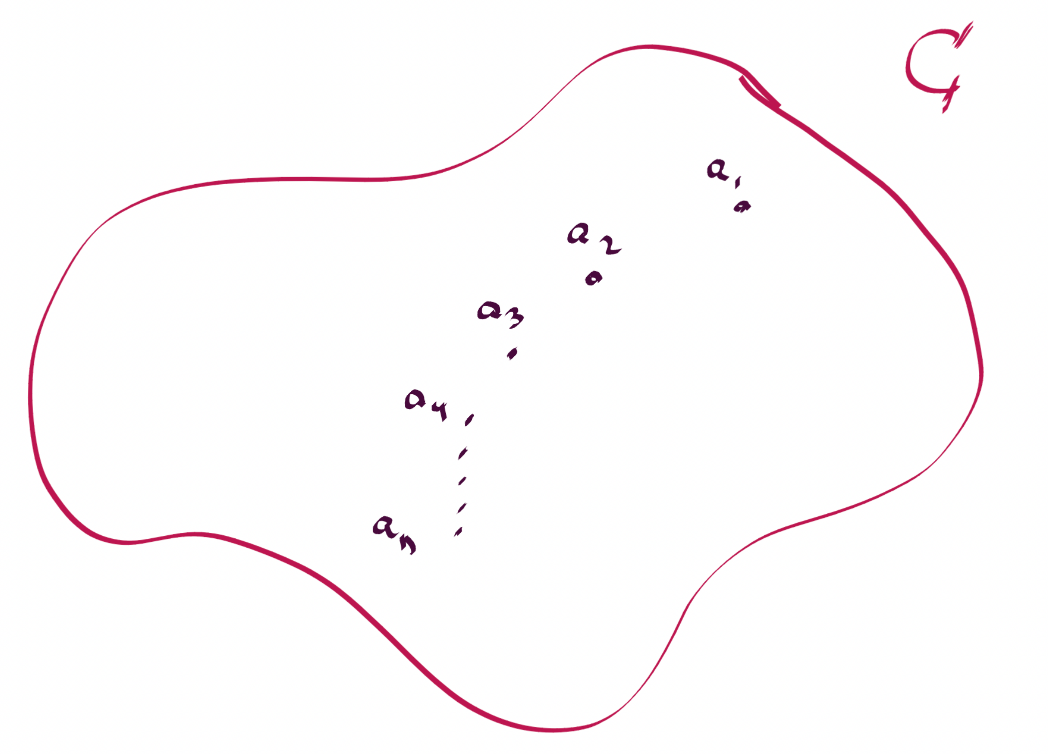
(Diagonal Proof) In a metric space, compact $\implies$ sequentially compact.

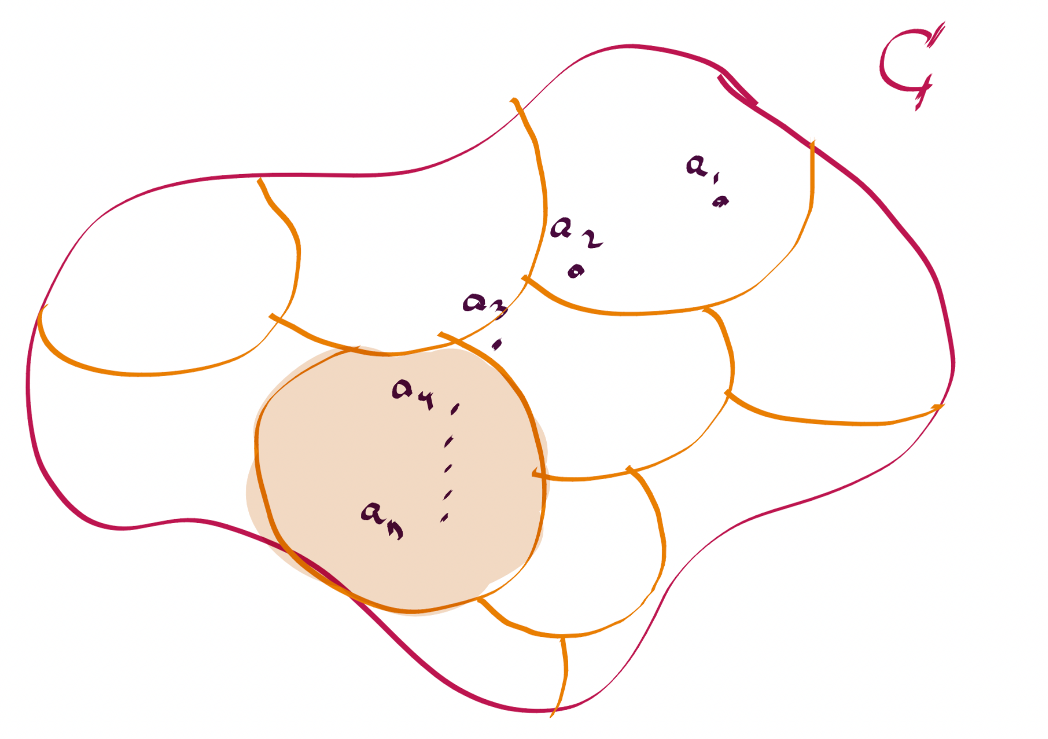
Tim did this proof in a way that **didn't explicitly require the intermediaries of completeness or totally boundedness** (that is, he came up with those intermediaries along the way, as they became necessary). It **did use, however, a diagonal argument.**

Here's how he does it:

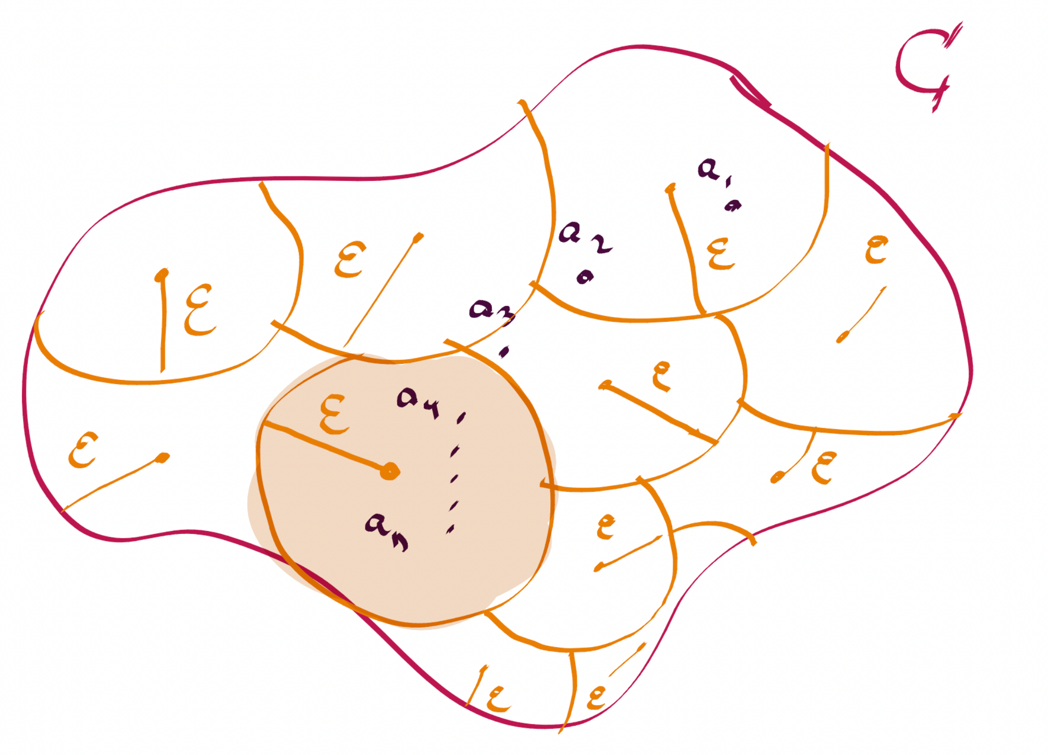
- So we have a sequence in a compact space, and we want to show it has a convergent subsequence.



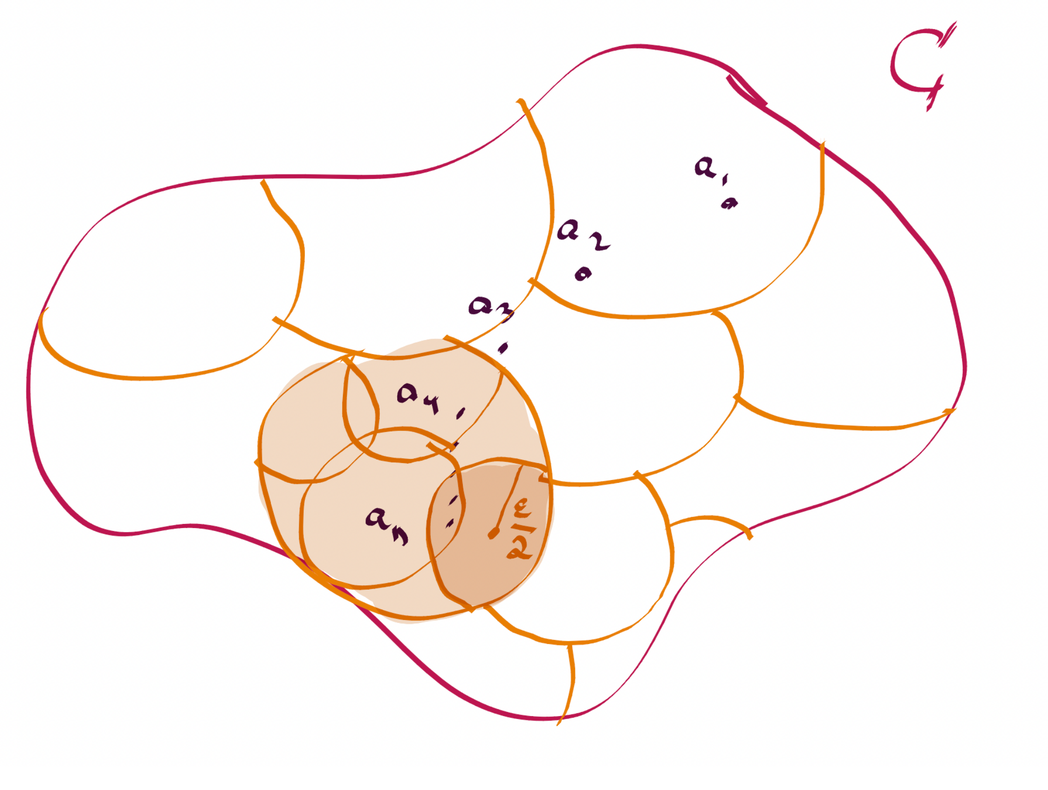
- There are infinitely many points in the sequence, but finitely elements in a subcover that still covers the full set. So, Tim mentions, this should cue a "pigeonhole" argument.



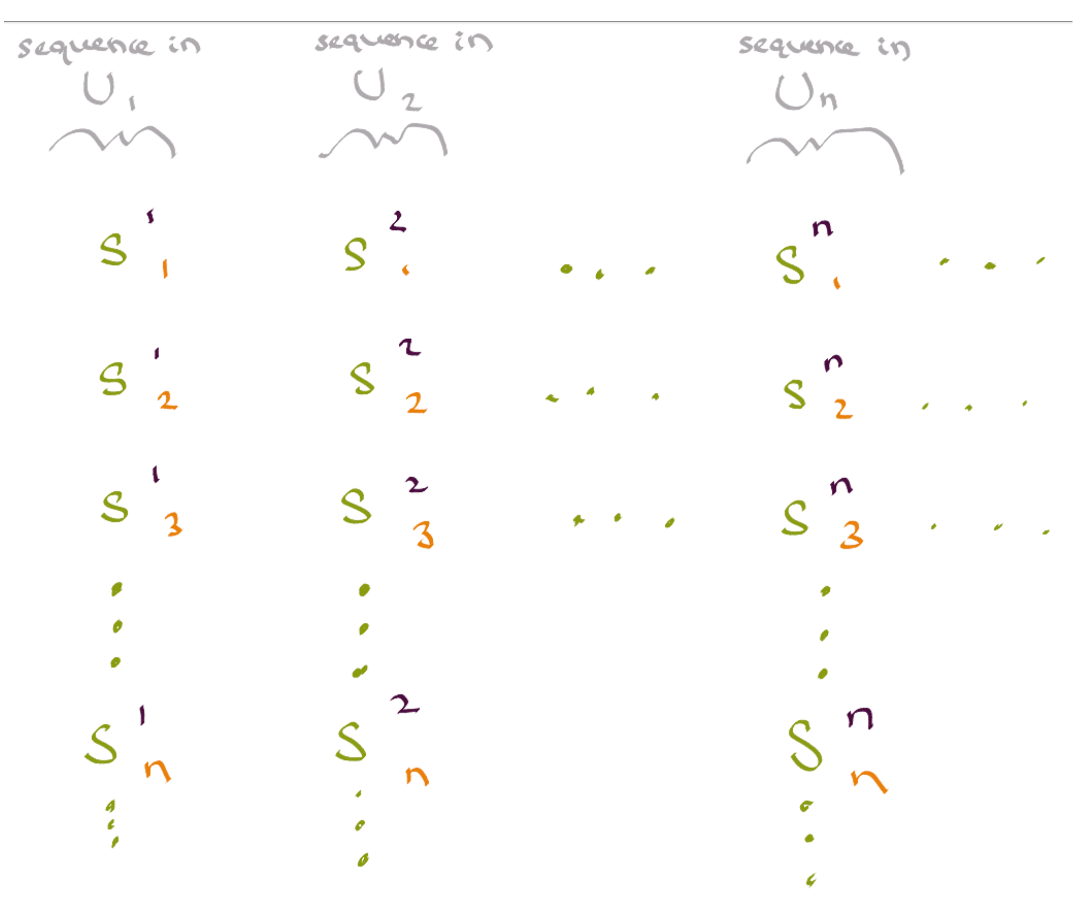
- That is, now it's motivated to fix an ϵ and choose an open cover that is centred around each point, and has that radius ϵ. So you get a finite subcover where each open set has radius ϵ. But, there are infinitely many points in a sequence. So then we should conclude there are infinitely many elements in the sequence in just one ball of radius ϵ.



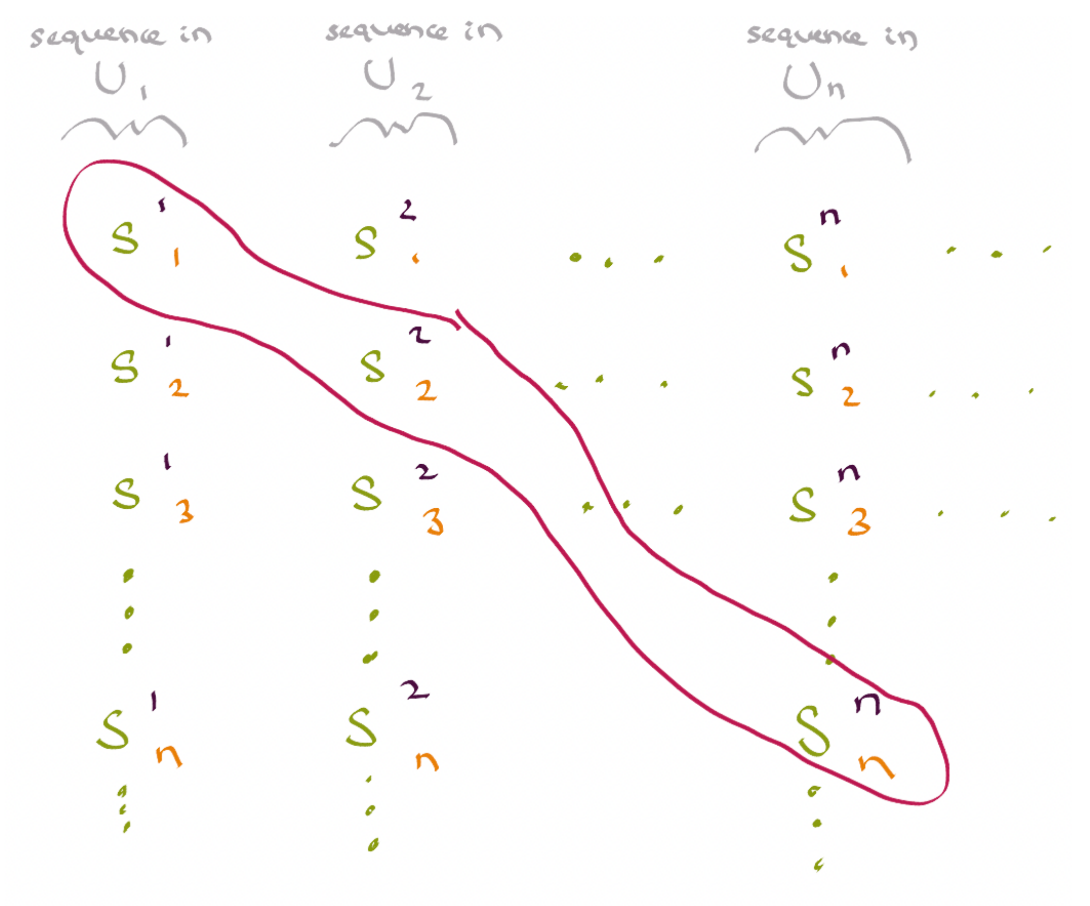
- So we have a ball B_1 with infinitely many items in the sequence. Let $U_1 = B_1$.
- Now we can cover the space C with some finite number of balls of radius $\frac{\epsilon}{2}$. We find the balls that have non-trivial intersection with U_1. One of those balls must have infinitely many items in the sequence (all of which were in U_1). Call it B_2. So we let $U_2 = U_1 \cap B_2$.



- We can keep going, finding subsequences in an increasingly small open set, building up a nested sequence of open sets $(U_1, U_2, U_3 ...)$.
- At that point, apparently, we use a diagonal argument to conclude the proof.
- We know each U_n has a subsequence of the original sequence, and the U_ns are all nested. Take the closure of each U_n, and then intersect them. The intersection of nested infinitely many non-empty closed sets has a non-empty intersection. So there's some point p in all of them. We have good reason to believe that this point p is something we could construct a subsequence to converge to.
- But now we need to construct a sequence $(p_1, p_2 ...)$ that converges to p. And the way we do that is with diagonalization.
- So let each U_n have a subsequence $(s_1^n, s_2^n, s_3^n ...)$.



- Let's conjecture that taking the "diagonal" sequence $(s_1^1, s_2^2, s_3^3 ...)$ is one that converges to p.



- We know s_n^n is in U_n, and therefore in $\overline{U_n}$, and therefore in $\overline{B_n}$ (a subset of $\overline{U_n}$, by definition of B_n) .
- And we know p is in $\overline{U_n}$ and therefore in $\overline{B_n}$.
- So both s_n^n and p are in a closed ball of radius $\frac{1}{n}$. So they can be at most distance $\frac{2}{n}$ away from each other.

$$d(s_n^n, p) < \frac{2}{n}$$

- Thus, s_n^n converges to p. $\square$

Analysis

Note that we don't know ahead of time which point we are converging to, until we've built up the sequences. The centre of each ball B_n keeps shifting, and so we're just "following the infinities."