

## DYNAMIC ASSET ALLOCATION STRATEGIES USING A STOCHASTIC DYNAMIC PROGRAMMING APPROACH

GERD INFANGER

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**Abstract**

A major investment decision for individual and institutional investors alike is to choose between different asset classes, i.e., equity investments and interest-bearing investments. The asset allocation decision determines the ultimate risk and return of a portfolio. The asset allocation problem is frequently addressed either through a static analysis, based on Markowitz' mean-variance model, or dynamically but often myopically through the application of analytical results for special classes of utility functions, e.g., Samuelson's fixed-mix result for constant relative risk aversion. Only recently, the full dynamic and multi-dimensional nature of the asset allocation problem could be captured through applications of stochastic dynamic programming and stochastic programming techniques, the latter being discussed in various chapters of this book. The chapter reviews the different approaches to asset allocation and presents a novel approach based on stochastic dynamic programming and Monte Carlo sampling that permits one to consider many rebalancing periods, many asset classes, dynamic cash flows, and a general representation of investor risk preference. The chapter further presents a novel approach of representing utility by directly modeling risk aversion as a function of wealth, and thus provides a general framework for representing investor preference. The chapter shows how the optimal asset allocation depends on the investment horizon, wealth, and the investor's risk preference and how it therefore changes over time depending on cash flow and the returns achieved. The chapter demonstrates how dynamic asset allocation leads to superior results compared to static or myopic techniques. Examples of dynamic strategies for various typical risk preferences and multiple asset classes are presented.

**Keywords**

portfolio theory and applications, dynamic asset allocation, stochastic dynamic programming, stochastic programming

*JEL classification:* C61, D81, G1

## 1. Introduction

The major decision of an investor regarding his/her portfolio is to choose the allocation between different asset classes, especially between equity investments and interest-bearing investments. Strategic asset allocation determines the ultimate expected rate of return and risk of an investor's portfolio, see, e.g., [Blake, Lehmann and Timmermann \(1999\)](#). For individuals and institutional investors alike, it is usually a long-term decision and its analysis should include all financial aspects, e.g., wealth, current and future cash flows, and financial goals. Inflation, as well as liquidity considerations (to plan for the unexpected), should be part of the analysis.

In the long run, equity investments have grown at a faster rate than other assets such as bonds, T-bills, and inflation, see, e.g., [Siegel \(2002\)](#), [Constantinides \(2002\)](#), and [Dimson, Marsh and Staunton \(2002\)](#). However, in the short run, the risk is significant and may lead to negative returns. Even in the long run, equity investments can be quite risky, and as we have painfully observed during the recent downturn from 2000 till 2003 as well as in bear markets before, equity investments can quickly lose a significant portion of their value. Interest-bearing investments have exhibited lower returns in the long run, barely exceeding inflation, but with less risk, making them possibly a better vehicle for the short-term. It is clear that one needs to determine the right balance between the asset classes, and this balance depends on how much risk an investor is willing to assume and may change over time due to changes in wealth, needs, goals and market conditions. It is well known that strategic asset allocation has a far greater impact on the overall performance of an investor's portfolio than, for example, the selection of individual securities.

The question of how the length of the investment horizon impacts the optimal asset allocation is an important theoretical and practical question, which has been discussed in the academic literature for more than 30 years. The answers to this question vary significantly depending on the assumptions made. For example, [Levy \(1972\)](#) and [Gunthorpe and Levy \(1994\)](#) discussed portfolio performance and composition versus investment horizon in a mean-variance framework. Practitioners tend to recommend a larger allocation towards stocks as the investment horizon increases. This is often argued under the name of "time diversification", and is based on the argument that if stocks are distributed independently and identically in each time period according to a log normal distribution, the distribution over many periods as the product of log normals is also log normal and the mean and the variance of the logarithm of the return distribution grows proportionally with the length of the investment period. Since the standard deviation of the log returns then grows with the square root of the horizon, the probability of capital loss decreases as time increases, and stocks become more favorable as the time horizon increases. Using this argument with a reasonable assumption of a 10% mean and 15% volatility, annually, one may verify that after an investment horizon of about seven years, the return on stocks exceeded with 95% probability is positive. It would take 13 years to arrive at a positive return exceeded with 99% probability.

Time diversification often includes a broader view of human capital, suggesting that the capital represented by an individual's ability to work and generate wages should also enter the equation. Consequently, at the beginning of one's professional life, most of one's capital is the potential of generating future income through labor, which is considered to be not risky, while at the end of one's professional career, most capital lies in one's financial assets (in the retirement account, but also real estate, stocks, etc.), and the ability to work and to generate income becomes a lesser part of the total capital. Thus, it is argued that at the beginning of one's career, the small amount in financial assets should be invested in a more risky allocation and gradually reduced to a less risky allocation at retirement, when all assets are the financial assets generated during the lifetime. Based on time diversification arguments, a common rule of thumb used by practitioners is to invest in stocks the percentage resulting from  $100 - \text{age}$ , thus a 30 year old should invest about 70% of his portfolio into stocks, and a 60 year old about 40%. These arguments only use time as the explaining factor and the corresponding strategies do not react to investment performance.

Treynor (2003) on time diversification puts forward the question of how each year's investment choices influence the wealth accumulated at the end of one's career. This perspective concludes that the impact of each year's dollar gains and losses on terminal wealth depend on the riskless rate, and one should time diversify in such a way that the risk evaluated in terminal dollars should be constant over the investment horizon. Therefore, unless investors can predict different returns for each year, the money amount exposed to the stock market should be approximately constant over the life-time.

Samuelson (1969) in his landmark work addressed how much of his/her wealth an individual should either consume or invest at any period in time, assuming no bequest is to be left behind. He proves using a backward dynamic programming recursion that, given the choice of one risky asset and one risk free asset, for the returns of the risky asset distributed identically and independently (iid) over time, for all income generated through investment, and for individuals valuing their consumption in time according to a power utility function with respect to consumption and maximizing discounted expected utility over the lifetime, it is optimal to invest the same proportion of wealth into stocks in every period, independently of wealth. The same was proved by Merton (1969) in continuous time and later, see Merton (1990), extended to multiple risky assets and various bequest situations. This life-time portfolio and consumption selection prompted an apparent conflict between theoreticians and practitioners, since the advice from Samuelson and Merton is quite different from what financial practitioners tell their clients. The remarkable aspect of Samuelson's and Merton's result is that, under their assumptions about the market and under constant relative risk aversion, the consumption decisions and the investment decisions are independent of each other, and therefore the optimal investment decision is not only invariant with respect to investment horizon but also with respect to wealth. Thus, the result translates directly to the investment problem only, where one wants to maximize the utility of final wealth at the end of the investment horizon, by allocating and re-allocating at each period along the way. The

result follows directly from the utility function used, stipulating that the (relative) risk aversion of the individual is invariant with respect to wealth.

Optimal multi-period investment strategies based on maximizing expected utility have been analyzed by Mossin (1968). Mossin's research attempted to isolate the class of utility functions of terminal wealth which result in myopic utility (value) functions for intermediate wealth positions. For a discussion of utility functions, see Section 4 below. Myopic means that these functions are independent of returns beyond the current period. Thus, it would be sufficient to analyze only the current optimization problem to arrive at an optimal multi-period investment strategy. Mossin concluded that logarithmic functions (for general asset distributions) and power functions (for serially independent asset distributions) are completely myopic. He also contended that, if there is a riskless asset (whose return is known for the entire investment horizon), and a risky one, all utility functions for which the risk tolerance is linear in wealth (HARA) would lead to partial myopia as one would optimally invest in any period as if in all further periods the investment were to be made in the riskless asset, and complete myopia would exist if the risk-free rate were to be zero. However, Hakansson (1971) demonstrated that, for the HARA case, even when asset returns are serially independent, no myopic strategies are optimal, except for the highly restricted case of a complete absence of any restrictions on short sales and borrowing. A percentage margin requirement, an absolute limit on borrowing, or a reasonable lending constraint such that the borrowed money would have to be repaid would not lead to a myopic strategy. Thus, in the presence of such restrictions only the power and logarithmic utility functions would lead to myopia. Furthermore, if asset returns are serially correlated, only the logarithmic utility function would result in a myopic policy. Later, Cox and Huang (1999) presented an analytical solution based on diffusion processes for the consumption-investment problem with HARA utility function and wealth and consumption constrained to be nonnegative. Later, an approach using approximate analytical solutions has been developed by Campbell and Viceira (Campbell and Viceira (2002)), based on perturbations of known exact solutions. See also the chapter by Chacko and Neumar (2006) in this book.

Thus we can summarize, a logarithmic utility function results in a myopic portfolio strategy, both for serially dependent and independent asset return distributions; a power utility function results in a myopic strategy only in the case of serially independent asset returns distributions; and a HARA utility function results in a myopic strategy only for serially independent asset return distributions and only in a non-realistic setting of complete absence of borrowing and short-selling constraints. All other utility functions do not result in myopic investment strategies for any return distributions.

More recently, numerical dynamic portfolio optimization methods have been developed that permit one to determine the asset allocation strategy that maximizes an investor's expected utility. These new approaches are based on stochastic dynamic programming and stochastic programming and promise to accurately solve for various types of utility functions and asset return processes.

## 2. Approaches for dynamic asset allocation

The two major approaches successful in solving practical dynamic asset allocation problems are stochastic dynamic programming (stochastic control) and stochastic programming. As we have discussed above, asset allocation problems under restrictive assumptions can be solved analytically. In practice, fixed-mix strategies are commonly implemented and often lead to very good results.

### 2.1. Multi-stage stochastic programming

The stochastic programming approach can efficiently solve the most general models, where transaction costs need to be considered, and the returns distributions have general serial dependency. The stochastic programming approach also lends itself well to the more general asset liability management problem (ALM). Here, liabilities in addition to assets need to be considered. This problem is faced by pension funds and insurance companies. Besides assets, pension plans need to consider retirement obligations, which may depend on uncertain economic and institutional variables, and insurance companies need to consider uncertain pay-out obligations due to unforeseen and often catastrophic events. Asset liability models are most useful when both asset returns and liability pay-outs are driven by common, e.g., economic, factors. In this case ALM represents the only approach that can take into account directly the joint distribution of asset returns and liability cash flows. Lenders operating in the secondary mortgage market also face a certain kind of ALM problem, when deciding on re-financing their (pools of) mortgages by issuing a portfolio of bonds, callable or non-callable with various maturities. Here the assets are the mortgages bought from banks and the liabilities are the bonds issued, see, e.g., [Infanger \(1999\)](#).

Traditional stochastic programming uses scenario trees to represent possible future events. The trees may be constructed by a variety of scenario generation techniques, with the emphasis on keeping the resulting tree thin but representative of the event distribution in order to arrive at a computationally tractable problem. Often, in later decision stages of the model, only a very small number of scenarios is used as a representation of the distribution leading to very thin sub-trees. Thus, the emphasis is on obtaining a good first-stage solution rather than obtaining an entire accurate policy. Early applications of stochastic programming for asset allocation are discussed in [Mulvey and Vladimirov \(1992\)](#), formulating financial networks, [Golub et al. \(1995\)](#). Early applications of stochastic programming for dynamic fixed-income strategies are discussed in [Zenios \(1993\)](#), discussing the management of mortgage-backed securities, [Hiller and Eckstein \(1993\)](#), and [Nielsen and Zenios \(1996\)](#). Early practical applications of stochastic programming for asset liability management are reported in [Kusy and Ziemba \(1986\)](#) for a bank and in [Carino et al. \(1994\)](#) for an insurance company. [Ziemba \(2003\)](#) gives a summary of the stochastic programming approach for asset liability management. An approach based on partitioning the probability space and calculating deterministic

bounds was developed by [Frauendorfer \(1996\)](#), and used for bond management and asset liability management. The book edited by [Wallace and Ziemba \(2005\)](#) gives publicly available stochastic programming code. Stochastic programming software can be best used from within a modeling system, for example, GAMS ([Brooke, Kendrick and Meeraus, 1988](#)) provides DECIS ([Infanger, 1997](#)) as an integrated stochastic programming solver.

Monte Carlo sampling is an efficient way to go when representing multi-dimensional distributions. An approach, referred to as decomposition and Monte Carlo sampling, uses Monte Carlo (importance) sampling within a decomposition for estimating Benders cut coefficients and right-hand sides. This approach has been developed by [Dantzig and Glynn \(1990\)](#) and [Infanger \(1992\)](#). [Dantzig and Infanger \(1993\)](#) show how the approach of decomposition and importance sampling could be used for solving multi-period asset allocation problems. The success of the sampling within the decomposition approach depends on the type of serial dependency of the stochastic parameter processes, determining whether or not cuts can be shared or adjusted between different scenarios of a stage. [Infanger \(1994\)](#) and [Infanger and Morton \(1996\)](#) show that for serial correlation (in form of autoregressive processes) of stochastic parameters, unless in the right hand side of the (linear) program, cut sharing is difficult for more than three decision stage problems. However, for serially independent stochastic parameters, the approach is very efficient for solving problems with many decision stages.

Monte Carlo pre-sampling uses Monte Carlo sampling to generate a tree, much like the scenario generation methods referred to above, and then employs a suitable method for solving the sampled (and thus approximate) problem. [Infanger \(1999\)](#) used the pre-sampling approach for representing the mortgage funding problem. This approach combines optimization and simulation techniques to represent a 360 month problem by four decision stages (initial, after a year, after 5 years, and at the end of the horizon) that are subject to optimization and by pre-defined decision rules representing intermediate decisions. The paper also provides an efficient way to independently evaluate the solution strategy as a result from solving the multi-stage stochastic program to obtain a valid upper bound on the objective. The pre-sampling approach is general in terms of modeling and solving stochastic processes with serial dependency; however, it is limited in the number of decision stages, since problems with many decision stages become computationally difficult. Assuming a reasonable sample size for representing the decision tree, problems with up to four decision stages are meaningfully tractable. Thus, if one were to represent asset allocation problems with many time (rebalancing) periods, one needed to represent more than one time period in one decision stage and define rules as to how to manage the assets and liabilities between decision stages. For example, one could assume buy and hold between decision stages and allow for rebalancing at the decision stages. In many situations this is considered a sufficiently accurate approximation of all future recourse decisions. A stochastic programming approach using pre-sampling has been employed by [Collomb and Infanger \(2005\)](#) to analyze the impact of serial dependency on the solution of dynamic asset allocation problems. Instead of pre-defining rebalancing periods up-front, the rebalancing decision may be modeled as

depending on certain conditions occurring. For example, McLean, Zhao and Ziemba (2006) model portfolio rebalancing as conditioned on prices exceeding or going below certain levels, and McLean, Ziemba and Li (2005) model portfolio rebalancing when certain wealth goals are met.

In this book, addressing dynamic strategies using stochastic programming for large institutions, Kouwenberg and Zenios (2006) introduce stochastic programming models as they apply to financial decision making, Mulvey et al. (2006) and Ziemba (2006) discuss the application of stochastic programming for multinational insurance companies, Consigli, Cocco and Zenios (2006) discuss the application of stochastic programming for insurance products with guarantees, Edirishinghe (2006) discusses dynamic strategies for money management, and Consigli (2006) discusses the development of stochastic programming models for individual investors.

## 2.2. Stochastic dynamic programming

When the focus is on obtaining optimal policies and transaction costs are not the primary issue, stochastic dynamic programming proves to be a very effective approach. Stochastic dynamic programming based on Bellman's (1957) dynamic programming principle has been used for the derivation of the theoretical results obtained for the HARA class of utility functions discussed above. For general monotone increasing and concave utility functions, no analytical solutions are available. However, stochastic dynamic programming can be used as an efficient numerical algorithm when the state space is small, say, up to three or four state variables. This limitation in the number of state variables is well known as the "curse of dimensionality" in dynamic programming. Recently, new methods of value function approximations, see, e.g., De Farias and Van Roy (2003) show promise for problems with larger state spaces; however, it is unclear at this point in time how accurately these methods will approximate the solution of the asset allocation problem. Stochastic dynamic programming as a numerical algorithm has been used by Musumeci and Musumeci (1999), representing results with two asset classes, one stock index and a risk free asset, where in the dynamic programming procedure they condition on the amount of wealth invested in the risky asset. Earlier, Adaci (1996) (in a Ph.D. thesis supervised by George Dantzig and the author) conditioned on wealth and thus set the stage for multiple asset classes, but reported results also only for two asset classes and a few periods. Brennan, Schwartz and Lagnado (1998a, 1998b) proposed a dynamic model using discrete state approximations including four state variables.

In this chapter we develop an efficient approach for solving the asset allocation problem under general utility functions, many time periods (decision stages), and many asset classes. We next review single-period portfolio choice and draw the connections between Markovitz' mean-variance analysis and utility maximization. We then discuss the properties of various utility functions and present a general framework for modeling utility. Then we discuss multi-period portfolio choice and present a novel approach based on stochastic dynamic programming and Monte Carlo sampling.



### 3. Single-period portfolio choice

Markowitz (1952), in his seminal work, see also Markowitz and van Dijk (2006) in this book, pointed out that the returns of a portfolio are random parameters and that for the evaluation of a portfolio, one should consider both its expected returns and its risk, where for representing risk he used the portfolio's variance. His mean-variance analysis laid the foundations for modern finance and our understanding as to how markets work. Mean-variance analysis is referred to as modern portfolio theory, whereas post-modern portfolio theory considers extensions including skewed distributions and asymmetric risk measures.

Let  $x = (x_1, \dots, x_n)$  be the holdings of  $n$  asset classes under consideration for a portfolio in a certain time period, and let  $R = (R_1, \dots, R_n)$  denote the random rates of return of the asset classes, with mean returns  $r = (r_1, \dots, r_n)$ , and covariance matrix  $\Sigma = [\sigma_{i,j}]$ , where  $\sigma_{i,j} = E(R_i - r_i)(R_j - r_j)$  for  $i, j = 1, \dots, n$ . Markowitz's mean-variance model is usually stated as

$$\begin{aligned} \min \text{var}(R^T x) &= x^T \Sigma x, \\ e^T x &= 1, \\ r^T x &\geq \rho, \\ x &\geq 0, \end{aligned}$$

where one chooses the holdings that minimize the variance of the portfolio returns for a given level of expected return  $\rho$ . All portfolios selected by the mean-variance model lie on the mean-variance efficient frontier, i.e., increasing the expected return by moving from one efficient portfolio to another means risk (measured through variance) also increases, or smaller risk can only be accomplished by sacrificing expected return.

If a risk-free return is present, and for a single-period analysis this is usually assumed to be the case, then the efficient frontier is represented by the line that intersects the ordinate at the level of the risk-less rate of return and is tangent to the efficient frontier established without the risk-less asset. The portfolio at the point at which the so called market line intersects with the original efficient frontier is the market portfolio. The striking result of Tobin's two fund separation theorem (see Tobin (1958)) is that, if a riskless asset is present, all investors would choose the same portfolio of risky assets, namely the market portfolio, in combination with various amounts of the risk free asset. An investor who wishes to assume the market risk would choose the market portfolio. An investor who is more risk averse than the market would chose a positive fraction of the riskless asset and invest the remainder into the market portfolio. An aggressive investor bearing more risk than the market would borrow at the risk-free rate in addition to his funds at hand and invest the entire amount in the market portfolio, thereby leveraging his wealth. There would be no need for any portfolio other than the market portfolio. For a computation of Tobin's market line and which point on this line to choose, see, Ziemba, Parkan and Brooks-Hill (1974). Usually in a single-period analysis, cash (or money market) represented by the 30-day rate or the three-month rate is considered as

risk-free. There may still be inflation risk, and when considering a single-period horizon of one year, which is a typical investment horizon, the riskless rate may change. However, for a single-period analysis these risks are often considered small enough to be neglected.

Utility theory according to Bernoulli (1738) and Von Neumann and Morgenstern (1944), as a means of dealing with uncertainty, is a generally accepted concept in finance. An investor values different outcomes of a uncertain quantity according to his (Von Neumann–Morgenstern) utility function and maximizes expected utility. The Von Neumann–Morgenstern utility function of a risk-averse investor is a concave function of wealth.

The single-period Markowitz investment problem may also be stated as

$$\begin{aligned} \max \quad & r^T x - \frac{\lambda}{2} x^T \Sigma x, \\ e^T x &= 1, \\ x &\geq 0, \end{aligned}$$

trading off expected return and variance in the objective function, where the parameter  $\lambda$  represents the risk aversion of the investor. In this case the investor trades off linearly the expected return and variance of the portfolio, and  $\lambda$  determines how many units of expected return the investor is willing to give up for a decrease of variance by one unit. In terms of maximization of expected utility, the optimization is stated as

$$\begin{aligned} \max \quad & Eu(R^T x), \\ e^T x &= 1, \\ x &\geq 0, \end{aligned}$$

where  $u(W)$  is a concave utility function. We maximize the expected utility of wealth at the end of the period, where the initial wealth is normalized to one and the distribution of wealth at the end of the period is  $W = R^T x$ . Choosing  $u(W) = -\exp(-\lambda W)$  (an exponential utility function with risk aversion  $\lambda$ ), and assuming the asset returns are distributed as multivariate normal (i.e.,  $R = N(r, \Sigma)$ , with  $r$  as the vector of mean returns and  $\Sigma$  the covariance matrix), we can integrate using the exponential transform and obtain  $E(-\exp(-\lambda W)) = -\exp(\lambda E W - \frac{\lambda^2}{2} \text{var}(W))$ . The certainty equivalent wealth is defined as the fixed amount of wealth that has the same utility as the expected utility of the wealth distribution. Denoting the certainty equivalent wealth as  $W_c$ , we can evaluate  $-\exp(-\lambda W_c) = -\exp(\lambda E W - \frac{\lambda^2}{2} \text{var}(W))$  and obtain  $W_c = E W - \frac{\lambda}{2} \text{var}(W)$ . In the case of multivariate normally distributed asset returns and exponential utility, the mean-variance model maximizes certainty equivalent wealth and therefore indirectly expected utility of wealth. For exponential utility and multivariate normally distributed asset returns, maximizing expected utility of wealth and trading off mean versus variance are therefore equivalent.

For normally distributed returns the scope is even broader. For multivariate normally distributed returns and for any monotonically increasing concave utility function  $u(W)$ ,  $u'(W) > 0$ ,  $u''(W) < 0$ , the obtained optimal solution is mean-variance efficient, and therefore lies on the efficient frontier; see, e.g., Hanoch and Levy (1969). If asset returns are not multi-variate normally distributed, at least one choice of utility function would lead to the equivalent result with respect to mean-variance analysis:  $u(W) = W - \frac{\lambda}{2}(W - EW)^2$ , namely a quadratic utility function explicitly specifying utility as wealth minus risk aversion times half the square of the deviation around the mean. Summarizing, if the asset return distribution is defined by its first two cumulants only, and all higher cumulants are zero (normal distribution), any monotonically increasing concave utility function will result in a mean-variance efficient solution. If the asset return distribution is characterized by higher (than the first two) non-zero cumulants also, only a quadratic utility function gives a mean-variance efficient solution by explicitly considering only the first and second cumulant of the distribution. Thus, when asset returns are not normally distributed, the optimal solution of an expected utility maximizing problem, except for the quadratic utility case, does not necessarily lie on the mean-variance efficient frontier. Assuming asset returns to be normally distributed, therefore, is very convenient. Also, any linear combination of normally distributed random variables is a normally distributed random variable as well. A theory for choosing portfolios when asset returns have stable distributions is given by Ziemba (1974).

#### 4. Utility functions

A Von Neumann–Morgenstern utility function  $u(W)$  represents an investor's attitude towards risk. According to Arrow (1971) and Pratt (1964), the absolute and relative risk aversion coefficients, defined as

$$ARA(W) = -\frac{u''(W)}{u'(W)}, \quad RRA(W) = -W \frac{u''(W)}{u'(W)}$$

determine investor behavior. Thus, it is not the absolute value of the function but how strongly it bends that determines differences in investor choice. A linear utility function representing risk-neutral behavior would exhibit an absolute risk aversion coefficient of zero. Concave utility functions, representing risk-averse behavior, exhibit positive absolute risk aversion. When modeling investor choice, we are interested in how the risk aversion changes with wealth. The inverse of the risk aversion is referred to as the risk tolerance, i.e.,  $ART(W) = 1/ARA(W)$  and  $RRT(W) = 1/RRA(W)$ . Table 1 presents the absolute and relative risk aversion and risk tolerance for some commonly used utility functions.

The power utility function and the logarithmic utility function with  $\alpha = 0$  have relative risk aversion that is constant with respect to wealth. They are therefore also referred to as CRRA (constant relative risk aversion) utility functions. The exponential utility function exhibits constant absolute risk aversion with respect to wealth, and is therefore

Table 1  
Commonly used utility functions (HARA)

Type	Function	ARA	RRA	ART	RRT
Exponential	$u(W) = -\exp(-\lambda W)$	$\lambda$	$\lambda W$	$\frac{1}{\lambda}$	$\frac{1}{\lambda W}$
Power	$u(W) = \frac{W^{1-\gamma}-1}{1-\gamma}, \gamma > 1$	$\frac{\gamma}{W}$	$\gamma$	$\frac{W}{\gamma}$	$\frac{1}{\gamma}$
Generalized Log	$u(W) = \log(\alpha + W)$	$\frac{1}{\alpha+W}$	$\frac{W}{\alpha+W}$	$\alpha + W$	$\frac{\alpha+W}{W}$

said to be of type CARA (constant absolute risk aversion). These utility functions are part of and exhaust the HARA (hyperbolic absolute risk aversion) class, defined by the differential equation  $-u'(W)/u''(W) = a + bW$ , where  $a$  and  $b$  are constants; see e.g., [Rubinstein \(1976\)](#).

In a single-period investment problem of one risky asset and one risk-free asset, the amount invested in the risky asset is proportional to the absolute risk tolerance  $ART(W)$ , and the fraction of wealth invested in the risky asset is proportional to the relative risk tolerance  $RRT(W)$ . Thus CARA implies that the amount of wealth invested in the risky asset is constant with respect to wealth, and therefore the fraction of the wealth invested in the risky asset declines with increasing wealth proportional to  $1/W$ . In contrast, CRRA implies that the fraction of wealth invested in the risky asset is constant, and the amount of wealth invested into the risky asset increases proportional to  $W$ . Different utility functions exhibit different risk aversion as functions of wealth. [Kallberg and Ziemba \(1983\)](#) discuss the implications of various utility functions in a single-period setting, showing that for concave utility functions and normally distributed asset returns, when the average risk aversion is the same for two utility functions, then the portfolios are also very similar, irrespective of the specific utility functional form. They further argue that under these assumptions any concave increasing utility function could be locally approximated by a quadratic function. In a multi-period investment problem, the relationship between relative and absolute risk aversion and portfolio choice is not as straightforward, and complex investment strategies arise.

Since the risk aversion as a function of wealth reflects investor behavior, the choice of utility function is an important aspect of the modeling. In many situations the HARA class appears too restrictive, and other utility functions have been explored. [Bell \(1988\)](#) defined a class of utility functions satisfying a “one switch” rule. Comparing the preference between two gambles (say, portfolios) at different wealth levels, a switch occurs in that below a certain wealth level the first gamble is preferred and above that wealth level the other. The class of one-switch utility functions when maximizing utility thus would not lead to solutions where for even larger wealth levels, the first gamble would be preferred again, or gambles could switch in and out of preference. His one-switch utility functions include and exhaust the following types: the quadratic ( $u(W) = aW^2 + bW + c$ ), the double exponential (sumex) functions ( $U(W) = ae^{bW} + ce^{dW}$ ), the linear plus exponential ( $u(W) = aW + be^{cW}$ ), and the

linear times exponential ( $u(W) = (aW + b)e^{cW}$ ). Musumeci and Musumeci (1999) found a linear combination of a power and an exponential function attractive, suggesting  $u(W) = -W^{-1} - ae^{-bW}$ , and give two sets of numerical values for the parameters representing typical more or less risk-averse investors:  $a = 2.9556e-5$ ,  $b = 8.0e-8$  would represent the less, and  $a = 1.4778e-5$ ,  $b = 4.0e-8$  the more risk-averse investor class.

A variant of the quadratic utility function, where investors are concerned with the downside risk of wealth falling below a certain target  $W_d$ , is  $u(W) = W - \lambda_q/2 (\max(0, W_d - W))^2$ , reflecting a lower partial moment, where risk is not the total variance but the variance of that part conditioned on the wealth falling below the target. We also refer to this utility function as a quadratic downside risk utility function. The linear downside risk utility function,  $u(W) = W - \lambda_l \max(0, W_d - W)$ , reflects the first lower partial moment, and risk is the expected wealth conditioned on the wealth falling below the target  $W_d$ . The efficient frontier with risk as the first lower partial moment is also referred to as the “put–call efficient frontier”; see, e.g., Dembo and Mautser (2000). More generally, the additive function of linear and quadratic downside risk, i.e.,  $u(W) = W - \lambda_l (\max(0, W_d - W)) - \lambda_q/2 (\max(0, W_d - W))^2$ , penalizes downside risk, where for  $\lambda_l > 0$  the function is non-differentiable at  $W_d$ , leading to a jump in risk aversion right at the downside of the target.

It has been generally believed that individual investor utility should be of decreasing absolute risk aversion and increasing relative risk aversion, spanning the range between CARA seen as possibly too conservative for large levels of wealth and CRRA seen as possibly too aggressive for large wealth levels. A classical empirical study about investor preference, Friend and Blume (1975), concludes that for representing average investor preference constant relative risk aversion (CRRA) reflects a first approximation, but there are many deviations in that investors could be exhibiting either increasing or decreasing relative risk aversion. Markowitz (1994) argues, based on empirical data from a small survey, why the exponential utility function may be too conservative. Bodie and Crane (1997) found by conducting a survey that the proportion of total assets held in equities declined with age and increased with wealth of the respondents. A more recent study by Faig (2002) conducted in Europe confirms a wide variety of investor behavior and that decreasing relative risk aversion can be shown as empirically valid. A different approach of modeling investor preference, called prospect theory, has been put forward by Kahnemann and Tversky (1979), postulating that investors are less concerned about their wealth as an absolute number than they are concerned about changes in wealth. They construct a utility function from a number of linear segments, each representing the utility of positive and negative changes in wealth.

## 5. A general approach to modeling utility

Following the empirical evidence about possible investor preference, we propose to model directly in the space of risk aversion rather than first defining a certain type of

utility function (which may or may not fit well) and then estimating its parameters. We have found an efficient way of representing the utility function as a piecewise exponential function with  $K$  pieces, where each piece represents a certain absolute risk aversion  $\alpha_i$ , where  $i = 1, \dots, K$ .

Let  $\widehat{W}_i$ ,  $i = 1, \dots, K$ , be discrete wealth levels representing the borders of each piece  $i$ , such that below each  $\widehat{W}_i$  the risk aversion is  $\alpha_i$  and above  $\widehat{W}_i$  (till  $\widehat{W}_{i+1}$ ) the risk aversion is  $\alpha_{i+1}$ , for all  $i = 1, \dots, K$ . For each piece  $i$  we represent utility using the exponential function. Thus, for  $\widehat{W}_i \leq W \leq \widehat{W}_{i+1}$ ,  $u_i(W_i) = a_i - b_i \exp(-\alpha_i W_i)$  and the first derivative with respect to wealth is  $u'_i(W_i) = b_i \alpha_i \exp(-\alpha_i W_i)$ . The absolute risk aversions  $\alpha_i$  are computed in such a way that they represent the desired function of risk aversion versus wealth. We determine the coefficients of the exponential functions of each piece  $i$  by matching function values and first derivatives at the intersections  $\widehat{W}_i$ . Thus, at each wealth level  $\widehat{W}_i$ , representing the border between risk aversion  $\alpha_i$  and  $\alpha_{i+1}$ , we obtain the following two equations

$$\begin{aligned} a_i - b_i e^{-\alpha_i \widehat{W}_i} &= a_{i+1} - b_{i+1} e^{-\alpha_{i+1} \widehat{W}_i}, \\ b_i \alpha_i e^{-\alpha_i \widehat{W}_i} &= b_{i+1} \alpha_{i+1} e^{-\alpha_{i+1} \widehat{W}_i}, \end{aligned}$$

from which we calculate the coefficients  $a_{i+1}$  and  $b_{i+1}$  as

$$\begin{aligned} b_{i+1} &= b_i \frac{\alpha_i}{\alpha_{i+1}} e^{(\alpha_{i+1} - \alpha_i) \widehat{W}_i}, \\ a_{i+1} &= a_i - b_i \left( 1 - \frac{\alpha_i}{\alpha_{i+1}} \right) e^{-\alpha_i \widehat{W}_i}, \end{aligned}$$

where we set arbitrarily  $a_1 = 0$  and  $b_1 = 1$ .

The piecewise exponential function may span the whole range of attainable wealth levels. Starting from parameters  $a_1 = 0$  and  $b_1 = 1$  and given risk versions  $\alpha_i$ , we compute all parameters  $a_{i+1}$  and  $b_{i+1}$  for each  $i = 1, \dots, K$ .

To test the piecewise approximation, we could set the risk aversion  $\alpha_i$  to represent typical utility functions, for example, setting each  $\alpha_i = \alpha$  would result in constant absolute risk aversion (CARA) or setting  $\alpha_i = \gamma / \widehat{W}_i$  would represent constant relative risk aversion (CRRA).

We are now in a position to set the piecewise absolute risk aversions to fit the risk aversion of the investor. Typically, we would model investors as having decreasing absolute risk aversion but either increasing or decreasing relative risk aversion. For example, the relative risk aversion may be increasing or decreasing from  $\gamma_0$  at  $\widehat{W}_0$ , to  $\gamma_K$  at  $\widehat{W}_K$ . For a constant rate of change  $\Delta$  in relative risk aversion, this could be modeled by setting

$$\gamma_{i+1} = \gamma_i + \Delta$$

where

$$\Delta = \frac{\gamma_K - \gamma_0}{\widehat{W}_K - \widehat{W}_0}$$

and

$$\alpha_i = \gamma_i / \widehat{W}_i.$$

In addition we may wish to represent constant relative risk aversion below  $\widehat{W}_0$  and above  $\widehat{W}_K$ . In order to do so we append to the piecewise exponential representation on each side a CRRA piece represented by the power function  $u(W) = c \frac{W^{1-\gamma}-1}{1-\gamma} + d$ ,  $\gamma > 1$ , with its first derivative  $u'(W) = cW^{-\gamma}$ . We then calculate the parameters of the power function as

$$c = \alpha e^{\alpha \widehat{W}} \widehat{W}^\gamma,$$

$$d = -e^{\alpha \widehat{W}} - c \frac{\widehat{W}^{1-\gamma} - 1}{1-\gamma},$$

where in the formula for obtaining the parameters  $c_0$  and  $d_0$  for the lower CRRA piece we use  $\alpha = \alpha_1$  and  $\widehat{W} = \widehat{W}_0$  and for obtaining  $c_K$  and  $d_K$  for the upper CRRA piece we use  $\alpha = \alpha_K$  and  $\widehat{W} = \widehat{W}_K$ . The formulas arise from setting at the intersections at  $\widehat{W}_0$  and  $\widehat{W}_K$  the function value and first derivative of the power piece equal to the function value and first derivative of the adjacent exponential piece, respectively.

As result we obtain a smooth, monotonically increasing and concave utility function that approximates arbitrarily closely (depending on the number of pieces used) the function of relative risk aversion representing the investor. Figure 1 displays the function of absolute risk tolerance versus wealth, for CARA and CRRA as well as two examples of representations of increasing and decreasing relative risk aversion.

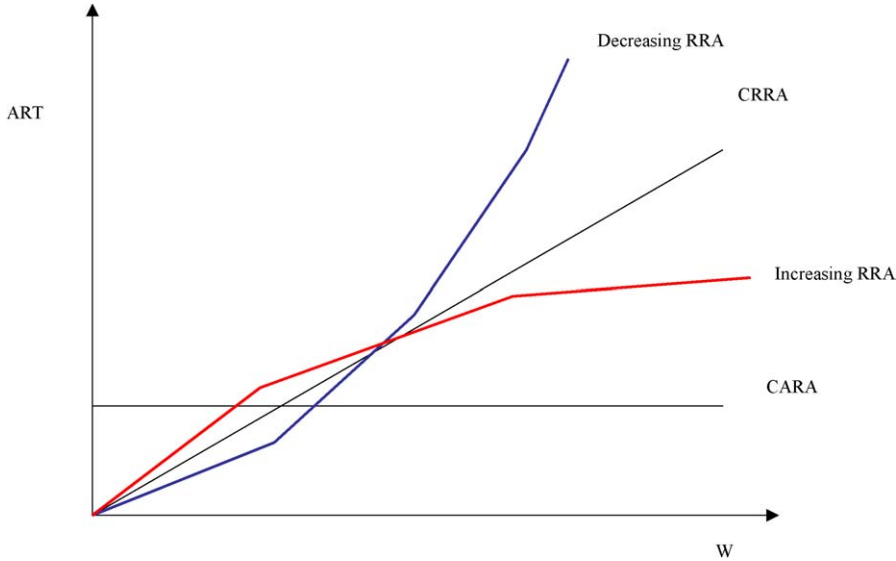


Fig. 1. Modeling risk aversion.

The piecewise exponential modeling represents a novel approach to representing an investor's risk aversion. We have found it easier to determine directly the risk aversion of an investor, e.g., by a questionnaire, as compared to determining the utility function by the certainty equivalent method or gain and loss equivalence method of comparing lotteries; see, e.g., [Keeney and Raiffa \(1976\)](#). Details of how to determine the risk aversion of an investor in this framework will be presented in a separate paper.

## 6. Dynamic portfolio choice

We now extend the single-period utility maximization model to a multi-period setting. Let  $t = 0, \dots, T$  be discrete time periods, with  $T$  the investment horizon. Let  $R_t$  be the random vector of asset returns in time periods  $t$ . Let  $y_t = (y_1, \dots, y_n)_t$  be the amount of money invested in the different asset classes  $i = 1, \dots, n$  at time  $t$ . Scalars  $W_0$  and  $s_t$ ,  $t = 0, \dots, T - 1$ , represent the initial wealth and possible cash flows (deposits positive and withdrawals negative) over time, respectively. The following problem states the multi-period investment problem:<sup>1</sup>

$$\begin{aligned} \max \quad & Eu(e^T y_T), \\ & e^T y_0 = W_0 + s_0, \\ & -R_{t-1}^T y_{t-1} + e^T y_t = s_t, \quad t = 1, \dots, T, \\ & y_t \geq 0, \quad W_0, s_0, \dots, s_{T-1} \text{ given}, \quad s_T = 0. \end{aligned}$$

At the beginning, the initial wealth plus the cash flow ( $W_0 + s_0$ ) are invested among the  $n$  asset classes. The returns of the investment plus the next cash flow are available for investment at the beginning of the next period. At the beginning of each period during the investment horizon, the investor can freely rearrange the portfolio. At the end of the investment horizon, the final wealth in the portfolio,  $W_T$ , is evaluated using the utility function  $u(W_T)$ . Short-selling and borrowing is ruled out, but could be easily introduced into the problem. Asset  $n$  could be a risk-free asset, but is treated like any other asset, since no distinction is necessary.

Instead of maximizing the utility of terminal wealth, we could maximize the discounted utilities of wealth in each period,

$$\max \sum_{t=1}^T \delta^{-t} u_t(e^T y_t),$$

where  $\delta$  represents the discount factor. This concept of an additive discounted utility represents a straightforward extension and proved very useful in controlling, say, downside risk in every period. As an extension, the Kahnemann and Tversky utility could be represented in such a way.

<sup>1</sup>  $T$  as a superscript always means transpose, while as a subscript  $T$  always denotes the last period of the investment horizon.



Defining  $x_t$ ,  $t = 0, T - 1$ , as the vector of fractions invested in each asset class in each period, we can write  $x_t = y_t / (W_t + s_t)$ , where we define the wealth available in each period (before adding cash flow) as  $W_t$ ,  $W_t = R_{t-1}x_{t-1}(W_{t-1} + s_{t-1})$ . We can then write the model as

$$\begin{aligned} \max E u(W_T), \\ e^T x_t = 1, \quad t = 0, T - 1, \\ W_{t+1} = R_t x_t (W_t + s_t), \quad t = 0, \dots, T - 1, \\ y_t \geq 0, \quad W_0, s_0, \dots, s_{T-1} \text{ given}, \quad s_T = 0. \end{aligned}$$

Here one can see that for serially independent asset returns wealth is a single state connecting one period with the next. Now we write the problem as the dynamic programming recursion

$$\begin{aligned} u_t(W_t) &= \max E u_{t+1}((W_t + s_t)R_t x_t), \\ e^T x_t &= 1, \\ A x_t &= b, \quad l \leq x_t \leq u, \end{aligned}$$

where  $u_T(W_T) = u(W)$ ,  $W_{t+1} = (W_t + s_t)R_t x_t$ ,  $W_0$  given.

One can now see that the multi-period problem is composed of a series of single-period utility maximization problems, where for the last period the actual utility function is used and for all other periods the implied (through the dynamic programming algorithm) “utility-to-go” utility function is employed. Referring to the Samuelson and Merton result discussed above, for the CRRA utility function, all implied utility-to-go functions are also of the type CRRA, with the same risk aversion coefficient  $\gamma$ . This results in the aforementioned fixed-mix strategies.

### 6.1. Dynamic stochastic programming and Monte Carlo sampling

In practice, we need to resort to Monte Carlo sampling to estimate the expected utility of the single-period utility maximizing problem of each period. Let  $R_t^\omega$ ,  $\omega \in S_t^i$ , and  $R_t^\omega$ ,  $\omega \in S_t^o$ ,  $t = 1, \dots, T - 1$ , be independent samples of the return distributions for each period  $t$ . The sample  $S_t^i$  includes the in-sample returns used for generating the single-period utility maximization problems and the sample  $S_t^o$  represents the out-of-sample return used for evaluating the obtained solution. Using two different samples, one for optimizing and the other for evaluating, prevents optimization bias. We represent the problem as

$$\begin{aligned} \hat{u}_t(W_t) &= \max \frac{1}{|S_t^i|} \sum_{\omega \in S_t^i} \hat{u}_{t+1}((W_t + s_t)R_t^\omega x_t), \\ e^T x_t &= 1, \\ A x_t &= b, \quad l \leq x_t \leq u, \end{aligned}$$

and we define  $\hat{u}_t(W_t) = \frac{1}{|S_t^o|} \sum_{\omega \in S_t^o} \hat{u}_{t+1}((W_t + s_t)R_t^\omega x_t)$ , where  $\hat{u}_T(W_T) = U(W)$ ,  $W_0$  given. We parameterize in  $W_t$  to carry out the dynamic programming recursion. Note  $\hat{\hat{u}}(\cdot)$  refers to the in-sample estimate, whereas  $\hat{u}(\cdot)$  represents the out-of-sample estimate of the utility-to-go function. Depending on the sample size used, the in-sample estimate  $\hat{\hat{u}}(\cdot)$  would include a significant amount of optimization bias that consequently would be carried forward between stages, whereas the out-of-sample estimate  $\hat{u}(\cdot)$  of the portfolio decision represents an independent evaluation without any optimization bias.

The dynamic optimization problem can now be solved using a backward dynamic programming recursion, conditioning on wealth. Starting at period  $T - 1$  we parameterize the wealth into  $K$  discrete wealth levels,  $W_{T-1}^k$ ,  $k = 1, \dots, K$ , and solve the period  $T - 1$  problem  $K$  times using sample  $S_{T-1}^i$ , and obtain solutions  $\hat{x}_{T-1}^k$ . We evaluate the obtained solutions by computing  $\hat{u}_{T-1}^k = \frac{1}{|S_{T-1}^o|} \sum_{\omega \in S_{T-1}^o} \hat{u}_T((W_{T-1}^k + s_{T-1})R_t^\omega x_t)$  and obtain for each parameterized value  $W_{T-1}^k$  a corresponding value of  $\hat{u}_{T-1}^k$ , which pairs represent  $K$  points of the Monte Carlo estimate of the value function ( $u_{T-1}(W_{T-1})$ ). We interpolate between those points, using an appropriate accurate interpolation technique, to obtain a smooth functional form. The value function  $u_{T-1}(W_{T-1})$  in period  $T - 1$  is the induced utility function for the period  $T - 2$  single-period optimization, and we repeat the process until all optimizations in period 1 are done. Finally, in period 0, the initial wealth is known and we conduct the final optimization using the period 1 value function as implied utility function  $u_1(W_1)$ . In each period in the backward recursion, we use a different independent sample of large size for the evaluation: thus, the sampling error is small and cancels out over the different rebalancing periods. The sampling-resampling procedure is a crucial part of the solution algorithm, because it prevents the dynamic recursion from carrying forward and accumulating sampling bias when solving for a large number of periods.

## 6.2. Serially dependent asset returns

In the case of serial dependency of asset returns, we can extend the model and consider the return  $R_t|R_{t-1}$  conditioned on the previous period return vector. For example, a vector autoregressive process (VAR(1)) of lag one would fit such a description. In this case we define

$$R_t = C + AR_{t-1} + \varepsilon,$$

where  $C$  is an intercept vector and  $A$  is an  $n \times n$  matrix of coefficients obtained from  $n$  least-squares regressions. The problem is stated as

$$\begin{aligned} u_t(W_t, R_{t-1}) &= \max E u_{t+1}((W_t + s_t)R_t|R_{t-1}x_t), \\ e^T x_t &= 1, \\ Ax_t &= b, \quad l \leq x_t \leq u, \end{aligned}$$

where  $u_T(W_T) = u(W)$ ,  $W_{t+1} = (W_t + s_t)R_t|R_{t-1}x_t$ ,  $W_0$ ,  $R_{-1}$  given.

A general lag one vector autoregressive process with  $n$  asset classes requires  $n + 1$  state variables and may exceed the limits of the stochastic dynamic programming approach. However, a restricted autoregressive process of lag one with a limited number of predicting variables may well be computationally tractable and statistically valid. Thus using, say, three predicting variables will lead to four state variables. The resulting dynamic program can still be solved accurately in reasonable time.

### 6.3. A fast method for normally distributed asset returns

For multivariate normally distributed asset returns, we can algorithmically take advantage of the fact that, for monotonically increasing concave utility functions, the optimal solution is mean-variance efficient. Instead of solving the recursive non-linear optimization problem, we can search a pre-computed set of mean-variance efficient solutions for the one maximizing the utility or value-to-go function. To maintain optimality in each stage, we need to ensure that the utility function,  $u_t(W_t)$ , induced from  $\max E u_{t+1}((W_t + s_t)R_t x_t)$  given the constraints, is also a monotonically increasing concave function, as is the utility function  $u(W)$  at the end of the investment horizon. The dynamic programming based proof of this theorem is omitted. Defining  $\eta_{k,t}$  as the return distribution of the  $k$ th allocation point  $(\mu_{k,t}, \sigma_{k,t})$  on the mean-variance efficient frontier in period  $t$ , i.e.,  $\eta_{k,t} = N(\mu_{k,t}, \sigma_{k,t})$ , we can write the dynamic programming recursion as

$$u_t(W_t) = \max_k E u_{t+1}((W_t + s_t)\eta_{k,t})$$

where  $u_T(W_T) = u(W)$ ,  $W_{t+1,k} = (W_t + s_t)\eta_{k,t}$ ,  $W_0$  given.

An effective search is used to speed-up the optimization by avoiding having to evaluate all distributions  $\eta_{k,t}$ . The work required for this recursion is independent of the number of assets and, like the general recursion above, linear in the number of stages.

This fast approach is very well suited for solving the problem with a restricted autoregressive return process, where the error terms of the restricted vector autoregression are assumed distributed as multivariate normal. Details of this approach will be discussed in a separate paper.

## 7. Numerical results

### 7.1. Data assumptions

For estimating the asset return distributions, we used historical monthly time series from January 1974 to December 2004 for stocks and from January 1986 to December 2004 for bonds, based on the data available from the Datastream Access database. Stocks were classified into US stocks (represented by the MSCI US index) and international stocks (represented by the MSCI EAFE&C index). Bonds were distinguished into US

Corporate Bonds (represented by the Salomon Brothers US Corp Bnd index) and US Gvnt Bonds (represented by the GP Morgan US Gov Bnd index). For cash we used 3 month Treasury notes (represented by the GP Morgan US 3M index). For obtaining the i.i.d. distributions, we estimated the correlation matrix using directly the historical data points for each time series from January 1986 to December 2004. We estimated separate correlation matrices for representing normally and lognormally distributed returns. The estimates for the correlation matrices and the means and standard deviations are presented in Table 2. However, for corporate bonds we lowered the historical expectation of 9.49% to 9.0% per annum, reflecting a more appropriate number, since expected corporate bond returns appeared as too high during this period and were overly dominating stocks.

Table 2  
Data estimates for normal and log normal distributions

Historical means and standard deviations					
	US stocks	Int stocks	Corp bnd	Gvnt bnd	Cash
Mean	10.80	10.37	9.49	7.90	5.61
STD	15.72	16.75	6.57	4.89	0.70
Regression-based means and standard deviations					
	US stocks	Int stocks	Corp bnd	Gvnt bnd	Cash
Mean	14.1	12.5	9.24	7.92	5.77
STD	15.72	16.75	6.57	4.89	0.70
Correlation matrix for normal model					
	US stocks	Int stocks	Corp bnd	Gvnt bnd	Cash
US stocks	1.00	0.601	0.247	0.062	0.094
Int stocks	0.601	1.00	0.125	0.027	0.006
Corp bnd	0.247	0.125	1.00	0.883	0.194
Gvnt bnd	0.062	0.027	0.883	1.00	0.27
Cash	0.094	0.006	0.194	0.27	1.00
Correlation matrix for log-normal model					
	US stocks	Int stocks	Corp bnd	Gvnt bnd	Cash
US stocks	1.00	0.609	0.236	0.05	0.083
Int stocks	0.609	1.00	0.124	0.02	-0.002
Corp bnd	0.236	0.124	1.00	0.884	0.195
Gvnt bnd	0.05	0.02	0.884	1.00	0.271
Cash	0.083	-0.002	0.195	0.271	1.00

In addition to using the historical mean returns directly, we also estimated a constant growth rate for each asset class. By fitting a constant growth rate to the data we accounted for the dips at the beginning and end of the time series of stock returns. Using a linear regression on the  $1 + \ln(R_{i,t}^C)$  compounded returns, the least square problem

$$\min(a_i + b_i t - 1 - \ln(R_{i,t}^C))^2$$

results in an intercept  $a_i$  and a coefficient  $b_i = (1 + \ln(\bar{r}_i))$ , from which the growth rate of  $\bar{r}_i$  for each asset class  $i$  can be obtained. (The latter could be used as forward-looking unconditional expected returns.) The standard deviations (vector  $\sigma$ ) were estimated directly from the monthly data. The historical time series in logarithmic representation as well as the linear function of estimated constant growth from the regression are represented in Figure 2 for US stocks, international stocks, and (for comparison) all stocks, represented by the MSCI World total return index, and in Figure 3 for corporate bonds, government bonds and cash.

Assuming either normal or lognormal asset returns distributions does not take into account properly the higher moments of the joint return distributions of the asset classes. We therefore used a bootstrapping procedure to capture the true co-movements of the asset classes. In order to do so, we subtracted from each historical time series its historical mean and divided by its historical standard deviation to obtain the vector of residuals  $\varepsilon_t$ . With  $\bar{r}^H$  and  $\sigma^H$  representing the historical vectors of means and standard deviations, respectively, and  $\varepsilon_t = (R_t - \bar{r}^H)/\sigma^H$ , the returns distribution is represented as  $R_t = \bar{r} + \sigma \varepsilon_t$ . For generating one yearly return observation via the bootstrapping scheme, we sampled 12 independent observations with replacement from the historical monthly residuals, and combined them into a corresponding yearly return by compounding, multiplying by the standard deviation, and adding the (forward looking unconditional) mean. For the discussion below we used the more conservative historical mean estimates, rather than the one obtained by the regression, and the variances were estimated directly from the monthly historical time series.

We remark that estimating the means of stock returns, given the data at hand, may not be very accurate. For example, our estimates for the annual means based on 374 monthly historical data have an estimated standard deviation of about 2.8%. Estimates of standard deviations and correlations are more accurate. The regression procedure provides a good way to estimate means. The regression results for stocks, while larger than the historical estimates, are within the confidence interval obtained for the historical mean estimates.

For the effect of estimation errors on optimal portfolio choice see, e.g., [Chopra and Ziemba \(1993\)](#) and on turnover [Chopra \(1993\)](#), showing that estimation errors in the means have a significantly larger effect than estimation errors in the variances and covariances. Simulating estimation error by adding zero mean i.i.d normal distributions to the data, and comparing certainty equivalent cash (wealth) in a single-period mean-variance optimization of 10 randomly selected stocks of the Dow Jones industrial average, for a risk tolerance of 50, errors in the mean estimate resulted in a 11 times larger loss of certainty equivalent cash as errors in the variances, and errors in

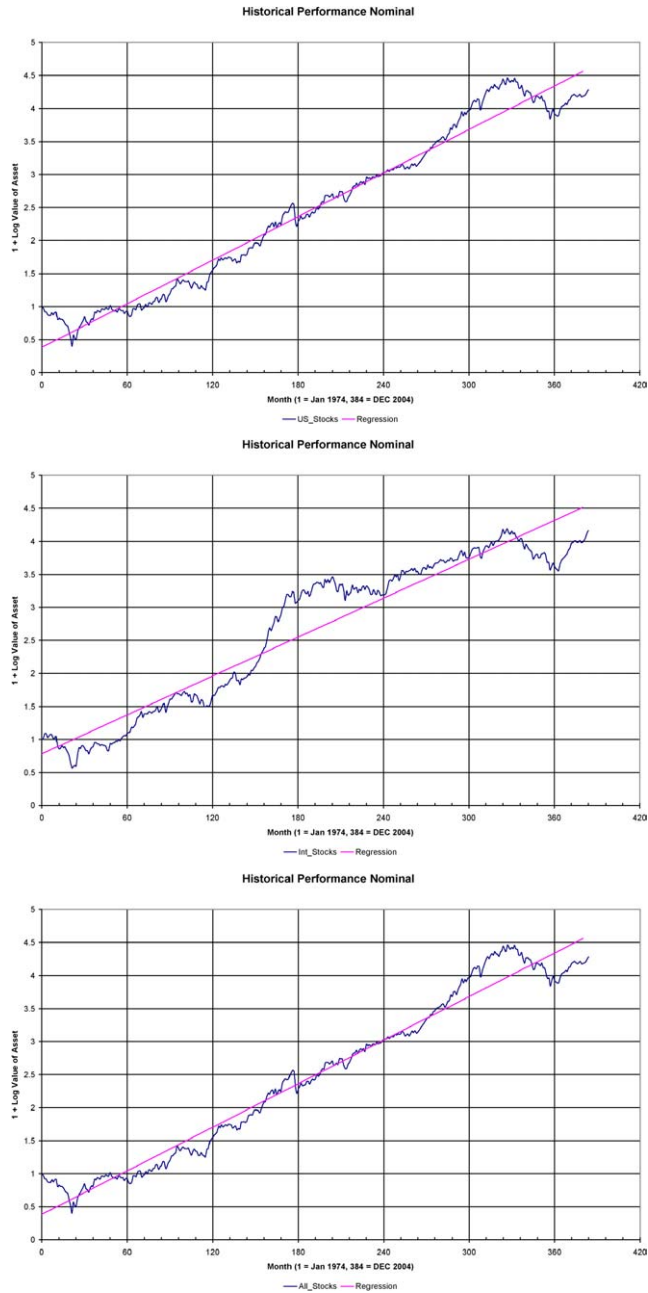


Fig. 2. Monthly stock performance, January 1974 – December 2004 (US stocks, international stocks, whole world).

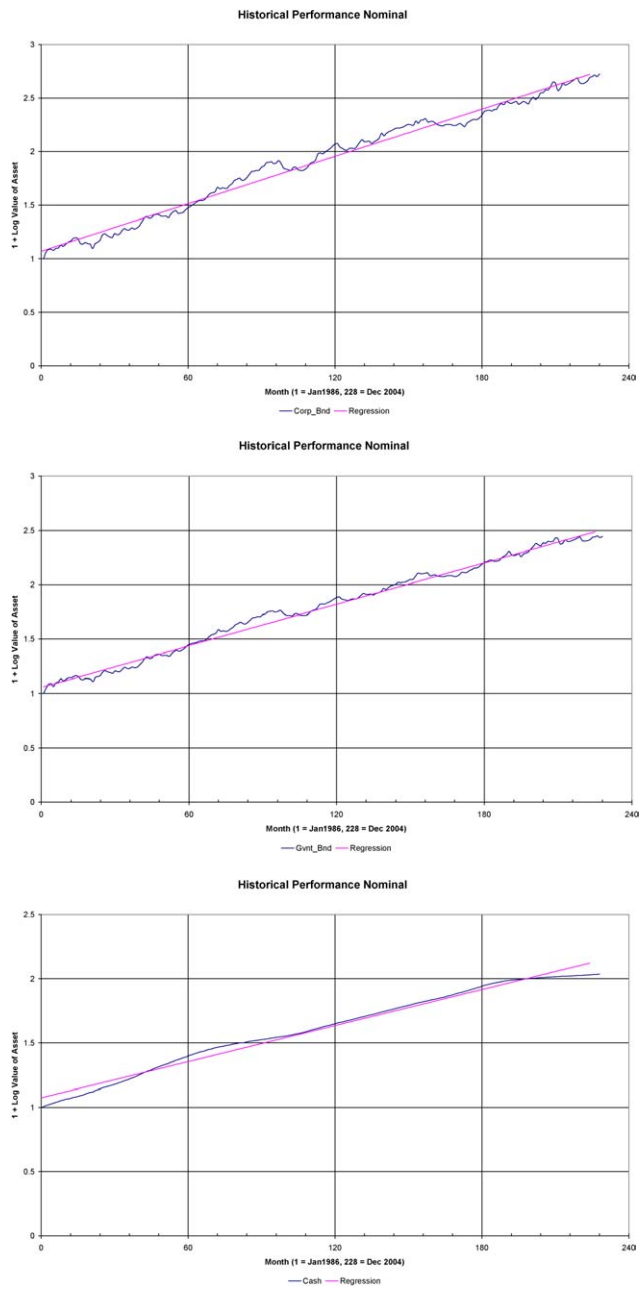


Fig. 3. Monthly bond performance, January 1986 – December 2004 (corporate bonds, government bonds, cash).

the variances resulted in 2 times larger loss than errors in the covariances. The effects were shown to increase with increasing risk tolerance and with increasing magnitude of errors. Already earlier, Kallberg and Ziemba (1981, 1984) concluded that errors in the mean estimate matter most. The effect of errors in the estimates of means relative to variances and covariances on certainty equivalent wealth obviously must increase with larger risk tolerance, since at a risk tolerance of zero, where only variances and covariances enter the optimization problem, exclusively errors in variances and covariances matter, and at an infinite risk tolerance, where only means enter the optimization problem, exclusively errors in mean estimates have an influence. Michaud (1989) showed that noisy forecasts in mean-variance optimization may lead to suboptimal portfolios, where assets with positive estimation error in the mean forecasts are significantly over-weighted and assets with negative estimation error are significantly under-weighted in the “optimal” portfolio, and proposed an approach based on re-sampling (bootstrapping) to counteract this effect. As a practical approach to counter estimation error, Connor (1997) proposed for linear regression based forecasting models to use Bayesian adjusted priors to shrink mean estimates with large observed estimation errors.

A different approach for obtaining unconditional means is to use an assumed mean-variance efficient portfolio, e.g., a broad index, and to infer the unconditional means from the standard deviations of returns, by viewing the portfolio as the optimal solution of a corresponding mean-variance problem with an appropriate risk aversion coefficient. This procedure of Grinold (1999) is called “grapes from wine”, and represents an efficient way to calibrate mean returns. Besides estimating standard deviations based on historical time series, also implied volatilities based on observed option prices could be used.

## 7.2. An investment example

In order to demonstrate dynamic investment policies obtained from our dynamic portfolio choice model, we discuss as an illustrative example a very typical investment situation. An investor has a current wealth of \$100k and plans to contribute \$15k per year for the next 20 years. What is the distribution of wealth at the end of the investment horizon, reflecting various reasonable assumptions about the investor’s risk aversion profile? We discuss four cases: (A) using the CARA utility function, (B) using increasing relative risk aversion but decreasing absolute risk aversion, (C) using decreasing relative risk aversion and decreasing absolute risk aversion, and (D) using a quadratic penalty of under-performing a target.

For the exponential utility function (case A), we assumed an absolute risk aversion coefficient of  $\lambda = 2$ . Figure 4 presents the optimal asset allocation (as a function of wealth) at various times, e.g., for one year (top), 10 years (middle) and 19 years (bottom) to go. The optimal asset allocation is not constant but varies significantly with the investment horizon and the wealth already accumulated. The case of one year to go is special because it shows the results for different wealth levels of a single-period optimization with the investor’s original utility function. All other period allocations



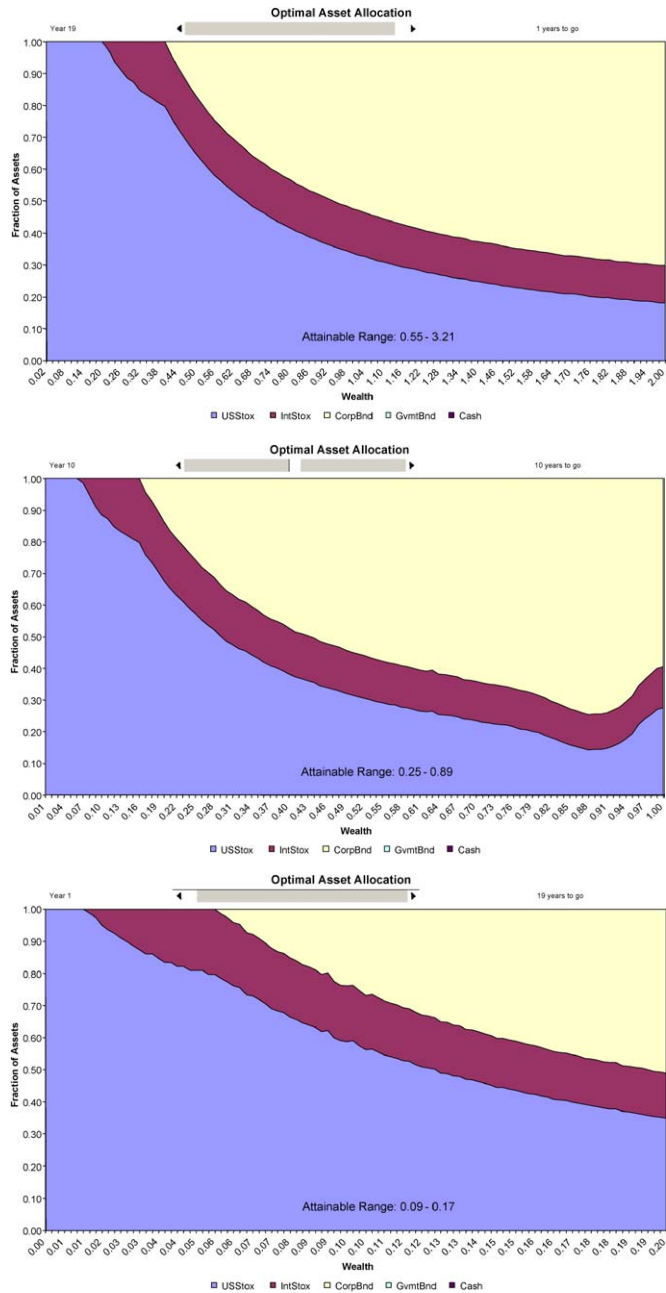


Fig. 4. Investment example: initial \$100k plus \$15k per year, 20-year horizon: exponential utility, investment strategy.

represent the result for using the implied utility function from the dynamic programming algorithm. In the last rebalancing period, the more money that is in the investor's account the less risk he/she takes, and accordingly the fraction of stocks decrease with increasing account value. When the account is under-performing a certain wealth level the optimal strategy prescribes to invest entirely in stocks, where the amount to be put internationally versus domestically varies with account value. This may be a significant strain on the investor, but the problem could be corrected by using a constraint restricting the maximum exposure to stocks. If there was no borrowing constraint, we know from analytical solutions for HARA policies that the optimal investment would prescribe for low wealth levels to borrow funds and use this money to buy more than 100% stocks, thus attempting to leverage the funds available. However, this would not be practical in a low-wealth situation, since such a strategy would quickly exceed margin requirements and thus would not be implementable. This demonstrates the importance of considering a borrowing constraint as part of the investment problem.

The dependence of the optimal asset allocation on wealth changes every year, as one can see by comparing the optimal asset allocation for only one year to go with the ones for 10 years to go and for 19 years to go. For very high wealth levels and long remaining horizons (e.g., 19 years to go) a small amount of cash enters the optimal portfolio. Figure 4 for the optimal asset allocation strategy also displays the attainable wealth range, obtained by simulation. The attainable range is defined by the wealth that is exceeded (left) or not exceeded (right) with 99.9% probability, respectively. For example, in year 19 the attainable range is between \$0.55 and \$3.21 million, in year 10 between \$0.25 and \$0.89 million, and in year 1 (after the first year of investment) between \$0.09 and \$0.17 million, including the cash flow at the beginning of the period.

Figure 5 (top) represents the expected value strategy, i.e., the optimal investment strategy implemented over time, assuming every year that the expected returns would be realized. At (the current) period zero, the optimal portfolio is 54% US stocks, 17% international stocks, and 29% corporate bonds. Cash and government bonds are not part of the initial portfolio. One can observe that from the optimal allocation at the outset of approximately 71% stocks and 29% bonds, the allocation changes gradually to approximately 37% stocks and 63% bonds at year 19, when the last portfolio decision is to be made. This path is similar to strategies that investment firms tend to prescribe their clients and recently have been implemented as life-cycle funds. The reasoning behind such strategies is to reduce stock allocation as the investment horizon shortens to prevent a possible significant loss without a good prospect of recovery before the end of the investment horizon. However, the strategies of practitioners and life-cycle funds are different to our dynamic strategy in that they do not react to prior performance. We can also view the dynamic strategy by starting from the expected value path, where stock allocation and risk is reduced as the remaining investment horizon shortens. In each period the stock allocation and the risk is reduced if the performance was better than expected (and thus the available wealth is larger), and the stock allocation and the risk is increased if the performance was worse than expected (and thus the wealth is smaller).

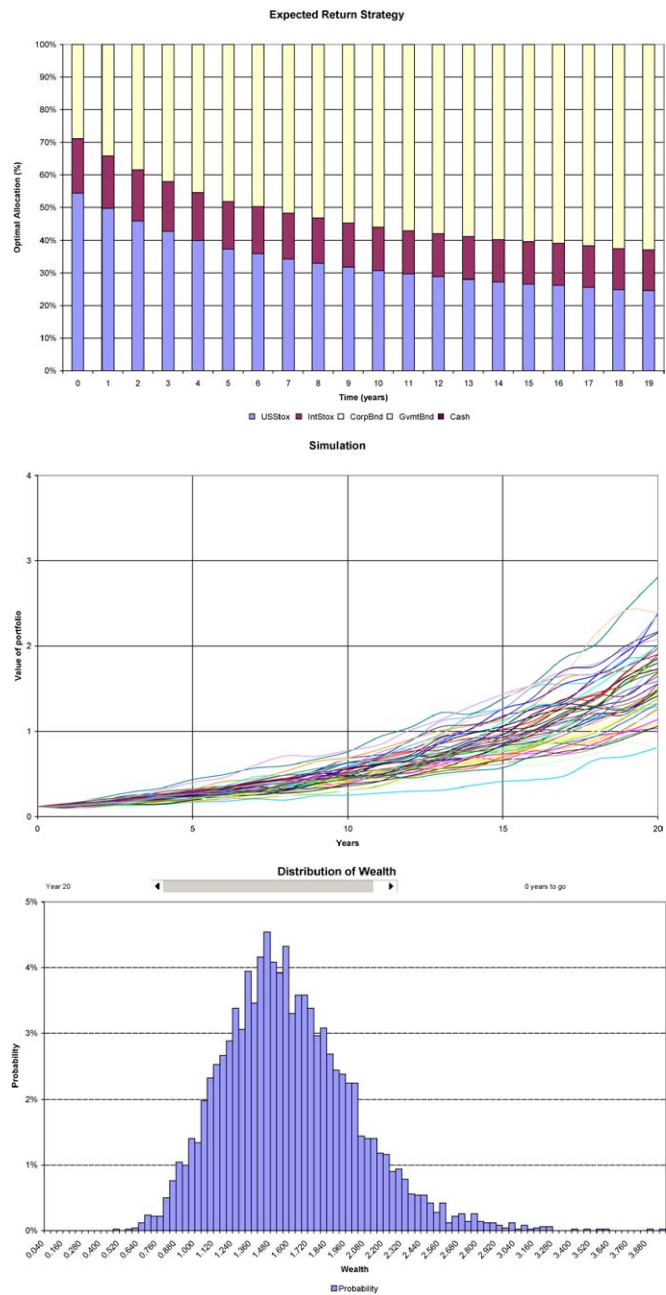


Fig. 5. Investment example: initial \$100k plus \$15k per year, 20-year horizon: exponential utility, simulations.

Table 3  
Out-of-sample simulation results

	CEW	Mean	Std	99%	95%
Exponential RA = 2	1.412	1.564	0.424	0.770	0.943
Increasing RRA	1.440	1.575	0.452	0.771	0.937
Decreasing RRA	1.339	1.498	0.436	0.865	0.998
Quadratic	0.982	1.339	0.347	0.911	1.006
Downside	1.349	1.481	0.352	0.865	0.997

Figure 5 depicts (in the middle) the out-of-sample simulations of wealth over time and (at the bottom) the resulting marginal distribution of wealth at the end of the investment horizon. Table 3 summarizes the statistics of wealth obtained at the end of the investment horizon. The mean wealth is \$1.564 million, with a standard deviation of \$0.424 million. With 99% probability a wealth of \$770,000 is exceeded and with 95% probability a wealth of \$943,000. The certainty equivalent wealth is \$1.412 million.

Alleged shortcomings of the exponential utility function (constant absolute risk aversion) include that very high stock allocations at very low wealth levels may lead to a too-high risk burden on the investor, and a too-small relative risk aversion at very high wealth levels may lead to overly conservative asset allocations. While both shortcomings can be compensated for with lower and upper bounds on the stock allocation, another way is to model the proper risk aversion of the investor directly.

For the increasing relative risk aversion utility function (case B), we assumed that at a wealth below  $W_L = 0.25$ , the relative risk aversion is  $\gamma = 2$ , increasing linearly with wealth to  $\gamma = 3.5$  at a wealth of  $W_U = 3.5$  and then remaining at that level for larger wealth levels. This profile of relative risk aversion was modeled using the piecewise exponential utility representation discussed above, using 200 exponential (CARA) pieces between  $W_L$  and  $W_U$ , and CRRA pieces for wealth levels below  $W_L$  and above  $W_U$ . Besides increasing relative risk aversion, the function exhibits decreasing absolute risk aversion, since the absolute risk aversion decreases from 8 at  $W_L = 0.25$  to 1 at  $W_U = 3.5$ , and the adjacent CRRA pieces also represent decreasing absolute risk aversion.

The results for increasing relative risk aversion are similar to the ones for the exponential case, but do not display the aforementioned shortcomings of the exponential utility function. The allocation of stocks at low levels of wealth are reasonable, with a maximum at about 64%, and the stock allocation at very high wealth levels does not become as conservative as in the exponential case. Again, the optimal strategies depend both on wealth level and remaining investment horizon, as Figure 6 presents for one year (top), ten years (middle) and 19 years (bottom) to go. The assumption of a relative risk aversion bounded between 2 and 3.5 is reflected in the flat response to changes in wealth levels. Figure 6 also presents the attainable wealth in year 19 between \$0.59 and \$3.44 million, in year 10 between \$0.27 and \$0.90 million, and in year 1 between \$0.10 and \$0.16 million.

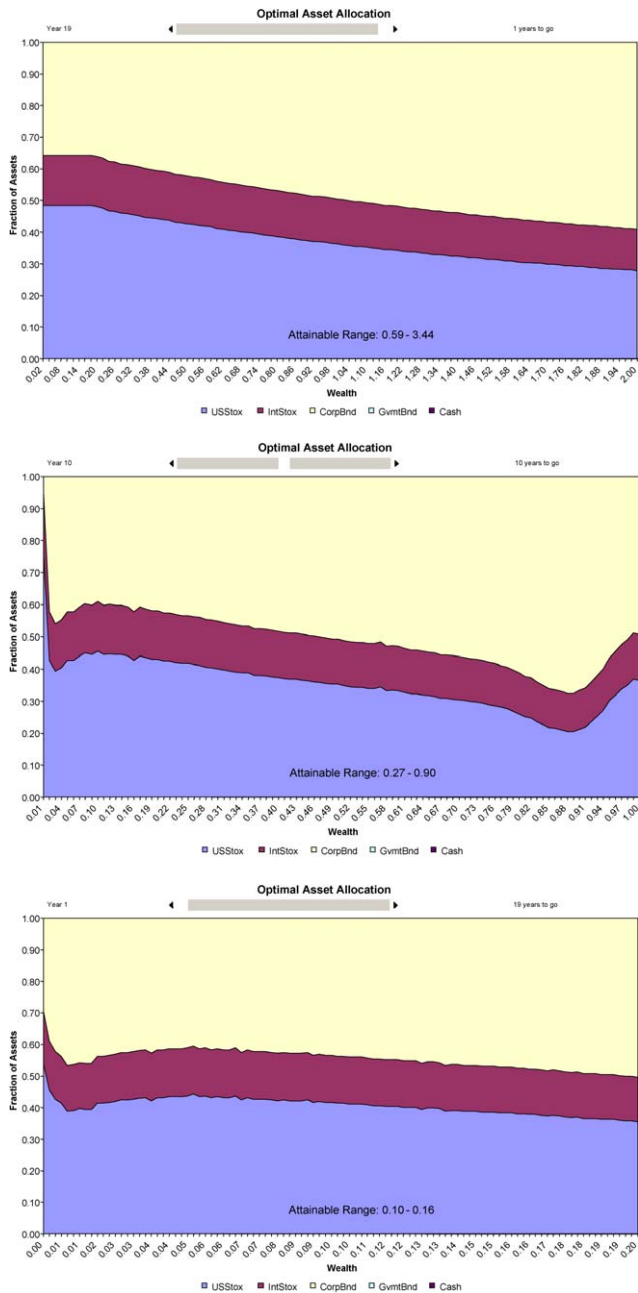


Fig. 6. Investment example: initial \$100k plus \$15k per year, 20-year horizon: increasing relative risk aversion, investment strategy.

Figure 7 represents the expected value strategy (top), the out-of-sample simulation of wealth (middle), and the marginal wealth distribution (bottom). The initial investment is 41% US stocks, 15% international stocks, and 44% corporate bonds. As in the exponential utility case above, government bonds and cash are not in the optimal initial portfolio. The expected value strategy calls for an initial optimal stock allocation of about 56%, to be decreased gradually to about 46% the last decision period. Again, we can describe the dynamic strategy in terms of deviation from the expected value strategy. The optimal dynamic strategy calls for a larger stock allocation and more risk compared to the expected value strategy in the case of under-performance (and therefore lower wealth) and smaller stock allocations and less risk in the case of out performance (and therefore larger wealth). The maximum and minimum stock allocation and the associated risk of the portfolio are within reasonable limits, which makes the increasing relative risk aversion strategy very practical for investors wishing to become more conservative at higher levels of wealth.

Not surprisingly, the out-of-sample results in Figure 7 (middle) are very similar to the exponential case in Figure 5 (middle) before. For the increasing relative risk aversion strategy, the mean wealth obtained at the end of the investment horizon is \$1.575 million with a standard deviation of \$0.452 million. With 99% probability a wealth of \$771,000 is exceeded and with 95% probability a wealth of \$937,000. The certainty equivalent wealth is \$1.44 million. The similar return results compared to the exponential utility case, combined with the lack of undue risk exposure, underlines the practicability of the increasing relative risk aversion strategy.

Next, as case (C), we discuss decreasing relative risk aversion, which implies also decreasing absolute risk aversion. We assume that the relative risk aversion is high at  $\gamma = 8.0$  below a wealth of  $W_L = 1.0$  and decreases linearly to a low relative risk aversion of  $\gamma = 1.01$  for wealth levels of  $W_H = 1.5$  and above. An investor exhibiting this utility function becomes increasingly less risk averse with increasing wealth, but is very risk averse at low levels of wealth. This type of utility function is for investors who want to hedge the downside, e.g., wish not to under-perform a wealth of about \$1 million at the end of the investment horizon. Figure 8 shows the optimal strategy for this utility function for one (top), 10 (middle) and 19 (bottom) years to go. The optimal asset allocation in the last decision period (one period to go) reflects the risk aversion profile: at small levels of wealth the stock allocation is small at about 21% and increases to about 87% stocks at higher levels of wealth. Again, the asset allocation changes with the remaining investment horizon and wealth, where the point of low stock allocations shifts towards higher wealth levels as the remaining investment horizon decreases and the change of allocation becomes less gradual. We also observe very conservative investments at very low wealth levels, but these are out of the range of wealth that can be reasonably obtained. The initial portfolio is 11% US stocks, 10% international stocks, 60% corporate bonds, and 19% government bonds. Cash is not part of the initial portfolio. The expected value strategy in Figure 9 (top) shows that the stock allocation and thus the risk increases with time starting from about 21% stocks at the initial investment up to 50% stocks at the last decision period in year 19. This reflects the investor's

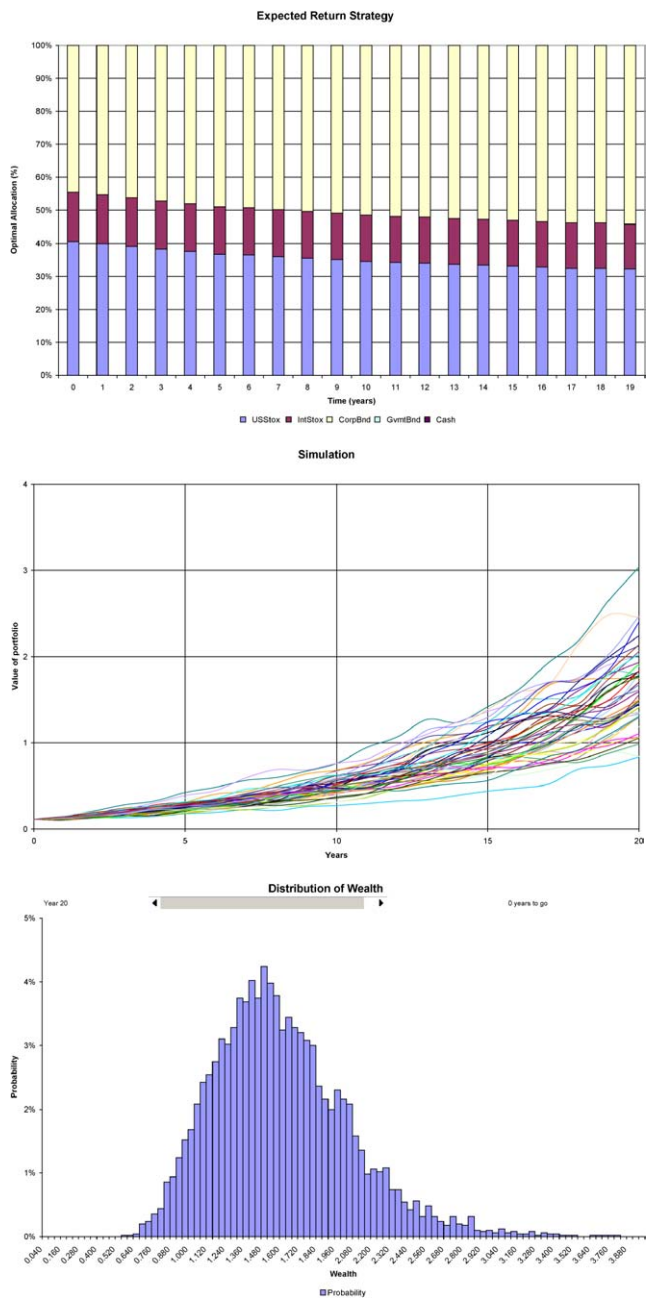


Fig. 7. Investment example: initial \$100k plus \$15k per year, 20-year horizon: increasing relative risk aversion, simulations.

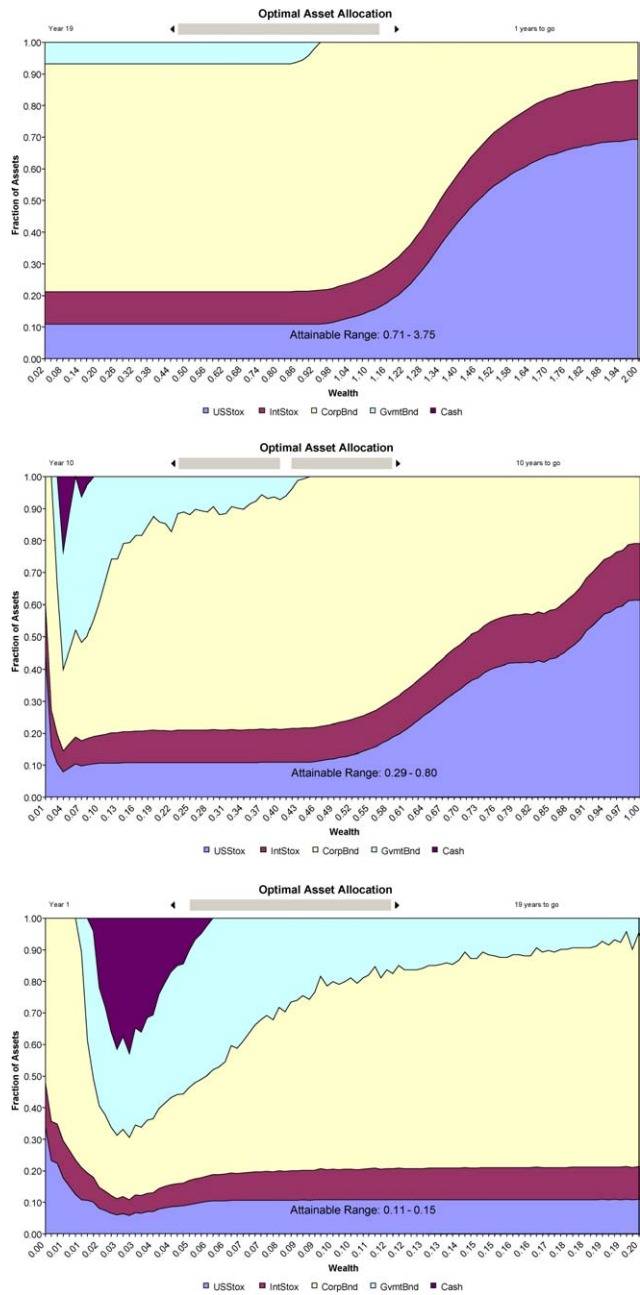


Fig. 8. Investment example: initial \$100k plus \$15k per year, 20-year horizon: decreasing relative risk aversion, investment strategy.



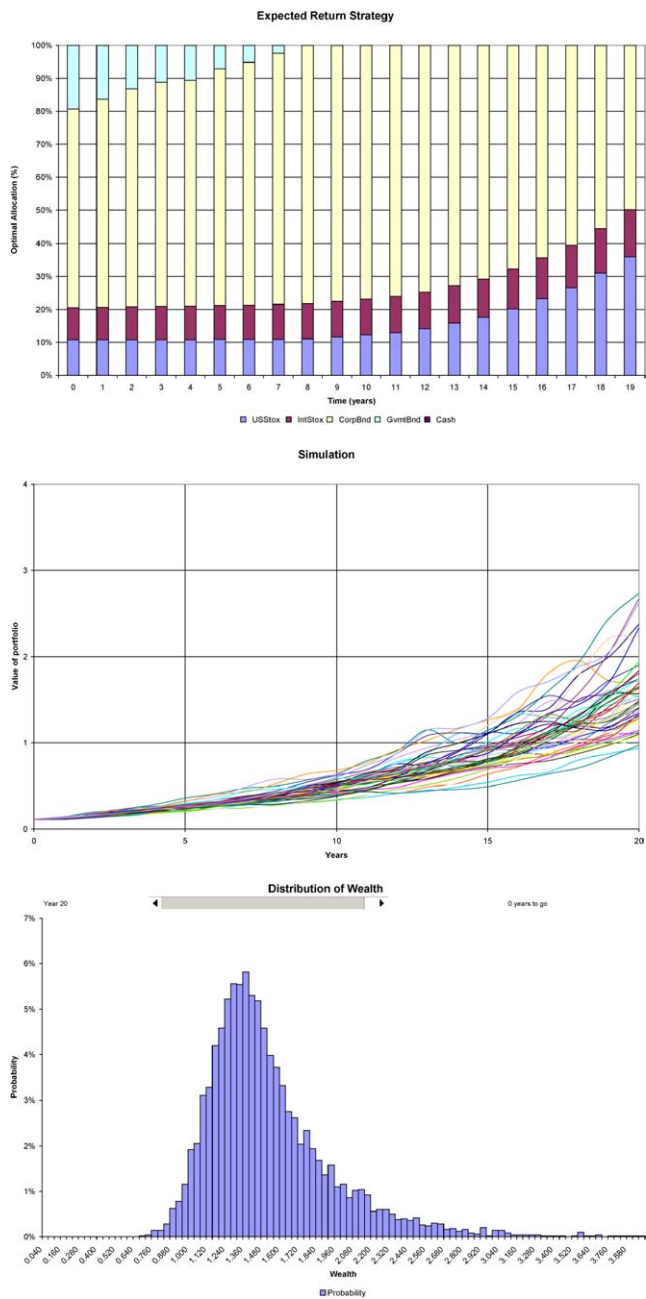


Fig. 9. Investment example: initial \$100k plus \$15k per year, 20-year horizon: decreasing relative risk aversion, simulations.

profile as a time path, where he/she is careful at low levels of wealth, but becomes increasingly less risk averse as wealth grows over time. Relative to the expected value path, the strategy prescribes in each period to increase the stock allocation and the risk if the performance was better than expected (and more wealth than expected is available) and to reduce the stock allocation and risk if under-performance occurred (and less wealth than expected is available). Figure 8 presents the attainable wealth in year 19 between \$0.71 and \$3.75 million, in year 10 between \$ 0.29 and \$0.80 million, and in year 1 between \$0.11 and \$0.15 million.

Figure 9 (in the middle) gives the out-of-sample simulation results, and (at the bottom) the marginal distribution of terminal wealth. In both views, one can observe that the downside is more protected than in the previous cases. The mean wealth obtained at the end of the investment horizon is \$1.498 million with a standard deviation of \$0.436 million. With 99% probability a wealth of \$866,000 is exceeded and with 95% probability a wealth of \$998,000. These are reasonable out-of-sample results for an investor wishing to protect his/her downside below \$1 million. The certainty equivalent wealth is \$1.339 million (Table 3).

The optimal dynamic strategies for increasing and decreasing relative risk aversion behave in a mirrored way. Assuming a crash in the stock market occurred, we can deduce that an investor with increasing relative risk aversion would react by increasing the stock allocation in order to make up for the loss, while the investor with decreasing relative risk aversion would reduce his/her stock allocation in order to further protect the downside. And we know that an investor with constant relative risk aversion (CRRA) would re-balance after the crash back to his/her original asset allocation. We may use this theoretical behavior to help infer the type of utility function that is most appropriate for an investor.

Case (D) reflects risk as a quadratic penalty of under-performing a target wealth. We assumed a target wealth of \$1M and traded off risk and expected return using a risk aversion of  $\lambda_q/2 = 1000$ . This is an extreme case reflecting an investor wishing to obtain a wealth of \$1 million very badly and therefore being prepared to forsake a significant part of the upside. Figure 10 presents the dynamic investment strategy for one (top), ten (middle) and 19 (bottom) years to go. Looking at the last rebalancing period, the optimal strategy reflects an increasing stock allocation for higher levels of wealth, starting slightly below the target wealth. This “critical” wealth level is where the target can still be reached with very high probability. Reducing wealth from large values, the closer we are to this critical wealth level, the more conservative the investment becomes, up to putting almost the entire portfolio into cash. For wealth levels above the target, stock allocations increase with increasing wealth up to 100%. For wealth levels below the critical level, the strategy becomes more risky with decreasing wealth, with stock allocations also rising up to 100%. This behavior reflects exactly the risk aversion represented by the quadratic downside utility function. For wealth levels above the target, the linear term (representing zero risk aversion) is dominant and leads to increased stock allocations up to 100 percent. For wealth levels below the critical point, the quadratic part of the utility function is dominant. Penalizing under-performance quadratically leads to

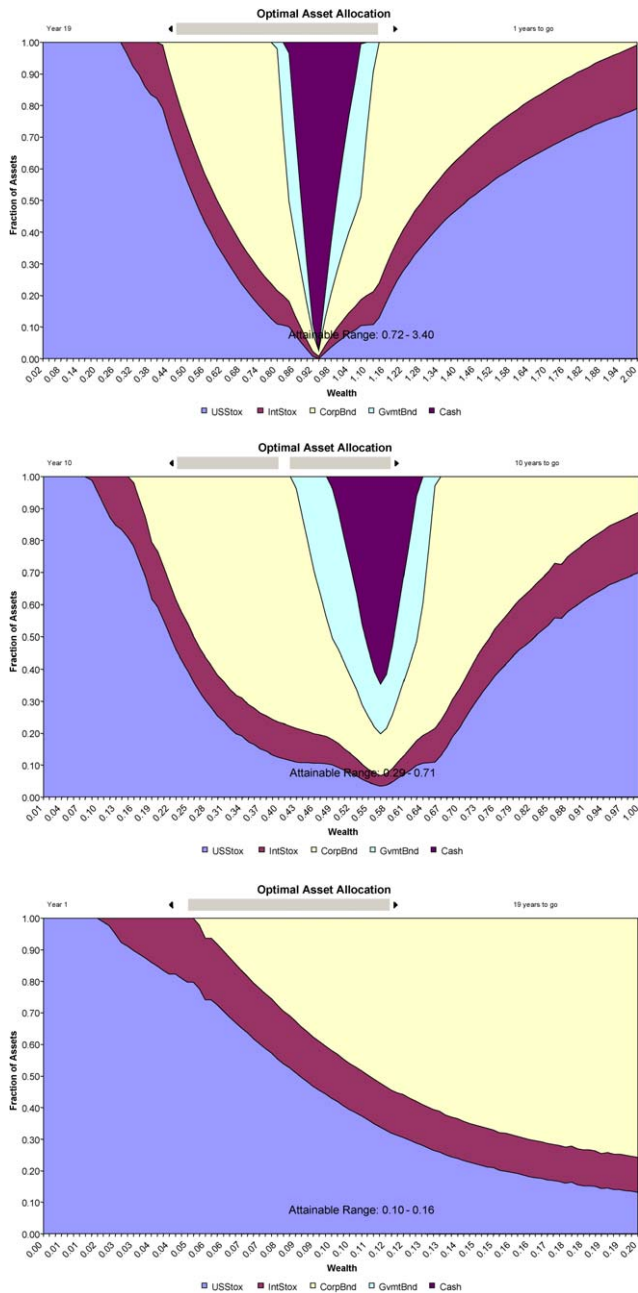


Fig. 10. Investment example: initial \$100k plus \$15k per year, 20-year horizon: quadratic utility, investment strategy.

a relative risk aversion that decreases with larger under-performance and thus increases with larger wealth. Therefore, the quadratic utility function reflects increasing relative risk aversion below the critical wealth and decreasing relative risk aversion above. With longer remaining investment horizons the critical point of reaching the target shifts to the left, and the allocation at that point becomes less conservative. With ten years to go, the critical wealth level is at about \$0.58 million and the allocation is at about 65% cash. Interestingly, with 19 years to go, the reachable wealth levels are all below the critical point and the strategy is entirely one of decreasing relative risk aversion, calling for stock allocations from about 60% at low levels of wealth following a bad first year to 30% after an out-performing first year. Figure 10 represents the attainable wealth in year 19 between \$0.72 and \$3.40 million, in year 10 between \$0.29 and \$0.71 million, and in year 1 between \$0.10 and \$0.16 million.

The initial portfolio is 36% US stocks, 14% international stocks, and 50% corporate bonds. Government bonds and cash are not part of the initial optimal portfolio. The expected return strategy in Figure 11 (top) confirms this result. Starting at an initial stock allocation of 50%, the stock allocation is reduced gradually through year 13 and then increased from year 14 on. Given the savings rate and the results from the investments, the investor first starts in an under-performing state, then achieves the critical wealth level during year 14, and ends above the critical wealth.

The out-of-sample simulation results in Figure 11 (middle) show good downside protection and the marginal probability chart (bottom) is very steep on the left reflecting the downside protection the quadratic downside risk function is said to provide. However, even the assumption of a very large risk aversion coefficient does not lead to a zero probability of wealth below \$1 million. The mean wealth obtained at the end of the investment horizon is \$1.339 million with a standard deviation of \$0.347 million. With 99% probability a wealth of \$911,000 is exceeded and with 95% probability a wealth of \$1.006 million. Thus, the target wealth of \$1 million is exceeded with larger than 95% probability. The statistics in Table 3 give a certainty equivalent wealth of \$998,000 as the lowest of all four utility functions, reflecting again the low emphasis on the upside displayed by the quadratic downside utility function.

### 7.3. The performance of dynamic strategies

Next we compare dynamic strategies obtained from using the different utility functions with a number of fixed-mix strategies, i.e., (1) US stocks only, (2) cash only, (3) all asset classes equally weighted, (4) risk averse, (5) medium, and (6) risk prone. Except for case (3), all fixed-mix strategies are mean-variance efficient and obtained from solving the single-period Markowitz problem. Figure 12 presents the efficient frontier and the efficient fixed-mix portfolios. We intentionally use the five representative mean-variance optimal fixed-mix strategies for comparison, because such portfolios are commonly held in practice. Many aggressive investors hold stocks-only portfolios, and many very conservative investors keep their funds entirely in money market accounts. Investment firms usually offer fund-of-funds portfolios, such as averse (often called

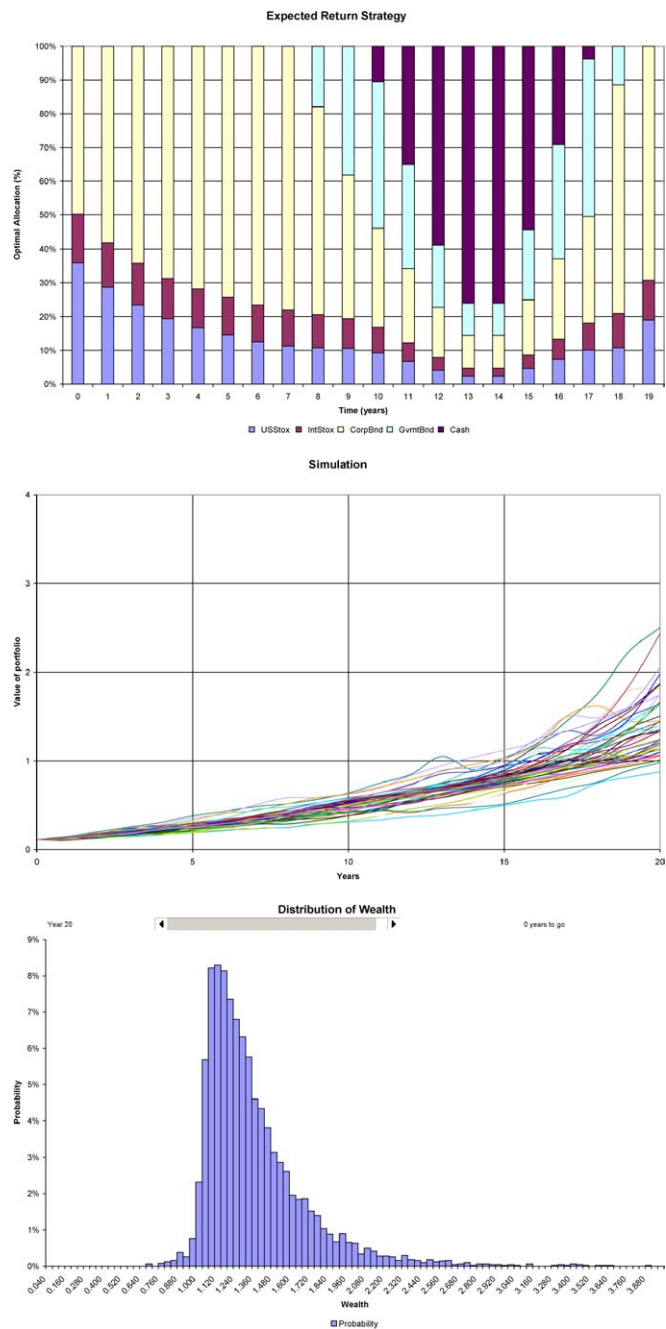


Fig. 11. Investment example: initial \$100k plus \$15k per year, 20-year horizon: quadratic utility, simulations.

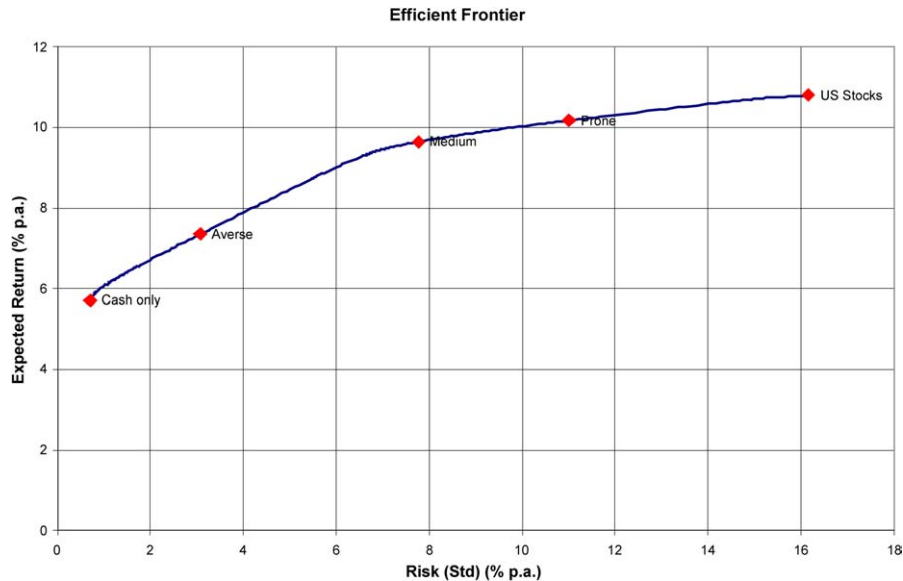


Fig. 12. Efficient frontier and efficient portfolios for fixed-mix strategies.

Table 4  
Out-of-sample simulation results for various fixed-mix strategies

	Mean	Std	99%	95%
US stocks	1.825	1.065	0.469	0.660
Cash	0.868	0.019	0.822	0.834
Equally weighted	1.349	0.301	0.799	0.920
Averse	1.098	0.110	0.869	0.930
Medium	1.538	0.407	0.825	0.975
Prone	1.663	0.639	0.677	0.852

conservative), medium (often referred to as dynamic), and prone (often referred to as aggressive), which are sold to investors allegedly according to their risk profile. The equally weighted strategy represents a non-efficient portfolio for comparison.

Table 4 represents the out-of-sample simulation results for the various fixed-mix strategies. Obviously, the largest expected wealth of \$1.825 million is obtained by the US Stocks portfolio, and the smallest expected wealth of \$0.868 million by the cash-only portfolio. More interestingly, the cash-only portfolio exceeds a wealth of \$822,000 with 99% probability, and the medium portfolio exceeds a wealth of \$825,000 with 99% probability. The expected wealth of the cash-only portfolio is a mere \$868,000 compared to the expected wealth of the medium portfolio of \$1.538 million.

Table 5  
Out-of-sample simulation results for CEW and improvement

	Exponential RA = 2		Increasing RRA		Decreasing RRA		Quadratic	
	CEW	Impr. (%)	CEW	Impr. (%)	CEW	Impr. (%)	CEW	Impr. (%)
Dynamic	1.41193		1.44004		1.35163		0.98204	
US stocks	1.28811	9.61	1.34365	7.17	0.68918	96.12	0.87639	12.06
Cash	0.86732	62.79	0.86728	66.04	0.86596	56.08	0.86628	13.36
Equally weighted	1.27082	11.10	1.28235	12.30	1.17981	14.56	0.96254	2.03
Averse	1.08667	29.93	1.08747	32.42	1.0605	27.45	0.97207	1.03
Medium	1.40415	0.55	1.4292	0.76	1.34326	0.62	0.97051	1.19
Prone	1.38933	1.63	1.43378	0.44	1.0925	23.72	0.93699	4.81

The risk-averse strategy exhibits the largest wealth of \$869,000 exceeded with 99% probability, with an expected wealth of \$1.098 million at the end of the investment horizon. In comparison, the dynamic strategy of decreasing relative risk aversion exhibits a very similar wealth of \$865,000 exceeded with 99% probability, but a significantly larger expected wealth of \$1.339 million. This represents an increase of 21.95% expected wealth at the same level of (downside) risk. The medium strategy exhibits the largest wealth of \$975,000 exceeded with 95% probability among all fixed-mix strategies. The best dynamic strategy on this measure is the quadratic downside risk strategy, which exceeds \$1.006 million with 95% probability. However, the quadratic strategy returns a significantly lower expected wealth of \$1.339 million, compared with \$1.538 million for the medium fixed-mix strategy. The extra downside protection is expensive on the upside. There is no dynamic strategy matching the 95% exceeded wealth of the medium fixed-mix strategy, and therefore we cannot make a statement of out-performance at the same level of 95% exceeded wealth given the strategies at hand.

In order to further evaluate the dynamic strategies, we compared the certainty equivalent wealth (CEW) of the dynamic strategies with the corresponding certainty equivalent wealth of each fixed-mix strategy. Table 5 presents the results in terms of certainty equivalent wealth and in the percentage improvement obtained when using the optimal dynamic strategy for each utility function compared to each of the fixed-mix strategies.

A certainty equivalent wealth (CEW) of \$1.41193 million is the maximum obtained from a dynamic strategy in the case of the exponential utility function with risk aversion of  $\lambda = 2$ . This utility represents a fairly aggressive investor. The medium strategy would be the best fixed-mix strategy with an CEW of \$1.40415 million. The improvement of the dynamic strategy versus the best fixed-mix strategy is 0.55%. The prone strategy would also fit reasonably well this investor, and results in a CEW of \$1.38933 million. The improvement of the dynamic strategy versus the prone strategy is 1.63%. The improvements in CEW of the dynamic strategy versus US stocks, cash and averse are 9.61, 62.79 and 29.93%, respectively.

Not surprisingly, the situation is very similar for the increasing relative risk aversion utility function. The dynamic strategy results in a certainty equivalent wealth of

\$1.44004 million. The prone strategy is the best fixed-mix strategy with a CEW of \$1.43378 million, resulting in an improvement of the dynamic strategy versus prone of 0.44%. The medium strategy is surpassed by the dynamic strategy by 0.76%. US stocks, cash only and risk averse are surpassed by the dynamic strategy by 7.17, 66.04 and 32.42%, respectively.

In the case of decreasing relative risk aversion, the dynamic strategy yields a CEW of \$1.35163 million. The best fixed-mix strategy is the medium, with a CEW of \$1.34326 million. The improvement of the dynamic strategy versus the medium strategy is 0.62%. The improvements of the dynamic strategy versus US stocks, cash-only, risk-averse and risk-prone are 95.12, 56.08, 27.45, and 23.72%, respectively, showing that none of these strategies are a particularly good fit for this type of utility function.

For the quadratic downside utility function, the dynamic strategy has a CEW of \$0.98204 million, while the best fixed-mix performance is the risk-averse strategy with a CEW of \$0.97207 million. Thus, the improvement of the dynamic strategy is 1.05% versus the best fixed-mix strategy. The improvements over US stocks, cash-only, medium and risk-prone are 12.06, 13.36, 1.19 and 4.81%, respectively.

The equally weighted strategy was never the best fixed-mix strategy for any utility function. For each utility function, the best fixed-mix strategy could be improved upon by about a half percent by the dynamic strategy, except for the quadratic utility function for which the improvement is about one percent. These results confirm that the improvement in certainty equivalent wealth can be significant. However, it also suggests that from a practical perspective a well-selected or numerically optimized fixed-mix strategy can be close in performance to a dynamic strategy when measured in terms of CEW. A possible explanation is that, when we used the historical mean return estimates, the difference in the mean returns between stock and corporate bonds was rather small, and the corresponding mean-variance efficient frontier rather flat. Using the mean estimates from the regression would result in an increased mean return for stocks, and consequently make the efficient frontier steeper. Thus, the benefits of dynamic asset allocation versus fixed-mix strategies would increase.

#### *7.4. Dynamic strategies for hedging downside risk*

In the next investment example, we consider an investment horizon of 10 years. An initial capital of \$1 million is to be invested, and there are no further payments made in or out of the account. The capital should be invested so that a minimum return of 2% per year is exceeded with 99% probability, and the expected return should be as large as possible. Such an investment situation is often faced by institutional investors. Time horizons may vary. While the problem can be stated as a chance-constrained optimization problem, we will explore empirically whether downside risk protection can be achieved by utility maximization using the quadratic downside risk or the decreasing relative risk aversion utility function.

For the quadratic function we set the target wealth at \$1.22 million reflecting the 2% per year minimum return to be exceeded with 99% probability. As in the investment



example above, we set the risk aversion at  $\lambda_q/2 = 1000$ , in order to heavily penalize any under-performance with respect to the target. For the decreasing relative risk aversion function, we set a high relative risk aversion of  $\gamma = 8$  below the level  $W_L = \$1.22$  million (the target wealth) and a low relative risk aversion of  $\gamma = 1.01$  above the wealth level  $W_H = \$1.5$  million, and assume the relative risk aversion increases linearly in between. For both utility functions, we solved the 10-year horizon problem assuming yearly, quarterly and monthly rebalancing. For quarterly rebalancing a 40-stage problem needed to be solved, and for the monthly rebalancing a 120-stage problem.

Table 6 gives the results for expected wealth, wealth exceeded with 99% probability, and minimum wealth (defined as the level exceeded with 99.9% probability), as well as expected return, return exceeded with 99% probability and minimum return (exceeded with 99.9% probability). Using yearly rebalancing, both utility functions lead to larger returns than 2% per annum with at least 99% probability. The quadratic downside utility function resulted in a very conservative strategy, and even the minimum returns (exceeded with 99.9% probability) met the 2% per year requirement. The expected returns from both strategies are 10.35% per year for the quadratic downside and 10.42% for the decreasing RRA utility function, indicating that the downside-risk-protected strategies are not overly costly on the upside. The results were rather similar for yearly, quarterly and monthly rebalancing for both utility functions; however, more frequent rebalancing allowed for somewhat better downside protection and larger expected returns.

For the quadratic downside risk strategy, Figure 13 represents the optimal asset allocation for yearly rebalancing with respect to wealth for one (top), five (middle) and nine years to go. For all attainable wealth levels (achieved with 99.9% probability), the optimal asset allocation is entirely on the right side of the critical wealth (with the most conservative asset allocation) and thus exhibits decreasing relative risk aversion. The better the performance (and the larger the wealth in the account), the more risk should be assumed and the larger should be the stock allocation. The attainable wealth in year 9 lies between \$1.16 and \$7.35 million, in year 5 between \$0.96 and \$3.54 million, and in

Table 6  
Out-of-sample simulation results for downside risk protection

Rebal.	Periods	Exp. wealth	99% wealth	Min. wealth	Exp. ret. (%)	99% ret. (%)	Min. ret. (%)
Quadratic downside risk							
yearly	10	2.67628	1.31957	1.22778	10.35	2.81	2.07
quarterly	40	2.79864	1.28774	1.22500	10.84	2.56	2.05
monthly	120	2.71229	1.36413	1.26250	10.49	3.15	2.36
Decreasing RRA							
yearly	10	2.69504	1.25000	1.10000	10.42	2.26	0.96
quarterly	40	2.78378	1.26714	1.05000	10.78	2.40	0.49
monthly	120	2.76007	1.32083	1.10000	10.69	2.82	0.96

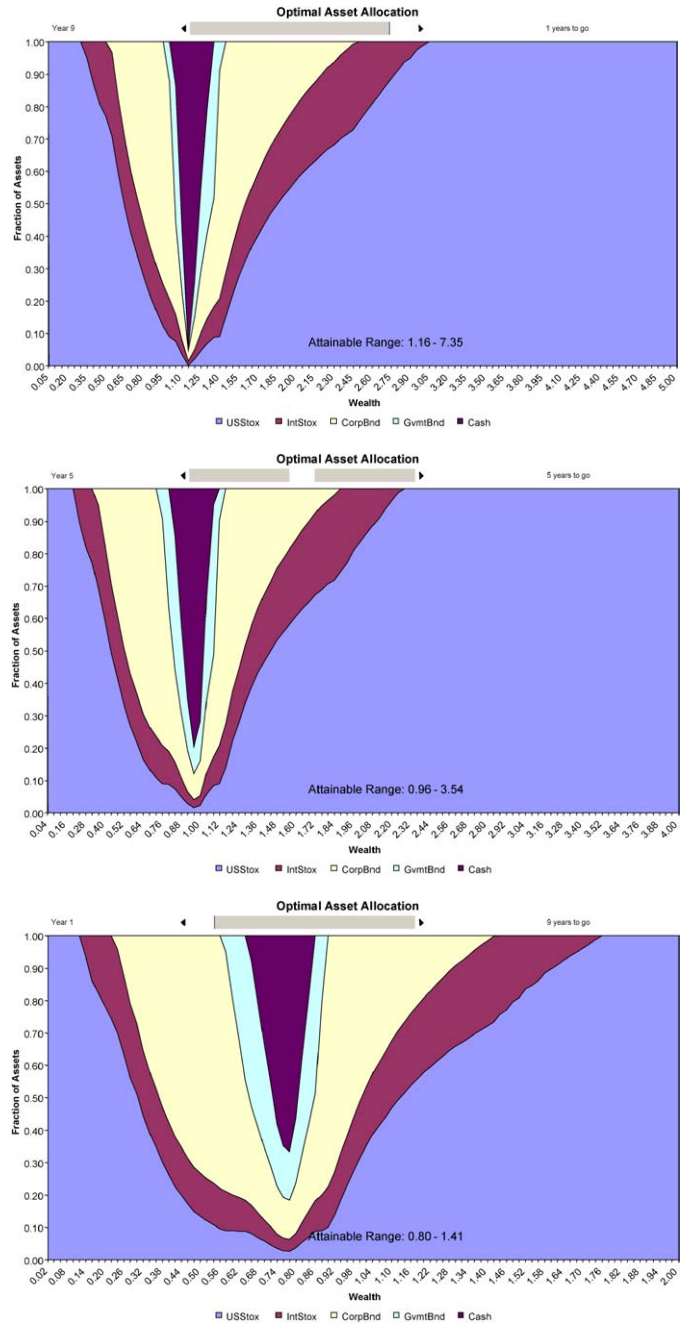


Fig. 13. Investment example: 10-year horizon: downside risk, quadratic, investment strategy.

Table 7  
Out-of-sample simulation results for downside risk protection

Quadratic dynamic downside risk						
Period	Exp. wealth	99% wealth	Min. wealth	Exp. ret. (%)	99% ret. (%)	Min. ret. (%)
1	1.10008	0.8814	0.8	10.01	−11.86	−20.00
2	1.21161	0.89267	0.825	10.07	−5.52	−9.17
3	1.33354	0.92828	0.86	10.07	−2.45	−4.90
4	1.47004	0.97231	0.9	10.11	−0.70	−2.60
5	1.62245	1.01714	0.9616	10.16	0.34	−0.78
6	1.79528	1.07636	1	10.24	1.23	0.00
7	1.98402	1.1296	1.05333	10.28	1.76	0.74
8	2.19368	1.18286	1.12	10.32	2.12	1.43
9	2.42128	1.25091	1.16071	10.32	2.52	1.67
10	2.67628	1.31957	1.22778	10.35	2.81	2.07

year 1 between \$0.80 and \$1.41 million. Figure 14 (top) represents the expected value strategy (assuming expected returns are realized in every period), (middle) the simulations of wealth over time, and (bottom) the marginal distribution of wealth at the end of the investment horizon for yearly rebalancing. The expected value strategy prescribes to start with an initial asset allocation of 60% stocks and 40% corporate bonds, and then to gradually increase the stock allocation to almost 100% in the last rebalancing period. The marginal distribution is steep on the left side, representing a limited downside, and the minimum wealth (exceeded with 99.9%) of \$1.22778 million translates into a minimal annual return of 2.07%.

How well this strategy protects the downside along the way is presented in Table 7, which exhibits the expected wealth, the wealth exceeded with 99% probability, and the minimum wealth (that is exceeded with 99.9% probability), along with corresponding measures for the returns in each year of the investment horizon. The downside protection during the initial periods is rather weak, with the returns exceeded with 99% probability in year one resulting as −11.86% and the minimum return resulting as −20.00%. Only in year five is nominal capital preserved with greater than 99% probability and only in year six with 99.9% probability. In year eight the target return of 2% per year is exceeded with 99% probability, and in year 10 with 99.9% probability. The expected returns increase monotonically as time passes, but only marginally from 10.01% per year in year one to 10.35% in year ten.

We next compare these results to the results of a strategy where the downside is controlled dynamically in every period.

7.5. Downside risk protection at every period

Especially institutional investors are evaluated with regard to their portfolio’s performance on a yearly basis. Given the ongoing evaluations, institutional managers may

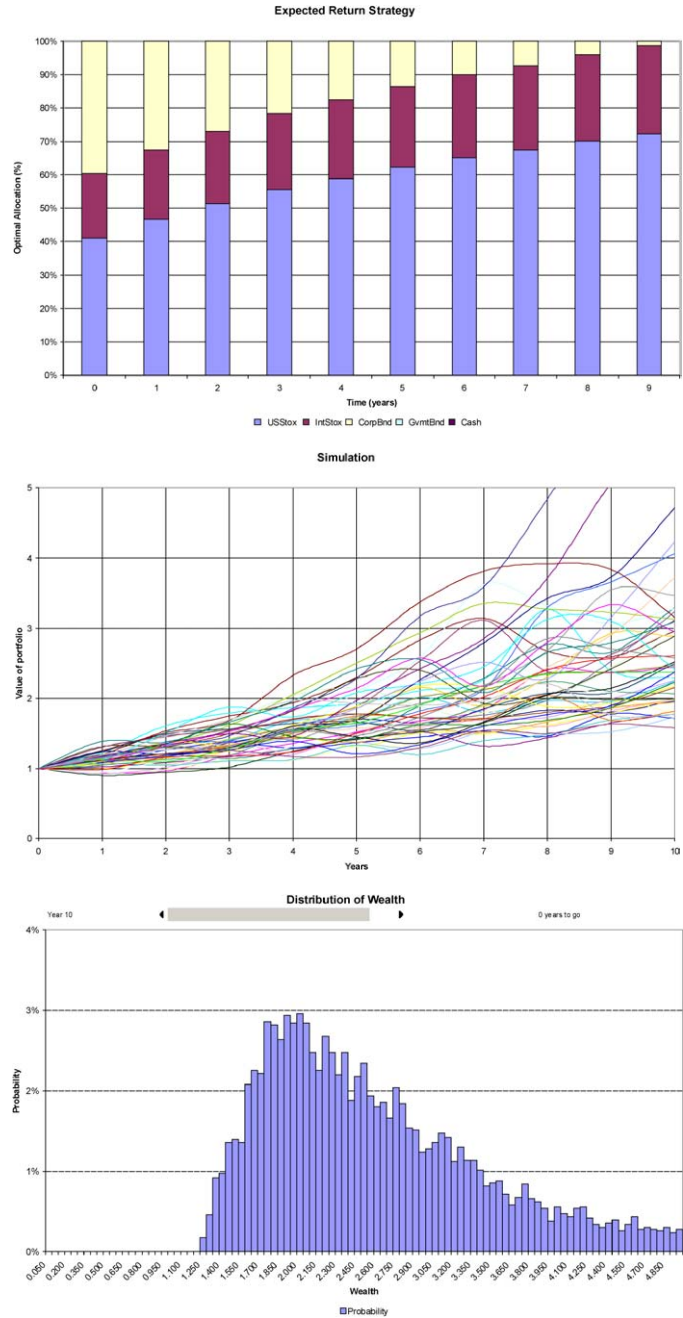


Fig. 14. Investment example: 10-year horizon: downside risk, quadratic, simulations.

consider implementing a dynamic strategy with limited downside risk not only at the end of the investment horizon but also at every year along the way. Again, using a 10-year investment horizon, we compute a strategy using a discounted additive quadratic downside utility function at every year. For the dynamic quadratic downside risk utility function, the discount factor was chosen as 1% per year and the target wealth in each period was set as  $W_{d,t} = 1.02^t$ ,  $t = 1, 10$ , reflecting the desired minimum return of 2% per year. We set the risk aversion coefficient at every period as  $\lambda_l = 1000$  and  $\lambda_q/2 = 1000$ , in order to heavily penalize (linearly and quadratically) any under-performance with respect to the targets in addition to the quadratic downside utility function at the end of the investment horizon.

Figure 15 represents the strategies for one (top), five (middle) and nine (bottom) years to go. The optimal strategy at the last rebalancing period must be identical to that, where only terminal wealth is evaluated. This follows directly from the dynamic programming recursion. At all other years the dynamically controlled downside risk strategy is more conservative, e.g., at five years to go risk and stock allocation increases slower with wealth compared to the uncontrolled case (with downside risk protection only at the end of the investment horizon), and the most conservative allocation at the critical point is more conservative than in the uncontrolled case. For example, in the uncontrolled case at nine years to go, the cash position at the critical point is about 70%, whereas in the controlled case it is about 90%. With downside controls, the attainable wealth in year 9 lies between \$1.26 and \$5.36 million, in year 5 between \$1.13 and \$2.30 million, and in year 1 between \$1.01 and \$1.12 million. This is more narrow range than for the uncontrolled case, reflecting the loss on the upside induced by the protection on the downside.

Figure 16 (top) represents the expected value strategy (middle) the simulations of wealth, and (bottom) the marginal probability distribution of terminal wealth. The expected value strategy under downside control prescribes to start with a stock allocation of 5% and to increase it up to about 85% in year nine. This is significantly more conservative than in the uncontrolled case. Comparing the wealth simulations, one can readily observe that the downside is more protected along the way than in the uncontrolled case.

Table 8 presents the statistics for each year of the investment horizon. (The last line of year 10 represents the terminal statistics.) The marginal distribution of final wealth displays a minimum wealth (exceeded with 99.9% probability) of \$1.3125 million, which translates into a 2.76% annual return. Compared to the uncontrolled case above, the downside is much better protected along the way, e.g., in every period the return exceeded with 99% probability is greater than 2%, and the minimum return (exceeded with 99.9% probability) exceeds 2% from year three on. The expected return is smaller compared to the unconstrained case and increases monotonically from 6.58 to 9.22%, which confirms that the dynamically downside-controlled strategy is overall more conservative than when only the terminal distribution is considered in the optimization.

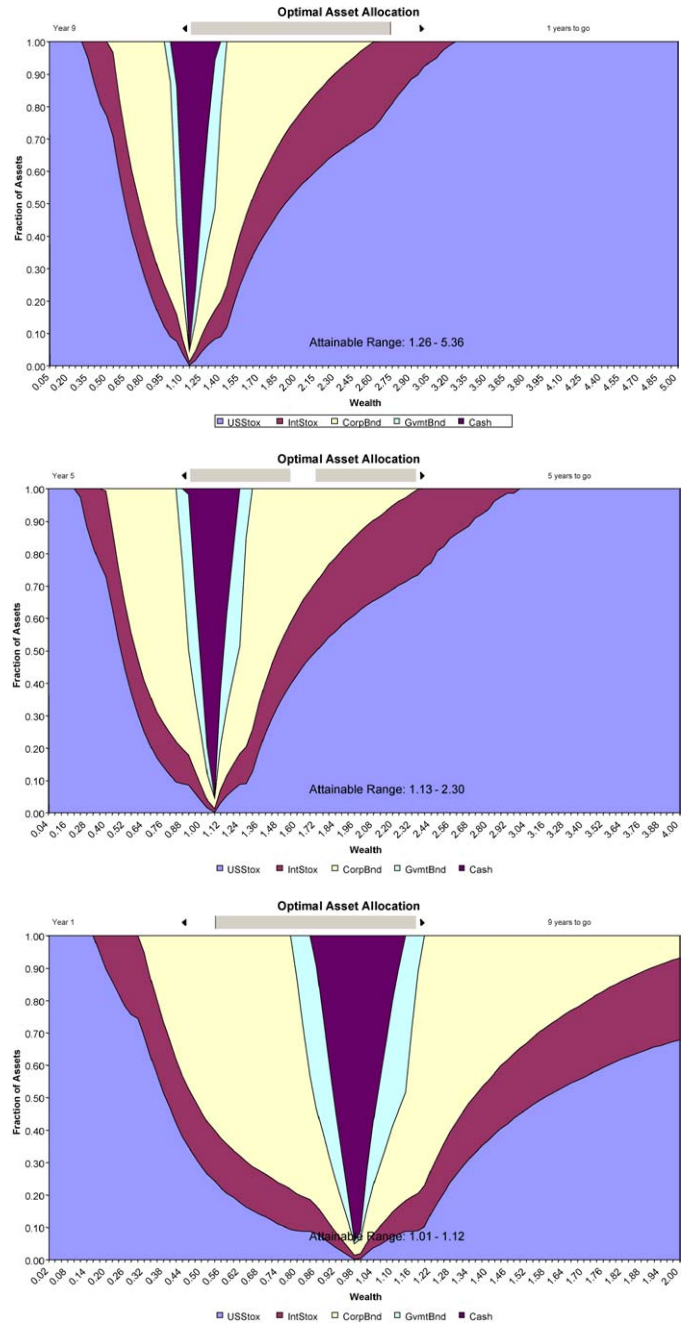


Fig. 15. Investment example: 10-year horizon: downside risk at each period, quadratic, investment strategy.

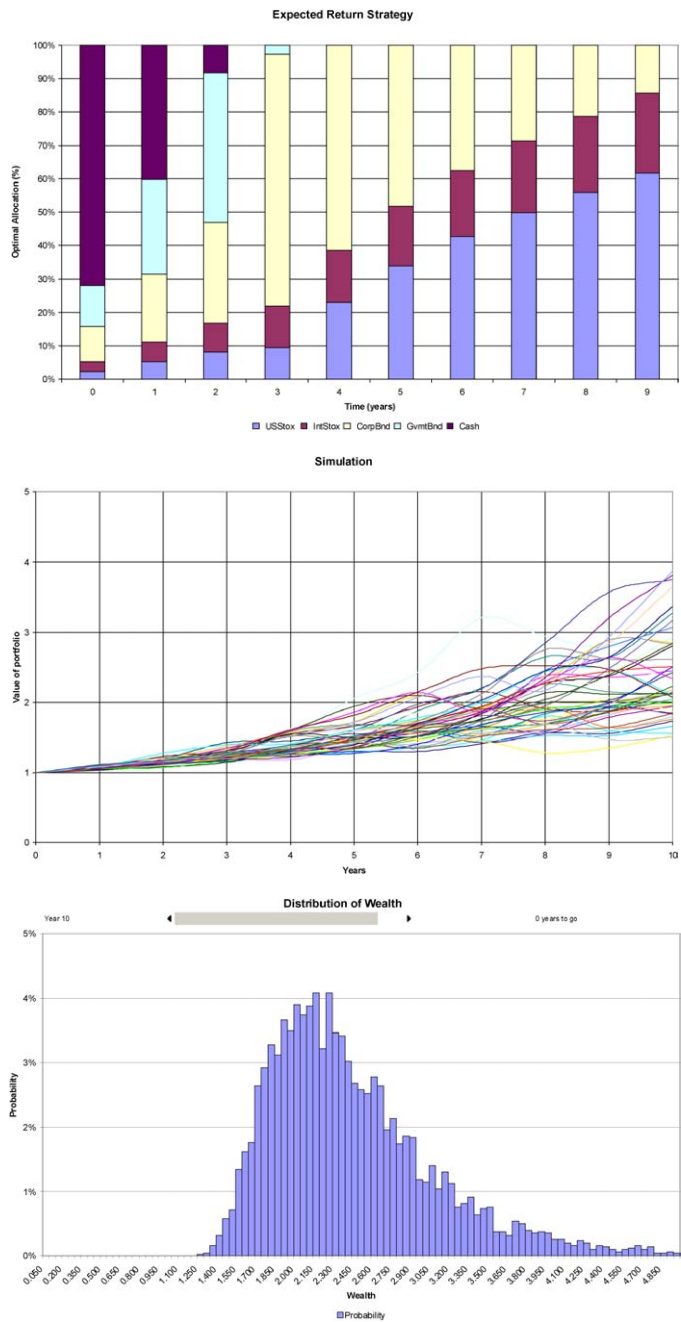


Fig. 16. Investment example: 10-year horizon: downside risk at each period, quadratic, simulations.

Table 8  
Out-of-sample simulation results for dynamic downside risk protection at every period

Quadratic dynamic downside risk						
Period	Exp. wealth	99% wealth	Min. wealth	Exp. ret. (%)	99% ret. (%)	Min. ret. (%)
1	1.06575	1.02225	1.00533	6.58	2.23	0.53
2	1.14561	1.06108	1.032	7.03	3.01	1.59
3	1.24271	1.09137	1.06222	7.51	2.96	2.03
4	1.35621	1.13641	1.095	7.91	3.25	2.29
5	1.48549	1.176	1.12667	8.24	3.30	2.41
6	1.63401	1.22638	1.152	8.53	3.46	2.39
7	1.79975	1.27385	1.2	8.76	3.52	2.64
8	1.9845	1.32121	1.23	8.94	3.54	2.62
9	2.18661	1.37778	1.26	9.08	3.62	2.60
10	2.41527	1.43966	1.3125	9.22	3.71	2.76

These are very encouraging results. All out-of-sample simulations were performed using bootstrapping from the historical monthly observations and thus represent possible fat tails in the asset returns distribution.

### 7.6. Computation times

For the optimization we used a sample size of  $|S_t^i| = 2000$  of annual returns for each period. The sample is based on bootstrapping from the monthly returns, thus we used 24,000 sampled monthly observations to compute the annual returns sample. In each period for the out-of-sample evaluation, we used a sample size of  $|S_t^o| = 5000$  of annual returns, also obtained by bootstrapping monthly returns. The computations were carried out using the fast method, pre-calculating the efficient frontier parameterized at 2000 points. After computing the optimal strategy, we simulated 5000 paths based on bootstrapping monthly returns. Table 9 presents the optimization time in CPU seconds (on a Pentium 4 3.4 MHz and 2 GB of RAM) for the different time horizons and utility functions used. It takes about 11 seconds to optimize a 20-period problem with yearly rebalancing, thus solving a 20-stage stochastic dynamic program. The computation time was about 8 seconds for a 10-year horizon with yearly rebalancing, about 17 seconds for quarterly rebalancing and 41 seconds for monthly rebalancing. In the latter, a 120 stage stochastic dynamic program was solved in 41 seconds.

The fast computations for yearly rebalancing permit us to carry out the calculations on-line. We have developed a commercial system, called WealthiOR, to help investment advisors find the best asset allocation strategies for their clients, individual and institutional. The system is web-based and the optimizations and simulations are carried out on-line in real time. The system is user friendly, easy to use, and also facilitates graphical output and downloadable tables for spreadsheets, providing all necessary sta-



Table 9  
Computation times in elapsed CPU seconds

Rebalancing	Horizon	Periods	Utility	
yearly	20	20	Exponential	Increasing RRA
			10.53	10.92
yearly	20	20	Quadratic	Decreasing RRA
			10.48	11.81
yearly	10	10	Quadratic	Decreasing RRA
			6.58	7.79
quarterly	10	40	15.85	17.06
monthly	10	120	41.35	41.35

tistical information. It permits an adviser to quickly devise customized (regarding risk preference and time horizon) optimized forward-looking dynamic strategies.

8. Comparison to multi-stage stochastic programming

Our results demonstrate that, using the stochastic dynamic programming approach, asset allocation problems with many asset classes and many time periods can be solved efficiently and accurately, allowing for a detailed discussion of the resulting investment strategies. The approach can also handle options and serial dependency of asset returns modeled by restricted autoregressive processes using a small number of predictors. Also, stochastic cash flows to represent uncertain liabilities can be handled by the approach to the extent that the serial correlation of the joint probability distribution of asset returns and cash flows can be modeled with a small number of state variables. This makes the approach also applicable to classes of the more general asset-liability problem. From a statistical perspective (considering the historical data available for estimating the processes and the number of parameters that need to be estimated), predictions based on too many autoregressive factors may lead to over-fitting, resulting in weak out-of-sample performance. Thus, from a practical perspective, predictions based on a limited number of factors (state variables) seem adequate, and thus the stochastic dynamic programming approach can be efficiently employed for solving a wide class of practical problems.

For general returns processes with many predictive variables, and when transaction costs must be considered, multi-stage stochastic programming can be employed successfully. For example, Collomb and Infanger (2005) explored the impact of serial correlation in asset returns on the optimal asset allocation strategy. The focus of the study was on active portfolio management on the asset class level in a multi-period look-ahead setting. For representing serial correlation, we estimated a lag-one vector autoregressive process for stocks, bonds and cash, and formulated a four-stage stochastic linear program representing the dynamic asset allocation problem for three annual

periods of look-ahead. Modeling the stochastic program, a typical scenario tree was represented by  $(50 \times 50 \times 40)$  scenarios leading to 100,000 scenarios, and the corresponding deterministic equivalent formulation of the stochastic program had several hundred thousand rows and columns. Such problems were typically solved in about 2 hours CPU time as deterministic equivalent linear programs and in about 20 minutes when solved using decomposition as implemented in Infanger's (1997) DECIS code. In comparison, the stochastic dynamic programming recursion with 2000 scenarios per period and 10 periods look ahead took only about 10 seconds.

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