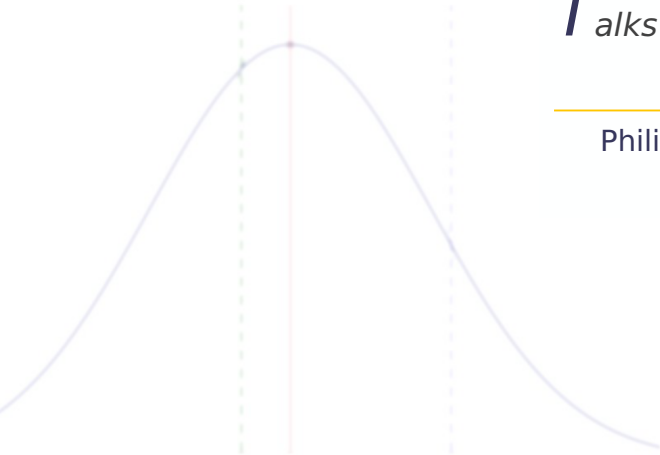


T_{alks}

Philipp Eisenhauer



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Sensitivity analysis

Philipp Eisenhauer

I draw on the material presented in:

- ▶ Saltelli, A., Tarantola, S., Campolongo, F., & Ratto, M. (2004). *Sensitivity analysis in practice: A guide to assessing scientific models*. John Wiley & Sons.
- ▶ Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., . . . Tarantola, S. (2008). *Global sensitivity analysis: The primer*. John Wiley & Sons.

Definitions

Uncertainty and sensitivity analysis study how the uncertainties in the the model input $\mathbf{X} = (X_1, \dots, X_K)$ affect the model's quantities of interest:

$$Y = f(\mathbf{X})$$

- ▶ uncertainty analysis quantifies the output variability
- ▶ sensitivity analysis describes the relative importance of each input in determining its variability

Sensitivity methods

- ▶ qualitative, e.g. Morris screening
- ▶ quantitative, e.g. variance-based methods

Sensitivity insights

- ▶ *factor prioritization*, determining most important model inputs
- ▶ *factor fixing*, identifying the least important factor which can be ignored
- ▶ *stability*, determining region of stability of for optimal decision

Selected issues

- ▶ computational cost
- ▶ deterministic vs. probabilistic
- ▶ independent vs. dependent
- ▶ global vs. local

Selected issues

- ▶ quantitative vs. qualitative
- ▶ interaction vs. additivity
- ▶ full model vs. surrogate

Sensitivity analysis in economics

- ▶ Harenberg, D., Marelli, S., Sudret, B., & Winschel, V. (2019). Uncertainty quantification and global sensitivity analysis for economic models. *Quantitative Economics*, 10(1), 1–41.

Notation

The model input vector $\mathbf{X} = (X_1, \dots, X_K) \in \mathbb{R}^K$. The quantity of interest y of the model $f(\cdot)$:

$$Y = f(\mathbf{X})$$

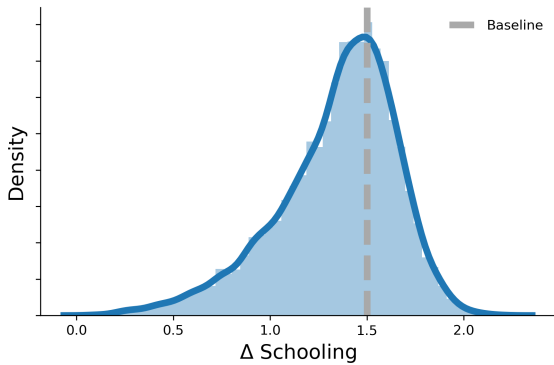
Following the literature, all parameters x_i are scaled to take on values in the interval $[0, 1]$, and the region of interest Ω is the K - dimensional unit hypercube.

- ▶ We collect all parameter in $\mathbf{x} = [x_1, \dots, x_K]$. x_i denotes one particular value for input parameter i and $\mathbf{x}_{\sim i} = [x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_K]$ as the complementary set of inputs.
- ▶ We use the notation x_i and \bar{x}_i to distinguish a random vector x_i generated from a joint probability density function $p(x_i, x_{\sim i})$ and a random vector \bar{x}_i generated from a conditional probability distribution $p(\bar{x}_i, x_{\sim i} \mid x_{\sim i})$.

Uncertainty propagation

- ▶ We sample from the distribution of input parameters and assess the distribution of the quantity of interest.

Figure: Uncertainty propagation



Qualitative

Morris method for independent and dependent factors.

- ▶ Morris, M. D. (1991). Factorial sampling plans for preliminary computational experiments. *Technometrics*, 33(2), 161–174.
- ▶ Ge, Q., & Menendez, M. (2017). Extending morris method for qualitative global sensitivity analysis of models with dependent inputs. *Reliability Engineering & System Safety*, 162, 28–39.

- ▶ The approach segments the model input ranges $[x_i^-, x_i^+]$ in I levels. Given I levels with K inputs, there are I^K points in the grid from which a subset of r points is drawn at random. For each of the r , the model is evaluated performing a series of OAT sensitivities.

$$ee_i = \frac{f(x_1, \dots, x_{i-1}, x_i + \Delta, x_{i+1}, \dots, x_K) - f(\mathbf{x})}{\Delta}$$

$$ee_i^{ind} = \frac{f(\bar{x}_i, \mathbf{x}_{\sim i}) - f(x_i, \mathbf{x}_{\sim i})}{\Delta}$$

$$ee_i^{dep} = \frac{f(\bar{x}_i, \bar{\mathbf{x}}_{\sim i}) - f(x_i, \mathbf{x}_{\sim i})}{\Delta}$$

$$\mu_i^j = \frac{1}{N} \sum_{r=1}^n |ee_{ir}^j|$$

$$\sigma_i^j = \frac{1}{N-1} \sum_{r=1}^n (ee_{ir}^j - \mu_i^j)^2$$

Quantitative

Alternatives

- ▶ variance-based
- ▶ moment-independent
- ▶ information-based

Variance-based methods

$$V[Y] = V_{X_i}[E_{X \sim i}[Y \mid X_i]] + E_{X_i}[V_{\mathbf{X} \sim i}[Y \mid X_i]] \quad (1)$$

Main effect

- ▶ We rank all based on the smallest conditional variance $V[Y | X_i = x_i]$ evaluated over all possible values x_i of X_i . Following Equation (1), this is equivalent to ranking factors by the largest $V_{X_i}[E_{\mathbf{x}_{\sim i}}[Y | X_i]]$ and so the main effect is defined as:

$$S_i^M = \frac{V_{X_i}[E_{\mathbf{x}_{\sim i}}[Y | X_i]]}{V[Y]}$$

Total effect

- ▶ We want to identify the factors that we can fix at their value without significantly reducing the output variance.

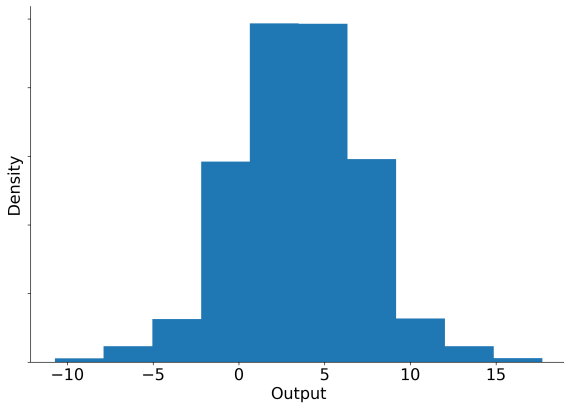
$$S_i^T = \frac{E_{\mathbf{x}_{\sim i}}[V_{X_i}[Y \mid \mathbf{X}_{\sim i}]]}{V[Y]} = 1 - S_{\sim i}^M$$

Ishigami function

$$f(\mathbf{x}) = \sin(x_1) + a \sin^2(x_2) + b x_3^4 \sin(x_1)$$

- ▶ The Ishigami function of y (?) is used as an example for uncertainty and sensitivity analysis methods, because it exhibits strong nonlinearity and nonmonotonicity.
- ▶ It also has a peculiar dependence on x_3 , as described by y (?).
- ▶ The independent distributions of the input random variables are usually: $x_i \sim \text{Uniform}[-\pi, \pi]$, for all $i = 1, 2, 3$.

Figure: Uncertainty propagation



Reference values

$$\text{var}(Y) = \frac{a^2}{8} + \frac{b\pi^4}{5} + \frac{b^2\pi^8}{18} + \frac{1}{2}$$

$$S_1 = \frac{1}{2} * \left(1 + \frac{b\pi^4}{5}\right)^2 \text{var}(Y)^{-1}$$

$$S_2 = \frac{a^2}{8} \text{var}(Y)^{-1}$$

$$S_3 = 0$$

Figure: Main effect of input 1

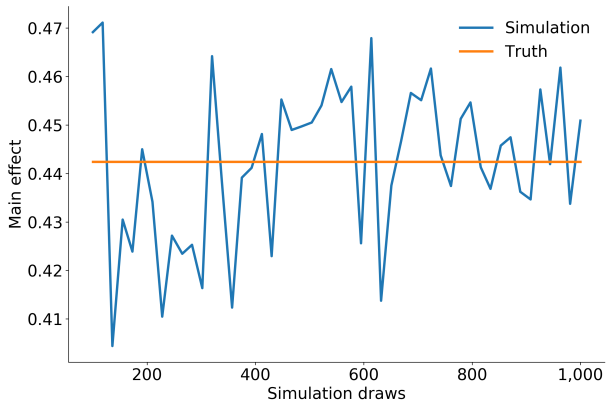
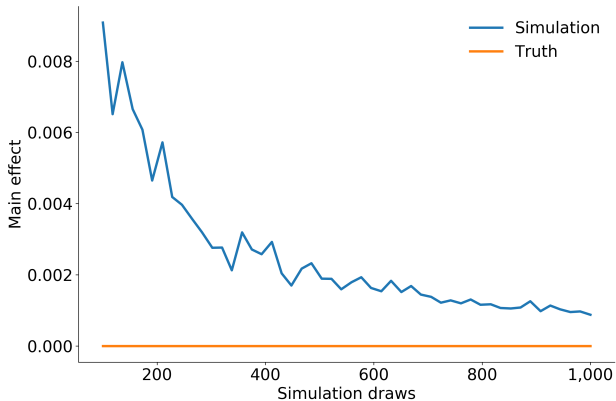


Figure: Main effect of input 3



Total effect

$$S_1^T = \left(\frac{1}{2} * \left(1 + \frac{b\pi^4}{5} \right)^2 + \frac{8b^2\pi^8}{225} \right) \text{var}(Y)^{-1}$$

$$S_2^T = \frac{a^2}{8} \text{var}(Y)^{-1}$$

$$S_3^T = \frac{8b^2\pi^8}{225} \text{var}(Y)^{-1}$$

Figure: Total effect of input 1

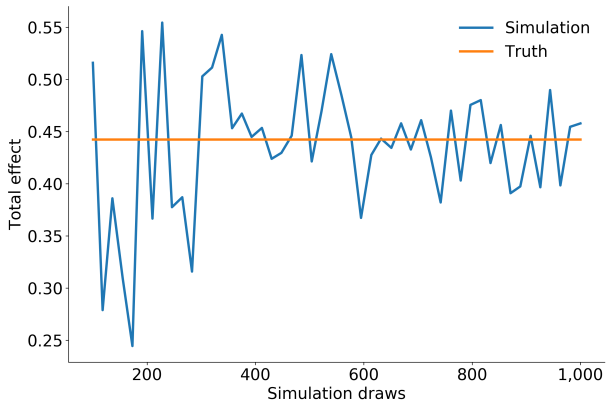
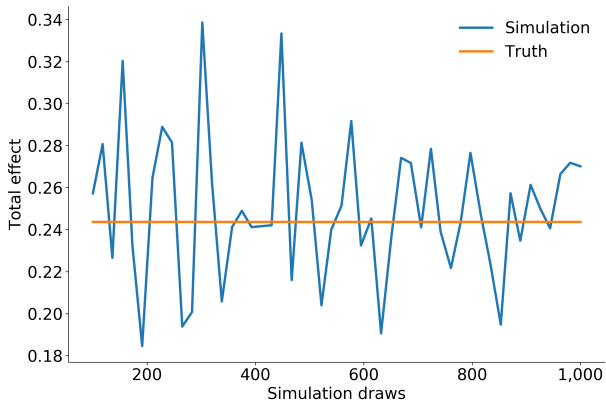


Figure: Total effect of input 3



Resources

Textbooks

- ▶ Saltelli, A., Tarantola, S., Campolongo, F., & Ratto, M. (2004). *Sensitivity analysis in practice: A guide to assessing scientific models*. John Wiley & Sons.
- ▶ Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., . . . Tarantola, S. (2008). *Global sensitivity analysis: The primer*. John Wiley & Sons.

Reviews

- ▶ Borgonovo, E., & Plischke, E. (2016). Sensitivity analysis: a review of recent advances. *European Journal of Operational Research*, 248(3), 869–887.

Seminal papers

- ▶ Saltelli, A., & Tarantola, S. (2002). On the relative importance of input factors in mathematical models: safety assessment for nuclear waste disposal. *Journal of the American Statistical Association*, 97(459), 702–709.
 - ▶ quantitative, brute force, dependent factors
- ▶ Morris, M. D. (1991). Factorial sampling plans for preliminary computational experiments. *Technometrics*, 33(2), 161–174.
 - ▶ qualitative, Morris screening

- ▶ Campolongo, F., Cariboni, J., & Saltelli, A. (2007). An effective screening design for sensitivity analysis of large models. *Environmental modelling & software*, 22(10), 1509–1518.
 - ▶ modification of Morris screening
- ▶ Homma, T., & Saltelli, A. (1996). Importance measures in global sensitivity analysis of nonlinear models. *Reliability Engineering & System Safety*, 52(1), 1–17.
 - ▶ total effect, factor screening
- ▶ Sobol, I. (1993). On sensitivity estimation for nonlinear mathematical models. *Math. Modelling & Comp. Exp.*
 - ▶ main effect, variance-based measure

- ▶ Ge, Q., & Menendez, M. (2017). Extending morris method for qualitative global sensitivity analysis of models with dependent inputs. *Reliability Engineering & System Safety*, 162, 28–39.
 - ▶ qualitative, Morris screening, dependent factors
- ▶ Kucherenko, S., Tarantola, S., & Annoni, P. (2012). Estimation of global sensitivity indices for models with dependent variables. *Computer Physics Communications*, 183(4), 937–946.
 - ▶ quantitative, variance-based measure, dependent factors

Appendix

References

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- Campolongo, F., Cariboni, J., & Saltelli, A. (2007). An effective screening design for sensitivity analysis of large models. *Environmental modelling & software*, 22(10), 1509–1518.
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- Homma, T., & Saltelli, A. (1996). Importance measures in global sensitivity analysis of nonlinear models. *Reliability Engineering & System Safety*, 52(1), 1–17.
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- Saltelli, A., Tarantola, S., Campolongo, F., & Ratto, M. (2004). *Sensitivity analysis in practice: A guide to assessing scientific models*. John Wiley & Sons.
- Sobol, & Levitan. (1999). On the use of.
- Sobol, I. (1993). On sensitivity estimation for nonlinear mathematical models. *Math. Modelling & Comp. Exp.*