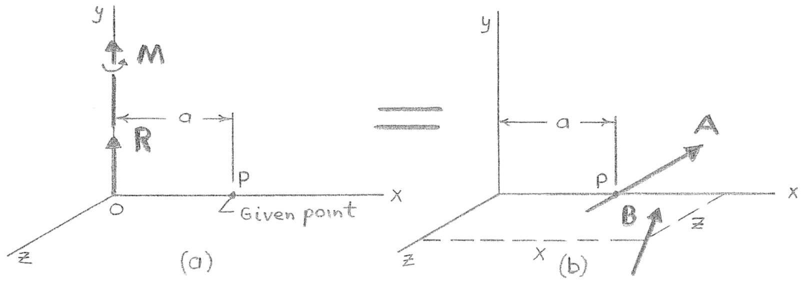
PROBLEM 3.140

Show that a wrench can be replaced with two perpendicular forces, one of which is applied at a given point.

SOLUTION



First, observe that it is always possible to construct a line perpendicular to a given line so that the constructed line also passes through a given point. Thus, it is possible to align one of the coordinate axes of a rectangular coordinate system with the axis of the wrench while one of the other axes passes through the given point.

See Figures *a* and *b*.

Have $\mathbf{R} = R\mathbf{j}$ and $\mathbf{M} = M\mathbf{j}$ and are known.

The unknown forces **A** and **B** can be expressed as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \quad \text{and} \quad \mathbf{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$$

The distance *a* is known. It is assumed that force **B** intersects the *xz* plane at $(x, 0, z)$. Then for equivalence

$$\Sigma F_x: \quad 0 = A_x + B_x \tag{1}$$

$$\Sigma F_y: \quad R = A_y + B_y \tag{2}$$

$$\Sigma F_z: \quad 0 = A_z + B_z \tag{3}$$

$$\Sigma M_x: \quad 0 = -zB_y \tag{4}$$

$$\Sigma M_y: \quad M = -aA_z - xB_z + zB_x \tag{5}$$

$$\Sigma M_z: \quad 0 = aA_y + xB_y \tag{6}$$

Since **A** and **B** are made perpendicular,

$$\mathbf{A}\cdot\mathbf{B} = 0 \quad \text{or} \quad A_xB_x + A_yB_y + A_zB_z = 0 \tag{7}$$

There are eight unknowns: $A_x, A_y, A_z, B_x, B_y, B_z, x, z$

But only seven independent equations. Therefore, *there exists an infinite number of solutions.*

PROBLEM 3.140 CONTINUED

Next consider Equation (4): $0 = -zB_y$

If $B_y = 0,$ Equation (7) becomes $A_xB_x + A_zB_z = 0$

Using Equations (1) and (3) this equation becomes $A_x^2 + A_z^2 = 0$

Since the components of **A** must be real, a nontrivial solution is not possible. Thus, it is required that $B_y \neq 0,$ so that from Equation (4), $z = 0.$

To obtain one possible solution, arbitrarily let $A_x = 0.$

(Note: Setting $A_y,\ A_z,$ or B_z equal to zero results in unacceptable solutions.)

The defining equations then become.

$$0 = B_x \qquad (1)'$$

$$R = A_y + B_y \qquad (2)$$

$$0 = A_z + B_z \qquad (3)$$

$$M = -aA_z - xB_z \qquad (5)'$$

$$0 = aA_y + xB_y \qquad (6)$$

$$A_yB_y + A_zB_z = 0 \qquad (7)'$$

Then Equation (2) can be written $A_y = R - B_y$

Equation (3) can be written $B_z = -A_z$

Equation (6) can be written $x = -\dfrac{aA_y}{B_y}$

Substituting into Equation (5)′,

$$M = -aA_z - \left(-a\frac{R - B_y}{B_y}\right)\left(-A_z\right)$$

or

$$A_z = -\frac{M}{aR}B_y \qquad (8)$$

Substituting into Equation (7)′,

$$\left(R - B_y\right)B_y + \left(-\frac{M}{aR}B_y\right)\left(\frac{M}{aR}B_y\right) = 0$$

PROBLEM 3.140 CONTINUED

or $$B_y = \frac{a^2R^3}{a^2R^2 + M^2}$$

Then from Equations (2), (8), and (3)

$$A_y = R - \frac{a^2R^3}{a^2R^2 + M^2} = \frac{RM^2}{a^2R^2 + M^2}$$

$$A_z = -\frac{M}{aR}\left(\frac{a^2R^3}{a^2R^2 + M^2}\right) = -\frac{aR^2M}{a^2R^2 + M^2}$$

$$B_z = \frac{aR^2M}{a^2R^2 + M^2}$$

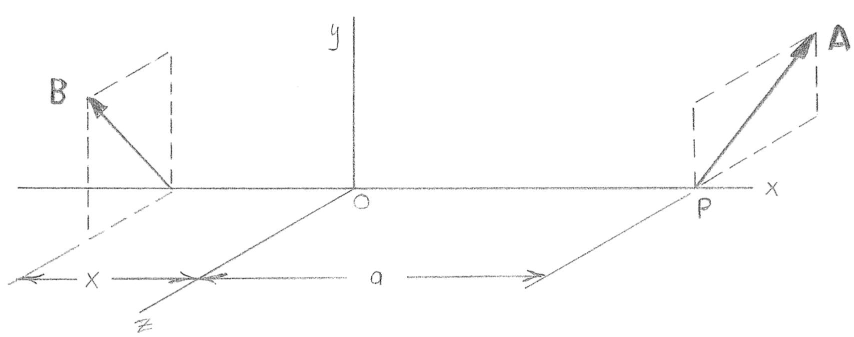
In summary

$$\mathbf{A} = \frac{RM}{a^2R^2 + M^2}\left(M\mathbf{j} - aR\mathbf{k}\right)$$

$$\mathbf{B} = \frac{aR^2}{a^2R^2 + M^2}\left(aR\mathbf{j} + M\mathbf{k}\right)$$

Which shows that it is possible to replace a wrench with two perpendicular forces, one of which is applied at a given point.

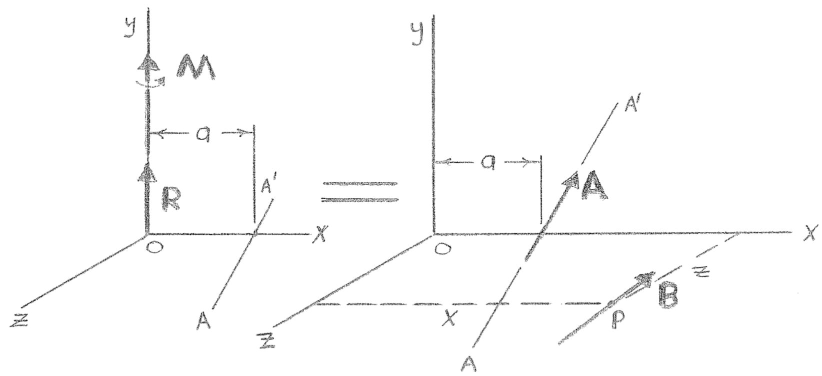
Lastly, if $R > 0$ and $M > 0$, it follows from the equations found for **A** and **B** that $A_y > 0$ and $B_y > 0$. From Equation (6), $x < 0$ (assuming $a > 0$). Then, as a consequence of letting $A_x = 0$, force **A** lies in a plane parallel to the *yz* plane and to the right of the origin, while force **B** lies in a plane parallel to the *yz* plane but to the left of the origin, as shown in the figure below.



PROBLEM 3.141

Show that a wrench can be replaced with two forces, one of which has a prescribed line of action.

SOLUTION



First, choose a rectangular coordinate system where one axis coincides with the axis of the wrench and another axis intersects the prescribed line of action (AA'). Note that it has been assumed that the line of action of force **B** intersects the *xz* plane at point $P(x,\ 0,\ z)$. Denoting the known direction of line AA' by

$$\boldsymbol{\lambda}_A = \lambda_x \mathbf{i} + \lambda_y \mathbf{j} + \lambda_z \mathbf{k}$$

it follows that force **A** can be expressed as

$$\mathbf{A} = A\boldsymbol{\lambda}_A = A\left(\lambda_x \mathbf{i} + \lambda_y \mathbf{j} + \lambda_z \mathbf{k}\right)$$

Force **B** can be expressed as

$$\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$$

Next, observe that since the axis of the wrench and the prescribed line of action AA' are known, it follows that the distance *a* can be determined. In the following solution, it is assumed that *a* is known.

Then, for equivalence

$$\Sigma F_x: \quad 0 = A\lambda_x + B_x \tag{1}$$

$$\Sigma F_y: \quad R = A\lambda_y + B_y \tag{2}$$

$$\Sigma F_z: \quad 0 = A\lambda_z + B_z \tag{3}$$

$$\Sigma M_x: \quad 0 = -zB_y \tag{4}$$

$$\Sigma M_y: \quad M = -aA\lambda_z + zB_x - xB_z \tag{5}$$

$$\Sigma M_z: \quad 0 = aA\lambda_y + xB_y \tag{6}$$

Since there are six unknowns $\left(A,\ B_x,\ B_y,\ B_z,\ x,\ z\right)$ and six independent equations, it will be possible to obtain a solution.

PROBLEM 3.141 CONTINUED

<u>*Case 1*</u>: Let $z = 0$ to satisfy Equation (4)

Now Equation (2) $$A\lambda_y = R - B_y$$

Equation (3) $$B_z = -A\lambda_z$$

Equation (6) $$x = -\frac{aA\lambda_y}{B_y} = -\left(\frac{a}{B_y}\right)\left(R - B_y\right)$$

Substitution into Equation (5)

$$M = -aA\lambda_z - \left[-\left(\frac{a}{B_y}\right)\left(R - B_y\right)\left(-A\lambda_z\right)\right]$$

$$\therefore \quad A = -\frac{1}{\lambda_z}\left(\frac{M}{aR}\right)B_y$$

Substitution into Equation (2)

$$R = -\frac{1}{\lambda_z}\left(\frac{M}{aR}\right)B_y\lambda_y + B_y$$

$$\therefore \quad B_y = \frac{\lambda_z aR^2}{\lambda_z aR - \lambda_y M}$$

Then $$A = -\frac{MR}{\lambda_z aR - \lambda_y M} = \frac{R}{\lambda_y - \frac{aR}{M}\lambda_z}$$

$$B_x = -A\lambda_x = \frac{\lambda_x MR}{\lambda_z aR - \lambda_y M}$$

$$B_z = -A\lambda_z = \frac{\lambda_z MR}{\lambda_z aR - \lambda_y M}$$

In summary

$$\mathbf{A} = \frac{P}{\lambda_y - \frac{aR}{M}\lambda_z}\boldsymbol{\lambda}_A \blacktriangleleft$$

$$\mathbf{B} = \frac{R}{\lambda_z aR - \lambda_y M}\left(\lambda_x M\mathbf{i} + \lambda_z aR\mathbf{j} + \lambda_z M\mathbf{k}\right) \blacktriangleleft$$

and $$x = a\left(1 - \frac{R}{B_y}\right) = a\left[1 - R\left(\frac{\lambda_z aR - \lambda_y M}{\lambda_z aR^2}\right)\right]$$

$$\text{or } x = \frac{\lambda_y}{\lambda_z}\frac{M}{R} \blacktriangleleft$$

Note that for this case, the lines of action of both **A** and **B** intersect the *x* axis.

PROBLEM 3.141 CONTINUED

<u>*Case 2*</u>: Let $B_y = 0$ to satisfy Equation (4)

Now Equation (2) $$A = \frac{R}{\lambda_y}$$

Equation (1) $$B_x = -R\left(\frac{\lambda_x}{\lambda_y}\right)$$

Equation (3) $$B_z = -R\left(\frac{\lambda_z}{\lambda_y}\right)$$

Equation (6) $aA\lambda_y = 0$ which requires $a = 0$

Substitution into Equation (5)

$$M = z\left[-R\left(\frac{\lambda_x}{\lambda_y}\right)\right] - x\left[-R\left(\frac{\lambda_z}{\lambda_y}\right)\right] \quad \text{or} \quad \lambda_z x - \lambda_x z = \left(\frac{M}{R}\right)\lambda_y$$

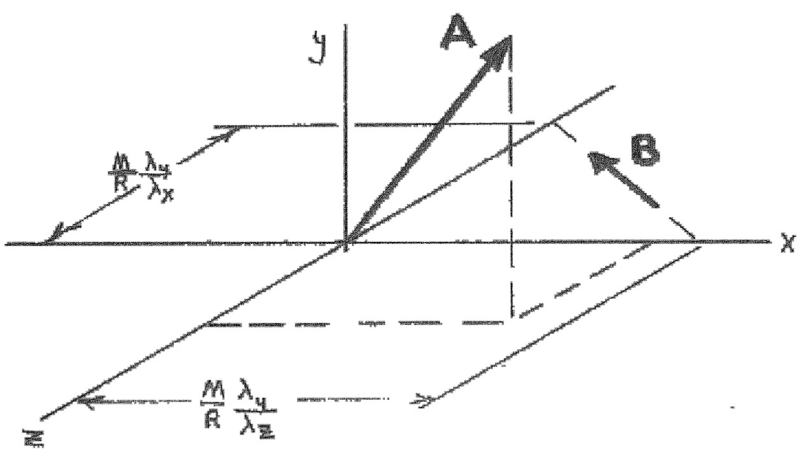
This last expression is the equation for the line of action of force **B**.

In summary

$$\mathbf{A} = \left(\frac{R}{\lambda_y}\right)\lambda_A$$

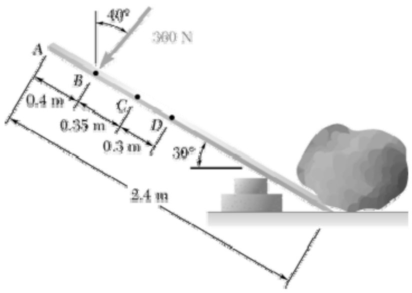
$$\mathbf{B} = \left(\frac{R}{\lambda_y}\right)\left(-\lambda_x\mathbf{i} - \lambda_z\mathbf{k}\right)$$

Assuming that $\lambda_x, \lambda_y, \lambda_z > 0$, the equivalent force system is as shown below.



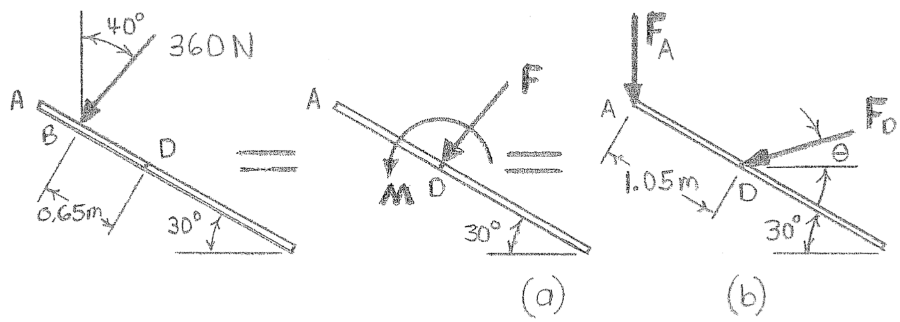
Note that the component of **A** in the *xz* plane is parallel to **B**.

PROBLEM 3.142



A worker tries to move a rock by applying a 360-N force to a steel bar as shown. (*a*) Replace that force with an equivalent force-couple system at *D*. (*b*) Two workers attempt to move the same rock by applying a vertical force at *A* and another force at *D*. Determine these two forces if they are to be equivalent to the single force of part *a*.

SOLUTION



(*a*) Have

$$\Sigma\mathbf{F}:\quad 360\text{ N}\left(-\sin 40^\circ\mathbf{i} - \cos 40^\circ\mathbf{j}\right) = -(231.40\text{ N})\mathbf{i} - (275.78\text{ N})\mathbf{j} = \mathbf{F}$$

$$\text{or } \mathbf{F} = 360\text{ N} \swarrow 50^\circ \blacktriangleleft$$

Have

$$\Sigma\mathbf{M}_D:\quad \mathbf{r}_{B/D} \times \mathbf{R} = \mathbf{M}$$

where

$$\mathbf{r}_{B/D} = -\left[(0.65\text{ m})\cos 30^\circ\right]\mathbf{i} + \left[(0.65\text{ m})\sin 30^\circ\right]\mathbf{j}$$

$$= -(0.56292\text{ m})\mathbf{i} + (0.32500\text{ m})\mathbf{j}$$

$$\therefore\quad \mathbf{M} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.56292 & 0.32500 & 0 \\ -231.40 & -275.78 & 0 \end{vmatrix}\text{ N}\cdot\text{m} = \left[(155.240 + 75.206)\text{ N}\cdot\text{m}\right]\mathbf{k}$$

$$= (230.45\text{ N}\cdot\text{m})\mathbf{k} \qquad \text{or } \mathbf{M} = 230\text{ N}\cdot\text{m} \circlearrowleft \blacktriangleleft$$

(*b*) Have

$$\Sigma\mathbf{M}_D:\quad \mathbf{M} = \mathbf{r}_{A/D} \times \mathbf{F}_A$$

where

$$\mathbf{r}_{A/D} = -\left[(1.05\text{ m})\cos 30^\circ\right]\mathbf{i} + \left[(1.05\text{ m})\sin 30^\circ\right]\mathbf{j}$$

$$= -(0.90933\text{ m})\mathbf{i} + (0.52500\text{ m})\mathbf{j}$$

PROBLEM 3.142 CONTINUED

$$\therefore \quad F_A \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.90933 & 0.52500 & 0 \\ 0 & -1 & 0 \end{vmatrix} \text{ N}\cdot\text{m} = \left[230.45 \text{ N}\cdot\text{m}\right]\mathbf{k}$$

or

$$\left(0.90933F_A\right)\mathbf{k} = 230.45\mathbf{k}$$

$$\therefore \quad F_A = 253.42 \text{ N}$$

or $\mathbf{F}_A = 253 \text{ N} \downarrow$ ◀

Have

$$\Sigma\mathbf{F}: \quad \mathbf{F} = \mathbf{F}_A + \mathbf{F}_D$$

$$-\left(231.40 \text{ N}\right)\mathbf{i} - \left(275.78 \text{ N}\right)\mathbf{j} = -\left(253.42 \text{ N}\right)\mathbf{j} + F_D\left(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right)$$

From

$$\mathbf{i}: \quad 231.40 \text{ N} = F_D\cos\theta \qquad (1)$$

$$\mathbf{j}: \quad 22.36 \text{ N} = F_D\sin\theta \qquad (2)$$

Equation (2) divided by Equation (1)

$$\tan\theta = 0.096629$$

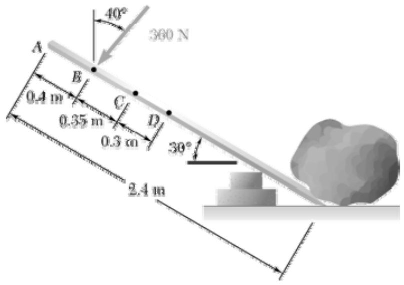
$$\therefore \quad \theta = 5.5193° \qquad \text{or} \qquad \theta = 5.52°$$

Substitution into Equation (1)

$$F_D = \frac{231.40}{\cos 5.5193°} = 232.48 \text{ N}$$

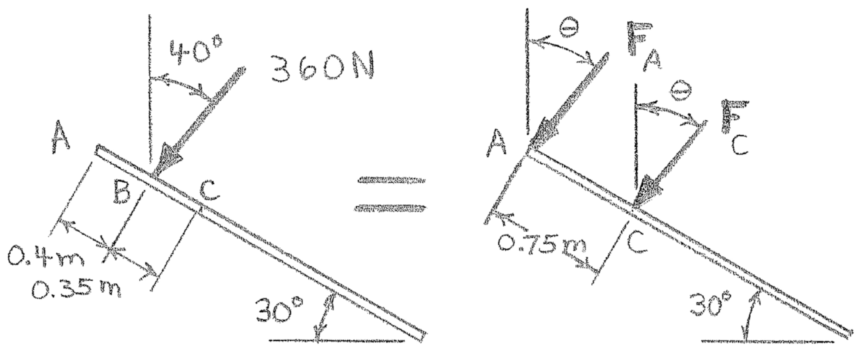
or $\mathbf{F}_D = 232 \text{ N} \measuredangle\ 5.52°$ ◀

PROBLEM 3.143



A worker tries to move a rock by applying a 360-N force to a steel bar as shown. If two workers attempt to move the same rock by applying a force at A and a parallel force at C, determine these two forces so that they will be equivalent to the single 360-N force shown in the figure.

SOLUTION



Have $\qquad \Sigma\mathbf{F}: \quad \mathbf{R} = \mathbf{F}_A + \mathbf{F}_C$

$$-\left[(360\text{ N})\sin 40°\right]\mathbf{i} - \left[(360\text{ N})\cos 40°\right]\mathbf{j} = -\left[(F_A + F_C)\sin\theta\right]\mathbf{i} - \left[(F_A + F_C)\cos\theta\right]\mathbf{j}$$

From $\qquad \mathbf{i}: \quad (360\text{ N})\sin 40° = (F_A + F_C)\sin\theta \qquad (1)$

$$\mathbf{j}: \quad (360\text{ N})\cos 40° = (F_A + F_C)\cos\theta \qquad (2)$$

Dividing Equation (1) by Equation (2),

$$\tan 40° = \tan\theta$$

$$\therefore \quad \theta = 40°$$

Substituting $\theta = 40°$ into Equation (1),

$$F_A + F_C = 360\text{ N} \qquad (3)$$

Have $\qquad \Sigma\mathbf{M}_C: \quad \mathbf{r}_{B/C} \times \mathbf{R} = \mathbf{r}_{A/C} \times \mathbf{F}_A$

where $\qquad \mathbf{r}_{B/C} = (0.35\text{ m})(-\cos 30°\mathbf{i} + \sin 30°\mathbf{j}) = -(0.30311\text{ m})\mathbf{i} + (0.175\text{ m})\mathbf{j}$

PROBLEM 3.143 CONTINUED

$$\mathbf{R} = (360\text{ N})(-\sin 40°\mathbf{i} - \cos 40°\mathbf{j}) = -(231.40\text{ N})\mathbf{i} - (275.78\text{ N})\mathbf{j}$$

$$\mathbf{r}_{A/C} = (0.75\text{ m})(-\cos 30°\mathbf{i} + \sin 30\mathbf{j}) = -(0.64952\text{ m})\mathbf{i} + (0.375\text{ m})\mathbf{j}$$

$$\mathbf{F}_A = F_A(-\sin 40°\mathbf{i} - \cos 40°\mathbf{j}) = F_A(-0.64279\mathbf{i} - 0.76604\mathbf{j})$$

$$\therefore \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.30311 & 0.175 & 0 \\ -231.40 & -275.78 & 0 \end{vmatrix} \text{N}\cdot\text{m} = F_A \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.64952 & 0.375 & 0 \\ -0.64279 & -0.76604 & 0 \end{vmatrix} \text{N}\cdot\text{m}$$

$$83.592 + 40.495 = (0.49756 + 0.24105)F_A$$

$$\therefore \quad F_A = 168.002\text{ N} \qquad \text{or} \qquad F_A = 168.0\text{ N}$$

Substituting into Equation (3),

$$F_C = 360 - 168.002 = 191.998\text{ N} \qquad \text{or} \qquad F_C = 192.0\text{ N}$$

or $\mathbf{F}_A = 168.0\text{ N} \measuredangle 50°$ ◀

$\mathbf{F}_C = 192.0\text{ N} \measuredangle 50°$ ◀