Generalizing an inequality in probability

Tim mentioned saw the following inequality proved as part of a research seminar.

The **problem and proof both generalize**, and in order to generalize it, you use unintuitive moves.

The unintuitive specialization/generalization moves used here are:

- **reducing reliance on constants**
- **adding symmetry**
- **proof-based generalizing**

(Specialized) Statement

Let Z be a symmetric random variable (that is, the probability that it equals x is the same as the probability that it equals -x).

Let u and v be real numbers with $u + v \geq 2$.

Then we have:

$$P[Z + u \geq 1] + P[Z + v \geq 1] \geq 1$$

(Specialized) Proof

$P[Z + v \geq 1] = P[Z \geq 1 - v] = P[Z \leq v - 1]$, where the last equality is by the symmetry of Z.

So we need to prove that $P[Z \geq 1 - u] + P[Z \leq v - 1] \geq 1$. But since $u + v \geq 2$, it follows that $1 - u \leq v - 1$. So either

- $Z \geq 1 - u$, or
- $Z \leq v - 1$.

So either

- $P[Z \geq 1 - u]$ is 1, or
- $P[Z \leq v - 1]$ is 1.

The result follows.

(Generalized) Statement

It turns out the statement can be generalized into a stronger one, and the generalized proof is a simpler one:

$$
\begin{gathered}
Z : \text{Symmetric Random Variable} \\
Y' : \text{Symmetric Random Variable} \\
\overline{==============} \\
P[Z + Y' \geq 0] \geq 1/2
\end{gathered}
$$

(Generalized) Proof

$Z + Y'$ is the sum of symmetric random variables, and therefore also symmetric. And a symmetric random variable is nonnegative with probability at least 1/2.

Generalization Process

Assume we start with the following statement, and not necessarily its proof:

$$
\begin{gathered}
Z : \text{Symmetric Random Variable} \\
u, v : \mathbb{R} \\
u + v \geq 2 \\
============== \\
P[Z + u \geq 1] + P[Z + v \geq 1] \geq 1
\end{gathered}
$$

We rewrite $u + v \geq 2$ as $\frac{u+v}{2} \geq 1$:

$$\begin{gathered} Z : \text{Symmetric Random Variable} \\ u, v : \mathbb{R} \\ (u+v)/2 \geq 1 \\ =============== \\ P[Z + u \geq 1] + P[Z + v \geq 1] \geq 1 \end{gathered}$$

We let Y be a random variable that takes the value u with probability $\frac{1}{2}$, and v with the probability $\frac{1}{2}$. Then, $\frac{u+v}{2} \geq 1$ means that the expected value of Y is given by $E[Y] = [\frac{u+v}{2}] \geq 1$ By repackaging u and v into a single random variable, we are, in a subtle way, **reducing reliance on constants**.

$$\begin{gathered} Z : \text{Symmetric Random Variable} \\ u, v : \mathbb{R} \\ Y : \text{Random Variable} := \{\, u, v \,\} \text{ with equal probability} \\ E[Y] \geq 1 \\ =============== \\ P[Z + u \geq 1] + P[Z + v \geq 1] \geq 1 \end{gathered}$$

We rewrite the goal to use Y instead of u and v. In particular, the law of total probability tells us $P[Z + Y \geq 1] = P[Z + u \geq 1]/2 + P[Z + v \geq 1]/2$.

$$\begin{gathered} Z : \text{Symmetric Random Variable} \\ u, v : \mathbb{R} \\ Y : \text{Random Variable} := \{\, u, v \,\} \text{ with equal probability} \\ E[Y] \geq 1 \\ =============== \\ P[Z + Y \geq 1] \geq 1/2 \end{gathered}$$

Then we **generalize** when we realize the first 1 in the law of total probability actually refers to the expectation of Y, also **reducing reliance on constants**. So we actually are just proving $P[Z + Y \geq EY] \geq 1/2$.

$$\begin{gathered} Z : \text{Symmetric Random Variable} \\ u, v : \mathbb{R} \\ Y : \text{Random Variable} := \{\, u, v \,\} \text{ with equal probability} \\ E[Y] \geq 1 \\ =============== \\ P[Z + Y \geq E[Y]] \geq 1/2 \end{gathered}$$

Then we can turn Y into a symmetric random variable by shifting it by EY. And then we can rewrite our goal using Y'. That is, we **add symmetry**.

$$
\begin{gathered}
Z : \text{Symmetric Random Variable} \\
u, v : \mathbb{R} \\
Y : \text{Random Variable} := \{ u, v \} \text{ with equal probability} \\
E[Y] \geq 1 \\
Y' = Y - Y' \\
Y' : \text{Symmetric Random Variable} \\
============== \\
P[Z + Y' \geq 0] \geq 1/2
\end{gathered}
$$

Then we know $Z + Y'$ is the sum of symmetric random variables, and therefore also symmetric. And a symmetric random variable is nonnegative with probability at least 1/2. So, we have that $P[Z + Y \geq 0] \geq 1/2$, which is what wee needed.

We then apply **proof-based generalization** when we realize that all we needed to know about Y' is that it is a symmetric random variable.

$$
\begin{gathered}
Z : \text{Symmetric Random Variable} \\
Y' : \text{Symmetric Random Variable} \\
============== \\
P[Z + Y' \geq 0] \geq 1/2
\end{gathered}
$$