## (Analysis) Either a graph or its complement is connected.

The **key idea** in this proof involves replacing the hypothesis with a weaker statement (and therefore **strengthening the entire statement**). That is, we go from:

- "there exists a vertex u and a vertex v with no path between them"
- "there exists a vertex u and a vertex v, where no path of length $\leq$ 2 exists between them"

How do we motivate that main idea?

### Taking inspiration from how humans solved it…

According to Tim's document when he first wrote up this proof, he did it immediately in his head.

When I first did the proof…

- I started by trying to draw out examples where both the graph and its complement were disconnected. But I slowly realized that the more "disconnected" I made one graph, the more "connected" I made the other. This solidified the "inverse" relationship between the connectedness of two graphs in my head (which perhaps a computer should automatically know given that one graph is the complement of another).
- And then I realized…even if a graph is just a *little* bit disconnected, the other graph is *super* connected (everything is at most length 2 away).
- So then I realized I should **strengthen the target to "the graph has diameter at most 3"**, mirroring this observation and intuition.
- That in turn hinted that I should **weaken the hypothesis analogously — the two disconnected vertices have no shared neighbours** .
- Filling in the gaps between this weakened hypothesis and strengthened conclusion was then quite straightforward.

### Taking inspiration from existing frameworks of conflict-guided reasoning…

This example fits into our framework of **strengthening in order to remove distracting assumptions** — we don't need to know there are no paths between two vertices…we just need to know they have no shared neighbours.

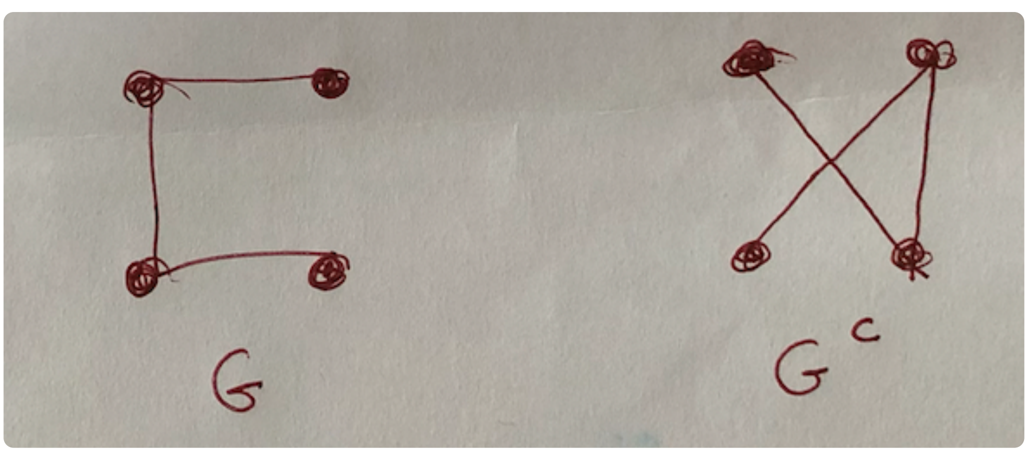
It also fits into the framework of **strengthening in existence problems to parameterize the construction** — it brings down the exponentially many ways you could construct a path to a more searchable size (e.g. we're only looking for paths of length 2).

Another framework this example falls into is the framework of **strengthening to balance bidirectionally** .

- That is, we started with the unidirectional statement:
  - $G$ is disconnected $\implies$ $G^c$ is connected
- And at the end, we have a bidirectional statement:
  - $G$ is disconnected $\iff$ $G^c$ is connected with diameter at most 3
  - or
  - $G$ has two vertices with no shared neighbours $\iff$ $G^c$ is connected
- How did we get there?
- Well, the initial implication doesn't have a weak enough hypothesis (disconnectedness of two vertices), so we weaken it (no shared neighbours between two vertices), and get the same conclusion.
- And then we realize the conclusion isn't strong enough (connectedness), so we strengthen it (connectedness with diameter at most three).

Notice that **we never strengthened to fail** and then learned from failure (which often gives meta-insights as well.). Rather, we strengthened to balance bidirectionality, which in turn helped us remove unnecessary assumptions and parameterize a construction.

- But, perhaps there is a way to motivate this with failure, in particular, **by failing to prove the converse** (if a graph is connected, its complement must be disconnected), we realize the statement isn't bidirectional, and so **we are compelled to strengthen to balance bidirectionality.**




### Next up…

In the next section on "automation", I discuss this learning-from-failure more, and discuss in general about how we might be able to leverage these theoretical frameworks to implement a practical algorithm to automate this proof.