

# *M*icroeconometrics

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Philipp Eisenhauer



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# Generalized method of moments

Philipp Eisenhauer

I heavily draw on the material presented in:

- ▶ Whitney Newey, course materials for 14.385 Non-linear Econometric Analysis, Fall 2007. MIT OpenCourseWare (<http://ocw.mit.edu>), Massachusetts Institute of Technology.

## General idea

- ▶ The generalized method of moments (GMM) is a general estimation principle, where the estimators are derived from so-called moment conditions. It provides a unifying framework for the comparison of alternative estimators.

## Structure

- ▶ Setup
- ▶ Identification
- ▶ Asymptotic distribution
- ▶ Testing

# Setup

## Notation

$\beta$	$p \times 1$	parameter vector
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$w_i$	$i = 1, \dots, n$	data points
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$g_i(w_i, \beta)$	$m \times 1$	moment
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- ▶ The GMM estimator is based on a model where, for the true parameter value  $\beta_0$  the moment conditions  $E[g_i(\beta_0)] = 0$  are satisfied.
- ▶ The estimator is formed by choosing  $\beta$  so that the sample average of  $g_i(\beta)$  is close to its zero population value.



The estimator is formed by choosing  $\beta$  so that the sample average of  $g_i(\beta)$  is close to its zero population value. Let

$$\hat{g}(\beta) = \frac{1}{n} \sum_{i=1}^n g_i(\beta)$$

- ▶ theoretical moments
- ▶ empirical moments

Let  $\hat{A}$  denote a  $m \times m$  positive semi-definite matrix, then the GMM estimator is given by

$$\hat{\beta} = \arg \min_{\beta} \hat{g}(\beta)' \hat{A} \hat{g}(\beta)$$

The GMM estimator chooses  $\hat{\beta}$  so the sample average  $\hat{g}(\beta)$  is close to zero.

## **Instrumental variables**

Let's work through an example on the blackboard.

## Unifying framework

Many other popular estimation strategies can be analyzed in a GMM setup.

Ordinary least squares  $E[x_i(y_i - x_i\beta_0)] = 0$

Instrumental variables  $E[z_i(y_i - x_i\beta_0)] = 0$

Maximum likelihood  $E[\partial \ln f(x_i, \beta_0)/\partial \beta] = 0$

If moments cannot be evaluated analytically then we have an application of the method of simulated moments.

## Distance and weighing matrix

Let's look at the role of the weighing matrix for a two dimensional example.

- ▶ identity matrix

$$Q(\beta) = \begin{pmatrix} g_1 & g_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = g_1^2 + g_2^2$$

► alternative

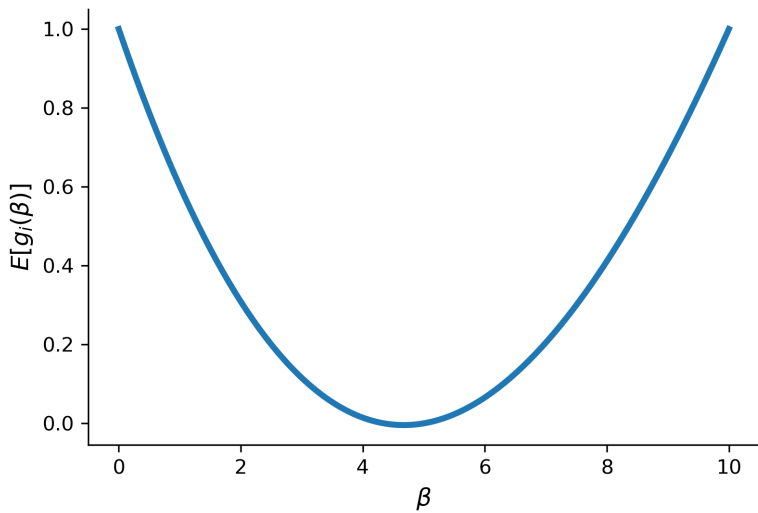
$$Q(\beta) = \begin{pmatrix} g_1 & g_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = 2g_1^2 + g_2^2$$

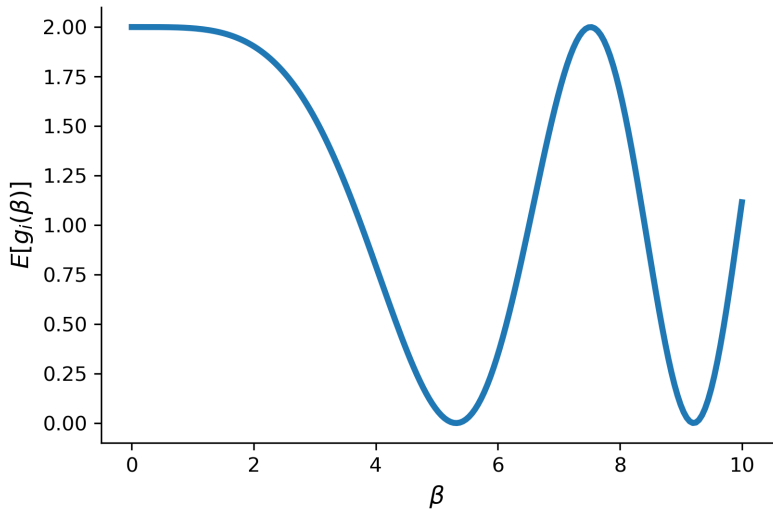
Our alternative attaches more weight to the first coordinate in the distance.

# Identification



The parameters  $\beta_0$  are identified if  $\beta_0$  is the only solution to  $E[g_i(\beta)] = 0$ .





- ▶ Necessary condition for identification is that  $m \geq p$ . When  $m \leq p$ , i.e. there are fewer equations to solve than parameters, there will typically be multiple solutions to the moment conditions.

- ▶ Let  $G = E[\partial g_i(\beta_0)/\partial \beta]$ . Rank condition is  $\text{rank}(G) = p$ . Necessary and sufficient for identification when  $g_i(\beta)$  is linear in  $\beta$ .
- ▶ In the general nonlinear case it is difficult to specify conditions for uniqueness of the solution to  $E[g_i(\beta)] = 0$ .

- ▶  $m = p$ , exact identification,  $\hat{g}(\hat{\beta}) = 0$  asymptotically
- ▶  $m > p$ , overidentification,  $\hat{g}(\hat{\beta}) > 0$  asymptotically

In the case of overidentification, the choice of  $A$  matters and affects the estimator's asymptotic distribution.

# **Asymptotic distribution**

## Asymptotic distribution

Under some regularity conditions, the GMM estimator has the following asymptotic distribution.

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} \mathbb{N}(0, V),$$

where  $V = (G'AG)^{-1}G'A\Omega AG(G'AG)^{-1}$  with  $G = E[\partial g_i(\beta_0)/\partial \beta]$  and  $\Omega = E[g_i(\beta_0)g_i(\beta_0)']$ .

$\Rightarrow$  asymptotic variance depends on the choice of the weighing matrix  $A$



The optimal weighing matrix  $A = \Omega^{-1}$  the asymptotic variance simplifies to

$$V = (G' \Omega^{-1} G)^{-1}$$

What makes a good moment?

- ▶ small  $\Omega$ , small sample variation of the moment
- ▶ large  $G$ , moment informative on true value

Figure: Weak identification

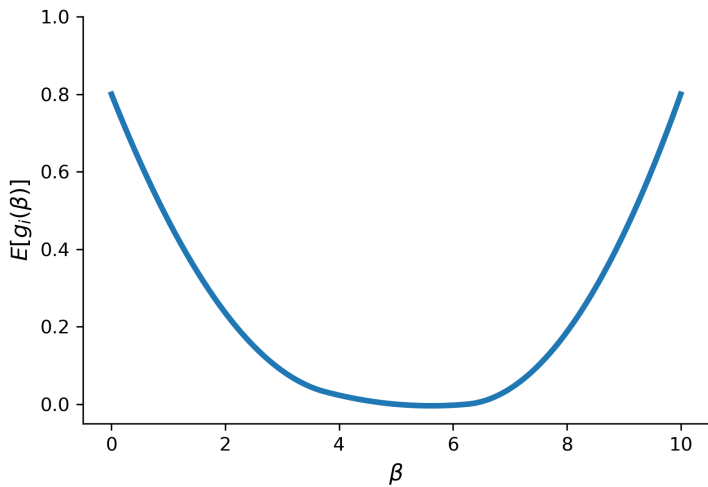
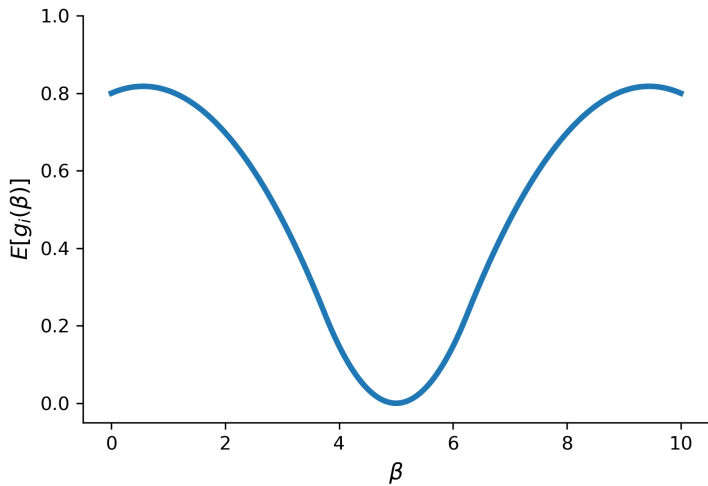


Figure: Sharp identification



## **Instrumental variables**

Let's continue our example on the blackboard.

# Testing

An important statistic for GMM is the test of overidentifying restrictions that is given by

$$T = n \hat{g}(\hat{\beta})' \hat{\Sigma}^{-1} \hat{g}(\hat{\beta})$$

which converges in distribution to

$$T \xrightarrow{d} \chi^2(m - p)$$

under  $H_0$  that the model is correctly specified.

Figure: Density of  $\chi^2(2)$

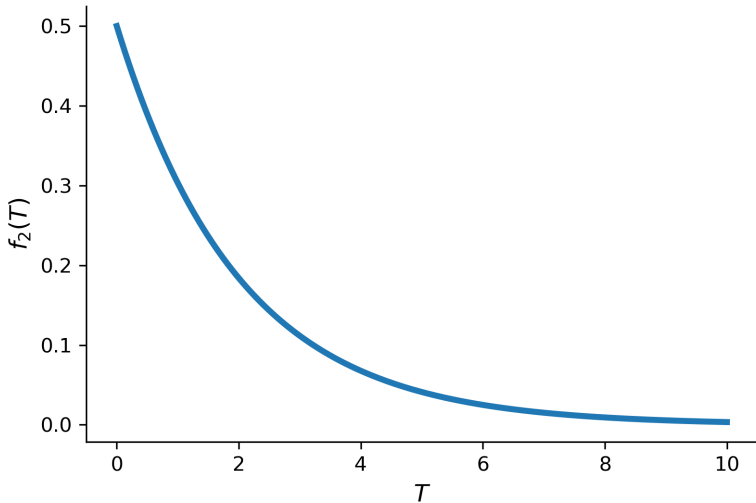
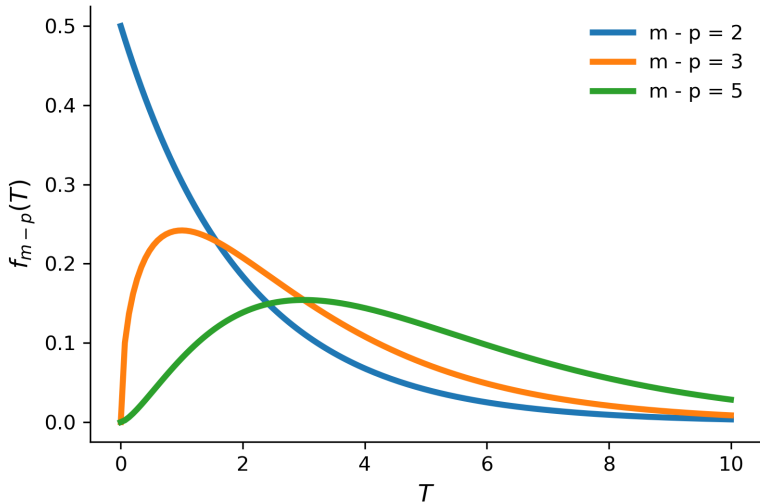


Figure: Density of  $\chi^2(m-p)$





# Wrapping up

## Feasible efficient GMM

The optimal weighing matrix depends on moment evaluations at  $\beta_0$  which is unknown.

- ▶ iterated feasible GMM
- ▶ continuously updating GMM

**Let's turn to our Jupyter notebooks ...**



# Appendix

# *References*

Davidson, R., & MacKinnon, J. G. (2003). *Econometric theory and methods*. New York: Oxford University Press.

Hall, A. A. (2005). *Generalized method of moments*. New York: Oxford University Press.