Particle feedback control convergence analysis

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Contents

1 Notations and assumptions

The goal of our work is to show convergence of the control to the "true control" under the temporal model discretization N.

1.1 Notations

1. We use $U_n : \{t_n, ..., T\} \to \mathbb{R}^d$ to denote the control process starting from time t_n and ends at time T. We use
$$\mathcal{U}_n := \{U_n | U_n : \{t_n, ..., T\} \to \mathbb{R}^d, U_n \text{ is } \mathcal{F}^M_{t_n}\text{-adapted}\}$$
to denote the collection of the admissible controls starting at time t_n.
2. We define the control at time t_n to be $u_n := U_n\big|_{t_n}$, the conditional distribution coming from a particle filter algorithm.
3. We define $\mu_n^N := \pi^N_{t_n|t_n}$ where the superscript means that the measure is obtained through the particle filter method, and so it is random.

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4. We use P_n^N to denote the transition operator (the prediction step) under the SGD-particle filter framework. And P_n the transition operator for the exact case (discrete model).

5. We use $\langle \cdot, \cdot \rangle$ to denote the deterministic L_2 inner product, i.e. if $f, g \in L^2([0,T]; \mathbb{R}^d)$, then

$$\langle f, g \rangle := \int_0^T f \cdot g \, dt \tag{1}$$

6. We define $J_N'^x(U_n) := \mathbb{E}[J_N'(U_n) | X_n = x]$. We then have $\mathbb{E}[J_N'^{X_n}(U_n)] := \int \mathbb{E}[J_N'(U_n) \big| X_n = s] \, d\mu_n^N(s)$. We remark that U_n is a process that starts from time t_n, and so X_n is essentially the initial condition of the diffusion process.

7. We define the distance between two random measures to be the following:

$$d(\mu, \nu) := \sup_{||f||_\infty \le 1} \sqrt{\mathbb{E}^\omega[|\mu^\omega f - \nu^\omega f|^2]} \tag{2}$$

where the expectation is taken over the randomness of the measure.

8. We use the total variation distance between two deterministic probability measures μ, ν:

$$d_{TV}(\mu, \nu) := \sup_{||f||_\infty \le 1} |\mu f - \nu f| \tag{3}$$

9. We use K_n to denote the total number of iterations taken in the SGD algorithm at time t_n; we use $\mathcal{N}$ to denote the total number of particles in the system. We use C to denote a generic constant which may vary from line to line.

10. Abusing the notation, we will denote $J_N'^x(U_n)$ in the following way where the argument U_n can be vector of any length $1 \le n \le N$:

$$J_N'^x(U_n)\big|_{t_i} := \mathbb{E}^{X_{t_n}=x}[f_u'(X_{t_i}, U_n\big|_{t_i}) + b_u'^T(X_i, U_n\big|_{t_i})Y_{t_i}] \tag{4}$$

1.2 Assumptions

1. We assume that J_N' satisfy the following strong condition: for any $x \in X$, there exist a constant $\lambda > 0$ such that for all $U, V \in \mathcal{U}_0$:

$$\lambda ||U - V||^2 \le \langle J_N'^x(U), J_N'^x(V), U - V \rangle \tag{5}$$

Notice that(6) will imply that such inequality is true for any $U_n, V_n \in \mathcal{U}_n$. And it can be seen from simply fixing all the $U_n|_{t_i}, V_n|_{t_i}$, $0 \le i \le n-1$ to be 0.

This is a very strong assumption, one should consider relaxing it to

$$\lambda ||U - V||^2 \le \mathbb{E}^\omega[\mathbb{E}^{\mu_{n,\cdot}}[\langle J_N'^x(U), J_N'^x(V), U - V \rangle]] \tag{6}$$

That is, this relation holds in expectation instead of point-wise

2. both b and σ are deterministic and in $C_b^{2,2}(\mathbb{R}^d \times \mathbb{R}^m; \mathbb{R}^d)$ in space variable x and control u.

3. $b, b_x, b_u, \sigma, \sigma_x, f_x, f_u$ are all uniformly lipschitz in x, u and uniformly bounded.

4. σ satisfies the uniform elliptic condition.

5. The initial condition $X_0 := \xi \in L^2(\mathcal{F}_0)$.

6. The terminal (Loss) function Φ is C^1 and positive, and Φ_x has at most linear growth at infinity.

7. We assume that the function g_n (related to the Bayesian step) takes has the following bound: there exist $0 < \kappa < 1$ such that

$$\kappa \le g_n \le \kappa^{-1}$$

2 Algorithm from [3]

The update step (exact) should take the following form:

$$U_n^{l+1,M}\big|_{t_i} = U_n^{l,M}\big|_{t_i} - \eta_l \mathbb{E}^{\pi_{t_n|t_n}}[J_N'^X(U_n^{l,M})]\Big|_{t_i}, \quad \forall n \le i \le N \tag{7}$$

where $X \sim \pi_{t_n|t_n}$.

We use $E^{\pi_{t_n|t_n}}[J_N'(U^{l,M})]$ in place of $E[J_N'^x(U^{l,M})|\mathcal{F}_{t_n}^M]$ for simplicity of notations, where the latter term was used in the original paper:

$$E[J_N'^x(U^{l,M})|\mathcal{F}_{t_n}^M] := \int \mathbb{E}[J_N'(U^{l,M})|X_n = x] \cdot p(x|\mathcal{F}_{t_n}^M)dx \tag{8}$$

The SGD version of the approximation for the algorithm takes the following form:

$$U_{t_i}^{l+1,M}|_{t_n} = U_{t_i}^{l,M}|_{t_n} - \rho j'^{(\hat{l},\hat{s})}(U)|_{t_n} \tag{9}$$

where $(\hat{l}, \hat{s})$ stands for the random sample value with the initial value x randomly drawn from the particle cloud.

With the optimal control at time n, which is $u_{t_n} = U_{t_i}^{K_n,M}|_{t_n}$, we carry out the prediction step which is followed by the standard analysis step.

We repeat the same process until we reach the terminal time.

3 Some descriptions on the exact control u_n^*

We realize that to use the particle filtering method, it will only make sense to consider a discretized model. The true control is obtained in the following way according to our algorithm:

1. At each time t_n we find the optimal control $U_n^* : \{t_n, ..., T\} \to \mathbb{R}^d$ control at time t_n based on the distribution $\pi_{t_n|t_n}^N$. Then, due to gradient decent algorithm and the assumption (6), we should have $\mathbb{E}^{\pi_{t_n|t_n}}[J_N'^x(\cdot)] : \mathcal{U} \to \mathbb{R}^d$ is strongly convex:

$$\lambda||U_n - V_n||_2^2 \le \langle \mathbb{E}^{\pi_{t_n|t_n}}[J_N'^x(U_n)] - \mathbb{E}^{\pi_{t_n|t_n}}[J_N'^x(V_n)], U_n - V_n\rangle, \quad \forall i \in n, ..., N \tag{10}$$

   Then, (7) converges since we are now in the standard finite dimensional convex optimization framework. And by optimality, we also have

$$\mathbb{E}^{\pi_{t_n|t_n}}[J_N'^x(U_n^*)]\Big|_{t_i} = 0, \quad \forall i \in n, ..., N \tag{11}$$

2. We define $u_n^* := U_{t_n}^*|_{t_n}$ to be the optimal control locally at time t_n and we do the prediction step to find the distribution of $\pi_{t_{n+1}|t_n}$ under the control u_n^*, and we do the analysis step to obtain the distribution $\pi_{t_{n+1}|t_{n+1}}$.

3. Repeat the two previous steps, and we will obtain a collection of the true model distribution, and the true model controls:

$$U_0^* := \{u_0^*, u_1^*, u_2^*, ..., u_{N-1}^*\} \tag{12}$$

4 Idea of proof

We first give the general idea of the proof, but before that we list some lemmas we need to use whose proof can be found in [1].

Lemma 4.1. *The following is true:*

$$\begin{aligned} \sup_{\mu \in \mathcal{P}(\mu)} d(S^{\mathcal{N}}\mu, \mu) &\le \frac{1}{\sqrt{\mathcal{N}}} \\ d(P_n^N\mu, P_n^N\nu) \le d(\mu, \nu),&\ \ d(P_n\mu, P_n\nu) \le d(\mu, \nu) \\ d(L_n\nu, L_n\mu) &\le 2\kappa^{-2}d(\nu, \mu) \end{aligned} \tag{13}$$

Here we describe the main idea of the proof for our feedback control algorithm.

$$\begin{aligned} d(\mu_{n+1}^{N,\cdot}, \mu_{n+1}) &\equiv d(L_n S^N P_n^N \mu_n^{N,\cdot}, L_n P_n \mu_n) \\ &\leq d(L_n S^N P_n^N \mu_n^{N,\cdot}, L_n S^N P_n \mu_n) + d(L_n S^N P \mu_n, L_n P_n \mu_n) \\ &\leq 2\kappa^{-2}\Big(\frac{2}{\sqrt{N}} + d(P_n^N \mu_n^{N,\cdot}, P_n \mu_n^{N,\cdot}) + d(S^N P_n \mu_n^{N,\cdot}, P_n \mu_n)\Big) \\ &\leq 2\kappa^{-2}\Big(\frac{3}{\sqrt{N}} + d(P_n^N \mu_n^{N,\cdot}, P_n \mu_n^{N,\cdot}) + d(\mu_n^{N,\cdot}, \mu_n)\Big) \end{aligned} \tag{14, 15}$$

Where in the above inequalities, we have used triangle inequalities and lemma 4.1.

Hence, if we can show that the inequality of the following form holds

$$d(P_n^N \mu_n^{N,\cdot}, P_n \mu_n^{N,\cdot}) \leq C_n d(\mu_n^{N,\cdot}, \mu_n) + \epsilon_n \tag{16}$$

for some constant C_n and ϵ_n that we can tune, then by recursion, we can show that by using (15) the convergence holds true.

Remark. *We point out that the difficulty lies in showing* (16). *Recall that the distance between two random measures is defined in* (2) *which involves in testing over all measurable function bounded by* 1. *However, we will see later that it is more desirable that we test against functions that are Lipschitz. Hence, since the underlying measure is a finite Borel probability measure, we want to identify the function first with a continuous function on a compact set (Lusin). Then we approximate this continuous function uniformly by a Lipschitz function since now the domain is compact.*

This way, we can roughly show that a form close to (16) *is true.*

Remark. *Notice that the first measure in* $d(P_n^N \mu_n^N, P\mu_n^N)$ *has two source of randomness: the randomness in* P_n *which comes from the SGD algorithm used to find the control, and the randomness in the measure* μ_n^N. *However, when we take the expectation, we do not distinguish the two.*

5 Proof

We first state the following lemma, *which says that regardless of the control, the probability of the particle* X_n *(obtained from the particle filter method) at any time* n *escaping from a very large region is very small.* And the reason why we want to have such result is that we want to restrict the particles to a compact subspace.

Lemma 5.1. *There exists* M *and constant* C, *such that under any admissible control* U_0

$$\mathbb{P}(\sup_{\substack{i\in\{1,\ldots,N\}\\ U_0\in\mathcal{U}}} |X_i| \geq M) \leq \frac{C}{M^2}, \quad X_i \sim \pi^N_{t_i|t_{i-1}} \ \ or \ \ X_i \sim \pi^N_{t_i|t_i} \tag{17}$$

Remark. *From this lemma, we also know that there is a compact set* $\mathcal{M}$ *with diameter* $diam(\mathcal{M}) \leq M$, *such that*

$$\mathbb{P}(\sup_{\substack{i\in\{1,\ldots,N\}\\ U_0\in\mathcal{U}}} |X_i| \geq diam(\mathcal{M})) \leq \frac{C}{M^2}, \quad X_i \sim \pi^N_{t_i|t_{i-1}} \ \ or \ \ X_i \sim \pi^N_{t_i|t_i} \tag{18}$$

We will use the following result extensively later

$$\mathbb{E}[\mathbf{1}_{\{|X_n|\geq M\}}] \leq \frac{C}{M^2}, \ \forall 1 \leq n \leq N \tag{19}$$

5.1 Main proof.

We point out here that P_n^N is random, and the randomness come from the SGD algorithm for finding the control u_n given the underlying measure. Hence, we consider $d^2(P_n^N \mu_n^{N,\cdot}, P_n \mu_n^{N,\cdot})$ where the expectation is taken over the randomness in the SGD.

Lemma 5.2. *For each $n = 0, 1, ..., N-1$, there exist M_n, L_n, δ_n, K_n such that the following inequality holds*

$$d(\mu_{n+1}^{N,\cdot}, \mu_{n+1}) \leq 2\kappa^{-2}\Big((1 + C\Delta t L_n M_n) d(\mu_n^{N,\cdot}, \mu_n) + \frac{C}{M_n} + \frac{CM_n}{K_n} + 2\delta_n + \frac{3}{\sqrt{N}}) \tag{20}$$

Proof. The key step is to estimate the quantity $d^2(P_n^N \mu_n^{N,\cdot}, P_n \mu_n^{N,\cdot})$, by (15). WLOG, we assume that the sup is realized by the function f with $||f||_\infty \leq 1$, then we have

$$d^2(P_n^N \mu_n^{N,\cdot}, P_n \mu_n^{N,\cdot}) = \mathbb{E}^\omega[|P_n^N \mu_n^{N,\cdot} f - P_n \mu_n^{N,\cdot} f|^2] \tag{21}$$

Notice that P_n^N is the prediction operator that uses the control u_n which carries the randomness from SGD, and P_n uses the control u_n^*. Then $P_n^N \mu_N^{N,\omega}$ is a random measure. And we comment that both u_n^* and μ_n are deterministic.

Without loss of generality, we use u_n^ω and $\mu_n^{N,\omega}$ to denote the random control and the random measure. (Even though the randomness can be different, we can concatenate $(\omega_1, \omega_2) := \omega$ to define them as ω in general.)

We have for the fixed randomness ω, and by Fubini's theorem

$$\begin{aligned} |P_n^N \mu_n^{N,\omega} f - P_n \mu_n^{N,\omega} f|^2 &= \Bigg|\mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbb{E}[\underbrace{f(X_n + b(X_n, u_n^\omega)\Delta t + \sigma(X_n)\Delta W_n)}_{f_1^\omega}|X_n = x]\Big] \\ &\quad - \mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbb{E}[\underbrace{f(X_n + b(X_n, u_n^*)\Delta t + \sigma(X_n)\Delta W_n)}_{f_2}|X_n = x]\Big]\Bigg|^2 \qquad (22) \\ &= \Bigg|\mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbb{E}[f_1^\omega - f_2|X_n = x]\Big]\Bigg|^2 \\ &= \underbrace{\Bigg|\mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{M}_n}\mathbb{E}[f_1^\omega - f_2|X_n = x]\Big]\Bigg|^2}_{A_1} + \underbrace{\Bigg|\mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{M}_n^c}\mathbb{E}[f_1^\omega - f_2|X_n = x]\Big]\Bigg|^2}_{A_2} \qquad (23) \end{aligned}$$

where the inner conditional expectation is taken with respect to ΔW_n.

Now, since we can pick $\mathcal{M}_n$ to be a large compact set containing the origin, with

$$\mathbb{P}(\sup_{n,U_0} |X_n| \geq diam(\mathcal{M}_n)) \leq \frac{C}{M_n^2} \tag{24}$$

To deal with A_1, A_2, we see that it is desirable that the function f has the *Lipchitz property*. However, it is only in general measurable. The strategy to overcome this difficulty is to first use the *Lusin's Theorem* to find a continuous identification $\tilde{f}$ with f on a large compact set, then on this compact set, we can approximate $\tilde{f}$ uniformly by a Lipchitz function.

We see that

$$A_1 \leq \mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{M}_n}\mathbb{E}[|f_1^\omega - f_2|^2|X_n = x]\Big] \tag{25}$$

Then, by taking expectation on both side over all the randomness in this quantity, we have

$$\mathbb{E}^\omega[A_1] \leq \mathbb{E}^\omega \mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{M}_n}\mathbb{E}[|f_1^\omega - f_2|^2]\Big] \tag{26}$$

We know that there exists a big compact $\mathcal{K}_n$ (so a large M_n) containing the origin such that

$$\mathbb{P}(\sup_{n,U_0} |X_n| \geq diam(\mathcal{K}_n)) \leq \frac{C}{M_n^2} \tag{27}$$

and a continuous $\tilde{f}^n$ with $\tilde{f}^n|_{\mathcal{K}_n} = f|_{\mathcal{K}_n}$ by *Lusin's theorem.*

And so we know that $\tilde{f}^n|_{\mathcal{K}_n \cap \mathcal{M}_n} = f|_{\mathcal{K}_n \cap \mathcal{M}_n}$. And we also have the following inequality:

$$\begin{aligned} \mathbb{E}^\omega \mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{M}_n}\mathbb{E}[|f_1^\omega - f_2|^2]\Big] &= \mathbb{E}^\omega \mathbb{E}^{\mu_n^{N,\omega}}\Big[(\mathbf{1}_{\mathcal{M}_n \cap \mathcal{K}_n} + \mathbf{1}_{\mathcal{M}_n \cap \mathcal{K}_n^c})\mathbb{E}[|f_1^\omega - f_2|^2]\Big] \qquad (28) \\ &\leq \mathbb{E}^\omega \mathbb{E}^{\mu_n^{N,\omega}}\Big[(\mathbf{1}_{\mathcal{M}_n \cap \mathcal{K}_n} + \mathbf{1}_{\mathcal{K}_n^c})\mathbb{E}[|f_1^\omega - f_2|^2]\Big] \qquad (29) \end{aligned}$$

Also, since both $\mathcal{K}_n$ and $\mathcal{M}_n$ are compact, $\mathcal{K}'_n := \mathcal{K}_n \cap \mathcal{M}_n$ is also compact with $diam(\mathcal{K}'_n) \leq M_n$. From Lemma 5.1, we know that there exist some constant C such that for any $\pi^N_{t_n|t_{n-1}}$, $\pi^N_{t_n|t_n}$ that one obtains from or particle filter-SGD algorithm, $X \sim \pi^N_{t_n|t_{n-1}}$ or $\pi^N_{t_n|t_n}$:

$$\mathbb{E}^{\omega}\Big[\mathbb{E}[\mathbf{1}_{\{X \in \mathcal{K}^c_n\}}]\Big] \leq \frac{C}{M_n^2} \tag{30}$$

Hence, we have that

$$(29) \leq E^{\omega}\mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{K}'_n}\mathbb{E}[|f_1^{\omega} - f_2|^2]\Big] + \frac{C}{M_n^2} \tag{31}$$

To deal with A_2, notice that $|f_1^{\omega} - f_2| \leq 2$ by the choice of f, we have the following.

$$\begin{aligned}
A_2 &\leq \mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{M}^c_n}\mathbb{E}[\big|f_1^{\omega} - f_2\big|^2\big|X_n = x]\Big] \\
\Rightarrow \mathbb{E}^{\omega}[A_2] &\leq 4\mathbb{E}^{\omega}[\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}^c_n}]] \\
&\leq \frac{C}{M_n^2} \qquad (32)
\end{aligned}$$

by Lemma 5.1.

To deal with A_1, we have by the density of the Lipchitz function there exists $||f^n - \tilde{f}^n||_{\mathcal{K}'_n,\infty} \leq \delta_n$ with Lipschitz constant L_n. We point out that L_n may depend on $\mathcal{K}'_n$, δ_n and the function $\tilde{f}|_{\mathcal{K}'_n}$. Now by taking the expectation on both sides and using the Lipchitz property, we have

$$\mathbb{E}^{\omega}[A_1] \leq (C\Delta t L_n)^2\,\underbrace{\mathbb{E}^{\omega}\Bigg[\mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{K}'_n}|u_n^{\omega} - u_n^*|^2\big|X_n = x\Big]\Bigg]}_{*} + \frac{C}{M_n^2} + 4\delta_n^2 \tag{33}$$

We realize that $*$ is the SGD optimization part of the algorithm in expectation, and we note that we have dropped the inner expectation. The expectation $\mathbb{E}^{\mu_n^{N,\omega}}[\cdot]$ mean that given the initial condition $X_n = x \in \mathbf{1}_{\mathcal{K}'_n}$, with $X_n \sim \mu_n^{N,\omega}$, one wants to find the difference in expectation between u_n and u_n^*. And the outer expectation $\mathbb{E}^{\omega}[\cdot]$ means averaging overall the randomness in both the measure and the SGD.

Now, by using (106) in Lemma 7.1, absorbing N in the constant C, we obtain the following

$$\mathbb{E}^{\omega}[A_1] \leq (C\Delta t L_n)^2 N M_n^2 \sup_{||q||\leq 1}\mathbb{E}^{\omega}[|\ \mu_n^{N,\omega}q - \mu_n q\ |^2] + \frac{CM_n^2}{K_n} + \frac{C}{M_n^2} + 4\delta_n^2 \tag{34}$$

By definition of the distance between two random measures, we have that :

$$\mathbb{E}[A_1] \leq (C\Delta t L_n)^2 N M_n^2 d^2(\mu_n^{N,\cdot}, \mu_n) + \frac{CM_n^2}{K_n} + \frac{C}{M_n^2} + 4\delta_n^2 \tag{35}$$

$$\Rightarrow \sqrt{\mathbb{E}[A_1]} \leq C\Delta t L_n M_n d(\mu_n^{N,\cdot}, \mu_n) + \frac{CM_n}{\sqrt{K_n}} + \frac{C}{M_n} + 2\delta_n \tag{36}$$

Since $\sqrt{\mathbb{E}[A_2]} \leq \frac{C}{M_n}$, we have that

$$\begin{aligned}
(15) &\leq 2\kappa^{-2}\Big(\frac{3}{\sqrt{N}} + C\Delta t L_n M_n d(\mu_n^{N,\cdot}, \mu_n) + \frac{CM_n}{\sqrt{K_n}} + \frac{C}{M_n} + 2\delta_n + \frac{2}{M_n} + d(\mu_n^{N,\cdot}, \mu_n)\Big) \\
\Rightarrow d(\mu_{n+1}^{N,\cdot}, \mu_{n+1}) &\leq 2\kappa^{-2}\Big((1 + C\Delta t L_n M_n)d(\mu_n^{N,\cdot}, \mu_n) + \frac{C}{M_n} + \frac{CM_n}{\sqrt{K_n}} + 2\delta_n + \frac{3}{\sqrt{N}}) \qquad (37)
\end{aligned}$$

where in (37), we have merged $\sqrt{N}$ into C. □

Remark. *(Lusin's theorem requires the underlying measure to be finite Borel regular, and in this case we are looking at the measure $\tilde{\mu}$ defined as follows: for $A \subset \mathbb{R}^n$, $\tilde{\mu}(A) = \mathbb{P}(\{\omega\Big|$ there exists n, U_0 such that $X_n(\omega) \in A\})$. $\tilde{\mu}$ is clearly a probability measure induced on the Polish space $\mathbb{R}^n$, and so it is tight by the inverse implication of the Prokhorov's theorem (or we can use the fact that all finite Borel measures defined on a complete metric space is tight). And so it is inner regular; since now $\tilde{\mu}$ is also clearly locally finite, it also implies the outer regularity.)*

Theorem 5.1: Convergence

By taking $\mu_0^N = \mu^0$, there exist $\{M_n | M_n \in \mathbb{R}, n = 0, 1, ..N-1\}$, $\{L_n | L_n \in \mathbb{R}, n = 0, 1, ..N-1\}$ and $\{\delta_n | \delta_n \in \mathbb{R}, n = 0, 1, ..N-1\}$ such that

$$d(\mu_N^{N,\cdot}, \mu_N) \leq \sum_{i=0}^{N-1} (2\kappa^{-2})^i \prod_{j=0}^{i-1} C_{N-j} \left(\frac{C}{M_{N-i}} + 2\delta_{N-i} + \frac{CM_{N-i}}{\sqrt{K_{N-i}}} + \frac{3}{\sqrt{\mathcal{N}}} \right) \tag{38}$$

where $C_j := 1 + C\Delta t L_j M_j$.
Then, for any $M > 0$, we have by picking $\{M_n\}$, $\{K_n\}$, $\mathcal{N}$ large enough and $\{\delta_n\}$ small enough, then the following hold

$$d(\mu_N^{N,\cdot}, \mu_N) \leq \frac{C}{M} \tag{39}$$

for some fixed constant C which depend only on κ.

Proof. See Appendix. □

Remark. *Notice that in Theorem 5.1, it is natural to have terms that depend on $\frac{1}{K_n}$ and $\frac{1}{\mathcal{N}}$. The presence of M_n and δ_n are due to technical difficulties. M_n basically gives the growth of the particles in the worst case scenario (we want our domain to be compact), while L_n and δ_n comes from the Lipchitz approximation for the test function f.*

6 Numerical Examples

Problem 1 *(The general setup) The cost functional is given by*

$$J[u] = \frac{1}{2}\int_0^T \sum_{i=1}^d \mathbb{E}[(y^i - y^{i,*})^2]dt + \frac{1}{2}\int_0^T \sum_{i=1}^d u^{i,2}(t)dt + \frac{1}{2}\sum_{i=1}^{d-1}(y_T^i)^2, \;\; K = U \tag{40}$$

The forward process is given by

$$dy^i(t) = u^i(t) - r^i(t)dt + \sigma u^i(t)dW_t \tag{41}$$

And one needs to find $u \in K$ such that

$$J(u^*) = \min_{u \in K} J(u)$$

where $u^{i,*}$ is the optimal control of this problem.

6.0.1 Construction of exact solutions.

An interesting fact of such example is that one can construct a time deterministic exact solution which depend only on x_0.

For simplicity, we let $d = 1$. By simplifying (40), we have

$$J[u] := \frac{1}{2}\int_0^T \mathbb{E}[y_t^2] - 2y^*\mathbb{E}[y_t^2] + y_t^{*,2} + u^2 dt + \frac{1}{2}\mathbb{E}[y_T^2] \tag{42}$$

Then, we define:

$$\begin{aligned} x_t := \mathbb{E}[y_t] &= \mathbb{E}[y_0 + \int_0^T u^i(t) - r^i(t)dt + \int_0^T \sigma u^i(t)dW_t] \\ &= \mathbb{E}[y_0 + \int_0^T u^i(t) - r^i(t)dt] \end{aligned} \tag{43}$$

Hence, we see that

$$
\begin{aligned}
\mathbb{E}[y_t^2] &= \mathbb{E}[(y_0 + \int_0^T u^i(t) - r^i(t)dt + \int_0^T \sigma u^i(t)dW_t)^2] \\
&= \mathbb{E}[\big(y_0 + \int_0^t u^i(t) - r^i(t)dt\big)^2] + \sigma^2 \mathbb{E}[\int_0^t u_t^2 dt] \\
&= x_t^2 + \sigma^2 \int_0^t u_t^2 dt
\end{aligned} \tag{44}
$$

And (44) is true because all the terms are deterministic in time given x_0. Also, we observe that

$$
\begin{aligned}
\mathbb{E}[y_T^2] &= \mathbb{E}[\Big(y_0 + \int_0^T u(t) - r(t)dt + \int_0^T \sigma u(t)dW_t\Big)^2] \\
&= x_T^2 + \sigma^2 \int_0^T u_t^2 dt
\end{aligned} \tag{45}
$$

As a result, we see that now (42) takes the form:

$$
J[u] := \frac{1}{2}\int_0^T x_t^2 - 2x_t x_t^* + x_t^{*,2} + (\sigma^2+1)u_t^2 dt + \frac{1}{2}\sigma^2 \int_0^T \int_0^t u_s^2 ds dt + \frac{1}{2}x_T^2 \tag{46}
$$

Notice that we cannot find $J'(u)$ directly using (46), however, by doing a simple integration by part, we have

$$
J[u] := \frac{1}{2}\int_0^T x_t^2 - 2x_t x_t^* + x_t^{*,2} + (\sigma^2+1)u_t^2 dt + \frac{1}{2}\sigma^2 \int_0^T (T-t)u_t^2 dt + \frac{1}{2}x_T^2 \tag{47}
$$

As a result, we have the following standard deterministic control problem:

$$
J[u] = \frac{1}{2}\int_0^T \underbrace{x_t^2 - 2x_t x_t^* + x_t^{*,2} + (\sigma^2+1)u_t^2 dt + \frac{1}{2}\sigma^2 (T-t)u_t^2}_{2R_t} dt + \frac{1}{2}x_T^2 \tag{48}
$$

$$
\frac{dx_t}{dt} = \underbrace{u_t - r_t}_{b}, \qquad x_{t_0} = x_0 \tag{49}
$$

Then, one can form the following Hamiltonian

$$
H(x,p,u) = bp + (2R) \tag{50}
$$

Then, we have

$$
\frac{\partial}{\partial x}H = \dot{p}, \quad p_T = x_T \tag{51}
$$

$$
\frac{\partial}{\partial u}H = 0 \tag{52}
$$

$$
\frac{dx_t}{dt} = u_t - r_t \qquad x_{t_0} = x_0 \tag{53}
$$

which gives us the following:

$$
\dot{p} = x_t - x_t^*, \quad p_T = x_T \tag{54}
$$

$$
\frac{dx_t}{dt} = u_t - r_t \qquad x_{t_0} = x_0 \tag{55}
$$

$$
u_t = -p_t \Big/ \Big(\sigma^2(T-t) + (1+\sigma^2)\Big) \tag{56}
$$

Then, by letting

$$
\dot{p} = x_t - x_t^* := t
$$

with $x_0 = 0$, we have the following solution according to this setup.

$$r_t := \frac{-t^2/2}{\beta_t},\ x^* = t + (\frac{T^2}{2\sigma^2} - \frac{X_T}{\sigma^2})\alpha_t,\ u_t^1 = \frac{-t^2/2 + T^2/2 - X_T}{\beta_t} \tag{57}$$

where

$$\alpha_t = \ln \frac{(1+\sigma^2) + \sigma^2 T}{(\sigma^2+1) + \sigma^2(T-t)}\ ,\ \beta_t = (1+\sigma^2) + \sigma^2(T-t),\ X_T = \frac{T^2}{2}D$$

with $D = \ln(1 + \frac{\sigma^2 T}{1+\sigma^2}) / \big(\sigma^2 + \ln(1 + \frac{\sigma^2 T}{1+\sigma^2})\big)$.

This setup however, has an analytic form only when $t_0 = 0, x_0 = 0$. If one ought to find an another exact form by following the trajectory of y_t in this setup (with the same r_t, x_t^*,) one will have to solve the following Coupled forward-backward ODE.

$$\frac{dx_t}{dt} = u_t - r_t \qquad x_n = y_{t_n} \tag{58}$$

$$\dot{p} = x_t - x_t^*, \quad p_T = x_T^{t_n, y_{t_n}} \tag{59}$$

$$\tag{60}$$

with $u_t = -p_t / \big(\sigma^2(T-t) + (1+\sigma^2)\big)$. As a result, we have

$$\frac{dx_t}{dt} = -p_t / \big(\sigma^2(T-t) + (1+\sigma^2)\big) - r_t \qquad x_n = y_{t_n} \tag{61}$$

$$\dot{p} = x_t - x_t^*, \quad p_T = x_T^{t_n, y_{t_n}} \tag{62}$$

That is, we need to solve the above coupled FBODE. Then, seeing that $p_t = x_T^{t_n, y_{t_n}} + \int_{t_n}^T x_s - r_s ds$. Writing $a_t := 1/\big(\sigma^2(T-t) + (1+\sigma^2)\big)$, we have

$$\frac{dx_t}{dt} = -a_t X_T - a_t \int_{t_n}^T (x_s - x_s^*) ds, \qquad x_{t_n} = y_{t_n} \tag{63}$$

To solve (63) numerically, we do a numerical discretization:

$$x_{t_{n+1}} - x_{t_n} = -a_{t_n} X_T \Delta t - a_{t_n} (\Delta t)^2 \sum_{i=n}^{N-1} (x_{t_i} - x_{t_i}^*)$$

$$\Rightarrow -a_{t_n}(\Delta t)^2 \sum_{i=n}^{N-1} x_{t_i}^* = x_{t_n} - x_{t_{n+1}} - a_{t_n}(\Delta t)^2 \sum_{i=n}^{N-1} x_{t_i} - a_{t_n} X_T, \quad x_{t_n} = y_{t_n} \tag{64}$$

We can put (64) into a large linear system, and solve it numerically.

6.0.2 Testing

We set the total number of discretization to be $N = 50$. We also set the following parameters $\sigma = 0.1$, r_t, x_t^* to be (57). To test our SGD algorithm (without the particle filtering part, that is assuming that we are able to observe the exact signal), we compare it with the result obtained from solving (64).

6.0.3 Testing convergence

We also present here the plot of the convergence. For such experiment, we study the error decay behavior of the algorithm in both the Sup and the L_2 norm with respect to the number of particles used. Each result is an average of 50 independent tests.

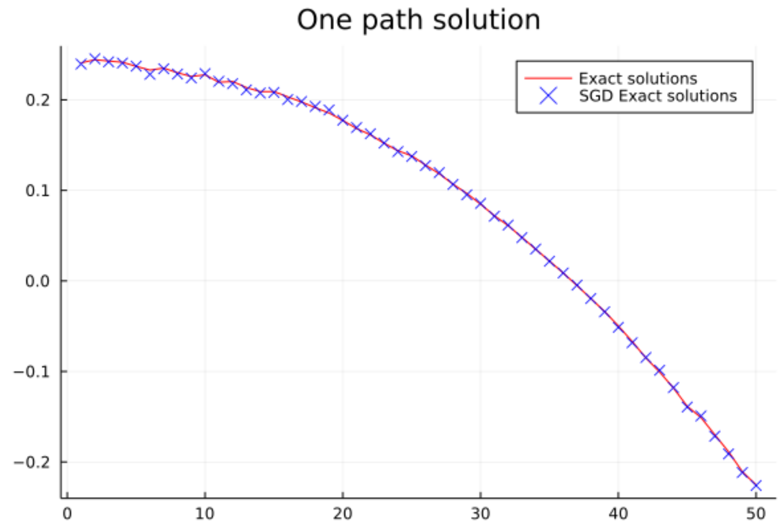


Figure 1: Comparing the exact solutions and the SGD solution with exact signal oberservation.

7 Appendix

Proof for Lemma 5.1

Proof. We start with time t_0.

Step 1. Starting from $X_0 \sim \xi$ with $\mathbb{E}[\xi^2] \leq C_0$, and by fixing an arbitrary control u_0 we have for the prediction step:

$$\begin{aligned} |X_1^-|^2 &\leq \mathbb{E}[|X_0 + b(X_0, u_0)\Delta t + \sigma(X_0)\Delta W_0|^2] \\ &\leq \mathbb{E}[(1+\Delta t)X_0^2 + (1+\frac{1}{\Delta t})b^2(\Delta t)^2] + C_\sigma^2 \Delta t \\ &\leq (1+\Delta t)C_0^2 + (C_b^2(\Delta t + 1) + C_\sigma^2)\Delta t \\ &:= C_0^- \end{aligned} \tag{65}$$

Step 2. We denote the distribution $\mathcal{L}(X_1^-) \sim \pi_{t_1|t_0}$, then the particle method will do a random resampling from such distribution, and obtain a random distribution

$$\frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \delta_{x_i(\omega)} := \pi_{t_1|t_0}^N \tag{66}$$

Hence, we have for $X \sim \pi_{t_1|t_0}^N$, take expectation where the expectation is taken over all randomness in the measure

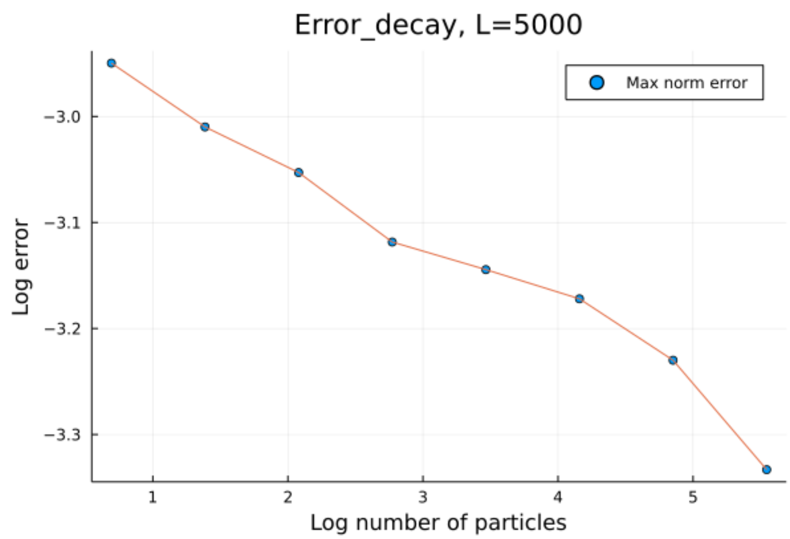
$$\begin{aligned} \mathbb{E}[X^2] &= \mathbb{E}\big[\mathbb{E}[X^2|\mathcal{G}_1^-]\big] \qquad (67) \\ &= \frac{\mathcal{N}}{\mathcal{N}} \mathbb{E}[\mathbb{E}[x_1^2|\mathcal{G}_1^-]] \\ &= \mathbb{E}[\tilde{X}^2] \qquad (68) \end{aligned}$$

where $x_i \sim \pi_{t_1|t_0}$ are i.i.d random samples, $\mathcal{G}_0$ contains the sampling randomness, and $\tilde{X} \sim \pi_{t_1|t_0}$. The conditional expectation is meant to show that all the particles ξ are conditionally independent (since there are other randomness accumulated in the history if we want to apply this argument recursively.) Thus, by (65)

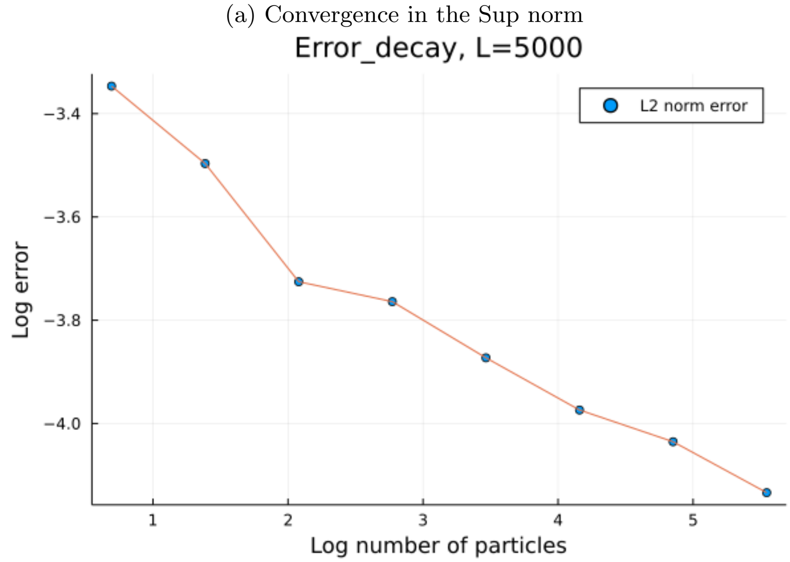
$$\mathbb{E}[\tilde{X}^2] \leq C_0^- \tag{69}$$

Step 3. We now have the random measure $\pi_{t_1|t_0}^N$, and we proceed to the analysis step. We have by definition

$$X_1 \sim \frac{g(x)d\pi_{t_1|t_0}^N(x)}{\int g(x)d\pi_{t_1|t_0}^N(x)} := \tilde{\pi}_{t_1|t_1}^N \tag{70}$$



(a) Convergence in the Sup norm



(b) Convergence in the L_2 norm

where $\pi^N_{t_1|t_0}(x)$ is the distribution of the terminal state X_1^- from the previous step. We give an estimate over $\mathbb{E}[|X_1|^2]$:

$$\begin{aligned}\mathbb{E}[|X_1|^2] &\leq (\frac{1}{\kappa})^2 \mathbb{E}[\int x^2 d\pi^N_{t_1|t_0}(x)] \\ &\leq (\frac{1}{\kappa})^2 C_0^- \\ &:= C_1 \end{aligned} \tag{71}$$

Step 4. Now we again apply the random sampling step

$$\frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \delta_{x_i(\omega)} := \pi^N_{t_1|t_1} \tag{72}$$

where $x_i(\omega) \sim \tilde{\pi}^N_{t_1|t_1}$. Then, for $X \sim \pi^N_{t_1|t_1}$, we have

$$\mathbb{E}[X^2] = \mathbb{E}\big[\mathbb{E}[X^2 \big| \mathcal{G}_1]\big] \tag{73}$$
$$= \frac{\mathcal{N}}{\mathcal{N}} \mathbb{E}[\mathbb{E}[x_i^2 \big| \mathcal{G}_1]]$$
$$= \mathbb{E}[\tilde{X}^2] \tag{74}$$

where $\tilde{X} \sim \tilde{\pi}^N_{t_1|t_1}$ and $\mathcal{G}_1$ is the Filtration that builds on $\mathcal{G}_1^-$ and the randomness of the current sampling. Then, by (71), we have

$$\mathbb{E}[X^2] \le C_1, \quad X \sim \tilde{\pi}^N_{t_1|t_1} \tag{75}$$

And this completes all the estimates for the first time-stepping.

Hence, after one time step, we have

$$C_1 = \kappa^{-2}(1 + \Delta t)C_0 + C\Delta t \tag{76}$$

which means that by applying the same argument, we will have the following recursion in general:

$$C_{n+1} = \kappa^{-2}(1 + \Delta t)C_n + C\Delta t \tag{77}$$

As a result, by picking arbitrary u_n, using this same argument repeatedly until N, we have that for all $n = 1, ..., N$:

$$C_n = (\kappa^{-2}(1 + \Delta t))^n C_0 + \sum_{i=0}^{n-1} (\kappa^{-2}(1 + \Delta t))^i C\Delta t \tag{78}$$

And we notice that C_n is increasing in n. As a result, we know that for any $X_n \sim \pi^N_{t_n|t_{n-1}}, \pi^N_{t_n|t_n}$, we have that

$$\mathbb{E}[|X_n|^2] \le C_N \tag{79}$$

Hence, by the Chebyshev's inequality, we have

$$\mathbb{P}(|X_n| \ge M) \le \frac{C_N}{M^2}, \quad \forall n \in \{1, 2, ..., N\} \tag{80}$$

then we have that

$$\mathbb{P}(\sup_n |X_n| \ge M) \le \frac{C_N}{M^2} \tag{81}$$

By noticing that since the control values are arbitrarily picked, we have that

$$\mathbb{P}(\sup_{n, U_0} |X_n| \ge M) \le \frac{C}{M^2}, \quad X_n \sim \pi^N_{t_n|t_{n-1}} or \;\; \pi^N_{t_n|t_n} \tag{82}$$

□

Proof for Theorem 5.1

Proof. With C_n defined as

$$C_n := 1 + C\Delta t L_n M_n \tag{83}$$

And by using (37) repeatedly, we obtain the following result:

$$d(\mu_N^{N,\cdot}, \mu_N) \le \sum_{i=0}^{N-1} (2\kappa^{-2})^i \prod_{j=0}^{i-1} C_{N-j} \Big(\frac{C}{M_{N-i}} + \frac{CM_{N-i}}{K_{N-i}} + 2\delta_{N-i} + \frac{3}{\sqrt{\mathcal{N}}}\Big) + (2\kappa^{-2})^N \prod_{j=0}^{N-1} C_{N-j} d(\mu_0^N, \mu_0) \tag{84}$$
$$\le \sum_{i=0}^{N-1} (2\kappa^{-2})^i \prod_{j=0}^{i-1} C_{N-j} \Big(\frac{C}{M_{N-i}} + \frac{CM_{N-i}}{K_{N-i}} + 2\delta_{N-i} + \frac{3}{\sqrt{\mathcal{N}}}\Big) \tag{85}$$

Since we know that $d(\mu_0^N, \mu_0) = 0$. Now, we just need to show that (85) vanishes when K_l, N gets large and δ_i gets small, $i \in \{0, 1, ..., N\}$. Notice that M_l comes from the domain truncation for each time step and δ_l comes from the uniform approximation which are free to choose. The choice of δ_l will potentially determine the value of L_n.

We fix $M_N := NM, \delta_N := \frac{1}{NM}$ where N is the number of time discretization and M is potentially a large number.

Then, we define δ_l, M_l through the following:

$$(2\kappa^{-2})^{i+1} \prod_{j=0}^{i} C_{N-j} 2\delta_{N-i-1} = (2\kappa^{-2})^{i} \prod_{j=0}^{i-1} 2\delta_{N-i} \tag{86}$$

$$(2\kappa^{-2})^{i+1} \prod_{j=0}^{i} C_{N-j} \frac{C}{M_{N-i-1}} = (2\kappa^{-2})^{i} \prod_{j=0}^{i-1} C_{N-j} \frac{C}{M_{N-i}} \tag{87}$$

Here we define $C_{N+1} \equiv 1$.

Notice that should iterate (**??**) and (87) iteratively, since defining δ_i will lead to the lipschitz constant L_i at stage i, which is needed for the definition for C_i.

Then we have that

$$\begin{aligned}\sum_{i=0}^{N-1} (2\kappa^{-2})^{i} \prod_{j=0}^{i-1} C_{N-j} \frac{C}{M_{n-i}} &\leq N \frac{C}{NM} \\ &\leq \frac{C}{M}\end{aligned} \tag{88}$$

And we also have

$$\begin{aligned}\sum_{i=0}^{N-1} (2\kappa^{-2})^{i} \prod_{j=0}^{i-1} C_{N-j} 2\delta_{n-i} &\leq N \frac{1}{NM} \\ &\leq \frac{1}{M}\end{aligned} \tag{89}$$

By picking K_{N-i} to be large, we then can have

$$\begin{aligned}\sum_{i=0}^{N-1} (2\kappa^{-2})^{i} \prod_{j=0}^{i-1} C_{N-j} \frac{CM_{N-i}}{K_{N-i}} &\leq N \frac{1}{NM} \\ &\leq \frac{1}{M}\end{aligned} \tag{90}$$

Last but not least, by taking $\mathcal{N}$ so large such that

$$\left(\sum_{i=0}^{N-i} (2\kappa^{-2})^{i} \prod_{j=0}^{i-1} C_{N-j}\right) \frac{3}{\sqrt{\mathcal{N}}} \leq \frac{1}{M} \tag{91}$$

we can see that (85) converges to 0 by taking M to be very large. □

7.1 Optmization proofs

Now we describe the difference between u_n and u_n^*. Let

$$\mathcal{G}_k := \{\Delta W_n^i, x^i\}_{i=0}^{k-1}$$

We can see that knowing $\mathcal{G}_K$ essentially means that we know the control U^K in the SGD framework, since according to our scheme the control is $\mathcal{G}_K$ measurable.

Should fix the notation later, it is probably more preferable to use $\mathcal{G}_k^n$ to mean that we are collecting past information until time k at time stage n.

Lemma 7.1. *Under a fixed temporal discretization number N, with the particle cloud $\mu^{N,\omega}$, a deterministic u_n^* and a compact domain $\mathcal{K}'_n$, (such that $\mathbb{E}^\omega\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}'^c_n}] \le \frac{C}{M_n^2}$ and $diam(\mathcal{K}'^c_n) \le M_n$), we have for any iteration number K, the following hold*

$$\mathbb{E}^\omega\left[\mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbf{1}_{\mathcal{K}'_n}|u_n^\omega - u_n^*|^2\big|X_n = x\right]\right] \le CM_n^2 \sup_{||q||_\infty \le 1} \mathbb{E}^\omega[|\ \mu_n^{N,\omega}q - \mu_n q\ |^2] + \frac{C}{M_n} + \frac{CM_n^2}{K} \tag{92}$$

Remark: The value of $\sup_{||q||_\infty\le 1}\mathbb{E}^\omega[|\ \mu_n^{N,\omega}q - \mu_n q\ |^2]$ depends on $\mu_n^{N,\omega}$ which is obtained from the previous step, and it does not depend on the current M_n. As a result, we can see that, as long as

$$\sup_{||q||_\infty \le 1} \mathbb{E}^\omega[|\ \mu_n^{N,\omega}q - \mu_n q\ |^2] \to 0$$

(92) can be made arbitrarily small on any compact domain $\mathcal{K}'_n$ And this indicates the point-wise convergence for U at any time t_n.

Proof.

$$U_n^{K+1} = U_n^K - \eta_k j_n'^{x^k}(U_n^K) \tag{93}$$
$$U_n^* = U_n^* - \eta_k \mathbb{E}^{\mu_n}[J_N'^x(U_n^*)] \tag{94}$$

where x^k is drawn from the current distribution $\mu_n^{N,\omega}$ and $\mathbb{E}^{\mu_n}[J_N'^x(U_n^*)] = 0$ by the optimality condition. Take the difference between (93) and (94), square both sides and take conditional expectation $\mathbb{E}[\cdot|\mathcal{G}_K]$, and this conditional expectation is taken with respect to two randomness:

1. The randomness coming from the selection on the initial point x_n^k.
2. The randomness coming from the pathwise approximated Brownian motion used for FBSDE.
3. The randomness coming from the accumulation of the past particle sampling.

We know from our previous work that given a fixed initial point x_0,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^{x_0}}[j'^{x_0}(U^K)|\mathcal{G}_K] &= \mathbb{E}^{\mathbb{P}^{x_0}}[J_N'^{x_0}(U^K)|\mathcal{G}_K]\\ &= J_N'^{x_0}(U^K)\end{aligned} \tag{95}$$

Hence, we can write $\mathbb{E}[j'^x(U^K)|\mathcal{G}_K] = \mathbb{E}^{\mu_n^{N,\omega}}[J_N'^x(U^K)|\mathcal{G}_K]$, which can be seen from the following

$$\mathbb{E}[j'^x(U^K)|\mathcal{G}_K] = \mathbb{E}^{\mu_n^{N,\omega}}[\mathbb{E}^{\mathbb{P}^x}[j'^x(U^K)\Big|\mathcal{G}_K]] \tag{96}$$
$$= \mathbb{E}^{\mu_n^{N,\omega}}[J_N'^x(U^K)|\mathcal{G}_K] \tag{97}$$

Then, by taking the square norm on both sides, multiply by an indicator function $\mathbf{1}_{\mathcal{K}'_n}$ and take conditional expectation $\mathbb{E}[\cdot|\mathcal{G}_K]$, noticing that U_n^K is $\mathcal{G}_K$ measurable and U^* is deterministic in this case, we get

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{\mathcal{K}'_n}||U_n^{K+1} - U_n^*||^2\Big|\mathcal{G}_K] &= \mathbb{E}[\mathbf{1}_{\mathcal{K}'_n}||U_n^K - U_n^*||^2\Big|\mathcal{G}_K] - 2\eta_k\langle\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}'_n}J_N'^x(U_n^K)\Big|\mathcal{G}_K] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}'_n}J_N'^x(U_n^*)], U_n^K - U_n^*\rangle\\
&+ \eta_K^2\mathbb{E}[\mathbf{1}_{\mathcal{K}'_n}||j'^x(U_n^K) - \mathbb{E}^{\mu_n}[J_N'^x(U_n^*)]||\Big|\mathcal{G}_K] \qquad (98)\\
&= \mathbb{E}[\mathbf{1}_{\mathcal{K}'_n}||U_n^K - U_n^*||^2\Big|\mathcal{G}_K] - 2\eta_k\mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{K}'_n}\langle J_N'^x(U_n^K) - J_N'^x(U_n^*)\\
&+ J_N'^x(U_n^*) - \mathbb{E}^{\mu_n}[J_N'^x(U_n^*)], U_n^K - U_n^*\rangle\Big|\mathcal{G}_K\Big] + \eta_k^2\mathbb{E}^{\mu_n^N}[\mathbf{1}_{\mathcal{K}'_n}||j'^x(U_n^K) - \mathbb{E}^{\mu_n}[J_N'^x(U_n^*)]||]\\
&= \mathbb{E}[\mathbf{1}_{\mathcal{K}'_n}||U_n^K - U_n^*||^2\Big|\mathcal{G}_K] - 2\eta_k\mathbb{E}^{\mu_n^{N,\omega}}[\langle\mathbf{1}_{\mathcal{K}'_n}J_N'^x(U_n^K) - \mathbf{1}_{\mathcal{K}'_n}J_N'^x(U^*), U_n^K - U_n^*\rangle\Big|\mathcal{G}_K]\\
&- 2\eta_k\langle\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}'_n}J_N'^x(U_n^*)] - \mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}'_n}\mathbb{E}^{\mu_n}[J_N'^x(U_n^*)]], \mathbf{1}_{\mathcal{K}'_n}(U_n^K - U_n^*)\rangle\\
&+ \eta_k^2\mathbb{E}[\mathbf{1}_{\mathcal{K}'_n}||j'^x(U_n^K) - \mathbb{E}^{\mu_n}[J_N'^x(U_n^*)]||\Big|\mathcal{G}_K]\\
&\le (1-\lambda\eta_k)\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}'_n}||U_n^K - U_n^*||^2\Big|\mathcal{G}_K] + \frac{\eta_k}{\lambda}\underbrace{||\mathbb{E}^{\mu_n^{N,\omega}}\Big[\mathbf{1}_{\mathcal{K}'_n}J_N'^x(U_n^*) - \mathbf{1}_{\mathcal{K}'_n}\mathbb{E}^{\mu_n}[J_N'^x(U_n^*)]\Big]||^2}_{**}\\
&+ \eta_k^2\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}'_n}C|x_i|^2 + C]
\end{aligned}$$

where in the last line we used Lemma 7.5.

Recall that $\mathbb{E}^{\mu_n}[J_N'^x(U^*)] = 0$, we then have

$$
\begin{aligned}
** &\le ||\mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\right] - \mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbb{E}^{\mu_n}[J_N'^x(U^*)]\right] + \mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbb{E}^{\mu_n}[J_N'^x(U^*)]\right] - \mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbf{1}_{\mathcal{K}_n'}\mathbb{E}^{\mu_n}[J_N'^x(U^*)]\right]||^2 \qquad (99)\\
&\le ||\mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\right] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)] + \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)] - \mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbb{E}^{\mu_n}[J_N'^x(U^*)]\right]||^2\\
&\le (1+\epsilon)||\mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\right] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)]||^2 + (1+\frac{1}{\epsilon})||\mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)] - \mathbb{E}^{\mu_n}[J_N'^x(U^*)]||^2\\
&\le C||\mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\right] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)]||^2 + \frac{C}{M_n} \qquad (100)
\end{aligned}
$$

Then we take expectation on both sides over the randomness and we have

$$
\begin{aligned}
\mathbb{E}[\mathbf{1}_{\mathcal{K}_n'}||U^{K+1} - U^*||^2] &\le (1-\lambda\eta_k)\mathbb{E}^{\mu_n^N}[||U^k - U^*||^2] + \frac{\eta_k}{\lambda}\mathbb{E}^{\omega}[C||\mathbb{E}^{\mu_n^{N,\omega}}\left[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\right] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)]||^2\\
&\quad + \frac{C}{M_n}] + \eta_k^2 C M_n^2\\
&\le \frac{||U^0 - U^*||^2}{K} + C\mathbb{E}^{\omega}[||\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)||_2^2] + \frac{C}{M_n} + \frac{CM_n^2}{K} \qquad (101)
\end{aligned}
$$

Notice that for the control U^* we know that for for a fixed x, $J_N'^x(U^*)$ is uniformly bounded:

$$
\begin{aligned}
||\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)]||_2^2 &= \sum_{i=n}^{N}\Delta t\left|\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\big|_i] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\big|_i]\right|^2\\
&\le \sum_{i=n}^{N}\Delta t\sup_{j\in\{n,\dots,N\}}\left|\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}_n'} J_N'(U^*)(x)\big|_j] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'(U^*)(x)\big|_j]\right|^2\\
&\le \sup_{j\in\{n,\dots,N\}}\left|\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\big|_j] - \mathbb{E}^{\mu_n}[\mathbf{1}_{\mathcal{K}_n'} J_N'^x(U^*)\big|_j]\right|^2 \qquad (102)
\end{aligned}
$$

However, since by Lemma 7.5:

$$
\sup_{j\in\{n,\dots,N\}}\left|J_N'^x(U^*)\big|_j\right|^2 \le C|x|^2 + C \qquad (103)
$$

we have that

$$
\mathbf{1}_{\mathcal{K}_n'}\sup_{j\in\{n,\dots,N\}}\left|J_N'^x(U^*)\big|_j\right|^2 \le C|M_n|^2\ \mathbf{1}_{\mathcal{K}_n'}|q(x)| \qquad (104)
$$

for some $q(x)$, where $||q(x)||_\infty \le 1$. As a result, we see that

$$
\begin{aligned}
(101) &\le CM_n^2\mathbb{E}^{\omega}\left[\left|\mathbb{E}^{\mu_n^{N,\omega}}[q(x)] - \mathbb{E}^{\mu_n}[q(x)]\right|^2\right] + \frac{CM_n^2}{K} \qquad (105)\\
&\le CM_n^2\sup_{||q||_\infty\le 1}\mathbb{E}^{\omega}[|\ \mu_n^{N,\omega}q - \mu_n q\ |^2] + \frac{CM_n^2}{K} \qquad (106)
\end{aligned}
$$

Thus, we have

$$
\mathbb{E}^{\omega}\left[\mathbb{E}^{\mu_n^{N,\omega}}[\mathbf{1}_{\mathcal{K}_n'}\sup_n|u^{K+1} - u^*|^2]\right] \le CM_n^2\sup_{||q||_\infty\le 1}\mathbb{E}^{\omega}[|\ \mu_n^{N,\omega}q - \mu_n q\ |^2] + \frac{C}{M_n} + \frac{CM_n^2}{K} \qquad (107)
$$

where we have absorbed the constant term N in C. □

We point out that the result (106) says that, the difference between U^{K+1} and U^* under the expectation restricting to a compact set will be very small when the number of iterations becomes large and the distribution μ_n^N gets "closer" to μ_n in some sense.

7.2 Other auxiliary lemmas

Lemma 7.2. *For any control $U_n \in \mathcal{U}$, we have*

$$\sup_n \mathbb{E}^{\mu_n}[|J_N'^{X_n}(U)\big|_n|^2] < \infty \tag{108}$$

Proof. By definition, we have

$$J_N'^{x_n}(U)\big|_n = b_u'(X_n^{x_n}, u_n)Y_n^{N,x_n} + f_u'(X_n^{x_n}, u_n)$$

then, since we know that

$$\mathbb{E}[\sup_n |Y_n^{N,x_n}|^2] \leq C|x_n|^2 + C \tag{109}$$

and that b_u', f_u' are uniformly bounded, we have that

$$|J_N'^{x_n}(U)\big|_n|^2 \leq C|x_n|^2 + C \tag{110}$$

Now, we have that

$$\begin{aligned} \mathbb{E}^{\mu_n}[|J_N'^{X_n}(U)\big|_n|^2] &\leq C\mathbb{E}^{\mu_n}[|x_n|^2] + C \\ &\leq C \end{aligned} \tag{111}$$

by the bound we have in the previous lemma. $\square$

As a quick corollary, we know that

$$\sup_n |\mathbb{E}^{\mu_n}[J_N'^{X_n}(U)\big|_n]|^2 \leq C \tag{112}$$

and such result tells us that the gradient that we compute at each time step will not blow up as we change the measure and the underlying initial distribution for x_n.

Without loss of generality we prove the following result

Lemma 7.3. *For any $U_n \in \mathcal{U}_n$, we have the following result.*

$$\Big|J_N'^{x}(U_n^*)\Big|^2 \leq C|x|^2 + C \tag{113}$$

Proof. Recall the definition of $J_N'^{x_n}(U_n)$, since the control does not appear in the diffusion term, we have

$$J_N'^{x}(U_n)\big|_j := \mathbb{E}\Big[b_u(X_n^{N,x}\big|_j, U_n\big|_j)Y_j^{N,x} + f_u'\big|_j\Big],\ j \in \{n, ..., N\} \tag{114}$$

Then, we have

$$|J_N'^{x}(U_n)\big|_j|^2 \leq 2C\mathbb{E}\Big[\sup_{j\in\{n,...,N\}} |Y_j^{N,x}|^2\Big] + C \tag{115}$$

$$\leq 2C(C|x|^2 + C) + C \tag{116}$$

$$\leq C|x|^2 + C$$

where (116) holds due to the uniform Lipschitz assumption on b and f, and holds due to the estimates *Theorem 4.2.1* and *Theorem 5.3.3* in [2]. Taking sup over j on the lefthand side, and we conclude. $\square$

This is a copy and paste from the previous writeup

Lemma 7.4. *Under Assumption (a) − (e), for any $u^N \in K_N$ we have*

$$\sup_{0\leq n\leq N} \mathbb{E}[|Y_n|^2] \leq C, \quad \sup_{0\leq n\leq N} \mathbb{E}[|Z_n|^2] \leq CN$$

for some positive constants C.

Proof. For notational simplicity, let $\big(\alpha(u_t,x),\gamma(u_t,x)\big)$ denote $\big(f_x(u_t,x),r_x(u_t,x)\big)$, then we know that by assumption there exists a constant C such that $||\alpha(u_t,x)||_\infty \le C, ||\gamma(u_t,x)||_\infty \le C$.

Now (Y_n,Z_n) are defined recursively in the following way:

$$Y_n = Y_{n+1} + h(\alpha_n Y_{n+1} + \gamma_n) \tag{117}$$

$$Z_n = \frac{Y_{n+1}\Delta W_{n+1}}{h} \tag{118}$$

and we have the following simple estimate on (118):

$$\begin{aligned}\mathbb{E}[|Y_n|^2] &\le \mathbb{E}[(1+\epsilon)(1+\alpha_n h)|Y_{n+1}|^2] + (1+\frac{1}{\epsilon})\mathbb{E}[|\gamma_n|^2h^2] \\ &\le (1+Ch)\mathbb{E}[|Y_{n+1}|^2] + Ch\end{aligned}$$

Then, by the discrete Gronwall inequality

$$\sup_{0\le n\le N}\mathbb{E}[|Y_n|^2] \le C\mathbb{E}[|Y_N|^2] + C \tag{119}$$

Then, since the terminal condition Φ_x also satisfies the Lipschitz condition, we have

$$\sup_{0\le n\le N}\mathbb{E}[|Y_n|^2] \le C \tag{120}$$

With such inequality, we also derive an estimate on Z_n:

$$\begin{aligned}\mathbb{E}[|Z_n|_2^2] &\equiv \mathbb{E}[|\frac{Y_n\Delta W_{n+1}}{h}|_2^2] \\ &\le \frac{2}{h}(\sup_n \mathbb{E}[|Y_n|_2^2] + 1) \\ &\le CN \end{aligned} \tag{121}$$

□

Now with the estimate above, we are ready for the following lemma.

Lemma 7.5.

$$\mathbb{E}[||j'^x(U_n)||^2] \le C|x|^2 + C \tag{122}$$

Proof. Recall the definition of $j'^x(U_n)$, we have

$$||j'^x(U_n)||^2 \le C\sup_{j\in\{n,\ldots,N\}}|Y_j^x|^2 + C \tag{123}$$

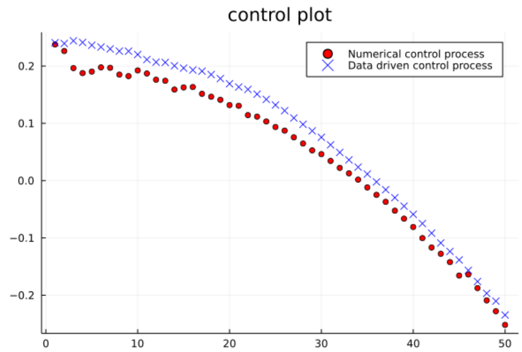
taking expectation on both side and use Lemma 7.4, we have that

$$\begin{aligned}\mathbb{E}[||j'^x(U_n)||^2] &\le C\mathbb{E}[\sup_{j\in\{n,\ldots,N\}}|Y_j^x|^2] + C \\ &\le C|x|^2 + C\end{aligned} \tag{124}$$

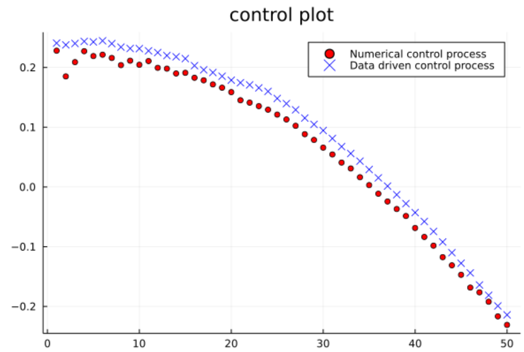
where (7.5) follows again from estimates *Theorem 4.2.1* and *Theorem 5.3.3* in [2]. □

7.2.1 Convergence tests, single plots

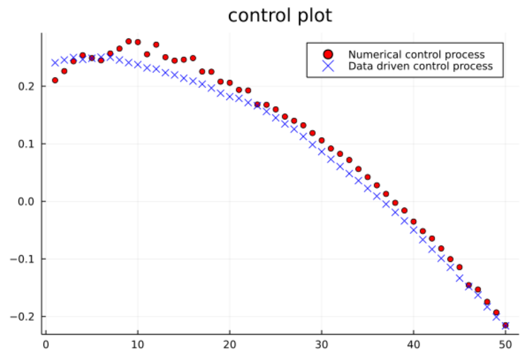
In this subsection, we selectively pick some values for S and compare their behavior



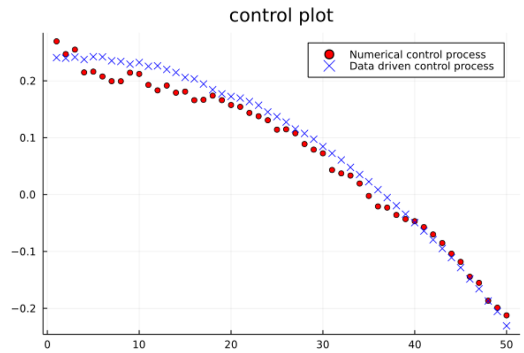
(a) $S = 2$



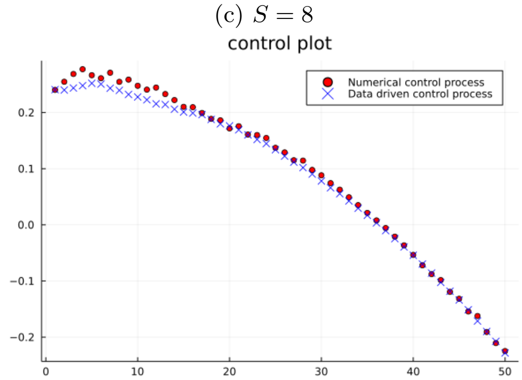
(b) $S = 4$



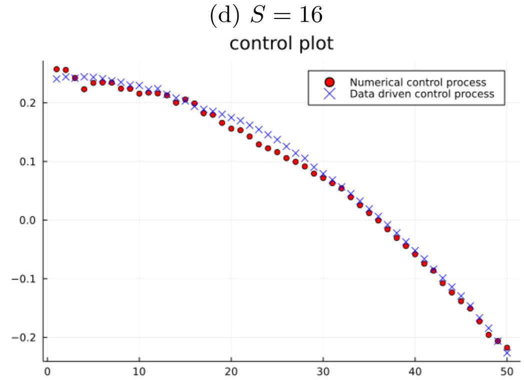
(c) $S = 8$



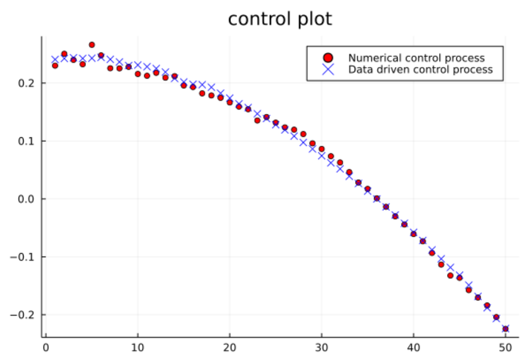
(d) $S = 16$



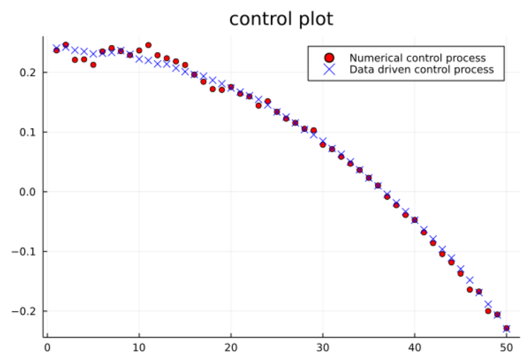
(e) $S = 32$



(f) $S = 64$



(g) $S = 128$



(h) $S = 512$

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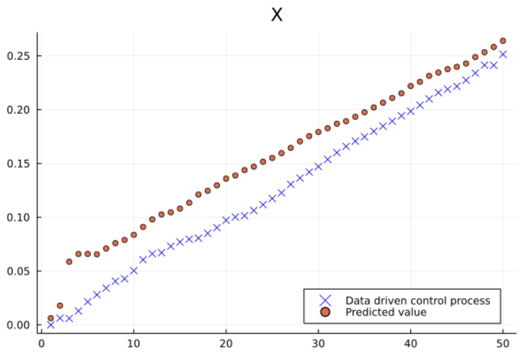
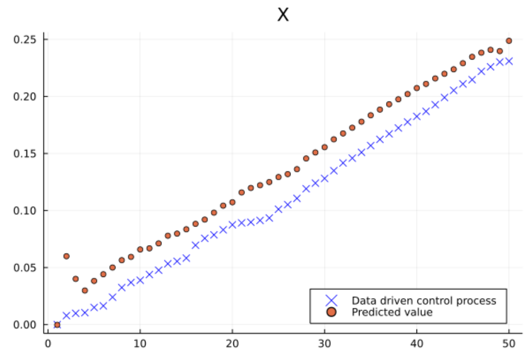
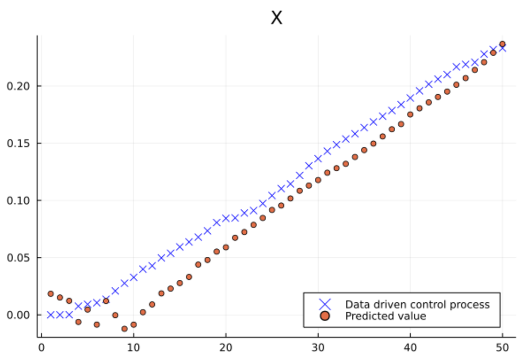


(a) $S = 2$



(b) $S = 4$



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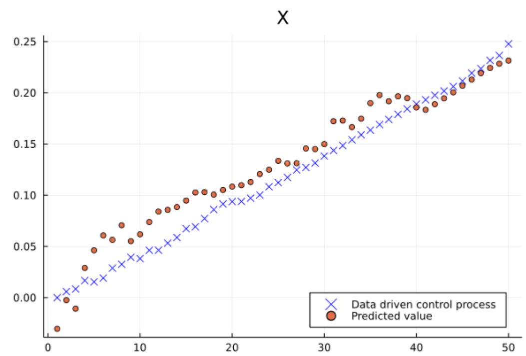

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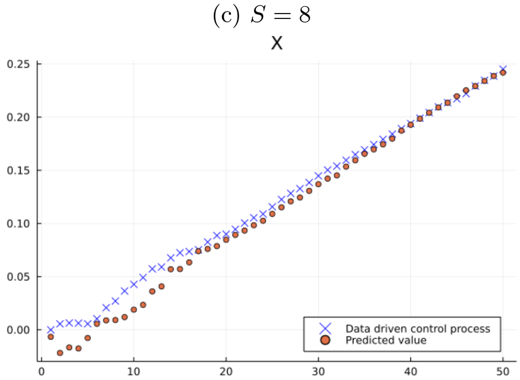

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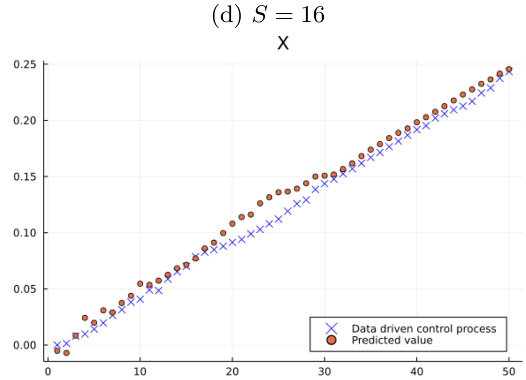

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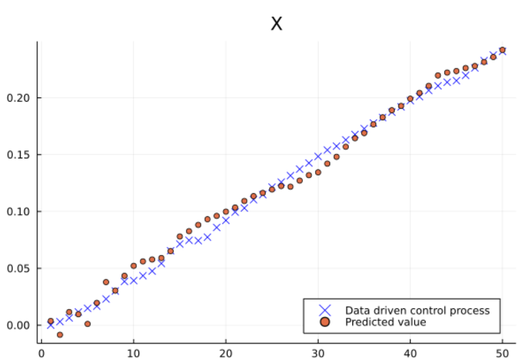
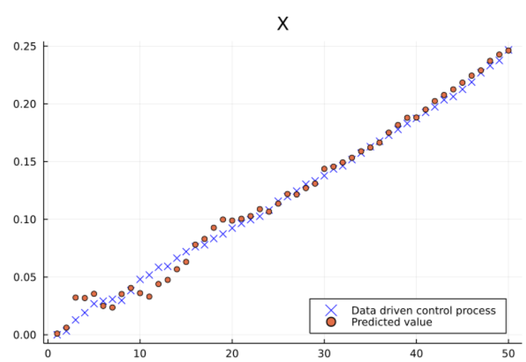


(g) $S = 128$



(h) $S = 512$

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