(Proof) Consider a sequence (a_n) in the reals. Prove the set of all its subsequential limits is closed.

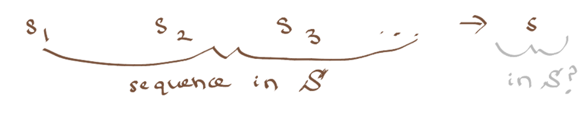
So you have a set $S \subseteq \mathbb{R}^* := \mathbb{R} \cup \{-\infty, \infty\}$ of all the possible limits any subsequence could tend to.

- For example, for the sequence $(a_n) = (0, 1, 0, 1, 0, 1...)$ has a set of subsequential limits $S = \{0, 1\}$.
- The sequence $(a_n) = (1, 2, 3...)$ has the set of subsequential limits $S = \{\infty\}$
- Consider an enumeration of rational numbers $(r_1, r_2, r_3 \dots)$. Consider the sequence $(a_n) = (r_1, r_1, r_2, r_1, r_2, r_3 \dots)$. That sequence has a set of subsequential limits $S = \mathbb{R}^*$.

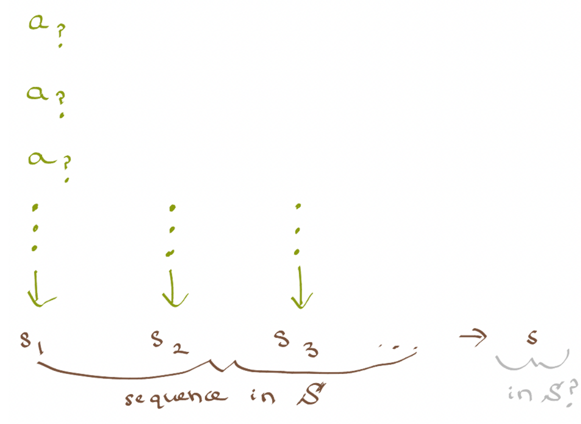
In general, to prove S is closed, we need that given any sequence (s_n) in S that converges to a limit s....

$$s_1 \quad s_2 \quad s_3 \quad \rightarrow \quad s$$

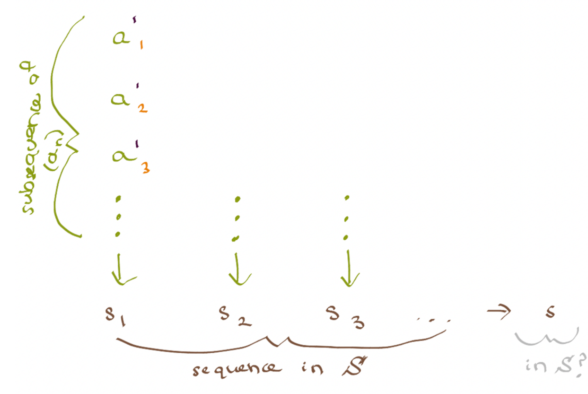
...s is also in S.



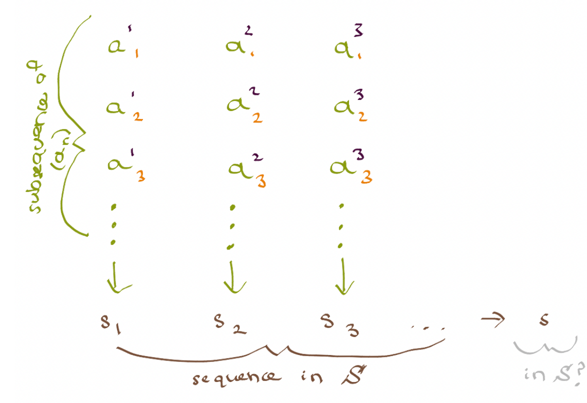
We know that for each element s_i in the set of subsequential limits, there must be some subsequence of (a_n) that converges to it.



So let's call $(a_1^1, a_2^1, a_3^1 \dots)$ the subsequence that converges to s_1 .

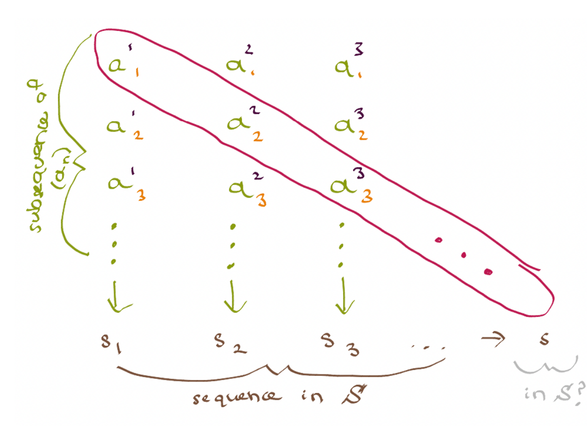


And in general, let's call $(a_1^i, a_2^i, a_3^i \dots)$ the subsequence that converges to s_i .



Now, to show $s \in S$ (that is, that s belongs to the set of subsequential limits of (a_n)), we need to find some subsequence of (a_n) that converges to s. But right now we just have that $(s_n) \to s$...but those s_n are not members of the sequence (a_n).

Well, **we notice that the closer we get to the bottom-right of the square we built, the closer we are getting to** s (I'd say this is the key idea of the proof). And so, we notice that the sequence (a_k^k) might have a shot at converging to s.



(Note that in order for this to be a subsequence of the initial sequence, we would need that a_{k+1}^{k+1} always comes after a_k^k in the initial sequence... but we would be able to do this by removing elements from the beginning of each "column" subsequence. That is, if a_k^k, which is in the k^{th} "column" subsequence, occurs at the i^{th} position in the initial sequence, then we can lop of the first i elements of the $(k+1)^{st}$ "column" subsequence, and we will get the result we desire.)

So given that conjecture, how do we prove it?

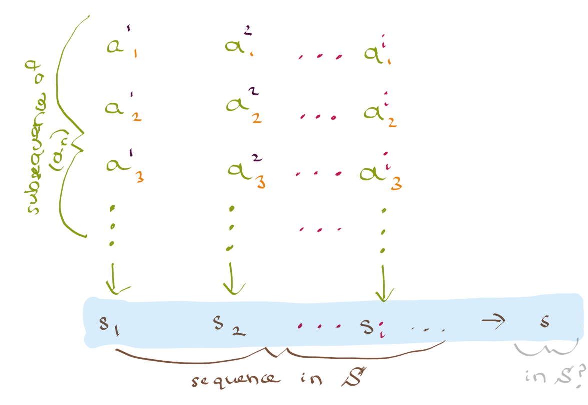
We want to show the following gets arbitrarily small as k gets big.

$$|a_k^k - s|$$

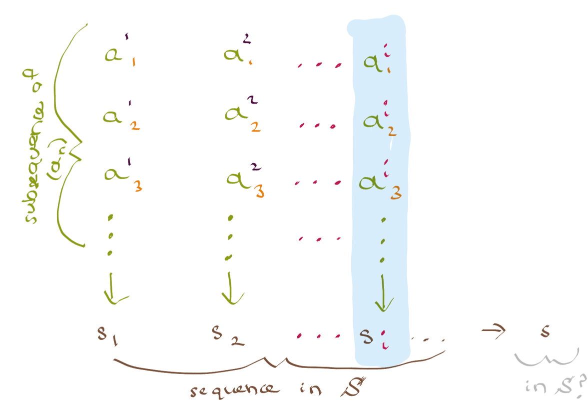
So let's use triangle inequality to split it up:

$$|a_k^k - s| \leq |a_k^k - s_k| + |s_k - s|$$

So we want to know how fast s_k is converging to its limit (this tells us an **upper bound on $|s_k - s|$**).



To get a bound on $|a_k^k - s_k|$, it's easier to consider holding a column constant. That is, we can look at the i^{th} column of our table, and look at how fast $(a_k^i) = (a_1^i, a_2^i, a_3^i \dots)$ is converging to its limit. This tells us an upper bound on $|a_k^i - s_i|$. As a consequence, by considering the k^{th}column, we get an **upper bound on $|a_k^k - s_k|$**.

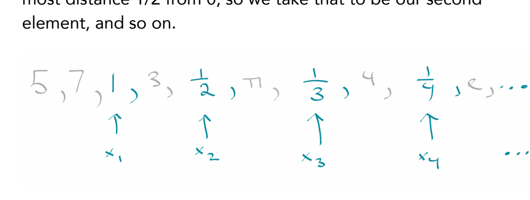


But it appears we don't know how fast each of those sequences converge...until we realize , we can make one simplifying assumption: we can construct our converging sequence to approach its limit as quickly as we want it to. That is, we can always ensure the n^{th} element of a sequence is within $\frac{1}{n}$ of its limit, by skipping ahead.

- For example, suppose we had a sequence that converged to 0.

$$5, 7, 1, 3, \tfrac{1}{2}, \pi, \tfrac{1}{3}, 4, \tfrac{1}{4}, e, \dots$$

- Then we know at one point it is at most distance 1 from 0, so we take that to be our first element, then at another point it is at most distance 1/2 from 0, so we take that to be our second element, and so on.



- We have power over which elements to choose, we just need to make sure they're all subsequences of (a_n).
- In general, we have that, for any sequence $(x_n) \to x$, we can remove elements of the sequence to create a new (x_n) that ensures:

$$|x_n - x| \leq \frac{1}{n}$$

So let's make that simplifying assumption for each column-sequences.

- We know $|a_k^i - s_i| \leq \frac{1}{k}$, and so specializing to consider the k^{th} column instead of the i^{th} column, we get $|a_k^k - s_k| \leq \frac{1}{k}$.

So we have:

$$|a_k^k - s| \leq |a_k^k - s_k| + |s_k - s|$$

$$|a_k^k - s| \leq \frac{1}{k} + |s_k - s|$$

Then taking the limit:

$$\lim_{k \to \infty} |a_k^k - s| = 0$$

So, indeed, we have that a_k^k gets arbitrarily close to s, thus we have a subsequence that converges to s, so any arbitrarily limit point s of the set of subsequential limits S is in the set, so S is closed. □

Analysis

There are **three big non-routine ideas** in this proof:

1. **Conjecturing (a_n^n) is the sequence** that works.
2. Using the **triangle inequality** to bound the expression $|a_k^k - s|$ by $|a_k^k - s_k| + |s_k - s|$.
3. **Prescribing the rate of convergence** for each subsequence.

1. How do we motivate the conjecture that (a_n^n) would be a sequence that works?

Both this conjecture and use of triangle inequality are, I believe, intertwined, and thus motivated by the same observation (see below).

2. How do we motivate the use of the triangle inequality?

- How do we motivate using triangle inequality in this instance? Is it something about the "countable by countable" square? What about the "countable by countable" square that we build up that screams "diagonal argument" to humans...and how do we translate that to a "trigger" for a computer algorithm?
- Well, the key idea of the proof is that the closer we get to the bottom-right of the square, the closer we get to the element s. How do we see that?
  - Moving **downwards** to a sequence like $(a_{999}^1, a_{999}^2, a_{999}^3, \dots)$ gets us closer to the sequence $(s_1, s_2, s_3 \dots)$ which we already know converges to s.
  - Moving to the **right** along that sequence gets us closer to the limit s.
  - So, the intuition is, moving **diagonally** downward and to the right, we should get what we need.
- I showed this problem to my friend , and he mentioned that "The **triangle inequality argument is really just a formalization of the intuitive idea** (I think!). The triangle inequality is often used when you have something gets close to one thing, which is getting close to another thing, which is exactly what's going on here!"
- Formally, our **algorithm can notice we have some $(a_k^i) \to s_k$ and $(s_k) \to s$ in the hypothesis, and when it sees something like this in the hypothesis, and something like $|a_k^i - s|$ in the target, it should strongly syntactically match with triangle inequality**.
- That is, in general, whenever the proof state looks like

$$a_n \to a$$

$$b_n \to b$$

$$\dots.$$

$$|a_n - b| \leq \dots$$

....Then it might be a **good cue to (1) use the triangle inequality and then (2) use a diagonal argument, in this case considering a_n^n.**

3. How do we motivate prescribing the rate of convergence for each subsequence?

This does seem to involve some strengthening the hypothesis...instead of considering a sequence

"Adding symmetry" — when we have a bunch of objects, we want them to behave the same (like we did in the dice problem.)

We have infinitely many sequences and you want each column to behave the same way....