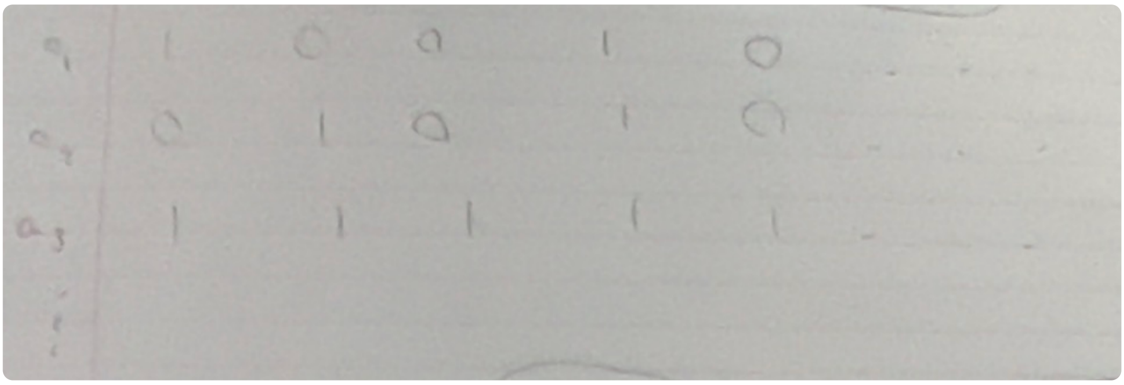
(Proof) Show the space of all infinite binary sequences is compact.

(The relevant topology is the one induced by the metric where two binary sequences are distance $\frac{1}{n}$ away when their first disagreement is at the n^{th} digit. This is equivalent to the product topology resulting from taking the discrete metric on $\{0, 1\}$.)

So we want to show every sequence has a convergent subsequence.

So first we are given an infinite sequence (of binary sequences…). That is, our sequence is $(a_1, a_2, a_3 \dots)$ where a_1 itself is a binary sequence, a_2 itself is a binary sequence, and so on.



To find a subsequence that converges, we need to find a subsequence such that…

- …at some point, all elements are within distance 1/2 of the limit sequence.
- …at some point, all elements are within distance 1/3 of the limit sequence
- … and so on.

That is, we need to find a subsequence such that…

- …at some point, all binary sequences have the same first digit.
- …at some point, all binary sequences have the same first two digits.
- … and so on.

How do we do this? When I was solving this, I weakened the statement to just focus on a subpart.

- Can I create a subsequence such that all items in that subsequence share the same first digit?
  - Well, yes. If there are only finitely many binary sequences that start with "1", there are infinitely many that start with "0", and so we can create a subsequence where every element starts with "0". And, if there are only finitely many binary sequences that start with "0", then we create a subsequence where every element starts with "1."
- Now, restrict to that subsequence. Can I further create a subsequence where after some point, all items share the same second digit?
  - Yes, by the same logic.
  - And note that in this restricted subsequence, since they all already shared the same first digit, all items in this subsequence share the first two digits.

This can be more formally characterized as an induction argument:

- Base Case: We can find a subsequence (of a sequence of binary sequences) such that the first digit of all elements in the sequence are the same.
- Inductive Step: Given a sequence (of binary sequences) such that the first m digits of all elements are the same, we can find a subsequence such that the first $m+1$ digits are the same.

At this point, I realize I can construct my convergent subsequence $(s_1, s_2, s_3 \ldots)$ to converge to the element $s = (s_{1,1}, s_{2,2}, s_{3,3} \ldots)$ like so:

- Let s_1 be the first binary sequence given in the our initial sequence (that is, a_1). If the subsequence does converge to some limit sequence, is guaranteed to be within distance 1 of s.
- Let s_2 be the first binary sequence given in the subsequence restricted to ones that all share the same first digit, such that it occurs after s_1 in the initial sequence. It is guaranteed to be within distance $\frac{1}{2}$ of s.
- Let s_3 be the first binary sequence (occurring after s_2 in the

initial sequence) given in the subsequence restricted to ones that all share the same two digits. It is guaranteed to be within distance $\frac{1}{3}$ of s.

- ... and so on.

And now, we've created a subsequence $(s_1, s_2, s_3 \ldots)$ that must converge to the element $s = (s_{1,1}, s_{2,2}, s_{3,3} \ldots)$.

- Proof: Let $\epsilon > 0$. Pick any $N > \frac{1}{\epsilon}$. Then if $n \geq N$, we know s_n shares the first N digits with s, and therefore is within distance $1/N$ of s, and therefore is is within distance ϵ of s. $\square$

Analysis

Note that we have another feature that seems to be common across many proofs with diagonal arguments. We want to prove a "there exists a limit" statement, but we don't get to immediately choose what limit that sequence converges to. We have to build it up.