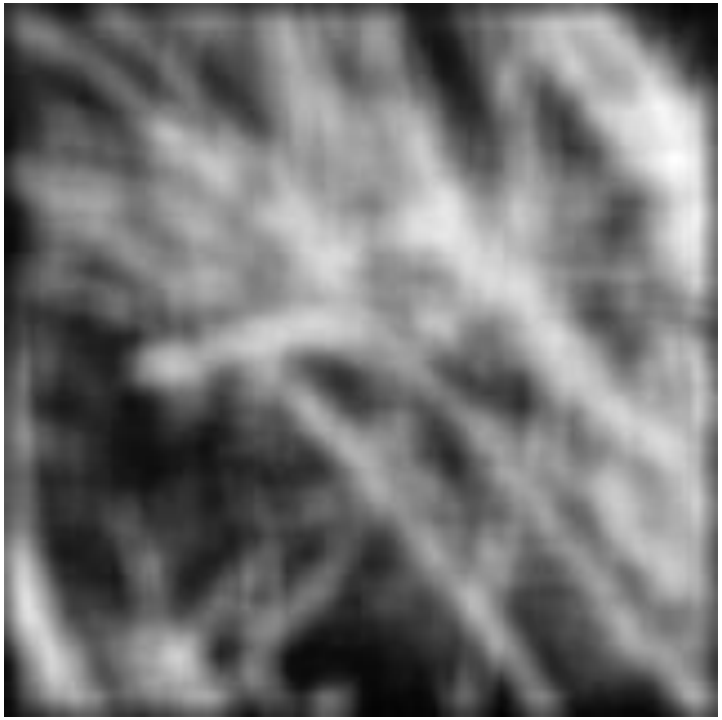
$$G = H \otimes F$$

Smoothing by averaging



depicts box filter:
white = high value, black = low value

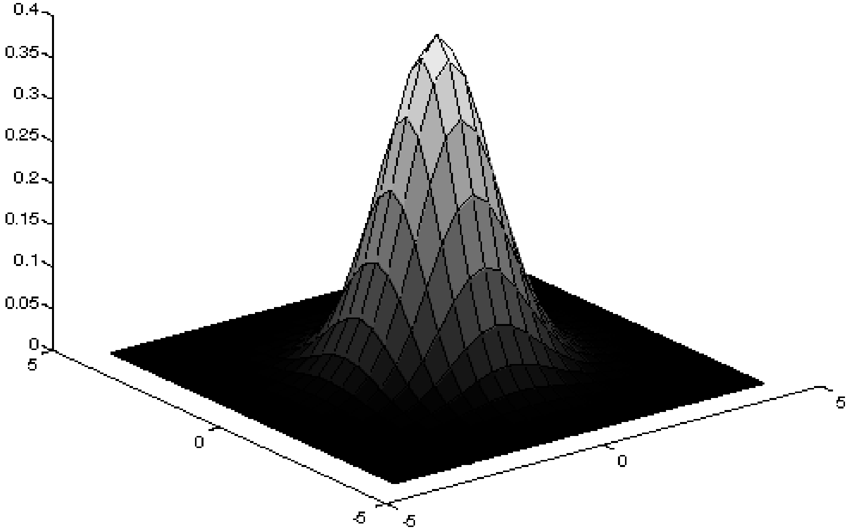
original



filtered

Squares aren't smooth…

- Smoothing with an average actually doesn't compare at all well with a defocussed lens
- Most obvious difference is that a single point of light viewed in a defocussed lens looks like a fuzzy blob; but the averaging process would give a little square.
- More about "impulse" responses later…



D. Forsyth

Gaussian filter

- What if we want nearest neighboring pixels to have the most influence on the output?

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

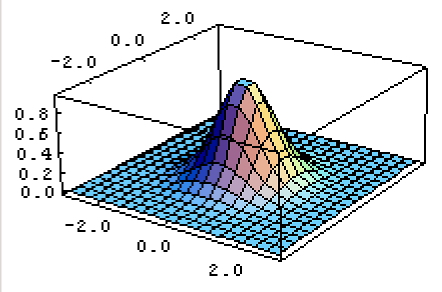
$F[x, y]$

$$\frac{1}{16}\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline\end{array}$$

$H[u, v]$

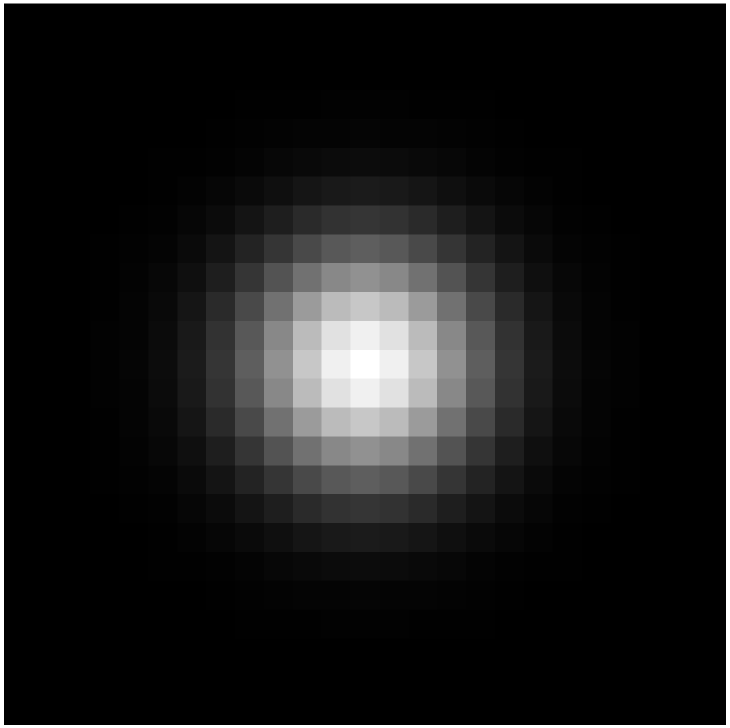
This kernel is an approximation of a Gaussian function:

$$h(u, v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{\sigma^2}}$$



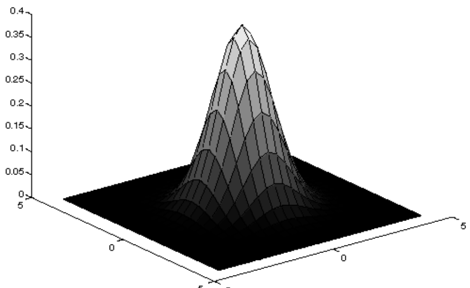
Source: S. Seitz

An Isotropic Gaussian



The picture shows a smoothing kernel proportional to

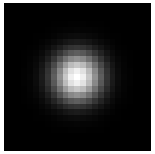
$$\exp\left(-\frac{(x^2 + x^2)}{2\sigma^2}\right)$$

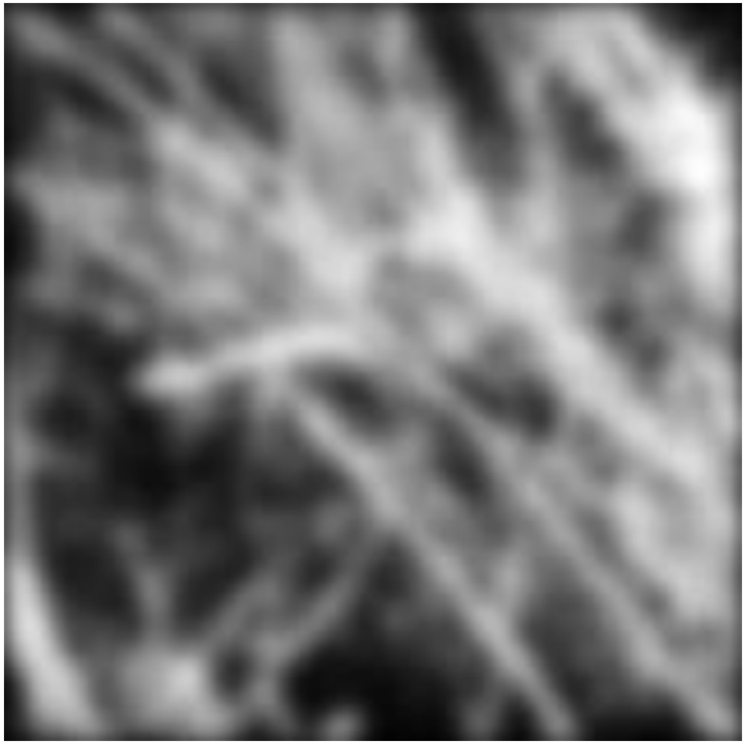


(which is a reasonable model of a circularly symmetric fuzzy blob)

D. Forsyth

Smoothing with a Gaussian





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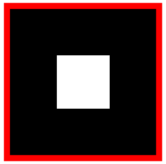
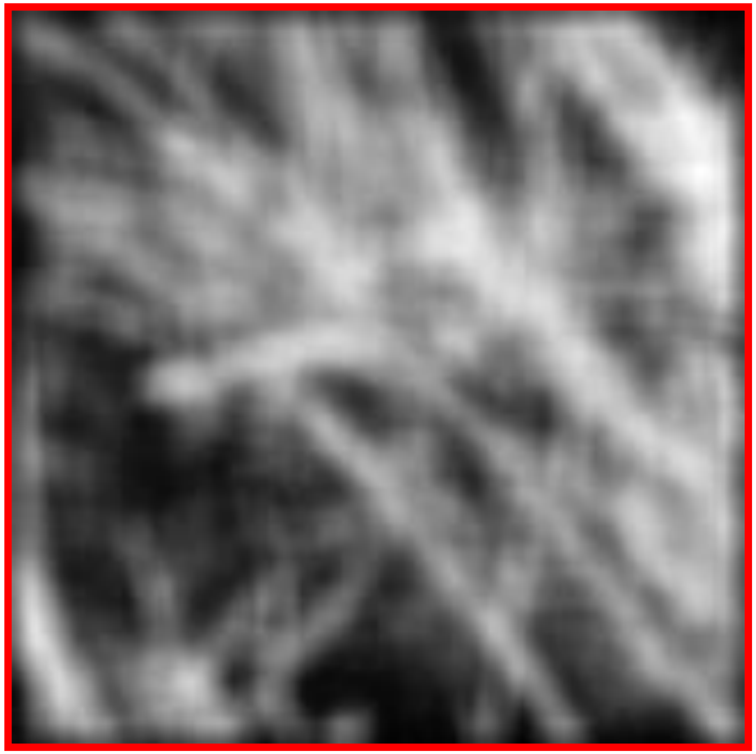
Smoothing with not a Gaussian

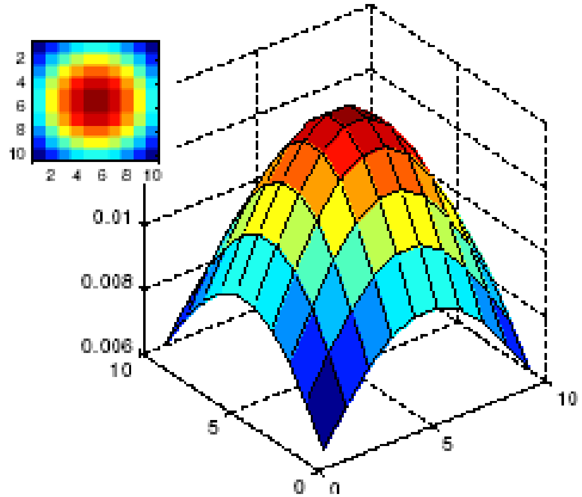




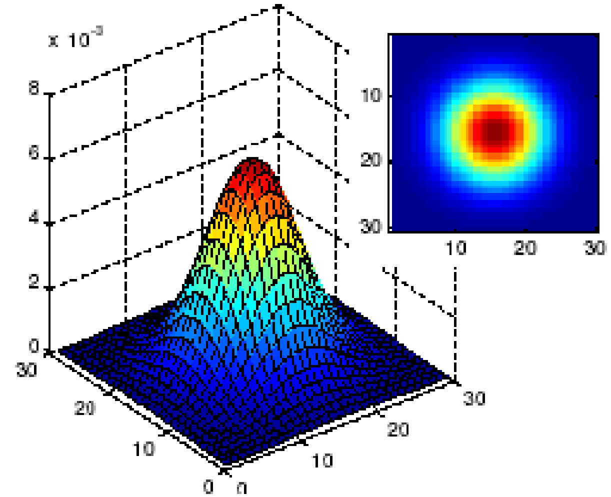
D. Forsyth

Gaussian filters

- What parameters matter here?
- **Size** of kernel or mask
  - Note, Gaussian function has infinite support, but discrete filters use finite kernels



σ = 5 with 10 x 10 kernel

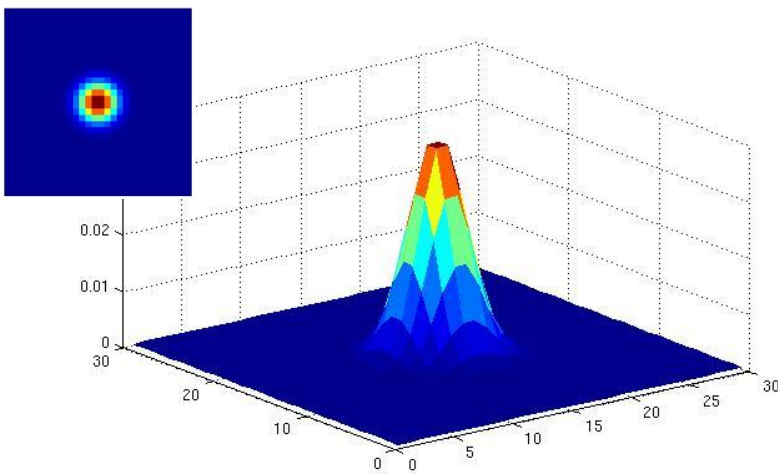


σ = 5 with 30 x 30 kernel

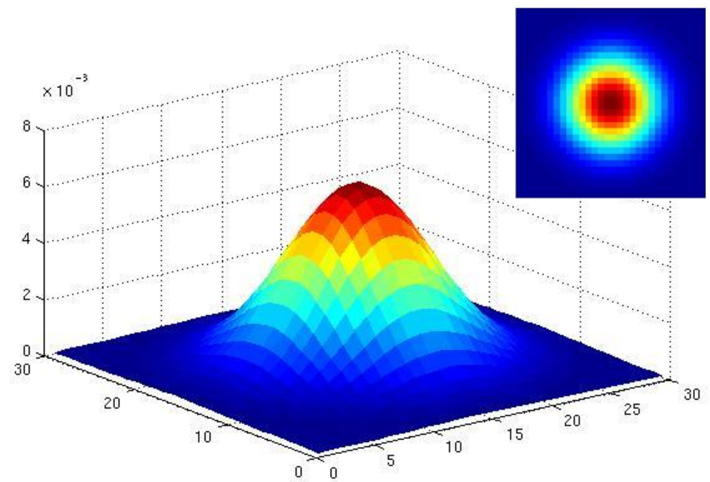
K. Grauman

Gaussian filters

- What parameters matter here?
- **Variance** of Gaussian: determines extent of smoothing



σ = 2 with
30 x 30
kernel



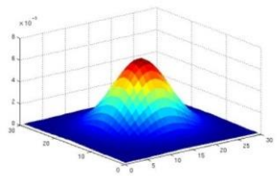
σ = 5 with
30 x 30
kernel

K. Grauman

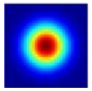
Matlab

```
>> hsize = 10;
>> sigma = 5;
>> h = fspecial('gaussian', hsize, sigma);
```

```
>> mesh(h);
```



```
>> imagesc(h);
```



```
>> outim = imfilter(im, h);
>> imshow(outim);
```

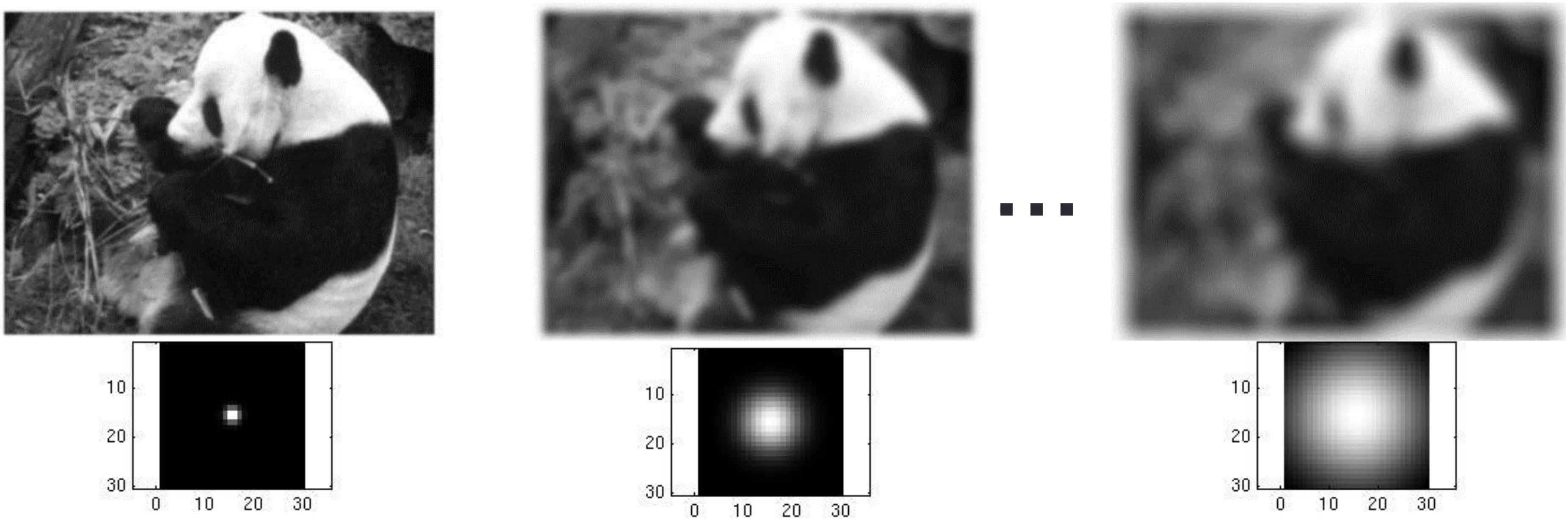
→

outim

K. Grauman

Smoothing with a Gaussian

Parameter σ is the “scale” / “width” / “spread” of the Gaussian kernel, and controls the amount of smoothing.

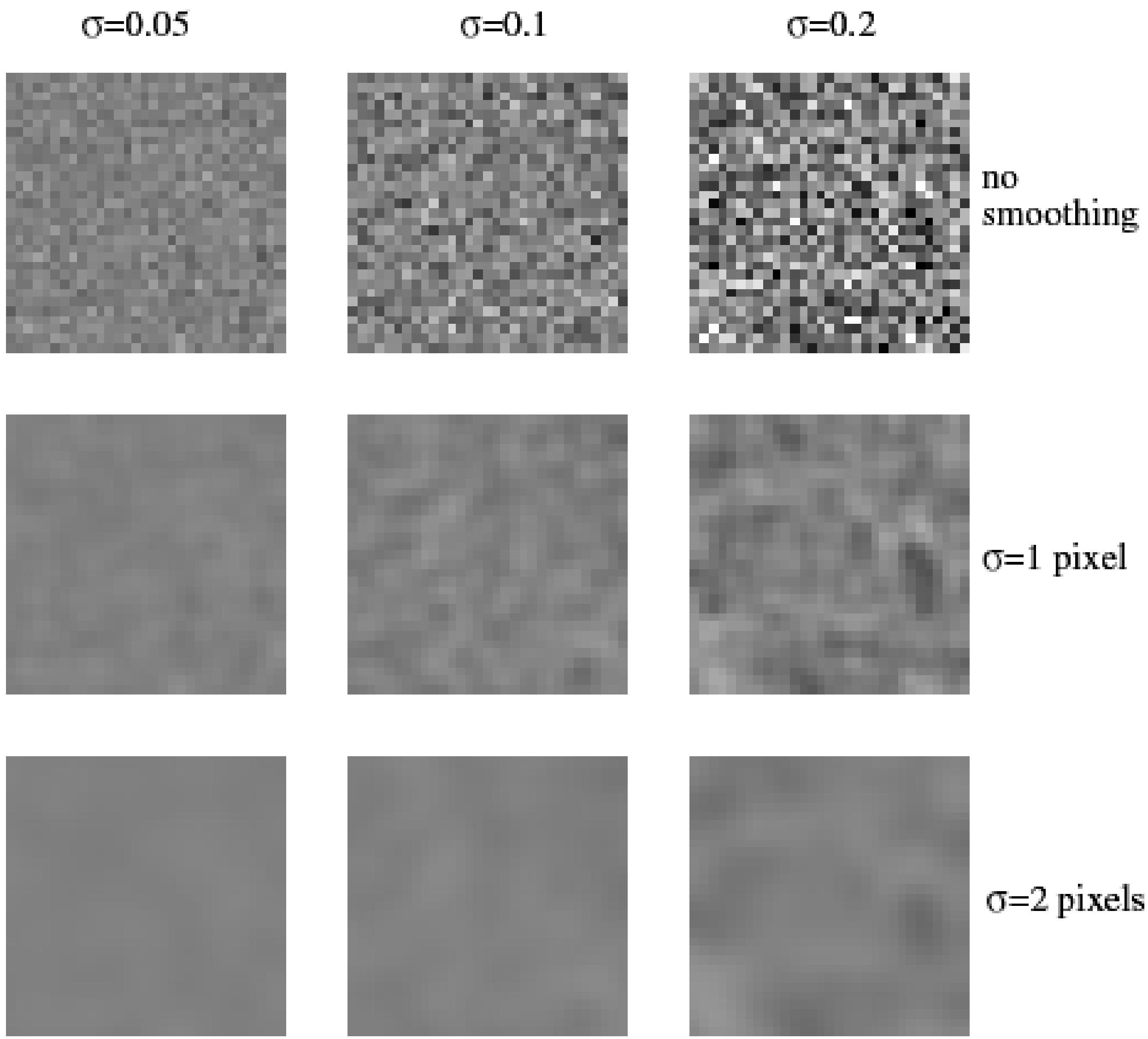



```
for sigma=1:3:10
  h = fspecial('gaussian`, fsize, sigma);
  out = imfilter(im, h);
  imshow(out);
  pause;
end
```

K. Grauman

Keeping the two Gaussians straight…

More Gaussian noise (like earlier) σ →



Wider Gaussian smoothing kernel σ →

And now some linear intuition…

An operator H (or system) is *linear* if two properties hold ($f1$ and $f2$ are some functions, a is a constant):

- Superposition (things sum):

  $H(f1 + f2) = H(f1) + H(f2)$ *(looks like distributive law)*

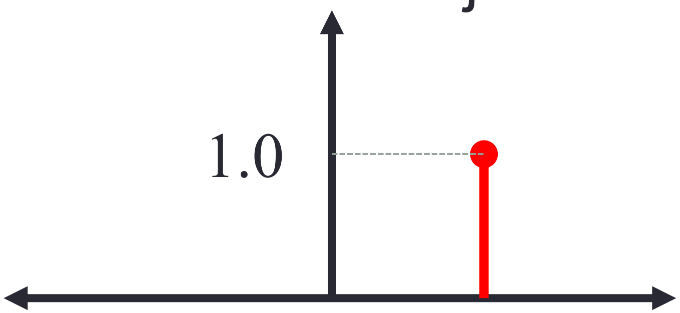
- Scaling (constant scales):

  $H(a \cdot f1) = a \cdot H(f1)$

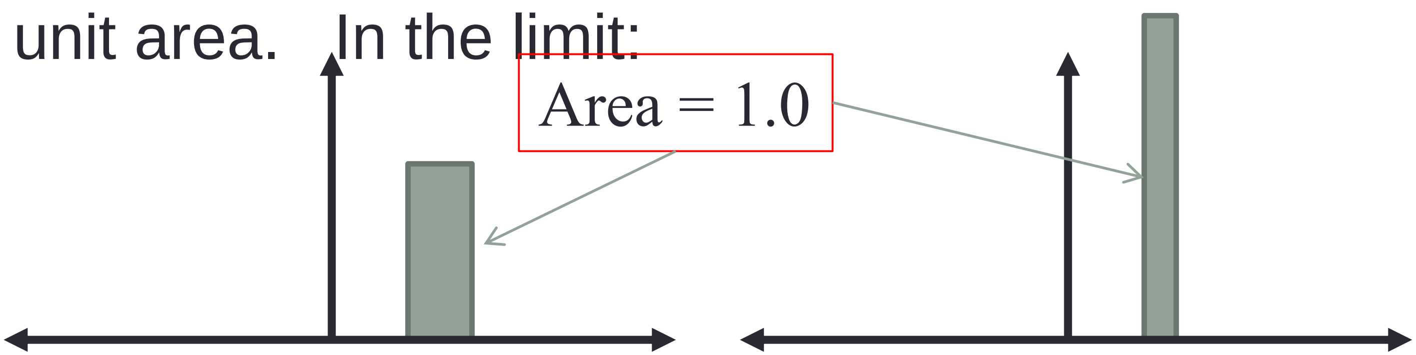
Because it is sums and multiplies, the "filtering" operation we were doing are linear.

An impulse function…

- In the discrete world, and *impulse* is a very easy signal to understand: it’s just a value of 1 at a single location.

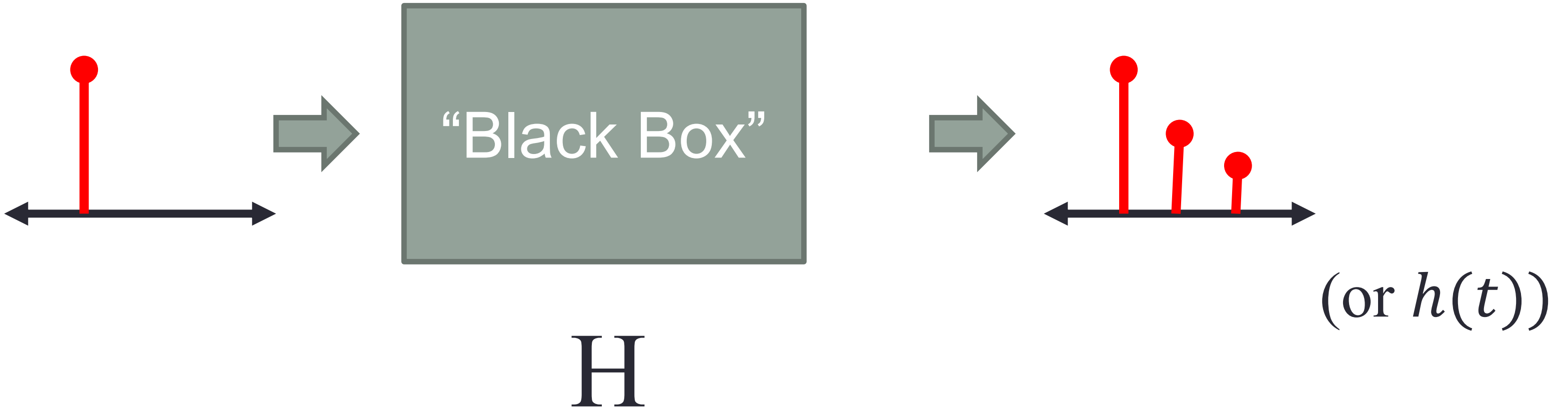


- In the continuous world, an *impulse* is an idealized function that is very narrow and very tall so that it has a unit area. In the limit:



An impulse response

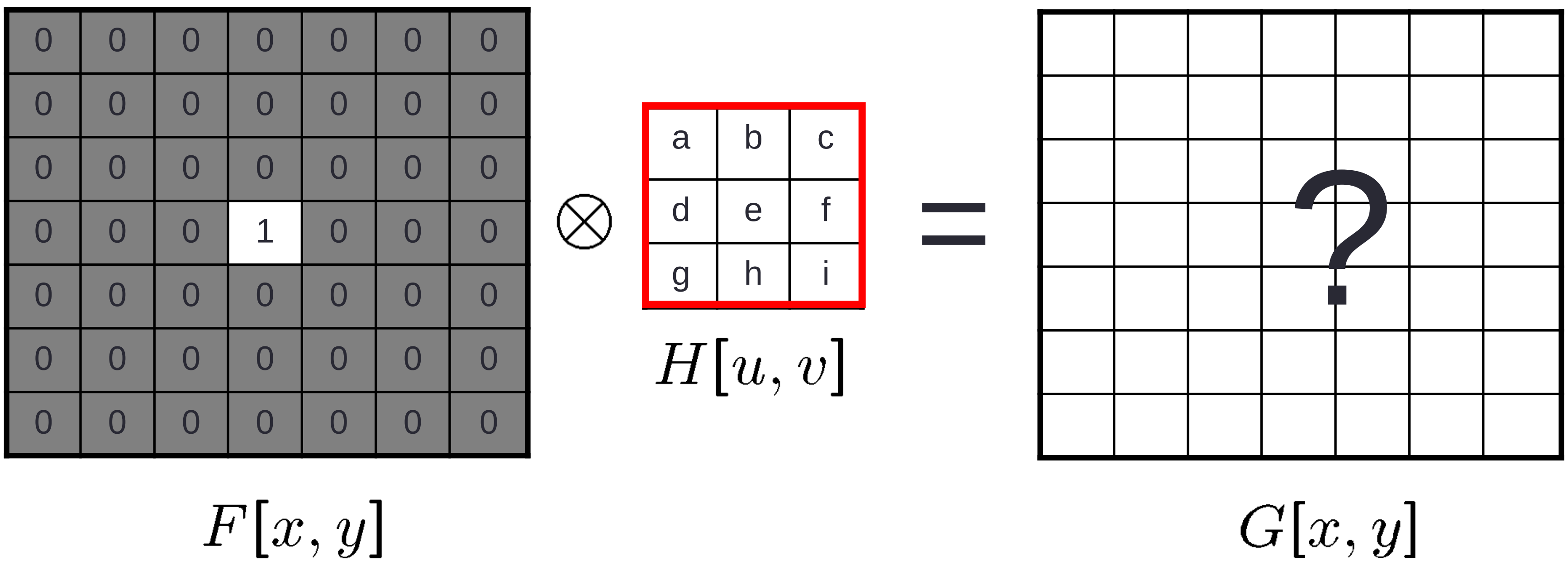
- If I have an unknown system and I "put in" an impulse, the response is called the impulse response. (Duh?)



- So if the black box is linear you can describe H by $h(x)$. Why?

Filtering an impulse signal

What is the result of filtering the impulse signal (image) *F* with the arbitrary kernel *H*?



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Filtering an impulse signal

0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

$F[x, y]$

$\otimes$

a	b	c
d	e	f
g	h	i

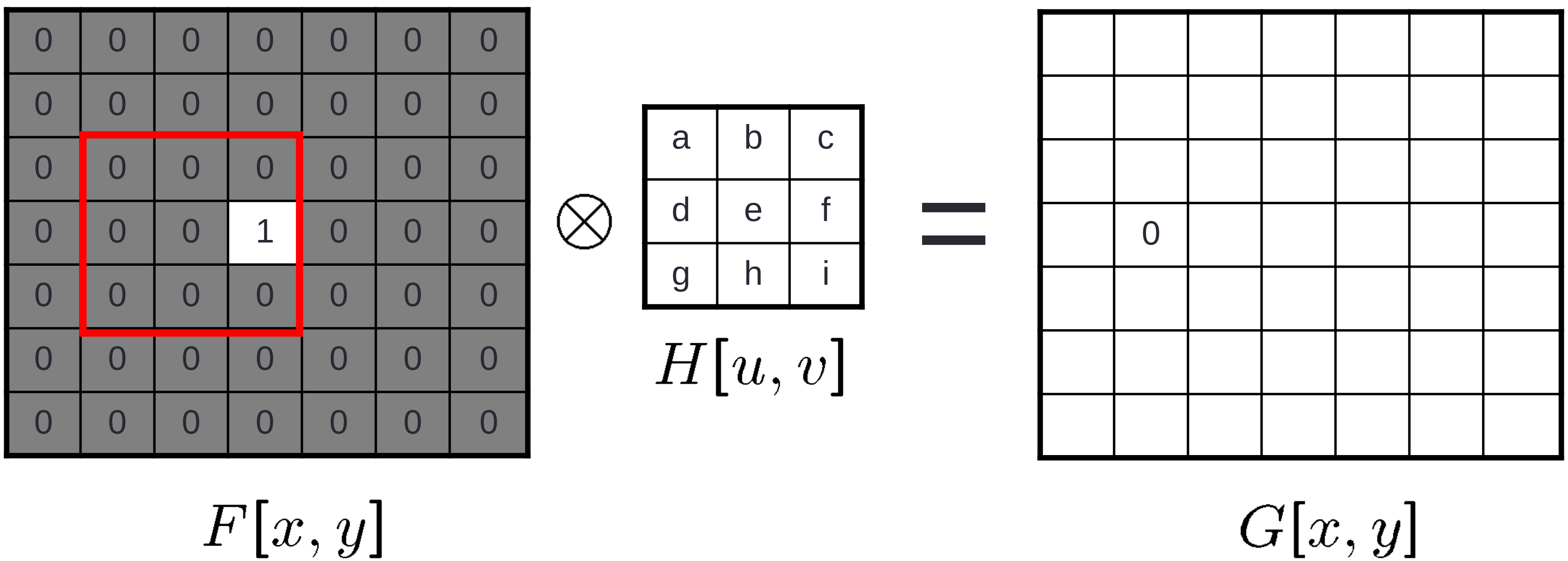
$H[u, v]$

$=$

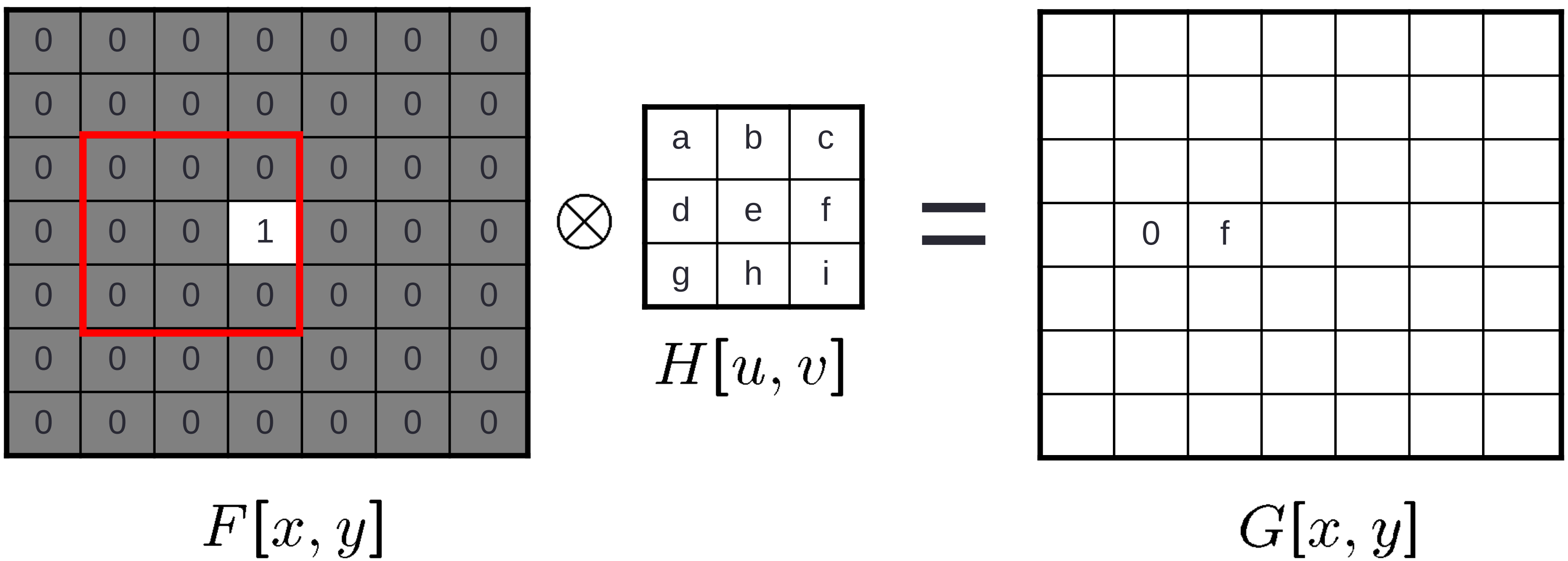
	0					

$G[x, y]$

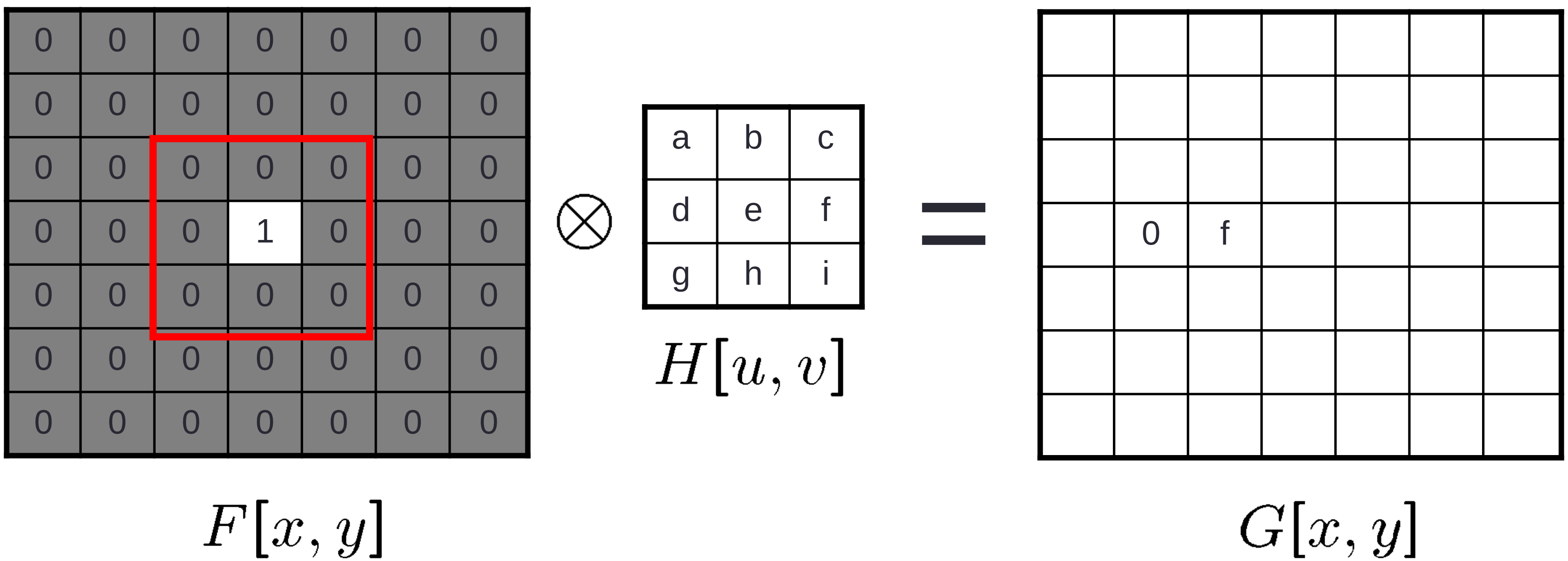
Filtering an impulse signal



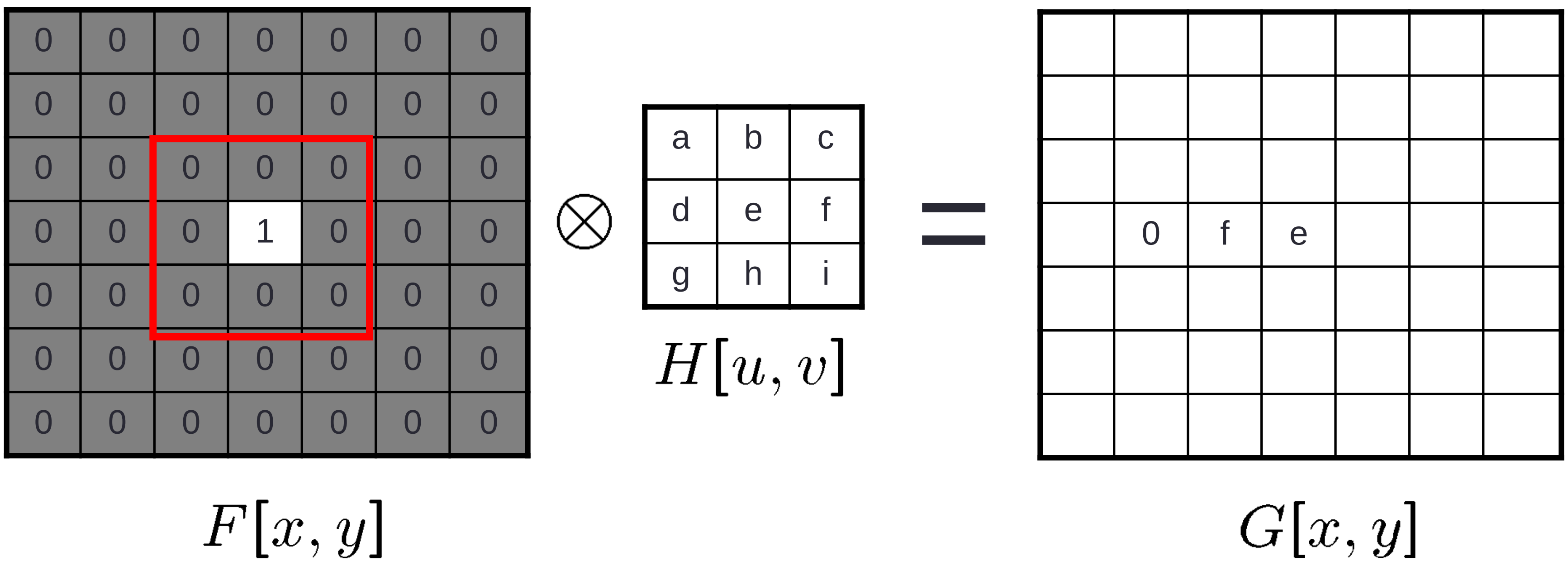
Filtering an impulse signal



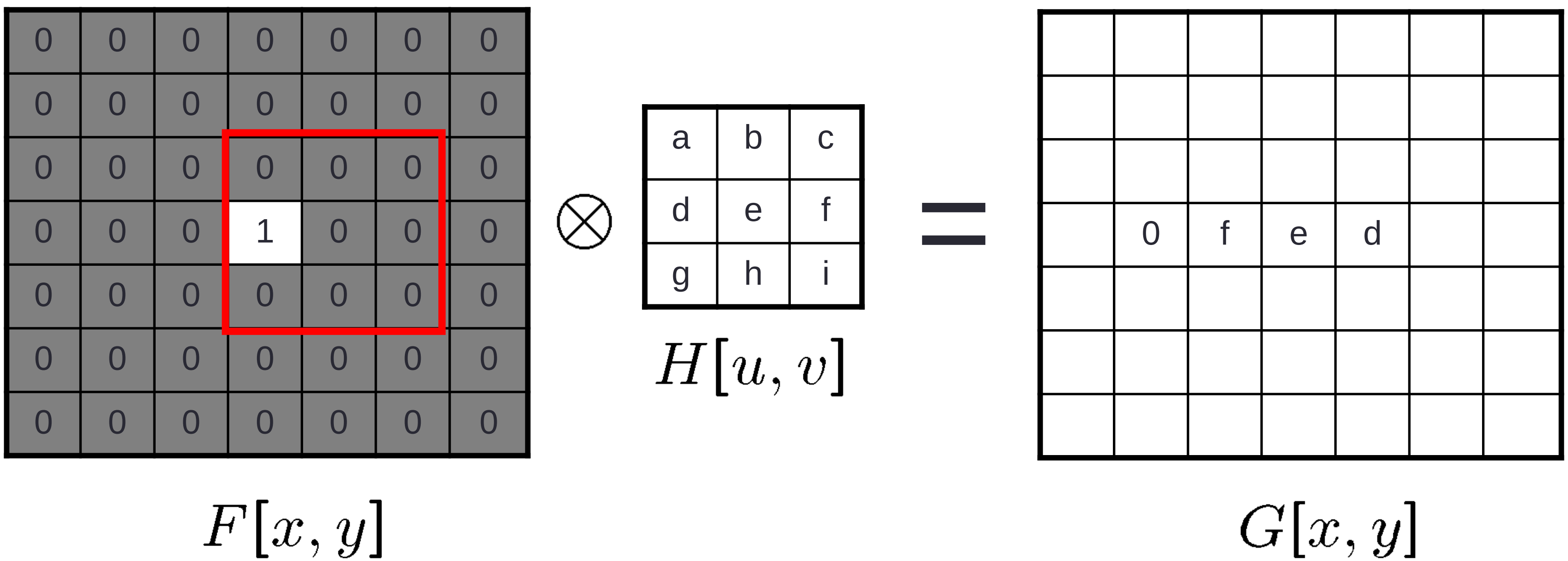
Filtering an impulse signal



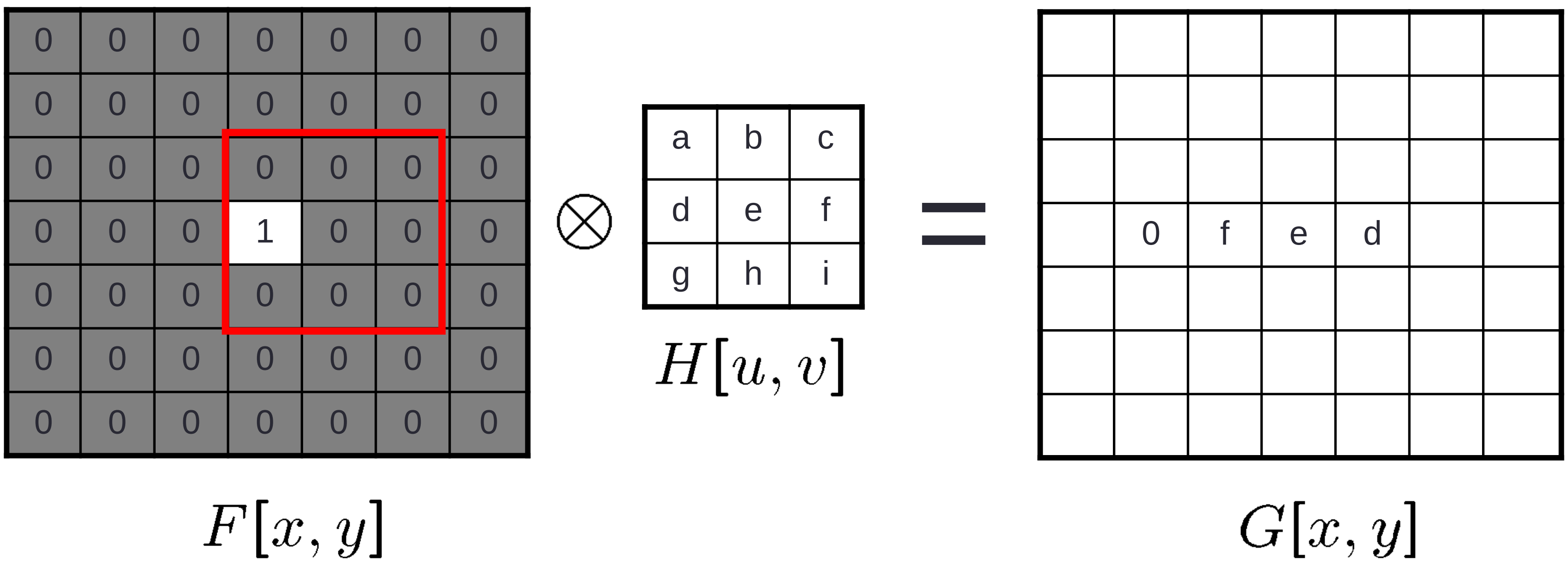
Filtering an impulse signal



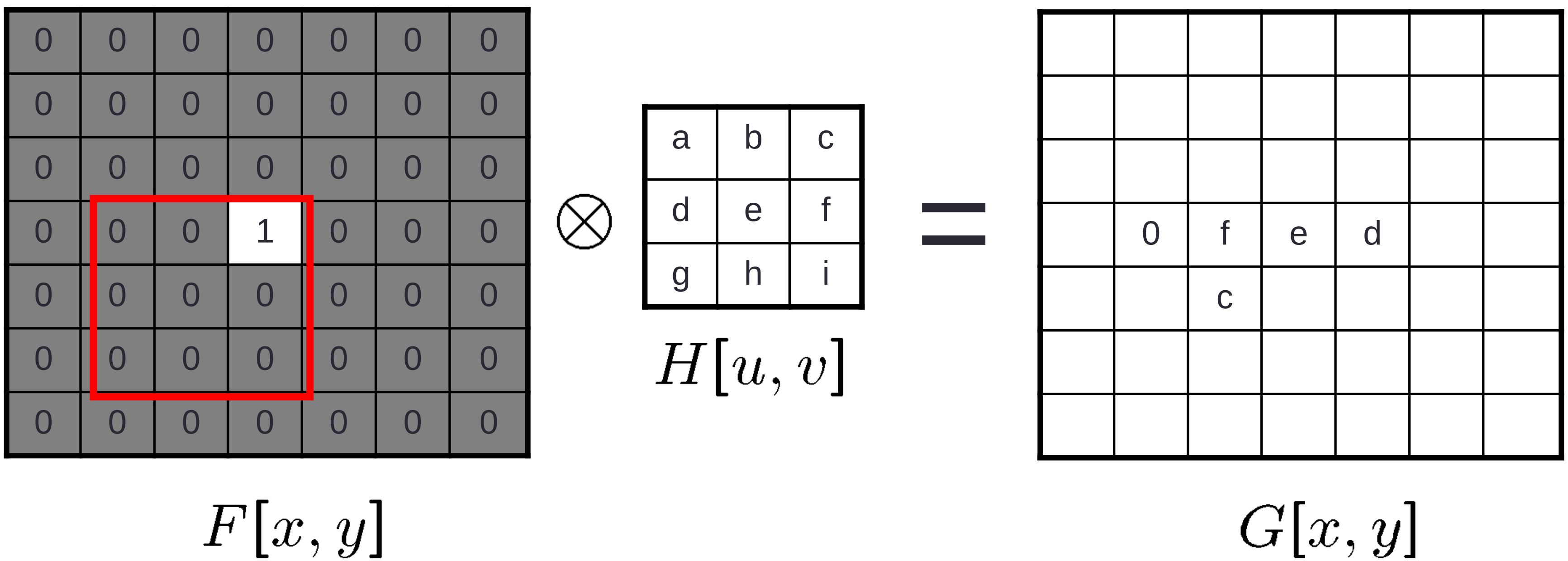
Filtering an impulse signal



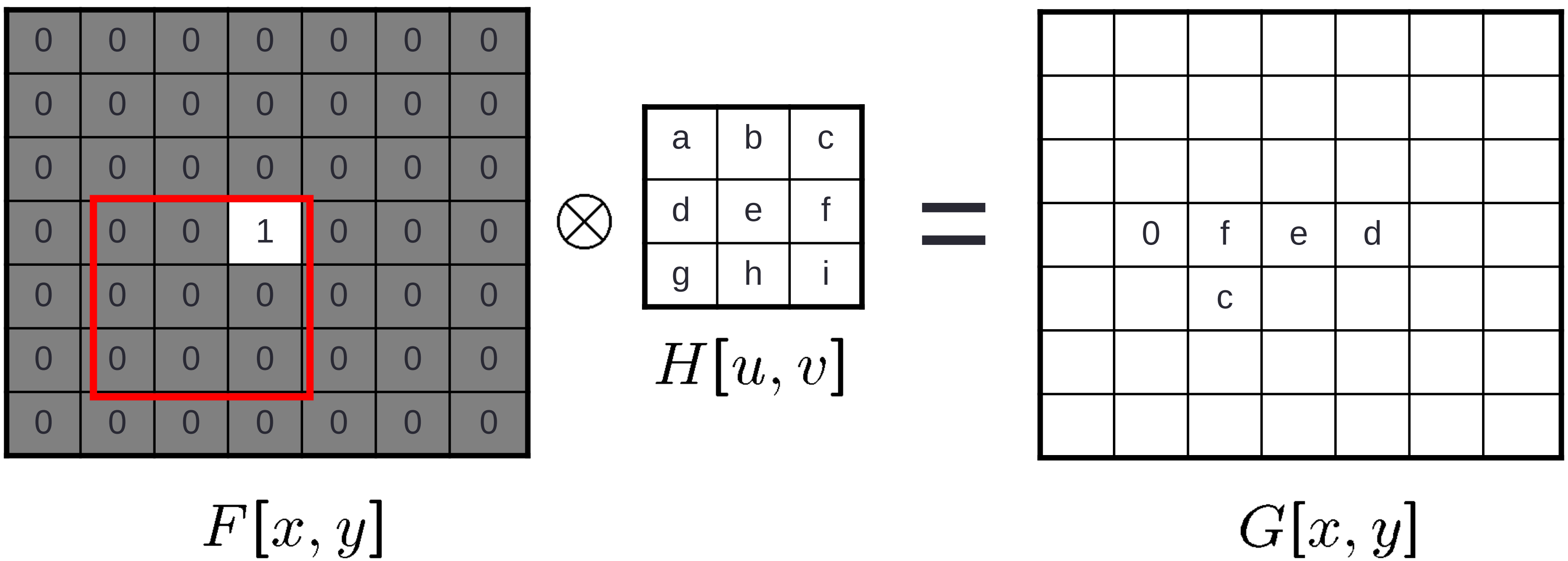
Filtering an impulse signal



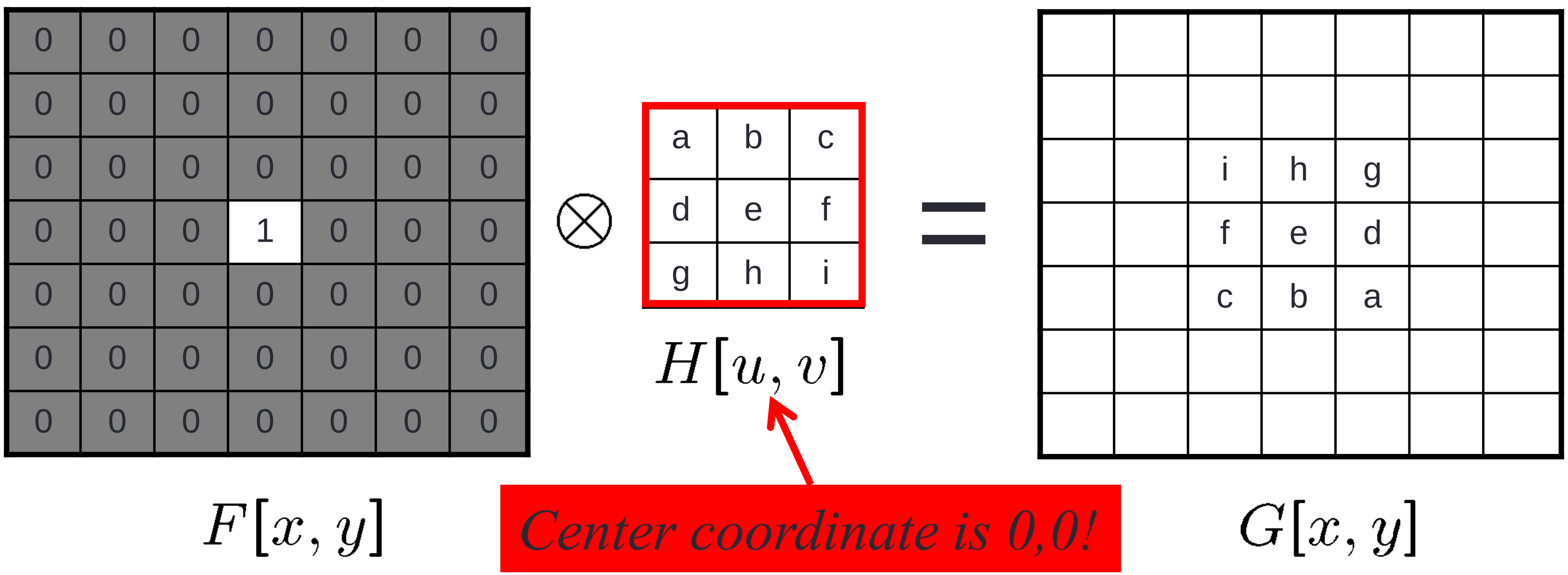
Filtering an impulse signal



Filtering an impulse signal



"Filtering" an impulse signal



If you just "filter" meaning slide the kernel over the image you get a ***reversed*** response.

Convolution

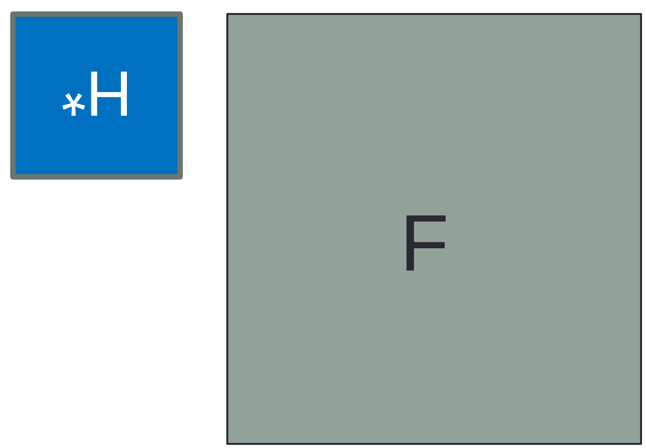
- Convolution:
  - Flip where the filter is applied in both dimensions (bottom to top, right to left)
  - Then apply cross-correlation

Centered at zero!

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v] F[i-u, j-v]$$

$$G = H * F$$

Notation for convolution operator



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One more thing…

- **Shift invariant:**
    - Operator behaves the same everywhere, i.e. the value of the output depends on the pattern in the image neighborhood, not the position of the neighborhood.

Properties of convolution

- Linear & shift invariant
- Commutative:

  f * g = g * f
- Associative

  (f * g) * h = f * (g * h)
- Identity:

  unit impulse e = […, 0, 0, 1, 0, 0, …]. f * e = f
- Differentiation: $$\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g$$

We'll use this later!

Convolution vs. correlation

Convolution

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v]F[i-u,j-v]$$

$$G = H \star F$$

Cross-correlation

$$G[i,j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} H[u,v]F[i+u,j+v]$$

$$G = H \otimes F$$

For a Gaussian or box filter, how will the outputs differ?

If the input is an impulse signal, how will the outputs differ?

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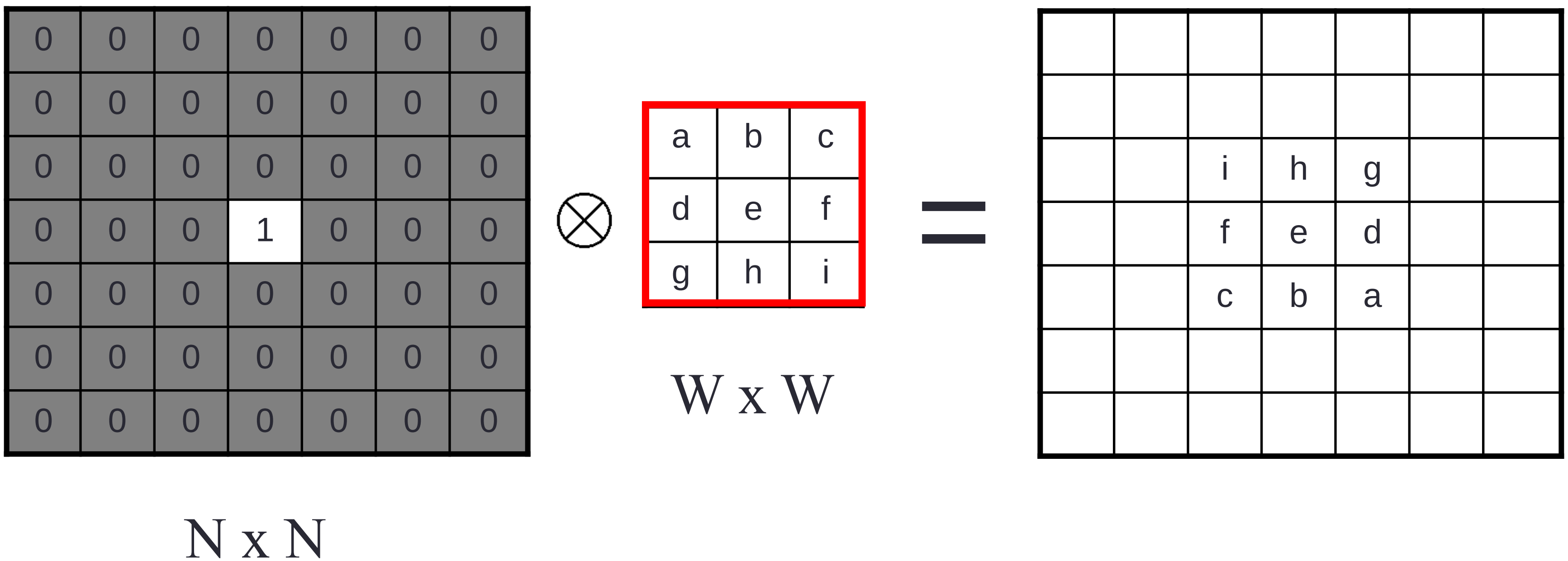
Computational Complexity

- If an image is *NxN* and a kernel (filter) is WxW, how many multiplies do you need to compute a convolution?



- You need N*N*W*W = N^2W^2
  - which can get big (ish)

Separability

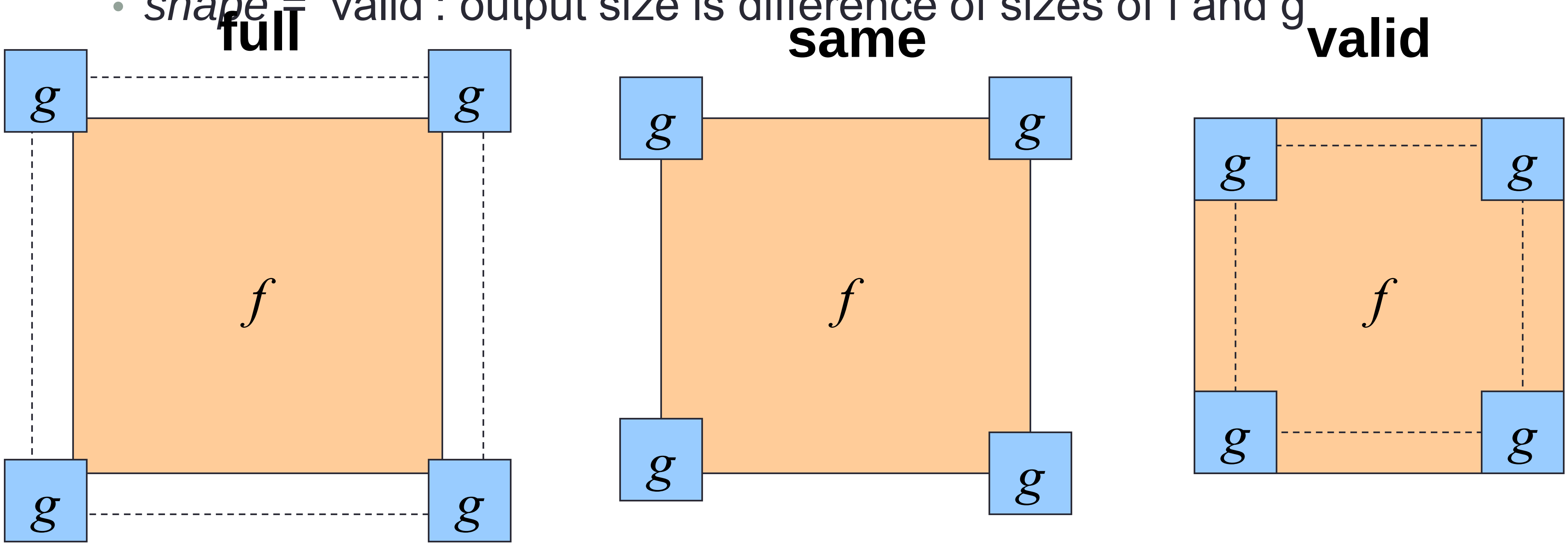
- In some cases, filter is separable, meaning you can get the square kernel by convolving a single column vector by some row vector:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

- To apply to an image you:
  - Convolve all rows
  - Convolve all resulting columns
- This used to be *very* important – instead of N*N*W*W it's N*N*W*2
  - So if your Kernel is a 31x31 filter you save a factor of 15

Boundary issues

- What is the size of the output?
- Old MATLAB: filter2(g, f, *shape*)
  - *shape* = ‘full’: output size is sum of sizes of f and g
  - *shape* = ‘same’: output size is same as f
  - *shape* = ‘valid’: output size is difference of sizes of f and g



Source: S. Lazebnik

Boundary issues

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - **clip filter (black)**

Source: S. Marschner

Boundary issues

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - **wrap around**

Source: S. Marschner

Boundary issues

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - wrap around
    - **copy edge**

Source: S. Marschner

Boundary issues

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods:
    - clip filter (black)
    - wrap around
    - copy edge
    - **reflect across edge**

Source: S. Marschner

Boundary issues

- What about near the edge?
  - the filter window falls off the edge of the image
  - need to extrapolate
  - methods (new MATLAB):
    - clip filter (black): imfilter(f, g, 0)
    - wrap around: imfilter(f, g, ‘circular’)
    - copy edge: imfilter(f, g, ‘replicate’)
    - reflect across edge: imfilter(f, g, ‘symmetric’)

Source: S. Marschner

Predict the filtered outputs

$$* \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = ?$$

$$* \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = ?$$

$$* \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = ?$$

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Practice with linear filters

Original

0	0	0
0	1	0
0	0	0

?

Source: D. Lowe

Practice with linear filters

Original

0	0	0
0	1	0
0	0	0

Filtered
(no change)

Source: D. Lowe

Practice with linear filters

Original

0	0	0
0	0	1
0	0	0

?

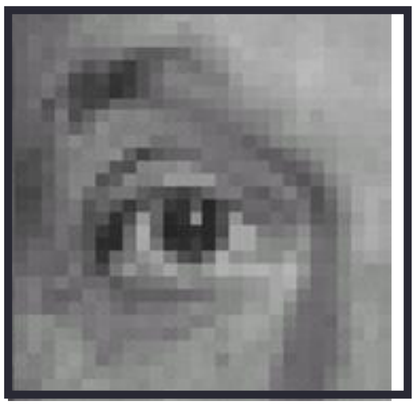
Source: D. Lowe

Practice with linear filters

Original

0	0	0
0	0	1
0	0	0

Center coordinate is 0,0!



Shifted left by 1 pixel with correlation

Source: D. Lowe

Practice with linear filters

Original

$$\frac{1}{9}\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

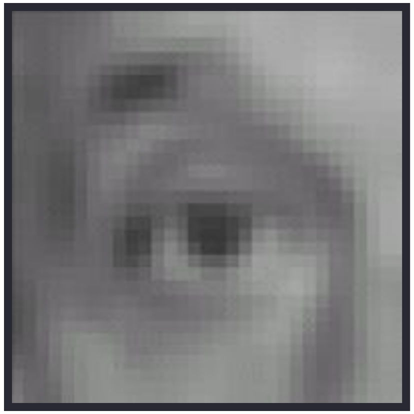
?

Source: D. Lowe

Practice with linear filters

Original

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}$$



Blur (with a box filter)

Source: D. Lowe

Practice with linear filters

Original

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

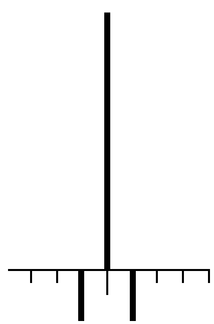
?

Source: D. Lowe

Practice with linear filters

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Original

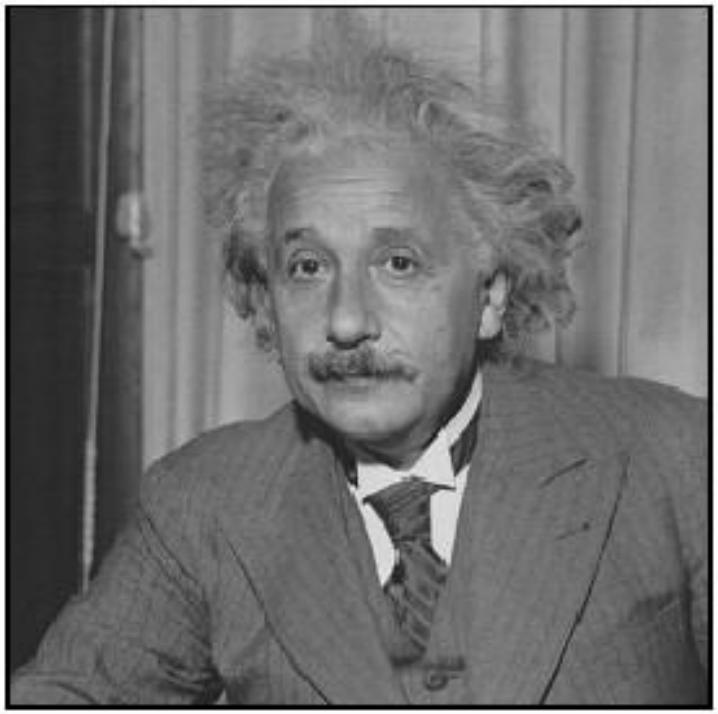


Sharpening filter

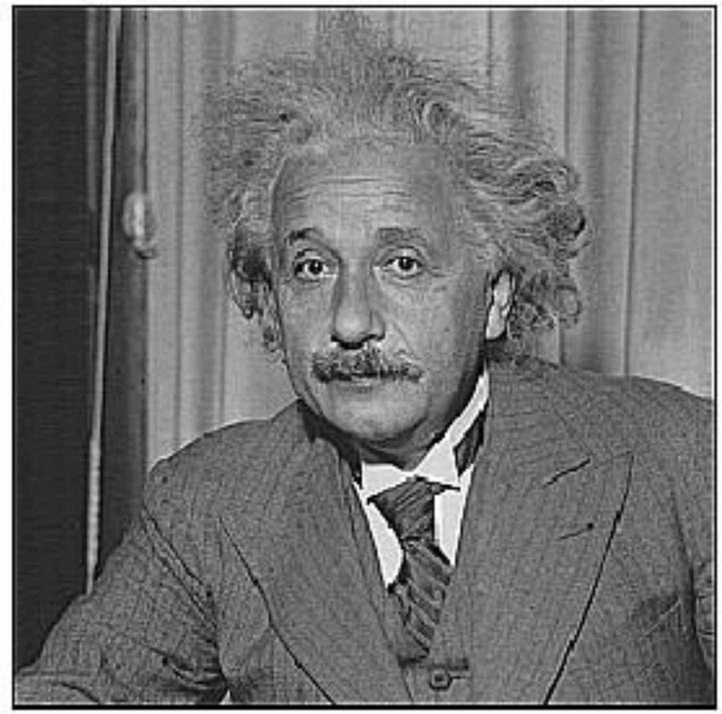
- Accentuates differences with local average

Source: D. Lowe

Filtering examples: sharpening



before



after

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Effect of smoothing filters

5x5

Additive Gaussian noise

Salt and pepper noise

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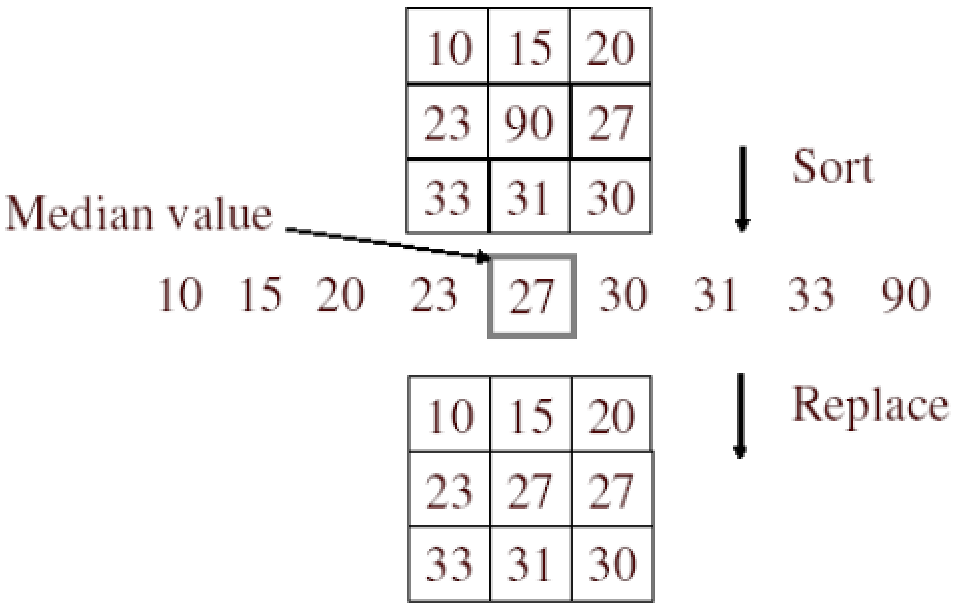
Median filter



- No new pixel values introduced
- Removes spikes: good for impulse, salt & pepper noise
- Linear?

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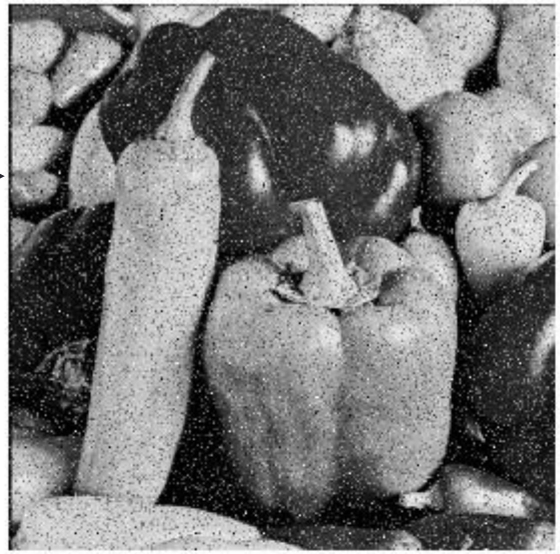
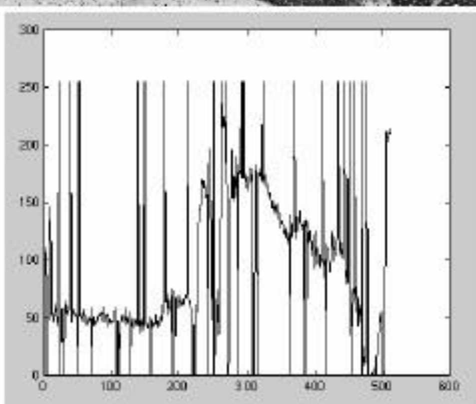
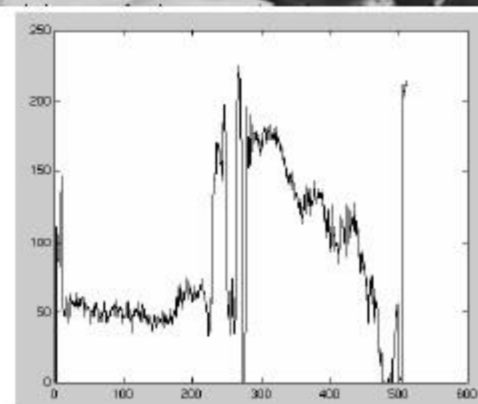
Median filter

Salt and pepper noise



Median filtered



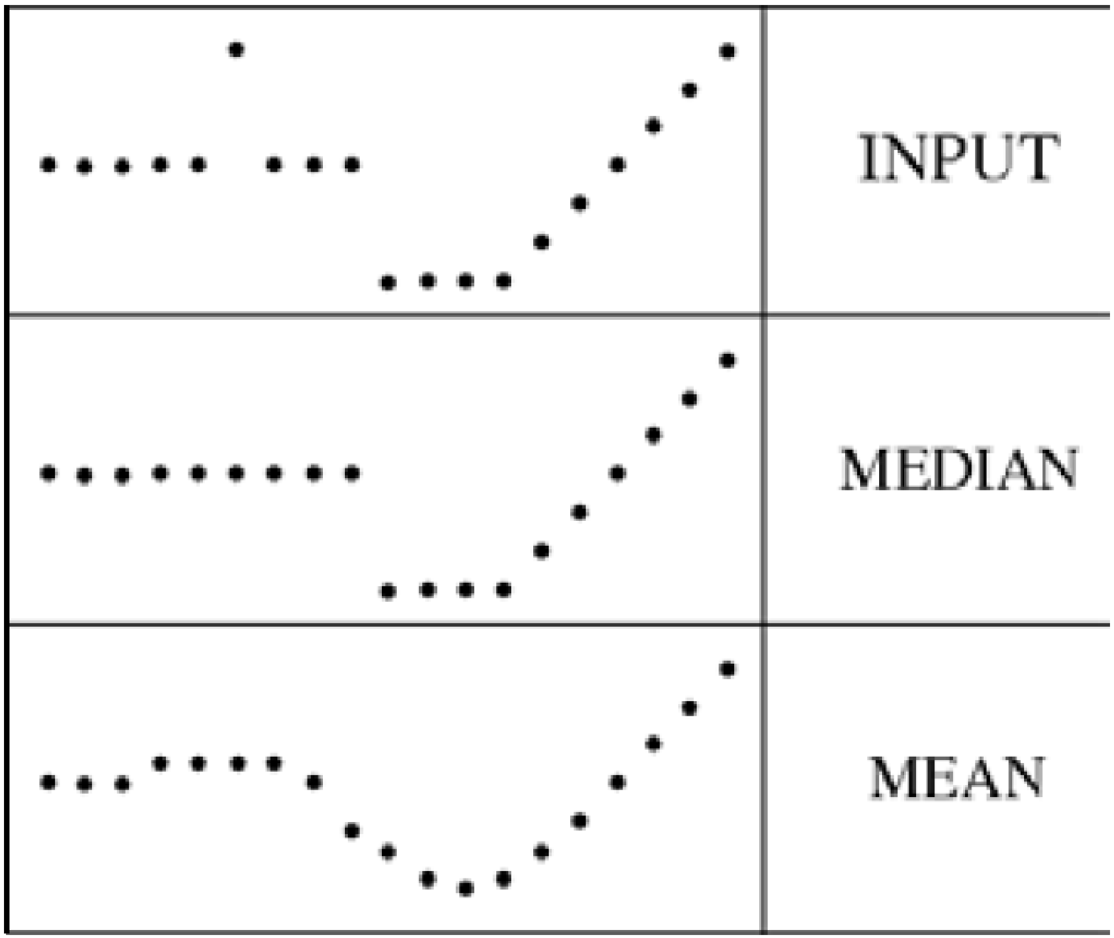


Plots of a row of the image

Source: M. Hebert

Median filter

- Median filter is edge preserving



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Exercise

- Design a kernel for:
  - Horizontal derivative
  - Vertical derivative
  - Second derivative

- What If:
  - H is the max/min operator?
  - We use an affine operator?

Homework For Next Class

- Read slide set
- Prepare 3 questions
  - Option 1: on things that you don't understand
  - Option 2: on things that you think would make good exam questions
- Do it individually
- Write them down and cut them into 3 strips: 1 per questions and distribute them.
- BRING ENOUGH LAPTOPS to access the class.