(Proof) In a metric space, compact $\implies$ sequentially compact.

(Actually, in a metric space, sequentially compact $\iff$ compact, but we just focus on one direction here.)

This proof breaks into multiple parts. We need to prove that given a metric space X:

- X is compact $\implies$ X is complete and totally bounded
- X is complete and totally bounded $\implies$ X is sequentially compact

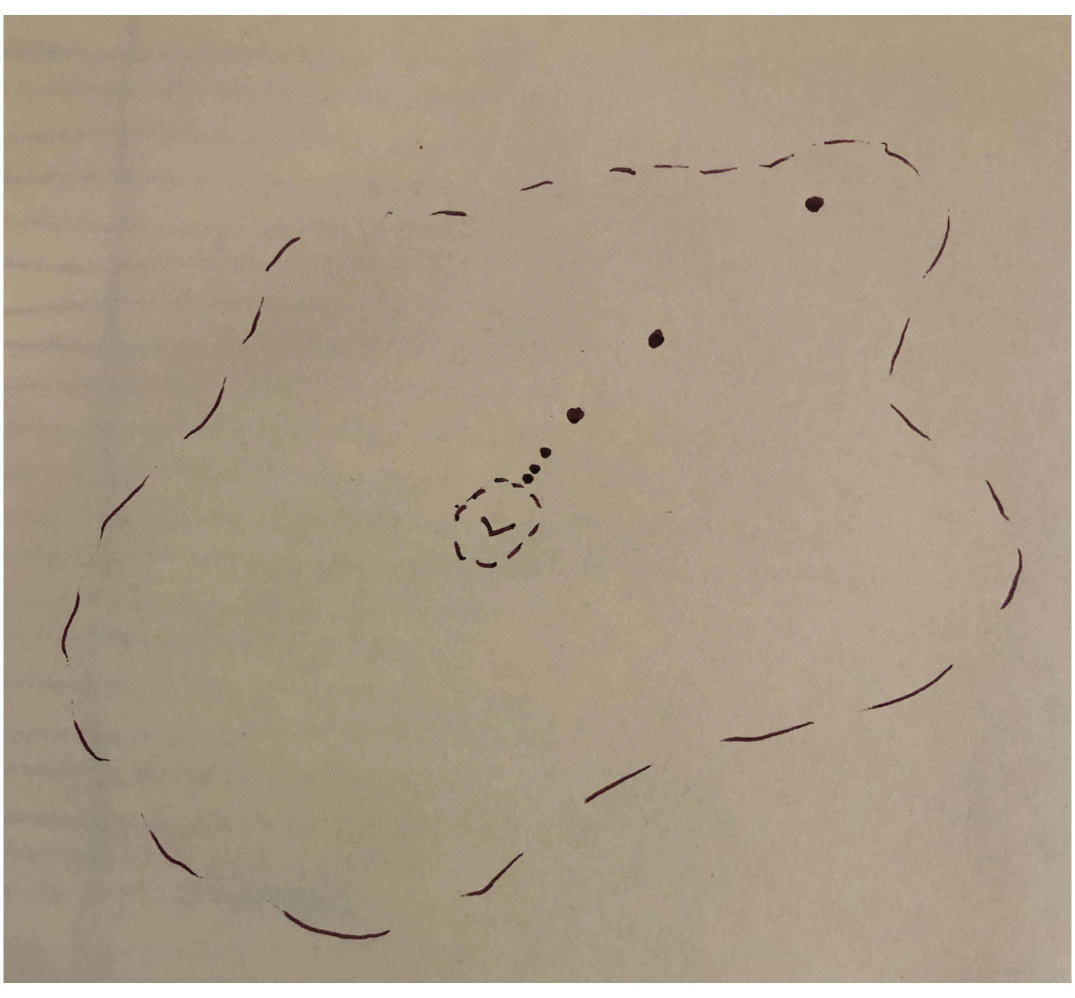
Proof

Compact $\implies$ totally bounded.

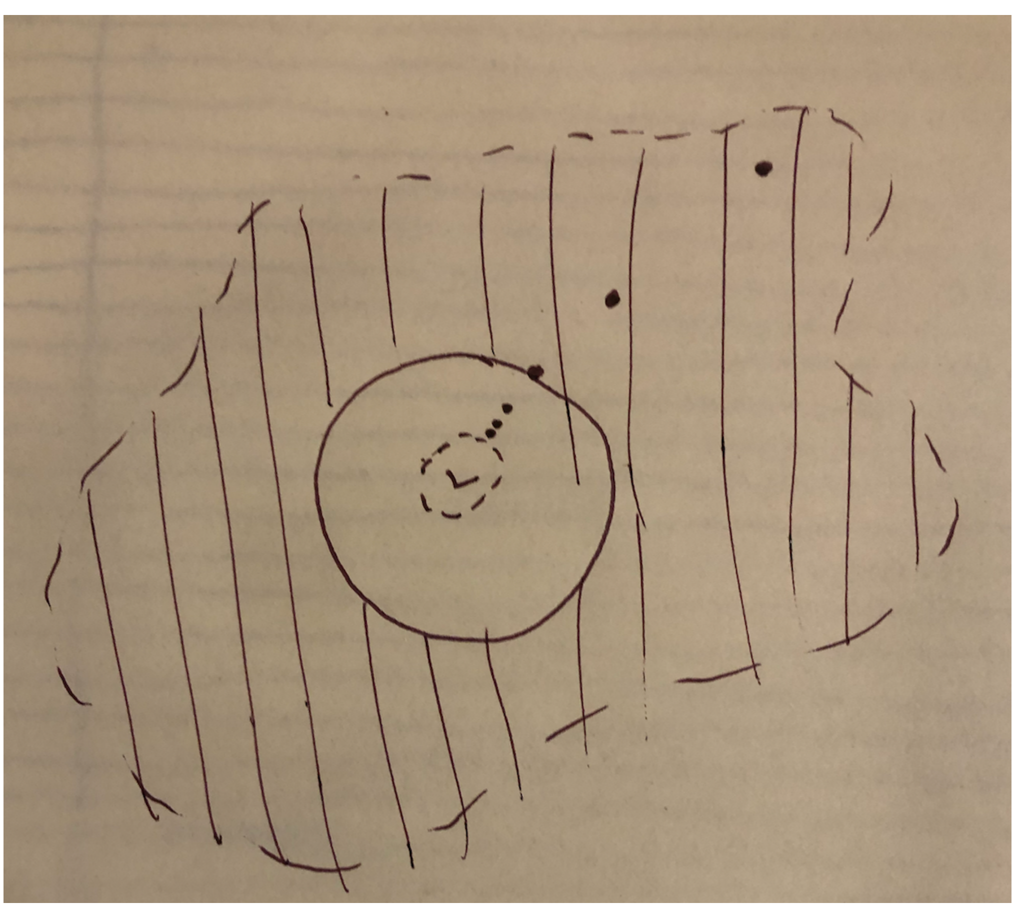
- Create an open cover with balls of radius ϵ, for any $\epsilon > 0$. That open cover has a finite subcover — that is, the space is covered with finitely many balls of size ϵ. ◻

Compact $\implies$ complete.

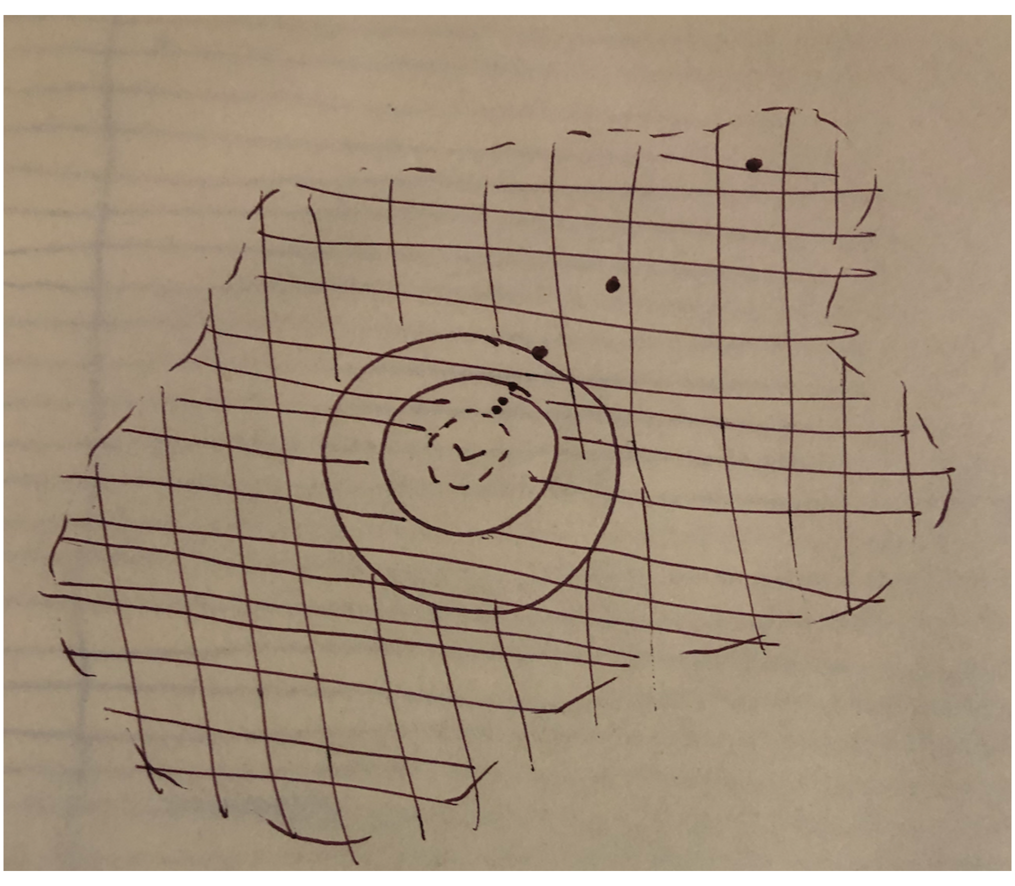
- Let's prove not complete $\implies$ not compact.
- By non-completeness, we have there's some Cauchy sequence that converges to a point Loutside the space (that is, to a point in the completion of the space, but not in the space itself).



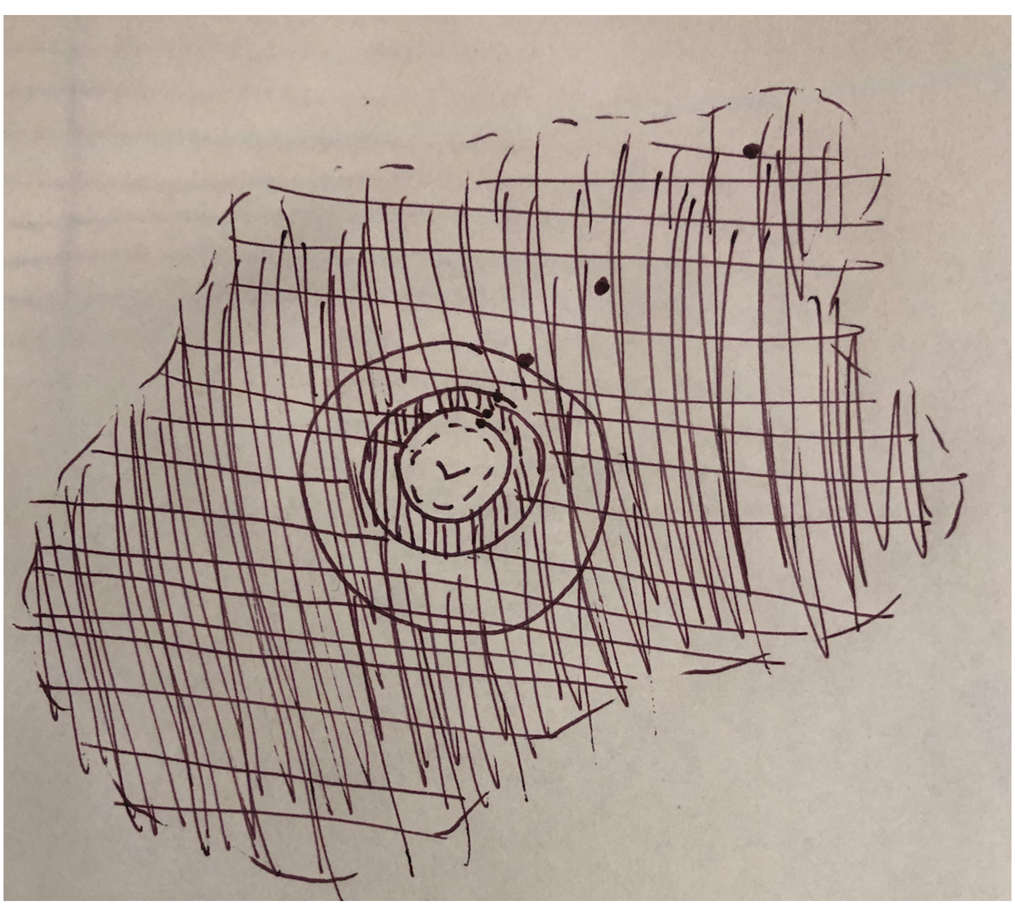
- So now we want to use that L to find an open cover with no finite subcover.
    - So take a closed ball of radius 1 around L. Its complement in X is an open set inside X — call it U_1.

    - Then take a closed ball of radius 1/2 around L. Its complement in X is an open set inside X — call it U_2.

    - Then take a closed ball of radius 1/3 around L. Its complement in X is an open set inside X — call it U_3.

    - Repeat to create the collection of open sets $\{U_i\}_{n=1}^{\infty}$.
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        - Note that $\{U_i\}_{n=1}^{\infty}$ forms an open cover, because we include all points that in the space distance $d \in (0, \infty)$ from L. That is, we include all points except L itself (and L is not in the space, so the open cover doesn't have to include L).
        - But note that any finite subset of these will not cover the whole space — because any finite number of these sets misses infinitely many points in the sequence (the points in the sequence get arbitrarily close to L, but any finite number of these sets is some positive distance away from L.)
    - So, this open cover has no finite subcover.
- So, any incomplete space is not compact. ◻

Complete & totally bounded $\implies$ sequentially compact.

- By total boundedness, we have that every sequence has a Cauchy subsequence.
- By completeness, we have that every Cauchy sequence converges.
- So every sequence has a convergent subsequence. ◻