Maximal ideals — where is the conflict-guided reasoning?

The most unintuitive step of the "maximal ideals" proof can, I believe, be motivated by a learning-from-failure/conflict-guided process.

- The biggest logical leap in the proof, I think, is figuring out we can construct J as

$$J := \{rx + i : r \in R, i \in I\}.$$

- So how do we motivate that step?

Fredy suggested in a slack message that this could be achieved by forward reasoning from the target, or finding *necessary* conditions. That is, what *must* such a set J contain?

- (Take a risk) Well it must contain x, and all multiples of x. So we can tentatively set, for any $r \in R$:

$$J = \{rx : r \in R\}$$

- (Fail) But that risky move (achieved by the risky "forward reasoning from target") will turn out to fail — we can't prove J contains I.
- (Learn from failure) So, it might be a good idea to change J to more obviously contain elements in I:

$$J := \{rx + i : r \in R, i \in I\},$$

  so we've learned from our failures.

This turns out to be reasoning forward from target where the "necessary" surprisingly turns out to be "sufficient" — we can notice now we already have enough to prove our theorem.

In fact, we didn't need our set J to be a minimal set containing I and x. We just needed it to be an ideal containing I and x, and for $I \subsetneq J \subsetneq R$. And this J works perfectly (and does happen to be minimal, but we don't need to prove that.)

We often see that "reasoning forward from the target" and "conflict-guided reasoning" are closely related (as Fabian pointed out earlier in the fall, and Tim pointed out recently).

- I think we have come to realize that the link is as follows. Reasoning forward from target can be an "unsafe" move if you're constructing things from necessary rather than sufficient conditions. And "unsafe" moves are the ones we can fail at, and therefore, learn from.

So this is one example of a case where "reasoning forward from the target" is an unsafe move that enables learning from failure. What more general patterns might this example exhibit?

- I wonder if "reasoning forward from target" or "finding necessary conditions" is typically helpful in situations where one component turns out not to be necessary. That is, we don't end up necessarily needing to prove that J is the *smallest* ideal containing I and x. And so once we've put together a few necessary conditions in a clean way, we end up with something sufficient.