Can you generate all Mobius transformations by inverses and translations?

We know all Mobius transformations can be generated by the following.

- inverses: $z \mapsto 1/z$
- translations: $z \mapsto z + b$
- scalings: $z \mapsto az$

The problem: can you generate all Mobius transformations using only the following?

- inverses: $z \mapsto 1/z$
- translations: $z \mapsto z + b$

Answer

The gut instinct of everyone I asked (undergraduates, PhD students, and Tim) was initially "no, you can't generate all Mobius transformations using this pair." But as they went about proving it, they realized the answer was "yes, this pair is enough." And in fact, it isn't true for any other pair in that set of three.

You can generate the scaling function $z \mapsto az$ by choosing b such that $-b^2 = a$, and then creating a function which…

- first **inverts** the input
- then **translates** that result by b
- then **inverts** the result
- then **translates** that result by $-1/b$
- then **inverts** the result
- then **translates** that result by b

The resulting function maps $z \mapsto az$.

Why it's interesting – proving the negation

This is a problem where, in the process of trying to disprove it, people often end up proving it. So, "proving the negation" has some value.

That is, you can prove that you can't get a scaling using just one inversion or translation, then you can't get a scaling using any two of them, and then you can't get a scaling using any three…but ultimately you can using six.

Why it's interesting – persisting through a semi-brute-force search

One person wrote that the answer seemed to be no, and began to try to disprove it…and in the process of disproving it actually ended up coming up with a function was getting close to the 6-function-composition-chain that is the result. But, they gave up partway through…as they started to chain together four functions, they thought it seemed it was betting a bit ridiculous, and gave up.
This poses an interesting issue for us, I think. It does seem that we can prove that you can't get a scaling using just one inversion or translation, then you can't get a scaling using any two of them, and then you can't get a scaling using any three… but by the time you get to five and you still haven't disproved the statement, it seems it's reasonable to give up. We want to figure out how to encourage a program (or human students) to keep going with this process.

Why it's interesting – weakening to build the existential

In this problem, in constructing the function, it helps to build a slightly more specialized function. (That is, we make the goal more specialized, solving a weaker problem). This is what Tim does in the following process.

Tim’s process

We want to use inversions and translations to create a Mobius transformation that scales by an arbitrary a. Remember, a Mobius transformation is uniquely determined by where it sends 3 points, so we are looking for any map that sends:

$$\begin{gathered}0 \mapsto 0 \\ 1 \mapsto a \\ \infty \mapsto \infty\end{gathered}$$

Tim, instead of trying to find the above map, **weakens** by dropping one of the requirements: “I’m going to try to find a non-trivial map that sends…”

$$\begin{gathered}0 \mapsto 0 \\ \infty \mapsto \infty\end{gathered}$$

What motivated him, he said, is not necessarily that it’s a weakening that “breaks the problem down into easier chunks”, but rather that Mobius scalings are precisely the Mobius maps that fix 0 and infinity.

We need to create an f by slowly chaining together inversions and translations.

So at any given point, you can invert or translate. He first decides to invert: $f(z) = \frac{1}{z}$. (There is a symmetry here – a correct scaling function can be achieved by either inverting or translating first, so it’s ok to make an arbitrary first choice). He notices this sends

$$\begin{gathered}0 \mapsto \infty \\ \infty \mapsto 0\end{gathered}$$

He then decides to translate by 1. A possible motivation: once you’ve inverted, there’s no choice but to translate (inverting again just undoes what you’ve done previously). So you can choose to translate by the “simplest” option there is here: 1. Tim ends up with $f(z) = \frac{1}{z} + 1$ and notes this sends

$$\begin{gathered}0 \mapsto \infty \\ \infty \mapsto 1\end{gathered}$$

He then decides to invert. Motivation: inverting here is again a forced choice. If instead you translated by something, that could have been absorbed into the previous translation. By inverting, Tim gets $f(z) = \frac{1}{\frac{1}{z}+1}$ and notes this sends

$$\begin{gathered} 0 \mapsto 0 \\ \infty \mapsto 1 \end{gathered}$$

He then decides to translate by -1 to bring ∞ mapping back to 0 (because later he wants to invert again, and his ultimate goal is to fix ∞): $f(z) = \frac{1}{\frac{1}{z}+1} - 1$.

$$\begin{gathered} 0 \mapsto -1 \\ \infty \mapsto 0 \end{gathered}$$

He then inverts again to fix ∞: $f(z) = \frac{1}{\frac{1}{\frac{1}{z}+1}-1}$.

$$\begin{gathered} 0 \mapsto -1 \\ \infty \mapsto \infty \end{gathered}$$

He then translates again by 1 to get 0 mapping back to 0. $f(z) = \frac{1}{\frac{1}{\frac{1}{z}+1}-1} + 1$

$$\begin{gathered} 0 \mapsto 0 \\ \infty \mapsto \infty \end{gathered}$$

And that's the "structure" of the final function:

$$f(z) = \frac{1}{\frac{1}{\frac{1}{z}+1} - 1} + 1$$

Tim says "it wasn't too scary that I was doing 6 operations", because the inversion is fixed, so the only choice is "by what do you translate each time." So, he made only 3 "brute-force" choices. He said he was "getting there greedily."

As it turns out, he could have also translated first, and gotten this slightly different function, which still fixes al the same values:

$$f(z) = \frac{1}{\frac{1}{\frac{1}{z+1}-1} + 1}$$

An Issue

However at this point, the function $f(z) = \frac{1}{\frac{1}{\frac{1}{z}+1} - 1} + 1$ maps 1 to 1.

$$\begin{gathered} 0 \mapsto 0 \\ 1 \mapsto 1 \\ \infty \mapsto \infty \end{gathered}$$

So we need to change our choice of translations so that 1 maps to an arbitrary a.

Tim says though, that now that he has a non-trivial map that sends infinity to infinity and zero to zero, it "fills me with confidence that I can get something else."

Generalizing

A slightly more general function that also fixes 0 and infinity is the following. For any $b \in \mathbb{C}$:

$$f(z) = \frac{1}{\frac{1}{\frac{1}{z}+b} - b^{-1}} + b$$

It sends

$$\begin{gathered} 0 \mapsto 0 \\ 1 \mapsto -b^2 \\ \infty \mapsto \infty \end{gathered}$$

You can arrive at this generalized function by first generalizing one of the translations by 1 to a translation by arbitrary b, recognize a failure to fix 0 or ∞, learn from failure, and adjust accordingly. This process, repeated a few times, yields the above function.

A final change of variables

At this point, there should be some routine move that realizes that if we want a function that scales by a, we should find a b such that $a = -b^2$. And then we do have that

$$f(z) = \frac{1}{\frac{1}{\frac{1}{z}+b} - b^{-1}} + b$$

Sends

$$\begin{gathered} 0 \mapsto 0 \\ 1 \mapsto a \\ \infty \mapsto \infty \end{gathered}$$

And so we're done. ☐