Why is backwards-reasoning from the hypothesis helpful?

Definition

First, a quick primer on what exactly "backwards reasoning from the hypothesis" means:

- This means we are trying to find conditions that imply the hypothesis.
- We usually do this by negating the hypothesis and seeing what happens. This works because of an implication being equivalent to its contrapositive: we can reason backward from the hypothesis (find P such that P $\implies$ Q) by reasoning forward from the negation of the hypothesis (finding P such that ¬Q $\implies$ ¬P).

A case study

Consider the following problem: In metric spaces, sequential compactness $\iff$ compactness.

Tim noticed that a useful technique here is reasoning backwards from hypothesis, or, equivalently, negating the hypothesis and seeing what follows.

So I applied that in proving the theorem myself, and this is what happened.

The tricky part, Tim points out, is spotting the relevance of the useful intermediary properties: completeness and totally boundedness.

- How do we decide "totally boundedness" is relevant (a helpful intermediary step for proving sequential compactness $\iff$ compactness)?
    - So you want to prove sequentially compact $\implies$ compact.
    - That is, you have an open cover, and you need to use a sequence to find a finite subcover.
    - So, we want to explore what's so special about sequences in non-sequentially-compact spaces . (That is, we **negate the hypothesis and see what follows**).
        - So, let's try to find a sequence with no convergent subsequence.
        - This could be the sequence (1,2,3,4…) in the reals.
        - Well, so it seems that if we find a sequence such that there exists an ϵ every term in the sequence is ϵ apart, we definitely won't get a convergent sequence.
        - That is, if we are in a non-compact space, it seems the thing that causes the sequence to have no convergent subsequence is that it might take infinitely many balls of radius ϵ to cover the space.
        - So, we come up with the condition of "totally boundedness" — for all $\epsilon > 0$, we only have finitely many radius-$\epsilon > 0$ balls covering the space.
- And now it's natural to ask…is that "totally boundedness" the only feature of a space that is needed to make it compact? In answering this, we ultimately come up with the condition of completeness. In detail…
- How do we decide "completeness" is relevant (a helpful intermediary step for proving compactness $\implies$ sequential compactness)?
    - In a space that is "totally bounded", we wonder…does every sequence has a convergent subsequence?
    - Well, we know that in a "totally bounded" space, every sequence eventually ends up in an arbitrarily small ϵ ball…but we realize that this doesn't mean every sequence converges (e.g. consider (0,1) which satisfies this condition). We realize that the closest we can get is that every sequence has a Cauchy subsequence.
    - So, if we have that every sequence has a Cauchy subsequence, and we want to prove that every sequence has a convergent subsequence, it is natural to try to prove the statement "every Cauchy sequence converges."
    - And so, the condition for completeness comes in.
    - (N.B. This completeness condition emerges while proving the direction sequential compactness $\implies$ compactness. However, completeness isn't strictly necessary for proving that direction…but it is quite helpful in proving that compactness $\implies$ sequential compactness. And so, we might consider (as humans do) transferring lemmas used in the proof of one theorem for use in proof of the converse.)

Summary of the case study

So what did we actually do to solve the above problem?

- Our hypothesis is that the metric space is "sequentially compact", that is "every sequence has a convergent subsequence."
- We then reason backward from that. That is, we want to find X such that X $\implies$ sequentially compact.
- So, we can do this by looking at the contrapositive: ¬ sequentially compact $\implies$ ¬ X.
- We find that a space that is not sequentially compact (might) imply there is a sequence with no Cauchy subsequence.
- But then we realize that implication is not fully true — rather, a space that is not sequentially compact $\implies$ there is a sequence with no Cauchy subsequence, *or*, the space is not complete.
- When we take the contrapositive, we find that totally bounded & complete $\implies$ sequentially compact. And in fact, compact $\iff$ totally bounded and complete $\iff$ sequentially compact.

And so, finding the conditions that implied the hypothesis (totally boundedness and completeness) also ended up being implied by the hypothesis, and ended up being the key intermediary steps to prove the conclusion.