(Proof) Either a graph or its complement is connected.

Tim initially motivated this proof — here I go through the proof with an eye towards highlighting how strengthening the statement is helpful.

The Proof

We want to prove that if G is not connected, then the complement of G is connected.

So, we know we have two vertices u and v in G with no path between them.

Here comes the main idea: we know there's no path at all in G between u and v, but let's **weaken that assumption** (and therefore **strengthen the statement**):

- instead of assuming there is *no path of any length* between u and v…
- …let's assume there is *no path of length 1 or 2* between u and v.

This weakening, as of now, is completely unmotivated. (Another way to do this proof involves strengthening the target instead, from *G is connected* to *G is connected with diameter at most 3*. But I just focus on this weakening here.)

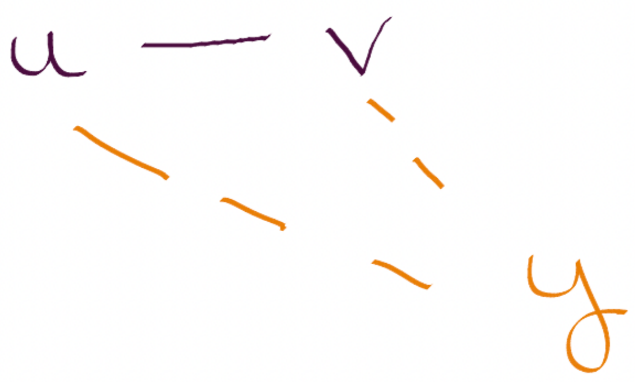
Now, we know that for any vertex in G, it may be a neighbour to u in G, and it may be a neighbour of v in G, but it can't be a neighbour of both u and v in G (or else there will be a path of length 2 in G between u and v).

So, now we want to prove that in G^c, any two vertices x and y are connected by a path.

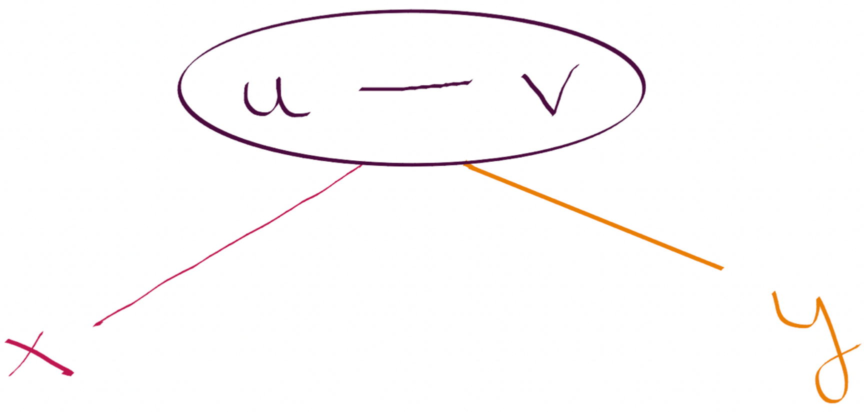
- We know u and v aren't neighbours in G, so they are neighbours in G^c.

- We also notice that x can't be neighbours with both u and v in G, so it's non-neighbours with at least one of them in G, so it's neighbours with at least one of them in G^c.

- Likewise, y is neighbours with at least one of u and v in G^c.



- So, x connects to the u-v-connected-component, which connects to y.



So, x and y are connected in G^c. □