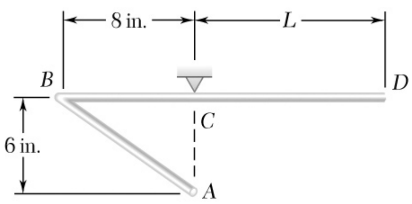
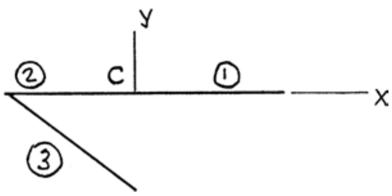
PROBLEM 5.28



The homogeneous wire *ABCD* is bent as shown and is attached to a hinge at *C*. Determine the length *L* for which the portion *BCD* of the wire is horizontal.

SOLUTION



First note that for equilibrium, the center of gravity of the wire must lie on a vertical line through *C*. Further, because the wire is homogeneous, its center of gravity will coincide with the centroid of the corresponding line.

Thus $\Sigma M_C = 0$, which implies that $\overline{x} = 0$

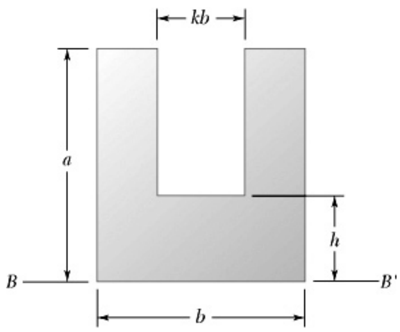
or $$\Sigma \overline{x}_i L_i = 0$$

Hence $$\frac{L}{2}(L) + (-4 \text{ in.})(8 \text{ in.}) + (-4 \text{ in.})(10 \text{ in.}) = 0$$

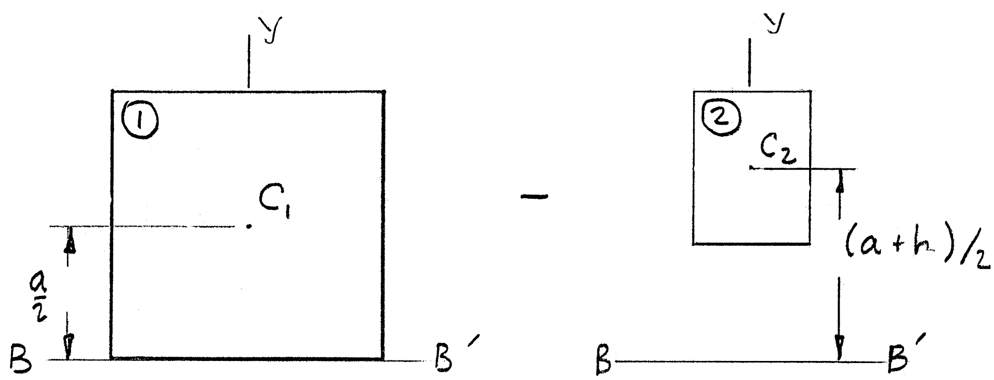
or $$L^2 = 144 \text{ in}^2 \qquad \text{or} \quad L = 12.00 \text{ in.} \blacktriangleleft$$

PROBLEM 5.29

Determine the distance h so that the centroid of the shaded area is as close to line BB' as possible when (*a*) $k = 0.2$, (*b*) $k = 0.6$.



SOLUTION



Then $$\bar{y} = \frac{\Sigma yA}{\Sigma A}$$

or $$\bar{y} = \frac{\frac{a}{2}(ab) - \left[\frac{(a+h)}{2}\right]\left[kb(a-h)\right]}{ba - kb(a-h)}$$

$$= \frac{1}{2}\frac{a^2(1-k) + kh^2}{a(1-k) + kh}$$

Let $c = 1 - k$ and $\zeta = \dfrac{h}{a}$

Then $$\bar{y} = \frac{a}{2}\frac{c + k\zeta^2}{c + k\zeta} \qquad (1)$$

Now find a value of ζ (or h) for which $\bar{y}$ is minimum:

$$\frac{d\bar{y}}{d\zeta} = \frac{a}{2}\frac{2k\zeta(c + k\zeta) - k\left(c + k\zeta^2\right)}{(c + k\zeta)^2} = 0 \qquad \text{or} \qquad 2\zeta(c + k\zeta) - \left(c + k\zeta^2\right) = 0 \qquad (2)$$

PROBLEM 5.29 CONTINUED

Expanding (2) $\quad 2c\zeta + 2\zeta^2 - c - k\zeta^2 = 0 \quad$ or $\quad k\zeta^2 + 2c\zeta - c = 0$

Then

$$\zeta = \frac{-2c \pm \sqrt{(2c)^2 - 4(k)(c)}}{2k}$$

Taking the positive root, since $h > 0$ (hence $\zeta > 0$)

$$h = a\frac{-2(1-k) + \sqrt{4(1-k)^2 + 4k(1-k)}}{2k}$$

(*a*) $k = 0.2$:

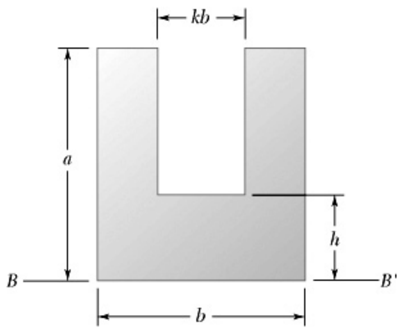
$$h = a\frac{-2(1-0.2) + \sqrt{4(1-0.2)^2 + 4(0.2)(1-0.2)}}{2(0.2)}$$

or $h = 0.472a$ ◀

(*b*) $k = 0.6$:

$$h = a\frac{-2(1-0.6) + \sqrt{4(1-0.6)^2 + 4(0.6)(1-0.6)}}{2(0.6)}$$

or $h = 0.387a$ ◀



PROBLEM 5.30

Show when the distance h is selected to minimize the distance $\overline{y}$ from line BB' to the centroid of the shaded area that $\overline{y} = h$.

SOLUTION

From Problem 5.29, note that Eq. (2) yields the value of ζ that minimizes h.

Then from Eq. (2)

We see
$$2\zeta = \frac{c + k\zeta^2}{c + k\zeta} \tag{3}$$

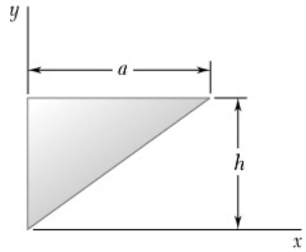
Then, replacing the right-hand side of (1) by 2ζ, from Eq. (3)

We obtain
$$\overline{y} = \frac{a}{2}(2\zeta)$$

But
$$\zeta = \frac{h}{a}$$

So
$$\overline{y} = h$$

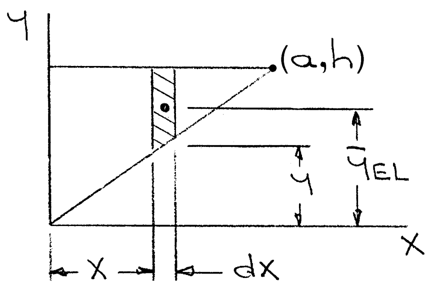
Q.E.D. ◀



PROBLEM 5.31

Determine by direct integration the centroid of the area shown. Express your answer in terms of a and h.

SOLUTION



For the element of area (EL) shown $\quad y = \dfrac{h}{a}x$

and
$$dA = (h - y)dx$$
$$= h\left(1 - \frac{x}{a}\right)dx$$

Then
$$x_{EL} = x$$
$$y_{EL} = \frac{1}{2}(h + y)$$
$$= \frac{h}{2}\left(1 + \frac{x}{a}\right)$$

Then area
$$A = \int dA = \int_0^a h\left(1 - \frac{x}{a}\right)dx = h\left(x - \frac{x^2}{2a}\right)\Big|_0^a = \frac{1}{2}ah$$

and
$$\int \bar{x}_{EL}dA = \int_0^a x\left[h\left(1 - \frac{x}{a}\right)dx\right] = h\left(\frac{x^2}{2} - \frac{x^3}{3a}\right)\Big|_0^a = \frac{1}{6}a^2h$$

$$\int \bar{y}_{EL}dA = \int_0^a \frac{h}{2}\left(1 + \frac{x}{a}\right)\left[h\left(1 - \frac{x}{a}\right)dx\right] = \frac{h^2}{2}\int_0^a\left(1 - \frac{x^2}{a^2}\right)dx$$
$$= \frac{h^2}{2}\left(x - \frac{x^3}{3a^2}\right)\Big|_0^a = \frac{1}{3}ah^2$$

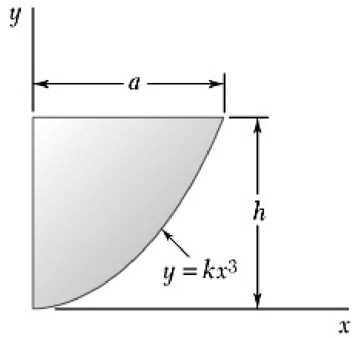
Hence
$$\bar{x}A = \int \bar{x}_{EL}dA$$
$$\bar{x}\left(\frac{1}{2}ah\right) = \frac{1}{6}a^2h$$

$$\bar{x} = \frac{1}{3}a \blacktriangleleft$$

$$\bar{y}A = \int \bar{y}_{EL}dA$$
$$\bar{y}\left(\frac{1}{2}ah\right) = \frac{1}{3}ah^2$$

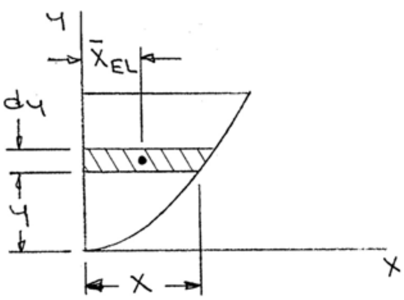
$$\bar{y} = \frac{2}{3}h \blacktriangleleft$$

PROBLEM 5.32



Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION



For the element (EL) shown

At $\quad x = a, y = h:\ h = ka^3 \quad$ or $\quad k = \dfrac{h}{a^3}$

Then $$x = \frac{a}{h^{1/3}} y^{1/3}$$

Now $$dA = xdy$$

$$= \frac{a}{h^{1/3}} y^{1/3}\ dy$$

$$\bar{x}_{EL} = \frac{1}{2}x = \frac{1}{2}\frac{a}{h^{1/3}} y^{1/3},\ \ \bar{y}_{EL} = y$$

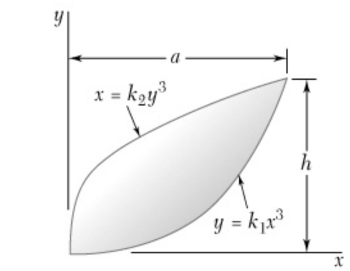
Then $$A = \int dA = \int_0^h \frac{a}{h^{1/3}} y^{1/3} dy = \frac{3}{4}\frac{a}{h^{1/3}}\left(y^{4/3}\right)\Big|_0^h = \frac{3}{4}ah$$

and $$\int \bar{x}_{EL} dA = \int_0^h \frac{1}{2}\frac{a}{h^{1/3}} y^{1/3}\left(\frac{a}{h^{1/3}} y^{1/3} dy\right) = \frac{1}{2}\frac{a}{h^{2/3}}\left(\frac{3}{5}y^{5/3}\right)\Bigg|_0^h = \frac{3}{10}a^2 h$$

$$\int \bar{y}_{EL} dA = \int_0^h y\left(\frac{a}{h^{1/3}} y^{1/3} dy\right) = \frac{a}{h^{1/3}}\left(\frac{3}{7}y^{7/3}\right)\Bigg|_0^h = \frac{3}{7}ah^2$$

Hence $$\bar{x}A = \int \bar{x}_{EL} dA:\ \bar{x}\left(\frac{3}{4}ah\right) = \frac{3}{10}a^2 h \qquad \bar{x} = \frac{2}{5}a \ \blacktriangleleft$$

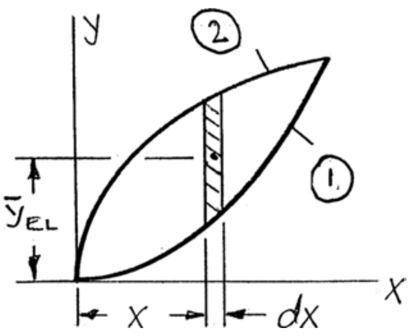
$$\bar{y}A = \int \bar{y}_{EL} dA:\ \bar{y}\left(\frac{3}{4}ah\right) = \frac{3}{7}ah^2 \qquad \bar{y} = \frac{4}{7}h \ \blacktriangleleft$$



PROBLEM 5.33

Determine by direct integration the centroid of the area shown. Express your answer in terms of *a* and *h*.

SOLUTION



For the element (EL) shown

At $\quad x = a, y = h: \quad h = k_1a^3 \quad$ or $\quad k_1 = \dfrac{h}{a^3}$

$$a = k_2h^3 \quad \text{or} \quad k_2 = \frac{a}{h^3}$$

Hence, on line 1

$$y = \frac{h}{a^3}x^3$$

and on line 2

$$y = \frac{h}{a^{1/3}}x^{1/3}$$

Then

$$dA = \left(\frac{h}{a^{1/3}}x^{1/3} - \frac{h}{a^3}x^3\right)dx \quad \text{and} \quad \bar{y}_{EL} = \frac{1}{2}\left(\frac{h}{a^{1/3}}x^{1/3} + \frac{h}{a^3}x^3\right)$$

$$\therefore\ A = \int dA = \int_0^a \left(\frac{h}{a^{1/3}}x^{1/3} - \frac{h}{a^3}x^3\right)dx = h\left(\frac{3}{4a^{1/3}}x^{4/3} - \frac{1}{4a^3}x^4\right)\Bigg|_0^a = \frac{1}{2}ah$$

$$\int \bar{x}_{EL}dA = \int_0^a x\left(\frac{h}{a^{1/3}}x^{1/3} - \frac{h}{a^3}x^3\right)dx = h\left(\frac{3}{7a^{1/3}}x^{7/3} - \frac{1}{5a^3}x^5\right)\Bigg|_0^a = \frac{8}{35}a^2h$$

$$\int \bar{y}_{EL}dA = \int_0^a \frac{1}{2}\left(\frac{h}{a^{1/3}}x^{1/3} + \frac{h}{a^3}x^3\right)\left(\frac{h}{a^{1/3}}x^{1/3} - \frac{h}{a^3}x^3\right)dx$$

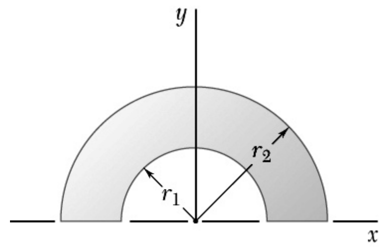
$$= \frac{h^2}{2}\int_0^a \left(\frac{x^{2/3}}{a^{2/3}} - \frac{x^6}{a^6}\right)dx = \frac{h^2}{2}\left(\frac{3}{5}\frac{x^{5/3}}{a^{5/3}} - \frac{1}{7}\frac{x^6}{a^6}\right)\Bigg|_0^a = \frac{8}{35}ah^2$$

From $\quad \bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}\left(\dfrac{ah}{2}\right) = \dfrac{8}{35}a^2h \quad$ or $\quad \bar{x} = \dfrac{16}{35}a$ ◀

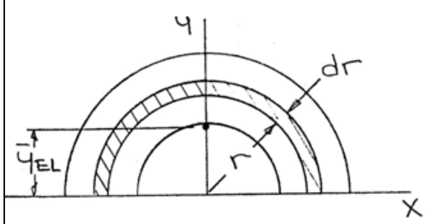
and $\quad \bar{y}A = \int \bar{y}_{EL}dA: \quad \bar{y}\left(\dfrac{ah}{2}\right) = \dfrac{8}{35}ah^2 \quad$ or $\quad \bar{y} = \dfrac{16}{35}h$ ◀

PROBLEM 5.34

Determine by direct integration the centroid of the area shown.



SOLUTION



First note that symmetry implies $\qquad \bar{x} = 0$ ◀

For the element (EL) shown

$$\bar{y}_{EL} = \frac{2r}{\pi} \quad \text{(Figure 5.8B)}$$
$$dA = \pi r\, d r$$

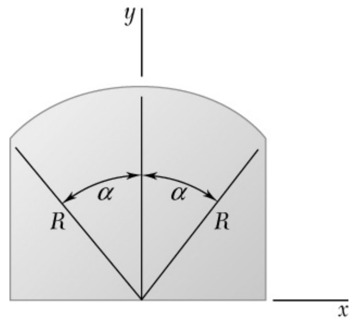
Then $\qquad A = \int dA = \int_{r_1}^{r_2} \pi r\, d r = \pi\left(\frac{r^2}{2}\right)\Bigg|_{r_1}^{r_2} = \frac{\pi}{2}\left(r_2^2 - r_1^2\right)$

and $\qquad \int \bar{y}_{EL} dA = \int_{r_1}^{r_2} \frac{2r}{\pi}\left(\pi r\, d r\right) = 2\left(\frac{1}{3}r^3\right)\Bigg|_{r_1}^{r_2} = \frac{2}{3}\left(r_2^3 - r_1^3\right)$

So $\qquad \bar{y}A = \int \bar{y}_{EL} dA: \quad \bar{y}\left[\frac{\pi}{2}\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)$

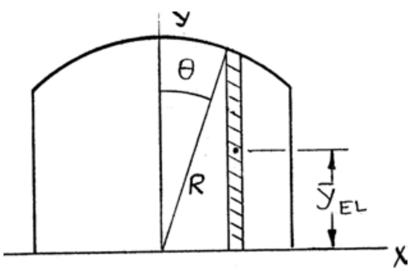
or $\quad \bar{y} = \dfrac{4}{3\pi}\dfrac{r_2^3 - r_1^3}{r_2^2 - r_1^2}$ ◀

PROBLEM 5.35



Determine by direct integration the centroid of the area shown.

SOLUTION



First note that symmetry implies $\bar{x} = 0$ ◀

For the element (EL) shown

$$y = R\cos\theta, \;\; x = R\sin\theta$$

$$dx = R\cos\theta\, d\theta$$

$$dA = ydx = R^2\cos^2\theta\, d\theta$$

Hence

$$A = \int dA = 2\int_0^{\alpha} R^2\cos^2\theta d\theta = 2R^2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\bigg|_0^{\alpha} = \frac{1}{2}R^2\left(2\alpha\sin 2\alpha\right)$$

$$\int \bar{y}_{EL}dA = 2\int_0^{\alpha}\frac{R}{2}\cos\theta\left(R^2\cos^2\theta d\theta\right) = R^3\left(\frac{1}{3}\cos^2\theta\sin\theta + \frac{2}{3}\sin\theta\right)\bigg|_0^{\alpha}$$

$$= \frac{R^3}{3}\left(\cos^2\alpha\sin\alpha + 2\sin\alpha\right)$$

But $\bar{y}A = \int \bar{y}_{EL}dA$ so

$$\bar{y} = \frac{\frac{R^3}{3}\left(\cos^2\alpha\sin\alpha + 2\sin\alpha\right)}{\frac{R^2}{2}\left(2\alpha + \sin 2\alpha\right)}$$

or

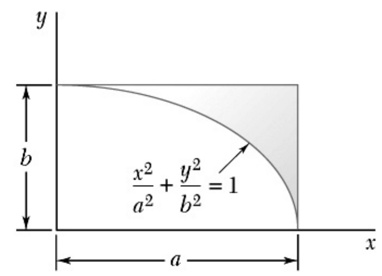
$$\bar{y} = \frac{2}{3}R\sin\alpha\frac{\left(\cos^2\alpha + 2\right)}{\left(2\alpha + \sin 2\alpha\right)}$$

Alternatively,

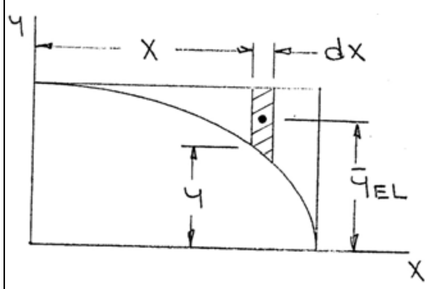
$$\bar{y} = \frac{2}{3}R\sin\alpha\frac{3 - \sin^2\alpha}{2\alpha + \sin 2\alpha}$$ ◀

PROBLEM 5.36

Determine by direct integration the centroid of the area shown.



SOLUTION



For the element (EL) shown

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

and
$$dA = (b - y)dx$$
$$= \frac{b}{a}\left(a - \sqrt{a^2 - x^2}\right)dx$$

$$\bar{x}_{EL} = x;\ \bar{y}_{EL} = \frac{1}{2}(y + b) = \frac{b}{2a}\left(a + \sqrt{a^2 - x^2}\right)$$

Then
$$A = \int dA = \int_0^a \frac{b}{a}\left(a - \sqrt{a^2 - x^2}\right)dx$$

To integrate, let $x = a\sin\theta$: $\sqrt{a^2 - x^2} = a\cos\theta,\ dx = a\cos\theta\, d\theta$

Then
$$A = \int_0^{\pi/2} \frac{b}{a}(a - a\cos\theta)(a\cos\theta d\theta)$$
$$= \frac{b}{a}\left[a^2\sin\theta - a^2\left(\frac{\theta}{2} + \sin\frac{2\theta}{4}\right)\right]\Bigg|_0^{\pi/2} = ab\left(1 - \frac{\pi}{4}\right)$$

and
$$\int \bar{x}_{EL}dA = \int_0^a x\left[\frac{b}{a}\left(a - \sqrt{a^2 - x^2}\right)dx\right] = \frac{b}{a}\left[\left(\frac{a}{2}x^2 + \frac{1}{3}\left(a^2 - x^2\right)^{3/2}\right)\right]\Bigg|_0^{\pi/2}$$
$$= \frac{1}{6}a^3b$$

$$\int \bar{y}_{EL}dA = \int_0^a \frac{b}{2a}\left(a + \sqrt{a^2 - x^2}\right)\left[\frac{b}{a}\left(a - \sqrt{a^2 - x^2}\right)dx\right]$$
$$= \frac{b^2}{2a^2}\int_0^a \left(x^2\right)dx = \frac{b^2}{2a^2}\left(\frac{x^3}{3}\right)\Bigg|_0^a = \frac{1}{6}ab^2$$

$$\bar{x}A = \int \bar{x}_{EL}dA:\quad \bar{x}\left[ab\left(1 - \frac{\pi}{4}\right)\right] = \frac{1}{6}a^2b \qquad \text{or } \bar{x} = \frac{2a}{3(4 - \pi)} \blacktriangleleft$$

$$\bar{y}A = \int \bar{y}_{EL}dA:\quad \bar{y}\left[ab\left(1 - \frac{\pi}{4}\right)\right] = \frac{1}{6}ab^2 \qquad \text{or } \bar{y} = \frac{2b}{3(4 - \pi)} \blacktriangleleft$$