Equations for the surface water - groundwater interaction.

Derivation of formula's for the cell-drain-resistance referred to as DL2020.

Version 0.6.0

0. Foreword

This document is the result of a long-term development of theory and practice that started in the early 1990-ties. It is not yet (2025) a well-completed report, but more a state of the art of the knowledge, the background and the derivation of the equations describing the surface water groundwater interaction boundary condition in modeling. It is intended for those who are interested in the use and background of the so-called De Lange (1997, 2020) formulas. In the upcoming time, explanations will be elaborated and extensions from new research will be added. The list of references contains the reviewed publications.

Acknowledgment to group (Jan van Bakel, Huite Bootsma, Jacco Hoogewoud, Paul Torfs).

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1. Introduction

1.1 The Robin boundary condition for a top-system of a groundwater model.

A boundary of a model is no more or less than a one-dimensional representation of the world outside the model domain. A calculation node in a model represents an area at the boundary, such as schematically represented by the circular connection-area between the cylinder and the box in the figure below, in which the interaction between the inside and outside world is described with area-averaged parameters being constants and variables.

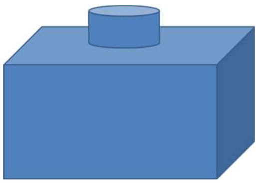


Figure 1.1 Scheme of a model boundary represented by the circular connection-area between the cylinder representing a part of the outside world and the rectangular box as the model domain.

In a model for saturated groundwater flow, the boundary condition represents the interaction between the groundwater flow domain simulated in the model domain and the outside world that occurs in the so-called top system containing a surface water and flow to or from an underlying regional aquifer, see the figure below.

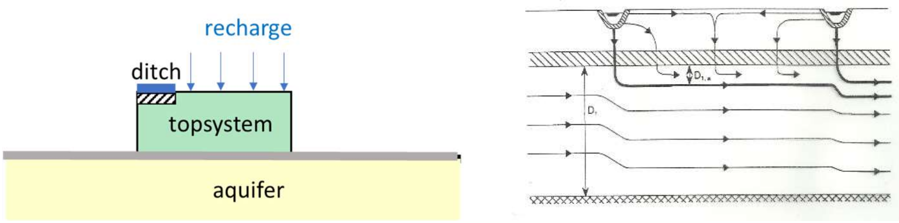


Figure 1.2 Scheme: Interaction between the model domain and the outside world (left), in a top-system containing flow to or from a surface water from or to an underlying regional aquifer (right).

Aim of present work is to derive equations for a Robin boundary condition, a linear relation between the potential difference and flux across the boundary surface represented by a calculation point. A well-known example of a Robin condition is Ohm's law and also a version of Darcy's law, in which the resistance as a constant is an important parameter. Using the principle of superposition, a combination of linear relations can well approximate non-linear behavior as observed in the field. Transient behavior can be simulated by using the method of successive steady states.

The main part of this work in chapters 2, 3 and 4 concerns the equations for resistance in the DL2020 formula-series as it occurs in the phreatic aquifer under the Dupuit assumption, so neglecting resistance from vertical flow in the aquifer. This assumption will be corrected by additional terms for complementary resistances using superposition in chapter 5.

1.2 Characterizing three-dimensional flow for use a conceptual model in a vertical plane.

The conceptual model used in the present work describes stationary, saturated flow to a surface water in a vertical plane. The two-dimensional vertical plane is conceptualized to occur along three-dimensional flow lines between surficial infiltration or seepage and a surface water. The third dimension, in the horizontal plane, is represented by a single parameter, the average distance between surface waters. In chapter 7 of the PhD thesis (De Lange,1996), it is shown that the average distance between surface waters in an area with an arbitrary distribution can well be represented by the inverse of the drainage density. The drainage density equals the length of the surface waters divided by the surface of the observed area. In surface hydrology, the drainage density is a well-known and often used parameter to characterize a complex drainage system, such as a river with multiple branches.

The idea of using a vertical section along flow lines and the derivation of the average distance between surface water will be illustrated at the hand of the pictures of the PhD thesis.

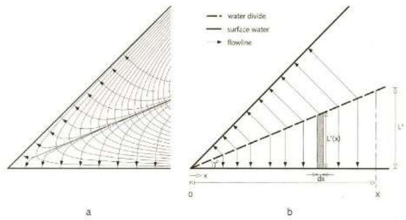


Figure 1.3 Path lines of infiltration water starting at the water divide flowing to surface waters in an angle (left); Vertical section of top-system between water divide and surface water (right).

In figure 1.3 left, the constant infiltration in the area between two surface waters forming an angle generates flow in a single aquifer that represents the top-system in figure 1.2. The path lines start at the water divide. Near the two surface waters the path lines form in virtually straight lines perpendicular to the direction of the surface water. The top system of figure 1.2 is conceptualized in a sectional model perpendicular to drain in figure 1.3 right. The patterns in figure 1.2 can be combined to surface waters in the shape of a diamond, figure 1.4 left. The pattern in a parallelogram, figure 1.4 right, includes zones with flow between parallel surface waters. Essentially, in this way many closed shapes of surface water can be covered by these two pattern, being flow to surface waters in a corner and between parallel surface waters.

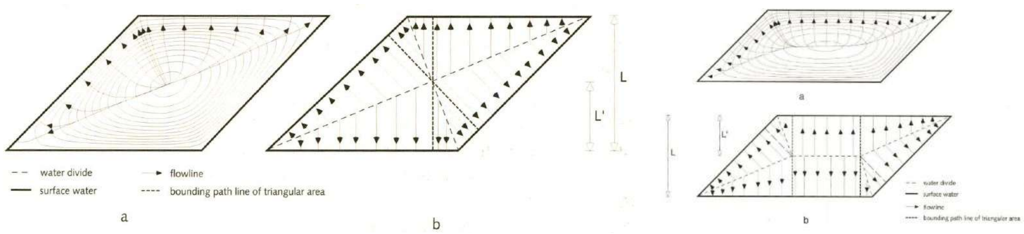


Figure 1.4 Path lines of infiltration water starting at the water divide flowing to surface waters in shape of a diamond (left) and parallellogram (right) schematized as in figure 1.3 in adjacent pictures.

In a case of open shapes of the surface waters, the flow concentrates to the endpoints. In this case the vertical section containing the top-system in figure 1.2 follows a radius as in flow to a well, figure 1.5.

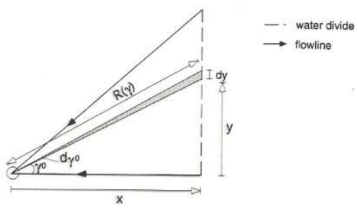
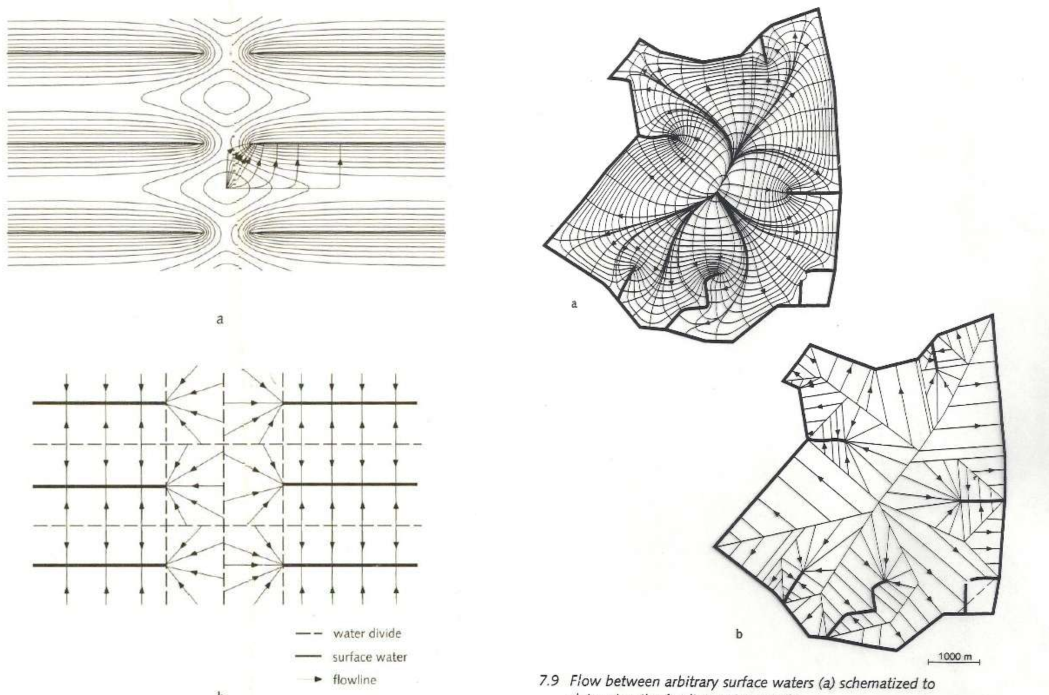


Figure 1.5 Scheme of vertical section of top-system between a straight water divide and well representing an end-point of a surface water.

The case of open surface waters is illustrated in figure 1.6 left, which shows that the radial flow is a reasonable representation of the flow to the endpoints of the surface waters.



7.9 Flow between arbitrary surface waters (a) schematized to determine the feeding resistance (b).

Figure 1.6 Path lines in the case of infiltration water starting at the water divide flowing to not continuous parallel surface waters (left) and in the case of surface waters in a field situation (right) schematized in adjacent pictures as in figure 1.3.

The distance between the surface waters grows linear with the length of the surface waters in an angle. Therefore, the average distance between the surface waters equals the surface of the triangular area divided by the length of its two sides. In general, as in the case of figure 1.6 right, every surface water has inflow from both sides. Therefore, inflow from one side as in the figures 1.2 – 1.4 can be translated as the surface water being effective over half its length. For a close shape of surface waters the average distance between surface waters can be derived directly from the drainage density as described above. The average distance between surface water is a major parameter in the equations describing the resistance in the Robin boundary condition, as it will follows in the derivations later on.

From the analysis of De Lange 1996, the radial flow to an endpoint is much less relevant for the resistance in the Robin condition than de lengths of the combined surface waters for the general drainage of an area, as shown in figure 1.6 right.

Therefore, for arbitrary shapes of surface waters the average distance between surface waters can be derived directly from the drainage density as described at the beginning of this section. By this relation the three-dimensional flow has been reduced to a two-dimensional flow in a vertical section in which the average distance between the surface waters is used.

Different surface water systems in 2D: Figure 1.7 shows three characteristic hydrological situations:
- At left: Polder system: seepage and dense drainage system
- Center: Extensive drainage system, change between seepage and infiltration, fall dry
- At right: High system: infiltration, deep groundwater levels, large distance between surface waters
Essentially, all three situations are determined by the area average distance between surface waters. In the present work, this distance results in a wide range of the resistance in the Robin boundary condition.

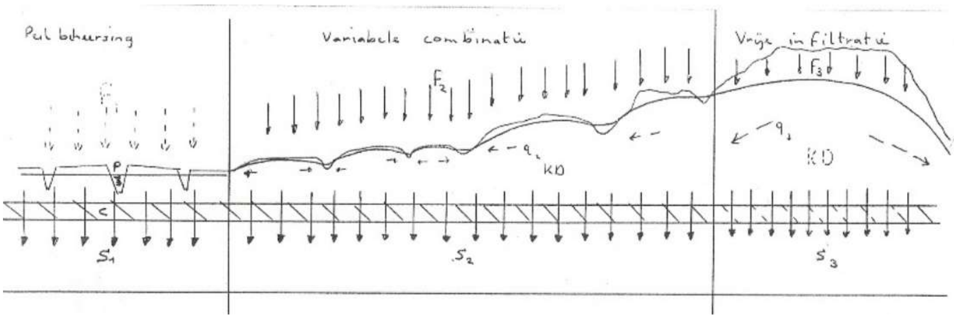


Figure 1.7 Different hydrological situations with varying density of surface water, see text,

In reality, the vertical section of a top-system is elongated as in Figure 1.8. The actual distance between ditches is in the order of tens of meters while the thickness is one to several meters. This shows the reason why the vertical section can be covered by multiple cells in a detailed model, which requires an extension of the equations for the resistance in the Robin boundary condition.

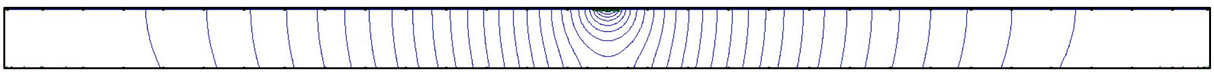


Figure 1.8 Vertical section: Flow between ditches at distance of tens of meters in a top-system with thickness of several meters. Blue lines are equipotential lines calculated by a AEM model.

1.3 The conceptual model for 2D groundwater flow between surface waters.

Aim of conceptual model = scheme to derive relation for flux and potential difference between aquifer and surface water at the model boundary, a Robin-type boundary condition. The conceptual model describes a so-called top-system, generally being the phreatic aquifer, figure 1.9, left.

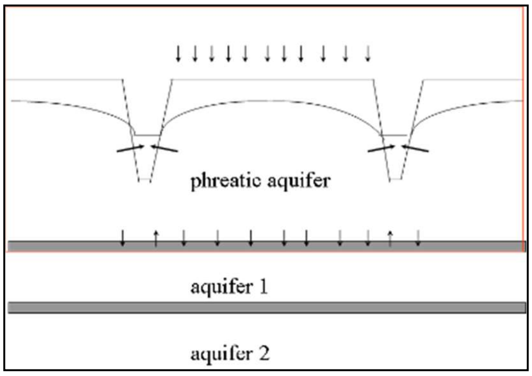


Figure 1.9 Concept of a top system in a vertical section: the phreatic aquifer of a multi layer system.

The relation for the boundary condition may describe multiple drainage pipes / surface waters in a single cell, figure 1.10 at left and also multiple cells between two surface waters figure 1.10 at right. In chapter 2, equations are derived for the first case while those for the second case are derived in chapter 3.

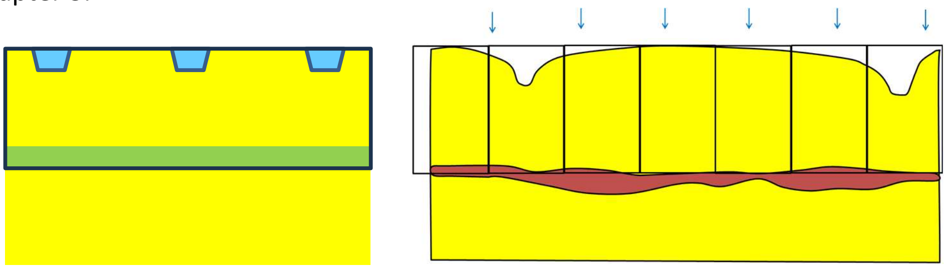


Figure 1.10 At left, multiple surface water in a cell the black rectangle. At right, o multiple cells between two surface waters.

In chapters 2 and 3 the scheme in figure 1.11 left is used to identify the type relation that is to be derived in the section. The scheme covers the flow between the water divides below the surface water and under the land surface as depicted in the scheme of figure 1.11 right. In the top-system the flow is assumed to be mainly local between the surface waters which causes the water divides. Figure 1.11 left is not yet related to a particular conceptual model. In each section in chapters 2 and 3, the picture changes with the derivation of equations as will be shown in section 1.4 below.

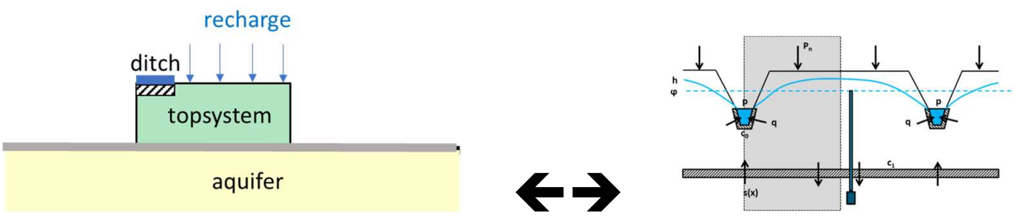


Figure 1.11. Scheme used in chapters 2 and 3 to identify the type relation that is to be derived in a section.

1.4 Steps in derivation and schemes of different conceptual models

Below a summary per section is given for the steps in the derivation of the equations.

<u>1.4.1 Chapter 2: At least one surface water in cell: cell larger than distance between surface waters: the *overall* boundary condition.</u>

Section 2.1 Mathematical definition and solution

This section presents the definition in mathematical terms of the flow problem, see figure 1.12. Two differential equations are presented, one for space section below the surface water x<0 in figure 1.12 right and one for the space section under land surface x>0 in figure 1.12 right. The boundary conditions are presented at the water divides at x=-B/2 and x=L/2 and at x=0 the condition for the connection of the two area are specified.

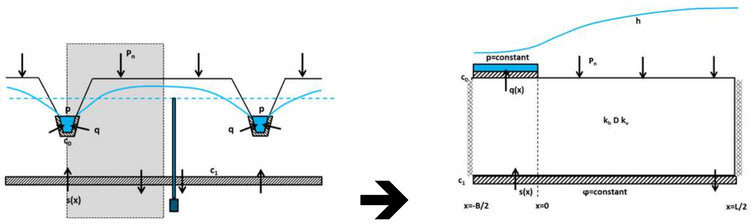


Figure 1.12 Schemes for the mathematical solution of the flow problem.

The resulting expressions are summarized in sub-section 2.1.4 and describe the so-called feeding resistance, the average head in both space sections and the entire space of the top-system between the two water divides.

Section 2.2 The resistance in the overall boundary condition on top of the first aquifer

This section contains the derivation of the constants in the Robin boundary condition on top of the first aquifer. This means that the flux equals the flux to or from the aquifer and the potential difference applies to the potential in the first aquifer. The top-system is lumped as a whole and in fact the boundary condition only supports to retrieve information on flow to/from and potential in the first aquifer.

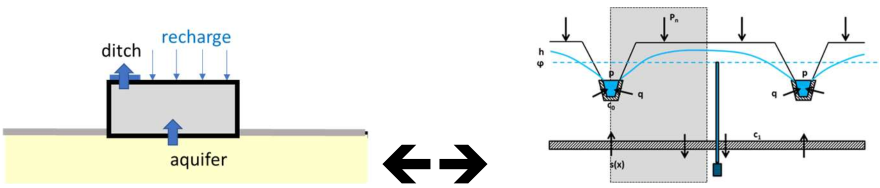


Figure 1.13 Schemes for the derivation of the general boundary condition on top of the first aquifer.

This result is to be considered as a step onto the values in the Robin condition derived in the following section. The resulting resistance contains the resistance of the leaky layer between the top-system and the first aquifer.

Section 2.3 The resistance in the overall boundary condition in phreatic aquifer

This section contains the derivation of the constants in the Robin boundary condition at the bottom of the surface water on top of the phreatic aquifer. This means that the flux equals the flux to or from the surface water and the potential difference applies to the head in the phreatic aquifer. The flow in the phreatic aquifer is lumped cell and in fact the boundary condition supports to retrieve information on the head in the phreatic aquifer.

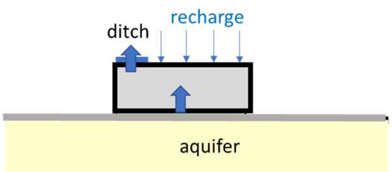


Figure 1.14 Scheme for the derivation of the general boundary condition in the phreatic aquifer.

This result is the Robin condition that can be used if the cell is larger than the distance between the surface waters. The resulting resistance does not contain the resistance of the leaky layer between the top-system and the first aquifer.

1.4.2 Chapter 3: Not a surface water in each cell; cell smaller than distance between surface waters: the *partial* boundary condition.

Section 3.1 The resistance in the partial boundary condition on top of the first aquifer

The focus is similar to that of section 2.2 but for cell smaller than distance between surface waters

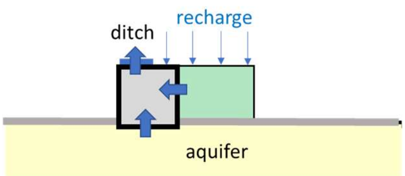


Figure 1.15 Schemes for the derivation of the partial boundary condition on top of the first aquifer.

This result is to be considered as a step onto the values in the Robin condition derived in the following section. The resulting resistance contains the resistance of the leaky layer between the top-system and the first aquifer.

Section 3.2 The resistance in the partial boundary condition in phreatic aquifer

The focus is similar to that of section 2.3 but for cell smaller than distance between surface waters

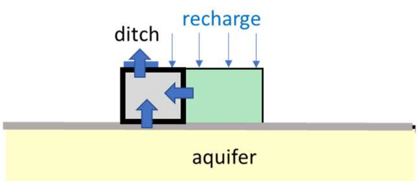
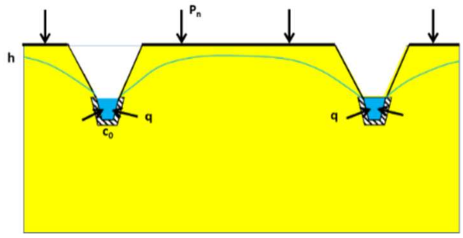


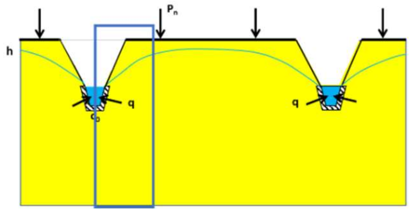
Figure 1.16 Scheme for the derivation of the partial boundary condition in the phreatic aquifer.

This result is the Robin condition that can be used if the cell is smaller than the distance between the surface waters. The resulting resistance does not contain the resistance of the leaky layer between the top-system and the first aquifer.

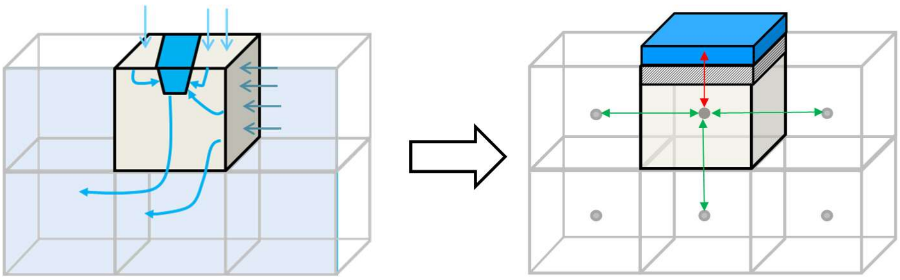
1.4.3 Chapter 4 Equations for single aquifer models – Ernst type equations

Scheme for derivation of equations.





1.4.4 Chapter 5 The complete set of equations for the cell-drain-resistance in numerical modeling.

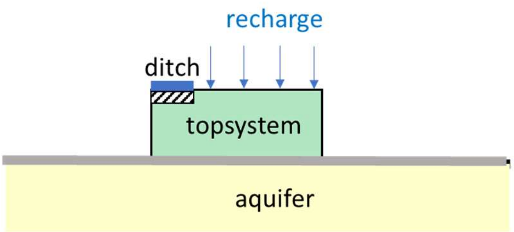


- Introduce background physics and mathematical derivation of horizontal and vertical resistance component of cell_drain_resistance. Derivation of total value by superposition.
- Describe the "how and why" of the different steps undertaken.
- Describe physical meaning of input parameters for a single cell in multi-layer system.

2 The Robin condition for horizontal flow in a cell with multiple surface waters

Boundary condition = Robin condition (Steward 2021)

2. 1 General solution from differential equation and boundary conditions



- cell = section ➔ use half-section for analytic solution

Describe:
Assumptions:
Horizontal flow: Dupuit Forchheimer: continuity of flow in 2-D plane is fulfilled (Strack paper), so:
- vertical flow is described with no resistance in the aquifer &
- the vertical resistance to flow is projected in resistance underlying leakage layer

➔ Mention radial resistance, vertical resistance

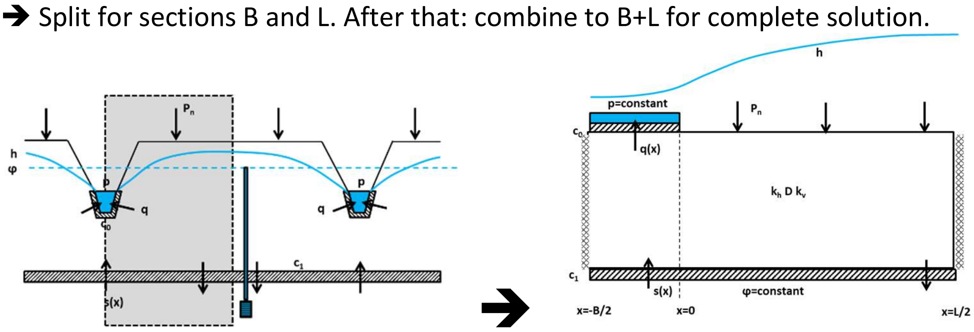
Symmetry: symmetry at center of surface water and in-between surface waters.

Potential in underlying aquifer constant relatively to phreatic head in vicinity of surface water (De Lange 1996)

Homogeneous aquifer; vertical anisotropy translated in vertical resistance.

$$c'_1 = c_1 + \frac{H}{k_v} \qquad (2.01)$$

Steps in derivation: explain why: 2 different Differential equations
➔ Split for sections B and L. After that: combine to B+L for complete solution.



Explain Figures....
Left = conceptual model with parameters
Right = "symbol" for type of topsystem: here, at grey boundary on top of aquifer.

2.1.1 Differential equation and boundary conditions

For both sections "B" and "L", the differential equation will be determined. The underlying physics is that an increase of a negative gradient in head is caused by flow into aquifer:

For the range B/2<x<0,

$$-k_x H \frac{d^2 h_x}{dx^2} = \frac{\phi - h_x}{c'_1} + \frac{p - h_x}{c_0}$$

So:

$$\frac{d^2 h_x}{dx^2} = \frac{h_x - \phi}{k_x H c'_1} + \frac{h_x - p}{k_x H c_0}$$

For the range 0<x<L/2, we have: (P_n is downward):

$$-k_x H \frac{d^2 h_x}{dx^2} = \frac{\phi - h_x}{c'_1} + P_n$$

So:

$$\frac{d^2 h_x}{dx^2} = \frac{h_x - \phi}{k_x H c'_1} - \frac{P_n}{k_x H}$$

The spreading-length describes how far a change in boundary condition spreads in the aquifer:

$$\lambda_B^2 = k_h H \frac{c_1' c_0}{c_1' + c_0} \quad (2.02)$$

$$\lambda_L^2 = k_h H c_1'; \quad (2.04)$$

The known heads and flux at upper and lower boundary are combined into a single parameter t:

$$t_B = \frac{p c_1' + \phi c_0}{c_1' + c_0} \quad (2.03)$$

$$t_L = \phi + P_n c_1' \quad (2.05)$$

With this, the differential equations become:

$\frac{d^2 h_x}{dx^2} = \frac{h_x - t_B}{\lambda_B^2}$ at -B/2<x<0 and $\frac{d^2 h_x}{dx^2} = \frac{h_x - t_L}{\lambda_L^2}$ at 0<x<L/2

The boundary conditions follow from symmetry at the left and right ends of the observed system and continuity of flow and head in the center between the land "L" part and surface water "B"part.

$$At \quad x = 0 \quad \rightarrow \quad h_{left} = h_{right} \quad and \quad \frac{dh_x}{dx}_{left} = \frac{dh_x}{dx}_{rght}$$

$$At \quad x = \frac{-B}{2} \quad and \quad x = \frac{L}{2} \quad \rightarrow \quad q = 0 \quad \rightarrow \quad \frac{dh_x}{dx} = 0$$

2.1.2 General solution

The general solution for the head for both differential equations is:

$$h_x - t_i = C_{1,i} e^{\frac{-x}{\lambda_i}} + C_{2,i} e^{\frac{x}{\lambda_i}} \quad for \quad i = B, L$$

and that for the derivative is:

$$\frac{dh_x}{dx} = -\frac{C_{1,i}}{\lambda_i} e^{\frac{-x}{\lambda_i}} + \frac{C_{2,i}}{\lambda_i} e^{\frac{x}{\lambda_i}} \quad for \quad i = B, L$$

For convenience, we define

$\alpha_B = e^{\frac{-B}{\lambda_B}}$ and $\alpha_L = e^{\frac{L}{\lambda_L}}$ (2.06) (2.07)

Solve at L/2 and -B/2 leads to:

$C_{1,B} = C_{2,B}\alpha_B$ and $C_{1,L} = C_{2,L}\alpha_L$

Use these and solve at x=0 for both h and its derivative leads to:

$$t_B + C_{2,B}(\alpha_B + 1) = t_L + C_{2,L}(\alpha_L + 1)$$
$$C_{2,B}(1 - \alpha_B)/\lambda_B = C_{2,L}(1 - \alpha_L)/\lambda_L$$

Solving these equations is straightforward

$C_{2,B} = C_{2,L}(1 - \alpha_L)/(1 - \alpha_B)\lambda_B/\lambda_L$ So: $C_{2,B} = \frac{\lambda_B(1-\alpha_L)}{\lambda_L(1-\alpha_B)} * C_{2,L}$

$$C_{2,L}(\alpha_L + 1) = t_B - t_L + C_{2,B}(\alpha_B + 1) = t_B - t_L + C_{2,L}(\alpha_B + 1)(1 - \alpha_L)/(1 - \alpha_B)\lambda_B/\lambda_L$$
$$C_{2,L} = \frac{t_B - t_L}{(\alpha_L + 1) - (\alpha_B + 1)(1 - \alpha_L)/(1 - \alpha_B)\lambda_B/\lambda_L}$$
$$= \frac{(t_B - t_L)(1 - \alpha_B)}{(\alpha_L + 1)(1 - \alpha_B) - (\alpha_B + 1)(1 - \alpha_L)\lambda_B/\lambda_L}$$
$$C_{2,L} = \frac{(t_B - t_L)(\alpha_B - 1)\lambda_L}{(\alpha_L + 1)(\alpha_B - 1)\lambda_L - (\alpha_B + 1)(\alpha_L - 1)\lambda_B}$$

$$C_{2,B} = \frac{\lambda_B(1 - \alpha_L)}{\lambda_L(1 - \alpha_B)} * \frac{(t_B - t_L)(\alpha_B - 1)\lambda_L}{(\alpha_L + 1)(\alpha_B - 1)\lambda_L - (\alpha_B + 1)(\alpha_L - 1)\lambda_B}$$
$$= \frac{(t_B - t_L)(\alpha_L - 1)\lambda_B}{(\alpha_L + 1)(\alpha_B - 1)\lambda_L - (\alpha_B + 1)(\alpha_L - 1)\lambda_B}$$

And results to

$C_{2,L} = (\alpha_B - 1)\lambda_L C_R$ and so $C_{1,L} = (\alpha_B - 1)\lambda_L C_R \alpha_L$

$C_{2,B} = (\alpha_L - 1)\lambda_B C_R$ and so $C_{1,B} = (\alpha_L - 1)\lambda_B C_R \alpha_B$

Where C_R is a constant:

$$C_R = \frac{(t_B - t_L)}{\lambda_L(\alpha_B - 1)(\alpha_L + 1) - \lambda_B(\alpha_B + 1)(\alpha_L - 1)}$$

Also, we may write:

$$t_B - t_L = \frac{pc_1' + \phi c_0}{c_1' + c_0} - (\phi + P_n c_1') = [\frac{p - \phi}{c_1' + c_0} - P_n]c_1'$$

By these expressions, the solution in terms of the phreatic head h is known for both zones:

For 0<x<L/2:

$$h_x = t_L + C_{1,L}e^{\frac{-x}{\lambda_L}} + C_{2,L}e^{\frac{x}{\lambda_L}}$$

For -B/2<x<0:

$$h_x = t_B + C_{1,B}e^{\frac{-x}{\lambda_B}} + C_{2,B}e^{\frac{x}{\lambda_B}}$$

2.1.3 Expressions for the average phreatic head

We derive the average over B/2 and L/2 by integration:

Integration over 0<x<L/2:

$$\overline{h}_x^L = \frac{2}{L}\int_0^{L/2} [t_L + C_{1,L}e^{\frac{-x}{\lambda_L}} + C_{2,L}e^{\frac{x}{\lambda_L}}]\, dx$$

$$\overline{h}_x^L = \frac{2}{L}\ [t_L x - \lambda_L C_{1,L}e^{\frac{-x}{\lambda_L}} + \lambda_L C_{2,L}e^{\frac{x}{\lambda_L}}]_0^{L/2}$$

$$\overline{h}_x^L = \frac{2}{L}\ [t_L \frac{L}{2} - \lambda_L C_{2,L}\alpha_L(e^{\frac{-L}{2\lambda_L}} - 1) + \lambda_L C_{2,L}(e^{\frac{L}{2\lambda_L}} - 1)]$$
$$= t_L + \frac{2}{L}\lambda_L C_{2,L}\ [-(e^{\frac{L}{2\lambda_L}} - e^{\frac{L}{\lambda_L}}1) + (e^{\frac{L}{2\lambda_L}} - 1)]$$

$$\overline{h}_x^L = t_L + [\lambda_L(\alpha_L - 1)C_{2,L}]\ \frac{2}{L}$$

Because

$$C_{2,L} = \frac{(t_B - t_L)(\alpha_B - 1)\lambda_L}{(\alpha_L + 1)(\alpha_B - 1)\lambda_L - (\alpha_B + 1)(\alpha_L - 1)\lambda_B}$$

substituted in the second term gives:

$$\lambda_L(\alpha_L - 1)C_{2,L}\frac{2}{L} = \frac{t_B - t_L}{\frac{L}{2\lambda_L}\frac{\alpha_L + 1}{\alpha_L - 1} + \frac{L}{B}\frac{\lambda_B^2}{\lambda_L^2}\frac{-B}{2\lambda_B}\frac{\alpha_B + 1}{\alpha_B - 1}} = \frac{[\frac{p - \phi}{c_1^{'} + c_0} - P_n]c_1^{'}}{\frac{L}{2\lambda_L}ctnh(\frac{L}{2\lambda_L}) + \frac{L}{B}\frac{c_0}{c_1^{'} + c_0}\frac{-B}{2\lambda_B}ctnh(\frac{-B}{2\lambda_B})}$$

And so:

$\lambda_L(\alpha_L - 1)C_{2,L}\frac{2}{L} = \frac{[p - \phi + P_n(c_1' + c_0)]c_1'}{(c_1' + c_0)F_L + \frac{c_0 L}{B}F_B}$ with $F_L = \frac{L}{2\lambda_L}ctnh(\frac{L}{2\lambda_L})$ and $F_B = \frac{-B}{2\lambda_B}ctnh(\frac{-B}{2\lambda_B})$

And in terms of h

$$\overline{h}_x^L = t_L + \frac{[p - \phi - P_n(c_1^{'} + c_0)]c_1^{'}}{(c_1^{'} + c_0)F_L + \frac{c_0 L}{B}F_B} = \phi + P_n c_1^{'} - \frac{[\phi - p + P_n(c_1^{'} + c_0)]c_1^{'}}{c_L^*}$$

$$\overline{h}_x^L = \phi - p + P_n c_1^{'} - \frac{[\phi - p + P_n(c_1^{'} + c_0)]c_1^{'}}{c_L^*} = \frac{(\phi - p)(c_L^* - c_1^{'})}{c_L^*} + \frac{P_n(c_L^* - c_1^{'} - c_0)]c_1^{'}}{c_L^*}$$

Integration over -B/2<x<0:

$$\overline{h}_x^B = \frac{2}{B}\int_{-B/2}^{0} [t_B + C_{1,B}e^{\frac{-x}{\lambda_B}} + C_{2,B}e^{\frac{x}{\lambda_B}}]\, dx$$

$$\overline{h}_x^B = \frac{2}{B}[t_B x - \lambda_B C_{1,B}e^{\frac{-x}{\lambda_B}} + \lambda_B C_{2,B}e^{\frac{x}{\lambda_B}}]_{-B/2}^{0}$$

$$\overline{h}_x^B = t_B + [\lambda_B(1 - \alpha_B)C_{2,B}]\ \frac{2}{B}$$

Rework right term at right hand side$\lambda_B(1-\alpha_B)C_{2,B}\frac{2}{B} = \frac{-(t_B-t_L)}{\frac{B\lambda_L^2}{L\lambda_B^2}\frac{L}{2\lambda_L}\frac{\alpha_L+1}{\alpha_L-1}+\frac{B}{2\lambda_B}\frac{\alpha_B+1}{\alpha_B-1}} =$

$$\frac{-(\frac{p-\phi}{c_1'+c_0}-P_n)c_1'}{\frac{B}{L}\frac{c_1'+c_0}{c_0}\frac{L}{2\lambda_L}ctnh(\frac{L}{2\lambda_L})+\frac{B}{2\lambda_B}ct\quad(\frac{B}{2\lambda_B})}$$

So:

$$\lambda_B(1-\alpha_B)C_{2,B}\frac{2}{B} = -\frac{p-\phi-P_n(c_1'+c_0)}{(c_1'+c_0)F_L+\frac{c_0L}{B}F_B} * \frac{L}{B}\frac{c_1'c_0}{(c_1'+c_0)}$$

In which:

$F_L = \frac{L}{2\lambda_L}ctnh(\frac{L}{2\lambda_L})$ (2.08)

and

$F_B = \frac{B}{2\lambda_B}ctnh(\frac{B}{2\lambda_B})$ (2.09)

We define:

$c_L^* = (c_1'+c_0)F_L + \frac{c_0L}{B}F_B$ (2.10)

In terms of h:

$$\overline{h}_x^B = t_B - \frac{p-\phi-P_n(c_1'+c_0)}{c_L^*} * \frac{c_0c_1'L}{B}(c_1'+c_0) = \frac{pc_1'+\phi c_0}{c_1'+c_0} - \frac{p-\phi-P_n(c_1'+c_0)}{(c_1'+c_0)} * \frac{c_1'}{c_L^*}\frac{c_0L}{B}$$

$$\overline{h}_x^B = \frac{pc_1'+\phi c_0}{c_1'+c_0} - [\frac{p-\phi}{(c_1'+c_0)}+P_n] * \frac{c_1'}{c_L^*}\frac{c_0L}{B} = \frac{pc_1'+\phi c_0}{c_1'+c_0} + [\frac{\phi-p}{(c_1'+c_0)}-P_n] * \frac{c_1'}{c_L^*}\frac{c_0L}{B}$$

The average head over -B/2<x<tot L/2 is:

$\bar{h}_x^{B+L} = \frac{B}{B+L}\bar{h}_x^B + \frac{L}{B+L}\bar{h}_x^L$ (2.11)

2.1.4 Summary of expressions for the general solution

$c'_1 = c_1 + \frac{H}{k_v}$ (2.01)

$\lambda_B^2 = k_hH\frac{c_1'c_0}{c_1'+c_0}$ and $t_B = \frac{pc_1'+\phi c_0}{c_1'+c_0}$ (2.02) (2.03)

$\lambda_L^2 = k_hHc_1'$; and$t_L = \phi + P_nc_1'$ (2.04) (2.05)

$\alpha_B = e^{\frac{-B}{\lambda_B}}$ and $\alpha_L = e^{\frac{L}{\lambda_L}}$ (2.06) (2.07)

$F_L = \frac{L}{2\lambda_L}ctnh(\frac{L}{2\lambda_L})$ (2.08)

and

$F_B = \frac{B}{2\lambda_B}ctnh(\frac{B}{2\lambda_B})$ (2.09)

We define:

$c_L^* = (c_1'+c_0)F_L + \frac{c_0L}{B}F_B$ (2.10)

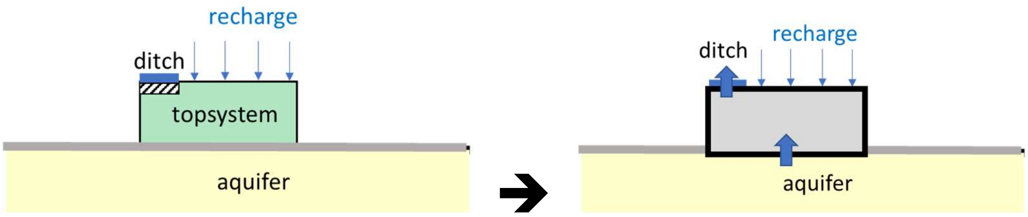
$\bar{h}_x^{B+L} = \frac{B}{B+L}\bar{h}_x^B + \frac{L}{B+L}\bar{h}_x^L$ (2.11)

$\overline{h}_x^L = \frac{(\phi-p)(c_L^*-c_1')}{c_L^*} + \frac{P_n(c_L^*-c_1'-c_0)]c_1'}{c_L^*}$ (2.12)

$$\overline{h}_x^B = \frac{pc_1' + \phi c_0}{c_1' + c_0} + [\frac{\phi - p}{(c_1' + c_0)} - P_n] * \frac{c_1'}{c_L^*}\frac{c_0 L}{B} \quad (2.13)$$

2.2 Robin condition for the boundary on top of the first aquifer

First step: derive expression for interaction between surface water and groundwater in the underlying aquifer from general solution. That is: in terms of change of flux from / to the underlying (regional) aquifer due to change in difference between surface water level and potential in underlying aquifer and including effect of recharge.



Explain Figures.... Left: boundary condition. Right: physical system

2.2.1 Boundary condition for sections "L " and "B" (land and water domains)

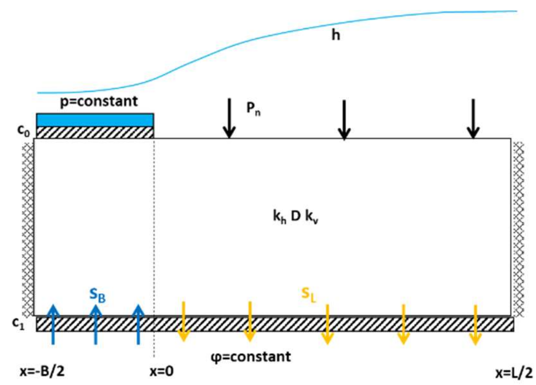


Figure: Mathematical definitions for the two sections.
We derive $\overline{s}$ for the average flux across the underlying leakage layer in the two sections.
We assume that h is at the top of the phreatic aquifer.

$$\overline{s}^B = \frac{\phi - \overline{h}^B}{c_1'} \text{ and } \overline{s}^L = \frac{\phi - \overline{h}^L}{c_1'} \quad (2.14)\ (2.15)$$

For L: Using

$$\overline{h}_x^L = \phi + P_n c_1' - \frac{[\phi - p + P_n(c_1' + c_0)]c_1'}{c_L^*}$$

Substitution of $\overline{h}^L$ leads to

$$\overline{s}^L = \frac{\phi - \overline{h}^L}{c_1'} = \frac{\phi - [\phi + P_n c_1' - \frac{[\phi - p + P_n(c_1' + c_0)]c_1'}{c_L^*}]}{c_1'}$$

$$= \frac{\phi - \phi - P_n c_1' + \frac{[\phi - p + P_n(c_1' + c_0)]c_1'}{c_L^*}}{c_1'}$$

$$\overline{s}^L = \frac{-P_n c_1' + \frac{\phi - p + P_n c_1'(c_1' + c_0)}{c_L^*}}{c_1'} = -P_n + \frac{\phi - p + P_n(c_1' + c_0)}{c_L^*} = \frac{\phi - (p + P_n(c_L^* - c_1' - c_0)}{c_L^*}$$

So:

$$\overline{s}^{L} = \frac{\phi - p}{c_L^*} - P_n \frac{(c_L^* - c_1^{'} - c_0)}{c_L^*}$$

Or:

$$\overline{s}^{L} = \frac{\varphi - p_L^*}{c_L^*} \qquad (2.16)$$

with $p_L^* = p + P_n(c_L^* - c_1^{'} - c_0)$ (2.17)
and c_L^* as given above.

For B: Using (2.13)

$$\overline{h}_x^{B} = \frac{pc_1^{'} + \phi c_0}{c_1^{'} + c_0} + [\frac{\phi - p}{(c_1^{'} + c_0)} + P_n] * \frac{c_1^{'}}{c_L^*} \frac{c_0 L}{B}$$

Substitution of $\overline{h}^{B}$ in (2.14) leads to

$$\overline{s}^{B} = \frac{\varphi - \overline{h}_x^{B}}{c_1^{'}} = \frac{\phi - \frac{pc_1^{'} + \phi c_0}{c_1^{'} + c_0}}{c_1^{'}} - [\frac{\phi - p}{(c_1^{'} + c_0)} + P_n] * \frac{c_0}{c_L^*} \frac{L}{B} = \frac{\phi - p}{(c_1^{'} + c_0)} - [\frac{\phi - p}{(c_1^{'} + c_0)} + P_n] * \frac{c_0}{c_L^*} \frac{L}{B}$$

$$\overline{s}^{B} = \frac{\phi - p}{(c_1^{'} + c_0)} - [\frac{\phi - p}{(c_1^{'} + c_0)} + P_n] * \frac{c_0}{c_L^*} \frac{L}{B}$$

Combine terms with (p-ϕ) and P_n :

$$\overline{s}^{B} = \frac{\phi - p}{c_1^{'} + c_0} * (1 - \frac{L}{B} \frac{c_0}{c_L^*}) - P_n \frac{c_0}{c_L^*} \frac{L}{B} = \frac{\phi - p}{c_B^*} - P_n \frac{c_0}{c_L^*} \frac{L}{B}$$

So:

$$\overline{s}^{B} = \frac{\phi - p}{c_B^*} - P_n \frac{c_0}{c_L^*} \frac{L}{B}$$

Which can be reworked to:

$$\overline{s}^{B} = \frac{\varphi - p_B^*}{c_B^*} \qquad (2.18)$$

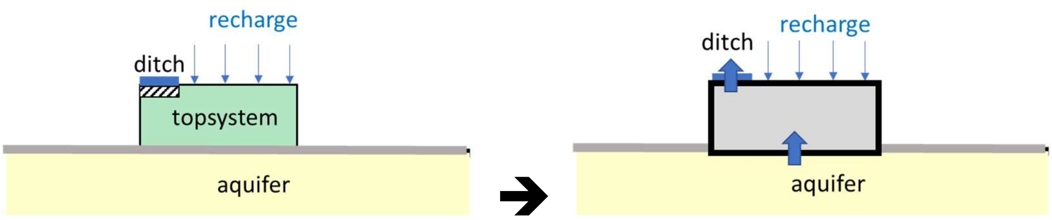
with

$$c_B^* = \frac{c_1^{'} + c_0}{1 - \frac{L c_0}{B c_L^*}} \qquad (2.19)$$

and

$$p_B^* = p + P_n c_B^* \frac{c_0}{c_L^*} \frac{L}{B} \qquad (2.20)$$

2.2.2 Boundary condition for the feeding resistance



The feeding resistance describes the interaction between the regional aquifer and the surface water including the impact of recharge. It applies to the entire domain B+L and includes the underlying leakage layer.

$$\bar{s}^{B+L} = \frac{\varphi - p^*_{B+L}}{c^*_{B+L}} \tag{2.21}$$

The expression will be derived using

$$\bar{s}^{B+L} = \frac{B}{B+L}\bar{s}^{B} + \frac{L}{B+L}\bar{s}^{L} \tag{2.22}$$

We substitute the expressions for s derived above:

$$(B+L)\frac{\varphi - p^*_{B+L}}{c^*_{B+L}} = \frac{\varphi - p^*_B}{c^*_B}B + \frac{\varphi - p^*_L}{c^*_L}L$$

The parameters p and ϕ are independent so we may split this equation into

$(B+L)\frac{\varphi}{c^*_{B+L}} = \frac{\varphi}{c^*_B}B + \frac{\varphi}{c^*_L}L$ and $(B+L)\frac{p^*_{B+L}}{c^*_{B+L}} = \frac{p^*_B}{c^*_B}B + \frac{p^*_L}{c^*_L}L$

So:

$$c^*_{B+L} = \frac{B+L}{\frac{B}{c^*_B} + \frac{L}{c^*_L}}$$

and

$$p^*_{B+L} = [\frac{p^*_B}{c^*_B}B + \frac{p^*_L}{c^*_L}L] * \frac{c^*_{B+L}}{B+L}$$

Work out using:

$c^*_B = \frac{c'_1 + c_0}{1 - \frac{Lc_0}{Bc^*_L}}$ and $p^*_B = p + P_n c^*_B \frac{c_0}{c^*_L}\frac{L}{B}$ and $p^*_L = p + P_n(c^*_L - c'_1 - c_0)$

$$\frac{B+L}{c^*_{B+L}} = \frac{B}{c^*_B} + \frac{L}{c^*_L} = \frac{B(1 - \frac{L}{B}\frac{c_0}{c^*_L})}{c'_1 + c_0} + \frac{L}{c^*_L} = \frac{B}{c'_1 + c_0} + \frac{L}{c^*_L}(1 - \frac{c_0}{c'_1 + c_0}) = \frac{B}{c'_1 + c_0} + \frac{L}{c^*_L}(\frac{c'_1}{c'_1 + c_0})$$

$$\frac{B+L}{c^*_{B+L}} = \frac{B}{c'_1 + c_0} + \frac{L}{c'_1 + c_0}\frac{c'_1}{c^*_L} = \frac{Bc^*_L + Lc'_1}{(c'_1 + c_0)c^*_L}$$

So:

$$c^*_{B+L} = \frac{c^*_L(c'_1 + c_0)(B+L)}{Bc^*_L + Lc'_1} \tag{2.23}$$

Combine p*-terms

$p^*_L = p + P_n(c'_1 + c_0 + c^*_L)$ $p^*_B = p - P_n c^*_B \frac{c_0}{c^*_L}\frac{L}{B}$

$$\frac{p^*_{B+L}(B+L)}{c^*_{B+L}} = \frac{p^*_B}{c^*_B}B + \frac{p^*_L}{c^*_L}L = \frac{p + P_n c^*_B \frac{c_0}{c^*_L}\frac{L}{B}}{c^*_B}B + \frac{p + P_n(c^*_L - c'_1 - c_0)}{c^*_L}L$$

$$\frac{p^*_{B+L}(B+L)}{c^*_{B+L}} = p\frac{B}{c^*_B} + P_n L\frac{c_0}{c^*_L} + p\frac{L}{c^*_L} + P_n L\frac{(c^*_L - c'_1 - c_0)}{c^*_L} = p(\frac{B}{c^*_B} + \frac{L}{c^*_L}) + P_n L(\frac{c^*_L - c'_1 - c_0}{c^*_L} + \frac{c_0}{c^*_L})$$

$$\frac{p^*_{B+L}(B+L)}{c^*_{B+L}} = p\frac{B+L}{c^*_{B+L}} + P_n L\frac{c^*_L - c'_1}{c^*_L}$$

Using:

$\frac{B+L}{c^*_{B+L}} = \frac{Bc^*_L + Lc'_1}{(c'_1 + c_0)c^*_L}$ and

And substitution gives:

$$p_{B+L}^{*} = p + P_n L \frac{c_L^{*} - c_1^{'}}{c_L^{*}} \frac{c_{B+L}^{*}}{B + L} = p + P_n (c_L^{*} - c_1^{'}) \frac{c_{B+L}^{*}}{c_L^{*}} \frac{L}{B + L}$$

$p_{B+L}^{*} = p + P_n (c_L^{*} - c_1^{'}) \frac{c_{B+L}^{*}}{c_L^{*}} \frac{L}{B+L}$ (2.24)

2.2.3 Summary of expressions for the feeding resistance.

$\overline{s}^{B+L} = \frac{\varphi - p_{B+L}^{*}}{c_{B+L}^{*}}$ (2.21)

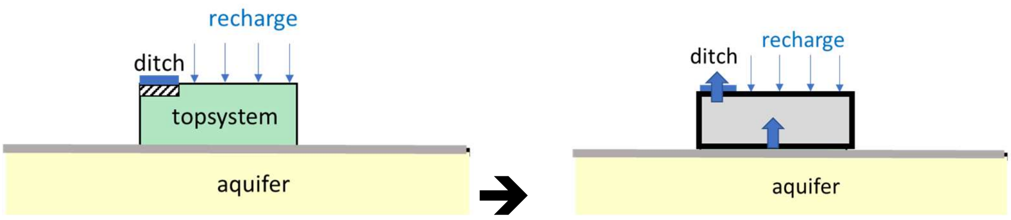
$\overline{s}^{B+L} = \frac{B}{B+L} \overline{s}^{B} + \frac{L}{B+L} \overline{s}^{L}$ (2.22)

$c_{B+L}^{*} = \frac{c_L^{*}(c_1^{'} + c_0)(B+L)}{B c_L^{*} + L c_1^{'}}$ (2.23)

$p_{B+L}^{*} = p + P_n (c_L^{*} - c_1^{'}) \frac{c_{B+L}^{*}}{c_L^{*}} \frac{L}{B+L}$ (2.24)

2.3 The Robin condition in the phreatic aquifer

Second step: derive expression in terms of phreatic head and flux to surface water by using the relation for the flux to and from the underlying aquifer from the head difference across the underlying resistance layer. That is: the flux across the lower boundary in terms of the difference between the average phreatic head and the potential in the underlying aquifer.



Aim q=fh)

Step 1: Work out ϕ in terms of h

$\bar{s}_{B+L} = \frac{\phi - \bar{h}_{B+L}}{c_1'} = \frac{\varphi - p^*_{B+L}}{c^*_{B+L}}$ leading to $\phi = \frac{c^*_{B+L}\bar{h}_{B+L} - c_1' p^*_{B+L}}{c^*_{B+L} - c_1'}$ and

Work out:

$$\bar{s}_{B+L} = \frac{\varphi - p^*_{B+L}}{c^*_{B+L}} = \frac{\frac{c^*_{B+L}\bar{h}_{B+L} - c_1' p^*_{B+L}}{c^*_{B+L} - c_1'} - p^*_{B+L}}{c^*_{B+L}} = \frac{c^*_{B+L}\bar{h}_{B+L} - c_1' p^*_{B+L} - (c^*_{B+L} - c_1')p^*_{B+L}}{(c^*_{B+L} - c_1')c^*_{B+L}}$$

$$= \frac{(\bar{h}_{B+L} - p^*_{B+L})}{(c^*_{B+L} - c_1')}$$

So:

$$\bar{s}_{B+L} = \frac{(\bar{h}_{B+L} - p^*_{B+L})}{(c^*_{B+L} - c_1')}$$

From previous:

$c^*_{B+L} = \frac{c_L^*(c_1' + c_0)(B+L)}{Bc_L^* + Lc_1'}$ and $p^*_{B+L} = p + P_n(c_L^* - c_1')\frac{c^*_{B+L}}{c_L^*}\frac{L}{B+L}$

Step 2: Derive expression for q:

Use water balance complete system: $\bar{q}^B B = \bar{s}^{B+L}(B+L) + P_n L$, so:

$$\bar{q}^B \frac{B}{B+L} = \frac{\bar{h}_{B+L} - [p + P_n(c_L^* - c_1')\frac{c^*_{B+L}}{c_L^*}\frac{L}{B+L}]}{(c^*_{B+L} - c_1')} + P_n\frac{L}{B+L}$$

$$\bar{q}^B \frac{B}{B+L} = \frac{\bar{h}_{B+L} - p}{(c^*_{B+L} - c_1')} - P_n\frac{L}{B+L}\frac{(c_L^* - c_1')}{(c^*_{B+L} - c_1')}\frac{c^*_{B+L}}{c_L^*} + P_n\frac{L}{B+L}$$

$$\bar{q}^B \frac{B}{B+L} = \frac{\bar{h}_{B+L} - p}{(c^*_{B+L} - c_1')} + P_n\frac{L}{B+L}[1 - \frac{c^*_{B+L}}{(c^*_{B+L} - c_1')}\frac{(c_L^* - c_1')}{c_L^*}]$$

$$\bar{q}^B = \frac{\bar{h}_{B+L} - p}{(c^*_{B+L} - c_1')\frac{B}{B+L}} + P_n\frac{L}{B}[1 - \frac{c^*_{B+L}}{(c^*_{B+L} - c_1')}\frac{(c_L^* - c_1')}{c_L^*}]$$

$$\bar{q}^{B}\frac{B}{B+L}=\frac{\bar{h}^{B+L}-p}{(c_{B+L}^{*}-c_{1}^{'})}+P_{n}\frac{L}{B+L}[\frac{(c_{B+L}^{*}-c_{1}^{'})-\frac{c_{B+L}^{*}}{c_{L}^{*}}(c_{L}^{*}-c_{1}^{'})}{(c_{B+L}^{*}-c_{1}^{'})}]\,\bar{q}^{B}\frac{B}{B+L}=\frac{\bar{h}_{B+L}-p}{(c_{B+L}^{*}-c_{1}')}+$$

$$P_{n}\frac{L}{B+L}[\frac{c_{B+L}^{*}-c_{1}'-c_{B+L}^{*}+\frac{c_{B+L}^{*}}{c_{L}^{*}}c_{1}'}{(c_{B+L}^{*}-c_{1}')}]\,\bar{q}^{B}\frac{B}{B+L}=\frac{\bar{h}^{B+L}-p}{(c_{B+L}^{*}-c_{1}^{'})}+P_{n}\frac{L}{B+L}[\frac{(\frac{c_{B+L}^{*}}{c_{L}^{*}}-1)c_{1}^{'}}{(c_{B+L}^{*}-c_{1}^{'})}]\,\bar{q}^{B}\frac{B}{B+L}=\frac{\bar{h}_{B+L}-p}{(c_{B+L}^{*}-c_{1}^{'})}+$$

$$\frac{P_{n}c_{1}^{'}\frac{L}{B+L}(\frac{c_{B+L}^{*}}{c_{L}^{*}}-1)}{(c_{B+L}^{*}-c_{1}^{'})}=\frac{\bar{h}_{B+L}-p_{celldrain}}{c_{celldrain}}$$

In terms of modeling parameters, we define:

$$\bar{q}^{B}\frac{B}{B+L}=\bar{q}_{cell}^{B+L}$$

Which results into:

$$\bar{q}_{cell}^{B+L}=\frac{\bar{h}_{B+L}-p_{cell\_L\_drain}}{c_{cell\_L\_drain}}$$

With

$c_{celdrain,B+L}=(c_{B+L}^{*}-c_{1}^{'})$

and

$$p_{celldrain,B+L}=p-P_{n}c_{1}^{'}\frac{L}{B+L}(\frac{c_{B+L}^{*}}{c_{L}^{*}}-1)=p+P_{n}c_{1}^{'}\frac{L}{B+L}(1-\frac{c_{B+L}^{*}}{c_{L}^{*}})$$

$$p_{celldrain,B+L}=p+P_{n}c_{1}^{'}\frac{L}{B+L}(1-\frac{c_{B+L}^{*}}{c_{L}^{*}})$$

Or, in explicit formula:

$$\bar{q}_{cell}^{B+L}=\frac{\bar{h}_{B+L}-p}{(c_{B+L}^{*}-c_{1}^{'})}-P_{n}\frac{c_{1}^{'}}{c_{L}^{*}}\frac{(c_{B+L}^{*}-c_{L}^{*})}{(c_{B+L}^{*}-c_{1}^{'})}\frac{L}{B+L}$$

Results:

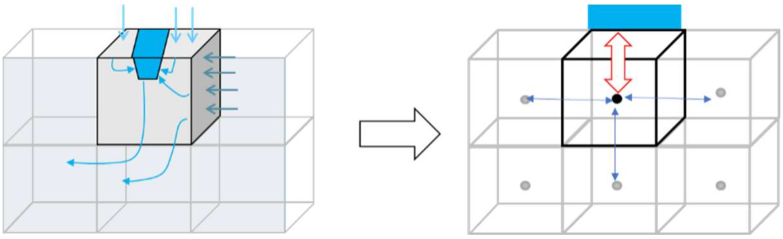
$$\bar{q}_{cell}^{B+L}=\frac{\bar{h}_{B+L}-p_{cell\_L\_drain}}{c_{cell\_L\_drain}}$$

$c_{celdrain,B+L}=(c_{B+L}^{*}-c_{1}^{'})$

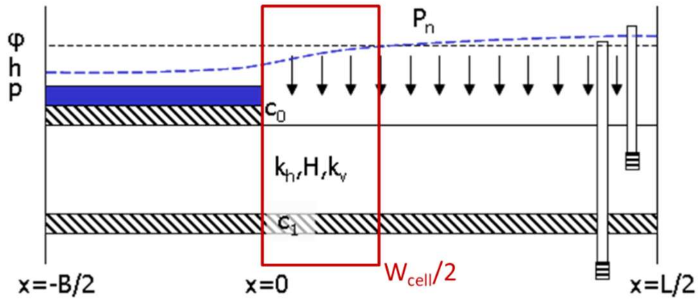
$$p_{celldrain,B+L}=p+P_{n}c_{1}^{'}\frac{L}{B+L}(1-\frac{c_{B+L}^{*}}{c_{L}^{*}})$$

3 The Robin condition for a cell smaller than the distance between the surface waters

3.1 Adapted solution for domain covering part of land surface



Scheme: Multiple cells in drain unit.



We consider a cell symmetrically around the center of the surface water, see figure above. Then, at both sides all distances are half. Differential equations and boundary conditions are equal to in section 2.1. The only difference is that integration over the land part occurs over $0<x<W_{cell}/2$.

Determine the average phreatic head in the cell:

$$\overline{h_x}^{W_{cell}} = \frac{2}{W_{cell}} \int_0^{W_{cell}/2} [t_L + C_{1,L} e^{\frac{-x}{\lambda_L}} + C_{2,L} e^{\frac{x}{\lambda_L}}]\, dx$$

$$\overline{h_x}^{W_{cel}} = \frac{2}{W_{cell}} \left[t_L x - \lambda_L C_{1,L} e^{\frac{-x}{\lambda_L}} + \lambda_L C_{2,L} e^{\frac{x}{\lambda_L}}\right]_0^{W_{cel}l/2}$$

With: $C_{1,L} = C_{2,L}\alpha_L$

$\overline{h_x}^{W_{cell}} = t_L + [\lambda_L (EE_{W_{cell}}) C_{2,L}]\ \frac{2}{W_{cell}}$

in which:

$$EE_{W_{cell}} = (e^{W_{Cell}/2\lambda_L} - 1) - e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L} - 1)$$

Next, we calculate: $\varphi - \overline{h}_{W_{cell}} = \varphi - (\varphi + P_n c_1{}' + [\lambda_L (EE_{W_{cell}}) C_{2,L}]\ \frac{2}{W_{cell}}) = -P_n c_1{}' - [\lambda_L (EE_{W_{cell}}) C_{2,L}]\ \frac{2}{W_{cell}}$

With

$\overline{s}_{W_{cell}} = \frac{\varphi - \overline{h}_{W_{cell}}}{c_1{}'}$ (3.01)

Water balance in cell:

$Q_{x=0,Wcell} = -(\overline{s}_{W_{cell}} + P_n) \frac{W_{cell}}{2}$ (3.02)

we may write:

$$Q_{x=0,Wcell} = -(\overline{s}_{W_{cell}} + P_n) \frac{W_{cell}}{2} = \frac{[\lambda_L (EE_{W_{cell}}) C_{2,L}]}{c_1'} \frac{2}{W_{cell}} \frac{W_{cell}}{2} = \frac{[\lambda_L (EE_{W_{cell}}) C_{2,L}]}{c_1'}$$

which is the flux entering across top and bottom of cell and excludes lateral flow

Similarly, for L we have, water balance over L:
$Q_{x=0,L} = -(\overline{s}_L + P_n)\frac{L}{2}$ (total flux at x=o) (3.03)

And use (2.15), (2.12):

$$Q_{x=0,L} = -(\overline{s}_L + P_n)\frac{L}{2} = \frac{[\lambda_L(\alpha_L - 1)C_{2,L}]}{c_1'}\frac{2}{L}\frac{L}{2} = \frac{[\lambda_L(\alpha_L - 1)C_{2,L}]}{c_1'}$$

We define the ratio between these fluxes E as:

$$E^{\#} = \frac{Q_{x=0,L}}{Q_{x=0,Wcell}} = \frac{(\alpha_L - 1)}{(EE_{W_{cell}})} = \frac{e^{L/\lambda_L} - 1}{(e^{W_{Cell}/2\lambda_L} - 1) - e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L} - 1)}$$

Then, we may write:

$$\overline{s}_{W_{cell}} = -P_n - Q_{x=0,W_{cell}}\frac{2}{W_{cell}} = -P_n - \frac{Q_{x=0,L}}{E^{\#}}\frac{2}{W_{cell}} = -P_n - \frac{-(P_n + s_L)}{E^{\#}}\frac{2}{W_{cell}}\frac{L}{2}$$
$$= \frac{\overline{s}_L + P_n(1-E)}{E} = \frac{\overline{s}_L}{E} + \frac{P_n(1-E)}{E}$$

In which we used:
$E = \frac{W_{cell}}{L}E^{\#} = \frac{e^{L/\lambda_L - 1}}{(e^{W_{Cell}/2\lambda_L} - 1) - e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L} - 1)}\frac{W_{cell}}{L}$ (3.05)

So, we derived:
$\overline{s}_{W_{cell}} = \frac{\overline{s}_L}{E} + \frac{P_n(1-E)}{E}$ (3.06)

Intermezzo:
In section 2.1 we derived:

$$\overline{h}_x^L = t_L + [\lambda_L(\alpha_L - 1)C_{2,L}]\ \frac{2}{L}$$

We may also interpret E as a constant between the average heads
$\overline{h}_x^L = t_L + [\lambda_L(\alpha_L - 1)C_{2,L}]\ \frac{2}{L}$ so $\overline{h}_x^L - t_L = [\lambda_L(\alpha_L - 1)C_{2,L}]\ \frac{2}{L}$
$\overline{h}_x^{W_{cell}} = t_L + [\lambda_L(EE_{W_{cell}})C_{2,L}]\ \frac{2}{W_{cell}}$ so $\overline{h}_x^{W_{cell}} - t_L = [\lambda_L(EE_{W_{cell}})C_{2,L}]\ \frac{2}{W_{cell}}$
Which leads to:
$\frac{\overline{h}_x^L - t_L}{\overline{h}_x^{W_{cell}} - t_L} = E$ or $(\overline{h}_x^{W_{cell}} - t_L)E = (\overline{h}_x^L - t_L)$ so $\overline{h}_x^{W_{cell}} = (\overline{h}_x^L - t_L(1-E))/E$
with $t_L = \phi + P_n c_1'$
Resulting into:
$\overline{h}_x^{W_{cell}} = \frac{\overline{h}_x^L}{E} - (\phi + P_n c_1')(\frac{1-E}{E})$ (3.04)
End intermezzo

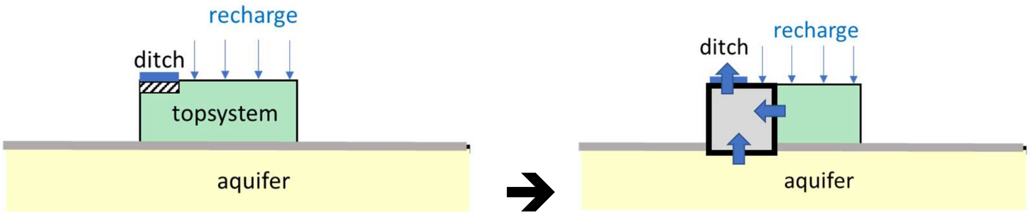
<u>Results overview</u>

$E = \frac{e^{L/\lambda_L - 1}}{(e^{W_{Cell}/2\lambda_L} - 1) - e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L} - 1)}\frac{W_{cell}}{L}$ (3.04)

$$\overline{h_x}^{W_{cell}} = \frac{\overline{h_x}^{L}}{E} - (\phi + P_n c_1^{'})(\frac{1-E}{E}) \qquad (3.05)$$

$$\bar{s}_{W_{cell}} = \frac{\bar{s}_L}{E} + \frac{P_n(1-E)}{E} \qquad (3.06)$$

3.2 Adapted Robin condition: the feeding resistance



Work out the expression for $s_{W_{cell}}$:

$$\frac{\bar{s}_L}{E} + \frac{P_n(1-E)}{E} = \frac{\phi - (p + P_n(c_L^* - c_1^{'} - c_0)}{Ec_L^*} + \frac{P_n(1-E)}{E}$$

$$= \frac{\phi - (p + P_n(c_L^* - c_1^{'} - c_0)}{Ec_L^*} + \frac{P_n c_L^* - P_n E c_L^*}{Ec_L^*}$$

$$\frac{\phi - p - P_n(c_L^* - c_1^{'} - c_0) + P_n c_L^* - P_n E c_L^*}{Ec_L^*} = \frac{\phi - p - P_n(Ec_L^* - c_1^{'} - c_0)}{Ec_L^*} = \frac{\phi - p_{Wcell}^*}{c_{Wcell}^*}$$

So:

$$\bar{s}_{Wcell} = \frac{\phi - p_{Wcell}^*}{c_{Wcell}^*} \qquad (3.07)$$

with

$$c_{Wcell}^* = Ec_L^* \qquad (3.08)$$

and

$$p_{Wcell}^* = p + P_n(c_{Wcell}^* - c_1^{'} - c_0) \qquad (3.09)$$

The term with P_n expresses the reduction of the recharge in the cell compared to that in between the surface waters. The **feeding resistance** can be found similarly as with L above, from:

$$\bar{s}_{\square}^{B+W_{cell}}(B + W_{cell}) = B\bar{s}^{B} + W_{cell}\bar{s}^{W_{cell}}$$

By definition, similar to (2.21) for B+L

$$\bar{s}_{\square}^{B+W_{cell}} = \frac{\varphi - p_{B+W}^*}{c_{B+W}^*} \qquad (3.10)$$

Use the expressions after the similar summation above at L: we simplify (index) W_{cell} to W:

$$(B+W)\frac{\varphi - p_{B+W}^*}{c_{B+W}^*} = \frac{\varphi - p_B^*}{c_B^*}B + \frac{\phi - p_W^*}{c_W^*}W$$

The parameters p and ϕ are independent so we may split this equation into

$$(B+W)\frac{\varphi}{c_{B+W}^*} = \frac{\varphi}{c_B^*}B + \frac{\varphi}{c_W^*}W \quad \text{and} \quad (B+W)\frac{p_{B+W}^*}{c_{B+W}^*} = \frac{p_B^*}{c_B^*}B + \frac{p_W^*}{c_W^*}W$$

So:

$$\frac{B+W}{c_{B+W}^*} = \frac{B}{c_B^*} + \frac{W}{c_W^*} \quad \text{and} \quad p_{B+L}^* = [\frac{p_B^*}{c_B^*}B + \frac{p_W^*}{c_W^*}W] * \frac{c_{B+W}^*}{B+W}$$

Work out using (2.19), (2.20)
$c_B^* = \frac{c_1^{'}+c_0}{1-\frac{Lc_0}{Bc_L^*}}$ and $p_B^* = p + P_n c_B^* \frac{c_0}{c_L^*}\frac{L}{B}$
and (3.08), (3.09)

$c_W^* = Ec_L^*$ and $p_W^* = p + P_n(c_W^* - c_1^{'} - c_0)$

First work out c*-term

$$\frac{B+W}{c_{B+W}^*} = \frac{B}{c_B^*} + \frac{W}{c_W^*} = \frac{B(1-\frac{L}{B}\frac{c_0}{c_L^*})}{c_1^{'}+c_0} + \frac{W}{Ec_L^*} = \frac{B}{c_1^{'}+c_0} + \frac{W}{c_L^*}(\frac{1}{E} - \frac{\frac{L}{W}c_0}{c_1^{'}+c_0})$$

$$= \frac{B}{c_1^{'}+c_0} + \frac{W}{c_L^*}(\frac{c_1^{'}+(1-\frac{L}{W}E)c_0}{E(c_1^{'}+c_0)})$$

$$\frac{B+W}{c_{B+W}^*} = \frac{B}{c_1^{'}+c_0} + \frac{W}{c_L^*}(\frac{c_1^{'}+(1-\frac{L}{W}E)c_0}{E(c_1^{'}+c_0)}) = \frac{Bc_W^* + W(c_1^{'}+(1-\frac{L}{W}E)c_0)}{(c_1^{'}+c_0)c_W^*}$$

So:

$$\frac{B+W}{c_{B+W}^*} = \frac{Bc_W^* + W(c_1^{'}+(1-\frac{L}{W}E)c_0)}{(c_1^{'}+c_0)c_W^*}$$

Or
$\frac{c_{B+W}^*}{B+W} = \frac{(c_1^{'}+c_0)c_W^*}{Bc_W^*+W(c_1^{'}+(1-\frac{L}{W}E)c_0)}$ (3.11)
Combine p*-terms

$$\frac{p_{B+W}^*(B+W)}{c_{B+W}^*} = \frac{p_B^*}{c_B^*}B + \frac{p_W^*}{c_W^*}W = \frac{p + P_n c_B^* \frac{c_0}{c_L^*}\frac{L}{B}}{c_B^*}B + \frac{p + P_n(c_W^* - c_1^{'} - c_0)}{c_W^*}W$$

$$\frac{p_{B+W}^*(B+W)}{c_{B+W}^*} = p\frac{B}{c_B^*} + P_n L\frac{c_0}{c_L^*} + p\frac{W}{c_W^*} + P_n W\frac{(c_W^* - c_1^{'} - c_0)}{c_W^*}$$

$$\frac{p_{B+W}^*(B+W)}{c_{B+W}^*} = p(\frac{B}{c_B^*} + \frac{W}{c_W^*}) + P_n W(\frac{c_W^* - c_1^{'} - c_0}{c_W^*} + \frac{Lc_0}{Wc_L^*})$$

$$\frac{p_{B+W}^*(B+W)}{c_{B+W}^*} = p\frac{B+W}{c_{B+W}^*} + P_n W(\frac{c_W^* - (c_1^{'} + (1-\frac{LE}{W})c_0)}{c_W^*})$$

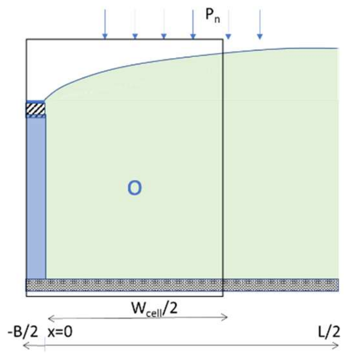
So

$$p_{B+W}^* = p + P_n W(\frac{c_W^* - (c_1^{'} + (1-\frac{LE}{W})c_0)}{c_W^*})\frac{c_{B+W}^*}{B+W}$$

$p_{B+W}^* = p + P_n \frac{W}{B+W}\frac{c_{B+W}^*}{c_W^*}(c_W^* - (c_1^{'} + (1-\frac{LE}{W})c_0))$ (3.12)

So, the Robin condition for (B+W)
$\overline{s}^{B+W} = \frac{\phi - p_{B+W}^*}{c_{B+W}^*}$ (3.10)
is yet derived.

Final Results



$$\bar{s}_{W_{cell}} = \frac{\varphi - \bar{h}_{W_{cell}}}{c_1{}'} \tag{3.01}$$

$$Q_{x=0,W cell} = -(\bar{s}_{W_{cell}} + P_n)\frac{W_{cell}}{2} \tag{3.02}$$

which is the flux entering across top and bottom of cell and excludes lateral flow

$$Q_{x=0,L} = -(\bar{s}_L + P_n)\frac{L}{2} \text{ (total flux at x=o)} \tag{3.03}$$

$$E = \frac{e^{L/\lambda_L - 1}}{(e^{W_{Cell}/2\lambda_L - 1}) - e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L - 1})}\frac{W_{cell}}{L} \tag{3.04}$$

$$\bar{h}_x^{W_{cell}} = \frac{\bar{h}_x^L}{E} - (\phi + P_n c_1^{'})(\frac{1-E}{E}) \tag{3.05}$$

$$\bar{s}_{W_{cell}} = \frac{\bar{s}_L}{E} + \frac{P_n(1-E)}{E} \tag{3.06}$$

$$\bar{s}_{W cell} = \frac{\phi - p^*_{W cell}}{c^*_{W cell}} \tag{3.07}$$

$$c^*_{W cell} = E c^*_L \tag{3.08}$$

$$p^*_{W cell} = p + P_n(c^*_{W cell} - c_1^{'} - c_0) \tag{3.09}$$

$$\bar{s}^{B+W_{cell}} = \frac{\varphi - p^*_{B+W}}{c^*_{B+W}} \tag{3.10}$$

$$\frac{c^*_{B+W}}{B+W} = \frac{(c_1^{'} + c_0)c^*_W}{B c^*_W + W(c_1^{'} + (1 - \frac{L}{W}E)c_0)} \tag{3.11}$$

$$p^*_{B+W} = p + P_n \frac{W}{B+W}\frac{c^*_{B+W}}{c^*_W}(c^*_W - (c_1^{'} + (1 - \frac{LE}{W})c_0)) \tag{3.12}$$

3.3 Adapted, modified Robin condition: the phreatic-leakage resistance

3.3.1 The cell-drain resistance in a detailed model with implicit lateral flow (iGrOw):

Case 1: The flux to the surface water is derived from the water balance between water divides:

The expression for the cell-drain resistance is formulated in terms of the average head in the cell. The cell-drain resistance is defined by:

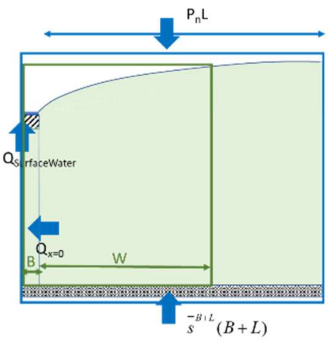
$$\bar{q}_{\square}^{B+W} = \frac{\bar{h}_{Wcell}^{B+W} - p_{cd,Wcell}^{*}}{c_{cd,Wcell}^{*}} \qquad (3.13)$$

The expression for the cell-drain resistance is derived from the total flux to the surface water:

$$Q_{surface-wat} = \bar{q}_{\square}^{B+W}(B+W) = \frac{\bar{h}_{Wcell}^{B+W} - p_{cd,Wcell}^{*}}{c_{cd,Wcell}^{*}}(B+W)$$

The expression for the cell-drain "entire" resistance (index e) is formulated in terms of the average head in the cell and is defined by:

$$\bar{q}_{\square}^{B+W} = \frac{\bar{h}_{Wcell}^{B+W} - p_{cd,Wcell}^{*}}{c_{cd,eWcell}^{*}} \qquad (3.13e)$$



$$Q_{surface-water} = \bar{s}_B B + (\bar{s}_L + P_n)L$$

Using the expression for s (2.22) derived above:

$$Q_{surface-water} - P_n L = \frac{\phi - p_{B+L}^{*}}{c_{B+L}^{*}}(B+L)$$

We may write for the lower boundary flux in the cell:

$$\bar{s}^{B+W_{cell}} = \frac{\phi - \bar{h}_{Wcell}^{B+W}}{c_1'}$$

In the expression for s, we replace ϕ by $\bar{h}_{Wcell}$ using this expression and (3.10):

$$\bar{s}^{B+W_{cell}} = \frac{\phi - p_{B+W}^{*}}{c_{B+W}^{*}} = \frac{\phi - \bar{h}_{Wcell}^{B+W}}{c_1'} \text{ leading to: } \phi = \frac{c_{B+Wl}^{*}\bar{h}_{Wcell}^{B+W} - c_1' p_{B+W}^{*}}{c_{B+W}^{*} - c_1'}$$

Rework the Robin condition for "$Q_{surfacewater} - P_n L$" from ϕ to h:

$$\frac{Q_{surfacewater} - P_n L}{B+L} = \frac{\phi - p^*_{B+L}}{c^*_{B+L}} = \frac{\dfrac{c^*_{B+W}\overline{h}^{B+W}_{Wcell} - c_1^{'} p^*_{B+W}}{c^*_{B+W} - c_1^{'}} - p^*_{B+L}}{c^*_{B+L}}$$

$$= \frac{c^*_{B+W}\overline{h}^{B+W}_{Wcell} - c_1^{'} p^*_{B+W} - p^*_{B+L}(c^*_{B+W} - c_1^{'})}{c^*_{B+L}(c^*_{B+W} - c_1^{'})}$$

$$\frac{Q_{surfacewater} - P_n L}{B+L} = \frac{c^*_{B+W}\overline{h}^{B+W}_{Wcell} - c_1^{'} p^*_{B+W} - p^*_{B+L}(c^*_{B+W} - c_1^{'})}{c^*_{B+L}(c^*_{B+W} - c_1^{'})}$$

$$= \frac{\overline{h}^{B+W}_{Wcell} - \dfrac{c_1^{'} p^*_{B+W} + p^*_{B+L}(c^*_{B+W} - c_1^{'})}{c^*_{B+W}}}{(c^*_{B+W} - c_1^{'})\dfrac{c^*_{B+L}}{c^*_{B+W}}}$$

Which has the shape of the required relation between Q and h in the cell

$$\frac{Q_{surfacewater} - P_n L}{B+L} = \frac{\overline{h}^{B+W}_{Wcell} - p^0_{(B+W)}}{(c^*_{B+W} - c_1^{'})\dfrac{c^*_{B+L}}{c^*_{B+W}}}$$

Work out the p^0-term:

$$p^0_{(B+W)} = \frac{c_1^{'} p^*_{B+W} + p^*_{B+L}(c^*_{B+W} - c_1^{'})}{c^*_{B+W}} = p^*_{B+L} + \frac{c_1^{'}}{c^*_{B+W}}(p^*_{B+W} - p^*_{B+L})$$

We derived before (2.24):

$$p^*_{B+L} = p + P_n(c^*_L - c_1^{'})\frac{c^*_{B+L}}{c^*_L}\frac{L}{B+L}$$

and (3.12)

$$p^*_{B+W} = p + P_n\frac{W}{B+W}\frac{c^*_{B+W}}{c^*_W}(c^*_W - (c_1^{'} + (1 - \frac{LE}{W})c_0))$$

So:

$$p^0_{(B+W)} = (1 - \frac{c_1^{'}}{c^*_{B+W}})p^*_{B+L} + \frac{c_1^{'}}{c^*_{B+W}}p^*_{B+W}$$

$$p^0_{(B+W)} = (1 - \frac{c_1^{'}}{c^*_{B+W}})[p + P_n(c^*_L - c_1^{'})\frac{c^*_{B+L}}{c^*_L}\frac{L}{B+L}] + \frac{c_1^{'}}{c^*_{B+W}}[p + P_n\frac{W}{B+W}\frac{c^*_{B+W}}{c^*_W}(c^*_W - (c_1^{'} + (1 - \frac{LE}{W})c_0))]$$

$$p^0_{(B+W)} = p + P_n[(1 - \frac{c_1^{'}}{c^*_{B+W}})(c^*_L - c_1^{'})\frac{c^*_{B+L}}{c^*_L}\frac{L}{B+L}] + \frac{c_1^{'}}{c^*_{B+W}}\frac{W}{B+W}\frac{c^*_{B+W}}{c^*_W}(c^*_W - (c_1^{'} + (1 - \frac{LE}{W})c_0))]$$

Which we rewrite into

$$p^0_{(B+W)} = p + P_n\beta$$

Where

$$\beta = [(1-\frac{c_1^{'}}{c_{B+W}^*})(c_L^*-c_1^{'})\frac{c_{B+L}^*}{c_L^*}\frac{L}{B+L}]+\frac{c_1^{'}}{c_{B+W}^*}\frac{W}{B+W}\frac{c_{B+W}^*}{c_W^*}(c_W^*-(c_1^{'}+(1-\frac{LE}{W})c_0))]$$

And finally:

$$\beta = \frac{L}{B+L}\frac{(c_{B+W}^*-c_1^{'})}{c_{B+W}^*}\frac{(c_L^*-c_1^{'})}{c_L^*}c_{B+L}^*+\frac{W}{B+W}\frac{c_1^{'}}{c_W^*}(c_W^*-(c_1^{'}+(1-\frac{LE}{W})c_0))$$

Intermezzo

CHECK 1, we rework $\frac{c_1^{'}}{c_{B+W}^*}(p_{B+W}^*-p_{B+L}^*)$ to show that it vanishes for W➔L:

$$(p_{B+W}^*-p_{B+L}^*) = [p+P_n\frac{W}{B+W}\frac{c_{B+W}^*}{c_W^*}(c_W^*-(c_1^{'}+(1-\frac{LE}{W})c_0))]-[p+P_n(c_L^*-c_1^{'})\frac{c_{B+L}^*}{c_L^*}\frac{L}{B+L}]$$

$$\frac{c_1^{'}}{c_{B+W}^*}(p_{B+W}^*-p_{B+L}^*) = P_n[\frac{W}{B+W}\frac{c_1^{'}}{c_W^*}(c_W^*-(c_1^{'}+(1-\frac{LE}{W})c_0))-\frac{L}{B+L}\frac{c_{B+L}^*}{c_{B+W}^*}\frac{c_1^{'}}{c_L^*}(c_L^*-c_1^{'})]$$

For W➔L we may write

$$\frac{c_1^{'}}{c_{B+W}^*}(p_{B+W}^*-p_{B+L}^*) = P_n[\frac{L}{B+L}\frac{c_1^{'}}{c_L^*}(c_L^*-(c_1^{'}+(1-\frac{L1}{L})c_0))-\frac{L}{B+L}\frac{c_{B+L}^*}{c_{B+L}^*}\frac{c_1^{'}}{c_L^*}(c_L^*-c_1^{'})]=0$$

So, we derived:
For W➔L $\quad p_{(B+W)}^0 = p_{B+L}^*$

So, the meaning of the term

$$\frac{c_1^{'}}{c_{B+W}^*}(p_{B+W}^*-p_{B+L}^*)$$

is related to the distribution of P_n to the surface water and the underlying aquifer for cases in which the cell width W is smaller than the distance between surface waters L. The term vanishes for W≥L.
End of intermezzo

So, we derived the parameters in:

$$\frac{Q_{surfacewater}-P_nL}{B+L} = \frac{\overline{h}_{Wcell}^{B+W}-p_{(B+W)}^0}{(c_{B+W}^*-c_1^{'})\frac{c_{B+L}^*}{c_{B+W}^*}}$$

Now, use the definition of the surface water flux averaged over the cell:

$$(B+W)\overline{q}^{B+W} = Q_{surfacewater}$$

Combining latter two expressions:

$$\overline{q}^{B+W} = \frac{Q_{surfacewater}}{(B+W)} = \frac{\overline{h}_{Wcell}^{B+W}-(p+P_n\beta)}{(c_{B+W}^*-c_1^{'})\frac{c_{B+L}^*}{c_{B+W}^*}}\frac{B+L}{(B+W)}+P_n\frac{L}{B+W}$$

leads to:

$$\overline{q}^{B+W} = \frac{\overline{h}_{Wcell}^{B+W}-(p+P_n\beta)}{\frac{(B+W)}{(B+L)}(c_{B+W}^*-c_1^{'})\frac{c_{B+L}^*}{c_{B+W}^*}}+P_n\frac{L}{B+W}$$

Then, we can write the Robin condition as follows

$$\overline{q}^{B+W} = \frac{\overline{h}_{Wcell}^{B+W} - p}{c_{cde,B+W}} - P_n[\frac{\beta}{c_{cde,B+W}} - \frac{L}{B+W}] = \frac{\overline{h}_{Wcell}^{B+W} - p}{c_{cde,B+W}} - P_n\alpha_{P_n}$$

where

$\alpha_{P_n} = [\frac{\beta}{c_{cde,B+W}} - \frac{L}{B+W}]$

Or

$$\overline{q}^{B+W} + P_n\alpha_{P_n} = \frac{\overline{h}_{Wcell}^{B+W} - p}{c_{cde,B+W}} \quad (3.14e)$$

In which we derived the **cell-drain "entire" resistance**:

$$c_{cde,B+W} = \frac{(B+W)}{(B+L)}(c_{B+W}^{*} - c_1^{'})\frac{c_{B+L}^{*}}{c_{B+W}^{*}} \quad (3.15e)$$

And for W➔L, this becomes:

$$c_{cde,B+W} = \frac{(B+L)}{(B+L)}(c_{B+L}^{*} - c_1^{'})\frac{c_{B+L}^{*}}{c_{B+L}^{*}} = (c_{B+L}^{*} - c_1^{'}) = c_{celdrain,B+L}$$

Next, we derive the expression for α_{P_n}:

$$\frac{\beta}{c_{cde,Wcell}} = \frac{\frac{L}{B+L}\frac{(c_{B+W}^{*} - c_1^{'})}{c_{B+W}^{*}}\frac{(c_L^{*} - c_1^{'})}{c_L^{*}}c_{B+L}^{*} + \frac{W}{B+W}\frac{c_1^{'}}{c_W^{*}}(c_W^{*} - (c_1^{'} + (1 - \frac{LE}{W})c_0))}{\frac{(B+W)}{(B+L)}(c_{B+W}^{*} - c_1^{'})\frac{c_{B+L}^{*}}{c_{B+W}^{*}}}$$

$$\frac{\beta}{c_{cde,Wcell}} = \frac{\frac{L}{B+L}\frac{(c_{B+W}^{*} - c_1^{'})}{c_{B+W}^{*}}\frac{(c_L^{*} - c_1^{'})}{c_L^{*}}c_{B+L}^{*}}{\frac{(B+W)}{(B+L)}(c_{B+W}^{*} - c_1^{'})\frac{c_{B+L}^{*}}{c_{B+W}^{*}}} + \frac{\frac{W}{B+W}\frac{c_1^{'}}{c_W^{*}}(c_W^{*} - (c_1^{'} + (1 - \frac{LE}{W})c_0))}{c_{cde,Wcell}}$$

$$\frac{\beta}{c_{cde,Wcell}} = \frac{L}{B+W}\frac{(c_L^{*} - c_1^{'})}{c_L^{*}} + \frac{W}{B+W}\frac{c_1^{'}}{c_W^{*}}\frac{(c_W^{*} - (c_1^{'} + (1 - \frac{LE}{W})c_0))}{c_{cde,Wcell}}$$

$$\alpha_{P_n} = [\frac{\beta}{c_{cde,Wcell}} - \frac{L}{B+W}] = \frac{L}{B+W}\frac{(c_L^{*} - c_1^{'})}{c_L^{*}} + \frac{W}{B+W}\frac{c_1^{'}}{c_W^{*}}\frac{(c_W^{*} - (c_1^{'} + (1 - \frac{LE}{W})c_0))}{c_{cde,Wcell}} - \frac{L}{B+W}$$

$$\alpha_{P_n} = \frac{W}{B+W}\frac{c_1^{'}}{c_W^{*}}\frac{(c_W^{*} - (c_1^{'} + (1 - \frac{LE}{W})c_0))}{c_{cde,Wcell}} - \frac{L}{B+W}\frac{c_1^{'}}{c_L^{*}}$$

Finally, we derived the constant α_{P_n}:

$$\alpha_{P_n} = \frac{W}{B+W}\frac{c_1^{'}}{c_W^{*}}\frac{(c_W^{*} - (c_1^{'} + (1 - \frac{LE}{W})c_0))}{c_{cde,Wcell}} - \frac{L}{B+W}\frac{c_1^{'}}{c_L^{*}}$$

The second term on the right hand side expresses the part (ratio$\frac{c_1^{'}}{c_L^{*}}$) of the recharge over the land surfaceP_nL that flows to the surface water and averaged over the cell-width $B+W$.

For W➔L, the parameterα_{P_n} becomes (all W's including indices change to L, and E=1):

$$\alpha_{P_n,W\to L} = \frac{L}{B+L}\frac{c_1^{'}}{c_L^*} - \frac{L}{B+L}\frac{c_1^{'}}{c_L^*}\frac{(c_L^* - (c_1^{'} + (1-\frac{L1}{L})c_0))}{(c_{B+L}^* - c_1^{'})} = \frac{L}{B+L}\frac{c_1^{'}}{c_L^*}[1 - \frac{(c_L^* - c_1^{'})}{(c_{B+L}^* - c_1^{'})}]$$

$$\alpha_{P_n,W\to L} = \frac{L}{B+L}\frac{c_1^{'}}{c_L^*}[1 - \frac{(c_L^* - c_1^{'})}{(c_{B+L}^* - c_1^{'})}] = \frac{L}{B+L}\frac{c_1^{'}}{c_L^*}[\frac{(c_{B+L}^* - c_1^{'}) - (c_L^* - c_1^{'})}{(c_{B+L}^* - c_1^{'})}] = \frac{L}{B+L}\frac{c_1^{'}}{c_L^*}\frac{(c_{B+L}^* - c_L^*)}{(c_{B+L}^* - c_1^{'})}$$

Next, we compare this result to the Robin condition for B+L:

$$\overline{q}_{cell}^{B+L} = \frac{\overline{h}_{B+L} - p}{(c_{B+L}^* - c_1^{'})} - P_n\frac{c_1^{'}}{c_L^*}\frac{(c_{B+L}^* - c_L^*)}{(c_{B+L}^* - c_1^{'})}\frac{L}{B+L} = \frac{\overline{h}_{B+L} - p}{(c_{B+L}^* - c_1^{'})} - P_n\alpha_{P_n}^{BL}$$

Which leads to a similar constant with P_n named$\alpha_{P_N}^{BL}$:

$$\alpha_{P_n}^{BL} = \frac{c_1^{'}}{(c_{B+L}^* - c_1^{'})}\frac{L}{B+L}(\frac{c_{B+L}^*}{c_L^*} - 1) = \frac{L}{B+L}\frac{c_1^{'}}{c_L^*}\frac{(c_{B+L}^* - c_L^*)}{(c_{B+L}^* - c_1^{'})}$$

This result is equal to $\alpha_{P_n}$for W➔L.

Final Text, results

$\overline{q}_{⬚}^{B+W} = \frac{\overline{h}_{Wcell}^{B+W} - p_{d,B+W}^*}{c_{cde,B+W}^*}$ becomes $\overline{q}^{B+W} + P_n\alpha_{P_n} = \frac{\overline{h}_{Wcell}^{B+W} - p}{c_{cde,B+W}}$ with

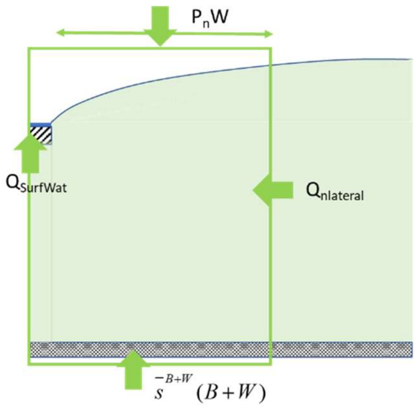
$c_{cde,B+W} = \frac{(B+W)}{(B+L)}(c_{B+W}^* - c_1^{'})\frac{c_{B+L}^*}{c_{B+W}^*}$ and

$$\alpha_{P_n} = \frac{W}{B+W}\frac{c_1^{'}}{c_W^*}\frac{(c_W^* - (c_1^{'} + (1-\frac{LE}{W})c_0))}{c_{cde,Wcell}} - \frac{L}{B+W}\frac{c_1^{'}}{c_L^*} \qquad (3.16e)$$

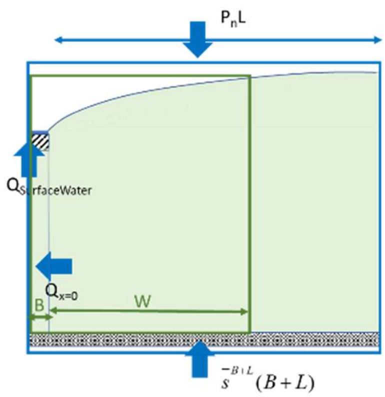
3.3.2 The cell-drain resistance in a regional cell-based model with explicit lateral flow (MODFLOW):

Case 2: The flux to the surface water is determined from the water balance of the cell (Figure at left). The lateral flow component is explicit in the derivation:

$$Q_{SurfWat} = \bar{s}^{B+W}(B+W) + P_n W + Q_{lateral}$$



IN STEAD OF



In this case, the expression for the cell-drain "cell" resistance (index c) is defined by:

$$\bar{q}^{B+W} - q_{lateral} = \frac{\bar{h}^{B+W}_{Wcell} - p^*_{cd,Wcell}}{c^*_{cdc,Wcell}} \qquad (3.13c)$$

Rewrite the water balance into:

$$\bar{s}^{B+W} = \frac{Q_{SurfWat} - P_n W - Q_{lateral}}{B+W} = \frac{Q_{B+W} - P_n W}{B+W}$$

From the Figure at right, it follows that:

$$Q_{SurfWat} = s^{B+L}(B+L) + P_n L$$

The lateral flux follows from left and right Figure:

$Q_{lat} = (P_n + \bar{s}^{L-W})(L-W)$ with: $\quad \bar{s}^{L-W}(L-W) = \bar{s}^{B+L}(B+L) - \bar{s}^{B+W}(B+W)$

So:

$$Q_{lat} = P_n(L-W) + \bar{s}^{B+L}(B+L) - \bar{s}^{B+W}(B+W)$$

Or

$$\bar{s}^{B+L}(B+L) = \bar{s}^{B+W}(B+W) + Q_{lat} - P_n(L-W)$$

Q_{lat} is calculated in MODFLOW and might be derived analytically. It is implicitly part of the analytic solution for flow and head in the cell. Next, we introduce Q_{B+W}:

$Q_{B+W} = Q_{SurfWat} - Q_{lat}$ so $Q_{SurfWat} = Q_{B+W} + Q_{lateral}$

So:

$$\bar{s}^{B+W} = \frac{Q_{B+W} - P_n W}{B+W} \frac{Q_{surfacewater} - P_n W - Q_{lat}}{B+L} = \frac{Q_{B+W} - P_n W}{B+L}$$

Before we derived:

$$\bar{s}^{B+W} = \frac{\phi - p^*_{B+W}}{c^*_{B+W}}$$

In the latter expression, we may replace ϕ by $\overline{h}_{Wcell}^{B+W}$ using 2 expressions for the lower boundary flux in the cell, that leads to a Robin condition with the phreatic head in the cell:
$\overline{s}^{B+W}{}_{cell} = \frac{\phi - p_{B+W}^*}{c_{B+W}^*} = \frac{\phi - \overline{h}_{Wcell}^{B+W}}{c_1^{'}}$ leading to: $\phi = \frac{c_{B+Wl}^* \overline{h}_{Wcell}^{B+W} - c_1^{'} p_{B+W}^*}{c_{B+W}^* - c_1^{'}}$

Replace and work out:

$$\overline{s}^{B+W} = \frac{\phi - p_{B+W}^*}{c_{B+W}^*} = \frac{\frac{c_{B+Wl}^* \overline{h}_{Wcell}^{B+W} - c_1^{'} p_{B+W}^*}{c_{B+W}^* - c_1^{'}} - p_{B+W}^*}{c_{B+W}^*}$$
$$= \frac{c_{B+Wl}^* \overline{h}_{Wcell}^{B+W} - c_1^{'} p_{B+W}^* - p_{B+W}^*(c_{B+W}^* - c_1^{'})}{c_{B+W}^*(c_{B+W}^* - c_1^{'})}$$

$$\overline{s}^{B+W} = \frac{c_{B+Wl}^* \overline{h}_{Wcell}^{B+W} - c_1^{'} p_{B+W}^* - p_{B+W}^*(c_{B+W}^* - c_1^{'})}{c_{B+W}^*(c_{B+W}^* - c_1^{'})}$$
$$= \frac{c_{B+Wl}^* \overline{h}_{Wcell}^{B+W} - c_1^{'} p_{B+W}^* - p_{B+W}^* c_{B+W}^* + p_{B+W}^* c_1^{'}}{c_{B+W}^*(c_{B+W}^* - c_1^{'})}$$

$$\overline{s}^{B+W} = \frac{c_{B+Wl}^* \overline{h}_{Wcell}^{B+W} - p_{B+W}^* c_{B+W}^*}{c_{B+W}^*(c_{B+W}^* - c_1^{'})} = \frac{\overline{h}_{Wcell}^{B+W} - p_{B+W}^*}{(c_{B+W}^* - c_1^{'})}$$

Or

$$\frac{Q_{B+W} - P_n W}{B+W} = \frac{c_{B+Wl}^* \overline{h}_{Wcell}^{B+W} - p_{B+W}^* c_{B+W}^*}{c_{B+W}^*(c_{B+W}^* - c_1^{'})} = \frac{\overline{h}_{Wcell}^{B+W} - p_{B+W}^*}{(c_{B+W}^* - c_1^{'})}$$

Or:

$$\frac{Q_{B+W} - P_n W}{B+W} = \frac{c_{B+Wl}^* \overline{h}_{Wcell}^{B+W} - p_{B+W}^* c_{B+W}^*}{c_{B+W}^*(c_{B+W}^* - c_1^{'})} = \frac{\overline{h}_{Wcell}^{B+W} - p_{B+W}^*}{(c_{B+W}^* - c_1^{'})}$$

$$q_{B+W} - P_n \frac{W}{B+W} - q_{lateral} = \frac{\overline{h}_{Wcell}^{B+W} - p_{B+W}^*}{(c_{B+W}^* - c_1^{'})}$$

We use:

$$p_{B+W}^* = p + P_n \frac{W}{B+W} \frac{c_{B+W}^*}{c_W^*} (c_W^* - (c_1^{'} + (1 - \frac{LE}{W}) c_0))$$

$$q_{B+W} - q_{lateral} = \frac{\overline{h}_{Wcell}^{B+W} - p - P_n \frac{W}{B+W} \frac{c_{B+W}^*}{c_W^*} (c_W^* - (c_1^{'} + (1 - \frac{LE}{W}) c_0))}{(c_{B+W}^* - c_1^{'})} + P_n \frac{W}{B+W}$$

$$q_{B+W} - q_{lateral} = \frac{\overline{h}_{Wcell}^{B+W} - p}{(c_{B+W}^* - c_1^{'})} + P_n \frac{W}{B+W} - P_n \frac{W}{B+W} \frac{c_{B+W}^*}{(c_{B+W}^* - c_1^{'})} \frac{(c_W^* - (c_1^{'} + (1 - \frac{LE}{W}) c_0))}{c_W^*}$$

$$q_{B+W} - q_{lateral} = \frac{\overline{h}_{Wcell}^{B+W} - p}{(c_{B+W}^* - c_1^{'})} + P_n \frac{W}{B+W} \frac{c_W^*(c_{B+W}^* - c_1^{'}) - c_{B+W}^*(c_W^* - (c_1^{'} + (1 - \frac{LE}{W}) c_0))}{(c_{B+W}^* - c_1^{'}) c_W^*}$$

$$q_{B+W} - q_{lateral} = \frac{\overline{h}_{Wcell}^{B+W} - p}{(c_{B+W}^{*} - c_1^{'})} + P_n \frac{W}{B+W} \frac{(c_{B+W}^{*} - c_1^{'}) - c_{B+W}^{*} + c_{B+W}^{*}(\frac{c_1^{'}}{c_W^{*}} + (1 - \frac{LE}{W})\frac{c_0}{c_W^{*}}))}{(c_{B+W}^{*} - c_1^{'})}$$

$$q_{B+W} - q_{lateral} = \frac{\overline{h}_{Wcell}^{B+W} - p}{(c_{B+W}^{*} - c_1^{'})} + P_n \frac{W}{B+W} \frac{c_1^{'}(c_{B+W}^{*} - c_W^{*}) + c_0(c_{B+W}^{*})(1 - \frac{LE}{W})}{c_W^{*}(c_{B+W}^{*} - c_1^{'})}$$

So, we derived:

$$q_{B+W} = \frac{\overline{h}_{Wcell}^{B+W} - p}{c_{cdc,B+W}} + P_n\beta + q_{lat} \qquad (3.14c)$$

In which the cell drain "cell" resistance is expressed by:

$$c_{cdc,B+W} = (c_{B+W}^{*} - c_1^{'}) \qquad (3.15c)$$

And with

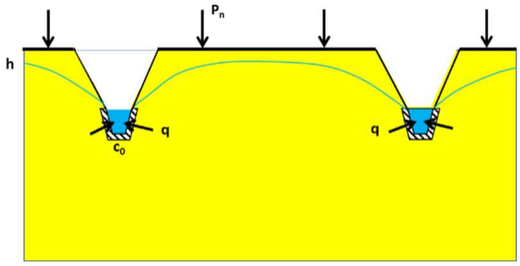
$$\beta = \frac{W}{B+W} \frac{c_1^{'}(c_{B+W}^{*} - c_W^{*}) + c_0(c_{B+W}^{*})(1 - \frac{LE}{W})}{c_W^{*}(c_{B+W}^{*} - c_1^{'})} \qquad (3.16c)$$

And q_{lat} is computed between nodes by MODFLOW

Chapter 4 Equations for Robin boundary condition in a single aquifer system

4.1 Introduction

In the following section result of multi-layer for single layer conditions is compared to classical formula (parabola version of Ernst).



4.2 Derivation of "classical" solution for single layer (aquifer)

In this case, the focus is on the expression for the resistance due to the flow in a single aquifer.

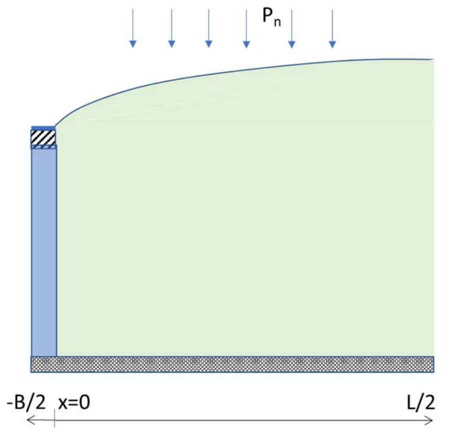


Figure 1 Scheme for derivation

Differential equation:

$\frac{d^2h_x}{dx^2} = \frac{h_x - p}{\lambda^2}$ for -B/2<x<0 and $\frac{d^2h_x}{dx^2} = \frac{-P_n}{kD}$ for 0<x<L/2

Boundary conditions:

$$x = 0 \quad \rightarrow \quad h_{left} = h_{right} \ and \ \frac{dh_x}{dx}_{left} = \frac{dh_x}{dx}_{right}$$

$$x = \frac{-B}{2} \quad en \quad x = \frac{L}{2} \quad \rightarrow \quad q = 0 \quad \rightarrow \quad \frac{dh_x}{dx} = 0$$

General solution:

$h_x - p = C_1 e^{\frac{-x}{\lambda_B}} + C_2 e^{\frac{x}{\lambda_B}}$ and $h_x = \frac{-P_n}{2k} x^2 + C_3 x + C_4$

$\frac{dh_x}{dx} = C_1 e^{\frac{-x}{\lambda_B}} + C_2 e^{\frac{x}{\lambda_B}}$ and $\frac{dh_x}{dx} = \frac{-P_n}{kD} x + C_3$

$$\lambda_B = \sqrt{k_h D c_0} \tag{4.1}$$

Solution of the integration constants:
From conditions at left and right boundary, it follows:
$C_1 = C_2 \alpha_B$

met $\alpha_B = e^{\frac{-B}{\lambda_B}}$ (4.2)

and $C_3 = \frac{P_n L}{2kD}$

From the conditions of continuity of flow and potential at x=0, it follows that

$C_2 = \frac{P_n L}{2kD} \frac{\lambda}{(1-\alpha_B)}$ and $C_4 = \frac{P_n L \lambda}{2} \frac{(1+\alpha_B)}{(1-\alpha_B)}$

The average head over -B/2<x<0, is found from integration:

$$\frac{\overline{h}_B}{C_2} = \frac{2}{B} \int_{\frac{-B}{2}}^{0} (e^{\frac{x}{\lambda_B}} + \alpha_B e^{\frac{-x}{\lambda_B}}) dx = \frac{2\lambda_B}{B} \int_{\frac{-B}{2}}^{0} (e^{\frac{x}{\lambda_B}} - \alpha_B e^{\frac{-x}{\lambda_B}})_{-B/2}^{0} =$$

$$= \frac{2\lambda_B}{B} [(1-\alpha_B) - (e^{\frac{-B}{2\lambda_B}} - \alpha_B e^{\frac{B}{2\lambda_B}})] = \frac{2\lambda_B}{B}(1-\alpha_B)$$

And using C_2:

$$\overline{h}_B = \frac{P_n}{KD} \frac{2}{B} \frac{L\lambda}{2} \frac{\lambda(1-\alpha_B)}{(1-\alpha_B)}$$

Work out:
$\overline{h}_B = c_0 \frac{P_n L}{B} \quad or \quad \frac{\overline{h}_B}{c_0} B = P_n L$ (4.3)

The average head over 0<x<L/2, is found from integration:

$$\overline{h}_L = \frac{2}{L} \int_0^{L/2} [\frac{-P_n}{kD} (\frac{x^2}{2} - \frac{L}{2} x) + C_4] \, dx = \frac{2}{L} [\frac{-P_n}{kD} (\frac{x^3}{6} - \frac{L}{2} \frac{x^2}{2}) + C_4 x]_0^{L/2} = \frac{P_n L^2}{12kD} + C_4$$

Work out C_4:

$$C_4 = \frac{P_n}{kD} \frac{\lambda_B L}{2} \frac{(1+\alpha_B)}{(1-\alpha_B)} = P_n c_0 \frac{L}{B} \frac{B}{2\lambda_B} ctnh(\frac{B}{2\lambda_B}) = P_n c_0 \frac{L}{B} F_B$$

So:
$\bar{h}_L = \frac{P_n L^2}{12kD} + P_n c_0 \frac{L}{B} F_B$ (4.4)

The average head over -B/2 tot L/2 is:

$\bar{h}_{B+L} = \frac{B}{B+L} \bar{h}_B + \frac{L}{B+L} \bar{h}_L$ (4.5)

The total, flux to the surface water $P_n L$ averaged over B+L is:

$$\overline{q}_{B+L} = \frac{P_n L}{B+L} \qquad (4.6)$$

De phreatic leakance is defined by:

$$c_{FL} = \frac{\bar{h}_{B+L}}{\overline{q}_{B+L}} \qquad (4.7)$$

Combining the expressions, we derive

$$c_{FL} = \frac{\bar{h}_{B+L}}{\overline{q}_{B+L}} = \frac{B}{LP_n}\bar{h}_B + \frac{1}{P_n}\bar{h}_L = \frac{B}{LP_n}c_0\frac{P_n L}{B} + \frac{1}{P_n}[\frac{P_n L^2}{12kD} + P_n c_0 \frac{L}{B}F_B]$$

Leading to:

$$c_{FL} = c_0 + \frac{L^2}{12kD} + c_0\frac{L}{B}F_B \qquad (4.8)$$

We define

$$B_{eff} = \frac{B}{F_B} \qquad (4.9)$$

Combine terms with c_0:

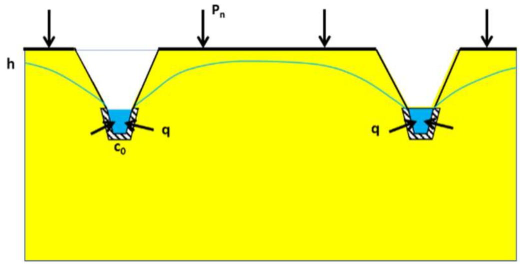
$$c_0 + c_0\frac{L}{B}F_B = c_0\frac{L+B_{eff}}{B_{eff}}$$

So:

$$c_{FL} = \frac{L^2}{12kD} + c_0\frac{L+B_{eff}}{B_{eff}} \qquad (4.10)$$

4.3 Approaching a single layer system: Infinite resistance of the leakage layer

Analytic expressions for the interaction between surface water and groundwater in a single aquifer have been derived in history (Ernst, etc.). Here, the results of former section will be compared to one of these solutions by using the case in which the resistance of the underlying layer approaches infinity.



Discuss: How reduce to single system & why

Combined resistance for parts "B" and "L" .

$$\frac{B+L}{c_{B+L}*} = \frac{B}{c*_B} + \frac{L}{c*_L}$$

Analysis of "L "part

$$c_L^* = (c_0 + c_1{'}) \cdot F_L + \left(c_0 \frac{L}{B}\right) F_B$$

$F_L = \frac{L}{2\lambda_L} \cdot ctnh\left(\frac{L}{2\lambda_L}\right) \qquad \lambda_L = \sqrt{c_{1,L}{'} \cdot k_h \cdot D_L}$

For $c_1 \rightarrow \infty$ we get:
- $c_1' >> c_0$ and so: ($c_1' + c_0 = c_1$).
- $L/2\lambda_L \rightarrow 0$

In Dwight 1959, we find the following series expansion:

$$c_1 \frac{L}{2\lambda_L} \cdot ctnh\left(\frac{L}{2\lambda_L}\right) = c_1[1 + \frac{(\frac{L}{2\lambda_L})^2}{3} - \frac{(\frac{L}{2\lambda_L})^4}{45} + \frac{2(\frac{L}{2\lambda_L})^6}{945} \dots]$$

For $L/2\lambda_L \rightarrow 0$, the higher order terms vanish (compared to the first and second term), so:

$$c_1 \frac{L}{2\lambda_L} \cdot ctnh\left(\frac{L}{2\lambda_L}\right) = c_1 + \frac{L^2}{12kD}$$

Then c^*_L becomes:

$$c_L^* = c_1 + \frac{L^2}{12kD} + c_0 \cdot \frac{L}{B} F_B$$

We define

$$B_{effectief} = \frac{B}{F_B}$$

And determine the resistance in the aquifer only:

$$c_{FL,L} = \frac{L^2}{12kD} + c_0 \cdot \frac{L}{B_{eff}}$$

Analysis of "B "part

$F_B = \frac{B}{2\lambda_B} \cdot ctnh\left(\frac{B_{hor}}{2\lambda_B}\right) \qquad \lambda_B = \sqrt{\frac{k_h \cdot D_B \cdot (c_{1,B}{'}) \cdot c_0}{c_1{'} + c_0}}$

For $c_1 \rightarrow \infty$, we find $\lambda_B = \sqrt{k_h D c_0}$

Look at expression for c^*_B:

$$c_B^* = \frac{c_1' + c_0}{1 - \frac{L}{B}\frac{c_0}{c_L^*}}$$

If $c_1 \rightarrow \infty$ in c^*_L the last term in the denominator vanishes compared to 1. So:

$c_B^* = c_1' + c_0$.

And determine the resistance in the aquifer only:

$$c_{FL,B} = c_0$$

In this case, the resistances are in positioned adjacent to each other; the y are in series when looking in the direction of the horizontal flow. So, combination is addition of resistances, leading to

$c_0 + c_0 \frac{L}{B} F_B = c_0 \frac{L+B_{eff}}{B_{eff}}$

So:

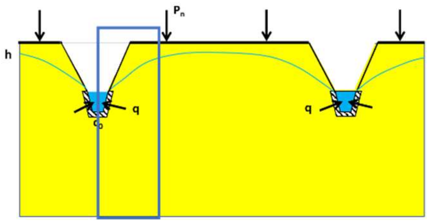
$$c_{FL} = \frac{L^2}{12kD} + c_0 \frac{L + B_{eff}}{B_{eff}}$$

Final text.
Which is equal to the result in section 4.2. So. The general formula derived in chapter 2.equals the single aquifer solution for a top-system above an impervious layer.

Summary: of results:

4.4 Derivation of "classical" solution for single layer (aquifer)

Analytic expressions for the interaction between surface water and groundwater in a single aquifer have been derived in history (Ernst, etc.). Here, the results of former section will be compared to one of these solutions by using the case in which the resistance of the underlying layer approaches infinity.



Differential equation, boundary conditions and solution for phreatic head have been derived in section 2.5. In this case we derive the average phreatic head in the cell by integration over the cell width:

$$\bar{h}_{Wcell} = \frac{2}{X}\int_0^{W_{cell}/2} [\frac{-P_n}{kD}(\frac{x^2}{2} - \frac{L}{2}x) + C_4]\, dx = \frac{2}{X}[\frac{-P_n}{kD}(\frac{x^3}{6} - \frac{L}{2}\frac{x^2}{2}) + C_4 x]_0^{W_{cell}/2}$$

$\bar{h}_{W_{cell}} = \frac{P_n}{8kD}(LW_{cell} - \frac{W_{cell}{}^2}{3}) + P_n c_0 \frac{L}{B} F_B$ (1.16)

The average phreatic head over the section -B/2 to X/2 is:

$\bar{h}_{W_{cell}} = \frac{B}{B+W_{cell}}\bar{h}_B + \frac{W_{cell}}{B+W_{cell}}\bar{h}_{W_{cell}}$ (1.17)

The total flux per unit length over the cell is:

$$\overline{q}_{B+W_{cell}} = \frac{P_n L}{B+W_{cell}} \tag{1.18}$$

Combine (1.17) and (1.18) gives:

$$c_{drainsys} = \frac{\bar{h}_{B+W_{cell}}}{\overline{q}_{B+W_{cell}}} = \frac{B}{P_n L}\bar{h}_B + \frac{X}{P_n L}\bar{h}_{W_{cell}}$$

Use (......): refer to section 2.5

$$c_{drainsys} = \frac{B}{P_n L}c_0\frac{P_n L}{B} + \frac{W_{cell}}{P_n L}[\frac{P_n}{8kD}(LW_{cell} - \frac{{W_{cell}}^2}{3}) + P_n c_0\frac{L}{B}F_B]$$

Leading to:

$$c_{drainsys} = c_0 + \frac{1}{8kD}\frac{W_{cell}}{L}(LW_{cell} - \frac{{W_{cell}}^2}{3}) + c_0\frac{W_{cell}}{B}F_B \tag{1.19}$$

Next, we separate in "entrance" and "horizontal" resistance:

The "horizontal" resistance is represented by the terms with kD:

$$c_{drainsys_{,hor}} = \frac{1}{8kD}\frac{W_{cell}}{L}(LW_{cell} - \frac{{W_{cell}}^2}{3})) \tag{1.20}$$

Work out:

$$\frac{1}{8}(LW_{cell} - \frac{{W_{cell}}^2}{3}) = \frac{{W_{cell}}^2}{12} + \frac{W_{cell}(L-W_{cell})}{8} \tag{1.21}$$

So:

$$c_{drainsys_{,hor}} = [\frac{{W_{cell}}^2}{12kD} + \frac{W_{cell}(L-W_{cell})}{8kD}]\frac{W_{cell}}{L} \tag{1.22}$$

Where W_{cell}/L indicates the part of the recharge P_n on the cell, the horizontal resistance

$$c_{celdrain\_hor} = \frac{{W_{cell}}^2}{12kD}\frac{W_{cell}}{L} \qquad W_{cell} \leq L \tag{1.23}$$

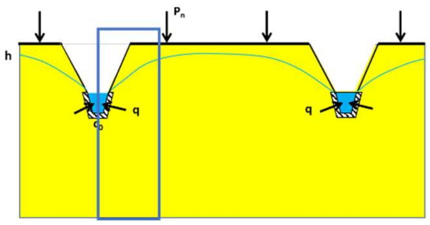
The part $(W_{cell}-X)/L$ represents the flux in the section $(L-W_{cell})$, lateral resistance:

$$c_{celdrain\_lat} = \frac{W_{cell}(L-W_{cell})}{8kD}\frac{W_{cell}}{L} = \frac{{W_{cell}}^2}{8kD}\frac{(L-W_{cell})}{L} \qquad W_{cell} \leq L \tag{1.24}$$

The entrance resistance is represented by the c_0 terms:

$$c_{drainsys_{,entree}} = c_0 + c_0\frac{W_{cell}}{B}F_B = c_0\frac{W_{cell}+B_{eff}}{B_{eff}} \qquad X \leq L \tag{1.26}$$

4.5 Assessing a single layer system from multi-layer system: Infinite resistance of leakage layer



Combined resistance for parts "B" and "W".

$$\frac{B+W}{c_{B+W}*} = \frac{B}{c*_B} + \frac{W}{c*_W}$$

Compare derivation in section 2.4, only the term in the with $c*_W$ is to be worked out.
Remember:

For $c_1 \rightarrow \infty$ we get:
- $c_1' >> c_0$ and so: $(c_1' + c_0 = c_1)$.
- $L/2\lambda_L \rightarrow 0$

$c_W^* = Ec_L^*$ with $c_L^* = (c_1' + c_0)F_L + \frac{c_0 L}{B} F_B$

$F_L = \frac{L}{2\lambda_L} ctnh(\frac{L}{2\lambda_L})$ and $F_B = \frac{B}{2\lambda_B} ctnh(\frac{B}{2\lambda_B})$;

And: for $c*_L$ we derived:

$$(c_1' + c_0)\frac{L}{2\lambda_L} ctnh(\frac{L}{2\lambda_L}) + \frac{c_0 L}{B}\frac{B}{2\lambda_B} ctnh(\frac{B}{2\lambda_B}) = (c_1' + c_0)\frac{L}{2\lambda_L}\frac{\alpha_L + 1}{\alpha_L - 1} + \frac{c_0 L}{B}\frac{B}{2\lambda_B}\frac{\alpha_B + 1}{\alpha_B - 1}$$

In which $\alpha_B = e^{\frac{-B}{\lambda_B}}$ and $\alpha_L = e^{\frac{L}{\lambda_L}}$

Work out E for $c_1 \rightarrow \infty$:

$$E = \frac{e^{L/\lambda_L} - 1}{(e^{W_{cell}/2\lambda_L} - 1) - e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L} - 1)}\frac{W_{cell}}{L}$$

$$= \frac{1 + \frac{L}{\lambda_L} - 1}{(1 + \frac{W_{cell}}{2\lambda_L} - 1) - (1 + \frac{L}{\lambda_L})(1 - \frac{W_{cell}}{2\lambda_L} - 1)}\frac{W_{cell}}{L}$$

$$E = \frac{1 + \frac{L}{\lambda_L} - 1}{(1 + \frac{W_{cell}}{2\lambda_L} - 1) - (1 + \frac{L}{\lambda_L})(1 - \frac{W_{cell}}{2\lambda_L} - 1)}\frac{W_{cell}}{L} = \frac{\frac{L}{\lambda_L}}{(\frac{W_{cell}}{2\lambda_L}) - (1 + \frac{L}{\lambda_L})(-\frac{W_{cell}}{2\lambda_L})}\frac{W_{cell}}{L}$$

$$= \frac{\frac{2\lambda_L}{W_{cell}}\frac{L}{\lambda_L}}{(1) + (1 + \frac{L}{\lambda_L})}\frac{W_{cell}}{L}$$

$E = \frac{2}{2 + \frac{L}{\lambda_L}} = 1 for \lambda_L \rightarrow \infty$

This result $E = 1 \quad for \lambda_L \rightarrow \infty$ is applicable to the term with F_B because the exponential terms in E and F_B are expressed in L and B respectively, so they are independent when taking the limit. However, this is not the case for E x F_L because the terms with L interact.

$$(c_1^{'}+c_0)\frac{L}{2\lambda_L}ctnh(\frac{L}{2\lambda_L})=(c_1^{'}+c_0)\frac{L}{2\lambda_L}\frac{\alpha_L+1}{\alpha_L-1}=(c_1^{'}+c_0)\frac{L}{2\lambda_L}\frac{e^{\frac{L}{\lambda_L}}+1}{e^{\frac{L}{\lambda_L}}-1}$$

Using

$$E=\frac{e^{L/\lambda_L}-1}{(e^{W_{Cell}/2\lambda_L}-1)-e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L}-1)}\frac{W_{cell}}{L}$$

We derive $c^*_w=Ec^*_L$, as follows:

$$(c_1^{'}+c_0)\frac{L}{2\lambda_L}\frac{e^{\frac{L}{\lambda_L}}+1}{e^{\frac{L}{\lambda_L}}-1}\frac{e^{L/\lambda_L}-1}{(e^{W_{Cell}/2\lambda_L}-1)-e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L}-1)}\frac{W_{cell}}{L}$$
$$=(c_1^{'}+c_0)\frac{W}{2\lambda_L}\frac{e^{L/\lambda_L}+1}{(e^{W_{Cell}/2\lambda_L}-1)-e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L}-1)}$$
$$(c_1^{'}+c_0)\frac{W}{2\lambda_L}\frac{(e^{W_{Cell}/2\lambda_L}-1)}{\frac{1}{(e^{L/\lambda_L}+1)}-\frac{e^{L/\lambda_L}}{(e^{L/\lambda_L}+1)}\frac{(e^{-W_{cell}/2\lambda_L}-1)}{(e^{W_{Cell}/2\lambda_L}-1)}}$$

Here, we have separated terms with L and W_{cell} for taking the limit of $\lambda_L\rightarrow\infty$:

The exponential series expansion reads:

$$e^{L/\lambda_L}=1+\frac{L}{\lambda_L}+\frac{L^2}{2!\,\lambda_L^2}+\frac{L^3}{3!\,\lambda_L^3}+\ldots=1+\frac{L}{\lambda_L}for\lambda_L\rightarrow\infty$$

For the limit the higher order terms vanish, so use only the first two terms in the following

$$(c_1^{'}+c_0)\frac{W}{2\lambda_L}\frac{(e^{W_{Cell}/2\lambda_L}-1)}{\frac{1}{(1+\frac{L}{\lambda_L}+1)}-\frac{1+\frac{L}{\lambda_L}}{(1+\frac{L}{\lambda_L}+1)}\frac{(e^{-W_{cell}/2\lambda_L}-1)}{(e^{W_{Cell}/2\lambda_L}-1)}}$$

Next, we use $\lambda_L\rightarrow\infty$:

$$(c_1^{'}+c_0)\frac{W}{2\lambda_L}\frac{(e^{W_{Cell}/2\lambda_L}-1)}{\frac{1}{(1+1)}-\frac{1}{(1+1)}\frac{(e^{-W_{cell}/2\lambda_L}-1)}{(e^{W_{Cell}/2\lambda_L}-1)}}=(c_1^{'}+c_0)\frac{W}{\lambda_L}\frac{(e^{W_{Cell}/2\lambda_L}-1)}{1-\frac{(e^{-W_{cell}/2\lambda_L}-1)}{(e^{W_{Cell}/2\lambda_L}-1)}}$$

Next, work out W-terms in the denominator:

$$1-\frac{(e^{-W_{cell}/2\lambda_L}-1)}{(e^{W_{Cell}/2\lambda_L}-1)}=1-\frac{(1-\frac{W}{2\lambda_L}-1)}{(1+\frac{W}{2\lambda_L}-1)}=1-\frac{(-\frac{W}{2\lambda_L})}{(\frac{W}{2\lambda_L})}=1+1=2$$

So:

$$(c_1^{'}+c_0)\frac{W}{\lambda_L}\frac{(e^{W_{Cell}/2\lambda_L}-1)}{1-\frac{(e^{-W_{cell}/2\lambda_L}-1)}{(e^{W_{Cell}/2\lambda_L}-1)}}=(c_1^{'}+c_0)\frac{W}{\lambda_L}\frac{(e^{W_{Cell}/2\lambda_L}-1)}{2}=(c_1^{'}+c_0)\frac{W}{2\lambda_L}(1+\frac{W}{2\lambda_L}-1)$$
$$=(c_1^{'}+c_0)[\frac{W}{2\lambda_L}]^2$$

Use

$\lambda_L=\sqrt{c_{1,L}{'}\cdot k_h\cdot D_L}$,

And use $c_1\rightarrow\infty$:

$$(c_1^{'} + c_0)[\frac{W}{2\lambda_L}]^2 = (c_1^{'} + c_0)\frac{W^2}{4c_1{'} \cdot k_h \cdot D_L} = \frac{W^2}{4k_h D_L}$$

This result is a factor 1/3 different from the horizontal resistance in section 4.2 and 4.3.

Chapter 5 The formulas and parameters for the cell-drain-resistance in numerical modeling.

The derivations of chapter 2 and 3 and the link to the Ernst-type formula’s in chapter 4 are used to set up a series of formulas that sum up to the resistance in the Robin boundary to be used in modeling, the so-called cell-drain-resistance. The formulas presented in this chapter apply to a single cell in a numerical model as sketched in figure 5.1. The cell-drain-resistance can straightforward be used in the conductance in MODFLOW.

Often, a top-system as described in chapter 1 is not very clear to define in the field. The field based properties for use in the formulas for the cell-drain-resistance do not one to one apply to the cell domain only, but may also involve a larger volume. This will be explained in section 5.5

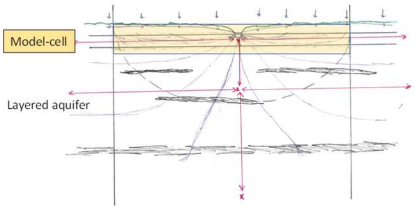


Figure 5.1 Sketch of a layered aquifer in which the uppermost model cell is arbitrary qua position and size. The red arrows depict interactions between calculation nodes.

The formulas are presented for the situation in which the cell is smaller than the distance between the surface waters see figure 5.2, because this is the most complete expression, which turns into the appropriate expression for the case the cell is equal or larger than the distance between surface waters. Also, the multi-later version is used which turns into a single layer version for infinite resistance of the leaky layer between top-system and first aquifer.

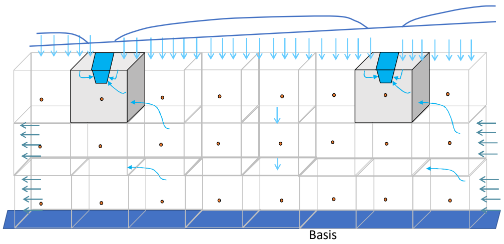


Figure 5.2 Scheme of a single cell in a numerical model for which the resistance in the Robin condition is derived.

The cell-drain-resistance applies to the term for the Robin boundary condition in a numerical model depicted by the red arrow in figure 5.3 at right.

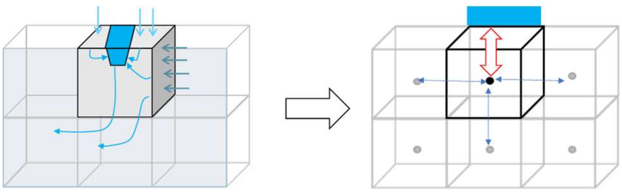


Figure 5.3 Scheme of a single cell in a numerical model for which the resistance in the Robin condition is derived.

In modelers practice, the basic idea of Ernst (1959) helps to understand the different components in the resistance. The concept of Ernst and later on of Van Bakel and De Lange use for sub-resistances being the horizontal, vertical, entrance and radial resistance, which will be presented first.

Definition of symbols (from Stromingen 2025)

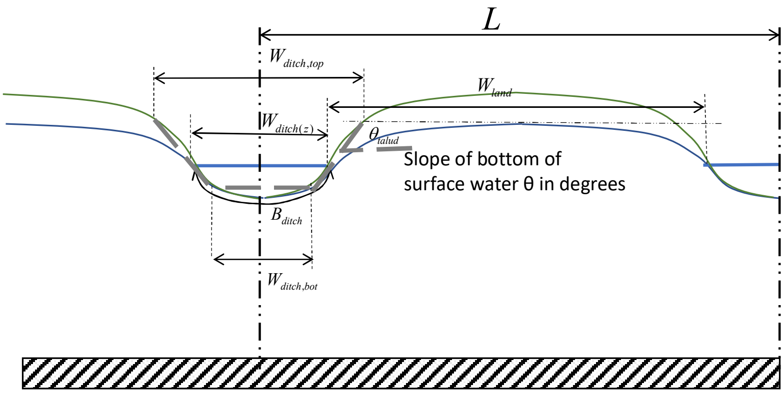


Figure 5.4 Definition of symbols

5.1 The combined entry and horizontal resistance (from CH 2 and 3)

<u>From section 3.3.2</u>

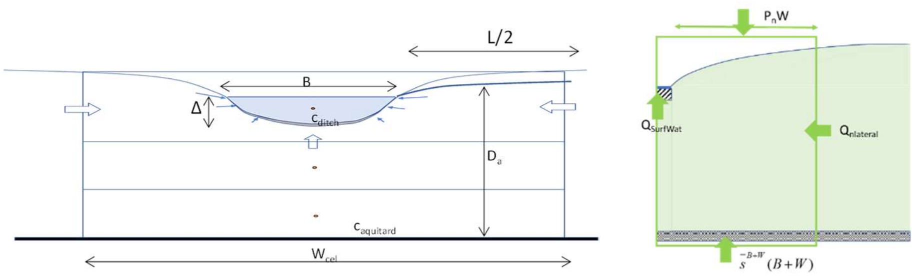


Figure 5.5 The combined entry resistance and horizontal resistance:

Robin boundary condition from chapter 3:

$$q_{B+W} = \frac{\overline{h}^{B+W}_{Wcell} - p}{c_{cDL2020,B+W}} + P_n\beta + q_{lat} \qquad (3.14c)$$

Where q_{lat} is computed between nodes of the cells and the recharge coefficient is:

$$\beta = \frac{W}{B+W}\,\frac{c_1'(c^*_{B+W} - c^*_W) + c_0(c^*_{B+W})(1 - \frac{LE}{W})}{c^*_W(c^*_{B+W} - c_1')} \qquad (3.16c)$$

In equation (3.14c) the "top-system resistance DL2020" $c_{cDl2020,B+W}$ includes the horizontal and entrance resistance and is expressed by:

$$c_{cDl2020,B+W} = (c^*_{B+W} - c_1') \qquad (3.15c)$$

<u>From section 3.2.</u>

$$c^*_{B+W} = \frac{(c_1' + c_0)c^*_W * (B+W)}{Bc^*_W + W(c_1' + (1 - \frac{L}{W}E)c_0)} \qquad (3.11)$$

$$c^*_W = Ec^*_L \qquad (3.08)$$

$$E = \frac{e^{L/\lambda_L - 1}}{(e^{W_{Cell}/2\lambda_L - 1}) - e^{L/\lambda_L}(e^{-W_{cell}/2\lambda_L - 1})} \frac{W_{cell}}{L} \qquad (3.04)$$

From analysis of $c_W^* = Ec_L^*$ the calculation can be simplified by

replace L by W_cell in F_L in (2.08) for L< W_cell

which automatically accounts for E.

<u>From section 2.1</u>

$$c_L^* = (c_1^{'} + c_0)F_L + \frac{c_0 L}{B} F_B \qquad (2.10)$$

$$F_L = \frac{L}{2\lambda_L} ctnh(\frac{L}{2\lambda_L}) \text{ and } F_B = \frac{B}{2\lambda_B} ctnh(\frac{B}{2\lambda_B}) \qquad (2.08)(2.09)$$

$$\lambda_B^2 = k_h H \frac{c'_1 c_0}{c'_1 + c_0} \text{ and } \lambda_L^2 = k_h H c'_1 \qquad (2.02)(2.04)$$

$$c'_1 = c_1 + \frac{H}{k_v} \qquad (2.01)$$

5.2 The vertical resistance

Strongly layered aquifer: low vertical conductivity, figure 5.6
Resistance along flow lines starting infiltration at land surface and ending at surface water: vertical resistance =< horizontal resistance

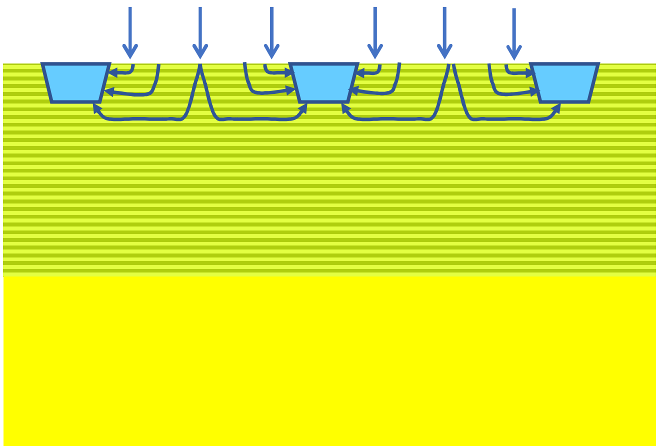


Figure 5.6 Resistance along flow lines: vertical resistance versus horizontal resistance

From comparisons and analyses with numerical model iGrOw, the following approach is sufficient.
Equations are derived for both the water (index B) and the land (index L) part.
The maximum horizontal maximum path lengths are determined by the physical property the characteristic length λ. From classical theory, the length over which groundwater flow is affected by a surface water equals 3λ, leading to:

$$B_{max} = \max(3\lambda_B, B) \text{ and } L_{max} = \max(3\lambda_L, L)$$

The maximum resistance in vertical direction is smaller than that in horizontal direction. The vertical resistance of the entire top-system is described by:

$$c_{vertical,basis} = \frac{H}{k_v}$$

The maximum resistance in vertical direction is less or equal to that in horizontal direction:

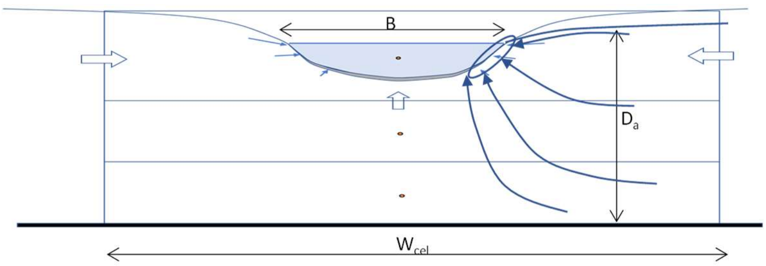
$c_{B,vert} = \max\,(B_{max}/k_h, c_{vertical,basis})$ and $c_{L,vert} = \max\,(L_{max}/k_h, c_{vertical,basis})$

Because the average (of all, figure 5.6) vertical path-line equals half of the largest path-line, the average vertical resistance also equals half of the entire. The total resistance is the sum of both:

$$c_{vertical} = \frac{c_{B,vert} + c_{L,vert}}{2}$$

5.3 The radial resistance

Flow into the surface water is contracted within the effective wetted perimeter Beff



input parameters $k_h, D_a, k_v, B, W_{cel}$, Ernst equation.

$$c_{rad} = \frac{W_{cel}}{\pi\sqrt{k_h k_v}} Ln(\frac{\pi D_a}{4B_{eff}})$$

$B_{eff} = \frac{B}{F_B}$ (see (4.8),(4.9))

$F_B = B/2\lambda_B + \frac{1}{1+B/2\lambda_B}$ (accurate approximation of 2.09)

$$\lambda_B = \sqrt{k_h D_a c_0}$$

5.4 The cell-drainage-resistance

The cell drainage resistance equals the sum of the above three resistances:

$$c_{cell\_drain\_resistancel} = c_{c_{cDL202,,B+W}} + c_{radial} + c_{vertical}$$

5.5 Assessing parameters for modeling from field situations

Often, a top-system as described in chapter 1 is not very clear to define in the field. The field based properties for use in the formulas for the cell-drain-resistance do not one to one apply to the cell domain only, but may also involve a larger volume. This will be explained next.

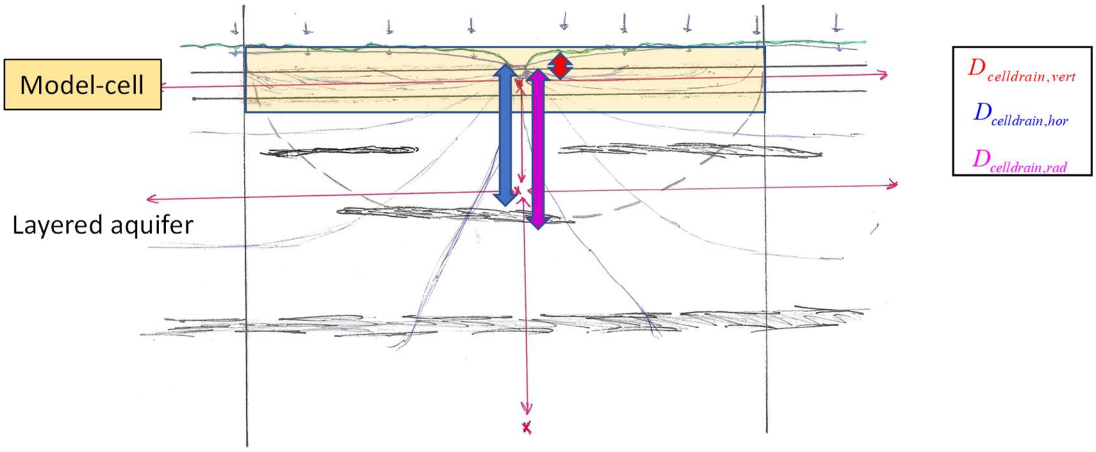


Figure 5.N Thicknesses (thick colored arrows) to be used in the different components of the cell-drain-resistance in the case the uppermost model cell is arbitrary qua position and size. The thin red arrows depict interactions between calculation nodes.

To be added.