(Proof) How many connected components in the preimage of an interval?

This conjecture came up as a subproblem over the course of my friend's research in differential geometry.

His conjecture was as follows:

- Suppose you have a continuous function from a compact metric space to the reals.

$$f : X \to \mathbb{R}$$

- Then, the preimage of an open interval has finitely many connected components.

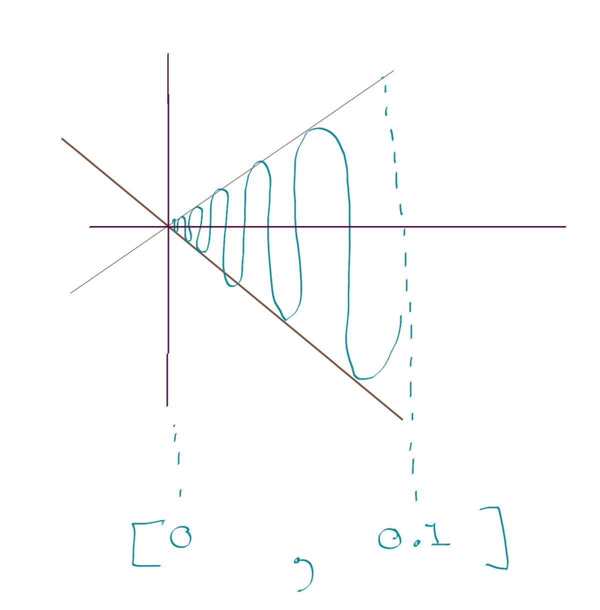
A failed attempt - conjecturing about the connectedness of compact sets

At one point, I made the naive conjecture: There cannot be a compact set with infinitely many disconnected components.

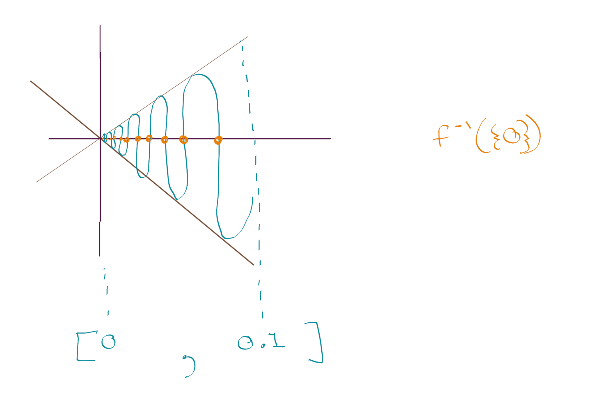
- But then I realized there was a counterexample: The set of all real numbers between 0 and 1 with decimal expansions involving only 4s and 7s. It's closed and bounded, and therefore compact. But I also think totally disconnected (?)

A failed attempt - thinking about a wiggly function

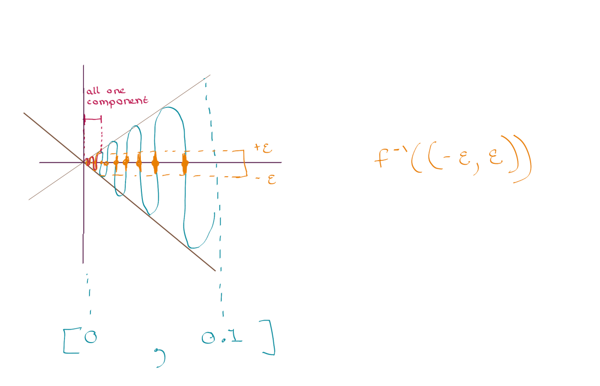
Then, I thought the function $x \sin \frac{1}{x}$ (but with the hole plugged in such that f(0)=0) might lead to a counterexample — that is, an example where the preimage of an open interval has *infinitely many* connected components.



Because I know that $f^{-1}(\{0\})$ has infinitely many connected components…



But then I realized once you take any interval around zero…the infinitely many components near the origin glob together as one connected component.

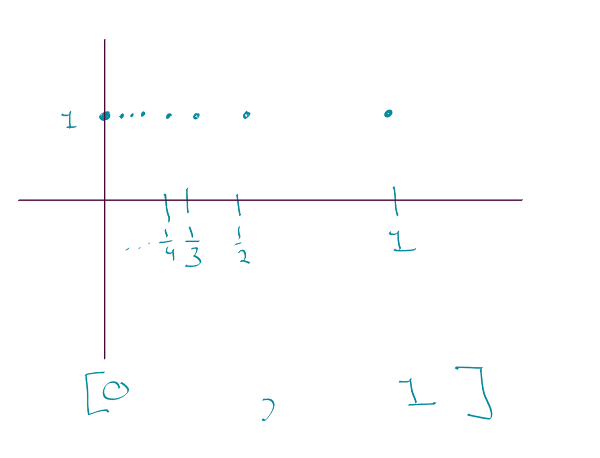


A failed attempt - proving the negation

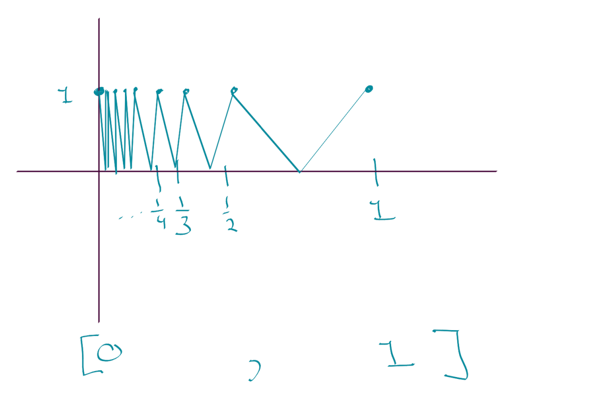
I tried to still take inspiration from the wiggly function, and create a similar-but-different function.

So I tried to create a function defined on the compact set $[0, 1]$.

And made sure it attained the value 1 at infinitely many points…



And then made it continuous at most points by adding a "sawtooth" pattern.



But now I still had to make it continuous at 0. But no matter what way I thought of modifying the function to make it continuous at 0 (e.g. multiplying it by x), every interval around 0 or 1 would have only finitely many connected components.

That is — I had the same issue I had with the wiggly function — a point's preimage may consist of infinitely many disconnected points… but as soon as you take an interval around that point, a bunch of the preimages glob together, and you end up with only finitely many connected components in the preimage.

Learning from the previous two failed attempts

At this point, my friend noticed that **our previous two attempts failed in the exact same way — in particular, the preimage of some point had a limit point…and there were issues around that point.**

Perhaps, we thought, the conjecture is true, and the key to proving it lies in reasoning about this limit point.

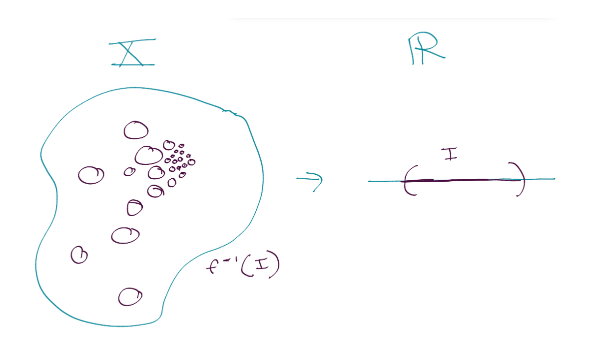
The two friends and myself who were solving the problem kept trying to take the interval around the limit point, for some reason. (We didn't think of not including that point in the interval at all…which would have gotten us the solution a lot quicker).

But in any case, since we were reasoning about a limit point…I was nudged towards using the *sequential definition of compactness*.

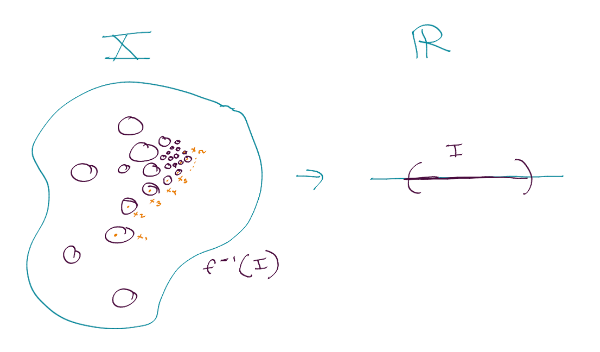
A failed attempt — Using sequential compactness

So, we start the proof afresh, by contradiction, and employing sequential compactness.

Suppose the preimage of some interval I has infinitely many connected components.

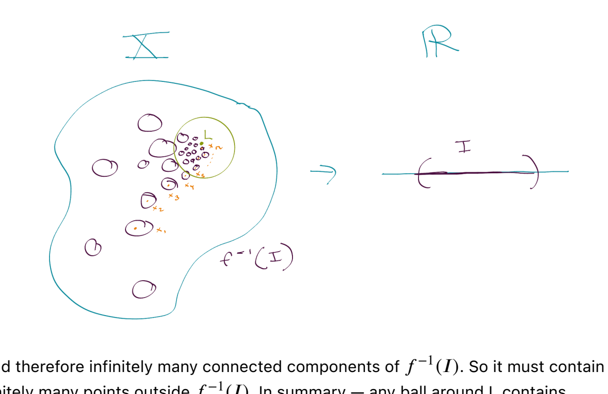


We create a sequence by taking one point from each connected component. That sequence exists in a compact space, so it at least has a convergent subsequence $x_n \to L$.

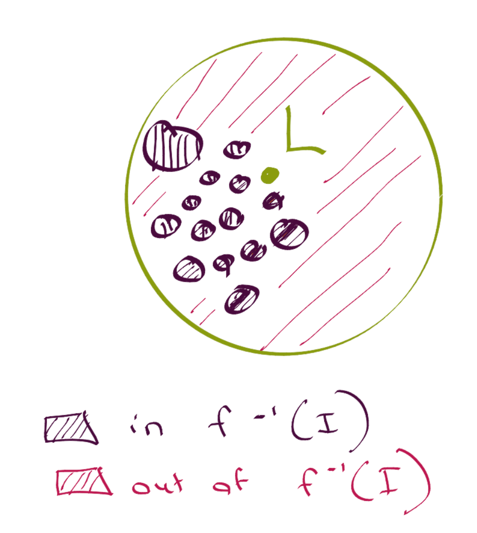


We always failed in the previous examples because if we did have infinitely many points in the preimage of a limit point L, it was always difficult to define f(L) such that f is still continuous. So from that I started thinking…maybe we can get some contradiction that $f(L)$ is not defined.

So we consider a ball around the limit "L". It contains infinitely many points in the sequence…



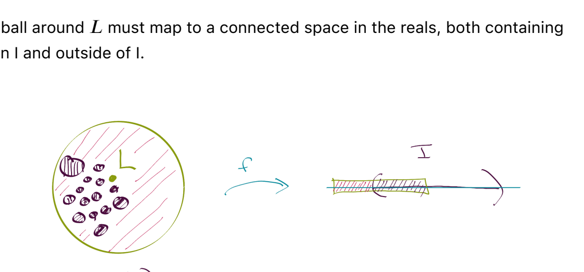
…and therefore infinitely many connected components of $f^{-1}(I)$. So it must contain infinitely many points outside $f^{-1}(I)$. In summary — any ball around L contains points inside and outside $f^{-1}(I)$. I tried to use this to find a contradiction.



We also know for continuous functions f:

$$f(\textit{connected space}) = \textit{connected space}$$

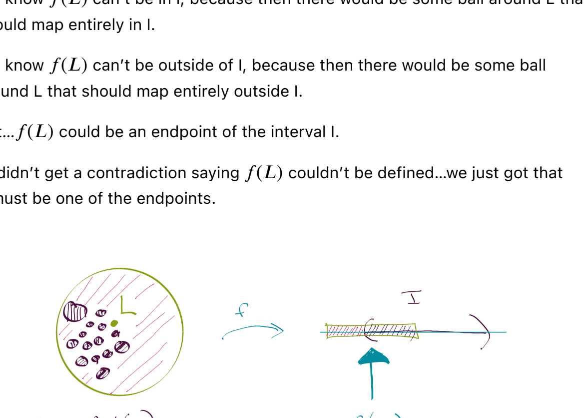
So any ball around L must map to a connected space in the reals, both containing points in I and outside of I.



So, now we want some contradiction saying $f(L)$ can't map anywhere.

- We know $f(L)$ can't be in I, because then there would be some ball around L that should map entirely in I.
- We know $f(L)$ can't be outside of I, because then there would be some ball around L that should map entirely outside I.
- But…$f(L)$ could be an endpoint of the interval I.

So we didn't get a contradiction saying $f(L)$ couldn't be defined…we just got that $f(L)$ must be one of the endpoints.



Another failed attempt — uniform continuity

We realized that if a function is continuous on a compact space, it is uniformly continuous…

- …and so for any given height difference ϵ, there is some δ so that no matter where you are in the domain of the function, as long as you only traverse a distance of δ in the domain, you will never exceed that height difference ϵ…
- …and so there's an upper bound on the slope of the function at any point…
- …and so the function can only "wiggle" so much because there's not enough space to wiggle a lot…
- …or, it can wiggle a lot, as long as the wiggles are getting smaller, because then the slope never exceeds the "maximum slope" given by uniform continuity.

But we weren't sure where to go with that either.

The solution — a counterexample

Then my friend realized that maybe the fact that this $f(L)$ can be defined on the interval endpoint is the key to constructing a counterexample.

For example, we take the real-valued function defined on $[0, 1]$:

$$f(x) = x \sin(1/x)$$

And, we take the preimage of the interval $(0, 1)$.

To make these two choices, we used our previous arguments advising us that….

- …we should choose a *function* such that its wiggles are getting smaller and smaller.
- ….we should choose an *interval* such that the point with infinitely many preimages (in this case $f(0)$) is one of the endpoints.

And so find we indeed have infinitely many disconnected components. So, the conjecture is false. □