The size of a coset of H is the same as the size of H

Let H be a subgroup of the group G.

We want to prove that $|any\ coset\ of\ H| = |H|$.

$$H : subgroup(G)$$

$$g_1 \in G - H$$

$$---------$$

$$|g_1 H| = |H|$$

We **reason forward from the target**. If the coset is the same size as H, we must also have the necessary condition that the coset has no "duplicates." (That is, the coset can't be any bigger than the size of H, given its definition — recall that $g_1 H := \{g_1 h | h \in H\}$. So the only option is for the coset to be smaller than the size of H, and that would require "duplicate" elements.)

So now we have:

$$H : subgroup(G)$$

$$g_1 \in G - H$$

$$---------$$

$$\nexists h_1, h_2 \in H, h_1 \neq h_2 \wedge g_1 h_1 = g_1 h_2$$

Now we find a contradiction. We **negate the conclusion** , and try to prove the opposite. If a duplicate element did exist…

$$H : subgroup(G)$$

$$g_1 \in G - H$$

$$\exists h_1, h_2 \in H, h_1 \neq h_2 \wedge g_1 h_1 = g_1 h_2$$

$$---------$$

$$False$$

…then we can tidy…

$$H : subgroup(G)$$

$$g_1 \in G - H$$

$$h_1, h_2 \in H$$

$$h_1 \neq h_2$$

$$g_1 h_1 = g_1 h_2$$

$$- - - - - - - --$$

$$False$$

…and we can cancel g_1 from both sides of $g_1 h_1 = g_1 h_2$ to get…

$$H : subgroup(G)$$

$$g_1 \in G - H$$

$$h_1 \neq h_2$$

$$h_1 = h_2$$

$$- - - - - - - --$$

$$False$$

And so we have our contradiction. □

Analysis

This an example of reasoning forward from the target in the case where the target does not have an existential quantifier.

Unfortunately, we end up with a statement that is actually not weaker, but equivalent. (The coset of H has a different size than H $\iff$ there is a duplicate element.)