PROBLEM 9.186

For the component described in the problem indicated, determine (*a*) the principal moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

Problem 9.150 and 9.172

SOLUTION

(*a*) From the solutions to Problem 9.150

$$I_x = 14.32 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_y = I_z = 18.62 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

From Problem 9.172 $\qquad I_{xy} = I_{zx} = 4.297 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2,\ I_{xz} = 0$

Substituting into Eq. (9.56) and using $\qquad I_y = I_z, \qquad I_{xy} = I_{zx}, \qquad I_{yz} = 0$

$$K^3 - \left(I_x + 2I_y\right)K^2 + \left[I_x\left(2I_y\right) + I_y^2 - 2\left(I_{xy}\right)^2\right]K - \left(I_x I_y^2 - 2I_y I_{xy}^2\right) = 0$$

$$K^3 - \left[14.32 \times 10^{-3} + 2\left(18.62 \times 10^{-2}\right)\right]K^2 + \left[\left(14.32 \times 10^{-3}\right)(2)\left(18.62 \times 10^{-3}\right) + \left(18.62 \times 10^{-3}\right)^2\right.$$
$$\left. - 2\left(4.297 \times 10^{-3}\right)^2\right]K - \left[\left(14.32 \times 10^{-3}\right)\left(18.62 \times 10^{-3}\right)^2 - 2\left(18.62 \times 10^{-3}\right)\left(4.297 \times 10^{-3}\right)^2\right] = 0$$

or $\qquad K^3 - 51.56 \times 10^{-3}K^2 + 0.84305 \times 10^{-3}K - 0.004277 \times 10^{-3} = 0$

Solving: $\qquad K_1 = 0.010022\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \qquad$ or $\quad K_1 = 10.02 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$\qquad K_2 = 0.018624\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \qquad$ or $\quad K_2 = 18.62 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$\qquad K_3 = 0.022914\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \qquad$ or $\quad K_3 = 22.9 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

(*b*) To determine the direction cosines $\lambda_x, \lambda_y, \lambda_z$ of each principal axis, use two of the equations of Equations (9.54) and Equation (9.57). Then

$\mathbf{K}_1$: Begin with Equations (9.54b) and (9.54c):

$$-I_{xy}(\lambda_x)_1 + (I_y - k_1)(\lambda_y)_1 - \cancelto{0}{I_{yz}}(\lambda_z) = 0$$

$$-I_{zx}(\lambda_x)_1 - I_{yz}(\lambda_y)_1 + (I_z - K_1)(\lambda_z) = 0$$

or $\qquad -4.297 \times 10^{-3}(\lambda_x)_1 + \left(18.62 \times 10^{-3} - 10.02 \times 10^{-3}\right)(\lambda_y)_1 = 0$

$$-4.297 \times 10^{-3}(\lambda_x)_1 + \left(18.62 \times 10^{-3} - 10.02 \times 10^{-3}\right)(\lambda_z)_1 = 0$$

PROBLEM 9.186 CONTINUED

$$\left(\lambda_y\right)_1 = \left(\lambda_z\right)_1 = 0.49965\left(\lambda_x\right)_1$$

$$\left(\lambda_x\right)_1^2 + 2\left[0.49965\left(\lambda_x\right)_1\right]^2 = 1$$

$$\left(\lambda_x\right)_1 = 0.81669$$

$$\left(\lambda_y\right)_1 = \left(\lambda_z\right)_1 = 0.40806$$

$$\left(\theta_x\right)_1 = 35.2°; \quad \left(\theta_y\right)_1 = \left(\theta_z\right)_1 = 65.9° \blacktriangleleft$$

$\mathbf{K}_2$: Begin with Equations (9.54a) and (9.54b):

$$\left(I_x - K_2\right)\left(\lambda_x\right)_2 - I_{xy}\left(\lambda_y\right)_2 - I_{zx}\left(\lambda_z\right)_2 = 0$$

$$-I_{xy}\left(\lambda_x\right)_2 + \left(I_y - K_2\right)\left(\lambda_y\right)_2 - \cancelto{0}{I_{yz}}\left(\lambda_z\right)_2 = 0$$

Substituting:

$$\left(14.32\times10^{-3} - 18.62\times10^{-3}\right)\left(\lambda_x\right)_2 - 4.297\times10^{-3}\left[\left(\lambda_y\right)_2 - \left(\lambda_z\right)_2\right] = 0 \qquad \text{(i)}$$

$$-4.297\times10^{-3}\left(\lambda_x\right)_2 + \left(18.62\times10^{-3} - 18.62\times10^{-3}\right)\left(\lambda_y\right)_2 = 0 \qquad \text{(ii)}$$

From (ii) $\qquad \left(\lambda_x\right)_2 = 0$

From (i) $\qquad \left(\lambda_y\right)_2 = -\left(\lambda_z\right)_2$

Substituting: $\qquad \left(\lambda_x\right)_2^2 + \left(\lambda_y\right)_2^2 + \left[-\left(\lambda_z\right)_2\right]^2 = 1$

$$\left(\lambda_y\right)_2 = \frac{1}{\sqrt{2}}$$

$$\left(\theta_x\right)_2 = 90.0°, \left(\theta_y\right)_2 = 45.0°, \left(\theta_z\right)_2 = 135.0° \blacktriangleleft$$

$\mathbf{K}_3$: Begin with Equations (9.54b) and (9.54c)

$$-I_{xy}\left(\lambda_x\right)_3 + \left(I_y - K_3\right)\left(\lambda_y\right)_3 + I_{yz}\left(\lambda_z\right)_3 = 0$$

$$-I_{zx}\left(\lambda_x\right)_3 - \cancelto{0}{I_{yz}}\left(\lambda_y\right) + \left(I_z - K_3\right)\left(\lambda_z\right)_3 = 0$$

Substituting:

$$-4.297\times10^{-3}\left(\lambda_x\right)_3 + \left(18.62\times10^{-3} - 22.9\times10^{-3}\right)\left(\lambda_z\right)_3 = 0$$

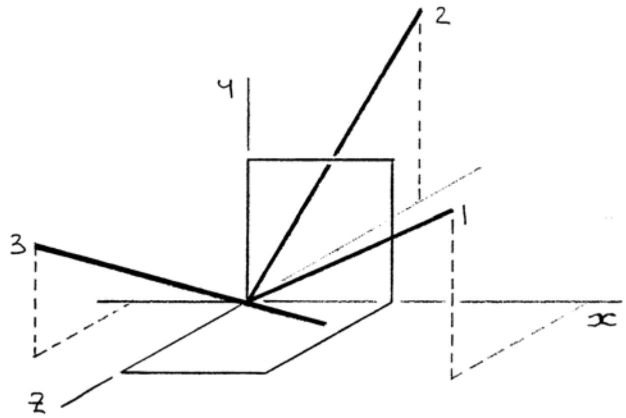
$$-4.297\times10^{-3}\left(\lambda_x\right)_3 + \left(18.62\times10^{-3} - 22.9\times10^{-3}\right)\left(\lambda_z\right)_3 = 0$$

Simplifying: $\qquad \left(\lambda_y\right)_3 = \left(\lambda_z\right)_3 = -\left(\lambda_x\right)_3$

$$\left(\lambda_x\right)_3^2 + 2\left[-\left(\lambda_x\right)_3\right]^2 = 1 \Rightarrow \left(\lambda_x\right)_3 = \frac{1}{\sqrt{3}} \qquad \text{and} \qquad \left(\lambda_y\right)_3 = \left(\lambda_z\right)_3 = -\frac{1}{\sqrt{3}}$$

$$\left(\theta_x\right)_3 = 54.7°, \left(\theta_y\right)_3 = \left(\theta_z\right)_3 = 125.3° \blacktriangleleft$$

PROBLEM 9.186 CONTINUED

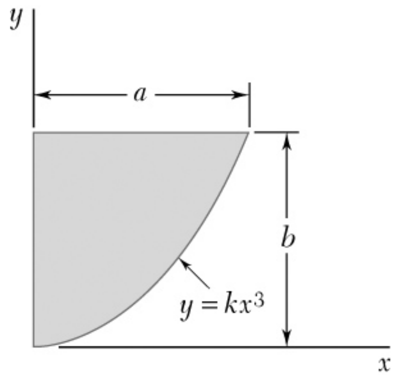


(c) Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set to obtain the direction cosines corresponding to the labeled axis, the negative root of Equation (i) must be chosen; that is:

$$(\lambda_x)_3 = -\frac{1}{\sqrt{3}}$$

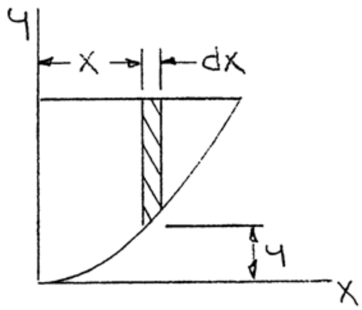
Then: $(\theta_x)_3 = 125.3°\quad (\theta_y)_3 = (\theta_z)_3 = 54.7°$ ◀

PROBLEM 9.187



Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $\quad x = a, y = b: \quad b = ka^3$

or $\quad k = \dfrac{b}{a^3}$

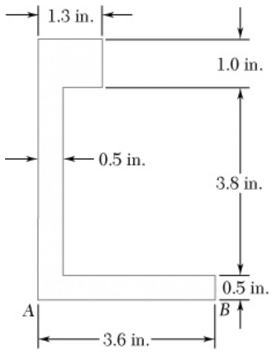
Then $\quad y = \dfrac{b}{a^3}x^3$

Now
$$dI_y = x^2 dA = x^2\left[(b - y)dx\right]$$
$$= bx^2\left(1 - \frac{x^3}{a^3}\right)dx$$

Then
$$I_y = \int dI_y = \int_0^a bx^2\left(1 - \frac{x^3}{a^3}\right)dx$$
$$= b\left[\frac{1}{3}x^3 - \frac{x^6}{6a^3}\right]_0^a$$

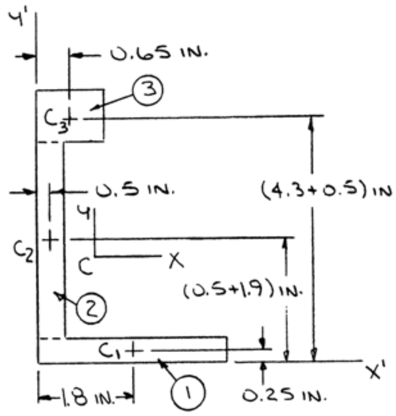
or $\quad I_y = \dfrac{1}{6}a^3 b$ ◀

PROBLEM 9.188



Determine the moments of inertia $\bar{I}_x$ and $\bar{I}_y$ of the area shown with respect to centroidal axes respectively parallel and perpendicular to side AB.

SOLUTION



First locate centroid C of the area.

	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$3.6 \times 0.5 = 1.8$	1.8	0.25	3.24	0.45
2	$0.5 \times 3.8 = 1.9$	0.25	2.4	0.475	4.56
3	$1.3 \times 1 = 1.3$	0.65	4.8	0.845	6.24
Σ	5.0			4.560	11.25

Then $\quad \bar{X}\Sigma A = \Sigma \bar{x}A: \quad \bar{X}(5 \text{ in}^2) = 4.560 \text{ in}^3$

or $\quad \bar{X} = 0.912 \text{ in.}$

And $\quad \bar{Y}\Sigma A = \Sigma \bar{y}A: \quad \bar{Y}(5 \text{ in}^2) = 11.25 \text{ in}^3$

or $\quad \bar{Y} = 2.25 \text{ in.}$

Now $\quad \bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3$

where
$$(I_x)_1 = \frac{1}{12}(3.6 \text{ in.})(0.5 \text{ in.})^3 + (1.8 \text{ in}^2)\left[(2.25 - 0.25)\text{in.}\right]^2$$
$$= (0.0375 + 7.20) \text{ in}^4 = 7.2375 \text{ in}^4$$
$$(I_x)_2 = \frac{1}{12}(0.5 \text{ in.})(3.8 \text{ in.})^3 + (1.9 \text{ in}^2)\left[(2.4 - 2.25) \text{ in.}\right]^2$$
$$= (2.2863 + 0.0428) \text{ in}^4 = 2.3291 \text{ in}^4$$
$$(I_x)_3 = \frac{1}{12}(1.3 \text{ in.})(1 \text{ in.})^3 + (1.3 \text{ in}^2)\left[(4.8 - 2.25 \text{ in.})\right]^2$$
$$= (0.1083 + 8.4533) \text{ in}^4 = 8.5616 \text{ in}^4$$

Then $\quad \bar{I}_x = (7.2375 + 2.3291 + 8.5616) \text{ in}^4 = 18.1282 \text{ in}^4$

or $\bar{I}_x = 18.13 \text{ in}^4$ ◀

PROBLEM 9.188 CONTINUED

Also $$\bar{I}_y = \left(I_y\right)_1 + \left(I_y\right)_2 + \left(I_y\right)_3$$

where $$\left(I_y\right)_1 = \frac{1}{12}(0.5\text{ in.})(3.6\text{ in.})^3 + \left(1.8\text{ in}^2\right)\left[(1.8 - 0.912)\text{in.}\right]^2$$

$$= (1.9440 + 1.4194)\text{in}^4 = 3.3634\text{ in}^4$$

$$\left(I_y\right)_2 = \frac{1}{12}(3.8\text{ in.})(0.5\text{ in.})^3 + \left(1.9\text{ in}^2\right)\left[(0.912 - 0.25)\text{ in.}\right]^2$$

$$= (0.0396 + 0.8327)\text{in}^4 = 0.8723\text{ in}^4$$

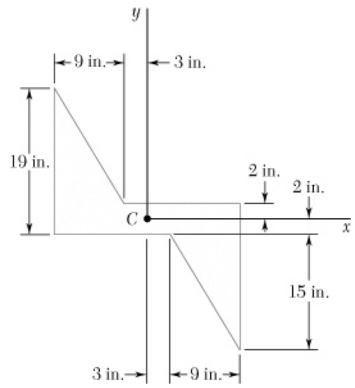
$$\left(I_y\right)_3 = \frac{1}{12}(1\text{ in.})(1.3\text{ in.})^3 + \left(1.3\text{ in}^2\right)\left[(0.912 - 0.65)\text{ in.}\right]^2$$

$$= (0.1831 + 0.0892)\text{in}^4 = 0.2723\text{ in}^4$$

Then $$\bar{I}_y = (3.3634 + 0.8723 + 0.2723)\text{in}^4 = 4.5080\text{ in}^4$$

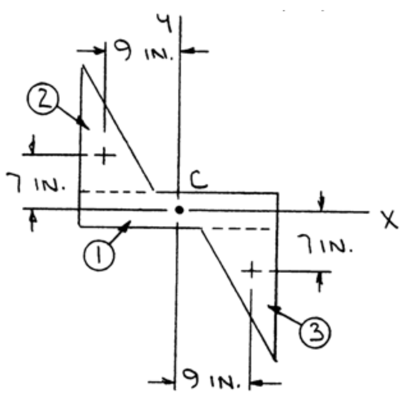
or $\bar{I}_y = 4.51\text{ in}^4$ ◀

PROBLEM 9.189



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



Have $$\bar{I}_{xy} = \left(\bar{I}_{xy}\right)_1 + \left(I_{xy}\right)_2 + \left(I_{xy}\right)_3$$

Now, symmetry implies

$$\left(\bar{I}_{xy}\right)_1 = 0$$

and for each triangle $$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

where, using the results of Sample Problem 9.6, $\bar{I}_{x'y'} = -\frac{1}{72}b^2h^2$ for both triangles. Note that the sign of $\bar{I}_{x'y'}$ is unchanged because the angles of rotation are $0°$ and $180°$ for triangles 2 and 3, respectively.

Now

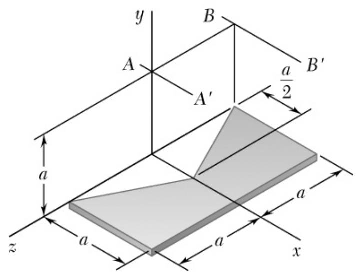
	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}\,\bar{y}A$, in^4
2	$\frac{1}{2}(9)(15) = 67.5$	–9	7	–4252.5
3	$\frac{1}{2}(9)(15) = 67.5$	9	–7	–4252.5
Σ				–8505

Then $$\bar{I}_{xy} = 2\left[-\frac{1}{72}(9\text{ in.})^2(15\text{ in.})^2\right] - 8505\text{ in}^4$$

$$= -9011.25\text{ in}^4$$

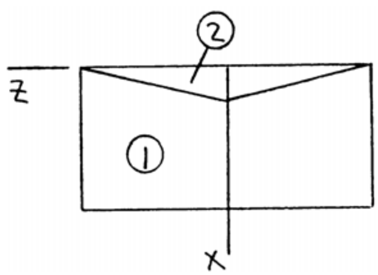
or $\bar{I}_{xy} = -9010\text{ in}^4$ ◀

PROBLEM 9.190



A piece of thin, uniform sheet metal is cut to form the machine component shown. Denoting the mass of the component by m, determine its moment of inertia with respect to (a) the x axis, (b) the y axis.

SOLUTION



First note $\quad \text{mass} = m = \rho V = \rho t A$

$$= \rho t\left[(2a)(a) - \frac{1}{2}(2a)\left(\frac{a}{2}\right)\right]$$

$$= \frac{3}{2}\rho t a^2$$

Also $\quad I_{\text{mass}} = \rho t I_{\text{area}}$

$$= \frac{2m}{3a^2} I_{\text{area}}$$

(a) Now

$$\bar{I}_{x,\text{area}} = (I_x)_{1,\text{area}} - 2(I_x)_{2,\text{area}}$$

$$= \frac{1}{12}(a)(2a)^3 - 2\left[\frac{1}{12}\left(\frac{a}{2}\right)(a)^3\right]$$

$$= \frac{7}{12}a^4$$

Then $\quad \bar{I}_{x,\text{mass}} = \frac{2m}{3a^2} \times \frac{7}{12}a^4$

or $\bar{I}_x = \frac{7}{18}ma^2$ ◀

(b) Have $\quad \bar{I}_{z,\text{ area}} = (I_z)_{1,\text{ area}} - 2(I_z)_{2,\text{ area}}$

$$= \frac{1}{3}(2a)(a)^3 - 2\left[\frac{1}{12}(a)\left(\frac{a}{2}\right)^3\right]$$

$$= \frac{31}{48}a^4$$

Then $\quad I_{z,\text{mass}} = \frac{2m}{3a^2} \times \frac{31}{48}a^4$

$$= \frac{31}{72}ma^2$$

PROBLEM 9.190 CONTINUED

Finally, $$I_{y,\text{ mass}} = \bar{I}_{x,\text{ mass}} + I_{z,\text{ mass}}$$

$$= \frac{7}{18}ma^2 + \frac{31}{72}ma^2$$

$$= \frac{59}{72}ma^2$$

or $I_y = 0.819\text{ ma}^2$ ◀