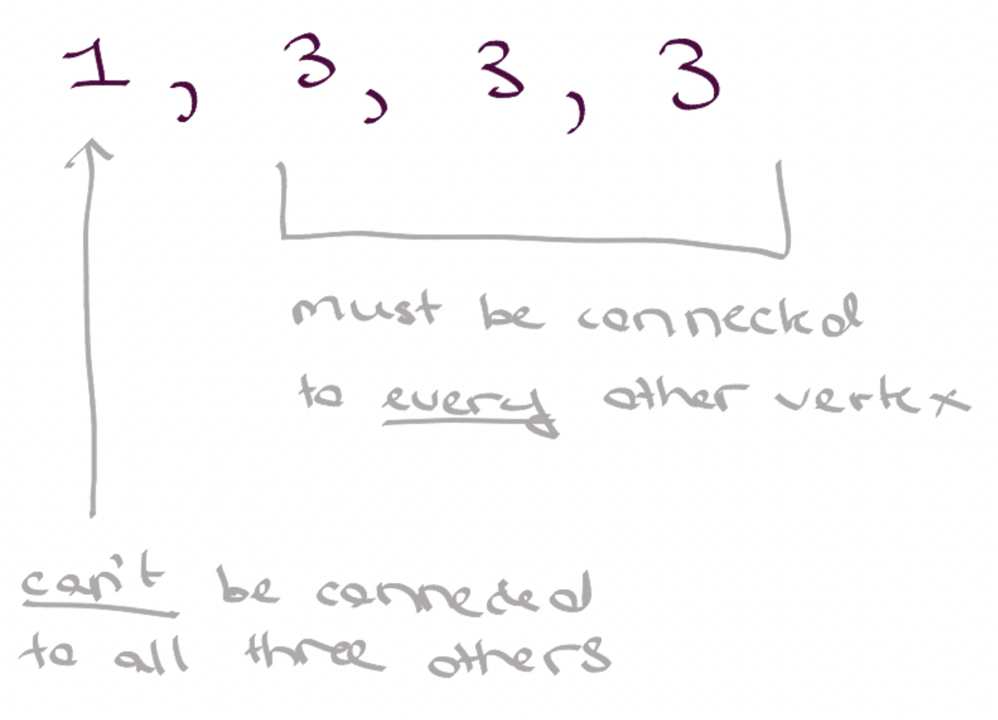
Is there a graph that has degree sequence 1,3,3,3?

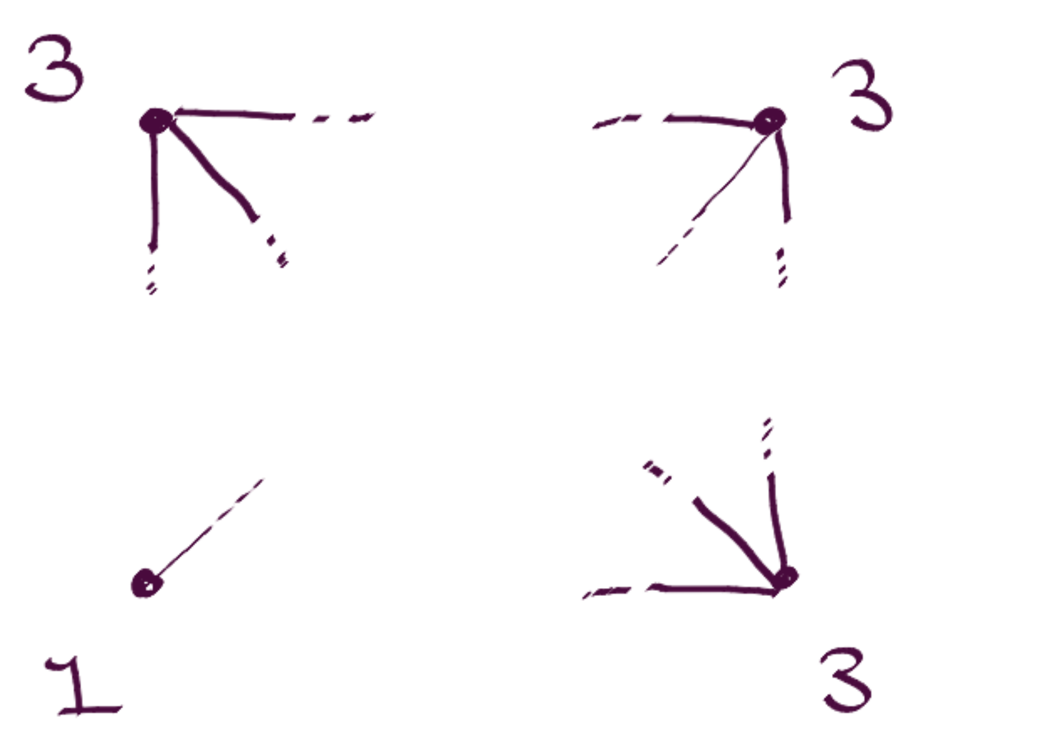
The short answer: no, because the three vertices of maximal degree 3 must be connected to all other vertices.



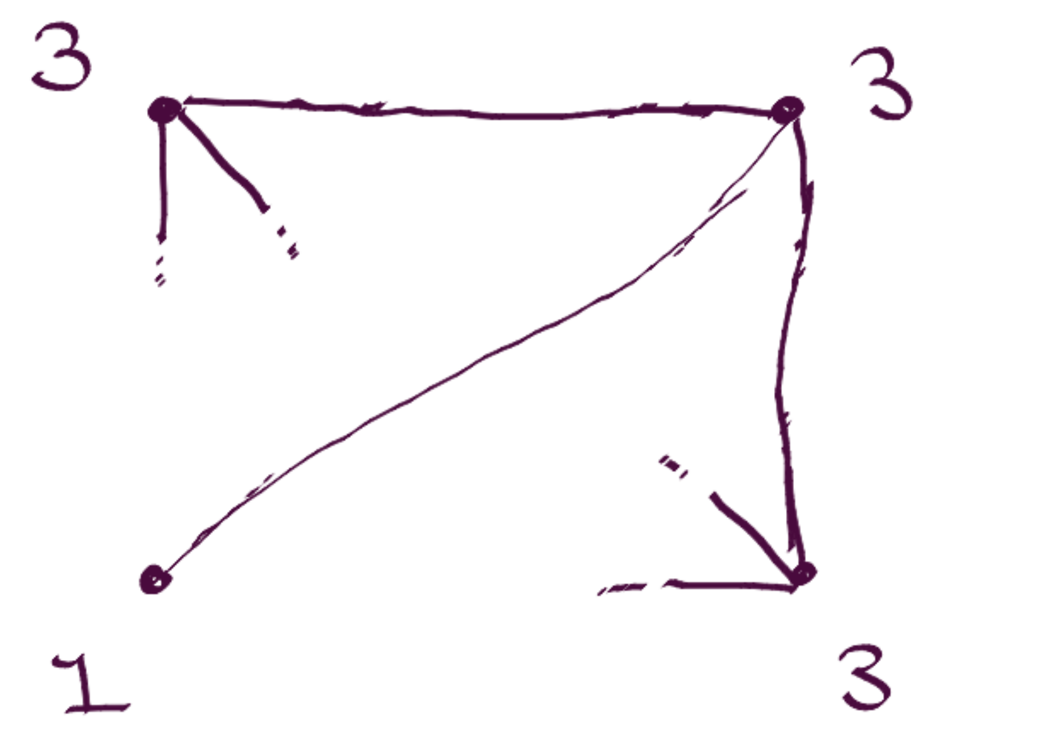
But how do we come to this conclusion in a motivated way?

Motivated Proof 1

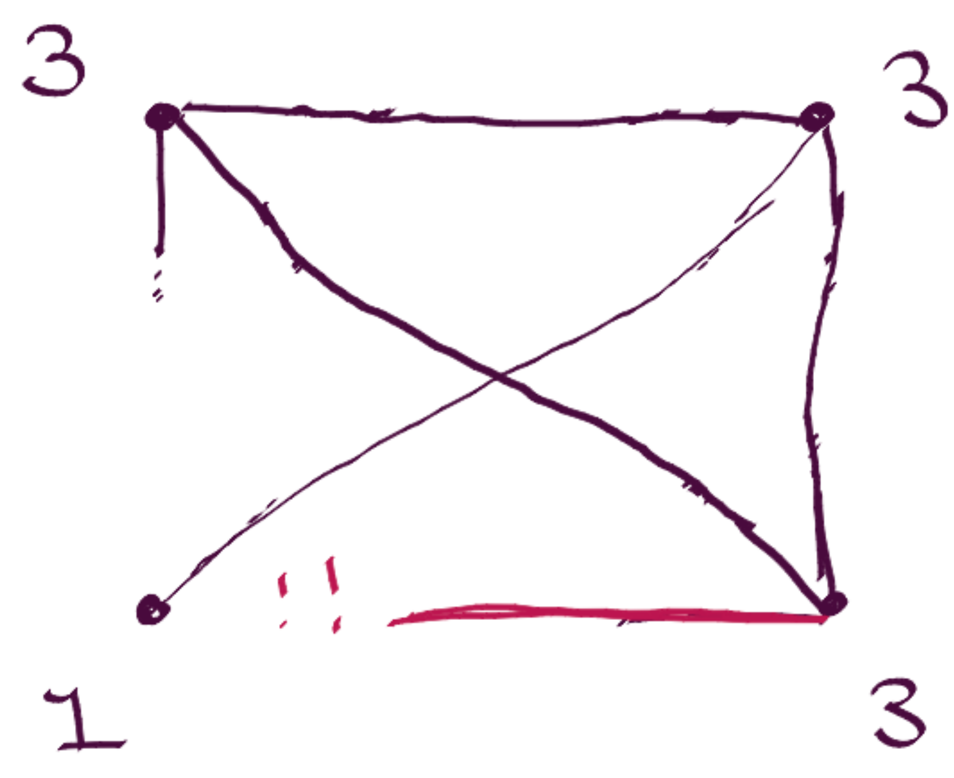
One way to arrive at this conclusion is the way I did, where you try to build up this graph **using forced moves**. That is, you start out with "stubs" of the edges protruding from each vertex…



…force-connect one of them to all others…



… and realize the degree-1 node needs to be connected to, but has already reached its full degree of 1.



And then you come up with the **conflict-inspired lemmas**:

- if a vertex has degree 3, it has to be connected to every other vertex.
- if a vertex has degree 1, it can be connected to at most one other vertex.

And now you can **generalize** this failure to the fact that there are at least two vertices of maximal degree (n-1) and therefore that should connect to everything, but one node that connects to at most one thing.

Motivated Proof 2

Another way to find the conclusion is to **immediately generalize** to a stronger statement:

$$\textit{Is there a graph with this degree sequence}: 1, n-1, n-1, n-1 \ldots ?$$

This makes it easier to syntax-match against previous lemmas, because you realize nodes with degree $n-1$ must be connected to everything.

This is similar to example where we had to prove "n is not prime $\implies 2^n - 1$ is not prime" — the syntax matching of the general statement "$x^n - 1$" was more helpful (because it matched to a library result) than something specific like "$2^6 - 1$".

The question is — how would we know to generalize to this? It seems one way is to:

- Learn from a forced-construction failure (as in Motivated Proof 1)
- To destroy instantiated variables when two variables are the same type (e.g. here we had the natural numbers 3 and n=4, so we might want to rewrite 3 as n-1). This seems a bit speculative though…

Takeaway

This example demonstrates **one way in which generalization is helpful: it lets you syntax match easier on theorems.**

- e.g. a result about a "vertex with degree n-1" should match with some result about a "maximal degree vertex"
- e.g. a result involving "$x^n - 1$" should match a library result that says it factors into $(x-1)(\ldots)$.

This example also demonstrates **one way in which I think specialization is helpful: it helps you work through a specific example, and often the construction that worked on that particular example easily generalizes** to the full proof.

- e,g. a degree sequence of 1,3,3,3 gives us something concrete to work with, and allowed you to implement forced constructions and fail, and then generalize the failure.