Optimization Methods. Seminar 1. Introduction.

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Survey for students

- Name
- Base chair
- Final project in Computer Science course
- Familiarity with LaTeX
- Expectations of this course
- Favorite course for 2 study-years and why
- Future directions: academia, industry, etc.

Expected outcome of this course

- Formalization of the problem: how to select an element from a set?
- Justification of the decision correctness
- Wide range of applications:
  - machine learning: classification, clustering, regression
  - molecular modeling
  - risk analysis
  - portfolio optimization
  - optimal control
  - sigmal processing
  - parameters estimation in statistic
  - many-many others[1]

[1] http://www.cvxpy.org/en/latest/examples/index.html

Short syllabus — general

Fall term (theory):

- Convex analysis
- Duality theory
- Optimality conditions

Spring term (practice):

- Methods for uncontrained optimization problems first and second orders
- Methods for constrained optimization problems
- Linear programming: simplex method and others
- Optimal methods
- ...

Fall term syllabus — detailed

- Lecture and seminar — once a week
- Two problem sets
- Final test in the end of term
  (and midterm test in the middle of the term)
- Oral exam in the end of term. Grading policy — average the following grades:
    - grade for the term
    - grade for the exam
- Minitests at the beginning of every seminar
- Homework after every seminar — LaTeX

Prerequisities

- Linear algebra
- Calculus
- Programming: Python (NumPy, SciPy, CVXPY) or MATLAB
- Selected topics of computational mathematics

Methodology

Main steps of usage optimization methods in solving real-world problems:

1. Define objective function
2. Define feasible set
3. Optimization problem statement and its analysis
4. Selection of the best algorithm for stated problem
5. Algorithm implementation and verification its correctness

Problem statement

$$
\begin{aligned}
&\min_{\mathbf{x} \in X} f_0(\mathbf{x}) \\
\text{s.t. } & f_i(\mathbf{x}) = 0, \; i = 1, \dots, p \\
& f_j(\mathbf{x}) \leq 0, \; j = p+1, \dots, m,
\end{aligned}
$$

- $\mathbf{x} \in \mathbb{R}^n$ — target variable
- $f_0(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ — objective function
- $f_j(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ — constraints functions

Example: selection items to invest money and defining how much money invest in every item:

- $\mathbf{x}$ — amount of money to invest in every item
- f_0 — total risk or revenue deviation
- f_j — budget constraints, min/max invest in item, minimal accepted revenue...

How to solve?

General case:

- NP-hard
- randomized algorithms: time vs stability

BUT some families of problems can be solved very fast!

- Linear programming
- Least squares problem
- Lowrank approximation
- Convex optimization

Linear programming

$$
\begin{aligned}
&\min_{\mathbf{x} \in \mathbb{R}^n} \mathbf{c}^\top \mathbf{x} \\
\text{s.t. } & \mathbf{a}_i^\top \mathbf{x} \leq c_i, \; i = 1, \dots, m
\end{aligned}
\tag{1}
$$

- no analytical solution
- effective algorithms exist
- developed technology
- simplex method for solving problem (1) is included in the TOP-10 algorithms of XX century[2]

[2] https://www.siam.org/pdf/news/637.pdf

Least squares problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2, \qquad (2)$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$

- analytical solution: $\mathbf{x}^* = (\mathbf{A}^\intercal \mathbf{A})^{-1} \mathbf{A}^\intercal \mathbf{b}$
- effective algorithms exist
- developed technology
- statistical interpretation

Lowrank approximation

$$\begin{aligned} &\min_{\mathbf{X} \in \mathbb{R}^{m \times n}} \|\mathbf{A} - \mathbf{X}\|_F \\ &\text{s.t. } \operatorname{rank}(\mathbf{X}) \le k \end{aligned} \tag{3}$$

Theorem (Eckart–Young, 1993)

Let $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ *be a singular value decomposition of matrix* $\mathbf{A}$, *where* $\mathbf{U} = [\mathbf{U}_k, \mathbf{U}_{r-k}] \in \mathbb{R}^{m \times r}$, $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_k, \ldots, \sigma_r)$, $\mathbf{V} = [\mathbf{V}_k, \mathbf{V}_{r-k}] \in \mathbb{R}^{n \times r}$ *and* $r = rank(\mathbf{A})$. *Then the solutuon of the problem* (3) *can be written as:*

$$\mathbf{X} = \hat{\mathbf{U}}\hat{\mathbf{\Sigma}}\hat{\mathbf{V}}^\top,$$

where $\hat{\mathbf{U}} \in \mathbb{R}^{m \times k}$, $\hat{\mathbf{\Sigma}} = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$, $\hat{\mathbf{V}} \in \mathbb{R}^{n \times k}$.

Algorithms to compute SVD are stable and fast simultaneously

Convex optimization

$$
\begin{aligned}
&\min_{\mathbf{x} \in \mathbb{R}^n} f_0(\mathbf{x}) \\
&\text{s.t. } f_i(\mathbf{x}) \leq b_i, \; i = 1, \ldots, m
\end{aligned}
\tag{4}
$$

- f_0, f_i — convex functions:

  $$f(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) \leq \alpha f(\mathbf{x}_1) + \beta f(\mathbf{x}_2),$$

  where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.
- no analytical solution
- effective algorithms exist
- recognition convex optimization problem is often difficult
- techniques to reformulate a problem to the form (4) exist

What problem is easier?

Search independent alphabet of the maximum cardinality

$$
\begin{aligned}
&\max_{\mathbf{x}\in\mathbb{R}^n} \sum_{i=1}^n x_i \\
\text{s.t. } & x_i^2 - x_i = 0 \quad i = 1, \dots, n \\
& x_i x_j = 0 \qquad \forall (i, j) \in \Gamma,
\end{aligned}
$$

where Γ is a set of pairs

Truss design

$$
\min -2 \sum_{i=1}^k \sum_{j=1}^m c_{ij} x_{ij} + x_{00}
$$

$$
\begin{aligned}
\text{s.t. } & \sum_{i=1}^k x_i = 1 \\
& \lambda_{\min}(\mathbf{A}) \geq 0,
\end{aligned}
$$

where $\mathbf{A} =$

$$
\begin{bmatrix}
x_1 & \dots & \dots & \sum\limits_{j=1}^m b_{pj} x_{1j} \\
\vdots & \ddots & & \vdots \\
 & & x_k & \sum\limits_{j=1}^m b_{pj} x_{kj} \\
\sum\limits_{j=1}^m b_{pj} x_{1j} & \dots & \sum\limits_{j=1}^m b_{pj} x_{kj} & x_{00}
\end{bmatrix}
$$

Why convexity is so important?

R. Tyrrell Rockafellar (1935 —)

The great watershed in optimization is not between linearity and non-linearity, but convexity and non-convexity.

- Local optimum is global
- Necessary optimality condition is sufficient

Questions:

- Is any convex optimization problem solved efficiently?
- Could non-convex optimization problems be solved efficiently?

Recap

- Admistrative issues
- Objective of the Optimization Methods course
- General optimization problem statement
- Classical optimization problems