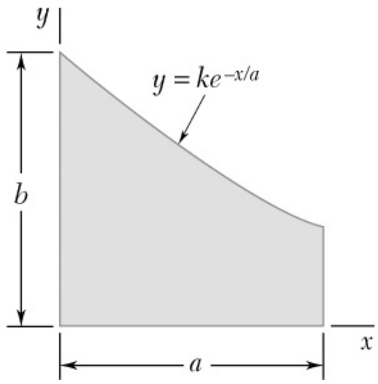
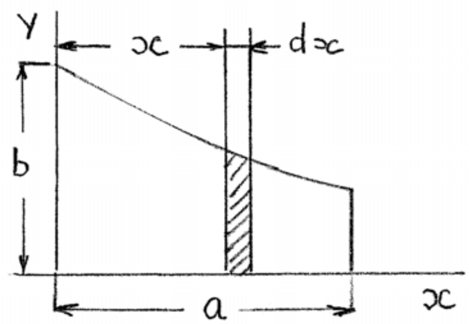


PROBLEM 9.11

Determine by direct integration the moment of inertia of the shaded area with respect to the x axis.

SOLUTION



At $$x = 0,\ y = b:\quad b = ke^0 = k$$

Then $$y = be^{-\frac{x}{a}}$$

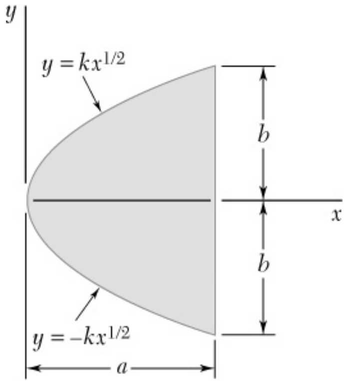
Now $$dI_x = \frac{1}{3}y^3 dx = \frac{b^3}{3}\left(e^{-\frac{x}{a}}\right)^3 dx$$

Then $$I_x = \int dI_x = \int_0^a \frac{b^3}{3}\left(e^{-\frac{x}{a}}\right)^3 dx$$

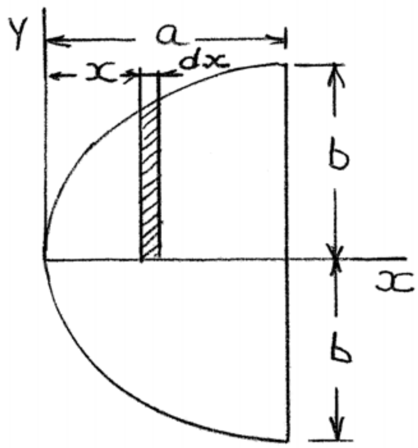
$$= \frac{b^3}{3}\int e^{-\frac{3x}{a}} dx = \frac{b^3}{3}\left(\frac{-a}{3}\right)e^{-\frac{3x}{a}}\bigg|_0^a = -\frac{b^3 a}{9}\left(e^{-3} - e^0\right)$$

$$= \frac{ab^3}{9}(0.95021) = 0.10558ab^3$$

or $I_x = 0.1056ab^3$ ◀

	**PROBLEM 9.12** Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $\quad x = a, \quad y = b: \quad b = ka^{\frac{1}{2}} \quad$ or $\quad k = \dfrac{b}{\sqrt{a}}$

Then $$y = \frac{b}{\sqrt{a}} x^{\frac{1}{2}}$$

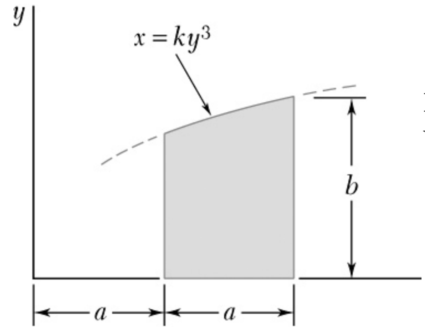
Now $$dI_y = x^2 dA, \qquad dA = y dx$$

$$dI_y = x^2 y dx = \frac{b}{\sqrt{a}} x^{\frac{5}{2}} dx$$

Then $$I_y = \int dI_y = 2\int_0^a \frac{b}{\sqrt{a}} x^{\frac{5}{2}} dx = \frac{4}{7}\frac{b}{\sqrt{a}} x^{\frac{7}{2}}\Big|_0^a$$

$$= \frac{4}{7}\frac{b}{a^{\frac{1}{2}}} a^{\frac{7}{2}}$$

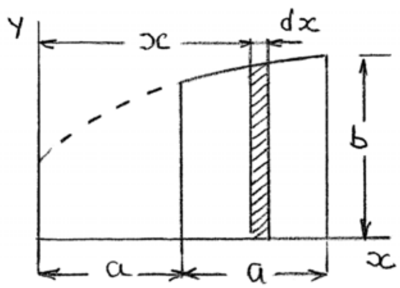
or $\quad I_y = \dfrac{4}{7}a^3 b$ ◀



PROBLEM 9.13

Determine by direct integration the moment of inertia of the shaded area with respect to the y axis.

SOLUTION



At $\quad x = 2a, \quad y = b: \quad 2a = kb^3 \qquad$ or $\qquad$ s

Then $$x = \frac{2a}{b^3}y^3$$

or $$y = \frac{b}{(2a)^{\frac{1}{3}}}x^{\frac{1}{3}}$$

Now $$I_y = \int x^2 dA \qquad dA = ydx$$

Then $$I_y = \int_a^{2a} x^2 \frac{b}{(2a)^{\frac{1}{3}}}x^{\frac{1}{3}}dx$$

$$= \frac{b}{(2a)^{\frac{1}{3}}}\int_a^{2a} x^{\frac{5}{3}}dx$$

$$= \frac{b}{(2a)^{\frac{1}{3}}}\frac{3}{10}x^{\frac{10}{3}}\Bigg|_a^{2a}$$

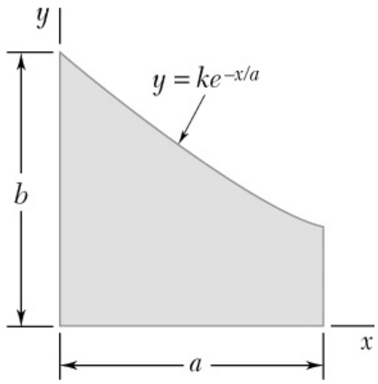
$$= \frac{3b}{10(2a)^{\frac{1}{3}}}\left[(2a)^{\frac{10}{3}} - a^{\frac{10}{3}}\right]$$

$$= \frac{3ba^3}{10(2)^{\frac{1}{3}}}\left(2^{\frac{10}{3}} - 1^{\frac{10}{3}}\right)$$

$$= 2.1619a^3b$$

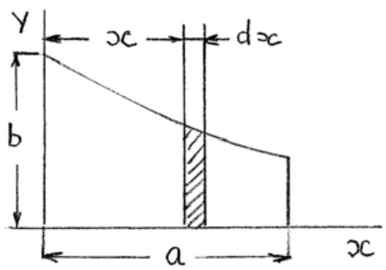
or $I_y = 2.16a^3b$ ◀

PROBLEM 9.14



Determine by direct integration the moment of inertia of the shaded area with respect to the *y* axis.

SOLUTION



At $$x = 0, \quad y = b: \quad b = ke^0 = k$$

Then $$y = be^{-\frac{x}{a}}$$

Now $$dI_y = x^2 dA = x^2 y dx$$

$$= x^2 b e^{-\frac{x}{a}} dx$$

Then $$I_y = \int dI_y = \int_0^a bx^2 e^{-\frac{x}{a}} dx = b\int_0^a x^2 e^{-\frac{x}{a}} dx$$

Use integration by parts

$$u = x^2 \qquad dv = e^{-\frac{x}{a}} dx$$

$$du = 2x dx \qquad v = -ae^{-\frac{x}{a}}$$

Then $$I_y = \int_0^a x^2 e^{-\frac{x}{a}} dx = b\left[-ax^2 e^{-\frac{x}{a}} \Big|_0^a - \int_0^a \left(-ae^{-\frac{x}{a}} \right) 2x dx \right]$$

$$= b\left(-a^3 e^{-1} + 2a\int_0^a xe^{-\frac{x}{a}} dx \right)$$

Again use integration by parts:

$$u = x \qquad dv = e^{-\frac{x}{a}} dx$$

$$du = dx \qquad v = -ae^{-\frac{x}{a}}$$

PROBLEM 9.14 CONTINUED

Then $\quad \int_0^a xe^{-\frac{x}{a}}dx = -axe^{-\frac{x}{a}}\Big|_0^a - \int_0^a\left(-ae^{-\frac{x}{a}}\right)dx$

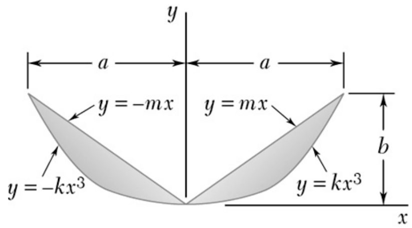
$$= -a^2e^{-1} - a^2e^{-\frac{x}{a}}\Big|_0^a = -a^2e^{-1} - a^2e^{-1} + a^2e^0$$

$$= -2a^2e^{-1} + a^2$$

Finally, $\quad I_y = b\left[-a^3e^{-1} + 2a\left(-2a^2e^{-1} + a^2\right)\right] = ba^3\left(-5e^{-1} + 2\right)$

$$= 0.1606ba^3$$

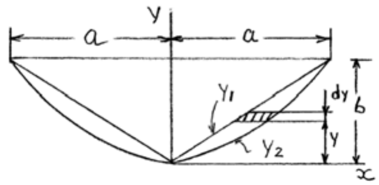
or $I_y = 0.1606ba^3$ ◀



PROBLEM 9.15

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the *x* axis.

SOLUTION



At $$x = a, \qquad y_1 = y_2 = b$$

$$y_1\!: \quad b = ma \quad \text{or} \quad m = \frac{b}{a}$$

$$y_2\!: \quad b = ka^3 \quad \text{or} \quad k = \frac{b}{a^3}$$

Then $$y_1 = \frac{b}{a}x \quad \text{or} \quad x_1 = \frac{a}{b}y$$

$$y_2 = \frac{b}{a^3}x^3 \quad \text{or} \quad x_2 = \left(\frac{a}{b^{\frac{1}{3}}}\right)y^{\frac{1}{3}}$$

Now $$dA = (x_2 - x_1)dy = \left(\frac{a}{b^{\frac{1}{3}}}y^{\frac{1}{3}} - \frac{a}{b}y\right)dy$$

$$A = 2\int dA = 2\int_0^b\left(\frac{a}{b^{\frac{1}{3}}}y^{\frac{1}{3}} - \frac{a}{b}y\right)dy = 2\left[\frac{a}{b^{\frac{1}{3}}}\frac{3}{4}y^{\frac{4}{3}} - \frac{a}{b}\frac{1}{2}y^2\right]_0^b$$

$$= \frac{3ab}{2} - ab = \frac{1}{2}ab$$

Then $$dI_x = y^2 dA = y^2\left(\frac{a}{b^{\frac{1}{3}}}y^{\frac{1}{3}} - \frac{a}{b}y\right)dy$$

Now $$I_x = 2\int dI_x = 2a\int_0^b\left(\frac{y^{\frac{7}{3}}}{b^{\frac{1}{3}}} - \frac{y^3}{b}\right)dy = 2a\left[\frac{3}{10}\frac{y^{\frac{10}{3}}}{b^{\frac{1}{3}}} - \frac{y^4}{4b}\right]_0^b$$

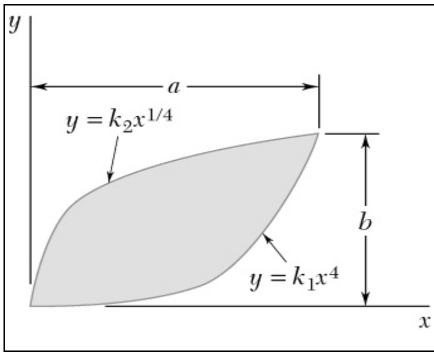
$$= 2a\left(\frac{3}{10}b^3 - \frac{b^3}{4}\right) = 2ab^3\left(\frac{3}{10} - \frac{1}{4}\right)$$

$$\text{or} \quad I_x = \frac{1}{10}ab^3 \blacktriangleleft$$

And $$k_x^2 = \frac{I_x}{A} = \frac{\frac{1}{10}ab^3}{\frac{1}{2}ab} = \frac{1}{5}b^2$$

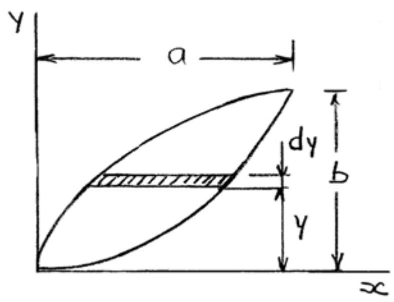
$$k_x = \frac{b}{\sqrt{5}} \blacktriangleleft$$

PROBLEM 9.16



Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.

SOLUTION



At $\quad x = a,\ y = b: \quad b = k_1 a^4 \qquad b = k_2 a^{\frac{1}{4}}$

or $$k_1 = \frac{b}{a^4} \qquad k_2 = \frac{b}{a^{\frac{1}{4}}}$$

Then $$y_1 = \frac{b}{a^4}x^4 \qquad y_2 = \frac{b}{a^{\frac{1}{4}}}x^{\frac{1}{4}}$$

and $$x_1 = \frac{a}{b^{\frac{1}{4}}}y^{\frac{1}{4}} \qquad x_2 = \frac{a}{b^4}y^4$$

Now $$A = \int (y_2 - y_1)\,dx = b\int_0^a \left(\frac{x^{\frac{1}{4}}}{a^{\frac{1}{4}}} - \frac{x^4}{a^4}\right)dx$$

$$= b\left[\frac{4}{5}\frac{x^{\frac{5}{4}}}{a^{\frac{1}{4}}} - \frac{1}{5}\frac{x^5}{a^4}\right]_0^a = \frac{3}{5}ab$$

Then $$I_x = \int y^2 dA \qquad dA = (x_1 - x_2)\,dy$$

$$I_x = \int_0^b y^2\left(\frac{a}{b^{\frac{1}{4}}}y^{\frac{1}{4}} - \frac{a}{b^4}y^4\right)dy$$

$$= a\left[\frac{4}{13}\frac{y^{\frac{13}{4}}}{b^{\frac{1}{4}}} - \frac{1}{7}\frac{y^7}{b^4}\right]_0^b$$

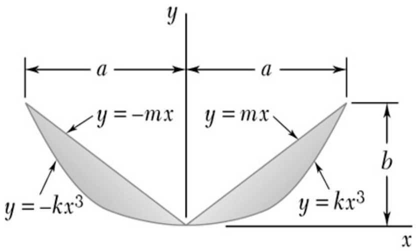
$$= ab^3\left(\frac{4}{13} - \frac{1}{7}\right)$$

or $I_x = \dfrac{15}{91}ab^3$ ◀

Now $$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{15}{91}ab^3}{\frac{3}{5}ab}} = \sqrt{\frac{25}{91}b^2} = 0.52414b$$

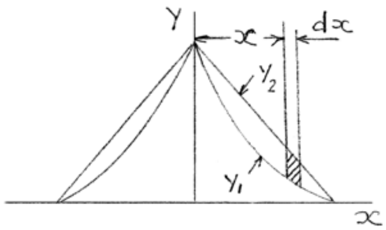
or $k_x = 0.524b$ ◀

PROBLEM 9.17



Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION



At $$x = a, \quad y_1 = y_2 = b$$

$$y_1: \quad b = ka^3 \qquad \text{or} \qquad k = \frac{b}{a^3}$$

$$y_2: \quad b = ma \qquad \text{or} \qquad m = \frac{b}{a}$$

Then $$y_1 = \frac{b}{a^3}x^3$$

$$y_2 = \frac{b}{a}x$$

Now $$dA = (y_2 - y_1)dx = \left(\frac{b}{a}x - \frac{b}{a^3}x^3\right)dx$$

$$A = \int dA = 2\frac{b}{a}\int_0^a \left(x - \frac{x^3}{a^2}\right)dx = 2\frac{b}{a}\left[\frac{1}{2}x^2 - \frac{1}{4a^2}x^4\right]_0^a$$

$$= 2\frac{b}{a}\left[\frac{a^2}{2} - \frac{1}{4a^2}a^4\right] = \frac{1}{2}ab$$

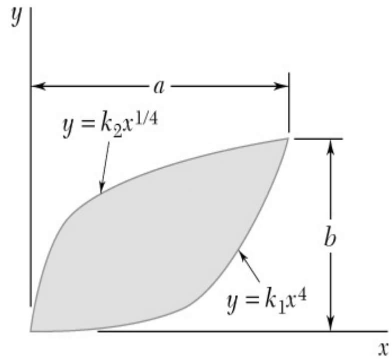
Now $$dI_y = x^2 dA = \frac{b}{a}\left[\left(x^3 - \frac{x^5}{a^2}\right)dx\right]$$

Then $$I_y = \int_0^a dI_y = 2\frac{b}{a}\int_0^a \left(x^3 - \frac{x^5}{a^2}\right)dx$$

$$= 2\frac{b}{a}\left[\frac{1}{4}x^4 - \frac{1}{6a^2}x^6\right]_0^a = 2\frac{b}{a}\left(\frac{a^4}{4} - \frac{1}{6}\frac{a^6}{a^2}\right)$$

$$= \frac{1}{6}a^3 b \qquad \text{or} \quad I_y = \frac{1}{6}a^3 b \blacktriangleleft$$

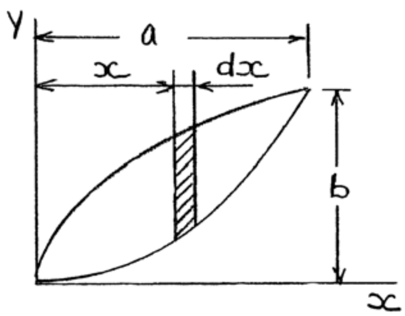
And $$k_y^2 = \frac{I_y}{A} = \frac{\frac{1}{6}a^3 b}{\frac{1}{2}ab} = \frac{1}{3}a^2 \qquad \text{or} \quad k_y = \frac{a}{\sqrt{3}} \blacktriangleleft$$



PROBLEM 9.18

Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the *y* axis.

SOLUTION



At $\qquad x = a, \; y = b: \qquad b = k_1 a^4 \qquad b = k_2 a^{\frac{1}{4}}$

or $\qquad k_1 = \dfrac{b}{a^4} \qquad k_2 = \dfrac{b}{a^{\frac{1}{4}}}$

Then $\qquad y_1 = \dfrac{b}{a^4}x^4 \qquad$ and $\qquad y_2 = \dfrac{b}{a^{\frac{1}{4}}}x^{\frac{1}{4}}$

Now
$$A = \int (y_2 - y_1)\,dx = b\int_0^a \left(\frac{x^{\frac{1}{4}}}{a^{\frac{1}{4}}} - \frac{x^4}{a^4}\right)dx$$

$$= b\left[\frac{4}{5}\frac{x^{\frac{5}{4}}}{a^{\frac{1}{4}}} - \frac{1}{5}\frac{x^5}{a^4}\right]_0^a = \frac{3}{5}ab$$

Now $\qquad I_y = \int x^2 dA \qquad dA = (y_2 - y_1)\,dx$

Then
$$I_y = \int_0^a x^2\left(\frac{b}{a^{\frac{1}{4}}}x^{\frac{1}{4}} - \frac{b}{a^4}x^4\right)dx$$

$$= b\int_0^a \left(\frac{x^{\frac{9}{4}}}{a^{\frac{1}{4}}} - \frac{x^6}{a^4}\right)dx$$

$$= b\left[\frac{4}{13}\frac{x^{\frac{13}{4}}}{a^{\frac{1}{4}}} - \frac{1}{7}\frac{x^7}{a^4}\right]_0^a$$

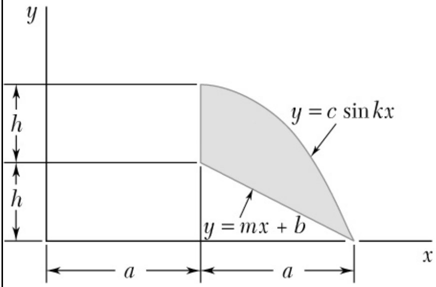
$$= b\left(\frac{4}{13}a^3 - \frac{1}{7}a^3\right)$$

or $\quad I_y = \dfrac{15}{91}a^3 b$ ◀

Now
$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{\frac{15}{91}a^3 b}{\frac{3}{5}ab}} = \sqrt{\frac{25}{91}}a = 0.52414a$$

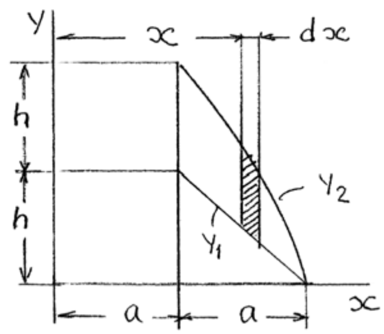
or $\quad k_y = 0.524a$ ◀

PROBLEMS 9.19 AND 9.20



P 9.19 Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the x axis.
P 9.20 Determine the moment of inertia and the radius of gyration of the shaded area shown with respect to the y axis.

SOLUTION



First determine constants m, b, and c

$$y_1: \quad \text{at} \quad x = 2a, \qquad y = 0$$

$$0 = m(2a) + b$$

At $$x = a, \qquad y = h$$

$$h = m(a) + b$$

Solving yields $$m = -\frac{h}{a} \qquad b = 2h$$

Then $$y_1 = -\frac{h}{a}x + 2h$$

$$y_2: \quad \text{at} \quad x = 2a, \qquad y = 0$$

$$0 = c\sin k(2a)$$

At $$x = a, \qquad y = 2h$$

$$2h = c\sin ka$$

Solving $$c\sin k(2a) = 0 \qquad c \neq 0$$

$$\sin k(2a) = 0, \;\; k(2a) = \pi, \;\; k = \frac{\pi}{2a}$$

Substitute k, $\quad 2h = c\sin ka \quad$ yields $\quad 2h = c\sin\frac{\pi}{2} \quad$ or $\quad c = 2h$

Then $$y_2 = 2h\sin\frac{\pi}{2a}x$$

To calculate the area of shaded surface, a differential strip parallel to the y axis is chosen to be dA.

$$dA = (y_2 - y_1)dx = \left[2h\sin\frac{\pi}{2a}x - \left(-\frac{h}{a}x + 2h\right)\right]dx$$

PROBLEMS 9.19 AND 9.20 CONTINUED

$$A = \int dA = \int_0^{2a}\left(2h\sin\frac{\pi}{2a}x - 2hx + \frac{h}{a}x\right)dx$$

$$= h\left[-\frac{4a}{\pi}\cos\frac{\pi}{2a}x - 2x + \frac{x^2}{2a}\right]_a^{2a}$$

$$= h\left[-\frac{4a}{\pi}\cos\frac{\pi}{2a}2a - 2(2a) + \frac{(2a)^2}{2a}\right]$$

$$-h\left[-\frac{4a}{\pi}\cos\frac{\pi}{2a}a - 2(a) + \frac{a^2}{2a}\right]$$

$$= h\left(\frac{4a}{\pi} - 4a + 2a\right) - h\left(-2a + \frac{a}{2}\right) = ah\left(\frac{4}{\pi} - \frac{1}{2}\right)$$

$$A = 0.77324ah$$

PROBLEM 9.19

Moment of inertia $\quad I_x = \int_a^{2a} dI_x$

where $\quad dI_x = \frac{1}{3}\left(y_2^3 - y_1^3\right)dx$

Now $\quad dI_x = \frac{1}{3}\left[\left(2h\sin\frac{\pi}{2a}x\right)^3 - \left(2h - \frac{h}{a}x\right)^3\right]dx$

$$= \frac{1}{3}\left[8h^3\sin^3\frac{\pi}{2a}x - h^3\left(2 - \frac{x}{a}\right)^3\right]dx$$

Then $\quad I_x = \frac{8h^3}{3}\int_{-a}^{2a}\sin^3\frac{\pi}{2a}xdx - \frac{h^3}{3}\int_a^{2a}\left(2 - \frac{x}{a}\right)^3 dx$

Now $\quad \int\sin^3\frac{\pi}{2a}xdx = \int\sin\frac{\pi}{2a}x\left(1 - \cos^2\frac{\pi}{2a}x\right)dx$

$$= \int\left(\sin\frac{\pi}{2a}x\right)dx - \int\left(\sin\frac{\pi}{2a}x\cos^2\frac{\pi}{2a}x\right)dx$$

$$= -\frac{2a}{\pi}\cos\frac{\pi}{2a}x + \frac{2a}{3\pi}\cos^3\frac{\pi}{2a}x$$

Then $\quad \int_a^{2a}\left(\sin^3\frac{\pi}{2}x\right)dx = -\frac{2a}{\pi}\left[\cos\frac{\pi}{2a}x - \frac{1}{3}\cos^3\frac{\pi}{2a}x\right]_a^{2a}$

$$= -\frac{2a}{\pi}\left(-1 + \frac{1}{3}\right) = \frac{4a}{3\pi}$$

PROBLEMS 9.19 AND 9.20 CONTINUED

And $$\int_a^{2a}\left(2-\frac{x}{a}\right)^3 dx = -\frac{a}{4}\left(2-\frac{x}{a}\right)^4\Bigg|_a^{2a}$$

$$= -\frac{a}{4}\left(2-\frac{2a}{a}\right)^4 + \frac{a}{4}\left(2-\frac{a}{a}\right)^4 = \frac{a}{4}$$

Then $$I_x = \frac{8h^3}{3}\left(\frac{4a}{3\pi}\right) - \frac{h^3}{3}\left(\frac{a}{4}\right) = \frac{h^3 a}{3}\left(\frac{32}{3\pi} - \frac{1}{4}\right)$$

$$I_x = 1.0484ah^3 \blacktriangleleft$$

and $$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{1.0484ah^3}{0.77324ah}} = 1.1644h$$

$$k_x = 1.164h \blacktriangleleft$$

PROBLEM 9.20

$$I_y = \int dI_y$$

$$dI_y = x^2 dA \qquad dA = (y_2 - y_1)dx$$

From Problem 9.19 $$y_1 = 2h - \frac{h}{a}x \qquad y_2 = 2h\sin\frac{\pi}{2a}x$$

Now $$dI_y = \left[2hx^2\sin\frac{\pi}{2a}x - h\left(2x^2 - \frac{x^3}{a}\right)\right]dx$$

Then $$I_y = \int_a^{2a} dI_y = h\int_a^{2a}\left(2x^2\sin\frac{\pi x}{2a} - 2x^2 + \frac{x^3}{a}\right)dx$$

Now using integration by parts

$$u = x^2 \qquad dv = \sin\frac{\pi}{2a}x dx$$

$$du = 2xdx \qquad v = -\frac{2a}{\pi}\cos\frac{\pi}{2a}x$$

Then $$\int x^2\sin\frac{\pi}{2a}xdx = x^2\left(-\frac{2a}{\pi}\cos\frac{\pi}{2a}x\right) - \int\left(-\frac{2a}{\pi}\cos\frac{\pi}{2a}x\right)2xdx$$

PROBLEM 9.20 CONTINUED

Now let $$u = x \qquad dv = \cos\frac{\pi}{2a}x\,dx$$

$$du = dx \qquad v = \frac{2a}{\pi}\sin\frac{\pi}{2a}x$$

Then $$\int x^2\sin\frac{\pi}{2a}x\,dx = -\frac{2a}{\pi}x^2\cos\frac{\pi}{2a}x + \frac{4a}{\pi}\left[x\left(\frac{2a}{\pi}\sin\frac{\pi}{2a}x\right) - \int\left(\frac{2a}{\pi}\sin\frac{\pi}{2a}x\right)dx\right]$$

Finally, $$I_y = 2h\left[\left(-\frac{2a}{\pi}x^2\cos\frac{\pi}{2a}x + \frac{8a^2}{\pi^2}x\sin\frac{\pi}{2a}x + \frac{4a^2}{\pi^2}\frac{2a}{\pi}\cos\frac{\pi}{2a}x\right) - \frac{1}{3}x^3 + \frac{1}{8a}x^4\right]_a^{2a}$$

$$= 2h\left[\frac{2a}{\pi}(2a)^2 - \frac{16a^3}{\pi^3} - \frac{(2a)^3}{3} + \frac{1}{8a}(2a)^4 - \frac{8a^2}{\pi^2}a + \frac{a^3}{3} - \frac{a^4}{8a}\right] = 1.5231a^3h$$

$$I_y = 1.523a^3h \blacktriangleleft$$

and $$k_y^2 = \frac{I_y}{A} = \frac{1.5231a^3h}{0.77324} = 1.4035a^2$$

$$k_y = 1.404a \blacktriangleleft$$