(Proof) In a metric space, sequentially compact $\implies$ compact.

(Actually, in a metric space, sequentially compact $\iff$ compact, but we just focus on one direction here.)

This proof breaks into multiple parts. We need to prove that given a metric space X:

- X is sequentially compact $\implies$ X is totally bounded
- X is sequentially compact and totally bounded $\implies$ X is compact

(Note that the use of the intermediary idea of "total boundedness" was inspired by backwards-reasoning from hypothesis, as detailed here. The following proof will just assume we've already discovered that intermediary idea.)

Proof

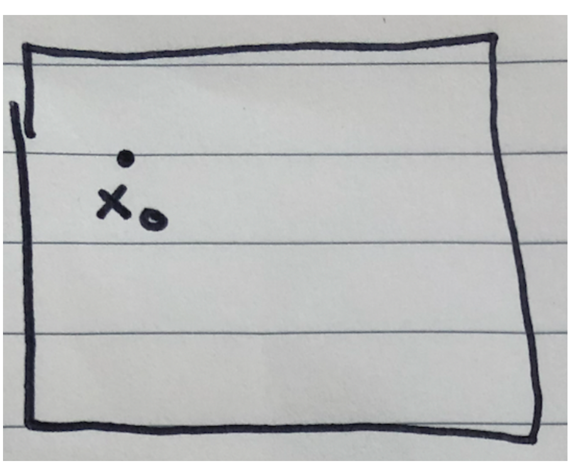
We prove that sequentially compact $\implies$ totally bounded.

Proof Idea

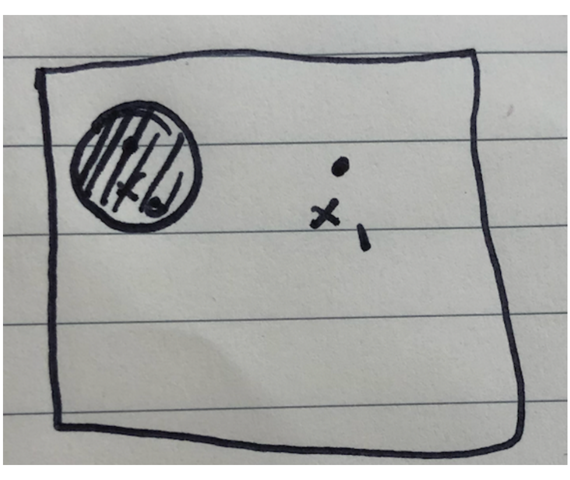
Let's prove the contrapositive: in a metric space that is not totally bounded, there exists a sequence with no convergent subsequence.

Because the space is not totally bounded, there exists some ϵ such that no finite cover with balls of radius ϵ exists.

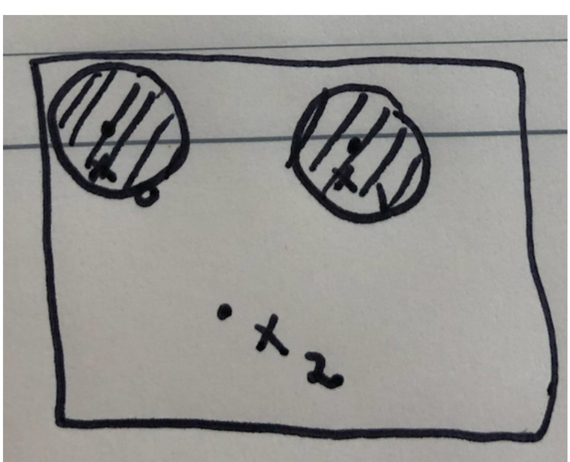
So create a sequence where the idea is to pick a point…



…exclude a ball of radius ϵ around it, and pick another point from the new allowable set…



…exclude a ball of radius ϵ around that, pick another point from the now smaller allowable set, and so on.



Why does that sequence exist?

- At any point in the sequence, you know the allowable set is nonempty (because you've only covered the set with finitely many balls of radius ϵ). So, you can definitely construct a sequence this way.

Why does that sequence have no convergent subsequence?

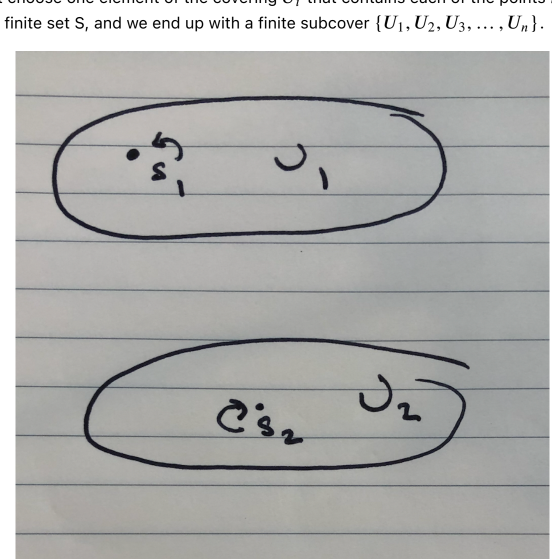
- Every point is at least epsilon away from every other point.

Then we prove that sequentially compact (and therefore totally bounded) $\implies$ compact.

So let's consider an arbitrary open cover. How can we select finitely many sets in that cover to still cover the space?

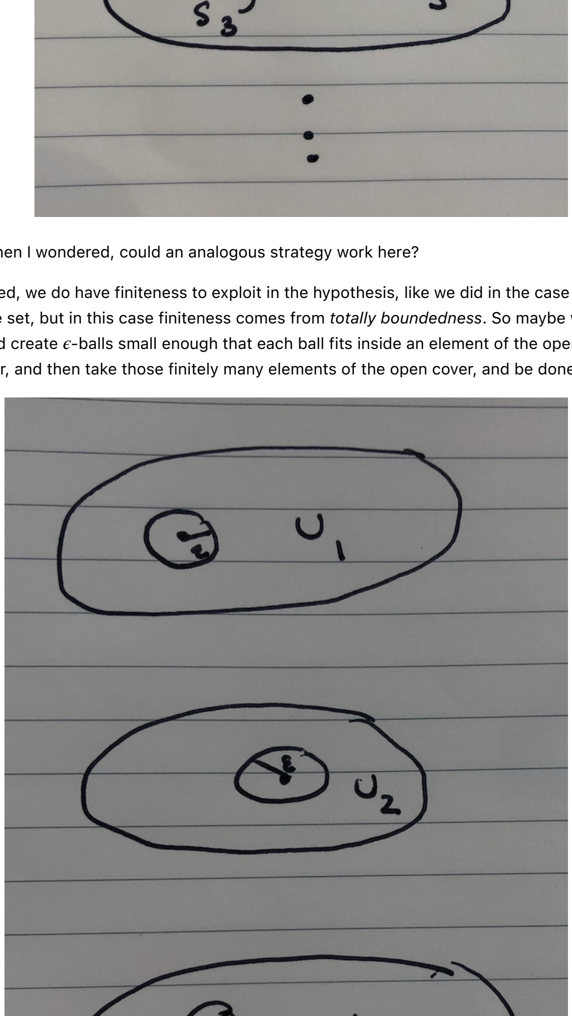
So I thought: how do we, in other instances, prove things are compact?

- I remembered that to prove a finite set $S = \{s_1, s_2, s_3, \dots, s_n\}$ is compact, we just choose one element of the covering U_i that contains each of the points s_i in the finite set S, and we end up with a finite subcover $\{U_1, U_2, U_3, \dots, U_n\}$.



So then I wondered, could an analogous strategy work here?

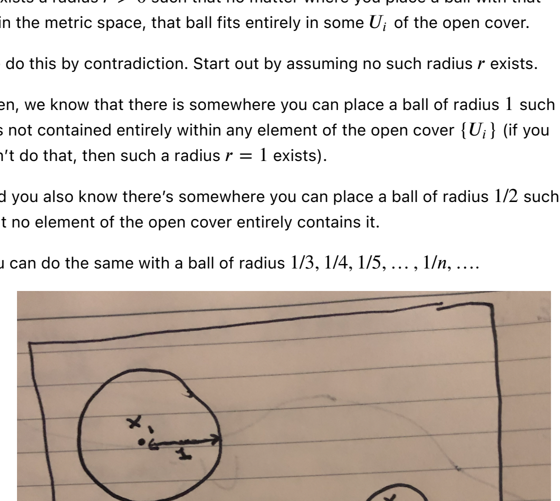
Indeed, we do have finiteness to exploit in the hypothesis, like we did in the case of a finite set, but in this case finiteness comes from *totally boundedness*. So maybe we could create ϵ-balls small enough that each ball fits inside an element of the open cover, and then take those finitely many elements of the open cover, and be done!



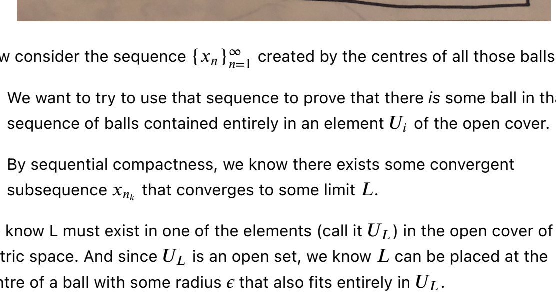
So it remains to prove that there is some ϵ small enough that each ball can fit inside an element of the open cover.

Formally, we want to prove that given any open covering $\{U_i\}$ of the metric space, there exists a radius $r > 0$ such that no matter where you place a ball with that radius in the metric space, that ball fits entirely in some U_i of the open cover.

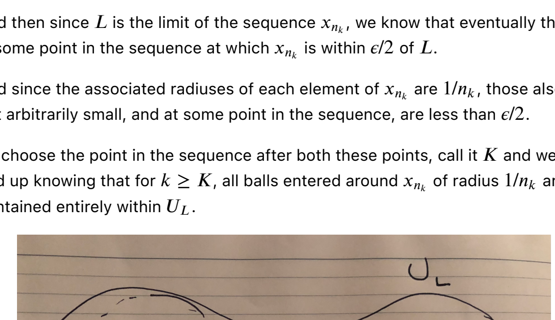
- We do this by contradiction. Start out by assuming no such radius r exists.
- Then, we know that there is somewhere you can place a ball of radius 1 such that it is not contained entirely within any element of the open cover $\{U_i\}$ (if you can't do that, then such a radius $r = 1$ exists).
- And you also know there's somewhere you can place a ball of radius $1/2$ such that no element of the open cover entirely contains it.
- You can do the same with a ball of radius $1/3, 1/4, 1/5, \dots, 1/n, \dots$.



- Now consider the sequence $\{x_n\}_{n=1}^{\infty}$ created by the centres of all those balls.
  - We want to try to use that sequence to prove that there *is* some ball in that sequence of balls contained entirely in an element U_i of the open cover.
  - By sequential compactness, we know there exists some convergent subsequence x_{n_k} that converges to some limit L.
- We know L must exist in one of the elements (call it U_L) in the open cover of the metric space. And since U_L is an open set, we know L can be placed at the centre of a ball with some radius ϵ that also fits entirely in U_L.



- And then since L is the limit of the sequence x_{n_k}, we know that eventually there is some point in the sequence at which x_{n_k} is within $\epsilon/2$ of L.
- And since the associated radiuses of each element of x_{n_k} are $1/n_k$, those also get arbitrarily small, and at some point in the sequence, are less than $\epsilon/2$.
- So choose the point in the sequence after both these points, call it K and we'll end up knowing that for $k \geq K$, all balls entered around x_{n_k} of radius $1/n_k$ are contained entirely within U_L.



- But we constructed these balls to be balls that never fit entirely within any element of the open cover.
- Contradiction.