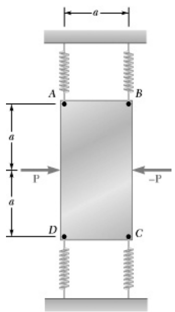
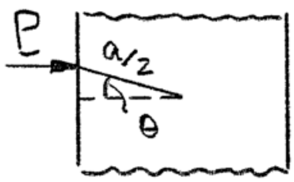


PROBLEM 10.91

The uniform plate *ABCD* of negligible mass is attached to four springs of constant k and is in equilibrium in the position shown. Knowing that the springs can act in either tension or compression and are undeformed in the given position, determine the range of values of the magnitude P of two equal and opposite horizontal forces **P** and **–P** for which the equilibrium position is stable.

SOLUTION

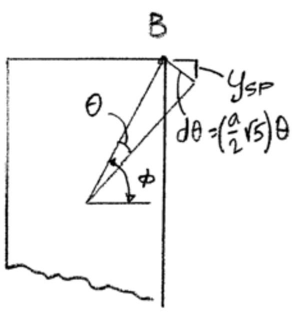


Consider a small clockwise rotation θ of the plate about its center.

Then $$V = 2V_P + 4V_{SP}$$

where $$V_P = P\left(\frac{a}{2}\cos\theta\right)$$

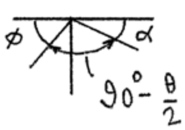
$$= \frac{1}{2}(Pa\cos\theta)$$



and $$V_{SP} = \frac{1}{2}ky_{SP}^2$$

Now $$d = \sqrt{\left(\frac{a}{2}\right)^2 + a^2}$$

$$= \frac{a}{2}\sqrt{5}$$



and $$\alpha = 180° - \left[\phi + \left(90° - \frac{\theta}{2}\right)\right]$$

$$= 90° - \left(\phi - \frac{\theta}{2}\right)$$

Then $$y_{SP} = \left[\left(\frac{a}{2}\sqrt{5}\right)\theta\right]\sin\alpha$$

$$= \frac{a}{2}\theta\sqrt{5}\sin\left[90° - \left(\phi - \frac{\theta}{2}\right)\right]$$

$$= \frac{a}{2}\theta\sqrt{5}\cos\left(\phi - \frac{\theta}{2}\right)$$

PROBLEM 10.91 CONTINUED

and $$V_{SP} = \frac{1}{2}k\left[\frac{a}{2}\theta\sqrt{5}\cos\left(\phi - \frac{\theta}{2}\right)\right]^2$$

$$= \frac{5}{8}ka^2\theta^2\cos^2\left(\phi - \frac{\theta}{2}\right)$$

$$\therefore\ V = Pa\cos\theta + \frac{5}{2}ka^2\theta^2\cos^2\left(\phi - \frac{\theta}{2}\right)$$

Then

$$\frac{dV}{d\theta} = -Pa\sin\theta + \frac{5}{8}ka^2\left[2\theta\cos^2\left(\phi - \frac{\theta}{2}\right)\right.$$

$$\left. + \theta^2\left(-\frac{1}{2}\right)\cos\left(\phi - \frac{\theta}{2}\right)\sin\left(\phi - \frac{\theta}{2}\right)\right]$$

$$= -Pa\sin\theta + \frac{5}{2}ka^2\left[2\theta\cos^2\left(\phi - \frac{\theta}{2}\right) + \frac{1}{2}\theta^2\sin(2\phi - \theta)\right]$$

$$\frac{d^2V}{d\theta^2} = -Pa\cos\theta + \frac{5}{2}ka^2\left[2\cos^2\left(\phi - \frac{\theta}{2}\right)\right.$$

$$-2\theta\left(-\frac{1}{2}\right)\cos\left(\phi - \frac{\theta}{2}\right)\sin\left(\phi - \frac{\theta}{2}\right) + \theta\sin(2\phi - \theta)$$

$$\left. -\frac{1}{2}\theta^2\cos(2\phi - \theta)\right]$$

$$= -Pa\cos\theta + \frac{5}{2}ka^2\left[2\cos^2\left(\phi - \frac{\theta}{2}\right) + \frac{3}{2}\theta\sin(2\phi - \theta)\right.$$

$$\left. -\frac{1}{2}\theta^2\cos(2\phi - \theta)\right]$$

$$\frac{d^2V}{d\theta^3} = Pa\sin\theta + \frac{5}{2}ka^2\left[4\left(-\frac{1}{2}\right)\cos\left(\phi - \frac{\theta}{2}\right)\sin\left(\phi - \frac{\theta}{2}\right) + \frac{3}{2}\sin(2\phi - \theta)\right.$$

$$\left. -\frac{3}{2}\theta\cos(2\phi - \theta) - \theta\cos(2\phi - \theta) + \frac{1}{2}\theta^2\sin(2\phi - \theta)\right]$$

$$= Pa\sin\theta + \frac{5}{2}ka^2\left[\frac{1}{2}\sin(2\phi - \theta) - \frac{5}{2}\theta\cos(2\phi - \theta)\right.$$

$$\left. +\frac{1}{2}\theta^2\sin(2\phi - \theta)\right]$$

PROBLEM 10.91 CONTINUED

When $\theta = 0$, $\dfrac{dV}{d\theta} = 0$ for all values of P.

For stable equilibrium when $\theta = 0$, require

$$\frac{d^2V}{d\theta^2} > 0: \quad -Pa + \frac{5}{2}ka^2\left(2\cos^2\phi\right) > 0$$

Now, when $\theta = 0$,

$$\cos\phi = \frac{\frac{a}{2}}{\frac{a}{2}\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\therefore \quad -Pa + 5ka^2\left(\frac{1}{5}\right) > 0$$

or

$$P < ka$$

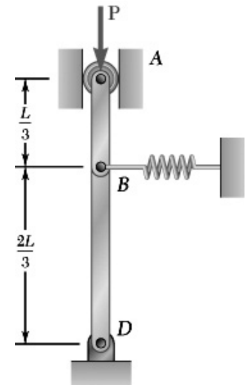
When $P = ka$ $\left(\text{for } \theta = 0\right)$:

$$\frac{dV}{d\theta} = 0$$

$$\frac{d^2V}{d\theta^2} = 0$$

$$\frac{d^3V}{d\theta^3} = \frac{5}{4}ka^2\sin 2\phi > 0 \Rightarrow \text{unstable}$$

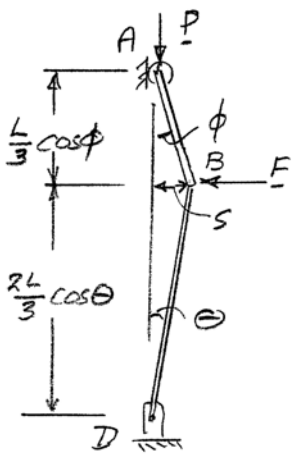
$\therefore$ Stable equilibrium for $0 \le P < ka$ ◀



PROBLEM 10.92

Two bars are attached to a single spring of constant k that is unstretched when the bars are vertical. Determine the range of values of P for which the equilibrium of the system is stable in the position shown.

SOLUTION



Spring: $$s = \frac{L}{3}\sin\phi = \frac{2L}{3}\sin\theta$$

For small values of ϕ and θ: $$\phi = 2\theta$$

$$V = P\left(\frac{L}{3}\cos\phi + \frac{2L}{3}\cos\theta\right) + \frac{1}{2}ks^2$$

$$= \frac{PL}{3}(\cos 2\theta + 2\cos\theta) + \frac{1}{2}k\left(\frac{2L}{3}\sin\theta\right)^2$$

$$\frac{dV}{d\theta} = \frac{PL}{3}(-2\sin 2\theta - 2\sin\theta) + \frac{2}{9}kL^2\sin\theta\cos\theta$$

$$= -\frac{PL}{3}(2\sin 2\theta + 2\sin\theta) + \frac{2}{9}kL^2\sin 2\theta$$

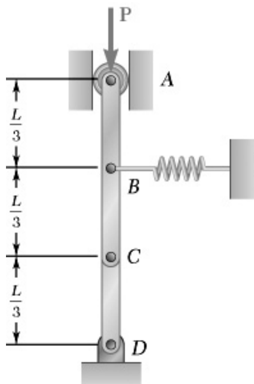
$$\frac{d^2V}{d\theta^2} = -\frac{PL}{3}(4\cos 2\theta + 2\cos\theta) + \frac{4}{9}kL^2\cos 2\theta$$

When $$\theta = 0: \quad \frac{d^2V}{d\theta^2} = -\frac{6PL}{3} + \frac{4}{9}kL^2$$

For stability: $$\frac{d^2V}{d\theta^2} > 0: \quad -2PL + \frac{4}{9}kL^2 > 0$$

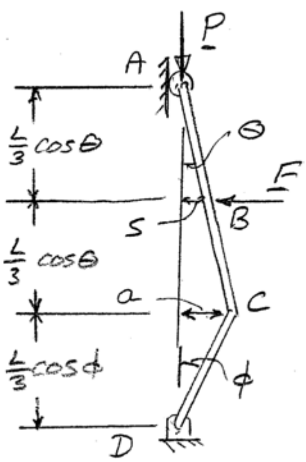
$$0 \le P < \frac{2}{9}kL \blacktriangleleft$$

PROBLEM 10.93



Two bars are attached to a single spring of constant k that is unstretched when the bars are vertical. Determine the range of values of P for which the equilibrium of the system is stable in the position shown.

SOLUTION



First note $$a = \frac{2L}{3}\sin\theta = \frac{L}{3}\sin\phi$$

and $$s = \frac{L}{3}\sin\theta$$

For small values of ϕ and θ: $$\phi = 2\theta$$

$$V = P\left(\frac{2L}{3}\cos\theta + \frac{L}{3}\cos\phi\right) + \frac{1}{2}ks^2$$

$$= \frac{PL}{3}\left(2\cos\theta + \cos 2\theta\right) + \frac{1}{2}k\left(\frac{L}{3}\sin\theta\right)^2$$

$$\frac{dV}{d\theta} = \frac{PL}{3}\left(-2\sin\theta - 2\sin 2\theta\right) + \frac{kL^2}{9}\sin\theta\cos\theta$$

$$= -\frac{2PL}{3}\left(\sin\theta + \sin 2\theta\right) + \frac{kL^2}{18}\sin 2\theta$$

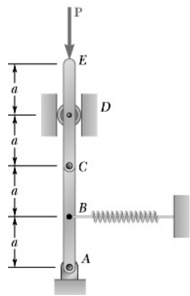
$$\frac{d^2V}{d\theta^2} = -\frac{2PL}{3}\left(\cos\theta + 2\cos 2\theta\right) + \frac{kL^2}{9}\cos 2\theta$$

When $$\theta = 0: \quad \frac{d^2V}{d\theta^2} = -2PL + \frac{kL^2}{9}$$

For stability: $$\frac{d^2V}{d\theta^2} > 0: \quad -2PL + \frac{kL^2}{9} > 0$$

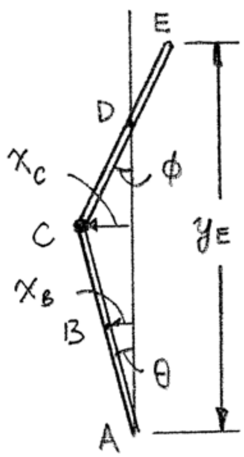
$$0 \le P < \frac{1}{18}kL \blacktriangleleft$$

PROBLEM 10.94



Bar AC is attached to a hinge at A and to a spring of constant k that is undeformed when the bar is vertical. Knowing that the spring can act in either tension or compression, determine the range of values of P for which the equilibrium of the system is stable in the position shown.

SOLUTION



Consider a small disturbance of the system defined by the angle θ.

Have

$$x_C = 2a\sin\theta = a\sin\phi$$

For small θ: $\quad 2\theta = \phi$

Now, the Potential Energy is

$$V = \frac{1}{2}kx_B^2 + Py_E$$

where $\quad x_B = a\sin\theta$

and $\quad y_E = y_C + y_{E/C}$

$$= 2a\cos\theta + 2a\cos\phi$$

$$= 2a\left(\cos\theta + \cos 2\theta\right)$$

Then $\quad V = \frac{1}{2}ka^2\sin^2\theta + 2Pa\left(\cos\theta + \cos 2\theta\right)$

and $\quad \frac{dV}{d\theta} = \frac{1}{2}ka^2\left(2\sin\theta\cos\theta\right) - 2Pa\left(\sin\theta + 2\sin 2\theta\right)$

$$= \frac{1}{2}ka^2\sin 2\theta - 2Pa\left(\sin\theta + 2\sin 2\theta\right)$$

$$\frac{d^2V}{d\theta^2} = ka^2\cos 2\theta - 2Pa\left(\cos\theta + 4\cos 2\theta\right)$$

For $\theta = 0$ and for stable equilibrium:

$$\frac{d^2V}{d\theta^2} > 0$$

or $\quad ka^2 - 2Pa\left(1 + 4\right) > 0$

PROBLEM 10.94 CONTINUED

or $$P < \frac{1}{10}ka$$

$$\therefore \quad 0 \le P < \frac{1}{10}ka \blacktriangleleft$$

Check stability for $$P = \frac{ka}{10}$$

$$\frac{d^3V}{d\theta^3} = -2ka^2 \sin 2\theta + 2Pa\left(\sin\theta + 8\sin 2\theta\right)$$

$$\frac{d^4V}{d\theta^4} = -4ka^2 \cos 2\theta + 2Pa\left(\cos\theta + 16\cos 2\theta\right)$$

Then, with $\theta = 0$ and $P = \dfrac{ka}{10}$

$$\frac{dV}{d\theta} = 0$$

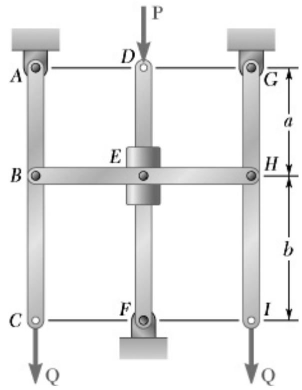
$$\frac{d^2V}{d\theta^2} = 0$$

$$\frac{d^3V}{d\theta^3} = 0$$

$$\frac{d^4V}{d\theta^4} = -4ka^2 + 2\left(\frac{1}{10}ka\right)(a)(1 + 16)$$

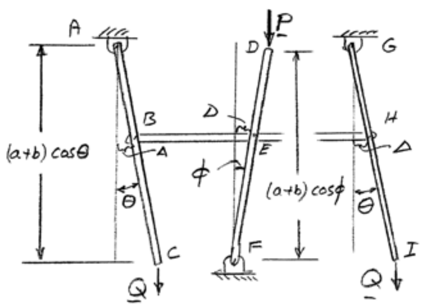
$$= -0.6ka^2 < 0 \Rightarrow \text{Unstable}$$

PROBLEM 10.95



The horizontal bar *BEH* is connected to three vertical bars. The collar at *E* can slide freely on bar *DF*. Determine the range of values of *P* for which the equilibrium of the system is stable in the position shown when $a = 300$ mm, $b = 400$ mm, and $Q = 90$ N.

SOLUTION



First note $\quad A = a\sin\theta = b\sin\phi$

For small values of θ and ϕ: $\quad a\theta = b\phi$

or
$$\phi = \frac{a}{b}\theta$$

$$V = P(a+b)\cos\phi - 2Q(a+b)\cos\theta$$

$$= (a+b)\left[P\cos\left(\frac{a}{b}\theta\right) - 2Q\cos\theta\right]$$

$$\frac{dV}{d\theta} = (a+b)\left[-\frac{a}{b}P\sin\left(\frac{a}{b}\theta\right) + 2Q\sin\theta\right]$$

$$\frac{d^2V}{d\theta^2} = (a+b)\left[-\frac{a^2}{b^2}P\cos\left(\frac{a}{b}\theta\right) + 2Q\cos\theta\right]$$

When $\theta = 0$:
$$\frac{d^2V}{d\theta^2} = (a+b)\left(-\frac{a^2}{b^2}P + 2Q\right)$$

Stability:
$$\frac{d^2V}{d\theta^2} > 0: \quad -\frac{a^2}{b^2}P + 2Q > 0$$

$$P < 2\frac{b^2}{a^2}Q \tag{1}$$

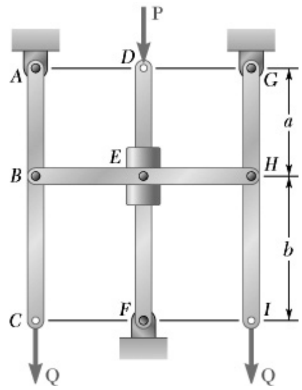
or
$$Q > \frac{a^2}{2b^2}P \tag{2}$$

With $\quad Q = 90$ N, $a = 300$ mm, and $b = 400$ mm

Equation (1):
$$P < 2\frac{(400\text{ mm})^2}{(300\text{ mm})^2}(90\text{ N}) = 320\text{ N}$$

For stability $\quad 0 \le P < 320$ N ◀

PROBLEM 10.96



The horizontal bar *BEH* is connected to three vertical bars. The collar at *E* can slide freely on bar *DF*. Determine the range of values of Q for which the equilibrium of the system is stable in the position shown when $a = 480$ mm, $b = 400$ mm, and $P = 600$ N.

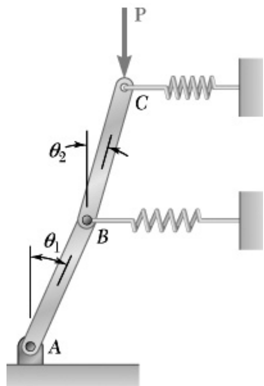
SOLUTION

Using Equation (2) of Problem 10.95 with $P = 600$ N, $a = 480$ mm, and $b = 400$ mm

Equation (2) $$Q > \frac{1}{2}\frac{(480 \text{ mm})^2}{(400 \text{ mm})^2}(600 \text{ N}) = 432 \text{ N}$$

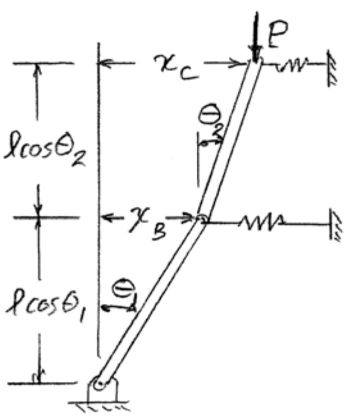
For stability $Q > 432$ N ◀

PROBLEM 10.97



Bars *AB* and *BC*, each of length *l* and of negligible weight, are attached to two springs, each of constant *k*. The springs are undeformed and the system is in equilibrium when $\theta_1 = \theta_2 = 0$. Determine the range of values of *P* for which the equilibrium position is stable.

SOLUTION



Have $$x_B = l\sin\theta$$

$$x_C = l\sin\theta_1 + l\sin\theta_2$$

$$y_C = l\cos\theta_1 + l\cos\theta_2$$

$$V = Py_C + \frac{1}{2}kx_B^2 + \frac{1}{2}kx_C^2$$

or $$V = Pl\left(\cos\theta_1 + \cos\theta_2\right) + \frac{1}{2}kl^2\left[\sin^2\theta_1 + \left(\sin\theta_1 + \sin\theta_2\right)^2\right]$$

For small values of θ_1 and θ_2:

$$\sin\theta_1 \approx \theta_1, \qquad \sin\theta_2 \approx \theta_2, \qquad \cos\theta_1 \approx 1 - \frac{1}{2}\theta_1^2, \qquad \cos\theta_2 \approx 1 - \frac{1}{2}\theta_2^2$$

Then $$V = Pl\left(1 - \frac{\theta_1^2}{2} + 1 - \frac{\theta_2^2}{2}\right) + \frac{1}{2}kl^2\left[\theta_1^2 + \left(\theta_1 + \theta_2\right)^2\right]$$

and $$\frac{\partial V}{\partial\theta_1} = -Pl\theta_1 + kl^2\left[\theta_1 + \left(\theta_1 + \theta_2\right)\right]$$

$$\frac{\partial V}{\partial\theta_2} = -Pl\theta_2 + kl^2\left(\theta_1 + \theta_2\right)$$

$$\frac{\partial^2 V}{\partial\theta_1^2} = -Pl + 2kl^2 \qquad \frac{\partial^2 V}{\partial\theta_2^2} = -Pl + kl^2$$

$$\frac{\partial^2 V}{\partial\theta_1\partial\theta_2} = kl^2$$

PROBLEM 10.97 CONTINUED

Stability Conditions for stability (see page 583).

For $\theta_1 = \theta_2 = 0$: $\dfrac{\partial V}{\partial \theta_1} = \dfrac{\partial V}{\partial \theta_2} = 0$ (condition satisfied)

$$\left(\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2}\right)^2 - \frac{\partial^2 V}{\partial \theta_1^2}\frac{\partial^2 V}{\partial \theta_2^2} < 0$$

Substituting,

$$\left(kl^2\right)^2 - \left(-Pl + 2kl^2\right)\left(-Pl + kl\right) < 0$$

$$k^2 l^4 - P^2 l^2 + 3Pkl^3 - 2k^2 l^4 < 0$$

$$P^2 - 3klP + k^2 l^2 > 0$$

Solving,

$$P < \frac{3 - \sqrt{5}}{2} kl \qquad \text{or} \qquad P > \frac{3 + \sqrt{5}}{2} kl$$

or

$$P < 0.382kl \qquad \text{or} \qquad P > 2.62kl$$

$$\frac{\partial^2 V}{\partial \theta_1^2} > 0: \quad -Pl + 2kl^2 > 0$$

or

$$P < \frac{1}{2} kl$$

$$\frac{\partial^2 V}{\partial \theta_2^2} > 0: \quad -Pl + kl^2 > 0$$

or

$$P < kl$$

Therefore, all conditions for stable equilibrium are satisfied when

$$0 \le P < 0.382kl \blacktriangleleft$$

PROBLEM 10.98

Solve Problem 10.97 knowing that $l = 400\text{ mm}$ and $k = 1.25\text{ kN/m}$.

SOLUTION

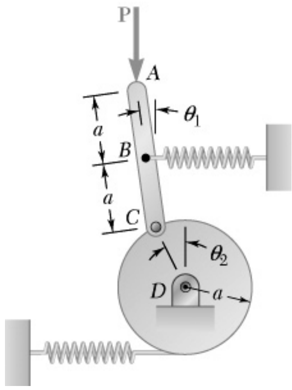
From the analysis of Problem 10.98 with

$$l = 400\text{ mm} \qquad \text{and} \qquad k = 1.25\text{ kN/m}$$

$$P < 0.382kl = 0.382(1250\text{ N/m})(0.4\text{ m}) = 191\text{ N}$$

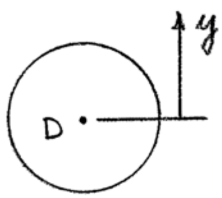
$$0 \le P < 191.0\text{ N} \blacktriangleleft$$

PROBLEM 10.99



Bar ABC of length $2a$ and negligible weight is hinged at C to a drum of radius a as shown. Knowing that the constant of each spring is k and that the springs are undeformed when $\theta_1 = \theta_2 = 0$, determine the range of values of P for which the equilibrium position $\theta_1 = \theta_2 = 0$ is stable.

SOLUTION



Have $$V = \frac{1}{2}k(a\theta_2)^2 + \frac{1}{2}k(a\sin\theta_1 + a\sin\theta_2)^2 + P(2a\cos\theta_1 + a\cos\theta_2)$$

Then $$\frac{\partial V}{\partial \theta_1} = ka^2(\sin\theta_1 + \sin\theta_2)\cos\theta_1 - 2Pa\sin\theta_1$$

$$= ka^2\left(\frac{1}{2}\sin 2\theta_1 + \cos\theta_1\sin\theta_2\right) - 2Pa\sin\theta_1$$

and $$\frac{\partial^2 V}{\partial \theta_1^2} = ka^2(\cos 2\theta_1 - \sin\theta_1\sin\theta_2) - 2Pa\cos\theta_1$$

$$\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} = ka^2\cos\theta_1\cos\theta_2$$

Also $$\frac{\partial V}{\partial \theta_2} = ka^2\theta_2 + ka^2(\sin\theta_1 + \sin\theta_2)\cos\theta_2 - Pa\sin\theta_2$$

$$= ka^2\theta_2 + ka^2\left(\sin\theta_1\cos\theta_2 + \frac{1}{2}\sin 2\theta_2\right) - Pa\sin\theta_2$$

and $$\frac{\partial^2 V}{\partial \theta_2^2} = ka^2 + ka^2(-\sin\theta_1\sin\theta_2 + \cos 2\theta_2) - Pa\cos\theta_2$$

When $$\theta_1 = \theta_2 = 0$$

$$\frac{\partial V}{\partial \theta_1} = 0 \qquad \frac{\partial^2 V}{\partial \theta_1 \partial \theta_2} = ka^2 \qquad \frac{\partial V}{\partial \theta_2} = 0$$

$$\frac{\partial^2 V}{\partial \theta_1^2} = ka^2 - 2Pa \qquad \frac{\partial^2 V}{\partial \theta_2^2} = ka^2 + ka^2 - Pa = 2ka^2 - Pa$$

PROBLEM 10.99 CONTINUED

Apply Equations 10.24

$$\frac{\partial V}{\partial \theta_1} = 0: \qquad \text{condition satisfied}$$

$$\frac{\partial V}{\partial \theta_2} = 0: \qquad \text{condition satisfied}$$

$$\left(\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2}\right)^2 - \frac{\partial^2 V}{\partial \theta_1^2}\frac{\partial^2 V}{\partial \theta_2^2} < 0: \ \left(ka^2\right)^2 - \left(ka^2 - 2Pa\right)\left(2ka^2 - Pa\right) < 0$$

or

$$k^2a^2 - \left(ka - 2P\right)\left(2ka - P\right) < 0$$

Expanding

$$k^2a^2 - 2ka^2 + 5kaP - 2P^2 < 0$$

or

$$2P^2 - 5kaP + k^2a^2 > 0$$

or

$$P < \frac{5 - \sqrt{17}}{4}ka \qquad \text{and} \qquad P > \frac{5 + \sqrt{17}}{4}ka$$

or

$$P < 0.21922ka \qquad \text{and} \qquad P > 2.2808ka$$

Also

$$\frac{\delta^2 V}{\delta \theta_1^2} > 0: \ ka^2 - 2Pa > 0 \qquad \text{or} \qquad \frac{\delta^2 V}{\delta \theta_2^2} > 0: \ 2ka^2 - Pa > 0$$

or

$$P < \frac{1}{2}ka \qquad \text{or} \qquad P < 2ka$$

$\therefore$ For stable equilibrium when $\theta_1 = \theta_2 = 0$:

$$0 \le P < 0.219ka \blacktriangleleft$$

PROBLEM 10.100

Solve Problem 10.99 knowing that $k = 10$ lb/in. and $a = 14$ in.

SOLUTION

From the solution to Problem 10.99, with $k = 10$ lb/in. and $a = 14$ in.

$$0 \le P < 0.21922ka$$

or

$$0 \le P < 0.21922(10 \text{ lb/in.})(14 \text{ in.})$$

or $0 \le P < 30.7$ lb ◀