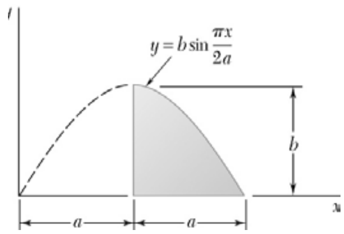
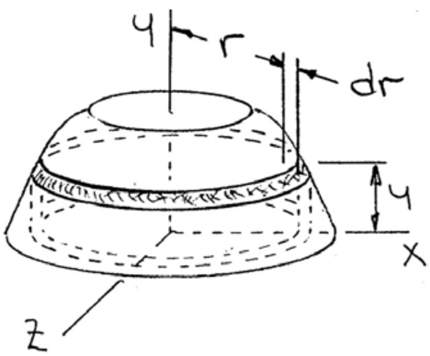


PROBLEM 5.123

Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the y axis. (*Hint:* Use a thin cylindrical shell of radius r and thickness dr as the element of volume.)

SOLUTION



First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a cylindrical shell of radius r and thickness dr.

Then $$dV = (2\pi r)(y)(dr), \quad \bar{y}_{EL} = \frac{1}{2}y$$

Now $$y = b\sin\frac{\pi r}{2a}$$

so that $$dV = 2\pi br\sin\frac{\pi r}{2a}dr$$

Then $$V = \int_a^{2a} 2\pi br\sin\frac{\pi r}{2a}dr$$

Use integration by parts with

$$u = r \qquad dv = \sin\frac{\pi r}{2a}dr$$

$$du = dr \qquad v = -\frac{2a}{\pi}\cos\frac{\pi r}{2a}$$

Then $$V = 2\pi b\left\{\left[(r)\left(-\frac{2a}{\pi}\cos\frac{\pi r}{2a}\right)\right]_a^{2a} - \int_a^{2a}\left(\frac{2a}{\pi}\cos\frac{\pi r}{2a}\right)dr\right\}$$

$$= 2\pi b\left\{-\frac{2a}{\pi}[(2a)(-1)] + \left[\frac{4a^2}{\pi^2}\sin\frac{\pi r}{2a}\right]_a^{2a}\right\}$$

$$V = 2\pi b\left(\frac{4a^2}{\pi} - \frac{4a^2}{\pi^2}\right)$$

$$= 8a^2b\left(1 - \frac{1}{\pi}\right)$$

$$= 5.4535\, a^2b$$

Also $$\int \bar{y}_{EL}dV = \int_a^{2a}\left(\tfrac{1}{2}b\sin\frac{\pi r}{2a}\right)\left(2\pi br\sin\frac{\pi r}{2a}dr\right)$$

$$= \pi b^2\int_a^{2a} r\sin^2\frac{\pi r}{2a}dr$$

PROBLEM 5.123 CONTINUED

Use integration by parts with

$$u = r \qquad dv = \sin^2 \frac{\pi r}{2a} dr$$

$$du = dr \qquad v = \frac{r}{2} - \frac{\sin \frac{\pi r}{a}}{\frac{2\pi}{a}}$$

Then

$$\int \overline{y}_{EL} dV = \pi b^2 \left\{ \left[(r)\left(\frac{r}{2} - \frac{\sin \frac{\pi r}{a}}{\frac{2\pi}{a}} \right) \right]_a^{2a} - \int_a^{2a} \left(\frac{r}{2} - \frac{\sin \frac{\pi r}{a}}{\frac{2\pi}{a}} \right) dr \right\}$$

$$= \pi b^2 \left\{ \left[(2a)\left(\frac{2a}{2} \right) - (a)\left(\frac{a}{2} \right) \right] - \left[\frac{r^2}{4} + \frac{a^2}{2\pi^2} \cos \frac{\pi r}{a} \right]_a^{2a} \right\}$$

$$= \pi b^2 \left\{ \frac{3}{2} a^2 - \left[\frac{(2a)^2}{4} + \frac{a^2}{2\pi^2} - \frac{(a)^2}{4} + \frac{a^2}{2\pi^2} \right] \right\}$$

$$= \pi a^2 b^2 \left(\frac{3}{4} - \frac{1}{\pi^2} \right)$$

$$= 2.0379 a^2 b^2$$

Now

$$\overline{y}V = \int \overline{y}_{EL} dV: \quad \overline{y}\left(5.4535 a^2 b\right) = 2.0379 a^2 b^2$$

or $\overline{y} = 0.374b$ ◀

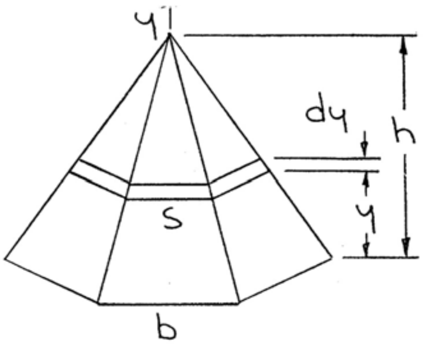
PROBLEM 5.124

Show that for a regular pyramid of height h and n sides ($n = 3, 4, \ldots$) the centroid of the volume of the pyramid is located at a distance $h/4$ above the base.

SOLUTION

Choose as the element of a horizontal slice of thickness dy. For any number N of sides, the area of the base of the pyramid is given by

$$A_{\text{base}} = kb^2$$



where $k = k(N)$; see note below. Using similar triangles, have

$$\frac{s}{b} = \frac{h-y}{h}$$

or
$$s = \frac{b}{h}(h-y)$$

Then
$$dV = A_{\text{slice}}dy = ks^2dy = k\frac{b^2}{h^2}(h-y)^2\,dy$$

and
$$V = \int_0^h k\frac{b^2}{h^2}(h-y)^2\,dy = k\frac{b^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h$$

$$= \frac{1}{3}kb^2h$$

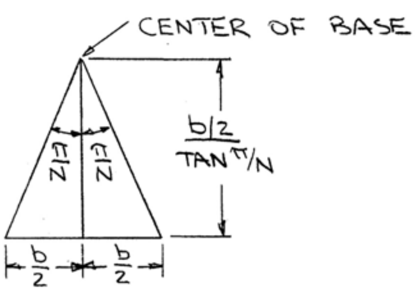
Also
$$\overline{y}_{EL} = y$$

so then
$$\int \overline{y}_{EL}dV = \int_0^h y\left[k\frac{b^2}{h^2}(h-y)^2\,dy\right] = k\frac{b^2}{h^2}\int_0^h\left(h^2y - 2hy^2 + y^3\right)dy$$

$$= k\frac{b^2}{h^2}\left[\frac{1}{2}h^2y^2 - \frac{2}{3}hy^3 + \frac{1}{4}y^4\right]_0^h = \frac{1}{12}kb^2h^2$$

Now
$$\overline{y}V = \int \overline{y}_{EL}dV: \quad \overline{y}\left(\frac{1}{3}kb^2h\right) = \frac{1}{12}kb^2h^2$$

$$\text{or}\ \ y = \frac{1}{4}h \ \ \text{Q.E.D.} \blacktriangleleft$$



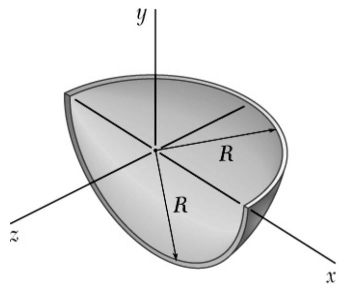
Note

$$A_{\text{base}} = N\left(\frac{1}{2} \times b \times \frac{\frac{b}{2}}{\tan\frac{\pi}{N}}\right)$$

$$= \frac{N}{4\tan\frac{\pi}{N}}b^2$$

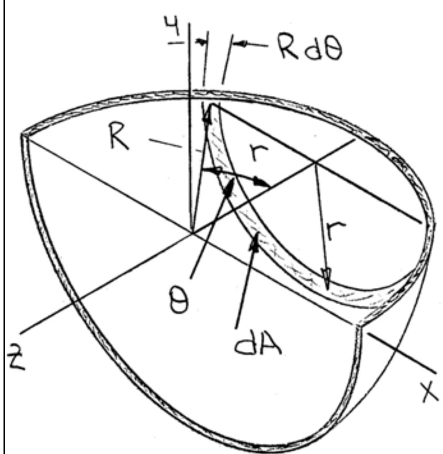
$$= k(N)\,b^2$$

PROBLEM 5.125



Determine by direct integration the location of the centroid of one-half of a thin, uniform hemispherical shell of radius *R*.

SOLUTION



First note that symmetry implies $\bar{x} = 0$ ◀

The element of area dA of the shell shown is obtained by cutting the shell with two planes parallel to the xy plane. Now

$$dA = (\pi r)(R d\theta)$$

$$\bar{y}_{EL} = -\frac{2r}{\pi}$$

where $r = R\sin\theta$

so that $dA = \pi R^2 \sin\theta d\theta$

$$\bar{y}_{\mathrm{EL}} = -\frac{2R}{\pi}\sin\theta$$

Then $A = \int_0^{\frac{\pi}{2}} \pi R^2 \sin\theta d\theta = \pi R^2 [-\cos\theta]_0^{\frac{\pi}{2}}$

$$= \pi R^2$$

and $\int \bar{y}_{EL} dA = \int_0^{\frac{\pi}{2}} \left(-\frac{2R}{\pi}\sin\theta\right)(\pi R^2 \sin\theta d\theta)$

$$= -2R^3\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

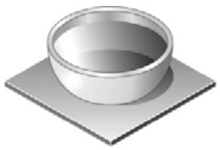
$$= -\frac{\pi}{2}R^3$$

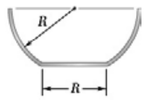
Now $\bar{y}A = \int \bar{y}_{EL} dA\text{:}\ \ \bar{y}(\pi R^2) = -\frac{\pi}{2}R^3$

or $\bar{y} = -\frac{1}{2}R$ ◀

Symmetry implies $\bar{z} = \bar{y}\ \therefore\ \bar{z} = -\frac{1}{2}R$ ◀

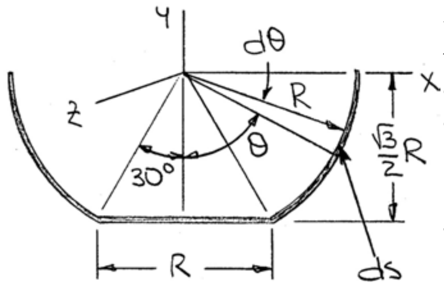
PROBLEM 5.126





The sides and the base of a punch bowl are of uniform thickness t. If $t << R$ and $R = 350$ mm, determine the location of the center of gravity of (a) the bowl, (b) the punch.

SOLUTION



(a) Bowl

First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

for the coordinate axes shown below. Now assume that the bowl may be treated as a shell; the center of gravity of the bowl will coincide with the centroid of the shell. For the walls of the bowl, an element of area is obtained by rotating the arc ds about the y axis. Then

$$dA_{\text{wall}} = (2\pi R\sin\theta)(Rd\theta)$$

and $(\bar{y}_{EL})_{\text{wall}} = -R\cos\theta$

Then $\quad A_{\text{wall}} = \int_{\pi/6}^{\pi/2} 2\pi R^2 \sin\theta d\theta = 2\pi R^2[-\cos\theta]_{\pi/6}^{\pi/2}$

$$= \pi\sqrt{3}R^2$$

and $\quad \bar{y}_{\text{wall}}A_{\text{wall}} = \int (\bar{y}_{EL})_{\text{wall}}\, dA$

$$= \int_{\pi/6}^{\pi/2}(-R\cos\theta)(2\pi R^2\sin\theta d\theta)$$

$$= \pi R^3\left[\cos^2\theta\right]_{\pi/6}^{\pi/2}$$

$$= -\frac{3}{4}\pi R^3$$

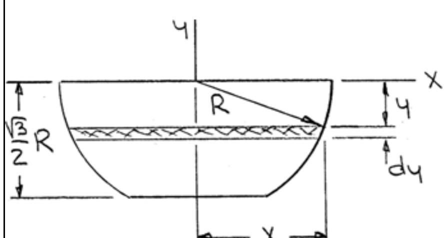
By observation $\quad A_{\text{base}} = \frac{\pi}{4}R^2, \quad \bar{y}_{\text{base}} = -\frac{\sqrt{3}}{2}R$

Now $\quad \bar{y}\Sigma A = \Sigma\bar{y}A$

or $\quad \bar{y}\left(\pi\sqrt{3}R^2 + \frac{\pi}{4}R^2\right) = -\frac{3}{4}\pi R^3 + \frac{\pi}{4}R^2\left(-\frac{\sqrt{3}}{2}R\right)$

or $\quad \bar{y} = -0.48763R \qquad R = 350 \text{ mm}$

$\therefore\ \bar{y} = -170.7 \text{ mm}$ ◀



(b) Punch

First note that symmetry implies $\bar{x} = 0$ ◀

$\bar{z} = 0$ ◀

and that because the punch is homogeneous, its center of gravity will coincide with the centroid of the corresponding volume. Choose as the element of volume a disk of radius x and thickness dy. Then

$$dV = \pi x^2 dy, \quad \bar{y}_{EL} = y$$

PROBLEM 5.126 CONTINUED

Now $\quad x^2 + y^2 = R^2 \quad$ so that $\quad dV = \pi\left(R^2 - y^2\right)dy$

Then $$V = \int_{-\sqrt{3}/2R}^{0} \pi\left(R^2 - y^2\right)dy = \pi\left[R^2y - \frac{1}{3}y^3\right]_{-\sqrt{3}/2R}^{0}$$

$$= -\pi\left[R^2\left(-\frac{\sqrt{3}}{2}R\right) - \frac{1}{3}\left(-\frac{\sqrt{3}}{2}R\right)^3\right] = \frac{3}{8}\pi\sqrt{3}R^3$$

and $$\int \overline{y}_{EL}dV = \int_{-\sqrt{3}/2R}^{0}(y)\left[\pi\left(R^2 - y^2\right)dy\right] = \pi\left[\frac{1}{2}R^2y^2 - \frac{1}{4}y^4\right]_{-\sqrt{3}/2R}^{0}$$

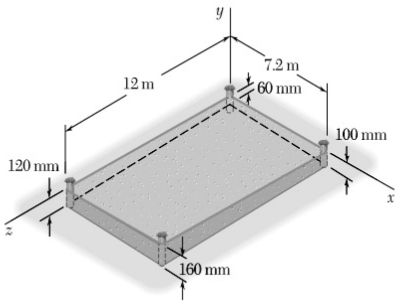
$$= -\pi\left[\frac{1}{2}R^2\left(-\frac{\sqrt{3}}{2}R\right)^2 - \frac{1}{4}\left(-\frac{\sqrt{3}}{2}R\right)^4\right] = -\frac{15}{64}\pi R^4$$

Now $$\overline{y}V = \int \overline{y}_{EL}dV: \quad \overline{y}\left(\frac{3}{8}\pi\sqrt{3}R^3\right) = -\frac{15}{64}\pi R^4$$

or $$\overline{y} = -\frac{5}{8\sqrt{3}}R \qquad R = 350 \text{ mm}$$

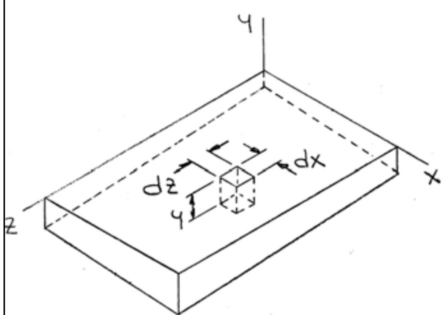
$$\therefore\ \overline{y} = -126.3 \text{ mm} \blacktriangleleft$$

PROBLEM 5.127



After grading a lot, a builder places four stakes to designate the corners of the slab for a house. To provide a firm, level base for the slab, the builder places a minimum of 60 mm of gravel beneath the slab. Determine the volume of gravel needed and the x coordinate of the centroid of the volume of the gravel. (*Hint:* The bottom of the gravel is an oblique plane, which can be represented by the equation $y = a + bx + cz$.)

SOLUTION



First, determine the constants a, b, and c.

At $\quad x = 0, z = 0: \; y = -60 \text{ mm}$

$$\therefore -60 \text{ mm} = a; \; a = -60 \text{ mm}$$

At $\quad x = 7200 \text{ mm}, z = 0: \; y = -100 \text{ mm}$

$$\therefore -100 \text{ mm} = -60 + b(7200)$$

$$b = -\frac{1}{180}$$

At $\quad x = 0, z = 12\,000 \text{ mm}: \; y = -120 \text{ mm}$

$$\therefore -120 \text{ mm} = -60 \text{ mm} + c(12000)$$

$$c = -\frac{1}{200}$$

It follows that $y = -60 - \frac{1}{180}x - \frac{1}{120}z$ where all dimensions are in mm.

Choose as the element of volume a filament of base $dx \times dz$ and height $|y|$. Then

$$dV = |y|\,dxdz, \quad \bar{x}_{EL} = x$$

or

$$dV = \left| -60 - \frac{1}{180}x - \frac{1}{200}z \right| dxdz$$

Then

$$V = \int_0^{12000}\int_0^{7200}\left(60 + \frac{1}{180}x + \frac{1}{200}z\right)dxdz$$

$$= \int_0^{12000}\left[60x + \frac{1}{360}x^2 + \frac{1}{200}xz\right]_0^{7200} dz$$

$$= \int_0^{12000}\left[(60)(7200) + \frac{(7200)^2}{360} + \frac{(7200)}{200}z\right]dz$$

$$= \left[576000z + \frac{36}{2}z^2\right]_0^{12000}$$

$$= 9.504 \times 10^9 \text{ mm}^3 = 9.50 \text{ m}^3$$

$$V = 9.50 \text{ m}^3 \blacktriangleleft$$

PROBLEM 5.127 CONTINUED

and $$\int \bar{x}_{EL} dV = \int_0^{12000} \int_0^{7200} x\left(60 + \frac{1}{180}x + \frac{1}{200}z\right) dxdz$$

$$= \int_0^{12000} \left[\frac{60}{2}x^2 + \frac{1}{540}x^3 + \frac{1}{400}x^2 z\right]_0^{7200} dz$$

$$= \int_0^{12000} \left(2246.4 \times 10^6 + 129\ 600z\right) dz$$

$$= \left[2246.4 \times 10^6 z - \frac{129\ 600}{2} z^2\right]_0^{12000}$$

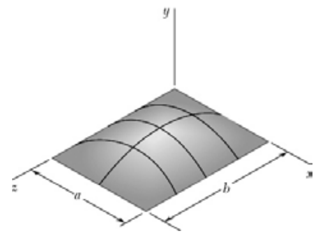
$$= 2.695 \times 10^{13} + 0.933 \times 10^{13}$$

$$= 3.63 \times 10^{13} \text{mm}^4 = 36.3\ \text{m}^4$$

Now $$\bar{x}V = \int \bar{x}_{EL} dV: \ \bar{x}\left(9.50\ \text{m}^3\right) = 36.3\ \text{m}^4$$

or $\bar{x} = 3.82$ m ◀

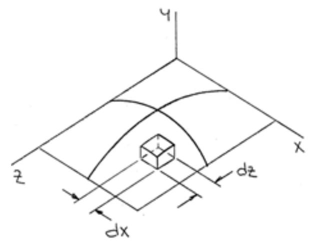
PROBLEM 5.128



Determine by direct integration the location of the centroid of the volume between the *xz* plane and the portion shown of the surface

$$y = \frac{16h\left(ax - x^2\right)\left(bz - z^2\right)}{a^2b^2}.$$

SOLUTION



First note that symmetry implies

$$\bar{x} = \frac{a}{2} \blacktriangleleft$$

$$\bar{z} = \frac{b}{2} \blacktriangleleft$$

Choose as the element of volume a filament of base $dx \times dz$ and height y. Then

$$dV = ydxdz, \bar{y}_{\text{EL}} = \frac{1}{2}y$$

or

$$dV = \frac{16h}{a^2b^2}\left(ax - x^2\right)\left(bz - z^2\right)dxdz$$

Then

$$V = \int_0^b \int_0^a \frac{16h}{a^2b^2}\left(ax - x^2\right)\left(bz - z^2\right)dxdz$$

$$V = \frac{16h}{a^2b^2}\int_0^b \left(bz - z^2\right)\left[\frac{a}{z}x^2 - \frac{1}{3}x^3\right]_0^a dz$$

$$= \frac{16h}{a^2b^2}\left[\frac{a}{2}\left(a^2\right) - \frac{1}{3}\left(a\right)^3\right]\left[\frac{b}{2}z^2 - \frac{1}{3}z^3\right]_0^b = \frac{8ah}{3b^2}\left[\frac{b}{2}\left(b\right)^2 - \frac{1}{3}\left(b\right)^3\right] = \frac{4}{9}abh$$

and

$$\int \bar{y}_{EL}dV = \int_0^b \int_0^a \frac{1}{2}\left[\frac{16h}{a^2b^2}\left(ax - x^2\right)\left(bz - z^2\right)\right]\left[\frac{16h}{a^2b^2}\left(ax - x^2\right)\left(bz - z^2\right)dxdz\right]$$

$$= \frac{128h^2}{a^4b^4}\int_0^b \int_0^a \left(a^2x^2 - 2ax^3 + x^4\right)\left(b^2z^2 - 2bz^3 + z^4\right)dxdz$$

$$= \frac{128h^2}{a^2b^4}\int_0^b \left(b^2z^2 - 2bz^3 + z^4\right)\left[\frac{a^2}{3}x^3 - \frac{a}{2}x^4 + \frac{1}{5}x^5\right]_0^a dz$$

$$= \frac{128h^2}{a^4b^4}\left[\frac{a^2}{3}\left(a\right)^3 - \frac{a}{2}\left(a\right)^4 + \frac{1}{5}\left(a\right)^5\right]\left[\frac{b^2}{3}Z^3 - \frac{b}{Z}Z^4 + \frac{1}{5}Z^5\right]_0^b$$

$$= \frac{64ah^2}{15b^4}\left[\frac{b^3}{3}\left(b\right)^3 - \frac{b}{2}\left(b\right)^4 + \frac{1}{5}\left(b\right)^5\right] = \frac{32}{225}abh^2$$

PROBLEM 5.128 CONTINUED

Now $$\bar{y}V = \int \bar{y}_{EL}dV: \quad \bar{y}\left(\frac{4}{9}abh\right) = \frac{32}{225}abh^2$$

or $\bar{y} = \dfrac{8}{25}h$ ◀