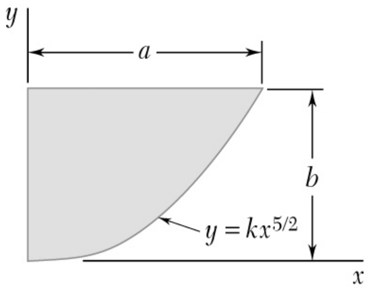
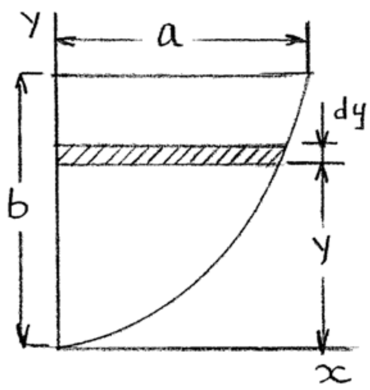


PROBLEM 9.1

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $x = a,\ y = b$: $b = ka^{\frac{5}{2}}$ or $k = \dfrac{b}{a^{\frac{5}{2}}}$

$$\therefore\ y = \frac{b}{a^{\frac{5}{2}}} x^{\frac{5}{2}} \quad \text{or} \quad x = \frac{a}{b^{\frac{2}{5}}} y^{\frac{2}{5}}$$

$$dI_y = \frac{1}{3} x^3 dy$$

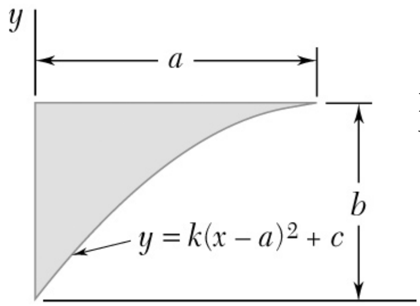
$$= \frac{1}{3} \frac{a^3}{b^{\frac{6}{5}}} y^{\frac{6}{5}} dy$$

Then $I_y = \dfrac{1}{3} \dfrac{a^3}{b^{\frac{6}{5}}} \displaystyle\int_0^b y^{\frac{6}{5}} dy$

$$= \frac{1}{3} \frac{5}{11} \frac{a^3}{b^{\frac{6}{5}}} y^{\frac{11}{5}} \Big|_0^b$$

$$= \frac{5}{33} \frac{a^3}{b^{\frac{6}{5}}} b^{\frac{11}{5}}$$

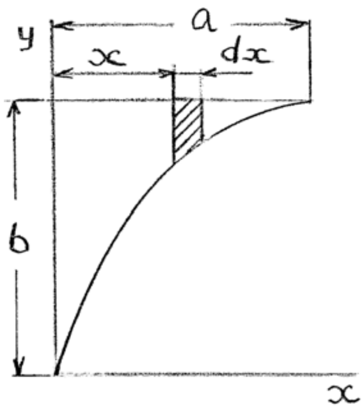
or $I_y = \dfrac{5}{33} a^3 b$ ◀



PROBLEM 9.2

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $\qquad x = 0,\ y = 0{:}\quad 0 = ka^2 + c$

$$k = -\frac{c}{a^2}$$

$$x = a, \quad y = b: \qquad b = c$$

$$\therefore k = -\frac{b}{a^2}$$

Then $\qquad y = -\frac{b}{a^2}(x-a)^2 + b$

$$= -\frac{b}{a^2}\left(x^2 - 2ax + a^2\right) + b$$

Now $\qquad dI_y = x^2 dA = x^2(ydx) = \left(-\frac{b}{a^2}x^4 + \frac{2b}{a}x^3 - bx^2 + bx^2\right)dx$

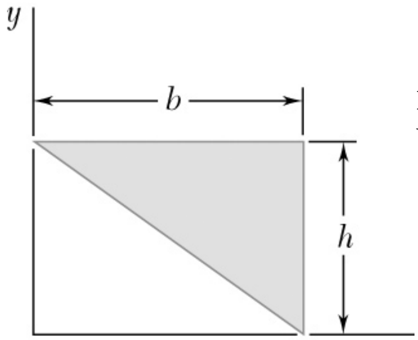
$$= \left(-\frac{b}{a^2}x^4 + \frac{2b}{a}x^3\right)dx$$

Then $\qquad I_y = \int dI_y = \int_0^a \left(-\frac{b}{a^2}x^4 + \frac{2b}{a}x^3\right)dx$

$$= b\left[-\frac{1}{a^2}\frac{x^5}{5} + \frac{2}{a}\frac{x^4}{4}\right]_0^a$$

$$= b\left(\frac{a^3}{5} + \frac{2a^3}{4}\right) = ba^3\left(\frac{1}{2} - \frac{1}{5}\right)$$

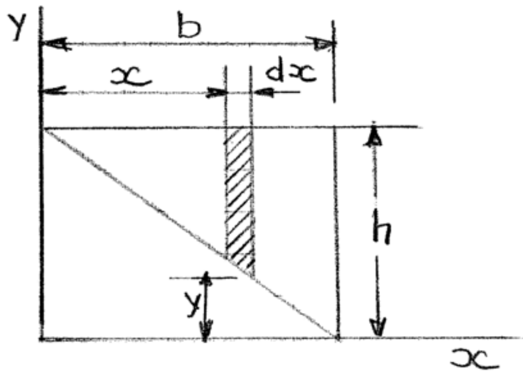
$$I_y = \frac{3a^3b}{10} \blacktriangleleft$$



PROBLEM 9.3

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION

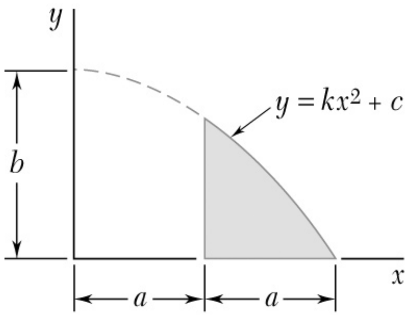


By observation
$$y = h - \frac{h}{b}x$$
$$= h\left(1 - \frac{x}{b}\right)$$

Now
$$dI_y = x^2 dA = x^2\left[(h - y)dx\right]$$
$$= x^2\left[h - h\left(1 - \frac{x}{b}\right)\right]dx$$
$$= \frac{hx^3}{b}dx$$

Then
$$I_y = \int dI_y = \int_0^b \frac{hx^3}{b}dx = \frac{hx^4}{4b}\bigg|_0^b = \frac{hb^4}{4b}$$

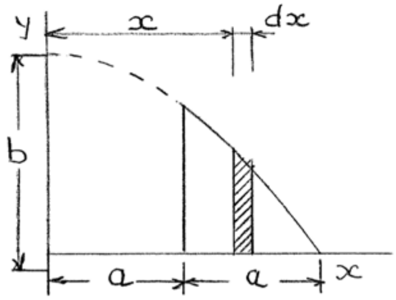
$$I_y = \frac{b^3h}{4} \blacktriangleleft$$



PROBLEM 9.4

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



Have $$y = kx^2 + c$$

At $$x = 0, y = b: \quad b = k(0) + c$$

or $$c = b$$

At $$x = 2a, \;\; y = 0: \quad 0 = k(2a)^2 + b$$

or $$k = -\frac{b}{4a^2}$$

Then $$y = -\frac{b}{4a^2}x^2 + b$$

$$= \frac{b}{4a^2}\left(4a^2 - x^2\right)$$

Then $$I_y = \int x^2 dA, \qquad dA = ydx = \frac{b}{4a^2}\left(4a^2 - x^2\right)dx$$

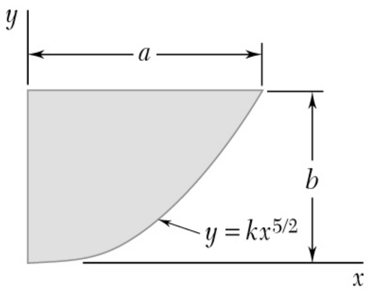
$$I_y = \int_a^{2a} x^2 dA = \frac{b}{4a^2}\int_a^{2a} x^2\left(4a^2 - x^2\right)dx$$

$$= \frac{b}{4a^2}\left[4a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_a^{2a}$$

$$= \frac{b}{3}\left(8a^3 - a^3\right) - \frac{b}{20a^2}\left(32a^5 - a^5\right)$$

$$= \frac{7a^3b}{3} - \frac{31a^3b}{20}$$

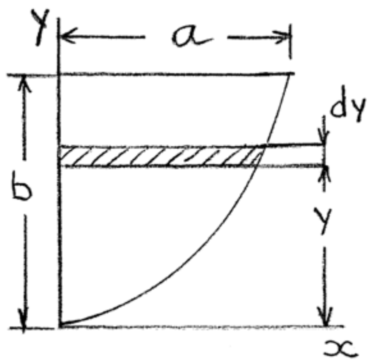
$$I_y = \frac{47}{60}a^3b \blacktriangleleft$$



PROBLEM 9.5

Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION



At $x = a,\ y = b:\quad b = ka^{\frac{3}{2}}$

or $k = \dfrac{b}{a^{\frac{3}{2}}}$

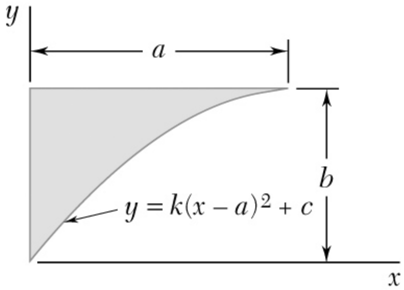
$\therefore\ y = \dfrac{b}{a^{\frac{3}{2}}}\, x^{\frac{3}{2}}$

$I_x = \int y^2 dA \qquad dA = xdy$

$= \int_0^b y^2 \left[\dfrac{a}{b^{\frac{2}{5}}}\, y^{\frac{2}{5}} dy \right]$

$= \dfrac{a}{b^{\frac{2}{5}}} \times \dfrac{5}{17}\, y^{\frac{17}{5}} \Big|_0^b = \dfrac{5a}{17} \dfrac{b^{\frac{17}{5}}}{b^{\frac{2}{5}}}$

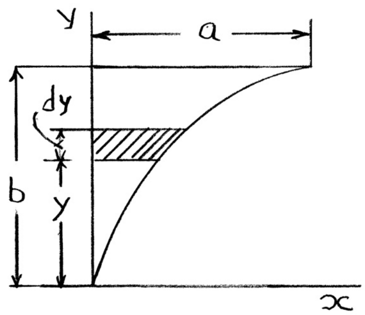
or $I_x = \dfrac{5}{17} ab^3$ ◀



PROBLEM 9.6

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



At $$x = 0,\ \ y = 0\!:\quad 0 = ka^2 + c$$

$$k = -\frac{c}{a^2}$$

$$x = a,\quad y = b \quad b = c$$

$$k = -\frac{b}{a^2}$$

Then $$y = b - \frac{b}{a^2}\left(x - a\right)^2$$

Now $$dI_x = y^2 dA = y^2\left(xdy\right)$$

From above $$\left(x - a\right)^2 = \frac{a^2}{b}\left(b - y\right)$$

Then $$x - a = a^2\sqrt{1 - \frac{y}{b}}$$

and $$x = a^2\sqrt{1 - \frac{y}{b}} + a$$

Then $$dI_x = ay^2\left(1 + \sqrt{1 - \frac{y}{b}}\right)dy$$

and $$I_x = \int dI_x = a\int_0^b y^2\left(1 + \sqrt{1 - \frac{y}{b}}\right)dy$$

$$= a\frac{y^3}{3}\bigg|_0^b + a\int_0^b y^2\left(\sqrt{1 - \frac{y}{6}}\right)dy$$

PROBLEM 9.6 CONTINUED

For the second integral use substitution

$$u = 1 - \frac{y}{b} \Rightarrow du = -\frac{1}{b}dy,\ y = b(1-u)$$

$$\begin{aligned} y = 0 \quad u = 1 \\ y = b \quad u = 0 \end{aligned}$$

Now $$\int_0^b y^2 \left(\sqrt{1 - \frac{y}{b}}\right) dy = -\int_0^b b^2 (1-u)^2 u^{\frac{1}{2}} du$$

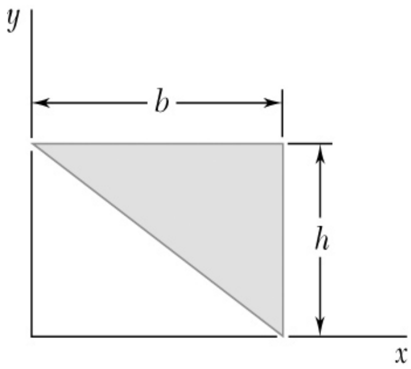
$$= -b^3 \int_1^0 \left(u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du = -b^3 \left(\frac{2}{3}u^{\frac{3}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{7}u^{\frac{7}{2}}\right)\Bigg|_1^0$$

$$= +b^3 \left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right) = b^3 \left(\frac{70 - 84 + 30}{105}\right) = \frac{16b^3}{105}$$

Then $$I_x = a\frac{b^3}{3} + \frac{16ab^3}{105} = \frac{51}{105}ab^3$$

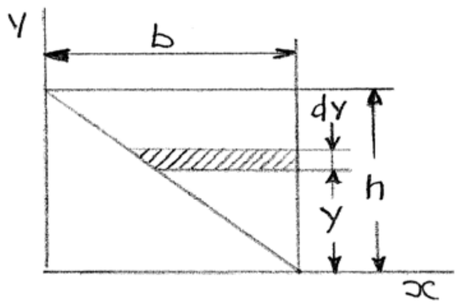
or $I_x = \dfrac{17}{35}ab^3$ ◀

PROBLEM 9.7



Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION



By observation

$$y = h - \frac{h}{b}x$$

$$= h\left(1 - \frac{x}{b}\right)$$

or

$$x = b\left(1 - \frac{y}{h}\right)$$

Now

$$dI_x = y^2 dA = y^2(b - x)dy$$

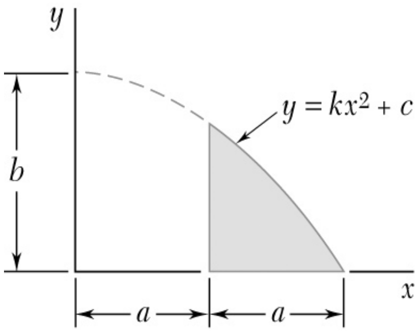
$$= y^2\left(b - b + \frac{by}{h}dy\right)$$

$$= \frac{by^3}{h}dy$$

Then

$$I_x = \int_0^h \frac{by^3}{h}dy = \frac{by^4}{4h}\bigg|_0^h = \frac{bh^4}{4h}$$

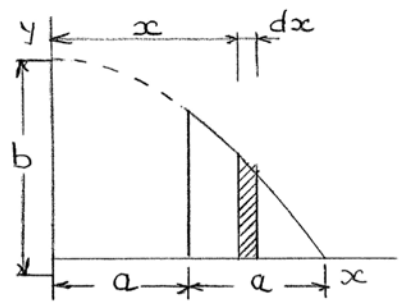
or $I_x = \dfrac{bh^3}{4}$ ◀



PROBLEM 9.8

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



Have $$y = kx^2 + c$$

At $$x = 0, \quad y = b: \quad b = k(0) + c$$

or $$c = b$$

At $$x = 2a, \quad y = 0: \quad 0 = k(2a)^2 + b$$

or $$k = -\frac{b}{4a^2}$$

Then $$y = \frac{b}{4a^2}\left(4a^2 - x^2\right)$$

Now $$dI_x = \frac{1}{3}y^3dx$$

$$= \frac{1}{3}\frac{b^3}{64a^6}\left(4a^2 - x^2\right)^3 dx$$

PROBLEM 9.8 CONTINUED

Then $$I_x = \int dI_x$$

$$= \frac{1}{3}\frac{b^3}{64a^6}\int_a^{2a}\left(4a^2 - x^2\right)^3 dx$$

$$= \frac{b^3}{192a^6}\int_a^{2a}\left(64a^6 - 48a^4x^2 + 12a^2x^4 - x^6\right)dx$$

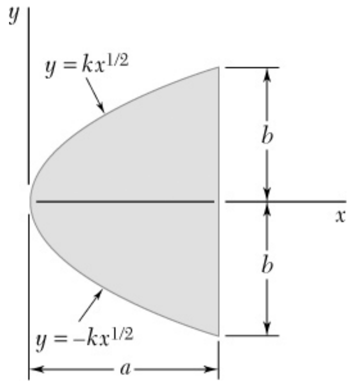
$$= \frac{b^3}{192a^6}\left[64a^6x - 16a^4x^3 + \frac{12}{5}a^2x^5 - \frac{x^7}{7}\right]_a^{2a}$$

$$= \frac{b^3}{192a^6}\left[64a^7(2-1) - 16a^7(8-1)\right.$$

$$\left.+ \frac{12}{5}a^7(32-1) - \frac{1}{7}(128-1)\right]$$

$$= \frac{ab^3}{192}\left(64 - 112 + \frac{372}{5} - \frac{127}{7}\right) = 0.043006ab^3$$

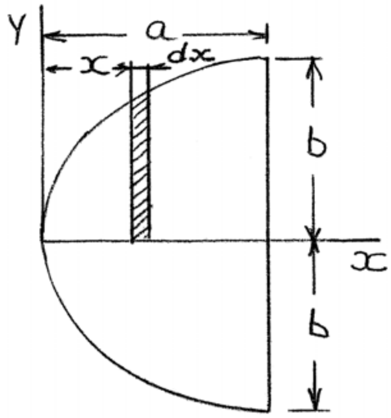
$$I_x = 0.0430ab^3 \blacktriangleleft$$



PROBLEM 9.9

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



At $$x = a,\ \ y = b\text{:}\quad b = ka^{\frac{1}{2}}$$

or $$k = \frac{b}{\sqrt{a}}$$

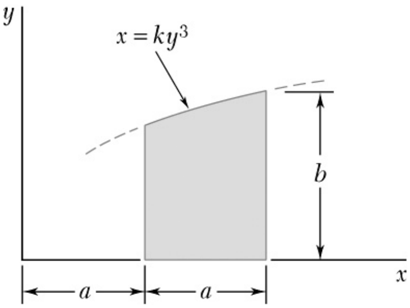
Then $$y = \frac{b}{\sqrt{a}}x^{\frac{1}{2}}$$

Now $$dI_x = \frac{1}{3}y^3dx = \frac{1}{3}\left(\frac{b}{\sqrt{a}}\right)^3 x^{\frac{3}{2}}dx$$

Then $$I_x = 2\int_0^a dI_x = 2\int_0^a \frac{1}{3}\left(\frac{b}{\sqrt{a}}\right)^3 x^{\frac{3}{2}}dx$$

$$= \frac{2}{3}\left(\frac{b}{\sqrt{a}}\right)^3 \frac{2}{5}x^{\frac{5}{2}}\Bigg|_0^a = \frac{4}{15}\frac{b^3}{a^{\frac{3}{2}}}a^{\frac{5}{2}}$$

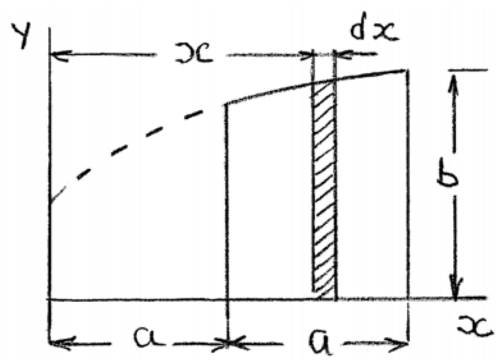
$$I_x = \frac{4}{15}ab^3 \blacktriangleleft$$



PROBLEM 9.10

Determine by direct integration the moment of inertia of the shaded area with respect to the *x* axis.

SOLUTION



At $$x = 2a,\ y = b: \qquad 2a = kb^3$$

or $$k = \frac{2a}{b^3}$$

Then $$x = \frac{2a}{b^3} y^3$$

or $$y = \frac{b}{(2a)^{\frac{1}{3}}} x^{\frac{1}{3}}$$

Now $$dI_x = \frac{1}{3} y^3 dx = \frac{1}{3} \frac{b^3}{2a} x dx$$

Then $$I_x = \int dI_x = \frac{1}{3} \frac{b^3}{2a} \int_a^{2a} x dx = \frac{1}{6} \frac{b^3}{a} \frac{1}{2} x^2 \Big|_a^{2a}$$

$$= \frac{b^3}{12a}\left(4a^2 - a^2\right)$$

$$I_x = \frac{1}{4} ab^3 \blacktriangleleft$$