There are n cars on a circular track.

Adding up the gas in their tanks, there's enough to get around the track exactly once. Show there exists a car that can complete one lap by collecting gas from the others on its way around.

Key Idea

First we prove that in any such scenario, there is one car that can make it to the next car.

Then, we use induction.

Full solution

First I thought of the weaker statement (reasoning forward from the target): is there even one car that can make it to the next car?

- I found it was true.
- Suppose the distance from car i to car $i+1$is d_i.
- Suppose the gas in car i is g_i.
- We know, because the total gas is the distance needed to get around once, that $\sum_{i=0}^{n} g_i = \sum_{i=0}^{n} d_i$.
- So, we have

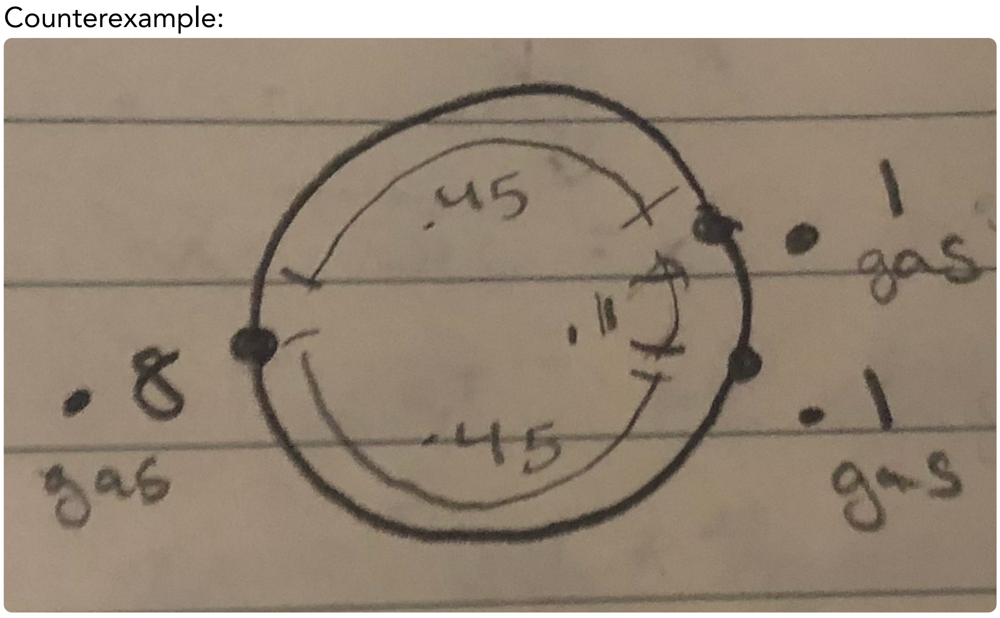
$$\sum_{i=0}^{n} g_i - \sum_{i=0}^{n} d_i = 0$$

$$\sum_{i=0}^{n} (g_i - d_i) = 0$$

$$\exists i \text{ such that } g_i \geq d_i$$

By the way, there are lots of *stronger (false)* conjectures to be made here, e.g.:

- Any car that can make it to the next car can make it all the way around the track.
  - Counterexample:

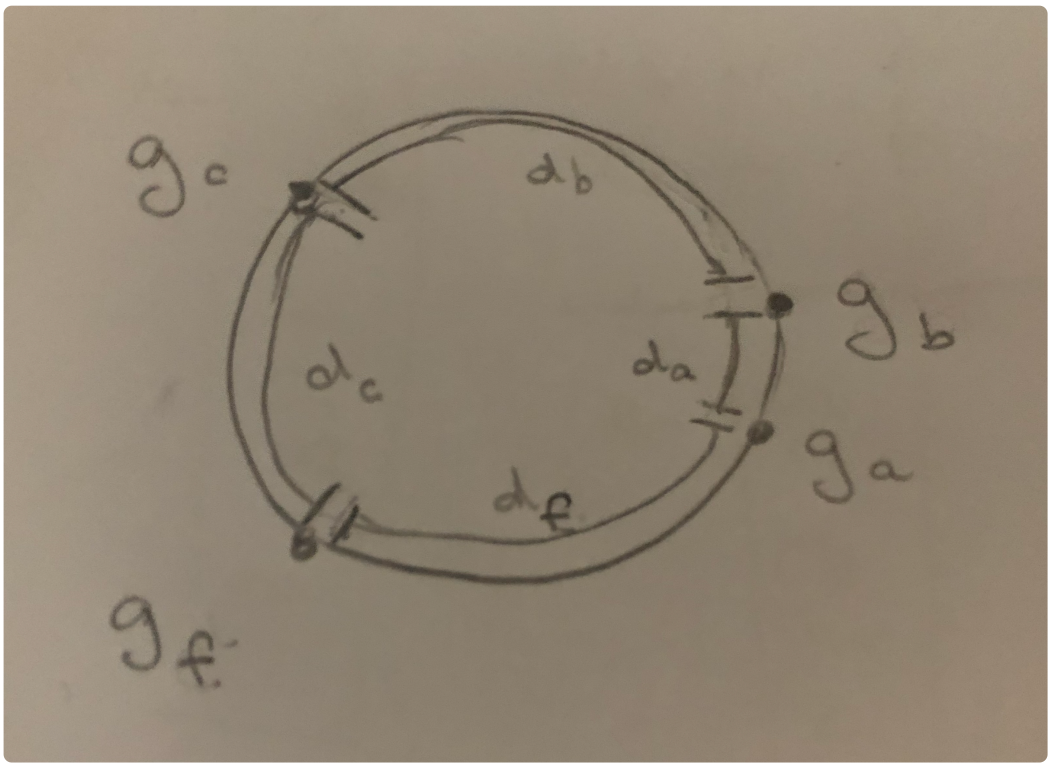


- It's always the car that has the most gas that works.
  - (Same counterexample)
- It's always the car closest to the next car that works.
  - (Same counterexample)

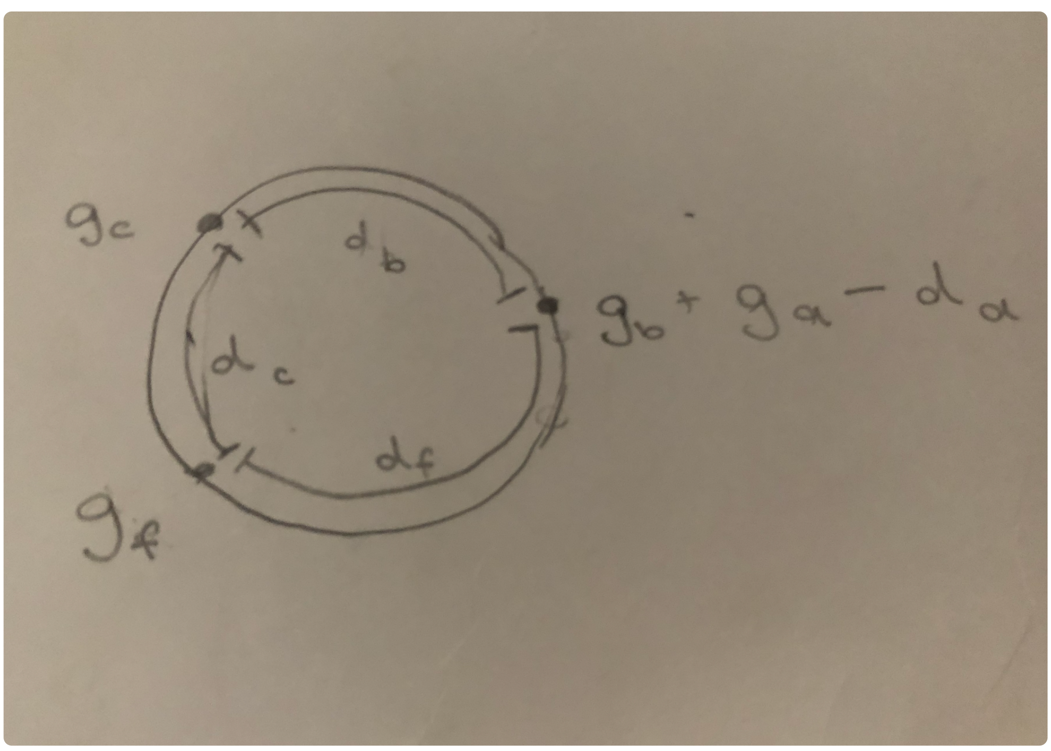
But I don't think that disproving any of these conjectures help aid proof discovery. So, these are some examples where there is not much to learn from failure (disproving a false, stronger statement).

In any case, now we induct.

- Suppose there is only 1 car on the track. Then it must have a full tank of gas, so it can make it all the way around.
- Suppose there are n+1 cars on the track. Then suppose car "a" is the one that can make it to the next car, car "b."



Consider the "shrunk" track where car a has been removed, the distance between car a and b has disappeared, and car b has the gas $g_b + g_a - d_a$ (the gas of cars a and b combined, minus the gas it took to go between them).



This new track has n cars, and satisfies all the conditions of the initial problem. So, there exists a car that can make it all the way around this smaller track.

And, if there's a car that can make it around this smaller track, it can make it around the bigger track. This is because we know it can make it to car a, and we know whatever gas it gets from car a will be enough to sustain it for its journey to car b, and now it has collected $g_b + g_a - d_a$, so it can proceed as it did in the smaller track.