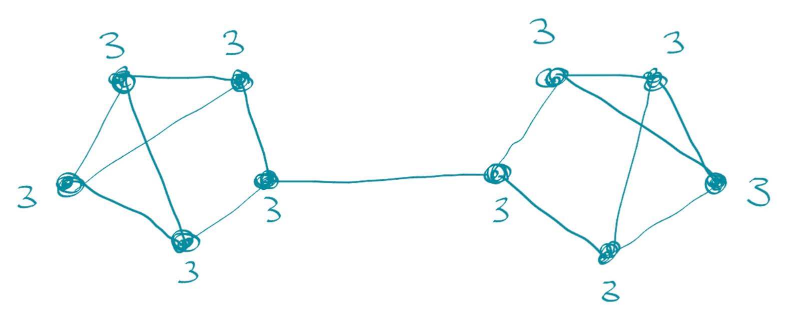
Prove an r-regular ($r \geq 2$), bipartite graph does not have a bridge.

(Reminder: a bridge is a “cut edge” of a graph — when removed, it disconnects the graph).

Trying to disprove a stronger statement

I tried to disprove the stronger statement: that any regular graph does not have a bridge. It took a while to find a counterexample, but I eventually did.



I then tried to figure out what “the most general” such counterexample could be. That is, what is the most general regular graph that has a bridge? But I struggled with coming up with a statement of the form:

- regular and (condition) => has bridge.

(Note: when we try to find “the most general” counterexample, we’re doing the thing Tim mentioned — “balancing” the implication to make it as bidirectional as possible. So, I’d be interested if we could figure something out here, and then see if it could inform the rest of the proof. In any case, I decided to finish the proof a different way.)

Switching proof techniques: trying proof by contradiction

I changed my focus, and tried to come up with a statement of the form:

- regular and has bridge => (some condition that contradicts bipartite)

I thought “Ok, consider a regular graph with a bridge. What do I know must be true about it?”

Well, I realized, it was r-regular, but when you remove the bridge, you end up with 2 disconnected subgraphs that have degree sequences that look roughly like:

- r, r, r…r-1
- r, r, r…r-1

Then I made a rather strong conjecture: do all graphs with degree sequences of the form “r, r, r, … r-1” have an odd cycle in it? (That would be sufficient to prove the full graph must be bipartite, because if the subgraph with degree sequence “r, r, r, … r-1” is not, then attaching a bridge and another subgraph to it will still keep it non-bipartite.)

As I tried to come up with examples with “r=2”, I **discovered a conflict-inspired lemma: we can’t actually have r be any even number** , because then we have an odd number of odd vertices in the graph (in particular, we have one vertex with degree r-1). So, we now have:

- **Lemma: r is odd**

…which further means that **$r \geq 3$** (before we had $r \geq 2$).

Now I tried to figure out if we know anything about the number of vertices n in this subgraph (the one that results from removing the bridge then taking the biggest subgraph). Since r is odd, we need an even number of vertices of degree r, and so adding in the one vertex of degree $r-1$, we have in this subgraph that:

- **Lemma: n is odd**

Now can we prove that subgraph is not bipartite? I know quite a lot of information about the degree sequence. One false start I had was thinking: is containing an odd cycle degree-sequence-invariant? That is, if one graph with a particular degree sequence has an odd cycle, does every one?

But then I asked a more general question: “what does a degree sequence for a bipartite graph look like”?

- We know, in a bipartite graph, we need the sum of degrees in partition A to equal the sum of degrees in partition B, and each to be an even number.
- In our graph, the total degree sum is

$$\textit{degrees of deg-r vertices} + \textit{degrees of deg-(r-1) vertices}$$

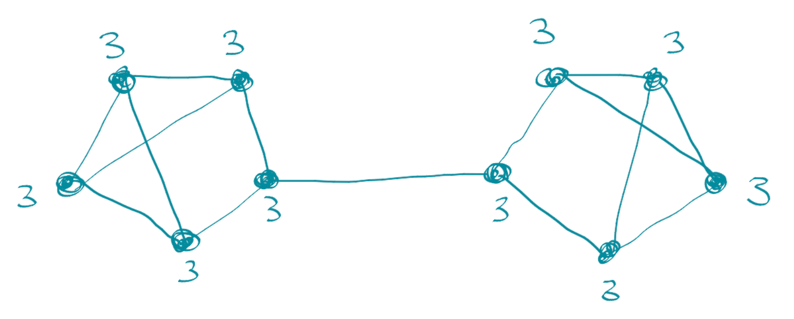
$$(n-1)r + (1)(r-1)$$

$$= nr - 1$$

- So each part in the partition should have the total degree sum of $\frac{nr-1}{2}$.
- Since n is odd and r is odd, nr is odd, so $nr-1$ is even. So, $\frac{nr-1}{2}$ is a whole number. No contradiction there. **So our conflict-inspired lemmas about the parity of r and n were not useful.**
- The actual way to finish the proof is as follows. Which vertices should be used to construct one side of this partition? You definitely can’t construct it using only the vertices of degree r, because that degree sum will be divisible by r, and $\frac{nr-1}{2}$ is not. But one side of this bipartition *must* have vertices of only degree r. Contradiction.

Analysis: proving a stronger, false statement guides a case analysis in proof by contrapositive

Still, even though some of our conflict-inspired lemmas were not useful, one of them was. When we tried to prove the stronger, false statement “every regular graph has no bridge”, we came up with a regular graph that does have a bridge…



…. and it helped us notice something about the degree sequences of such graphs.

And so, in general, **I’m wondering whether maybe the reason that proving a stronger, false statement is helpful because it provides some key “contradictory” property that guides a proof by contradiction/contrapositive**. In this case, the property that contradicted bipartiteness was the degree sequence of a subgraph of a particular graph.

I think this might be clear to see in the case where the “strength” of the statement comes from removing a hypothesis.

For example, suppose you try to prove a theorem using two methods.

Method 1: Try to prove a stronger, false statement.

For example, suppose you need to prove

```
regular & bipartite => has no bridge
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Then you might try to prove the stronger statement

```
regular => has no bridge
```

And find you can’t prove it because you get that there are regular graphs with bridges, and they all have a certain property R (in this case, the sum of the degree sequence of bridge-removed subgraphs).

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regular and has bridge => R
```

And then you find that that property R implies the graph is not bipartite.

```
regular and has bridge => R => not bipartite
```

Method 2: Do a proof by contrapositive.

This is the same conclusion you’d reach if you started with

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regular and bipartite => has no bridge
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And launched directly into proof by contrapositive.

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regular and has bridge => R => not bipartite.
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Then you’d start off considering regular graphs with bridges, just as we did in the “stronger, false” case.

The takeaway: failing to prove a stronger, false statement due to a property R is a “hint” that we should prove by contrapositive and use property R