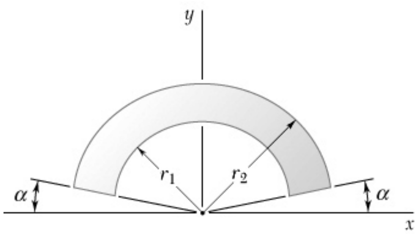
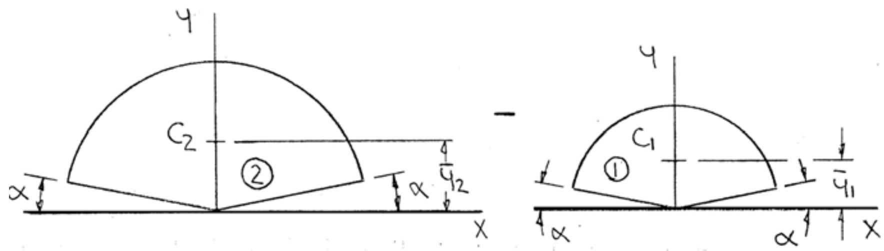


PROBLEM 5.10

Show that as r_1 approaches r_2, the location of the centroid approaches that of a circular arc of radius $(r_1 + r_2)/2$.

SOLUTION



First, determine the location of the centroid.

From Fig. 5.8A:

$$\bar{y}_2 = \frac{2}{3}r_2\frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\left(\frac{\pi}{2}-\alpha\right)} \qquad A_2 = \left(\tfrac{\pi}{2}-\alpha\right)r_2^2$$

$$= \frac{2}{3}r_2\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}$$

Similarly

$$\bar{y}_1 = \frac{2}{3}r_1\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)} \qquad A_1 = \left(\tfrac{\pi}{2}-\alpha\right)r_1^2$$

Then

$$\Sigma\bar{y}A = \frac{2}{3}r_2\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}\left[\left(\tfrac{\pi}{2}-\alpha\right)r_2^2\right] - \frac{2}{3}r_1\frac{\cos\alpha}{\left(\frac{\pi}{2}-\alpha\right)}\left[\left(\tfrac{\pi}{2}-\alpha\right)r_1^2\right]$$

$$= \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

and

$$\Sigma A = \left(\frac{\pi}{2}-\alpha\right)r_2^2 - \left(\frac{\pi}{2}-\alpha\right)r_1^2$$

$$= \left(\frac{\pi}{2}-\alpha\right)\left(r_2^2 - r_1^2\right)$$

Now

$$\bar{Y}\Sigma A = \Sigma\bar{y}A$$

$$\bar{Y}\left[\left(\frac{\pi}{2}-\alpha\right)\left(r_2^2 - r_1^2\right)\right] = \frac{2}{3}\left(r_2^3 - r_1^3\right)\cos\alpha$$

$$\bar{Y} = \frac{2}{3}\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}\frac{\cos\alpha}{\frac{\pi}{2}-\alpha}$$

PROBLEM 5.10 CONTINUED

Using Figure 5.8B, $\bar{Y}$ of an arc of radius $\frac{1}{2}(r_1 + r_2)$ is

$$\bar{Y} = \frac{1}{2}(r_1 + r_2)\frac{\sin\left(\frac{\pi}{2} - \alpha\right)}{\left(\frac{\pi}{2} - \alpha\right)}$$

$$= \frac{1}{2}(r_1 + r_2)\frac{\cos\alpha}{\left(\frac{\pi}{2} - \alpha\right)} \qquad (1)$$

Now
$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{(r_2 - r_1)\left(r_2^2 + r_1 r_2 + r_1^2\right)}{(r_2 - r_1)(r_2 + r_1)}$$

$$= \frac{r_2^2 + r_1 r_2 + r_1^2}{r_2 + r_1}$$

Let
$$r_2 = r + \Delta$$
$$r_1 = r - \Delta$$

Then
$$r = \frac{1}{2}(r_1 + r_2)$$

and
$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{(r + \Delta)^2 + (r + \Delta)(r - \Delta) + (r - \Delta)^2}{(r + \Delta) + (r - \Delta)}$$

$$= \frac{3r^2 + \Delta^2}{2r}$$

In the limit as $\Delta \to 0$ (i.e., $r_1 = r_2$), then

$$\frac{r_2^3 - r_1^3}{r_2^2 - r_1^2} = \frac{3}{2}r$$

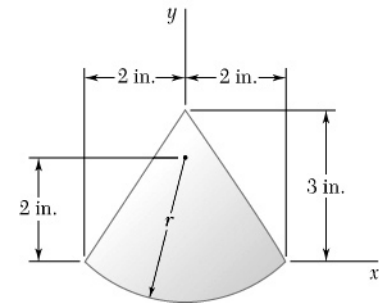
$$= \frac{3}{2} \times \frac{1}{2}(r_1 + r_2)$$

so that
$$\bar{Y} = \frac{2}{3} \times \frac{3}{4}(r_1 + r_2)\frac{\cos\alpha}{\frac{\pi}{2} - \alpha}$$
or
$$\bar{Y} = \frac{1}{2}(r_1 + r_2)\frac{\cos\alpha}{\frac{\pi}{2} - \alpha} \blacktriangleleft$$

Which agrees with Eq. (1).

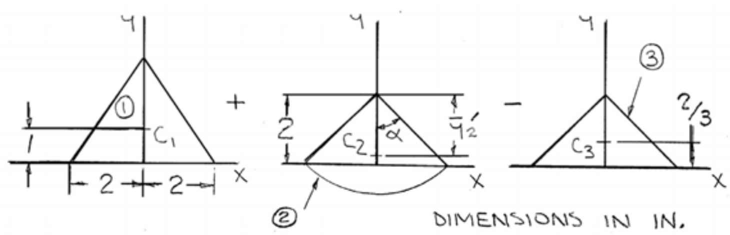
PROBLEM 5.11

Locate the centroid of the plane area shown.



SOLUTION

First note that symmetry implies $\bar{X} = 0$ ◀



$$r_2 = 2\sqrt{2}\text{ in.},\ \alpha = 45°$$

$$\bar{y}_2' = \frac{2r\sin\alpha}{3\alpha} = \frac{2\left(2\sqrt{2}\right)\sin\left(\frac{\pi}{4}\right)}{3\left(\frac{\pi}{4}\right)} = 1.6977\text{ in.}$$

	A, in^2	$\bar{y}$, in.	$\bar{y}A$, in^3
1	$\frac{1}{2}(4)(3) = 6$	1	6
2	$\frac{\pi}{4}\left(2\sqrt{2}\right)^2 = 6.283$	$2 - \bar{y}' = 0.3024$	1.8997
3	$-\frac{1}{2}(4)(2) = -4$	0.6667	−2.667
Σ	8.283		5.2330

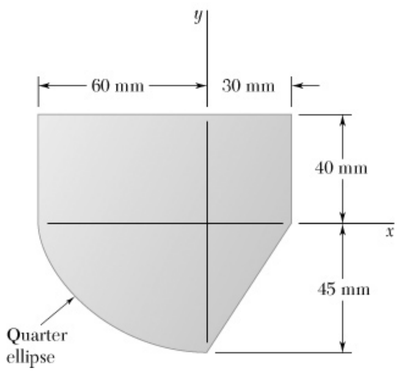
Then $$\bar{Y}\Sigma A = \Sigma\,\bar{y}A$$

$$\bar{Y}\left(8.283\text{ in}^2\right) = 5.2330\text{ in}^3$$

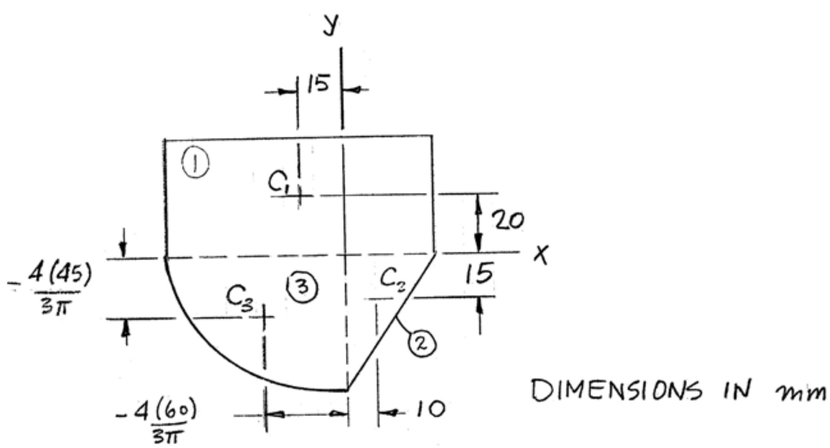
or $\bar{Y} = 0.632$ in. ◀

PROBLEM 5.12

Locate the centroid of the plane area shown.



SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$(40)(90) = 3600$	–15	20	–54 000	72 000
2	$\frac{\pi(40)(60)}{4} = 2121$	10	–15	6750	–10 125
3	$\frac{1}{2}(30)(45) = 675$	–25.47	–19.099	–54 000	–40 500
Σ	6396			–101 250	21 375

Then

$$\bar{X}A = \Sigma\bar{x}A$$

$$\bar{X}\left(6396\text{ mm}^2\right) = -101\,250\text{ mm}^3$$

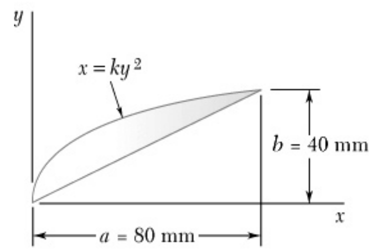
or $\bar{X} = -15.83$ mm ◀

and

$$\bar{Y}A = \Sigma\bar{y}A$$

$$\bar{Y}\left(6396\text{ mm}^2\right) = 21\,375\text{ mm}^3$$

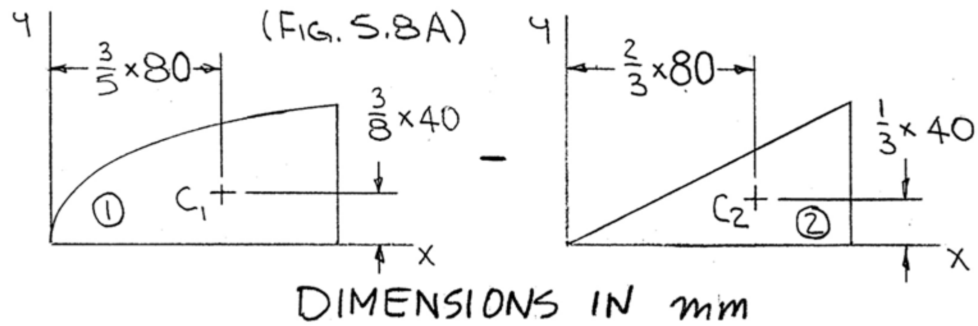
or $\bar{Y} = 3.34$ mm ◀



PROBLEM 5.13

Locate the centroid of the plane area shown.

SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{2}{3}(40)(80) = 2133$	48	15	102 400	32 000
2	$-\frac{1}{2}(40)(80) = -1600$	53.33	13.333	–85 330	–21 330
Σ	533.3			17 067	10 667

Then
$$\bar{X}\Sigma A = \Sigma \bar{X} A$$

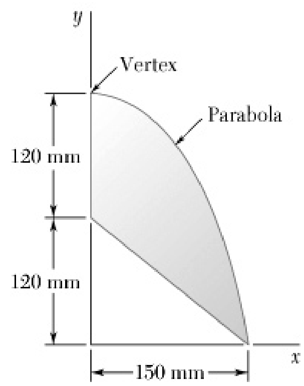
$$\bar{X}\left(533.3 \text{ mm}^2\right) = 17\,067 \text{ mm}^3 \qquad \text{or } \bar{X} = 32.0 \text{ mm} \blacktriangleleft$$

and
$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

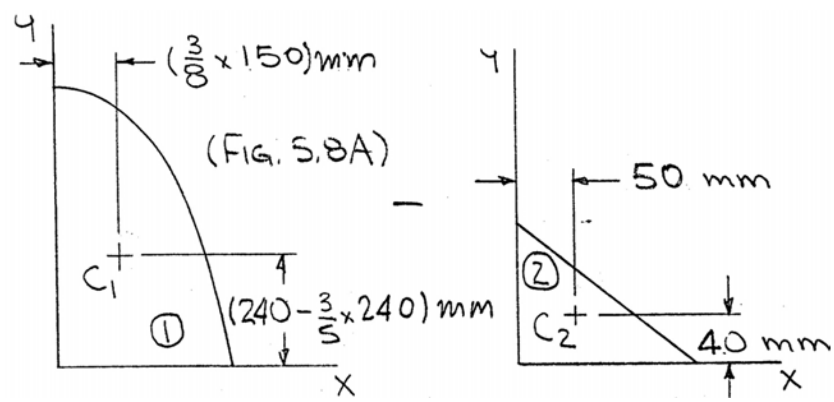
$$\bar{Y}\left(533.3 \text{ mm}^2\right) = 10\,667 \text{ mm}^3 \qquad \text{or } \bar{Y} = 20.0 \text{ mm} \blacktriangleleft$$

PROBLEM 5.14

Locate the centroid of the plane area shown.



SOLUTION



	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm^3	$\bar{y}A$, mm^3
1	$\frac{2}{3}(150)(240) = 24\,000$	56.25	96	1 350 000	2 304 000
2	$-\frac{1}{2}(150)(120) = -9000$	50	40	−450 000	−360 000
Σ	15 000			900 000	1 944 000

Then

$$\bar{X}\Sigma A = \Sigma \bar{x}A$$

$$\bar{X}\left(15\,000 \text{ mm}^2\right) = 900\,000 \text{ mm}^3$$

or $\bar{X} = 60.0$ mm ◀

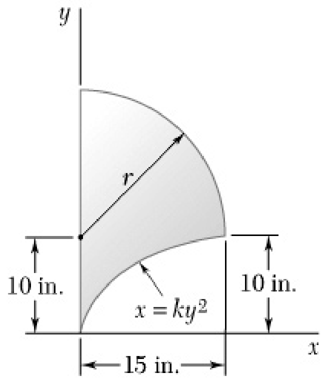
and

$$\bar{Y}\Sigma A = \Sigma \bar{y}A$$

$$\bar{Y}\left(15\,000 \text{ mm}^2\right) = 1\,944\,000$$

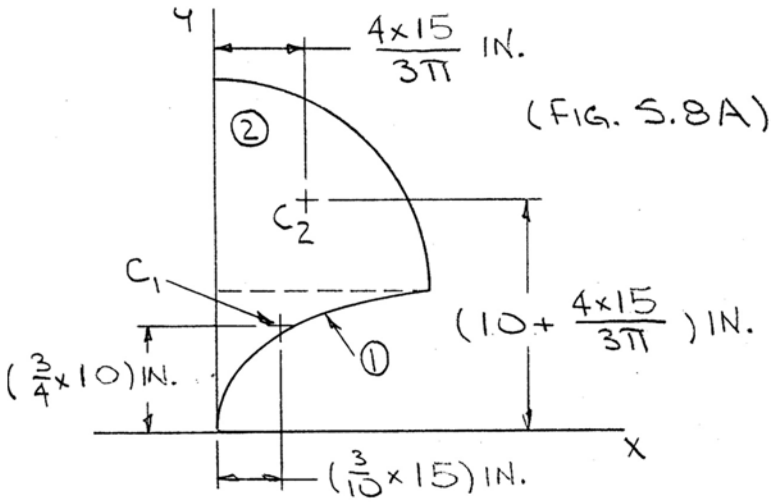
or $\bar{Y} = 129.6$ mm ◀

PROBLEM 5.15



Locate the centroid of the plane area shown.

SOLUTION



	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\frac{1}{3}(10)(15) = 50$	4.5	7.5	225	375
2	$\frac{\pi}{4}(15)^2 = 176.71$	6.366	16.366	1125	2892
Σ	226.71			1350	3267

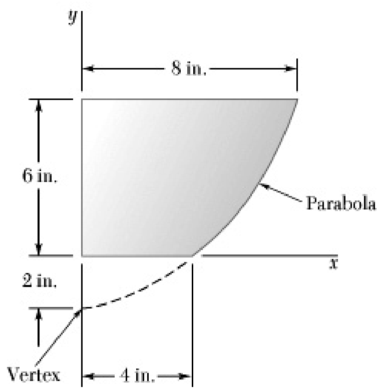
Then $$\bar{X}\Sigma A = \Sigma\bar{x}A$$

$$\bar{X}\left(226.71 \text{ in}^2\right) = 1350 \text{ in}^3$$ or $\bar{X} = 5.95$ in. ◀

and $$\bar{Y}\Sigma A = \Sigma\bar{y}A$$

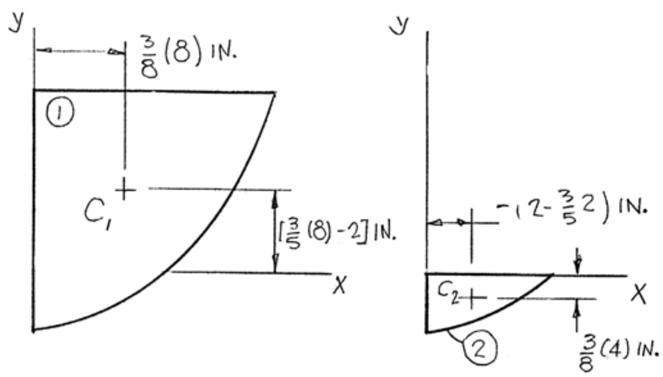
$$\bar{Y}\left(226.71 \text{ in}^2\right) = 3267 \text{ in}^3$$ or $\bar{Y} = 14.41$ in. ◀

PROBLEM 5.16



Locate the centroid of the plane area shown.

SOLUTION



	A, in^2	$\bar{x}$, in.	$\bar{y}$, in.	$\bar{x}A$, in^3	$\bar{y}A$, in^3
1	$\frac{2}{3}(8)(8) = 42.67$	3	2.8	128	119.47
2	$-\frac{2}{3}(4)(2) = -5.333$	1.5	–0.8	–8	4.267
Σ	37.33			120	123.73

Then

$$\bar{X}\Sigma A = \Sigma \bar{x} A$$

$$\bar{X}\left(37.33 \text{ in}^2\right) = 120 \text{ in}^3$$

or $\bar{X} = 3.21$ in. ◀

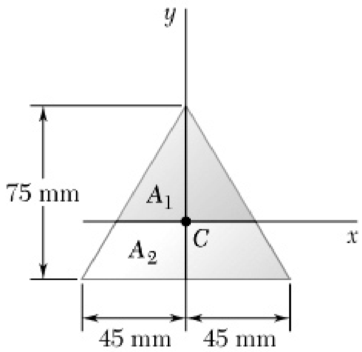
and

$$\bar{Y}\Sigma A = \Sigma \bar{y} A$$

$$\bar{Y}\left(37.33 \text{ in}^2\right) = 123.73 \text{ in}^3$$

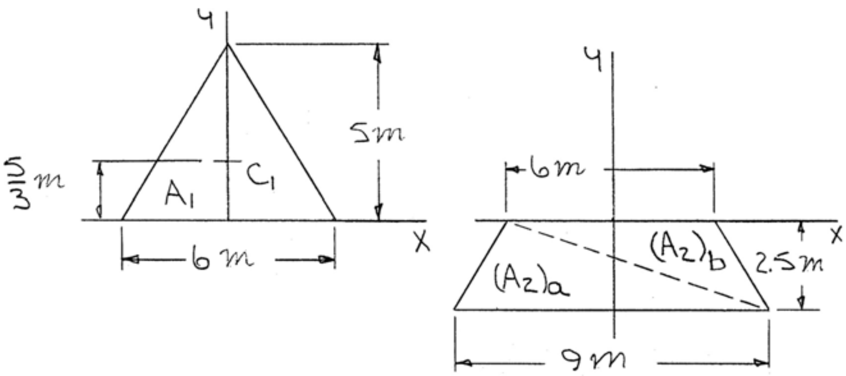
or $\bar{Y} = 3.31$ in. ◀

PROBLEM 5.17



The horizontal x axis is drawn through the centroid C of the area shown and divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the x axis, and explain the results obtained.

SOLUTION



Note that $$Q_x = \Sigma \bar{y} A$$

Then $$(Q_x)_1 = \left(\frac{5}{3}\text{ m}\right)\left(\frac{1}{2}\times 6\times 5\right)\text{m}^2 \qquad \text{or } (Q_x)_1 = 25.0\times 10^3\text{ mm}^3 \blacktriangleleft$$

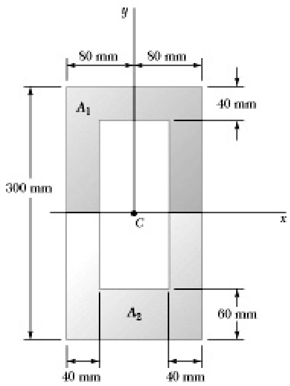
and $$(Q_x)_2 = \left(-\frac{2}{3}\times 2.5\text{ m}\right)\left(\frac{1}{2}\times 9\times 2.5\right)\text{m}^2 + \left(-\frac{1}{3}\times 2.5\text{ m}\right)\left(\frac{1}{2}\times 6\times 2.5\right)\text{m}^2$$

$$\text{or } (Q_x)_2 = -25.0\times 10^3\text{ mm}^3 \blacktriangleleft$$

Now $$Q_x = (Q_x)_1 + (Q_x)_2 = 0$$

This result is expected since x is a centroidal axis $(\text{thus } \bar{y} = 0)$

and $$Q_x = \Sigma \bar{y} A = \bar{Y}\Sigma A \quad (\bar{y} = 0 \Rightarrow Q_x = 0)$$

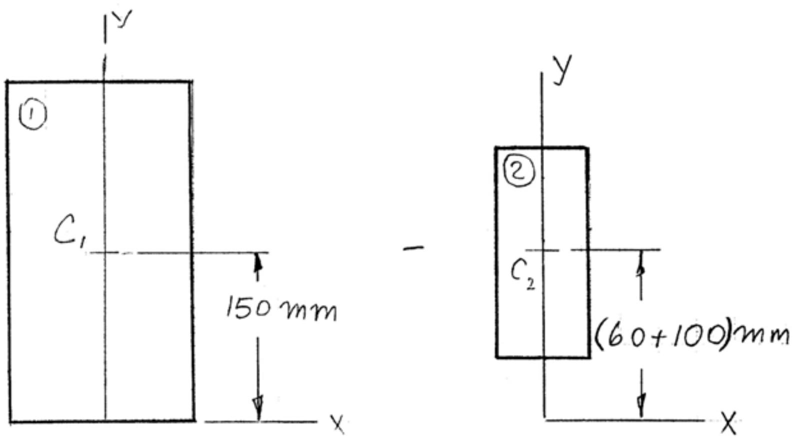


PROBLEM 5.18

The horizontal x axis is drawn through the centroid C of the area shown and divides the area into two component areas A_1 and A_2. Determine the first moment of each component area with respect to the x axis, and explain the results obtained.

SOLUTION

First, locate the position $\overline{y}$ of the figure.



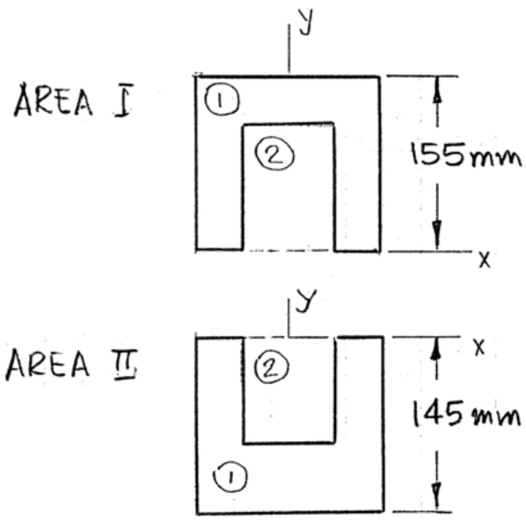
	A, mm^2	$\overline{y}$, mm	$\overline{y}A$, mm^3
1	$160 \times 300 = 48\ 000$	150	7 200 000
2	$-150 \times 80 = -16\ 000$	160	−2 560 000
Σ	32 000		4 640 000

Then $$\overline{Y}\Sigma A = \Sigma \overline{y} A$$

$$\overline{Y}\left(32\ 000 \text{ mm}^2\right) = 4\ 640\ 000 \text{ mm}^3$$

or $$\overline{Y} = 145.0 \text{ mm}$$

PROBLEM 5.18 CONTINUED



$$A_{\mathrm{I}}: \quad Q_{\mathrm{I}} = \Sigma \bar{y} A$$

$$= \frac{155}{2}(160 \times 155) + \frac{115}{2}\left[-(80 \times 115)\right]$$

$$= 1.393 \times 10^6 \text{ mm}^3 \quad ◀$$

$$A_{\mathrm{II}}: \quad Q_{\mathrm{II}} = \Sigma \bar{y} A$$

$$= -\frac{145}{2}(160 \times 145) - \left[-\frac{85}{2}(80 \times 85)\right]$$

$$= -1.393 \times 10^6 \text{ mm}^3 \quad ◀$$

$$\therefore \quad \left(\mathrm{Q}_{\text{area}}\right)_x = Q_{\mathrm{I}} + Q_{\mathrm{II}} = 0$$

Which is expected since $Q_x = \Sigma \bar{y} A = \bar{y} A$ and $\bar{y} = 0$, since x is a centroidal axis.