Diverging tuple (a-b, b-c, c-d, d-a)

Problem

You start with a tuple containing four (not all equal) integers (a_0, b_0, c_0, d_0).

You then change that tuple to

$$(a_1, b_1, c_1, d_1) = (a_0 - b_0, b_0 - c_0, c_0 - d_0, d_0 - a_0).$$

You then change that tuple to

$$(a_2, b_2, c_2, d_2) = (a_1 - b_1, b_1 - c_1, c_1 - d_1, d_1 - a_1).$$

Then you continue that process.

Prove that one term in the tuple gets arbitrarily large.

Solution

We can consider the sequence of 4d points:

$$(a_0, b_0, c_0, d_0)$$

$$(a_1, b_1, c_1, d_1)$$

$$(a_2, b_2, c_2, d_2)$$

$$\cdots$$

We notice that if one coordinate of the 4d point goes to infinity as we move that point along this sequence, then the point's distance from the origin needs to also go to infinity. (That is, we **forward-reason from the target to a necessary statement**).

So, we can construct the sequence of natural numbers…

$$S_0 = a_0^2 + b_0^2 + c_0^2 + d_0^2$$

$$S_1 = a_1^2 + b_1^2 + c_1^2 + d_1^2$$

$$S_2 = a_2^2 + b_2^2 + c_2^2 + d_2^2$$

$$\cdots$$

$$S_i = a_i^2 + b_i^2 + c_i^2 + d_i^2$$

…and try to prove that S_i should diverge to infinity. The proof is as follows.

- We try to find the difference between S_{i+1} and S_i:

$$S_{i+1} = a_{i+1}^2 + b_{i+1}^2 + c_{i+1}^2 + d_{i+1}^2$$

$$S_{i+1} = (a_i - b_i)^2 + (b_i - c_i)^2 + (c_i - d_i)^2 + (d_i - a_i)^2$$

$$S_{i+1} = 2(a_i^2 + b_i^2 + c_i^2 + d_i^2) - 2(a_i b_i + b_i c_i + c_i d_i + d_i a_i)$$

$$S_{i+1} = 2S_i - 2(a_i b_i + b_i c_i + c_i d_i + d_i a_i)$$

- We want to find some term that involves the pesky $a_i b_i + b_i c_i + c_i d_i + d_i a_i$.
- To do this, we first notice that we have an invariant:
  $\forall i > 0, a_i + b_i + c_i + d_i = 0$. So, we set $i > 0$, square both sides, and notice:

$$0 = (a_{i+1} + b_{i+1} + c_{i+1} + d_{i+1})^2$$

$$0 = (a_i + c_i)^2 + 2(a_i + c_i)(b_i + d_i) + (b_i + d_i)^2$$

$$0 = (a_i + c_i)^2 + 2(a_i b_i + b_i c_i + c_i d_i + d_i a_i) + (b_i + d_i)^2$$

$$-2(a_i b_i + b_i c_i + c_i d_i + d_i a_i) = (a_i + c_i)^2 + (b_i + d_i)^2$$

- Now we can plug that in to our earlier expression:

$$S_{i+1} = 2S_i + (a_i + c_i)^2 + (b_i + d_i)^2$$

$$S_{i+1} \geq 2S_i$$

- So, since our sum (that is a natural number) at least doubles each time, it must diverge. (We know that $S_1 \geq 1$ by the problem statement saying that not all terms a_0, b_0, c_0, d_0 are equal.)

Once we do that, we still haven't proven that one term gets arbitrarily large (because one term going to negative infinity could still cause the distance of the point from the origin to get arbitrarily large). But, once we remember that $\forall i > 0, a_i + b_i + c_i + d_i = 0$, we know one term must in fact diverge to infinity.