Optimization methods. Seminar 9. Conjugate functions

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Reminder

- Feasible direction cone
- Tangent cone
- Sharp extremum

Definition

Conjugacy again?

- Previously we introduced conjugate (dual) sets and in particular conjugate cones
- Today we consider conjugate (dual) functions
- Further we will introduce dual (conjugate) optimization problem

Definition

A function $f^* : \mathbb{R}^n \to \mathbb{R}$ is called conjugate function of function f and is defined as

$$f^*(\mathbf{y}) = \sup_{\mathbf{x} \in dom\ f} (\mathbf{y}^\mathsf{T}\mathbf{x} - f(\mathbf{x})).$$

Domain of f^* is a set of $\mathbf{y}$, such that the supremum is finite.

Properties and interpretations

- Conjugate function f^* is always convex as supremum of linear functions independently of convexity of f
- Young-Fenchel inequality:

$$\mathbf{y}^\mathsf{T}\mathbf{x} \leq f(\mathbf{x}) + f^*(\mathbf{y})$$

- If f is differentiable, then $f^*(\mathbf{y}) = \nabla f^\mathsf{T}(\mathbf{x}^*)\mathbf{x}^* - f(\mathbf{x}^*)$, where $\mathbf{x}^*$ is a supremum point.
- Geometrical interpretation

Examples

1. Linear function: $f(\mathbf{x}) = \mathbf{a}^\top \mathbf{x} + b$
2. Negative entropy: $f(x) = x \log x$
3. Indicator function of the set S: $I_S(x) = 0$ iff $x \in S$
4. Norm: $f(\mathbf{x}) = \|\mathbf{x}\|$.
5. Squared norm: $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$

Recap

- Conjugate functions
- Young-Fenchel inequality and other properties
- Examples