A Generalized Diagonal Argument (Lawvere's fixed point theorem)

The theorem

Fix sets A and B.

$$\exists f : B \to B \textit{ with no fixed point}$$

$$\implies$$

$$\nexists \phi : A \to (A \to B) \textit{ where } \phi \textit{ is surjective}$$

Applying the theorem: the powerset is bigger than the set

- Let A be any set.
- Let $B = \{0, 1\}$. Now we have:

$$\exists f : \{0,1\} \to \{0,1\} \textit{ with no fixed point}$$

$$\implies$$

$$\nexists \phi : A \to (A \to \{0,1\}) \textit{ where } \phi \textit{ is surjective}$$

- We know there exists a function f from $\{0, 1\}$ to $\{0, 1\}$ with no fixed point (the one that sends 0 to 1, and 1 to 0). So we can conclude

$$\nexists \phi : A \to (A \to B) \textit{ where } \phi \textit{ is surjective}$$

- We also know any function $A \to \{0, 1\}$ corresponds to an element of $\mathcal{P}(A)$. So, we can conclude:

$$\nexists \phi : A \to \mathcal{P}(A) \textit{ where } \phi \textit{ is surjective}$$

- So, the powerset of A is bigger than A. $\square$

Applying the theorem: there are uncountably many infinite binary sequences

- Let A be the set of natural numbers $\mathbb{N}$.

- Let $B = \{0, 1\}$. Now we have:

$$\exists f : \{0, 1\} \to \{0, 1\} \textit{ with no fixed point}$$

$$\Longrightarrow$$

$$\nexists \phi : \mathbb{N} \to (\mathbb{N} \to \{0, 1\}^{\mathbb{N}}) \textit{ where } \phi \textit{ is surjective}$$

- We know there exists a function f that sends a binary sequence to its "flipped" binary sequence (sending 1s to 0s, and 0s to 1s) and it has no fixed point. So we have:

$$\nexists \phi : \mathbb{N} \to (\mathbb{N} \to \{0, 1\}) \textit{ where } \phi \textit{ is surjective}$$

- We also know any function $\mathbb{N} \to \{0, 1\}$ corresponds to a binary-valued sequence.

$$\nexists \phi : \mathbb{N} \to \{0, 1\}^{\mathbb{N}} \textit{ where } \phi \textit{ is surjective}$$

- This is saying there can be no powerful listing function ϕ that lists every infinite binary sequence in a countable list. So, there are an uncountable number of them. $\square$

Interpreting the theorem

The function f always serves as a sort of flipping function (e.g., the function that takes in a binary sequence and flips all the digits).

The function ϕ always serves as some sort of function that is quite powerful, but often we're trying to prove is too powerful to exist (e.g. the function that subjects a set to its powerset, or a countable sequence containing all binary sequences).

So (I think) the theorem says that if you have a set that's so "disordered" you can create a flipping function, it's going to be difficult to reach all of its elements.

Generalizing the theorem

When A and B are sets…Lawvere's theorem seems kind of weak. Because I think that if you have any set B with more than 1 element, (I think…) you'll always be able to find a function with no fixed point.

But Lawvere's theorem is more general.

- Actually A could be any closed category, and B could be any closed **category** (including a different one, possibly?).

- And then, f and ϕ are **morphisms** in that category.

Applying the theorem to topology

While in set theory, the hypothesis about a function existing with no fixed point is almost always satisfied, it's not so in topology.

Let A be any topological space.

Let B be the unit disk (where we know that any continuous map from the unit disk to itself must have a fixed point).

In the category of topological spaces, the morphisms are continuous functions. So...

$$\exists \textit{ continuous } f : B \to B \textit{ with no fixed point}$$

$$\implies$$

$$\nexists \phi : A \to (A \to B) \textit{ where } \phi \textit{ is surjective}$$

But, such a function does not exist, so, we can't use this theorem to conclude anything. So, Lawvere's fixed point theorem doesn't trivially apply to any topological space.

Applying the theorem to vector spaces

Similarly, let A and B be the category of vector spaces, where morphisms are linear maps.

Linear maps always fix 0. So again, the hypothesis of the theorem is never satisfied, and so I imagine the Lawvere's fixed point theorem isn't very useful in this domain.

Applying the theorem to metric spaces

Now, let A and B be the category of metric spaces. Then, its morphisms are "nonexpanding maps", that is, continuous functions that do not increase pairwise distance between points in the space.

Then, I was thinking that the contraction mapping theorem might be relevant. By the contraction mapping theorem (a.k.a. Banach fixed-point theorem), if you have a map from a metric space to itself that, roughly, "decreases" distance between points (e.g. it can't be a translation), it must have a fixed point.

I was hoping that there might be a way to apply this to convergence of sequences...

- like in the proof that the infinite binary sequences are compact, or
- like in the proof that the set of subsequential limits of a sequence is closed.

I was thinking…maybe the limits could be the “fixed points” or something like that…

But I found I couldn’t figure out what the “flipping function” with the fixed point is for these examples. Nor could I figure out what the “surjective function” is.

So I was led to the following conclusion…

There are two different diagonalization arguments

I think there are two quite different arguments that people often just refer to as “diagonalization” arguments.

One of them is for counting:

- The vague sense of this argument is “we look at elements along a diagonal…we **flip** those elements…and we get an some element that appears nowhere on this list.”

One of them is for a convergence:

- The vague sense of this argument is “we look at elements along a diagonal…we **keep them the same**… and we get a sequence with the desirable properties.”

A friend suggested to me that **one of the strongest arguments for the non-equivalence of these two diagonalization techniques is precisely that — the continuous diagonalization arguments have no “flipping” or “negating” aspect to them.**. So I think these may warrant their own (perhaps related) generalization.

Valentin also suggested another distinction: **discrete diagonalization** is often used to refute the existence of an object, and as such is **universally quantified**, whereas **continuous diagonalization** is used to construct an object, and as such is **existentially quantified**. Valentin mentions that humans think about these two quantifiers quite differently, so it may make sense to treat these as different proof techniques.