Optimization Methods. Seminar 3. Projection of a point on a set, separation, support hyperplane.

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Reminder

- Affine hull and affine set
- Convex hull and convex set
- Conical hull and convex cone
- Operations that preserve convexity

Types interior of a set

Interior of a set

Interior of a set G consists of points from G such that:

$$\mathbf{int}\, G = \{\mathbf{x} \in G \mid \exists \varepsilon > 0, B(\mathbf{x}, \varepsilon) \subset G\},$$

where $B(\mathbf{x}, \varepsilon) = \{\mathbf{y} \mid \|\mathbf{x} - \mathbf{y}\| \leq \varepsilon\}$

Relative interior of a set

Relative interior of set G is called the following set:

$$\mathbf{relint}\, G = \{\mathbf{x} \in G \mid \exists \varepsilon > 0, B(\mathbf{x}, \varepsilon) \cap \mathbf{aff}\, G \subseteq G\}$$

Q: why do we need relative interior concept?

Examples

Find relative interior of the following sets

- $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}\}$
- $\{\mathbf{x} \in \mathbb{R}^n | \sum\limits_{i=1}^n \alpha_i x_i^2 \leq 1,\ \alpha_i > 0, i = 1, \dots, n\}$
- $\{\mathbf{x} \in \mathbb{R}^n | \sum\limits_{i=1}^n \alpha_i x_i^2 = 1,\ \alpha_i > 0, i = 1, \dots, n\}$
- $\{(x_1, x_2, x_3) \in \mathbb{R}^3 | -1 \leq x_1 \leq 1,\ -1 \leq x_2 \leq 1,\ x_3 = 0\}$

Projection of point on a set

Distance between a point and a set

Let d be a distance between point $\mathbf{a} \in \mathbb{R}^n$ and closed set $X \subset \mathbb{R}^n$ according to the norm $\|\cdot\|$:

$$d(\mathbf{a}, X, \|\cdot\|) = \inf\{\|\mathbf{a} - \mathbf{y}\| \mid \mathbf{y} \in X\}$$

Projection of a point on a set

Let $\pi_X(\mathbf{a}) \in X$ be a projection of a point $\mathbf{a} \in \mathbb{R}^n$ on a set $X \subset \mathbb{R}^n$ according to the norm $\|\cdot\|$:

$$\pi_X(\mathbf{a}) = \arg\min_{\mathbf{y} \in X} \|\mathbf{a} - \mathbf{y}\|$$

Q: is projection unique? If not, then in what case it is unique? How uniqueness of projected is related to the convexity of set?

Facts about projections

Projection criterion

A point $\pi_X(\mathbf{a}) \in X$ is a projection of a point $\mathbf{a}$ on a set $X \Leftrightarrow \|\mathbf{a} - \mathbf{x}\| \geq \|\mathbf{a} - \pi_X(\mathbf{a})\|, \; \forall \mathbf{x} \in X$.

Projection criterion for ℓ_2-norm

A point $\pi_X(\mathbf{a}) \in X$ is a projection of a point $\mathbf{a}$ on a set $X \Leftrightarrow \langle \pi_X(\mathbf{a}) - \mathbf{a}, \mathbf{x} - \pi_X(\mathbf{a}) \rangle \geq 0, \; \forall \mathbf{x} \in X$.

Examples

- Find projection on a ball $\{\mathbf{x} \in \mathbb{R}^2 | \|\mathbf{x}\|_* \leq 1\}$ in ℓ_1, ℓ_2 and ℓ_∞ norms
- Find projection on the affine set $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{A} \in \mathbb{R}^{m \times n}, rank(\mathbf{A}) = m\}$
- Find projection on the affine set $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \mathbf{x}_0 + \mathbf{S}\mathbf{y},\ \mathbf{S} \in \mathbb{R}^{n \times m},\ \mathbf{y} \in \mathbb{R}^m, rank(\mathbf{S}) = m\}$

Separation of a convex sets

Definitions

Let $X_1, X_2 \subset \mathbb{R}^n$ be arbitrary sets. They are called:

- separated, if $\exists \mathbf{p}, \beta : \langle \mathbf{p}, \mathbf{x}_1 \rangle \geq \beta \geq \langle \mathbf{p}, \mathbf{x}_2 \rangle$, $\forall \mathbf{x}_1 \in X_1$ and $\forall \mathbf{x}_2 \in X_2$.
- self-separated, if they are separated and $\exists \mathbf{x}_1^* \in X$ и $\exists \mathbf{x}_2^* \in X$: $\langle \mathbf{p}, \mathbf{x}_1^* \rangle > \langle \mathbf{p}, \mathbf{x}_2^* \rangle$
- strong separated if $\exists \mathbf{p} \neq 0$ и β:
  $$\inf_{\mathbf{x}_1 \in X_1} \langle \mathbf{p}, \mathbf{x}_1 \rangle > \beta > \sup_{\mathbf{x}_2 \in X_2} \langle \mathbf{p}, \mathbf{x}_2 \rangle$$
- strict separated if $\forall \mathbf{x}_1 \in X_1$ и $\forall \mathbf{x}_2 \in X_2$: $\langle \mathbf{p}, \mathbf{x}_1 \rangle > \langle \mathbf{p}, \mathbf{x}_2 \rangle$.

Separating hyperplane

Separating hyperplane for sets X_1, X_2 is a hyperplane $\{\mathbf{x} | \langle \mathbf{p}, \mathbf{x} \rangle = \beta\}$ such that $\langle \mathbf{p}, \mathbf{x}_1 \rangle \geq \beta$ for all $\mathbf{x}_1 \in X_1$ and $\langle \mathbf{p}, \mathbf{x}_2 \rangle \leq \beta$ for all $\mathbf{x}_2 \in X_2$

Facts about separation

Existence

If X_1 and X_2 be convex and disjoint sets, then there exists hyperplane that separates them.

Separation criterion for convex sets

Two convex sets such that at least one of them is open are disjoint if and only if there exists separating hyperplane.

Strict separation criterion

Two convex sets are strict separated if and only if distance between them is positive.

Examples

- Find separating hyperplane for sets X_1, X_2:
  $X_1 = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 x_2 > 1, x_1 > 0\}$,
  $X_2 = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 \leq 9 + \frac{4}{x_1 - 1}\}$.
- Criterion of consistency the system of strict linear inequalities $\mathbf{A}\mathbf{x} < \mathbf{b}$ in terms of non-intersection of affine set $\{\mathbf{b} - \mathbf{A}\mathbf{x} | \mathbf{x} \in \mathbb{R}^n\}$ and set $\{\mathbf{y} \in \mathbb{R}^m | y_i > 0\}$
- Example of two closed disjoint convex sets which are not strict separating
- Find separating hyperplane for sets
  $X_1 = \{\mathbf{x} \in \mathbb{R}^n | \|\mathbf{x}\|_2^2 \leq 1\}$ и
  $X_2 = \{\mathbf{x} \in \mathbb{R}^n | x_1^2 + \ldots + x_{n-1}^2 + 1 \leq x_n\}$.

Supporting hyperplane

Supporting hyperplane

A hyperplane $\{\mathbf{x} \in \mathbb{R}^n | \langle \mathbf{p}, \mathbf{x} \rangle = \beta\}$ is called supporting to the set X at boundary point $\mathbf{x}_0$, if $\langle \mathbf{p}, \mathbf{x} \rangle \geq \beta = \langle \mathbf{p}, \mathbf{x}_0 \rangle$ for all $\mathbf{x} \in X$.

Self-supporting hyperplane

A hyperplane $\{\mathbf{x} \in \mathbb{R}^n | \langle \mathbf{p}, \mathbf{x} \rangle = \beta\}$ is called self-supporting to the set X at point $\mathbf{x}_0$, if it is supporting and $\exists \tilde{\mathbf{x}} \in X$: $\langle \mathbf{p}, \tilde{\mathbf{x}} \rangle > \beta$.

Theorem about supporting hyperplane

There exists supporting hyperplane (self-supporting) at any boundary (relative boundary) point of convex set.

Examples

- Represent the set $\{(x_1, x_2) \in \mathbb{R}^2_+ | x_1 x_2 \geq 1\}$ as intersection of hyperplanes
- Construct supporting hyperplane to the set $X = \{(x_1, x_2) \in \mathbb{R}^2 | e^{x_1} \leq x_2\}$ в точке $\mathbf{x}_0 = (0, 1)$
- Find hyperplane which is supporting to the set $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 \geq x_1^2 + x_2^2\}$ and separating it from the point $\mathbf{x}_0 = (-5/4, 5/16, 15/16)$

Recap

- Interior and relative interior of convex set
- Projection of point on a set
- Separation of convex sets
- Supporting hyperplane