(Point-and-Click) Either a graph or its complement is connected.

We start out with this proof state:

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$$G \textit{ is connected} \vee G^c \textit{ is connected}$$

– –

A common trick when proving $P \vee Q$ is to prove the equivalent $\neg P \implies Q$. The point-and-click move here may be to **unfold one of the definitions of "or."**

– –

$$G \textit{ is not connected}$$

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$$G^c \textit{ is connected}$$

– –

Then we **unfold "not connected."**

– –

$$\exists \textit{ vertices } u, v \in V(G) \textit{ with no path between them in } G$$

====================================

$$G^c \textit{ is connected}$$

– –

Then we **unfold "connected"**.

– –

$$\exists \textit{ vertices } u, v \in V(G) \textit{ with no path between them in } G$$

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$$\forall \textit{ vertices } x, y \in V(G^c), \exists \textit{ path between them in } G^c$$

– –

Then we **tidy** (which maybe happens automatically after the previous move):

– –

$$\exists \textit{ vertices } u, v \in V(G) \textit{ with no path between them in } G$$

$$x, y \in V(G^c)$$

====================================

$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

Then we **specialize** — which involves parameterizing an object by its properties (perhaps "path" has some most-important-properties like "length", "start point", "end point", and "included point."). We should be allowed to initialize any natural number property (like length) to any natural number in the context, or, to any "sufficiently small number" like 1 through 5.

- This move could perhaps be motivated by trying to prove the converse (a graph is connected means its complement is disconnected), failing to do so since a counterexample exists, and then realizing the statement can be strengthened accordingly. I go into details on this in the "Analysis" and "Automation" documents.

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$$\exists \textit{ vertices } u, v \in V(G) \textit{ with no path of length } \leq 2 \textit{ between them in } G$$

$$x, y \in V(G^c)$$

====================================

$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

Then we **break into cases**.

– –

$$\exists \textit{ vertices } u, v \in G \textit{ with no path of length } 1 \textit{ between them in } G$$

$$\exists \textit{ vertices } u, v \in G \textit{ with no path of length } 2 \textit{ between them in } G$$

$$x, y \in V(G^c)$$

====================================

$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

Then we automatically **tidy again** (pulling out common terms to remove redundancy in the hypothesis).

– –

$$u, v \in V(G)$$

$$\textit{there is no path of length } 1 \textit{ between } u \textit{ and } v \textit{ in } G$$

$$\textit{there is no path of length } 2 \textit{ between } u \textit{ and } v \textit{ in } G$$

$$x, y \in V(G^c)$$

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$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

Then we **use a defining property of the complement** (that $V(G) = V(G^c)$) to find that vertices in the complement are the same as vertices in the graph.

- This could perhaps be motivated by increasing syntactic similarity between different hypotheses.
- This could also be motivated by just shortening the number of characters and lines in the proof context.

– –

$$u, v, x, y \in V(G) = V(G^c)$$

$$\textit{there is no path of length } 1 \textit{ between } u \textit{ and } v \textit{ in } G$$

$$\textit{there is no path of length } 2 \textit{ between } u \textit{ and } v \textit{ in } G$$

====================================

$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

Then we **unfold** the definition of a "path" — in particular a path of length 1 — motivated by the fact that the complement has to do with edges, not paths, and we we want to create syntactic similarity.

– –

$$u, v, x, y \in V(G) = V(G^c)$$

$$\nexists \textit{ edge } u - v \textit{ in } G$$

$$\textit{there is no path of length } 2 \textit{ between } u \textit{ and } v \textit{ in } G$$

====================================

$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

Then we **unfold** the definition of a "path" again — in particular a path of length 2.

- This could still be motivated by the fact that the complement has to do with edges, not paths, and we want to create syntactic similarity.
- Yet another motivation is that we would want to be able to use our other objects in the hypothesis — the "x" and "y" of a certain type (vertex of G) — so having a hypothesis that takes in that type (is quantified over that type) would be helpful.

– –

$$u, v, x, y \in V(G) = V(G^c)$$

$$\nexists \textit{ edge } u - v \textit{ in } G$$

$$\forall z \in V(G), \nexists \textit{ path u-z-v in } G$$

====================================

$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

Then we further **unfold** the definition of a path into its component edges.

- This could be motivated by the fact that we want to translate into something about G^c, and complements have to do with neighbours…again we're syntactically bringing this hypothesis closer together with the target and the other hypothesis that mentions edges…

– –

$$u, v, x, y \in V(G) = V(G^c)$$

$$\nexists \textit{ edge } u - v \textit{ in } G$$

$$\forall z \in V(G), \nexists \textit{ edge u-z in } G \vee \nexists \textit{ edge z-v in } G$$

====================================

$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

And then we translate statements about G into statements about G^c by **using another defining property of the complement**.

- This could again perhaps be motivated either by creating syntactic similarity, or removing the ugly $\nexists$ symbols.

– –

$$u, v, x, y \in V(G) = V(G^c)$$

$$\exists \textit{ edge } u - v \textit{ in } G^c$$

$$\forall z \in V(G^c), \exists \textit{ edge u-z in } G^c \vee \exists \textit{ edge z-v in } G^c$$

====================================

$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

Then we **specialize** the hypothesis to all the different cases it could apply to (that is, we further weaken the hypothesis):

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$$u, v, x, y \in V(G) = V(G^c)$$

$$\exists \textit{ edge } u - v \textit{ in } G^c$$

$$\exists \textit{ edge u-x in } G^c \vee \exists \textit{ edge x-v in } G^c$$

$$\exists \textit{ edge u-y in } G^c \vee \exists \textit{ edge y-v in } G^c$$

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$$\exists \textit{ path between } x \textit{ and } y \textit{ in } G^c$$

– –

And at this point we could apply a **routine move of case analysis** on the four cases (x can be connected to u or v, and y can be connected to u or v).

Finally, again using **routine moves** , we should find that in all cases, x and y are connected. □