G is a connected, acyclic graph $\implies$ G is connected and has $\leq$ n-1 edges

We present the proof as a 2 player game.

GOAL

The prover tries to prove “true”, and the skeptic tries to prove “false.”

SCORING

The game is semi-adversarial and semi-cooperative. That is, we want a game where:

- both players have an incentive to get to a proof
- the skeptic has an incentive to find any mistakes the prover makes
- the prover has the freedom to make as many mistakes as they want, and thus can explore specializations/generalizations without too much worry.

Thus, the game can be modelled using the following score system:

- If they prove “true”, the prover gets 10 points and the skeptic gets 10 points.
- For each time they prove “false”, the skeptic gets 1 point.

ALLOWABLE MOVES

The prover is allowed to

- Propose lemmas (either specializations or generalizations of previous lemmas).

- Retract a proposed lemma.
- Prove a proposed lemma (step by step).
- Retract a step in the proof of a proposed lemma.

In response to a proposed lemma or proof, the skeptic is allowed to:

- Provide potential counterexamples.
- Ask "why" the lemma holds/the proof is true.
- Grant the lemma/proof.
- Reject the lemma/proof.

THE GAME (AN EXAMPLE)

Here is the game, working at a high-level, almost exactly as it happened between a friend and I. I think each of these "human moves" we did breaks down into several "computer moves".

First, we have the starting proof state:

$$G : Graph(V, E)$$

$$connected(G)$$

$$acyclic(G)$$

$$|V| = n$$

$$---------------$$

$$connected(G) \wedge |E| \leq n - 1$$

Prover: So the graph is connected, by assumption.

Skeptic: Granted.

Current proof state:

$$---------------$$

$$G : Graph(V, E)$$

$$connected(G)$$

$$acyclic(G)$$

$$|V| = n$$

$$----------------$$

$$|E| \leq n - 1$$

Prover: So you claim the graph might not have at most n-1 edges? Fine, let's have it your way. Suppose there are at least n edges. I want to show that then, G has a cycle.

Skeptic: Granted. Let's assume there are at least n edges, and prove that G has a cycle.

Current proof state:

$$----------------$$

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

$$----------------$$

$$hasCycle(G)$$

Prover: Then we have a least as many edges as vertices.

Skeptic: Why?

Prover: We have that $|V| = n$ and $|E| \geq n$. So therefore $|E| \geq |V|$.

Skeptic: Granted.

Prover: Can I say that the average degree of a vertex is at least 2?

Skeptic: Why?

Prover: We have: $\sum_{v \in V} d(v) = 2|E| \implies \sum_{v \in V} d(v) \geq 2n \implies \frac{1}{n} \sum_{v \in V} d(v) \geq 2$.

Skeptic: Ok, granted.

Current proof state:

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

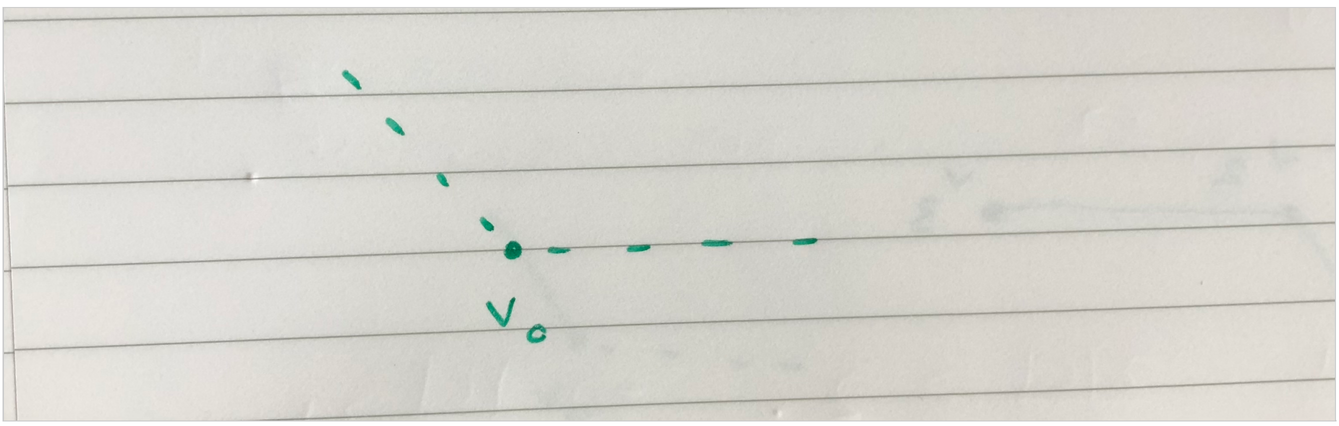
$$\frac{1}{n} \sum_{v \in V} d(v) \geq 2$$

$$hasCycle(G)$$

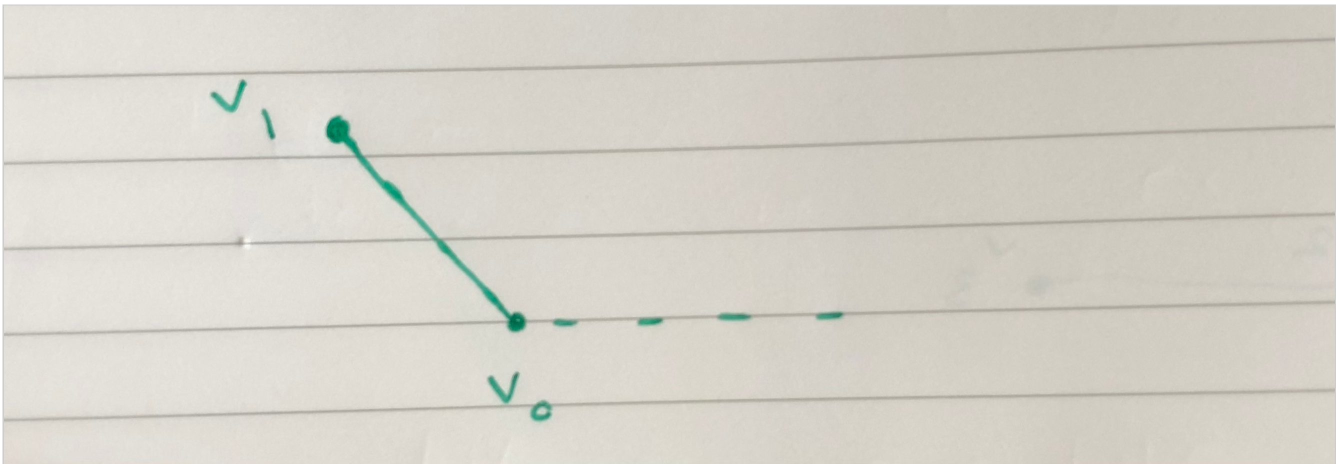
Prover: Ok…now I propose this specialized lemma. Each connected graph where every vertex has degree exactly 2 is a cycle.

Skeptic: I think I'm going to try to come up with a counterexample…(and then Skeptic tries a bit …and then gives up.) Ok…why do you think it's true?

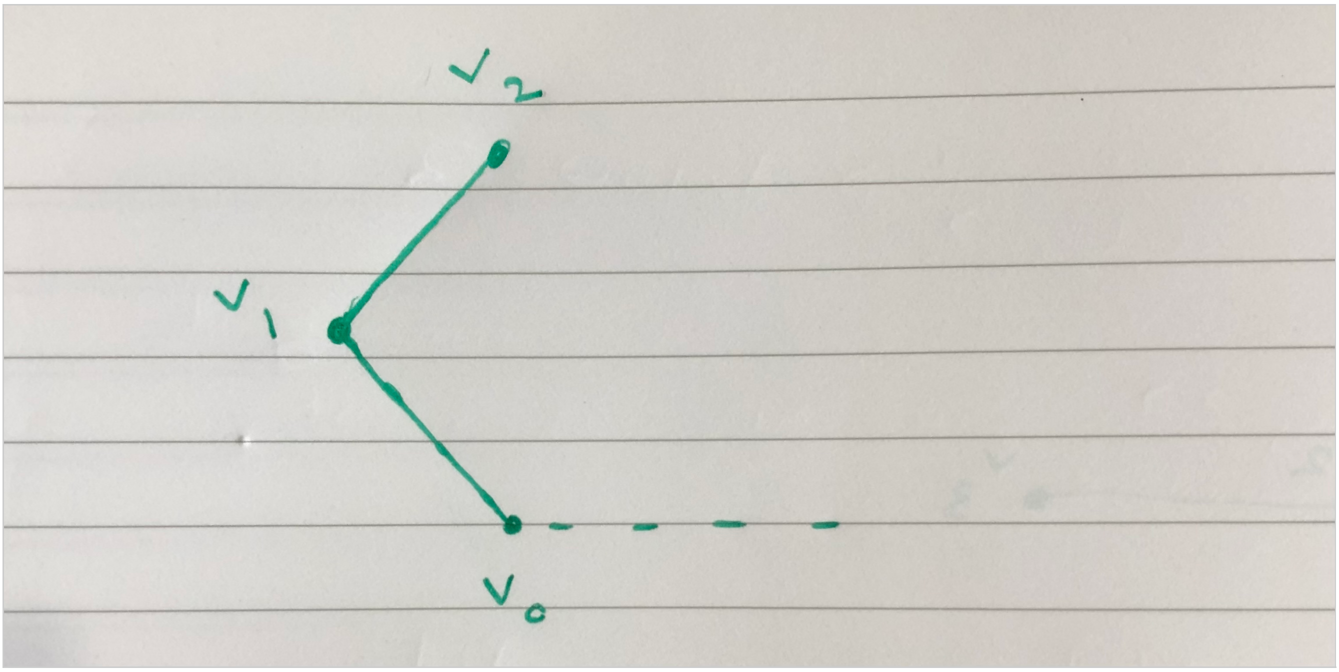
Prover: Start at the arbitrary vertex v_0. This vertex has exactly 2 neighbours by assumption…



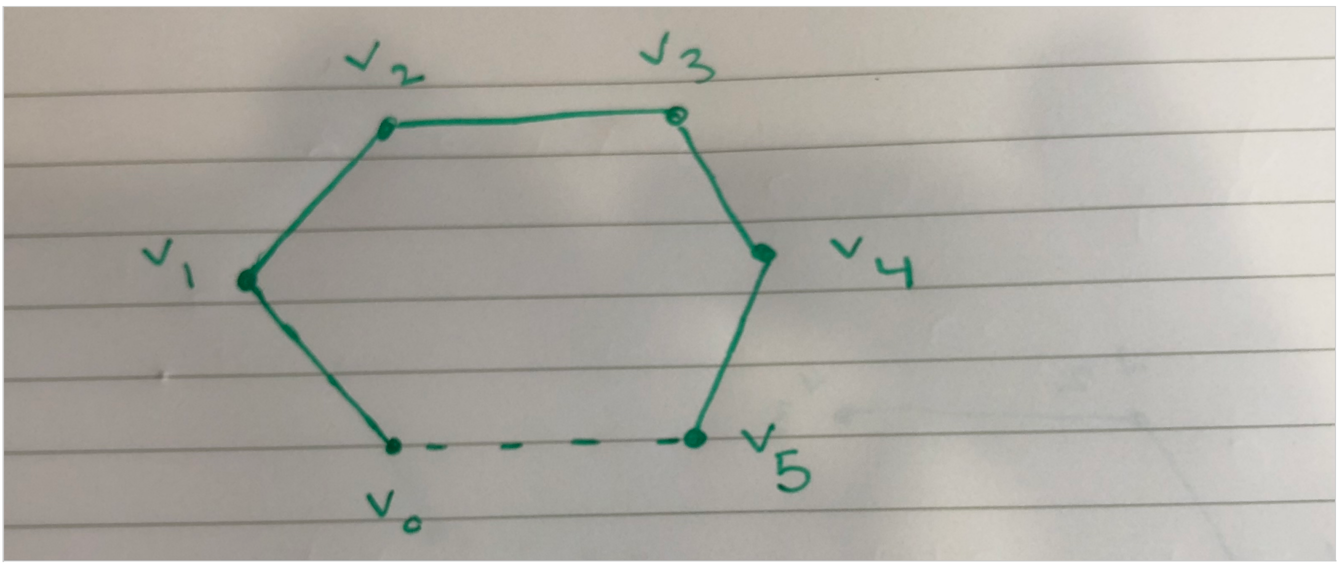
…so choose one neighbour arbitrarily and call it v_1.



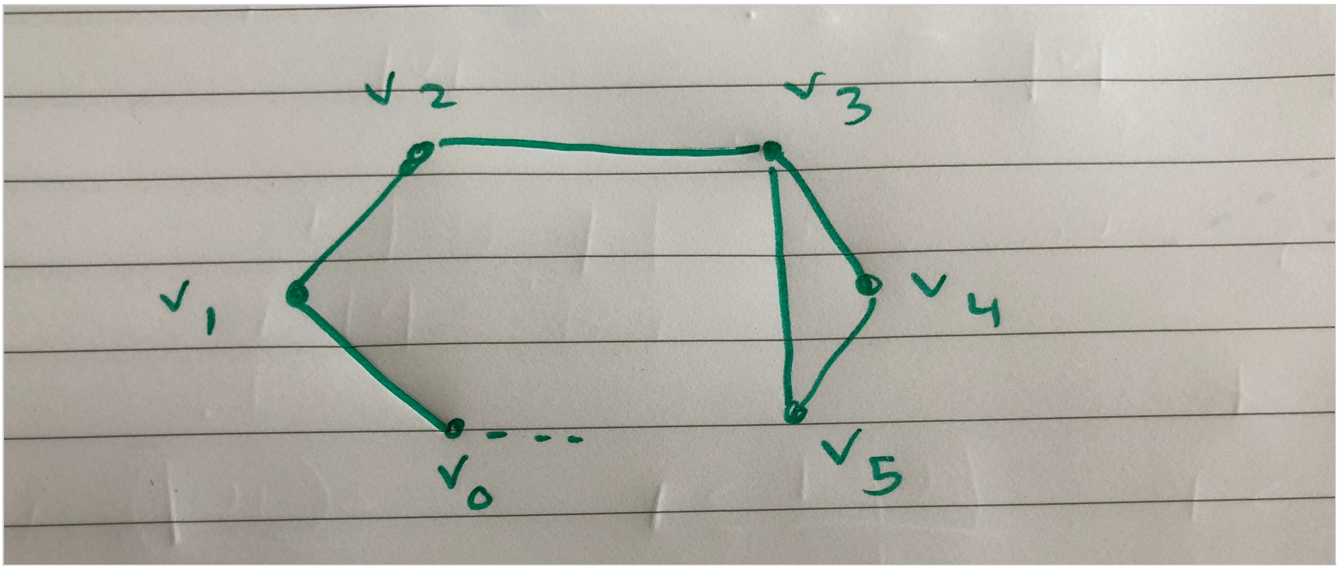
Now v_1 has exactly two neighbours: v_0 and some other vertex we can call v_2. So now go to v_2.



Keep repeating this process, visiting a vertex v_k and then naming its unvisited neighbour v_{k+1}. Eventually, we will have visited all vertices, because the graph is connected. And this process terminates, since there are finitely many vertices. And it has to terminate with v_n connecting back to v_0. So the graph is a cycle.



Skeptic: Why can't v_n connect back to another vertex, say, v_3?



Prover: Because we know v_3 already has two neighbours: v_2 and v_4. So, if v_3 has yet another neighbour, it will have degree 3. But we know each vertex has degree 2.

Skeptic. Ok, granted.

Current proof state:

$$-------------- --$$

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

$$\frac{1}{n} \sum_{v \in V} d(v) \geq 2$$

$$connected(G) \wedge \forall v \in V, d(v) = 2 \implies isCycle(G)$$

$$-------------- --$$

$$hasCycle(G)$$

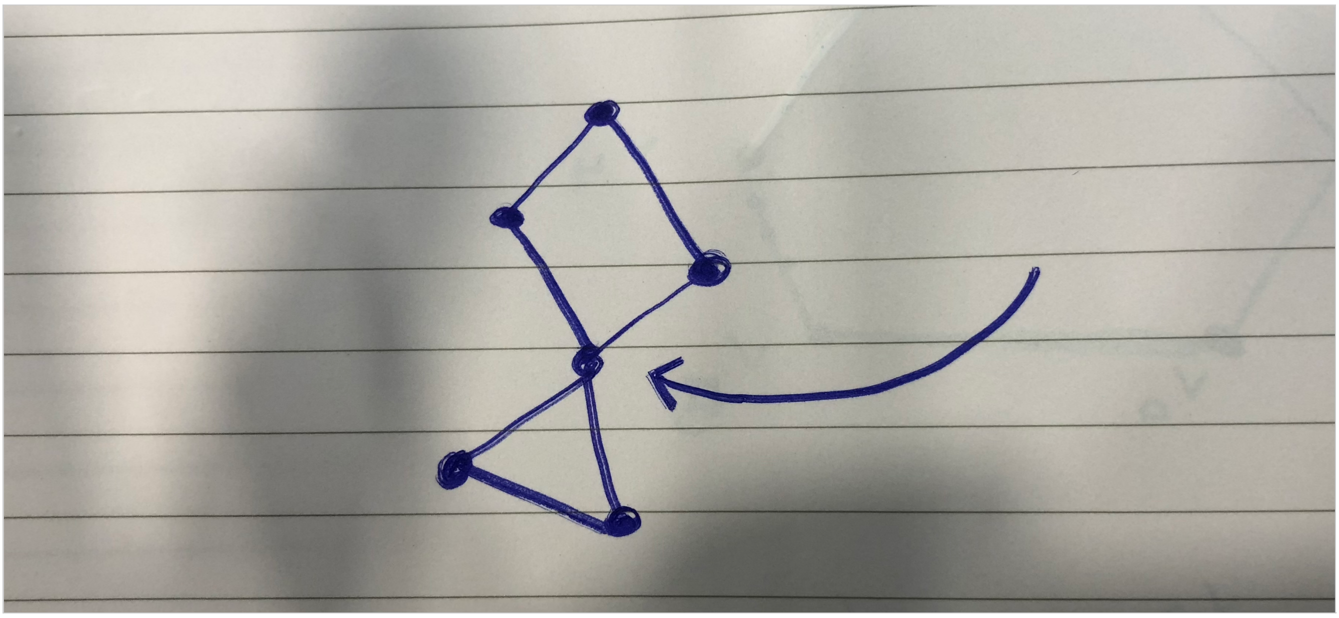
Prover: So now consider a slight generalization of that lemma. I want to prove that

$$connected(G) \wedge \forall v \in V, d(v) \geq 2 \implies hasCycle(G).$$

Skeptic: Ok, why is that true?

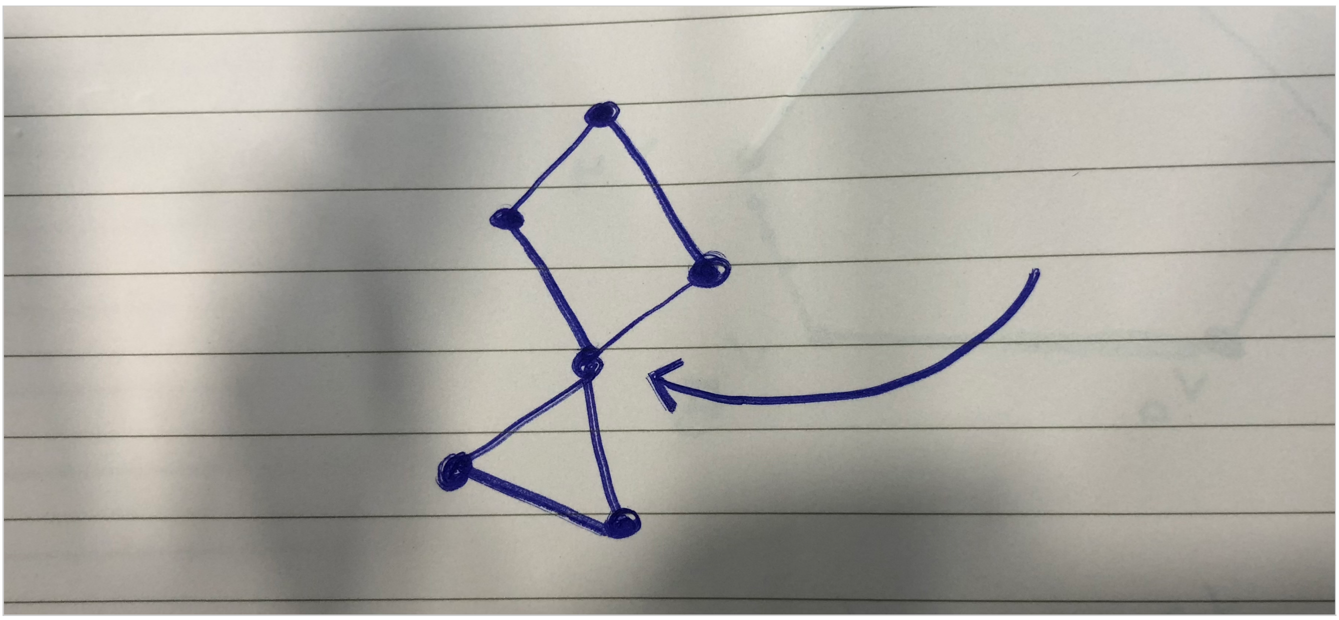
Prover: So we know each vertex has degree at least 2. So now you can remove edges from each vertex until each vertex has degree exactly 2. And then…

Skeptic: Wait. I don't know about that. Here's an example. How do you remove edges from the cut vertex on this graph so each vertex has degree exactly 2?

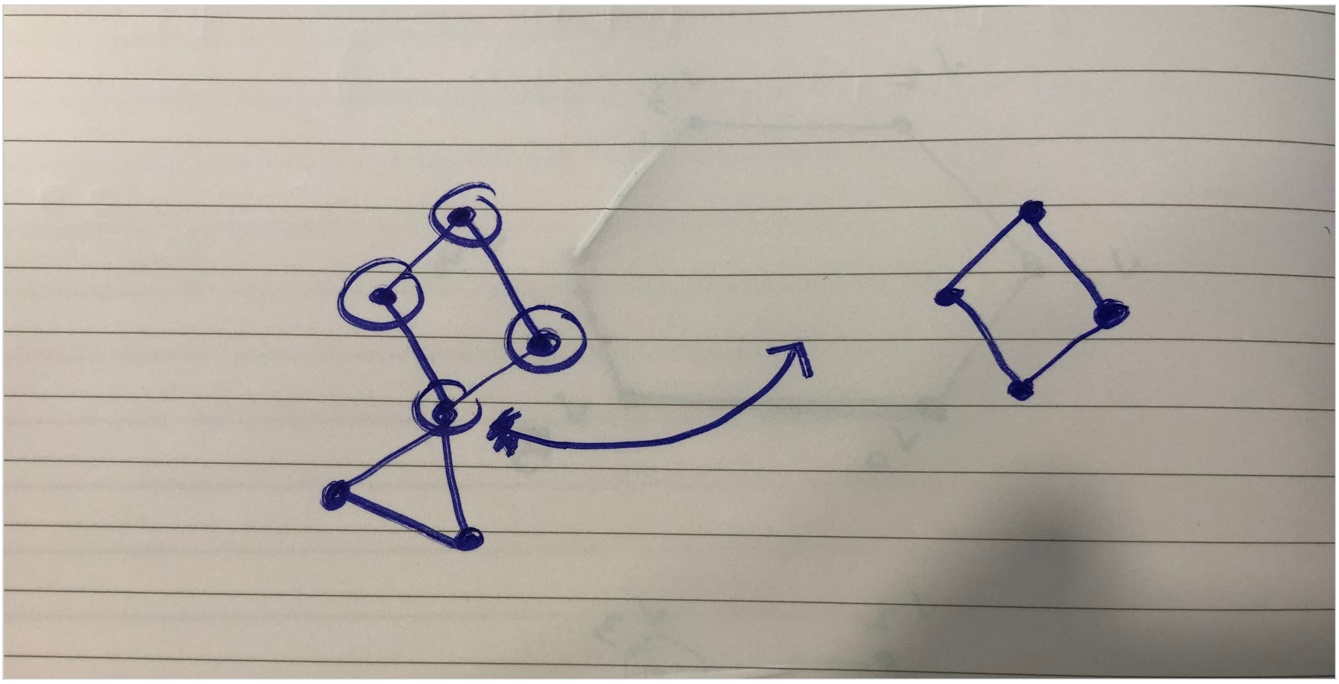


Prover: Ah ok I withdraw that proof step. Instead, remove vertices and edges until every vertex has degree exactly 2.

Skeptic: Here's an example. How do you do it?



Prover: Here.



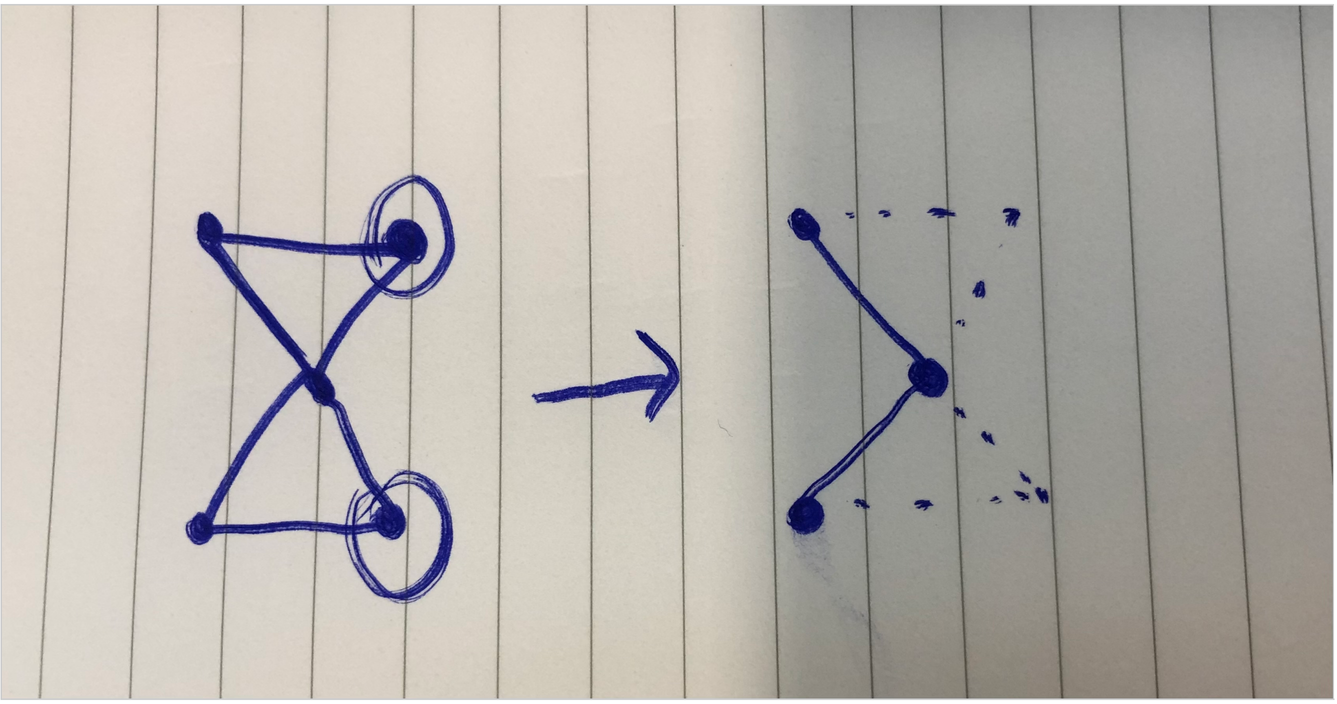
Skeptic: Ah ok. Now how do you prove you can always do that?

Prover: So suppose the vertex has k neighbours. Then remove k-2 neighbours.

Skeptic: *Any* k-2 neighbours? Or *particular* k-2 neighbours?

Prover: Let's say any.

Skeptic: Then here's an example. The middle vertex has 4 neighbours. If I remove the two indicated here, the graph will never be 2-regular.



Prover: Ah ok, then there *exists* k-2 neighbours that you can remove.

Skeptic: Which ones?

Prover: Ah ok I retract this line of reasoning. I still want to prove

$$connected(G) \land \forall v \in V, d(v) \geq 2 \implies hasCycle(G).$$

But I'm starting my proof over. Instead, pick one vertex arbitrarily: v_0. This vertex has at least 2 neighbours by assumption, so choose one neighbour arbitrarily and call it v_1. Now v_1 has at least 2 neighbours: v_0 and at least one other vertex we can call v_2. So now go to v_2. Keep repeating this process, visiting a vertex v_k and then naming one of its unvisited neighbours v_{k+1}. Eventually, this process terminates, since there are finitely many vertices, and therefore we can't keep having new unvisited vertices. And it has to terminate with v_n connecting back to a previously visited vertex, that is, some v_k where $k < n$. So the graph contains a cycle.

Skeptic: Granted.

Current proof state:

– – – – – – – – – – – – – – ––

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

$$\frac{1}{n} \sum_{v \in V} d(v) \geq 2$$

$$connected(G) \land \forall v \in V, d(v) \geq 2 \implies hasCycle(G)$$

– – – – – – – – – – – – – – ––

$$hasCycle(G)$$

Skeptic: But why doesn't this proof work if you assume that one vertex has degree less than 2?

Prover: Because, if, say v_3 has only degree 1 (is a leaf), then there is no unvisited vertex to go to.

Skeptic: Ok.

Prover: Oh but that gives me an idea! Ok, so the idea is to remove all degree 1 vertices, until you have a leaf-free subgraph. Removing all the leaves will give us a graph with only vertices with degree 2 or higher. Then, the lemma should apply to that subgraph, so we get that the subgraph has a cycle.

Skeptic: So hold on. First, is it true that if a subgraph has a cycle, then the graph definitely has a cycle?

Prover: Yes.

Skeptic: Ok…I can't come up with a counterexample. And in the process of trying, I see why it's true. So, granted.

Current proof state:

— — — — — — — — — — — — — — —

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

$$\frac{1}{n} \sum_{v \in V} d(v) \geq 2$$

$$connected(G) \wedge \forall v \in V, d(v) \geq 2 \implies hasCycle(G)$$

$$G' : Graph(V', E') := noLeaves(G)$$

$$hasCycle(G') \implies hasCycle(G)$$

— — — — — — — — — — — — — — —

$$hasCycle(G)$$

Prover: And so our problem now cleans up to:

Current proof state:

— — — — — — — — — — — — — — —

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

$$\frac{1}{n}\sum_{v \in V} d(v) \geq 2$$

$$connected(G) \wedge \forall v \in V, d(v) \geq 2 \implies hasCycle(G)$$

$$G' : Graph(V', E') := noLeaves(G)$$

$$---------------$$

$$hasCycle(G')$$

Prover: And now let's try to make our previous lemma apply. We first use a specialization of our previous lemma.

Current proof state:

$$---------------$$

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

$$\frac{1}{n}\sum_{v \in V} d(v) \geq 2$$

$$G' : Graph(V', E') := noLeaves(G)$$

$$connected(G') \wedge \forall v \in V', d(v) \geq 2 \implies hasCycle(G')$$

$$---------------$$

$$hasCycle(G')$$

Prover: And now our problem cleans up yet again.

Current proof state:

$$---------------$$

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

$$\frac{1}{n} \sum_{v \in V} d(v) \geq 2$$

$$G' : Graph(V', E') := noLeaves(G)$$

$$-------------$$

$$connected(G') \wedge \forall v \in V', d(v) \geq 2$$

Skeptic: So now having applied that lemma, you seem to be confident that G' is connected, and that every vertex in it has degree at least 2. How do you know those things?

Prover: Yes. First, removing a leaf will always keep the graph connected.

Skeptic: Granted. So G' is connected.

Current proof state:

$$-------------$$

$$G : Graph(V, E)$$

$$connected(G)$$

$$|V| = n$$

$$|E| \geq n$$

$$\frac{1}{n} \sum_{v \in V} d(v) \geq 2$$

$$G' : Graph(V', E') := noLeaves(G)$$

$$-------------$$

$$\forall v \in V', d(v) \geq 2$$

Prover: Second, removing a leaf also preserves the fact that the average degree is at least 2. This is because we start with $\frac{1}{n} \sum_{v \in V} d(v) = \frac{2|E|}{|V|} \geq 2 \implies \frac{|E|}{|V|} \geq 1$ $\implies |E| \geq |V|$, as before.

Then if we remove a leaf, we remove one vertex and one edge, so we get that in the new graph $\tilde{G}$ with vertex set $\tilde{V}$ and edge set $\tilde{E}$, then $|\tilde{E}| = |E| - 1$ and , $|\tilde{V}| = |V| - 1$, so we still have:

$$|E| - 1 \geq |V| - 1 \implies |\tilde{E}| \geq |\tilde{V}| \implies \frac{2|\tilde{E}|}{|\tilde{V}|} \geq 2 \implies \frac{1}{|\tilde{V}|} \sum_{v \in \tilde{V}} d(v) \geq 2.$$

Thus, even as we remove leaves, the average degree of the graph stays above 2. And in a connected graph (no degree 0 vertices) with no leaves (no degree 1 vertices) but average degree greater than 2 (the graph isn't empty), every vertex has degree greater than 2. So, the goal holds.

Skeptic: Granted. I think the proof is done.