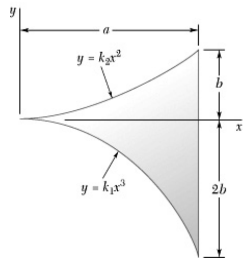
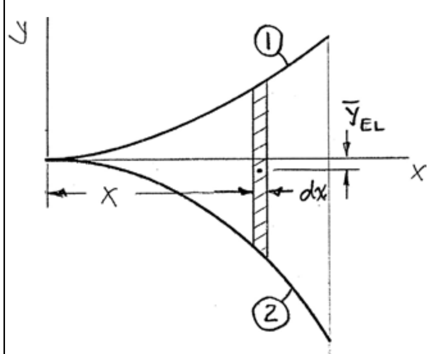
PROBLEM 5.37



Determine by direct integration the centroid of the area shown. Express your answer in terms of a and b.

SOLUTION



For the element (EL) shown on line 1 at

$$x = a, \; b = k_2 a^2 \qquad \text{or} \qquad k_2 = \frac{b}{a^2}$$

$$\therefore \quad y = \frac{b}{a^2} x^2$$

On line 2 at

$$x = a, \; -2b = k_1 a^3 \qquad \text{or} \qquad k_2 = \frac{-2b}{a^3}$$

$$\therefore \quad y = \frac{-2b}{a^3} x^3$$

$$dA = \left(\frac{b}{a^2} x^2 + \frac{2b}{a^3} x^3 \right) dx$$

Then

$$A = \int dA = \int_0^a \frac{b}{a^2}\left(x^2 + \frac{2x^3}{x} \right) dx = \frac{b}{a^2}\left(\frac{x^3}{3} + \frac{2x^4}{4a} \right)\Bigg|_0^a$$

$$= ab\left(\frac{1}{3} + \frac{1}{2} \right) = \frac{5}{6} ab$$

and

$$\int \overline{x}_{EL} dA = \int_0^a x\left(\frac{b}{a^2} x^2 + \frac{2b}{a^3} x^3 \right) dx = \frac{b}{a^2}\left(\frac{x^4}{4} + \frac{2x^5}{5a} \right)\Bigg|_0^a = a^2 b\left(\frac{1}{4} + \frac{2}{5} \right)$$

$$= \frac{13}{20} a^2 b$$

$$\int \overline{y}_{EL} dA = \int_0^a \frac{1}{2}\left(\frac{b}{a^2} x^2 - \frac{2b}{a^3} x^3 \right)\left[\left(\frac{b}{a^2} x^2 + \frac{2b}{a^3} x^3 \right) dx \right]$$

$$= \int_0^a \frac{1}{2}\left[\left(\frac{b}{a^2} x^2 \right)^2 - \left(\frac{2b}{a^3} x^3 \right)^2 \right] dx = \frac{b^2}{2a^4}\left(\frac{x^5}{5} - \frac{2}{7a^2} x^7 \right)\Bigg|_0^a$$

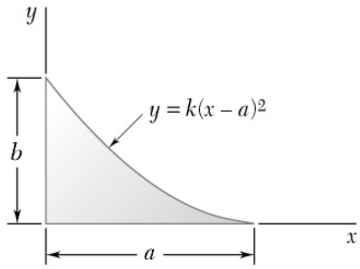
$$= b^2 a^5 \left(\frac{1}{10} - \frac{2}{7} \right) = -\frac{13}{70} ab^2$$

Then

$$\overline{x}A = \int \overline{x}_{EL} dA: \quad \overline{x}\left(\frac{5}{6} ab \right) = \frac{13}{20} a^2 b \qquad \text{or} \quad \overline{x} = \frac{39}{50} a \; ◀$$

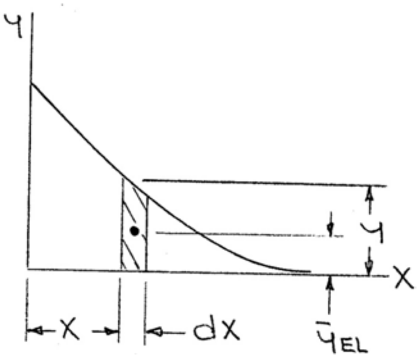
$$\overline{y}A = \int \overline{y}_{EL} dA: \quad \overline{y}\left(\frac{5}{6} ab \right) - \frac{13}{70} ab^2 \qquad \text{or} \quad \overline{y} = -\frac{39}{175} b \; ◀$$

PROBLEM 5.38



Determine by direct integration the centroid of the area shown. Express your answer in terms of a and b.

SOLUTION



At $$x = 0, y = b$$

$$b = k(0-a)^2 \qquad \text{or} \qquad k = \frac{b}{a^2}$$

Then $$y = \frac{b}{a^2}(x-a)^2$$

Now $$\bar{x}_{EL} = x, \; \bar{y}_{EL} = \frac{y}{2} = \frac{b}{2a^2}(x-a)^2$$

and $$dA = y\,dx = \frac{b}{a^2}(x-a)^2\,dx$$

Then $$A = \int dA = \int_0^a \frac{b}{a^2}(x-a)^2\,dx = \frac{b}{3a^2}\left[(x-a)^3\right]_0^a = \frac{1}{3}ab$$

and $$\int \bar{x}_{EL}\,dA = \int_0^a x\left[\frac{b}{a^2}(x-a)^2\,dx\right] = \frac{b}{a^2}\int_0^a \left(x^3 - 2ax^2 + a^2x\right)dx$$

$$= \frac{b}{a^2}\left(\frac{x^4}{4} - \frac{2}{3}ax^3 + \frac{a^2}{2}x^2\right) = \frac{1}{12}a^2b$$

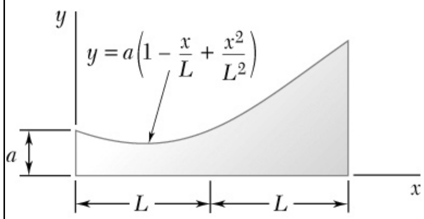
$$\int \bar{y}_{EL}\,dA = \int_0^a \frac{b}{2a^2}(x-a)^2\left[\frac{b}{a^2}(x-a)^2\,dx\right] = \frac{b^2}{2a^4}\left[\frac{1}{5}(x-a)^5\right]_0^a$$

$$= \frac{1}{10}ab^2$$

Hence $$\bar{x}A = \int \bar{x}_{EL}\,dA: \quad \bar{x}\left(\frac{1}{3}ab\right) = \frac{1}{12}a^2b \qquad \bar{x} = \frac{1}{4}a \blacktriangleleft$$

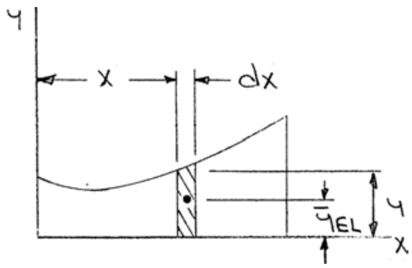
$$\bar{y}A = \int \bar{y}_{EL}\,dA: \quad \bar{y}\left(\frac{1}{3}ab\right) = \frac{1}{10}ab^2 \qquad \bar{y} = \frac{3}{10}b \blacktriangleleft$$

PROBLEM 5.39



Determine by direct integration the centroid of the area shown.

SOLUTION



Have $$\bar{x}_{EL} = x$$

$$\bar{y}_{EL} = \frac{1}{2}y = \frac{a}{2}\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)$$

$$dA = ydx = a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx$$

Then $$A = \int dA = \int_0^{2L} a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx = a\left[x - \frac{x^2}{2L} + \frac{x^3}{3L^2}\right]_0^{2L} = \frac{8}{3}aL$$

and $$\int \bar{x}_{EL}dA = \int_0^{2L} x\left[a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx\right] = a\left[\frac{x^2}{2} - \frac{x^3}{3L} + \frac{x^4}{4L^2}\right]_0^{2L}$$

$$= \frac{10}{3}aL^2$$

$$\int \bar{y}_{EL}dA = \int_0^{2L} \frac{a}{2}\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)\left[a\left(1 - \frac{x}{L} + \frac{x^2}{L^2}\right)dx\right]$$

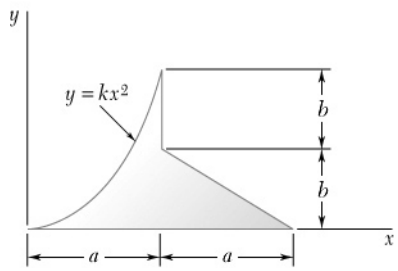
$$= \frac{a^2}{2}\int_0^{EL}\left(1 - 2\frac{x}{L} + 3\frac{x^2}{L^2} - 2\frac{x^3}{L^3} + \frac{x^4}{L^4}\right)dx$$

$$= \frac{a^2}{2}\left[x - \frac{x^2}{L} + \frac{x^3}{L^2} - \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^{2L} = \frac{11}{5}a^2L$$

Hence $$\bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}\left(\frac{8}{3}aL\right) = \frac{10}{3}aL^2 \qquad \bar{x} = \frac{5}{4}L \blacktriangleleft$$

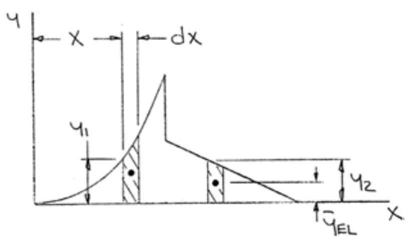
$$\bar{y}A = \int \bar{y}_{EL}dA: \quad \bar{y}\left(\frac{1}{8}a\right) = \frac{11}{5}a^2 \qquad \bar{y} = \frac{33}{40}a \blacktriangleleft$$

PROBLEM 5.40



Determine by direct integration the centroid of the area shown. Express your answer in terms of a and b.

SOLUTION



For $\quad y_1$ at $x = a,\ y = 2b \qquad 2b = ka^2 \qquad$ or $\ k = \dfrac{2b}{a^2}$

Then $$y_1 = \frac{2b}{a^2}x^2$$

By observation $$y_2 = -\frac{b}{a}(x + 2b) = b\left(2 - \frac{x}{a}\right)$$

Now $$\bar{x}_{EL} = x$$

and for $0 \le x \le a$:

$$\bar{y}_{EL} = \frac{1}{2}y_1 = \frac{b}{a^2}x^2 \qquad \text{and} \qquad dA = y_1 dx = \frac{2b}{a^2}x^2 dx$$

For $a \le x \le 2a$:

$$\bar{y}_{EL} = \frac{1}{2}y_2 = \frac{b}{2}\left(2 - \frac{x}{a}\right) \qquad \text{and} \qquad dA = y_2 dx = b\left(2 - \frac{x}{a}\right)dx$$

Then $$A = \int dA = \int_0^a \frac{2b}{a^2}x^2 dx + \int_a^{2a} b\left(2 - \frac{x}{a}\right)dx$$

$$= \frac{2b}{a^2}\left[\frac{x^3}{3}\right]_0^a + b\left[-\frac{a}{2}\left(2 - \frac{x}{a}\right)^2\right]_0^{2a} = \frac{7}{6}ab$$

and $$\int \bar{x}_{EL} dA = \int_0^a x\left(\frac{2b}{a^2}x^2 dx\right) + \int_a^{2a} x\left[b\left(2 - \frac{x}{a}\right)dx\right]$$

$$= \frac{2b}{a^2}\left[\frac{x^4}{4}\right]_0^a + b\left[x^2 - \frac{x^3}{3a}\right]_0^{2a}$$

$$= \frac{1}{2}a^2b + b\left\{\left[(2a)^2 - (a)^2\right] + \frac{1}{3a}\left[\left(2a^2\right) - (a)^3\right]\right\}$$

$$= \frac{7}{6}a^2b$$

PROBLEM 5.40 CONTINUED

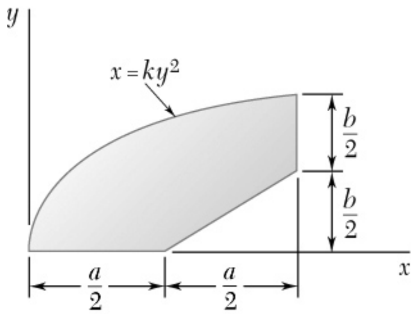
$$\int \bar{y}_{EL} dA = \int_0^a \frac{b}{a^2} x^2 \left[\frac{2b}{a^2} x^2 dx \right] + \int_0^{2a} \frac{b}{2}\left(2 - \frac{x}{a}\right)\left[b\left(2 - \frac{x}{a}\right) dx \right]$$

$$= \frac{2b^2}{a^4}\left[\frac{x^5}{5}\right]_0^a + \frac{b^2}{2}\left[-\frac{a}{3}\left(2 - \frac{x}{a}\right)^3\right]_a^{2a}$$

$$= \frac{17}{30} ab^2$$

Hence $\quad \bar{x}A = \int \bar{x}_{EL} dA: \quad \bar{x}\left(\frac{7}{6}ab\right) = \frac{7}{6}a^2 b \qquad \bar{x} = a$ ◀

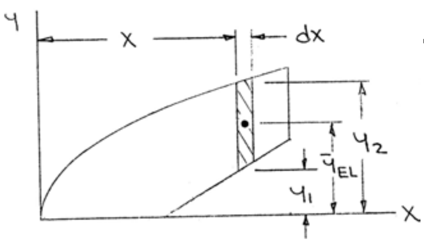
$$\bar{y}A = \int \bar{y}_{EL} dA: \quad \bar{y}\left(\frac{7}{6}ab\right) = \frac{17}{30}ab^2 \qquad \bar{y} = \frac{17}{35}b \;◀$$



PROBLEM 5.41

Determine by direct integration the centroid of the area shown. Express your answer in terms of a and b.

SOLUTION



For y_2 at $x = a,\ y = b$: $a = kb^2$ or $k = \dfrac{a}{b^2}$

Then $$y_2 = \frac{b}{\sqrt{a}}x^{1/2}$$

Now $$\bar{x}_{EL} = x$$

and for $0 \le x \le \dfrac{a}{2}$: $\bar{y}_{EL} = \dfrac{y_2}{2} = \dfrac{b}{2}\dfrac{x^{1/2}}{\sqrt{a}},\ dA = y_2 dx = b\dfrac{x^{1/2}}{\sqrt{a}}dx$

For $\dfrac{a}{2} \le x \le a$: $\bar{y}_{EL} = \dfrac{1}{2}(y_1 + y_2) = \dfrac{b}{2}\left(\dfrac{x}{a} - \dfrac{1}{2} + \dfrac{x^{1/2}}{\sqrt{a}}\right)$

$$dA = (y_2 - y_1)dx = b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx$$

Then $$A = \int dA = \int_0^{a/2} b\frac{x^{1/2}}{\sqrt{a}}dx + \int_{a/2}^{a} b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx$$

$$= \frac{b}{\sqrt{a}}\left[\frac{2}{3}x^{3/2}\right]_0^{a/2} + b\left[\frac{2}{3}\frac{x^{3/2}}{\sqrt{a}} - \frac{x^2}{2a} + \frac{1}{2}x\right]_{a/2}^{a}$$

$$= \frac{2}{3}\frac{b}{\sqrt{a}}\left[\left(\frac{a}{2}\right)^{3/2} + (a)^{3/2} - \left(\frac{a}{2}\right)^{3/2}\right]$$

$$+ b\left\{-\frac{1}{2a}\left[(a^2) - \left(\frac{a}{2}\right)^2\right] + \frac{1}{2}\left[(a) - \left(\frac{a}{2}\right)\right]\right\}$$

$$= \frac{13}{24}ab$$

PROBLEM 5.41 CONTINUED

and
$$\int \bar{x}_{EL}\,dA = \int_0^{a/2} x\left(b\frac{x^{1/2}}{\sqrt{a}}dx\right) + \int_{a/2}^{a} x\left[b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)\right]dx$$

$$= \frac{b}{\sqrt{a}}\left[\frac{2}{5}x^{5/2}\right]_0^{a/2} + b\left[\frac{2}{5}\frac{x^{5/2}}{\sqrt{a}} - \frac{x^3}{3a} + \frac{x^4}{4}\right]_{a/2}^{a}$$

$$= \frac{2}{5}\frac{b}{\sqrt{a}}\left[\left(\frac{a}{2}\right)^{5/2} + (a)^{5/2} - \left(\frac{a}{2}\right)^{5/2}\right]$$

$$+ b\left\{-\frac{1}{3a}\left[(a)^3 - \left(\frac{a}{2}\right)^3\right] + \frac{1}{4}\left[(a)^2 - \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{71}{240}a^2b$$

$$\int \bar{y}_{EL}\,dA = \int_0^{a/2} \frac{b}{2}\frac{x^{1/2}}{\sqrt{a}}\left[b\frac{x^{1/2}}{\sqrt{a}}dx\right]$$

$$+ \int_{a/2}^{a} \frac{b}{2}\left(\frac{x}{a} - \frac{1}{2} + \frac{x^{1/2}}{\sqrt{a}}\right)\left[b\left(\frac{x^{1/2}}{\sqrt{a}} - \frac{x}{a} + \frac{1}{2}\right)dx\right]$$

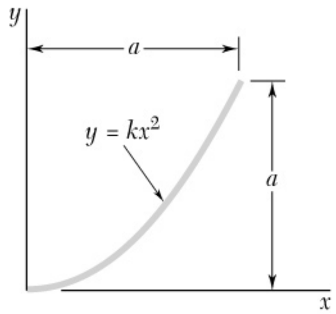
$$= \frac{b^2}{2a}\left[\frac{1}{2}x^2\right]_0^{a/2} + \frac{b^2}{2}\left[\left(\frac{x^2}{2a} - \frac{1}{3a}\left(\frac{x}{a} - \frac{1}{2}\right)^3\right)\right]_{a/2}^{a}$$

$$= \frac{b}{4a}\left[\left(\frac{a}{2}\right)^2 + (a)^2 - \left(\frac{a}{2}\right)^2\right] - \frac{b^2}{6a}\left(\frac{a}{2} - \frac{1}{2}\right)^3$$

$$= \frac{11}{48}ab^2$$

Hence $\bar{x}A = \int \bar{x}_{EL}\,dA$: $\quad \bar{x}\left(\frac{13}{24}ab\right) = \frac{71}{240}a^2b \qquad \bar{x} = \frac{17}{130}a = 0.546a$ ◀

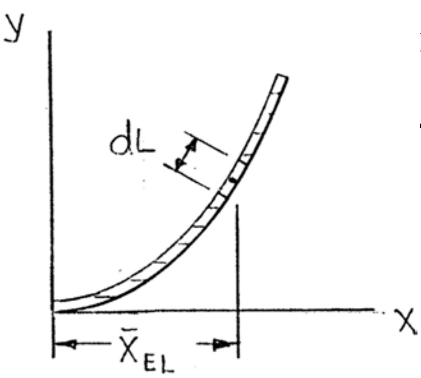
$\bar{y}A = \int \bar{y}_{EL}\,dA$: $\quad \bar{y}\left(\frac{13}{24}ab\right) = \frac{11}{48}ab^2 \qquad \bar{y} = \frac{11}{26}b = 0.423b$ ◀



PROBLEM 5.42

A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid. Express your answer in terms of a.

SOLUTION



First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line

Have at $\quad x = a,\ y = a:\ a = ka^2 \quad$ or $\quad k = \dfrac{1}{a}$

Thus $\quad y = \dfrac{1}{a}x^2 \quad$ and $\quad dy = \dfrac{2}{a}xdx$

Then $\quad dL = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}dx = \sqrt{1 + \left(\dfrac{2}{a}x\right)^2}dx$

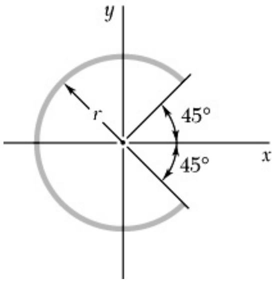
$$\therefore \quad L = \int dL = \int_0^a \sqrt{1 + \frac{4}{a^2}x^2}dx = \left[\frac{x}{2}\sqrt{1 + \frac{4x^2}{a^2}} + \frac{a}{4}\ln\left(\frac{2}{a}x + \sqrt{1 + \frac{4x^2}{a^2}}\right)\right]_0^a$$

$$= \frac{a}{2}\sqrt{5} + \frac{a}{4}\ln\left(2 + \sqrt{5}\right) = 1.4789a$$

$$\int \bar{x}_{EL}dL = \int_0^a x\left(\sqrt{1 + \frac{4x^2}{a^2}}dx\right) = \left[\frac{2}{3}\left(\frac{a^2}{8}\right)\left(1 + \frac{4}{a^2}x^2\right)^{3/2}\right]_0^a$$

$$= \frac{a^2}{12}\left(5^{3/2} - 1\right) = 0.8484a^2$$

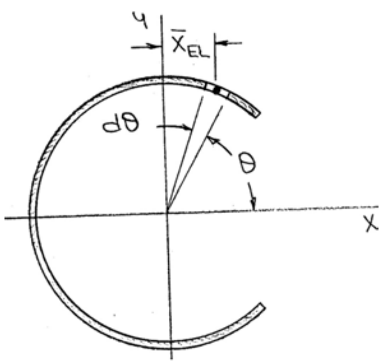
Then $\quad \bar{x}L = \int \bar{x}_{EL}dL: \quad \bar{x}(1.4789a) = 0.8484a^2 \qquad \bar{x} = 0.574a$ ◀



PROBLEM 5.43

A homogeneous wire is bent into the shape shown. Determine by direct integration the x coordinate of its centroid.

SOLUTION



First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line

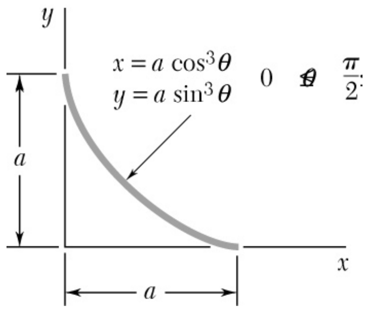
Now $\qquad \bar{x}_{EL} = r\cos\theta \qquad$ and $\qquad dL = r\,d\theta$

Then $$L = \int dL = \int_{\pi/4}^{7\pi/4} r\,d\theta = r\left[\theta\right]_{\pi/4}^{7\pi/4} = \frac{3}{2}\pi r$$

and $$\int \bar{x}_{EL}\,dL = \int_{\pi/4}^{7\pi/4} r\cos\theta\left(r\,d\theta\right)$$

$$= r^2\left[\sin\theta\right]_{\pi/4}^{7\pi/4} = r^2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = -r^2\sqrt{2}$$

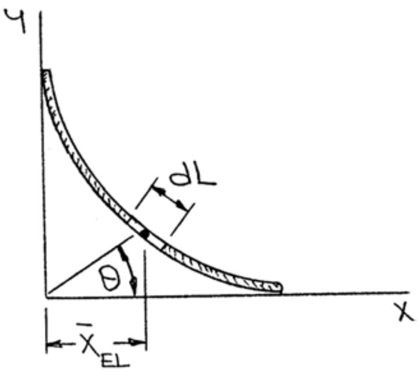
Thus $$\bar{x}L = \int \bar{x}\,dL: \quad \bar{x}\left(\frac{3}{2}\pi r\right) = -r^2\sqrt{2} \qquad\qquad \bar{x} = -\frac{2\sqrt{2}}{3\pi}r \ ◀$$



PROBLEM 5.44

A homogeneous wire is bent into the shape shown. Determine by direct integration the *x* coordinate of its centroid.

SOLUTION



First note that because the wire is homogeneous, its center of gravity coincides with the centroid of the corresponding line

Now $\quad \overline{x}_{EL} = a\cos^3\theta \quad$ and $\quad dL = \sqrt{dx^2 + dy^2}$

Where $\quad x = a\cos^3\theta: \quad dx = -3a\cos^2\theta\sin\theta d\theta$

$$y = a\sin^3\theta: \quad dy = 3a\sin^2\theta\cos\theta d\theta$$

Then
$$dL = \left[\left(-3a\cos^2\theta\sin\theta d\theta\right)^2 + \left(3a\sin^2\theta\cos\theta d\theta\right)^2\right]^{1/2}$$
$$= 3a\cos\theta\sin\theta\left(\cos^2\theta + \sin^2\theta\right)^{1/2} d\theta$$
$$= 3a\cos\theta\sin\theta d\theta$$

$$\therefore \quad L = \int dL = \int_0^{\pi/2} 3a\cos\theta\sin\theta d\theta = 3a\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}$$
$$= \frac{3}{2}a$$

and
$$\int \overline{x}_{EL} dL = \int_0^{\pi/2} a\cos^3\theta\left(3a\cos\theta\sin\theta d\theta\right)$$
$$= 3a^2\left[-\frac{1}{5}\cos^5\theta\right]_0^{\pi/2} = \frac{3}{5}a^2$$

Hence $\quad \overline{x}L = \int \overline{x}_{EL} dL: \quad \overline{x}\left(\frac{3}{2}a\right) = \frac{3}{5}a^2 \qquad \overline{x} = \frac{2}{5}a$ ◀

PROBLEM 5.44 CONTINUED

Alternative solution

$$x = a\cos^3\theta \;\Rightarrow\; \cos^2\theta = \left(\frac{x}{a}\right)^{2/3}$$

$$y = a\sin^3\theta \;\Rightarrow\; \sin^2\theta = \left(\frac{y}{a}\right)^{2/3}$$

$$\therefore\quad \left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1 \qquad \text{or} \qquad y = \left(a^{2/3} - x^{2/3}\right)^{3/2}$$

Then $$\frac{dy}{dx} = \left(a^{2/3} - x^{2/3}\right)^{1/2}\left(-x^{-1/3}\right)$$

Now $$\overline{x}_{EL} = x$$

and $$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \left\{1 + \left[\left(a^{2/3} - x^{2/3}\right)^{1/2}\left(-x^{-1/3}\right)\right]^2\right\}^{1/2} dx$$

Then $$L = \int dL = \int_0^a \frac{a^{1/3}}{x^{1/3}}dx = a^{1/3}\left[\frac{3}{2}x^{2/3}\right]_0^a = \frac{3}{2}a$$

and $$\int \overline{x}_{EL}dL = \int_0^a x\left(\frac{a^{1/3}}{x^{1/3}}dx\right) = a^{1/3}\left[\frac{3}{5}x^{5/3}\right]_0^a = \frac{3}{5}a^2$$

Hence $$\overline{x}L = \int \overline{x}_{EL}dL:\quad \overline{x}\left(\frac{3}{2}a\right) = \frac{3}{5}a^2 \qquad \overline{x} = \frac{2}{5}a \;\blacktriangleleft$$