FINITE GROUPS SUCH THAT ALL SUBGROUPS ARE NORMAL

It occurred to me recently, somewhat to my surprise, that I did not know the answer to the following simple question: if every subgroup of a finite group G is normal, does it follow that G is Abelian? In this document I want to tell the story of how I answered the question. It took me longer than it should have, but there may be some interest in what I did in order to get there in the end. As with some other documents I've posted recently, this will be an informal account. Also, I did not make careful notes of all my thoughts on this, so what follows is to some extent a rational reconstruction that makes me look as though I was more sensible and systematic than I actually was.

I started in what I think of as Gil Kalai mode. He was the one who once said that there is no difference between looking for a proof and looking for a counterexample, and I think I now have a rather precise understanding of when that is an appropriate thing to say. It's when we try to create a counterexample without losing any generality (or possibly we allow ourselves to lose generality just for convenience when we're pretty sure we'll be able to generalize back again if necessary). In such cases, we say things like "Well, I'll have to do this," but if all moves are indeed forced and our example ends up not working, then we have a proof. If on the other hand we genuinely lose generality by doing an unforced move, then we will no longer have a proof if our counterexample attempt fails, though we may get some of the way there by improving our understanding of what a counterexample could look like.

So I wanted to find a non-Abelian group G such that all subgroups are normal. But if G is non-Abelian, then I know for certain that it contains elements a and b such that $ab \neq ba$. I also know that the subgroup generated by a is normal, so bab^{-1} must be a power of a, and similarly aba^{-1} is a power of b. If we are given that $bab^{-1} = a^r$ and $aba^{-1} = b^s$, then it is tempting to look at the commutator $aba^{-1}b^{-1}$, since there will be two ways of rewriting it. Indeed, we find that it is equal both to b^{s-1} and to a^{1-r}, so we get that a power of a is equal to a power of b. Furthermore, these equal powers are not just the identity, since $aba^{-1}b^{-1} \neq e$ by the hypothesis that $ab \neq ba$.

This felt potentially like the basis for a contradiction, but I couldn't quite see how the argument would go, so I wondered about plugging in some numbers for r and s to see whether I could identify a problem that would arise. But for that I needed to decide what orders to give to a and b. Order 2 was not an option, since bab^{-1} can't be the identity, and it can't be a unless $ab = ba$. So the simplest option to try was $a^3 = b^3 = e$ and $aba^{-1} = b^{-1}$, $bab^{-1} = a^{-1}$.

But then the observation about the commutator gives that $b^{-2} = a^2$, which is equivalent to the statement $b = a^{-1}$, which contradicts the fact that $ab \neq ba$.

In general, if a has order t, and $a^u = b^v$, then if u is coprime to t, then we can find w such that $uw \equiv 1 \pmod{t}$, which gives us that $a = b^{vw}$, and once again a and b commute. So the failed example, though too specific to be a proof, does at least give us a lemma, which is that $r-1$ has a factor in common with the order of a, and of course we also get that $s-1$ has a factor in common with the order of b.

How can we achieve that as simply as possible? Well, it's tempting to stick with $aba^{-1} = b^{-1}$ and $bab^{-1} = a^{-1}$, which gives us that $a^2 = b^{-2}$. But the simplest positive integer greater than 2 with a factor in common with 2 is 4, so we could try $a^4 = b^4 = e$.

I noted that the group with these generators and relations has order 8, since the fact that $ba = ab^{-1}$ allows us to write every element in the form $a^u b^v$ and then the fact that $a^2 = b^{-2}$, which in this group implies that $a^2 = b^2$, allows us to take u to be 0 or 1. And the fact that $b^4 = e$ means that we can take b to be 0, 1, 2 or 3. I then found myself wondering about ab, and noted that it too seemed to generate a normal subgroup.

But by this time I was thinking that I didn't know many non-Abelian groups of order 8 – the dihedral group and the quaternion group are I think the only examples. This clearly wasn't the dihedral group, since the only elements of order 4 in that group commute with each other. However, it looked very like the quaternion group, and I quickly realized that it was indeed the quaternion group.

It is easy to check carefully that every subgroup of the quaternion group is normal. If it is trivial or has index 2, then we are done on general grounds. But the only subgroup of order 2 is the subgroup $\{1, -1\}$, which is clearly normal – indeed, it is the centre of the group.

In retrospect, I could probably have found the solution more quickly by means of a brute-force search through small non-Abelian groups, especially after noting that dihedral groups have non-normal subgroups. But the approach I actually took has some advantages: although I don't actually know how easy it is to find examples, I have constrained them to some extent, which might make it easier to find more examples.

A remark I meant to make earlier is that if H is normal in a group G and K is a subgroup of G that contains H, then H is normal in K. So there was no loss of generality in considering a group generated just by a and b. That is, if G is a group with the properties we want and a and b are two non-commuting elements, then the subgroup of G generated by a and b also has the properties we want. This is a situation where one property – containing two non-commuting elements –

is increasing and the other – having all subgroups normal – is decreasing, so it makes sense to choose a minimal example with the first property.