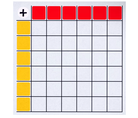
(Proof) Fill-in-the-dice

The Problem

This problem and illustrations are taken (and slightly modified) from a TED-ed video.

The problem is to place these 12 dice according to the following rules:

> Place one dice on the board so that the sum of the same possibilities on a character gamer of added dice.
>
> 1. On one die, no two pips can have a number greater than 6.
>
> 2. Every face of dice plus each face's opposite, totals number of dots.



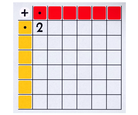
The Solution

Both the video creator and myself went about the solution in the same way... suggesting to me there's something "natural" about this approach.

We realize we have 36 possibilities to fill in...

...and 36 sums to achieve...

And then we notice we are restricted in some ways – we need to set a 2 exactly once, so we need a 1 on each die.

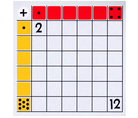


And then symmetry pushes us to find the other extreme...is there only one way to make a 12?

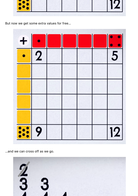
Yes, we realize, for example we could have either an 8 and 4, or a 9 and 3, although I'm choosing to ignore the 3 for now. Symmetry pushes us to place the 12 on the opposite side given a 1, and we're not there fully using the flexibility of the problem – that one die can be numbered up to 6.

So, we have the **unique move** of making compromising that one of the faces is a 4, and **3 faces** there has to be exactly one 9 on the other die.

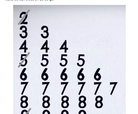
But now we get some extra values for free...



...and we can cross off as we go.



At this point I thought about adding another 1 or another 6 to the low valued die, but quickly realized that reasoning that would need to be more than one way to make a 2 or 10, and that **becomes** impossible only on 1 and one other die.



At this point, I felt stuck. So, I made another **unique move** — putting two 3s and two 2s on the low die. It worked out...

But I was suspicious that this unique move worked out so far so – could it have been better? Well, I felt I had to explore that. Unfortunately, there's a lot of possibilities (that is, places where I could hop back to if I reached a contradiction) – and I already created one branch by guessing there was a 4 on the low die.

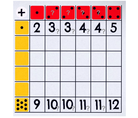
So I tried to change things – could there be just one 2 and three 3s? Yeilded — that would mean there are three ways to make an 11. So, I learned that there can be at most two 2s on the low die, and therefore at least two 3s.

But then could there be three 2s? Again, I felt/felt because that would mean there are three ways to make a 3. So I learned there are exactly two 3s and exactly two 2s on the low die.

Ubaa — this proof follows another proof heuristic we noticed – if you're trying to decide two things at once e.g. numbers on one die and numbers on another, it makes sense to start with the more restricted object, in this case the low die. This is what I'm been doing with this point in the proof, but I think I didn't realize that until later.

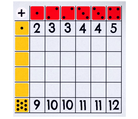
So now the low die is completely determined, which should make continuing on a bit easier. The remaining values to fill in are:

Could there be a 12? Well there has to be at least one, otherwise there is no other way to make the remaining 9. And there can't be two 3s, or else there are too many 9s, so we have figured another value...



Symmetry encourages us to do the same thing on the other side. (here can be only one then 10, so there can't be a 1, and it has there has to be exactly one 9.

And now we can update our list of values.



Now the only remaining numbers to add to the die are 8 or 5.

- Could there be two 5s? No, we felt/realized that gives us too many 5s, and there must be one on the low die.
- Could there be two 8s? No, we felt/realized that gives us too many 8s, and there must be at most one 5.

So now we know there is exactly one 5 and exactly one 8.

And now, the final have-been-set:

Uniqueness

It does turn out this is the only way. Geeslens the nerdmet 1-2-3-4-5-6 ways of creating this he asked dice with only positive integers to end up with the same probability distribution as normal dice.

That is, the restriction on it being the max value on one side wasn't even necessary, the problem gave's decision to add an extra hypothesis just helped reduce the search space.

It seems without being given this restriction, you might need to either use

- more forms, or
- a proof involving generating functions and cyclotomic polynomials. I haven't looked into this proof enough to see if "learning from failure" also helps guide this proof search.