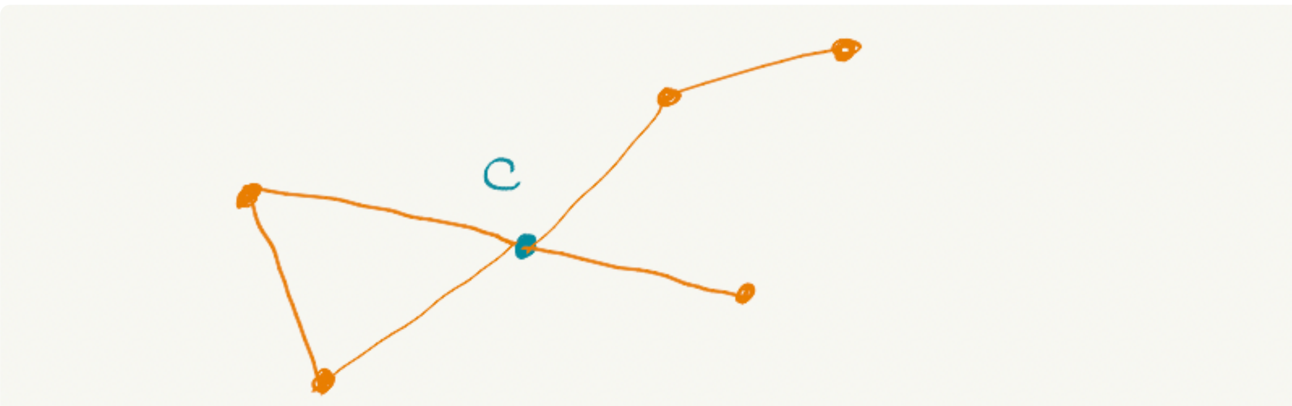
### On a connected graph (with at least 3 vertices) and no cut vertex, every two vertices are on a cycle.

Any two vertices on a cycle $\implies$ no cut vertex

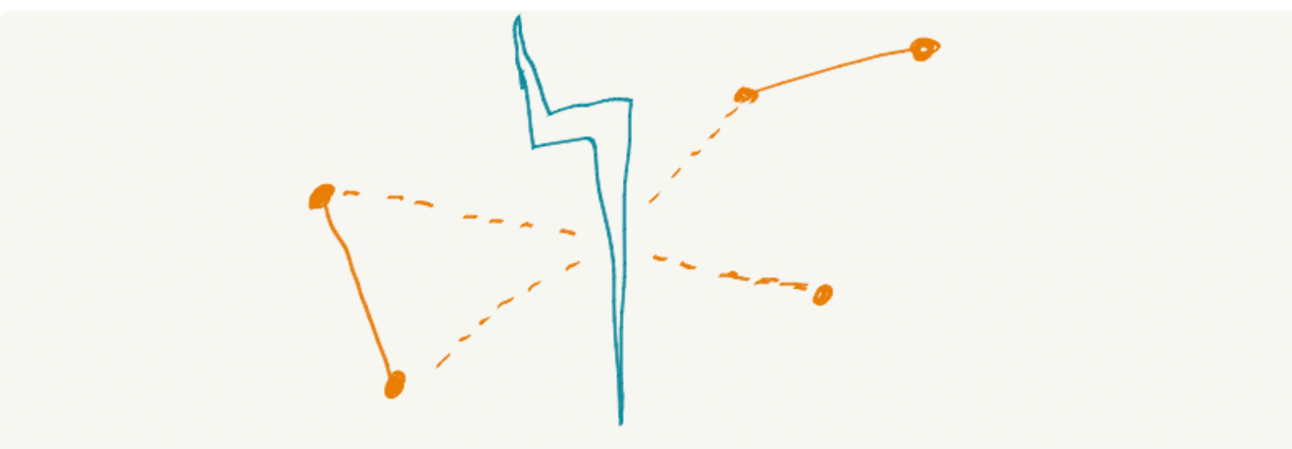
This is the simpler direction, included mostly just for completeness.

We have to prove a “there does not exist” statement, so we switch to proving by contrapositive.

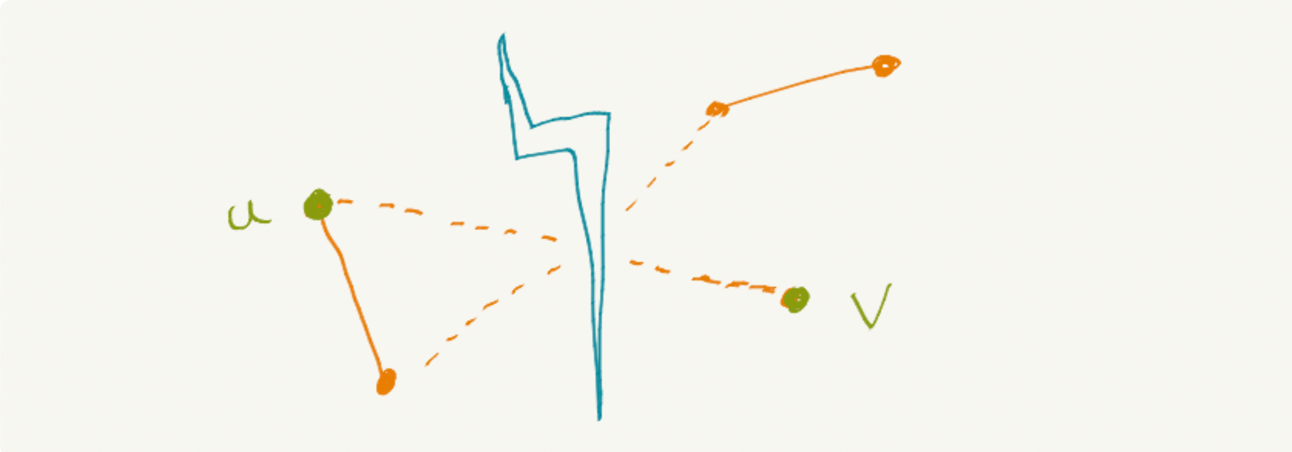
Suppose there was a cut vertex $c$.



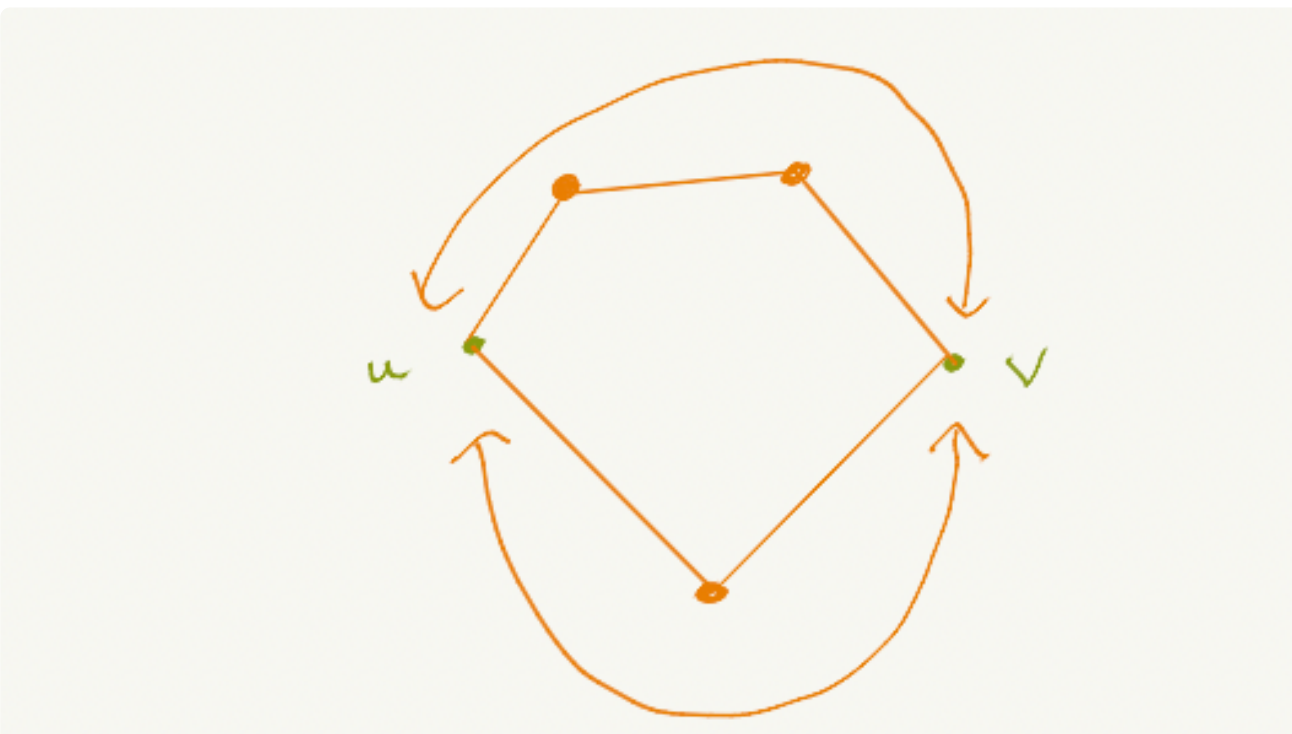

Then, removing it should disconnect the graph into at least two non-empty components.



We need to prove a statement about the existence of two metavariable vertices $u$ and $v$ not on a cycle. So it might make sense to instantiate those two metavariables from two of the components we have (using some sort of weak pattern matching for “two”). So, we have that the cut vertex $c$ splits the two vertices $u$ and $v$ such they are on disconnected components, and thus there is no path between them.




Can $u$ and $v$ have been on a cycle? We can go to **conflict-guided mode** and try to draw out cycles where removing a cut vertex disconnects two parts of the cycle. Indeed, we'll see that can't be the case, because in a cycle, there are always two different, and in fact vertex-disjoint, paths between any two vertices. So, we add a **conflict-inspired lemma: two vertices are on a cycle $\iff$ there are two vertex-disjoint paths between them**.




Now we know $u$ and $v$ cannot be on a cycle (which was what we had to prove), because if they were, there had to be two vertex-disjoint paths between them, which means there was a path between them not involving $c$ . Thus $c$ could not have been a cut vertex, a contradiction.

### No cut vertex $\implies$ any two vertices on a cycle

Let's consider an arbitrary pair of vertices $u$ and $v$.

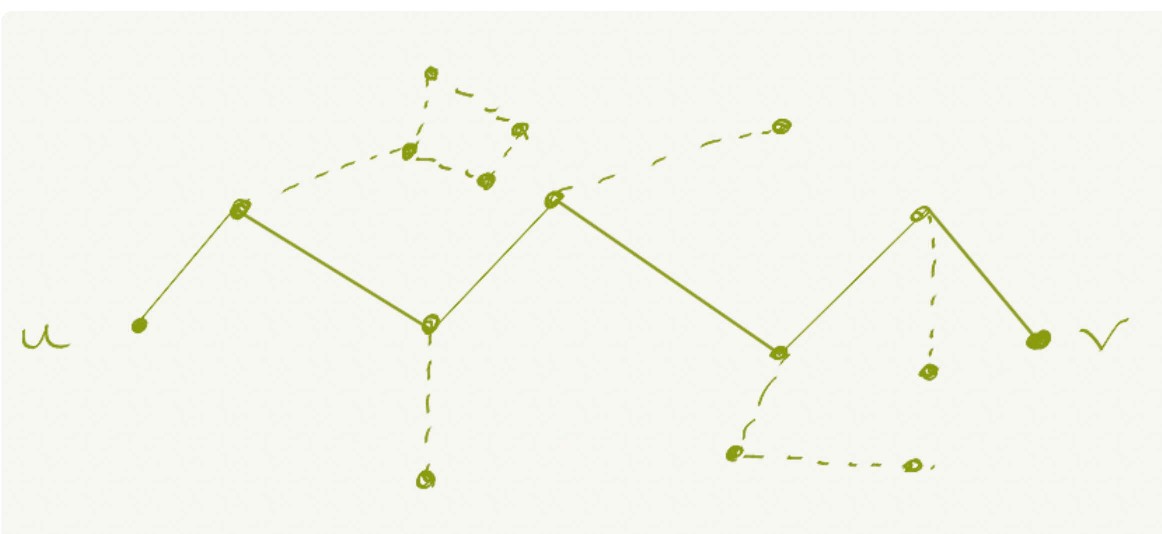
We want to come up with a cycle that contains both of them.

We then notice we don't have any information about cycles in the hypothesis, but we do have information about paths (when we unfold the definition of "cut vertex"). And we have a previous lemma that relates cycles to paths (two vertices are on a cycle $\iff$ there are two disjoint paths between them). So we extract that result from the library, and go about proving that there are two disjoint paths between $u$ and $v$.

Well first, is there a path between $u$ and $v$? Yes, because the graph is connected.


We then try to find if there is a disjoint path between $u$ and $v$, but all of the humans I've noticed try this problem tend not to get this after trying for a bit and end up trying to prove a weaker statement: Is there a different path between $u$ and $v$?

- Here, we switch into **conflict-guided mode, i.e. we try to find counterexamples to the claim, in the hope that all counterexamples will have something in common that we can conjecture, prove, and add as a lemma.**
- You might try to draw examples of graphs where there is only one path between two vertices $u$ and $v$.



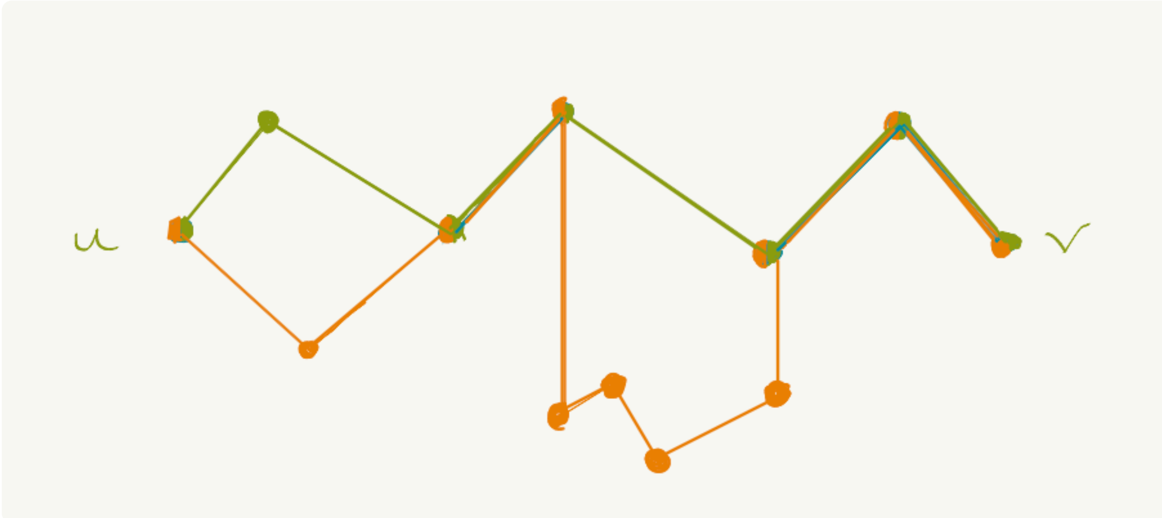

- But then you'll notice that you can't, and in particular, the vertices in between them become cut vertices.

- So then we create a **conflict-inspired lemma: all pairs of vertices must have at least two different paths between them.**
- Now how do we prove it? Well the only thing we have to work with is that there is no cut vertex, and that the conflict was drawn from intermediary vertices between $u$ and $v$.
  - Indeed, if there is only one path between two non-neighbouring vertices then it implies the intermediate vertices on the path between them are cut vertices, a contradiction.
  - But as we do the other part of this case analysis, another conflict comes up: what if the vertices $u$ and $v$ are neighbours? Again, we go into **conflict-guided mode** and draw out a bunch of examples of neighbouring vertices that have only one path between them, and try to get a graph with no cut vertices. Of course, it never happens. If there is only one path between neighbours $u$ and $v$, it seems to imply that at least one of the vertices $u$ or $v$ is a cut vertex. Why? A third vertex $x$ (there are at least three vertices) will then have either only one path to $u$ or only one path to $v$). We create a new **conflict-inspired lemma: at least one vertex of a cut-edge (or bridge) is a cut vertex** . We then prove it by recognizing that either removing $u$ cuts off the $x$-to-$v$-path, or removing $v$ cuts off the $u$-to-$v$-path.

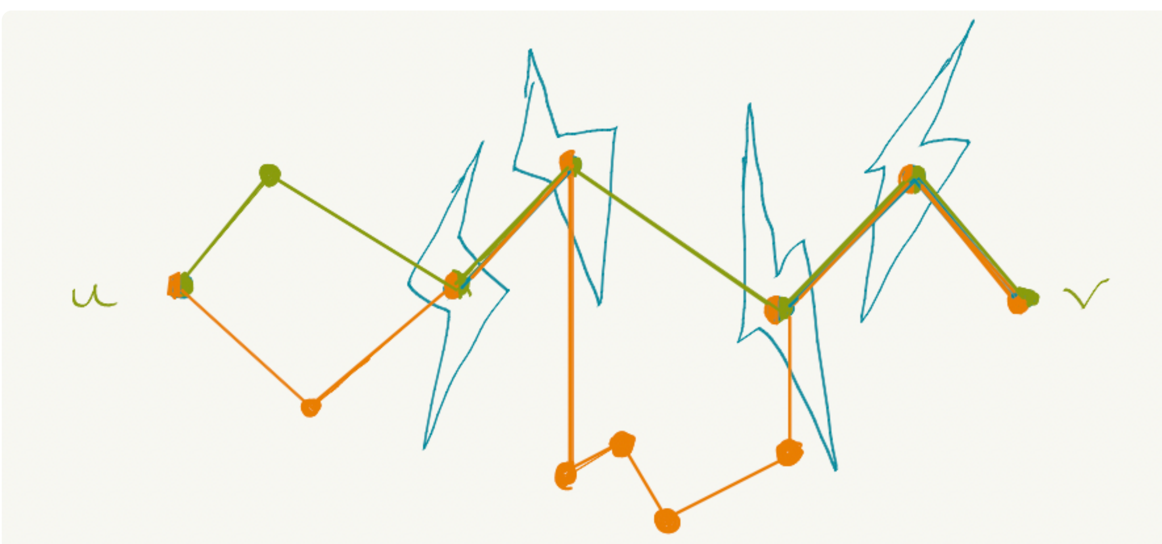
So now we have that there are two *different* paths between $u$ and $v$. But we need something stronger: that there are two *disjoint* paths between $u$ and $v$.

- If the paths do happen to be disjoint, we are done because we have a cycle with $u$ and $v$ (via the earlier lemma).

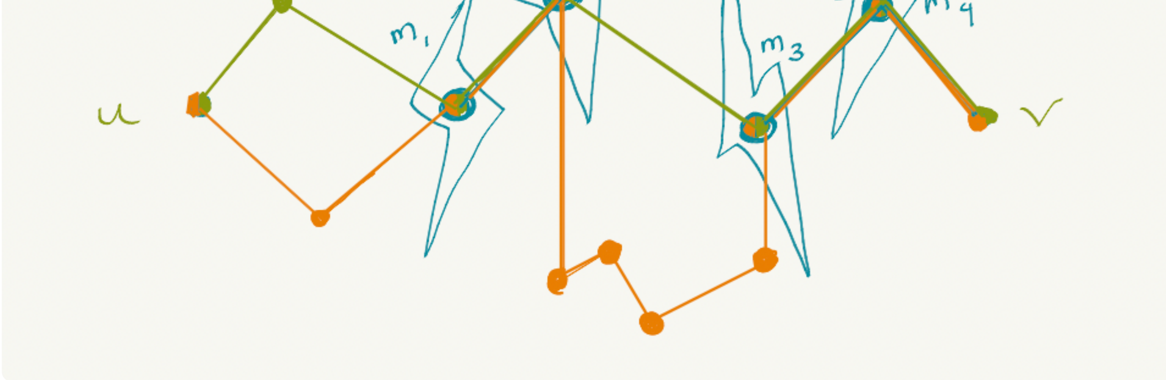
- So let's assume they're not disjoint.



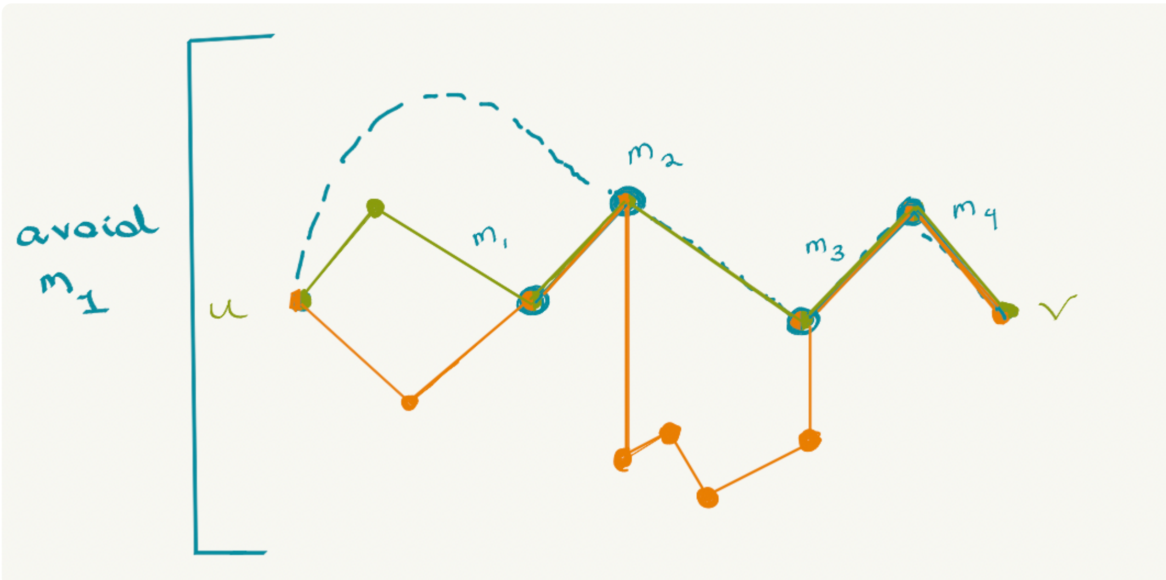

- How do we get $u$ and $v$ on a cycle? Again we switch into **conflict-guided mode** and draw out a bunch of examples of graphs where there are only two paths between $u$ and $v$ such that the paths are not disjoint. And then we try to see — is it possible that these two paths are all there is in the graph? (If so, then there are indeed only two non-disjoint paths between $u$ and $v$, and all our work has been in vain.) Luckily, these two paths can't be all there is in the graph, because we find that all these graphs do have a cut vertex, and in particular, they have a cut vertex at every point where the non-disjoint paths intersect.

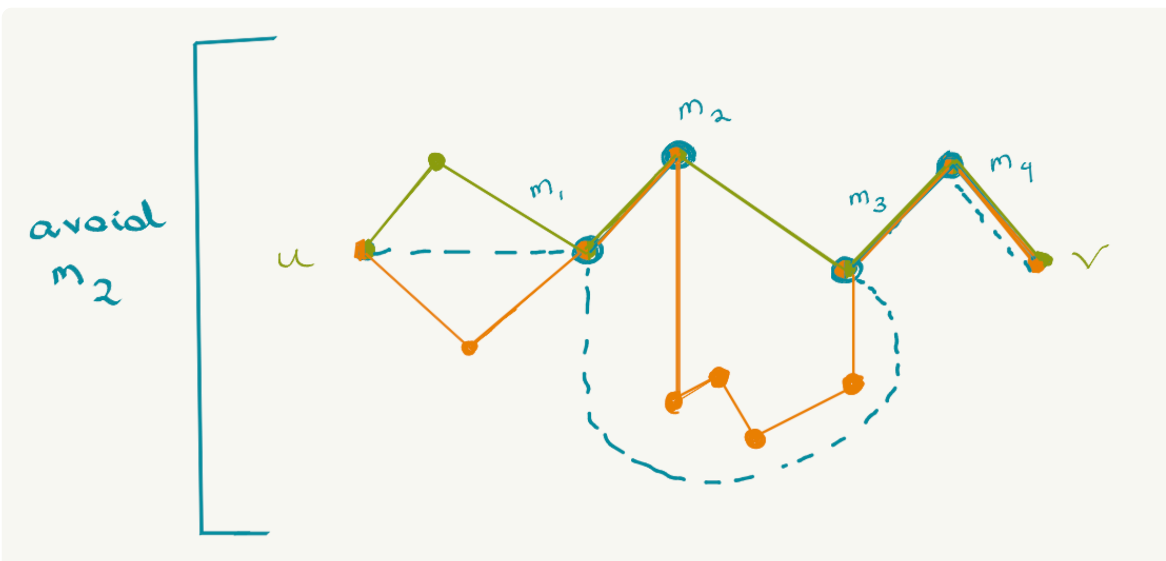


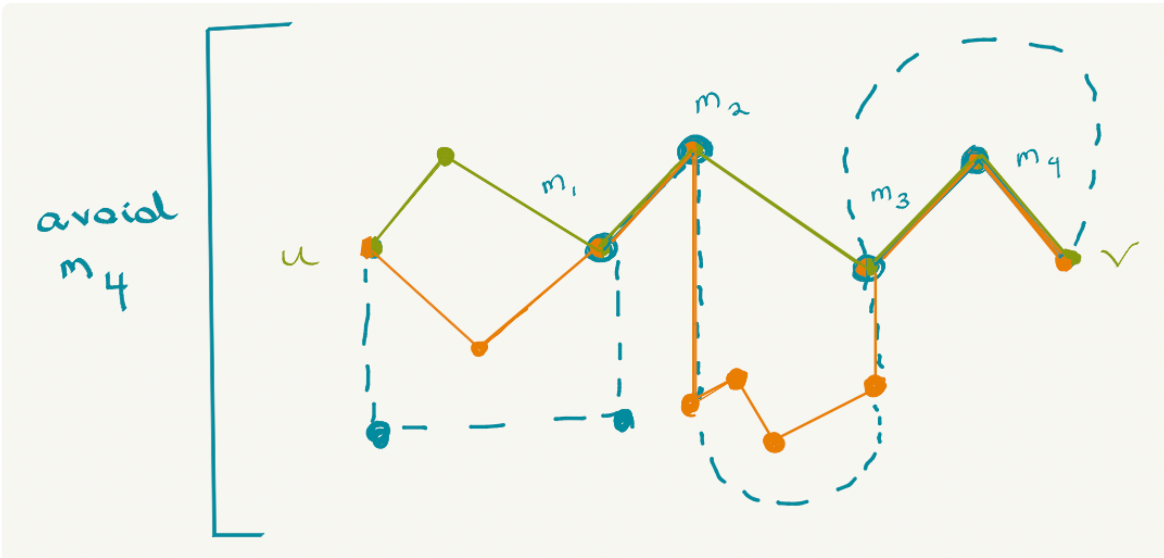

- Call any of those cut vertices $m_i$, with the one closest to $u$ on the initial path being $m_1$. (You can call $m_i$ the $i^{\text{th}}$ cut vertex encountered when traversing the initial path).




- So now we know that we couldn't have discovered all paths going from $u$ to $v$, because there has to be another path that avoids $m_1$, a path $p_2$ that avoids $m_2$, and so on.



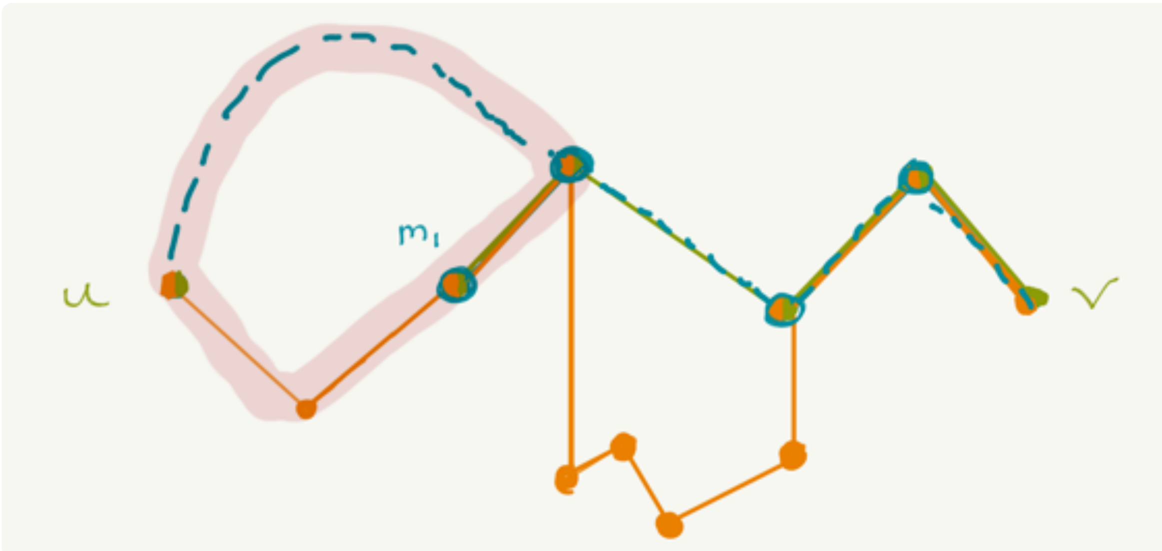




- And so now we have a **conflict-inspired lemma: If two paths between vertices intersect, there is a third path avoiding that intersection point.**

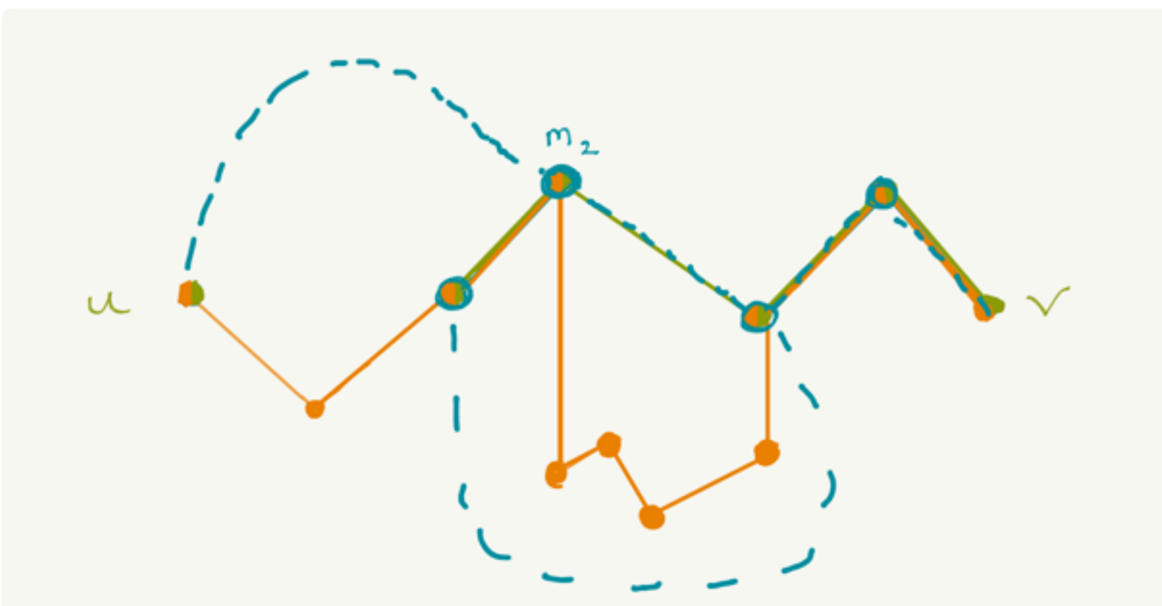
It's around this point that humans usually notice that while they can't immediately prove that there exists a cycle containing both $u$ and $v$ , they can prove that larger and larger cycles exist, until they prove the existence of a cycle containing both $u$ and $v$. How do humans typically incrementally create and enlarge this cycle? ("Enlarge" in this case means creating a cycle containing a point farther along the initial path to $v$). They typically go into **conflict-guided mode** and draw out a bunch of instances of us creating a cycle containing just $u$, and repeatedly enlarge the graph (and thus the cycle) until the graph has no cut vertex (at which point the cycle encompasses $v$). For example…

- Maybe we first use the path avoiding $m_1$ (which is now in the hypothesis as a path that exists)…

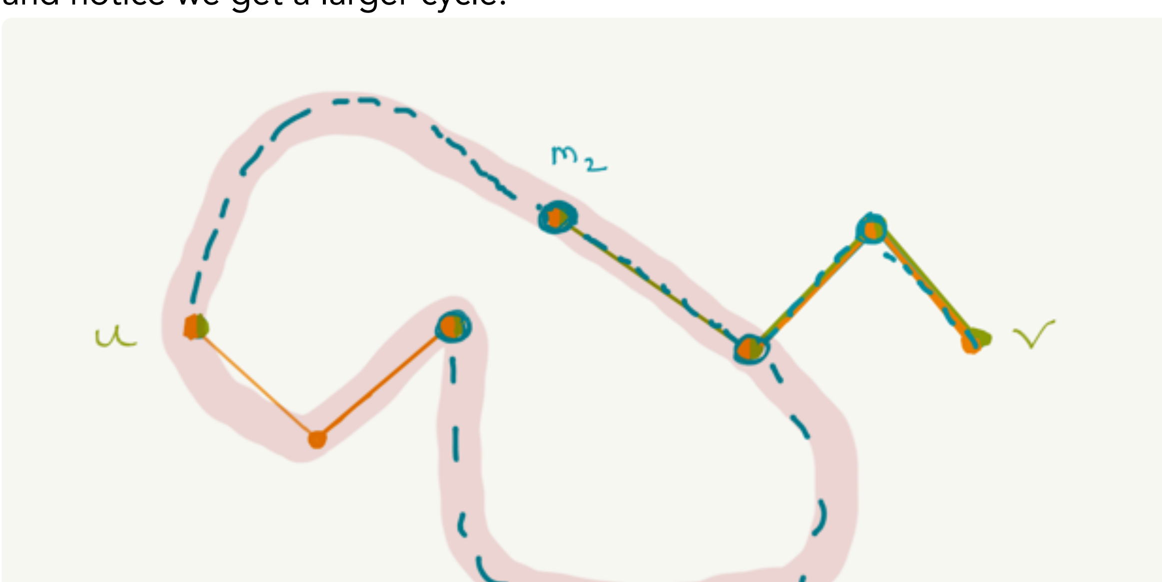
and notice we get a cycle (at this point, the human might not necessarily be too sure of how they ended up with a cycle here, and might not be convinced that they always will, but just notices it).



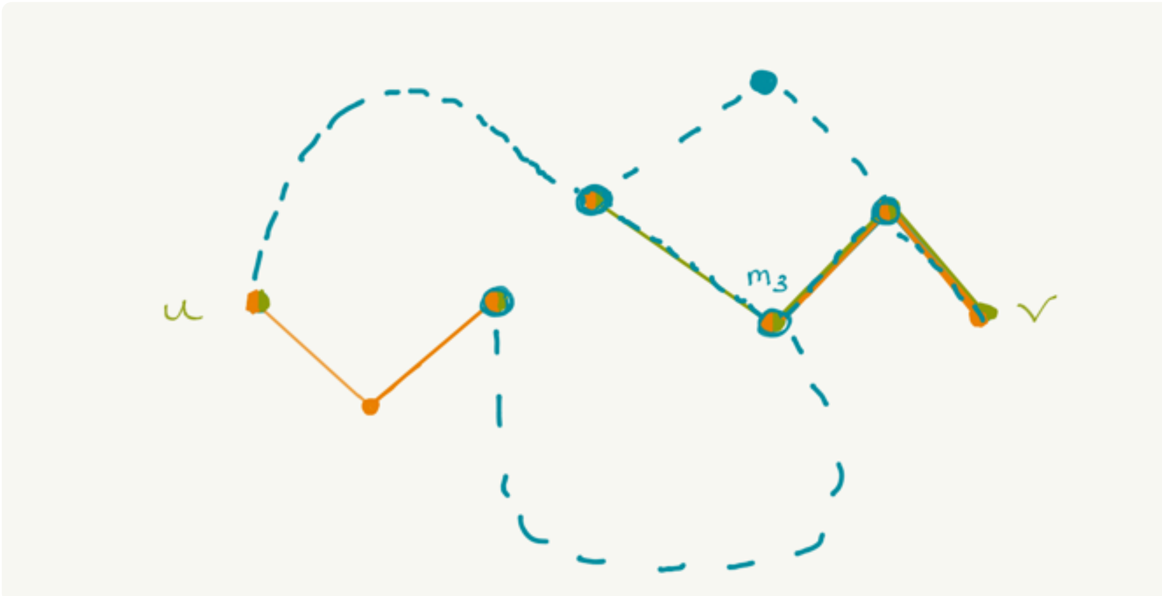

- Then we use the path avoiding $m_2$…



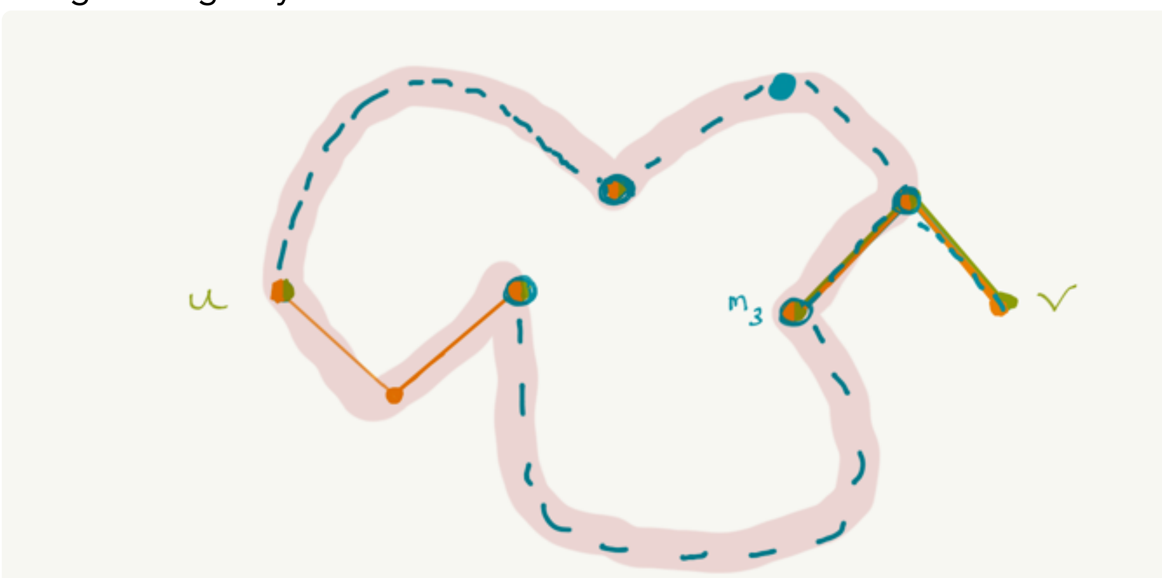

and notice we get a larger cycle.



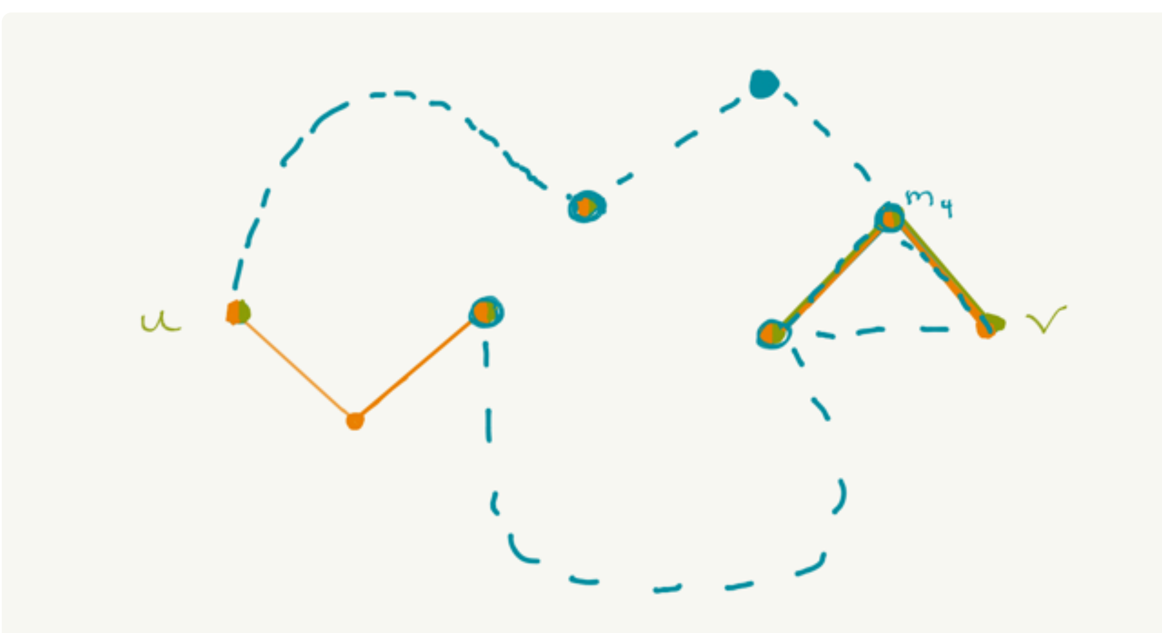
- Then we use the path avoiding $m_3$…



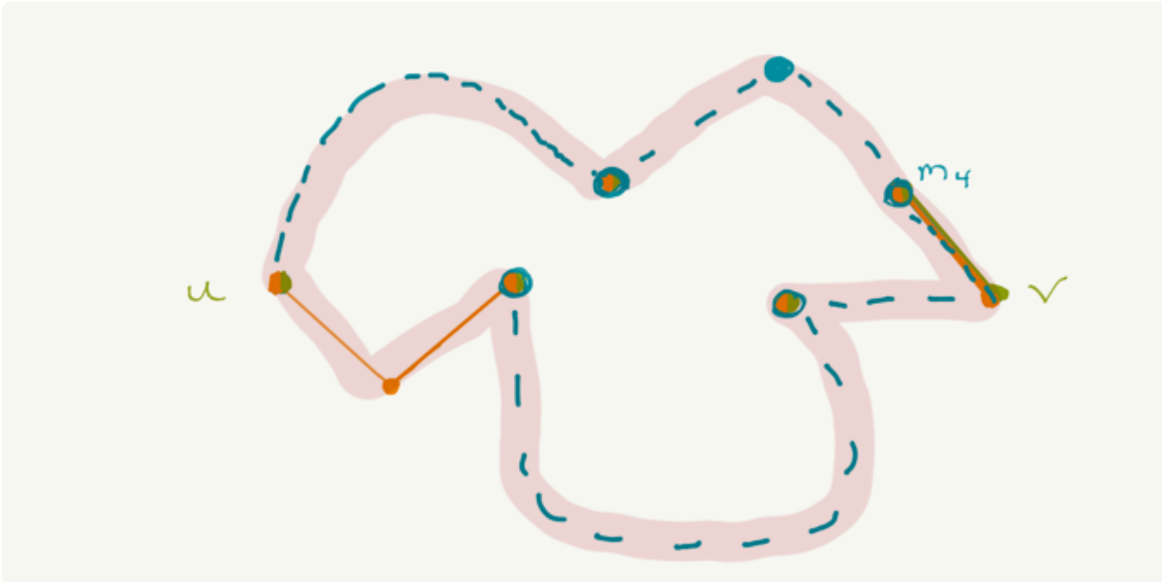

and get a larger cycle.




- Finally we use the path avoiding $m_4$…



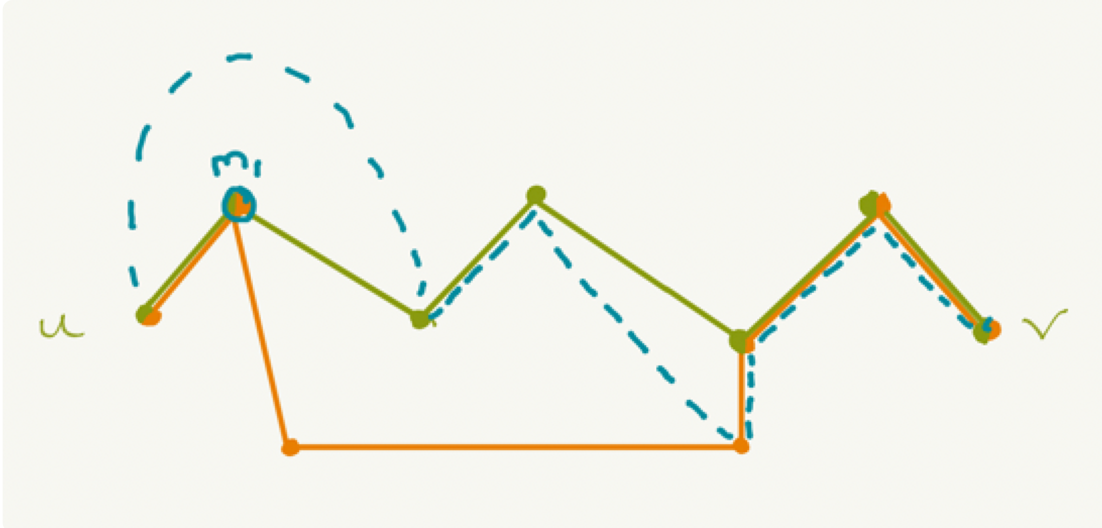
and get the desired cycle.



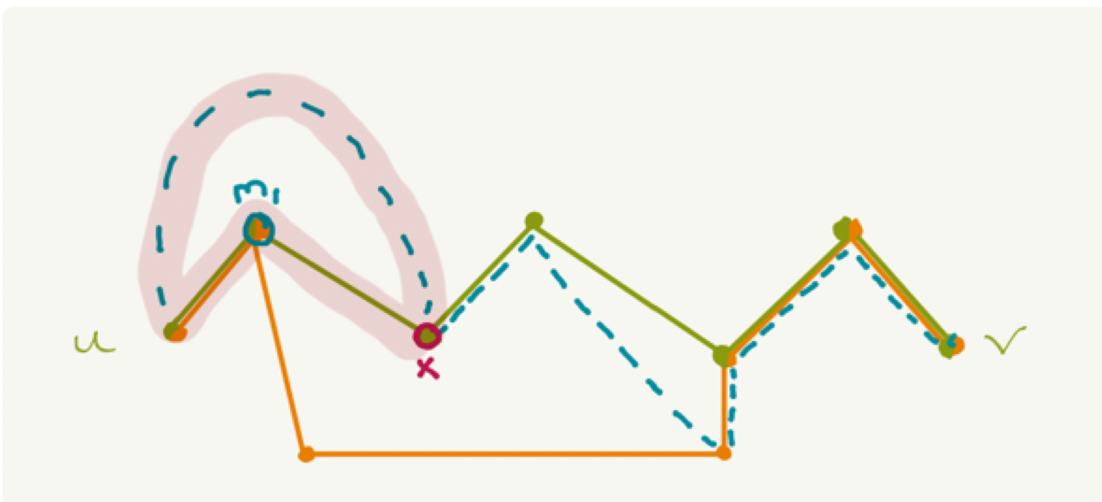

- But how do we formalize and generalize what we did in this one example? Suppose we were to repeat this process of finding cycles containing $u$ and $v$ numerous times. We would notice that typically:
  - We first find two different paths.


  - And then we identify $m_1$ (the first point of intersection along the initial $u$-to-$v$-path, besides $u$).

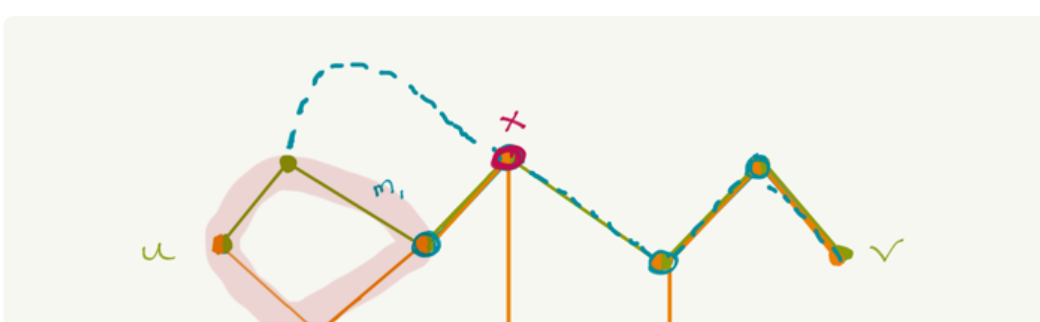

  - We'll find we often start by finding the path connecting $u$ to $v$ that avoids $m_1$. (We find in the previous example, and would in others, that the path that in $n^{th}$ step consistently "enlarges" the cycle containing $u$ is the path avoiding $m_n$, while the others do not. )



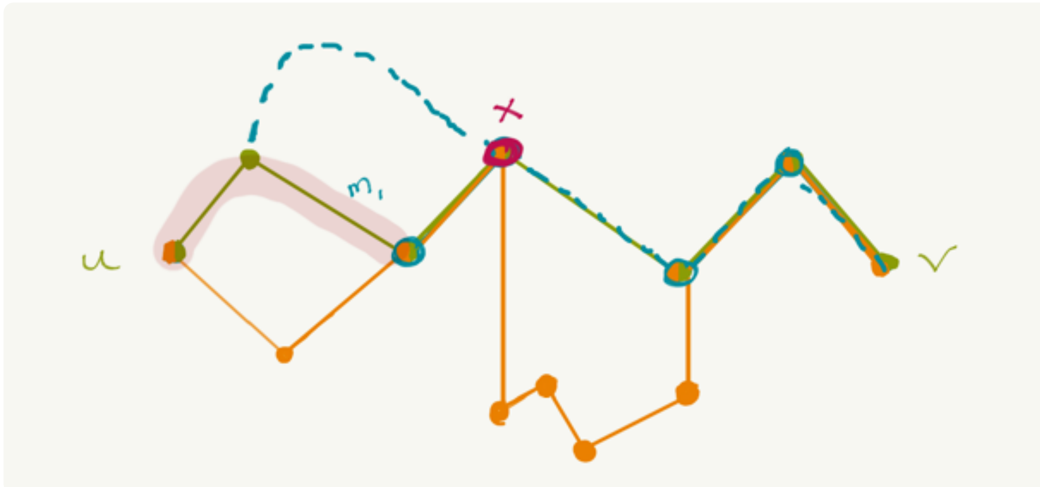

- We notice a particular vertex $x$ always plays a role —where $x$ is the vertex where the new $m_1$-avoidant-path first intersects the initial path from $u$ to $v$. In particular, we consistently notice there is a way to build a cycle containing $u$ and $x$.



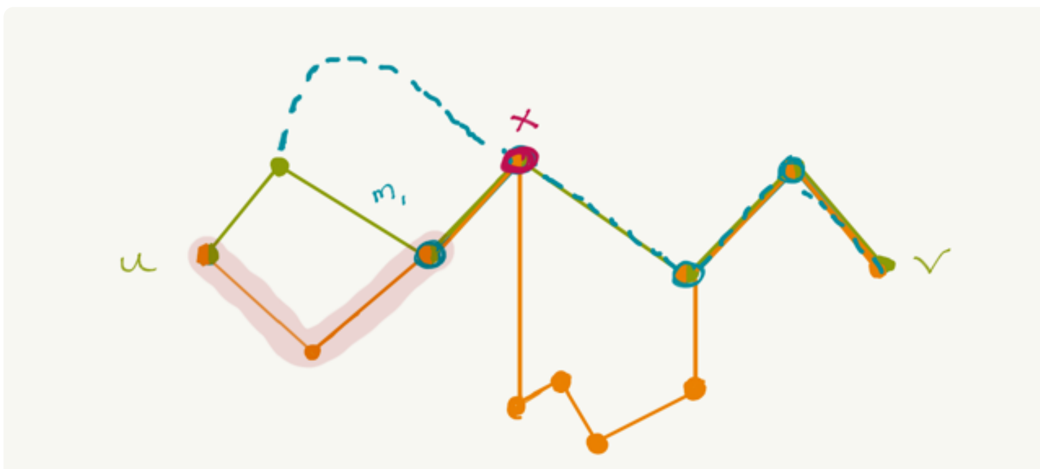

- So we again have a **conflict-inspired lemma: There are two disjoint paths from $u$ to any such $x$** . We can then prove this lemma by noticing that these two paths can be identified by the following process:
  - Suppose $u$ and $m_1$ are already on a cycle (We can go into conflict-driven mode to prove this base case at a later point, but I won't go into details here).
  - Call $c$ the cycle containing $u$ and $m_1$.



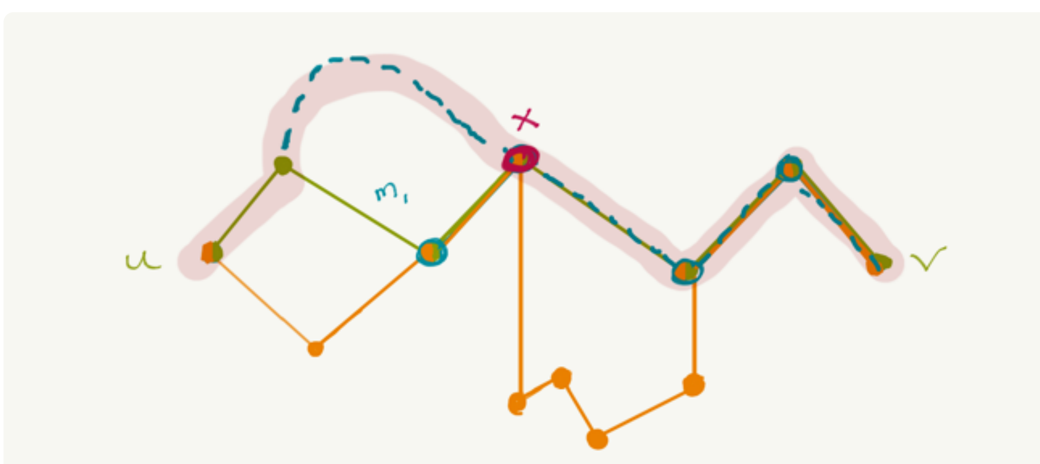
- We know $c$ breaks up into two disjoint paths. Call them $c_1$ …




…and $c_2$ (each connecting $u$ to $m_1$).

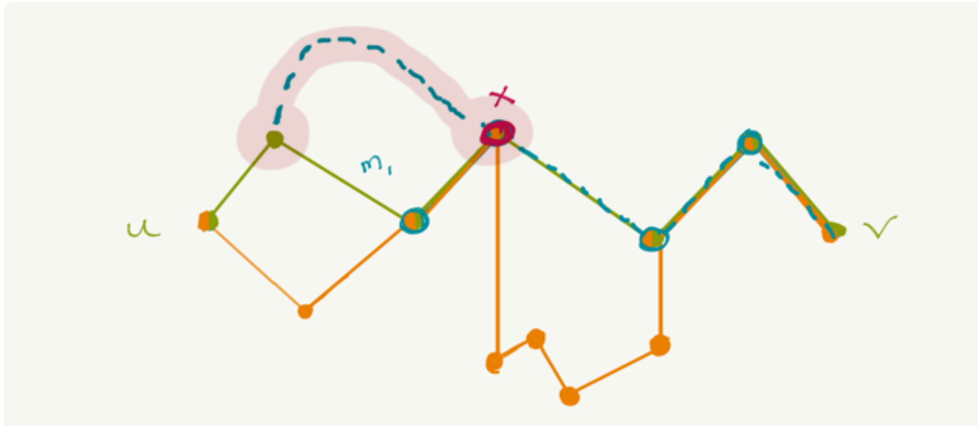



- Call $p$ the path avoiding $m_1$.

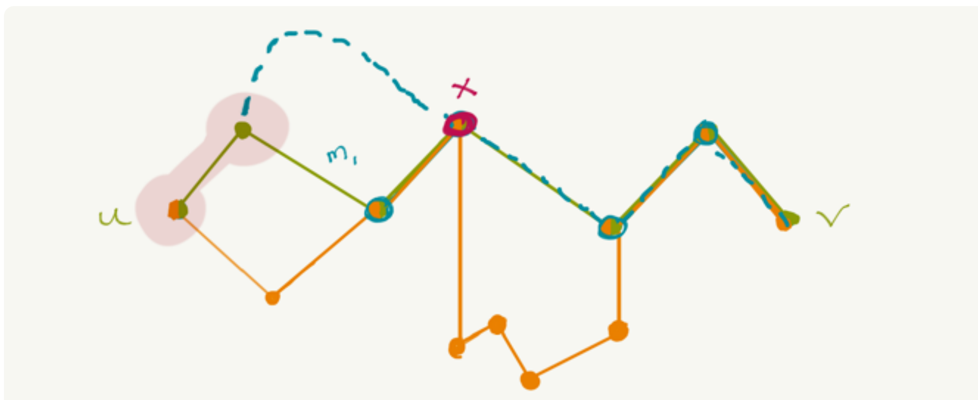



- So we create this new enlarged cycle by concatenating the following four paths:

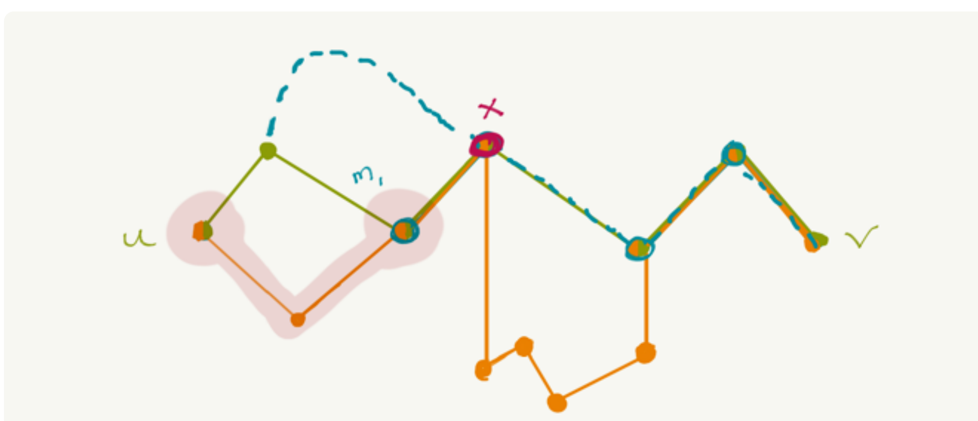
1. the sub-path of $p$ from $x$ to the vertex where it first intersects $c$. Suppose WLOG that this intersects $c$ along $c_1$ rather than $c_2$.



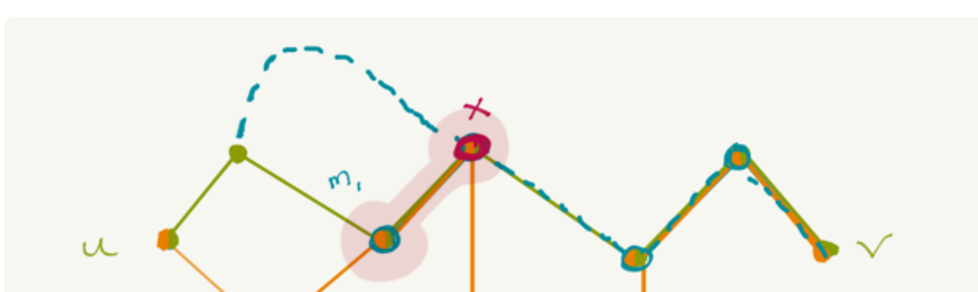

2. the path along $c_1$ from the previous intersection to $u$



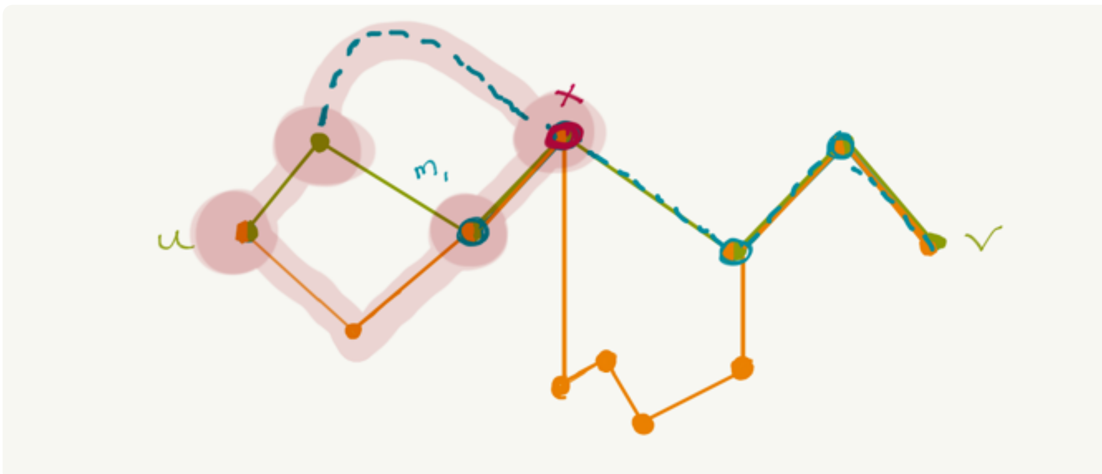

3. the path along $c_2$ from $u$ to $m_1$




4. the path from $m_1$ to $x$ along the initial $u - v$ path (remember both of these vertices were selected to be on that initial $u - v$ path)

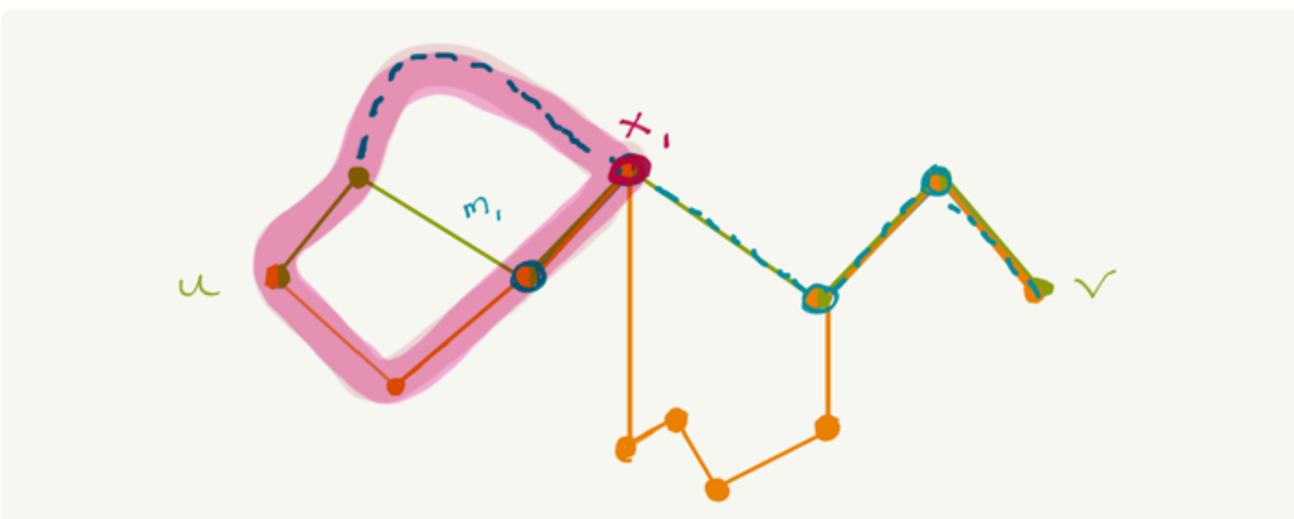


- It just remains to show these four paths are vertex-disjoint except where the paths are concatenated (which follows from the disjointness of $c_1$ and $c_2$, and the construction of $p$). Once we've proven disjointness, we've proven we have a cycle containing $u$ and $x$.

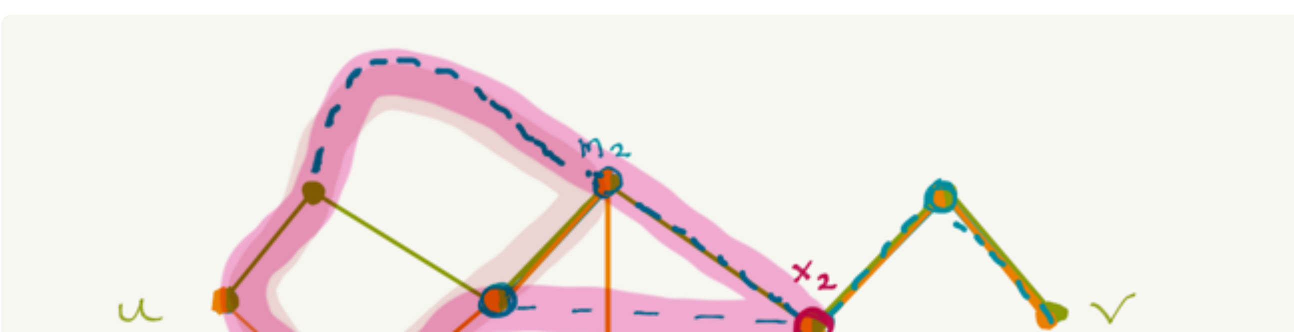



We can then repeatedly apply the above construction, updating $x$ each time. For example:

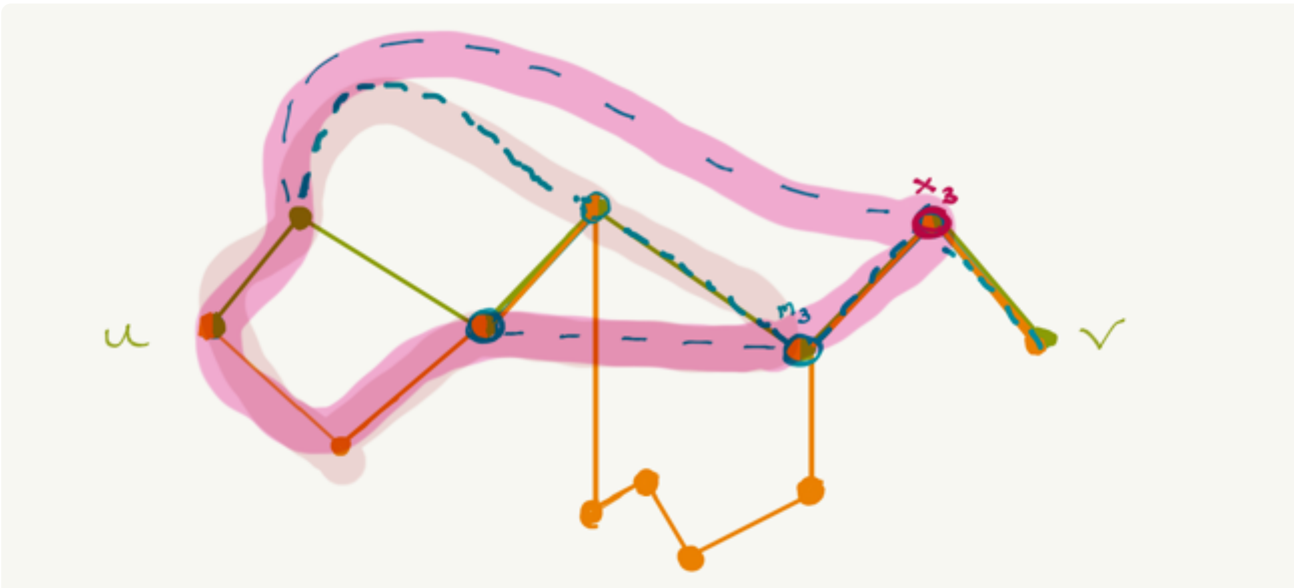
Our $x$ in this example moves from here:



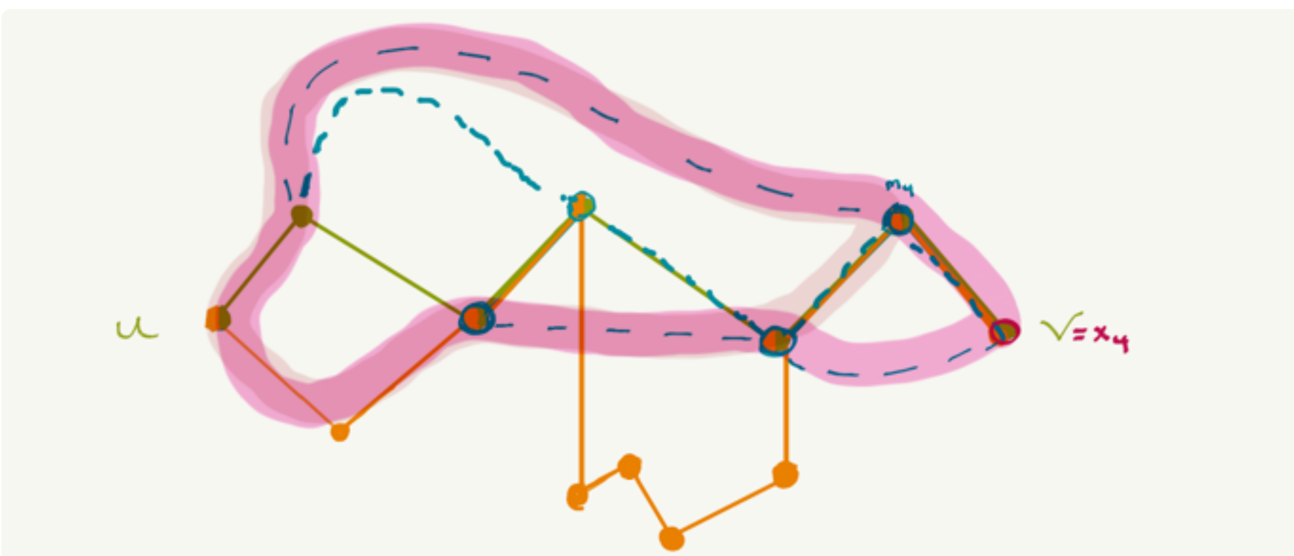

To here:



To here:




And finally to here:




Since we move $x$ a positive length down the initial $u - v$ path each iteration (as a reminder, $x_i$ is the point where the $m_i$-avoidant path first intersects the initial $u - v$ path), and the initial $u - v$ path has finite length, for some $i$, $x_i$ is eventually $v$. To prove this "positive length" lemma, we might again go into **conflict-guided mode**. We try to find examples of $u - v$ paths that avoid $m_i$, but that first intersect the initial $u - v$ path in at a vertex between $u$ and $m_i$ instead of the helpful area between $m_i$ and $v$. Of course, we can't find any, because $m_i$ always ends up being a cut-vertex in any such graph. So we have the **conflict-guided lemma: there exists a $u - v$ path that avoids $m_i$ that first intersects the initial $u - v$ path at a vertex**

**along that initial path between $m_i$ and $v$.** How do we prove it? Well, if such a path does attach between $u$ and $m_i$, yet another such path must exist, because in that new graph, $m_i$ is still a cut-vertex.

And there we have it - a cycle involving $u$ and $v$, as desired.