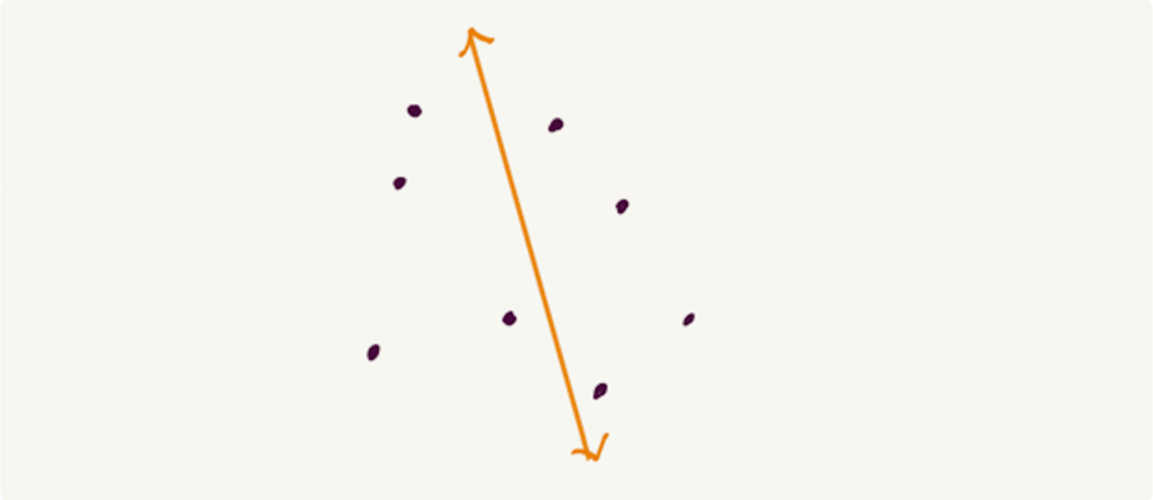
Partitioning points on a plane with a line

The Problem

Suppose you have n (an even number) of points on a plane.

Then there exists a line that divides them exactly in half, with n/2 on one side and n/2 on the other side of the line.

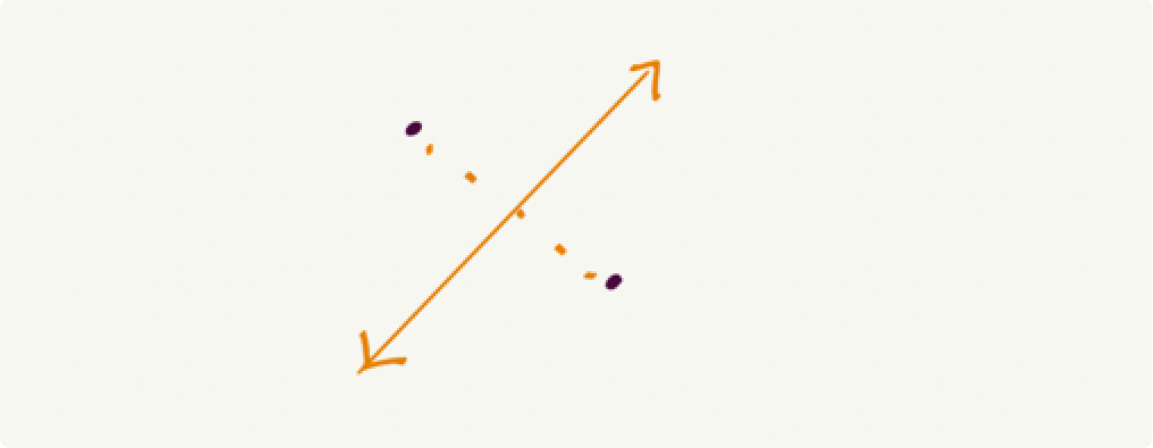


Why this problem?

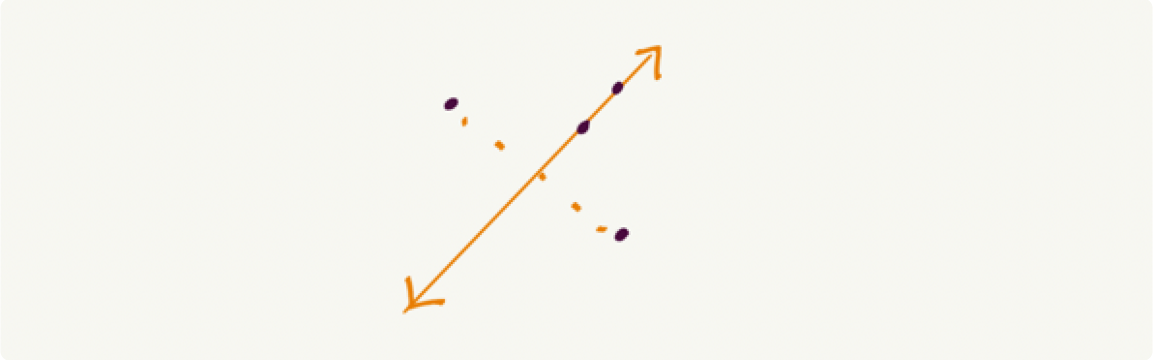
I think this is an example where *why learning from failure helps* is really clear.

Failure with induction

I first tried induction. It is clear why it's always possible to divide two points with a line (e.g. using a perpendicular bisector of the line segment between the points).



But it's difficult to use this line in a useful way when you have four points. So induction doesn't seem like the right way to go.



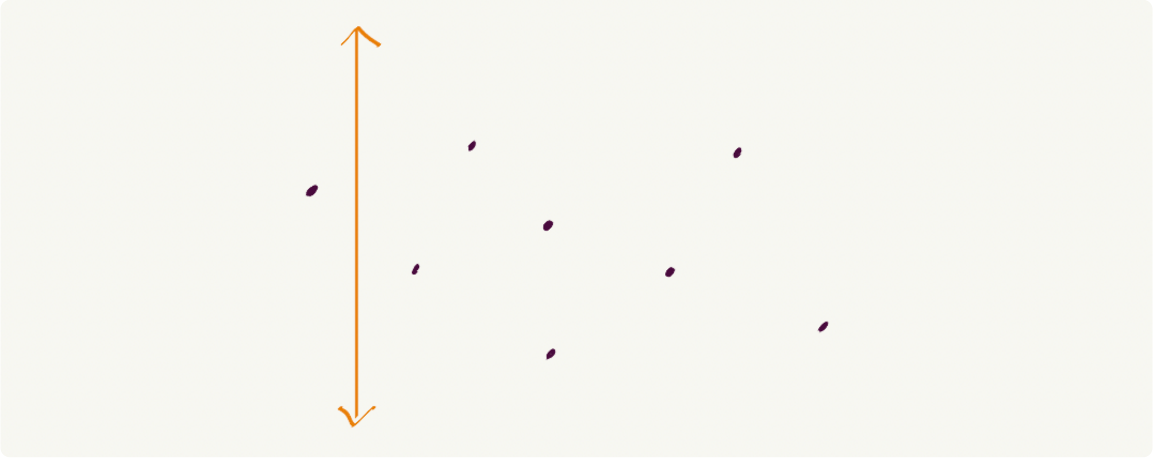
Success with contradiction

I then tried contradiction. Suppose there is no such dividing line. Then, every line in the plane has more points on one side than the other.

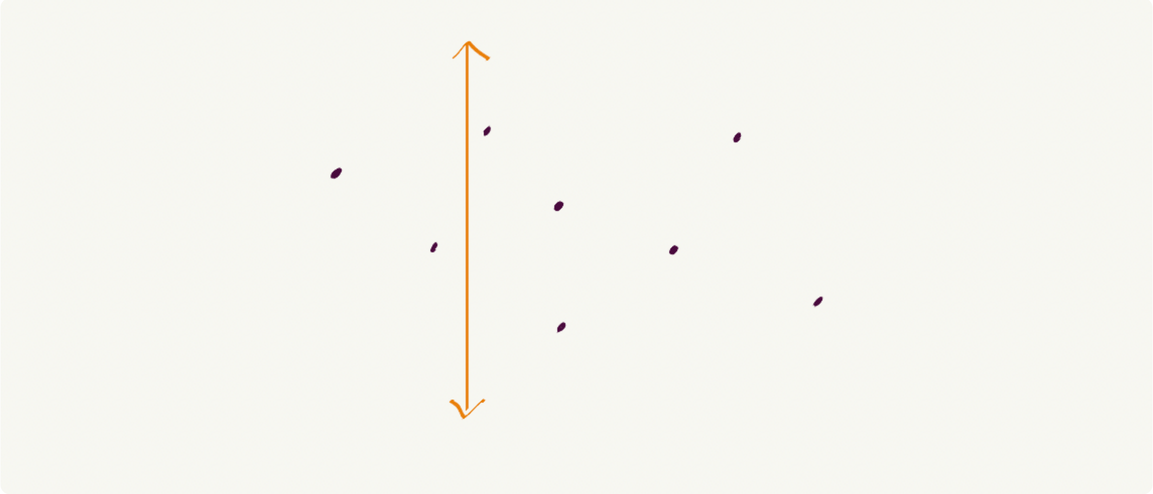
At this point, it helped me to *specialize*. If it's true for every line, then it's true for vertical lines. That is, everywhere I place a vertical line, it has more points on the left or right.

(So here you have more points on the right…)

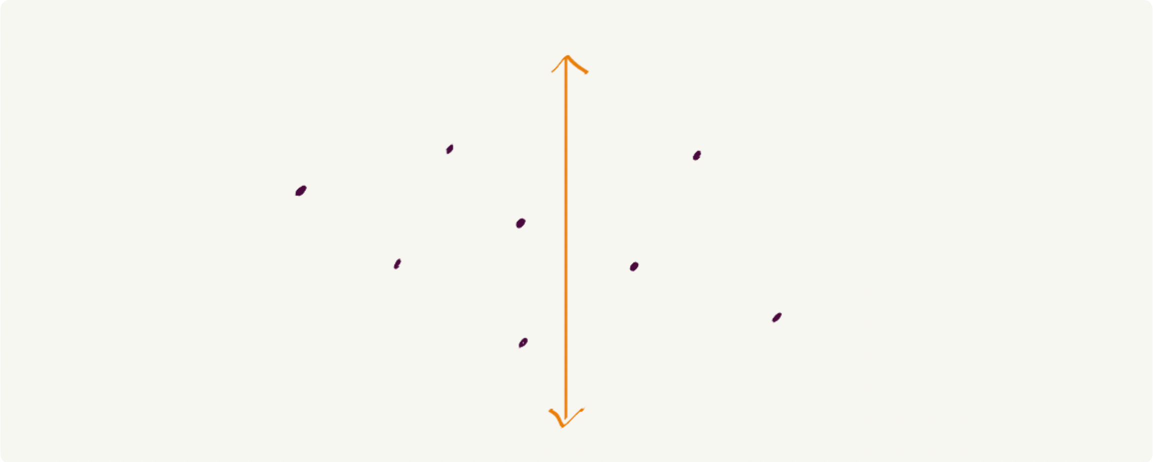
(And still, more points on the right…)



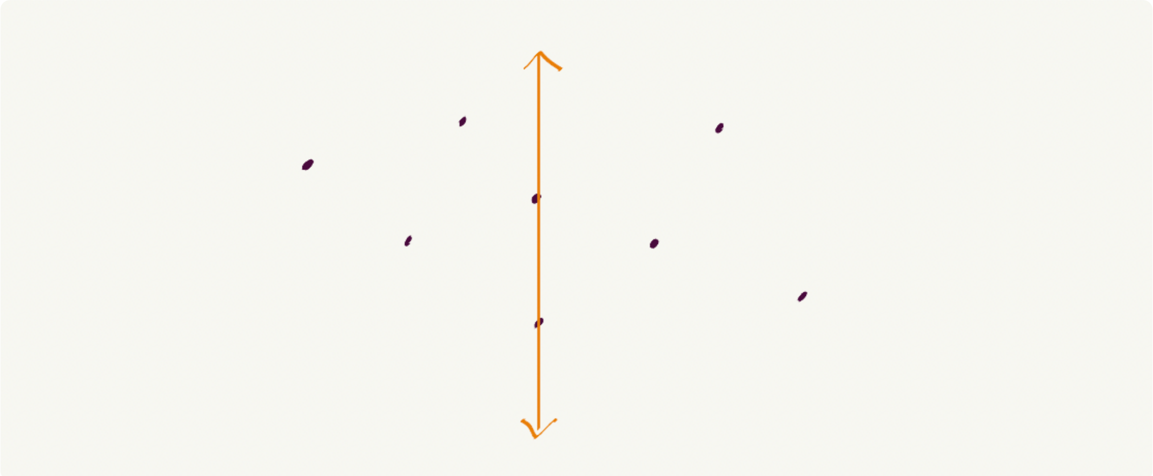
(And still, more points on the right…)



(And suddenly, more points on the left…)



And that implies that there exists one vertical line that intersects two points. That is, at least two points are collinear.



Now we can *generalize*: two points are collinear with a vertical line. But this is true for all lines. So no matter what slope of line I choose, two points are collinear along that line.

Now we want something that contradicts that there are n points. Motivated by the fact that that's the only contradiction I can get, I want to do some counting that shows that there are not. In particular, maybe using some syntax matching to look for integers or cardinalities, I find that the only other cardinality in the proof state is that there are uncountably infinitely many slopes. (My human-like induction told me that so many slopes meant that there were a lot of points, and this cardinality-matching is my best guess for what was going on).

Formally, I find that there can be at most $\binom{n}{2}$ lines such that every two points is contained in a line. So they create at most $\binom{n}{2}$ slopes. But I have more than $\binom{n}{2}$ slopes — I have infinitely many. Contradiction.

The learning from failure

I think what I implicitly did here was

- **specialize** the problem to vertical lines: there exists no vertical line that divides the plane such that there are n/2 points on one side and n/2 on the other.
- …and then **generalize** that with the conjecture: there exists no vertical line that divides the plane such that there are k points on one side, and n-k on the other (for all k).

But we **fail** at this generalization.

And so a natural next step is to ask the question: for which lines does it fail? And this directly leads to the **lemma that it fails for lines which intersect at least two points**.

This point of failure was crucial to the rest of the proof — determining exactly how many collinear points there are is key.