(Point-and-click) How many connected components in the preimage of an interval?

Here's my proposal for how a point-and-click system can prove this problem.

- **Click 'Prove the opposite'**: We have a "for all" conjecture to prove or disprove. It's easier to disprove it (its negation is an "exists" statement) so we start with that. We now have to find a continuous function from a compact space to the reals which has an interval with *infinitely many connected components in its preimage*.
- **Click 'specialize'**: We instead want to prove we can find such a function which has a *point* (interval of length 0) with infinitely many connected components in its preimage.
- **Click 'guess'**: We have a library of maybe thirty of so functions that a student would typically see in a calculus or analysis class. We choose $xsin(1/x)$ because it is both "continuous" and "wiggly" (which we know intuitively corresponds to a point having a lot of preimages). More formally, perhaps each function in this library of functions could have several "tags" (or lemmas concerning it) — one of the lemmas would be continuity, but I'm not yet sure what the lemma could be that captures its "wiggliness."
- **Click 'specialize'** : Choose the point 0. We now satisfy the specialization.
- **Click 'generalize'**: Now we want to go back to finding a function where the *interval* has infinitely connected components in its preimage. So we turn the point 0 into the interval $(-\epsilon, +\epsilon)$.
- **Click 'fail'**: At this point the human user might realize they failed, use routine moves to prove there are actually only finitely many components. And so, the 'failure' will add as a lemma that that particular function, while it has infinitely many preimages of a point, finitely many in an interval.
- **Click 'Prove the opposite'** The failure has now caused the point "L" (with infinitely many preimages) to enter the proof state as a potentially significant mathematical object. Now we try to prove the lemma using this "L".

- **Click 'unfold'**We might now be motivated to unfold to the sequential definition of compactness.
- **Apply routine moves** We now apply routine moves to try to find a contradiction regarding the definition of $f(L)$...
- **Click 'fail'**...but we ultimately find no contradiction can be achieved because the point L can be defined to be an endpoint of the interval I.
- **Click 'Prove the opposite'** It seems each time we fail, we are convinced what we are doing isn't possible, so we negate the statement. So here we do again.
- **Click 'specialize'** We use the function $x \sin(1/x)$ as before.
- **Click 'specialize'** And finally, we know we should make the point $L = 0$ an endpoint of the interval - so we specialize the interval to $(0, 1)$.

And we're done.