(Analysis) How many connected components in the preimage of an interval?

Where did we learn from failure?

We learned from failure a few times in this proof.

First conflict-guided subtask:

- (Made an unsound move) Guess a function (the wiggly function $\sin(1/x)$) and an interval (an interval around 0) to be a counterexample.
- (Fail) The guessed function isn't continuous at 0.
- (Learn a lesson) We want to try to modify this function so it is continuous at 0....

Second conflict-guided subtask:

- (Made an unsound move) Guess a modified function (the continuous wiggly function $x \sin(1/x)$) and an interval (an interval around 0) to be a counterexample.
- (Fail) Making the function continuous in this way now made it so the preimage of any interval around 0 has only finitely many connected components.
- (Learn a lesson) There seems to be a tension between making the function *continuous* near the points in the domain where you want the infinitely many connected components and actually having *infinitely many connected components*.

The conflict, we see, is centred around the point L in the domain where infinitely many disconnected components squeeze together. It seems once you make the function continuous near that point, the preimages all glob together. And if you try to make the preimages not all glob together, the function is no longer continuous. Either way, the “salient” part of the counterexample is that “limit point.” This knowledge feeds into the next iteration: we have some “limit point”, so we might want to think about “sequences” and therefore “sequential compactness.”

At this point, we try to *prove* the theorem since there seems to be a tension in constructing the counterexample. That is, we try to prove the conjecture by contradiction…and try to say there are infinitely many connected components, and exploit the tension of before to say then the function isn't continuous or defined around $f(L)$.

- (Made an unsound move) Unfold to the sequential compactness definition.
  - (Fail) No contradiction can be achieved — because L can be defined to be an endpoint of the interval.

  - (Learn a lesson) The conjecture can't be proved because L can be an endpoint…so maybe it can be disproved using this L.

Finally, we construct a counterexample to the conjecture — where the point with infinitely many preimages is the *endpoint* of the interval.

Why is learning from failure necessary in this proof?

Now, in retrospect, it seems silly that we were so close to the counterexample from the very beginning (using the wiggly function) but just needed to shift the interval ever-so-slightly.

But I think this process helps guide how a point-and-click system and eventually an algorithm might find both the function and the interval…