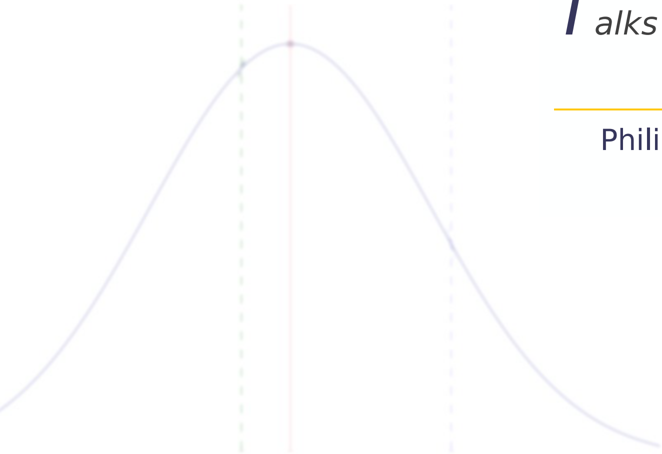

Talks

Philipp Eisenhauer

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Sensitivity analysis

Philipp Eisenhauer

I draw on the material presented in:

- Saltelli, A., Tarantola, S., Campolongo, F., & Ratto, M. (2004). *Sensitivity analysis in practice: A guide to assessing scientific models*. John Wiley & Sons.

- Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., . . . Tarantola, S. (2008). *Global sensitivity analysis: The primer*. John Wiley & Sons.

Definitions

Uncertainty and sensitivity analysis study how the uncertainties in the the model input $\mathbf{X} = (X_1, \ldots, X_K)$ affect the model's quantities of interest:

$$Y = f(\mathbf{X})$$

- uncertainty analysis quantifies the output variability
- sensitivity analysis describes the relative importance of each input in determining its variability

Sensitivity methods

- qualitative, e.g. Morris screening
- quantitative, e.g. variance-based methods

Sensitivity insights

- *factor prioritization*, determining most important model inputs
- *factor fixing*, identifying the least important factor which can be ignored
- *stability*, determining region of stability of for optimal decision

Selected issues

- computational cost
- deterministic vs. probabilistic
- independent vs. dependent
- global vs. local

Selected issues

- quantitative vs. qualitative
- interaction vs. additivity
- full model vs. surrogate

Sensitivity analysis in economics

- Harenberg, D., Marelli, S., Sudret, B., & Winschel, V. (2019). Uncertainty quantification and global sensitivity analysis for economic models. *Quantitative Economics*, *10*(1), 1–41.

Notation

The model input vector $\mathbf{X} = (X_1, \ldots, X_K) \in \mathbb{R}^K$. The quantity of interest y of the model $f(\cdot)$:

$$Y = f(\mathbf{X})$$

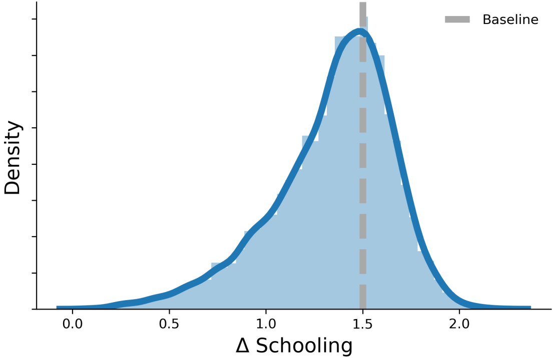
Following the literature, all parameters x_i are scaled to take on values in the interval $[0, 1]$, and the region of interest Ω is the K- dimensional unit hypercube.

- We collect all parameter in $\mathbf{x} = [x_1, \ldots, x_K]$. x_i denotes one particular value for input parameter i and $\mathbf{x}_{\sim i} = [x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_K]$ as the complementary set of inputs.
- We use the notation x_i and $\bar{x}_i$ to distinguish a random vector x_i generated from a joint probability density function $p(x_i, x_{\sim i})$ and a random vector $\bar{x}_i$ generated from a conditional probability distribution $p(\bar{x}_i, x_{\sim i} \mid x_{\sim i})$.

Uncertainty propagation

- We sample from the distribution of input parameters and assess the distribution of the quantity of interest.

Figure: Uncertainty propagation



Qualitative

Morris method for independent and dependent factors.

- Morris, M. D. (1991). Factorial sampling plans for preliminary computational experiments. *Technometrics*, *33*(2), 161–174.
- Ge, Q., & Menendez, M. (2017). Extending morris method for qualitative global sensitivity analysis of models with dependent inputs. *Reliability Engineering & System Safety*, *162*, 28–39.

- The approach segments the model input ranges $[x_i^-, x_i^+]$ in l levels. Given l levels with K inputs, there are l^n points in the grid from which a subset of r points is drawn at random. For each of the r, the model is evaluated performing a series of OAT sensitivities.

$$ee_i = \frac{f(x_1, \ldots, x_{i-1}, x_i + \Delta, x_{i+1}, \ldots, x_K) - f(\mathbf{x})}{\Delta}$$

$$
\begin{aligned}
ee_i^{ind} &= \frac{f(\bar{x}_i, \mathbf{x}_{\sim i}) - f(x_i, \mathbf{x}_{\sim i})}{\Delta} \\
ee_i^{dep} &= \frac{f(\bar{x}_i, \bar{\mathbf{x}}_{\sim i}) - f(x_i, \mathbf{x}_{\sim i})}{\Delta}
\end{aligned}
$$

$$\mu_i^j = \frac{1}{N}\sum_{r=1}^{n} |ee_{ir}^j|$$

$$\sigma_i^j = \frac{1}{N-1}\sum_{r=1}^{n}(ee_{ir}^j - \mu_i)^2$$

Quantitative

Alternatives

- variance-based
- moment-independent
- information-based

Variance-based methods

$$V[Y] = V_{X_i}[E_{X_{\sim i}}[Y \mid X_i]] + E_{X_i}[V_{\mathbf{X}_{\sim i}}[Y \mid X_i]] \tag{1}$$

Main effect

- We rank all based on the smallest conditional variance $V[Y \mid X_i = x_i]$ evaluated over all possible vales x_i of X_i. Following Equation (1), this is equivalent to ranking factors by the largest $V_{X_i}[E_{\mathbf{X}_{\sim i}}[Y \mid X_i]]$ and so the main effect is defined as:

$$S_i^M = \frac{V_{X_i}[E_{\mathbf{X}_{\sim i}}[Y \mid X_i]]}{V[Y]}$$

Total effect

- We want to identify the factors that we can fix at their value without significantly reducing the output variance.

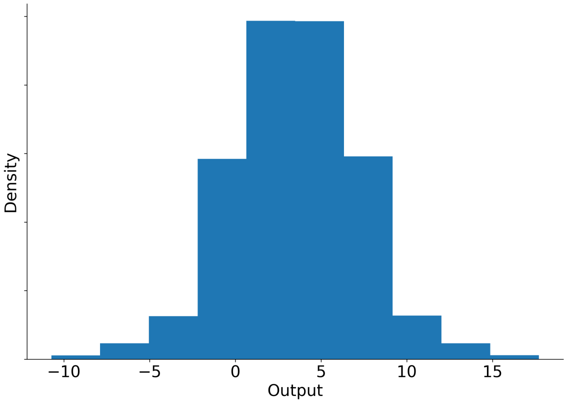
$$S_i^T = \frac{E_{\mathbf{X}_{\sim i}}[V_{X_i}[Y \mid \mathbf{X}_{\sim i}]]}{V[Y]} = 1 - S_{\sim i}^M$$

Ishigami function

$$f(\mathbf{x}) = \sin(x_1) + a\,\sin^2(x_2) + b\,x_3^4\,\sin(x_1)$$

- The Ishigami function of Ishigami and Homma (1990) is used as an example for uncertainty and sensitivity analysis methods, because it exhibits strong nonlinearity and nonmonotonicity.
- It also has a peculiar dependence on x_3, as described by Y. Sobol I. & Levitan (1999).
- The independent distributions of the input random variables are usually: $x_i \sim Uniform[-\pi, \pi]$, for all $i = 1, 2, 3$.

Figure: Uncertainty propagation



Reference values

$$\mathrm{var}(Y) = \frac{a^2}{8} + \frac{b\pi^4}{5} + \frac{b^2\pi^8}{18} + \frac{1}{2}$$

$$S_1 = \frac{1}{2} * \left(1 + \frac{b\pi^4}{5}\right)^2 \mathrm{var}(Y)^{-1}$$

$$S_2 = \frac{a^2}{8} \mathrm{var}(Y)^{-1}$$

$$S_3 = 0$$

Figure: Main effect of input 1

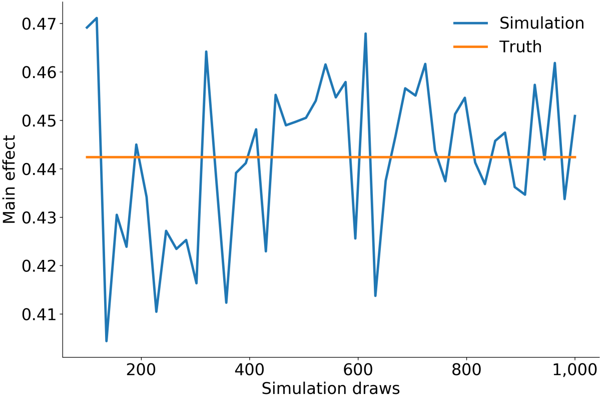
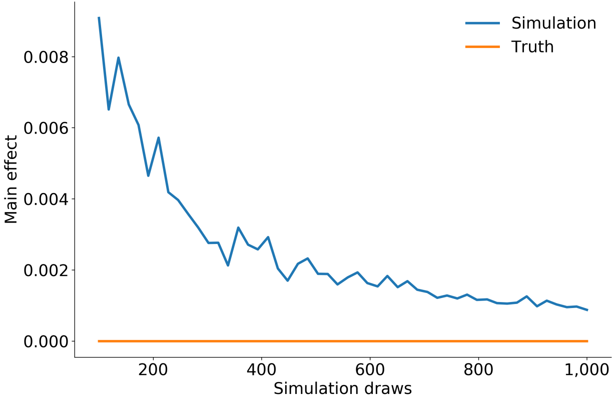


Figure: Main effect of input 3



Total effect

$$S_1^T = \left(\frac{1}{2} * \left(1 + \frac{b\pi^4}{5} \right)^2 + \frac{8b^2\pi^8}{225} \right) \operatorname{var}(Y)^{-1}$$

$$S_2^T = \frac{a^2}{8} \operatorname{var}(Y)^{-1}$$

$$S_3^T = \frac{8b^2\pi^8}{225} \operatorname{var}(Y)^{-1}$$

Figure: Total effect of input 1

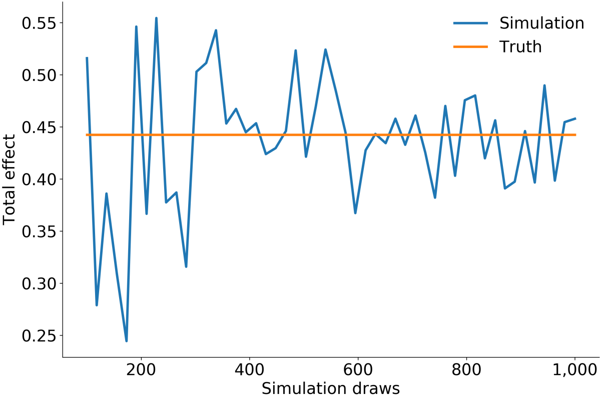
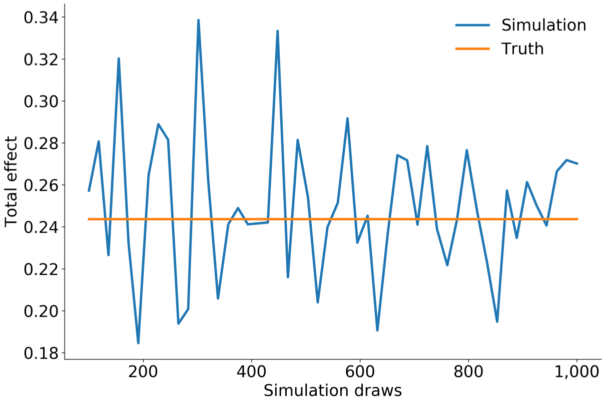


Figure: Total effect of input 3



Resources

Textbooks

- Saltelli, A., Tarantola, S., Campolongo, F., & Ratto, M. (2004). *Sensitivity analysis in practice: A guide to assessing scientific models*. John Wiley & Sons.
- Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., ... Tarantola, S. (2008). *Global sensitivity analysis: The primer*. John Wiley & Sons.
- Borgonovo, E. (2017). *Sensitivity analysis: An introduction for the management scientist*. Springer.

Reviews

- Borgonovo, E., & Plischke, E. (2016). Sensitivity analysis: a review of recent advances. *European Journal of Operational Research*, *248*(3), 869–887.

Seminal papers

- Saltelli, A., & Tarantola, S. (2002). On the relative importance of input factors in mathematical models: safety assessment for nuclear waste disposal. *Journal of the American Statistical Association*, *97*(459), 702–709.
  - quantitative, brute force, dependent factors
- Morris, M. D. (1991). Factorial sampling plans for preliminary computational experiments. *Technometrics*, *33*(2), 161–174.
  - qualitative, Morris screening

- Campolongo, F., Cariboni, J., & Saltelli, A. (2007). An effective screening design for sensitivity analysis of large models. *Environmental modelling & software*, *22*(10), 1509–1518.
  - modification of Morris screening
- Homma, T., & Saltelli, A. (1996). Importance measures in global sensitivity analysis of nonlinear models. *Reliability Engineering & System Safety*, *52*(1), 1–17.
  - total effect, factor screening
- Sobol, I. (1993). On sensitivity estimation for nonlinear mathematical models. *Math. Modelling & Comp. Exp*.
  - main effect, variance-based measure

- Ge, Q., & Menendez, M. (2017). Extending morris method for qualitative global sensitivity analysis of models with dependent inputs. *Reliability Engineering & System Safety*, *162*, 28–39.
  - qualitative, Morris screening, dependent factors
- Kucherenko, S., Tarantola, S., & Annoni, P. (2012). Estimation of global sensitivity indices for models with dependent variables. *Computer Physics Communications*, *183*(4), 937–946.
  - quantitative, variance-based measure, dependent factors

Appendix

References

Borgonovo, E. (2017). *Sensitivity analysis: An introduction for the management scientist*. Springer.

Borgonovo, E., & Plischke, E. (2016). Sensitivity analysis: a review of recent advances. *European Journal of Operational Research*, *248*(3), 869–887.

Campolongo, F., Cariboni, J., & Saltelli, A. (2007). An effective screening design for sensitivity analysis of large models. *Environmental modelling & software*, *22*(10), 1509–1518.

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Harenberg, D., Marelli, S., Sudret, B., & Winschel, V. (2019). Uncertainty quantification and global sensitivity analysis for economic models. *Quantitative Economics*, *10*(1), 1–41.

Homma, T., & Saltelli, A. (1996). Importance measures in global sensitivity analysis of nonlinear models. *Reliability Engineering & System Safety*, *52*(1), 1–17.

Ishigami, T., & Homma, T. (1990). An importance quantification technique in uncertainty analysis for computer models. In *Proceedings. first international symposium on uncertainty modeling and analysis* (pp. 398–403).

Kucherenko, S., Tarantola, S., & Annoni, P. (2012). Estimation of global sensitivity indices for models with dependent variables. *Computer Physics Communications*, *183*(4), 937–946.

Morris, M. D. (1991). Factorial sampling plans for preliminary computational experiments. *Technometrics*, *33*(2), 161–174.

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Saltelli, A., & Tarantola, S. (2002). On the relative importance of input factors in mathematical models: safety assessment for nuclear waste disposal. *Journal of the American Statistical Association*, *97*(459), 702–709.

Saltelli, A., Tarantola, S., Campolongo, F., & Ratto, M. (2004). *Sensitivity analysis in practice: A guide to assessing scientific models*. John Wiley & Sons.

Sobol, I. (1993). On sensitivity estimation for nonlinear mathematical models. *Math. Modelling & Comp. Exp*.

Sobol, Y., I. & Levitan. (1999). On the use of variance reducing multipliers in monte carlo computations of a global sensitivity index. *Computer Physics Communications*, *117*(1), 52–61.