(Analysis) Either a graph or its complement is connected.

The **key idea** in this proof involves replacing the hypothesis with a weaker statement (and therefore **strengthening the entire statement**). That is, we go from:

- "there exists a vertex u and a vertex v with no path between them"
- "there exists a vertex u and a vertex v, where no path of length ≤ 2 exists between them"

How do we motivate that main idea?

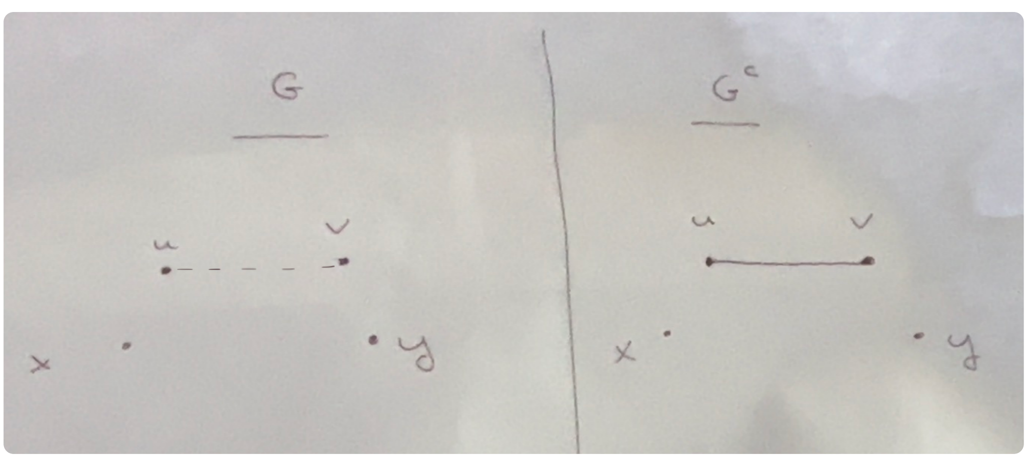
Taking inspiration from how humans solved it…

According to Tim's document when he first wrote up this proof, he did it immediately in his head. Interestingly, he later explained that he remembers doing it thinking of a different process, which gives you a better bound on the diameter:

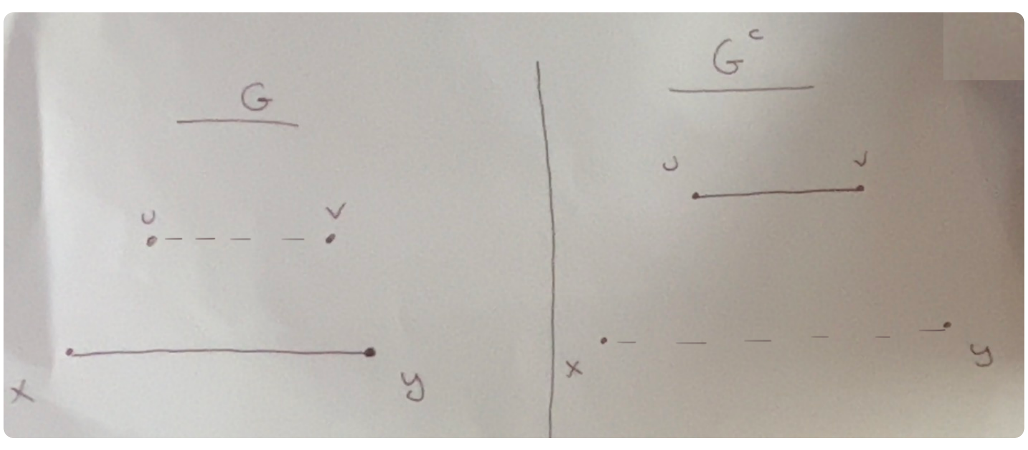
- if u and v are disconnected, then u and v are in different connected components
- So, in the complement, any two vertices in different connected components are connected by an edge
- So in the complement, any two vertices x and y are either…
  - in different connected components in G, which means they are connected by an edge in G^c
  - in the same connected component in G, which means they are both connected to some vertex "w" in the other component in G^c, so their diameter is 2.

Tim then proposed another way of doing it that does involve learning from failure…

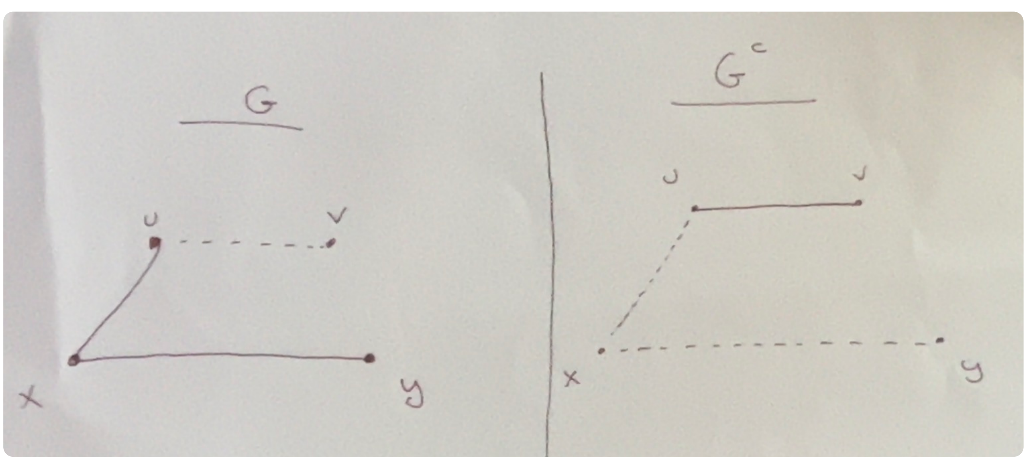
- You know u and v are disconnected in the graph.
- You are trying to find edges between x and y in the complement.
- So you are trying to find non-edges between x and y in the graph.
- Where are the non-edges?
- Well definitely there is a non-edge between u and v in G, so there's an edge between them in G^c. (Dashed lines below indicate that we know there can't be an edge there.)



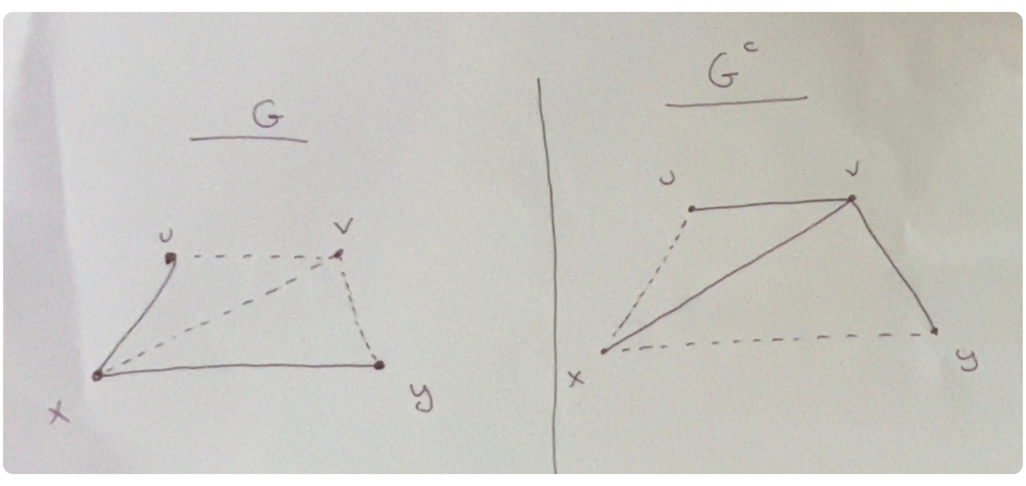
- Ok, so now suppose we're trying to build a counterexample — a graph where both G and G^c are disconnected (or in particular, where G is connected, and x and y are disconnected in G^c).
- Suppose there is a non-edge between x and y in G. Then they are immediately connected in G^c so we have no counterexample…so let's consider the case where there is an edge between x and y in G.



- Can there be an edge between x and u in G? Well, right now if we leave G as it is, then G^c would be connected, so we do need to add an edge somewhere to G. WLOG you'd need to add an edge between x and u in G(that is, you could add edges x-u, y-u, x-v, y-v, but these are all symmetric.)



- (Note that we really just have to consider the 4-vertex subgraph containing x, u, v, and y).
- Well for there to be no path from u to v in the graph G , there must be no path in this induced 4-vertex subgraph between u and v in G . So, we can't have the edges v-x or v-y in G, which means we must have them in G^c .



- But oh no — now using only necessary conditions, we got that G^c was connected.
- So in this process, we fail to find a counterexample, and in the process, find a proof.

When I first did the proof…

- I started by trying to draw out examples where both the graph and its complement were disconnected. But I slowly realized that the more "disconnected" I made one graph, the more "connected" I made the other. This solidified the "inverse" relationship between the connectedness of two graphs in my head (which perhaps a computer should automatically know given that one graph is the complement of another).

- And then I realized…even if a graph is just a *little* bit disconnected, the other graph is *super* connected (everything is at most length 2 away).
- So then I realized I should **strengthen the target to "the graph has diameter at most 3"**, mirroring this observation and intuition.
- That in turn hinted that I should **weaken the hypothesis analogously — the two disconnected vertices have no shared neighbours** .
- Filling in the gaps between this weakened hypothesis and strengthened conclusion was then quite straightforward.

Taking inspiration from existing frameworks of conflict-guided reasoning…

This example fits into our framework of **strengthening in order to remove distracting assumptions** — we don't need to know there are no paths between two vertices…we just need to know they have no shared neighbours.

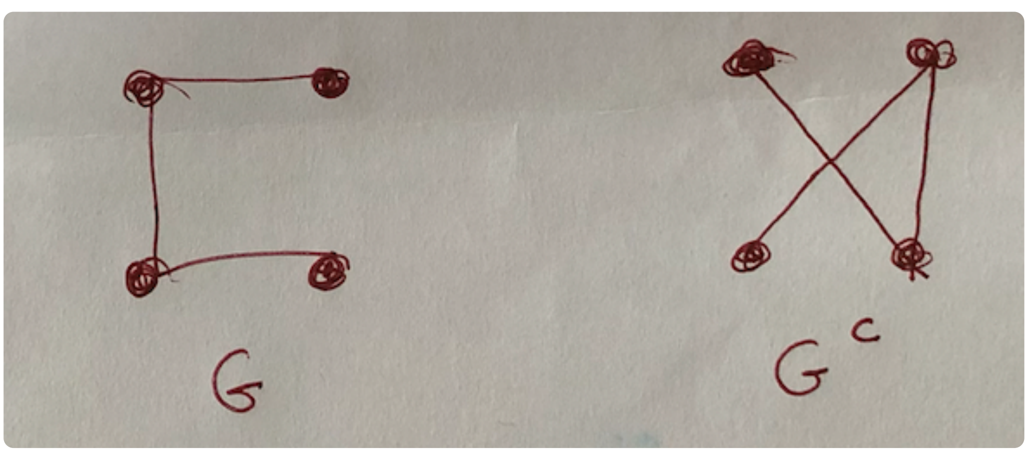
It also fits into the framework of **strengthening in existence problems to parameterize the construction** — it brings down the exponentially many ways you could construct a path to a more searchable size (e.g. we're only looking for paths of length 2).

Another framework this example falls into is the framework of **strengthening to balance bidirectionally** .

- That is, we started with the unidirectional statement:
  - G is disconnected $\implies$ G^c is connected
- And at the end, we end up with bidirectional statements.
  - G is disconnected $\iff$ G^c is connected with diameter at most 2
    - Proof:
    - The forward direction we proved.
    - To prove the backward direction, take the contrapositive of the other strengthening we proved: G has two vertices with no shared neighbours $\implies$ G^c is connected. The contrapositive is: G is disconnected $\implies$ Any two vertices in G^c share a neighbour. So, G is disconnected $\implies$ G^c is connected with diameter at most 2.
  - or
  - G has two vertices with no shared neighbours $\iff$ G^c is connected
    - Proof:
    - The forward direction we proved.
    - To prove the backward direction, take the contrapositive of the other strengthening we proved: G is disconnected $\implies$ G^c is connected with diameter at most 2. So, G has two vertices whose shortest path has length greater than two $\implies$ G^c is connected. So, G has two vertices with no shared neighbors $\implies$ G^c is connected.
- How did we get there?
- Well, the initial implication doesn't have a weak enough hypothesis (disconnectedness of two vertices), so we weaken it (no shared neighbours between two vertices), and get the same conclusion.
- And then we realize the conclusion isn't strong enough (connectedness), so we strengthen it (connectedness with diameter at most three).

Notice that **we never strengthened to fail**, neither in Tim's method nor in mine. (In Tim's method, we did learn from failure, but never strengthened). And in my method, we strengthened to balance bidirectionality, which in turn helped us remove unnecessary assumptions and parameterize a construction.

- But, perhaps there is a way to motivate this method with failure, in particular, **by failing to prove the converse** (if a graph is connected, its complement must be disconnected), we realize the statement isn't bidirectional, and so **we are compelled to strengthen to balance bidirectionality.**



Next up…

In the next section on "automation", I discuss this learning-from-failure more, and discuss in general about how we might be able to leverage these theoretical frameworks to implement a practical algorithm to automate this proof.