PROBLEM 9.171

Determine the mass moment of inertia of the machine component of Problems 9.138 and 9.157 with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-4\mathbf{i} + 8\mathbf{j} + \mathbf{k})/9$.

SOLUTION

From Problem 9.138:

$$I_x = 4.212 \text{ kg}\cdot\text{m}^2$$

$$I_y = 7.407 \text{ kg}\cdot\text{m}^2$$

$$I_z = 3.7055 \text{ kg}\cdot\text{m}^2$$

From Problem 9.157:

$$I_{xy} = -0.19312 \text{ kg}\cdot\text{m}^2$$

$$I_{yz} = 0.30987 \text{ kg}\cdot\text{m}^2$$

$$I_{zx} = 2.25956 \text{ kg}\cdot\text{m}^2$$

Now

$$\boldsymbol{\lambda}_{OL} = \frac{1}{9}(-4\mathbf{i} + 8\mathbf{j} + \mathbf{k})$$

Eq. (9.46):

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

$$= \left[4.212\left(-\frac{4}{9}\right)^2 + 7.407\left(\frac{8}{9}\right)^2 + 3.7055\left(\frac{1}{9}\right)^2\right.$$

$$- 2(-0.19312)\left(-\frac{4}{9}\right)\left(\frac{8}{9}\right) - 2(0.3098)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right)$$

$$\left.- 2(2.25956)\left(\frac{1}{9}\right)\left(-\frac{4}{9}\right)\right] \text{kg}\cdot\text{m}^2$$

$$= (0.832 + 5.85244 + 0.04575 - 0.15259 - 0.061195 + 0.22317) \text{ kg}\cdot\text{m}^2$$

$$= 6.73957 \text{ kg}\cdot\text{m}^2$$

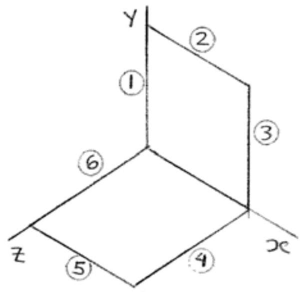
$$I_{OL} = 6.74 \text{ kg}\cdot\text{m}^2 \blacktriangleleft$$

PROBLEM 9.172

For the wire figure of the problem indicated, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = \left(-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}\right)/7$.

Problem 9.150

SOLUTION



Mass of each leg is identical:

$$m = \left(\frac{W/L}{g}\right)L$$

$$= \frac{0.041\text{ lb/ft}\left(1.5\text{ ft}\right)}{32.2\text{ ft/s}^2} = 0.00190994\text{ lb}\cdot\text{s}^2/\text{ft}$$

Also, $\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{z'x'} = 0$ for each leg,

and $\quad \bar{x}_1 = \bar{x}_6 = 0 \qquad \bar{y}_4 = \bar{y}_5 = \bar{y}_6 = 0 \qquad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$

Now $\quad I_{xy} = \Sigma\left(\cancelto{0}{I_{x'y'}} + m\bar{x}\,\bar{y}\right) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3$

$$= \left(0.00190994\text{ lb}\cdot\text{s}^2/\text{ft}\right)\left[\left(0.75\right)\left(1.5\right) + \left(1.5\right)\left(0.75\right)\right]\text{ft}^2$$

$$= 0.0042974\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 4.2974\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_{yz} = 0$$

$$I_{zx} = \Sigma\left(I_{z'x'} + m\bar{z}\,\bar{x}\right) = m_4\bar{z}_4\,\bar{x}_4 + m_5\bar{z}_5\bar{x}_5$$

$$= \left(0.00190994\text{ lb}\cdot\text{s}^2/\text{ft}\right)\left[\left(0.75\right)\left(1.5\right) + \left(1.5\right)\left(0.75\right)\right]\text{ft}^2$$

$$= 0.0042974\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 4.2974\times 10^{-3}\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

PROBLEM 9.172 CONTINUED

From Problem 9.150 $$I_x = 14.32 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_y = I_z = 18.62 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

Now $$\lambda_{OL} = \frac{1}{7}\left(-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}\right) \text{ and then}$$

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2\cancelto{0}{I_{yz}}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x \qquad [\text{Eq. } (9.46)]$$

$$= \left\{\left[\left(14.32\times10^{-3}\right)\left(-\frac{3}{7}\right)^2 + \left(18.62\times10^{-3}\right)\left[\left(\frac{-6}{7}\right)^2 + \left(\frac{2}{7}\right)^2\right] - 2\left(4.2974\times10^{-3}\right)\left[\left(-\frac{3}{7}\right)\left(\frac{-6}{7}\right)\right]\right.\right.$$

$$\left. - 2\left(4.2974\times10^{-3}\right)\left[\left(\frac{2}{7}\right)\left(\frac{-3}{7}\right)\right]\right\}\text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= \left(2.6302\times10^{-3} + 15.20\times10^{-3} - 3.1573\times10^{-3} + 1.05242\times10^{-3}\right)\text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 15.725\times10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

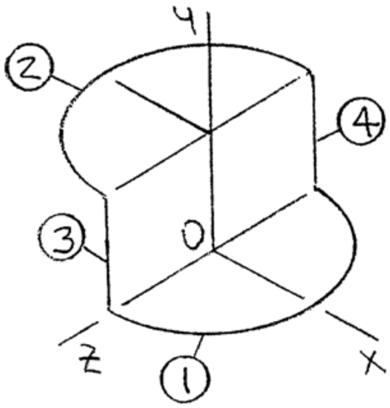
or $I_{OL} = 15.73\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.173

For the wire figure of the problem indicated, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = (-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})/7$.

Problem 9.149

SOLUTION



First compute the mass of each component

Have $$m = \rho_{st} V = mAL$$

$$= \left(7850 \text{ kg/m}^3\right)\left[\pi\left(0.0015 \text{ m}\right)^2\right]L$$

$$= \left(0.055488L\right) \text{ kg/m}$$

Then $$m_1 = m_2 = 0.055488 \text{ kg/m}\left(\pi \times 0.36 \text{ m}\right)$$

$$= 0.062756 \text{ kg}$$

$$m_3 = m_4 = 0.055488 \text{ kg/m}\left(0.36 \text{ m}\right)$$

$$= 0.019976 \text{ kg}$$

Now observe that the centroidal products of inertia $\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{z'x'} = 0$ for each component.

Also $$\bar{x}_3 = \bar{x}_4 = 0, \qquad \bar{y}_1 = 0, \qquad \bar{z}_1 = \bar{z}_2 = 0$$

Then

$$I_{xy} = \Sigma\left(\cancelto{0}{I_{x'y'}} + m\bar{x}\,\bar{y}\right) = m_2\bar{x}_2\bar{y}_2$$

$$= \left(0.062756 \text{ kg}\right)\left(-\frac{2 \times 0.36 \text{ m}}{\pi}\right)\left(0.36 \text{ m}\right) = -5.1777 \times 10^{-3} \text{ kg}\cdot\text{m}^2$$

$$I_{yz} = \Sigma\left(\cancelto{0}{I_{y'z'}} + m\bar{y}\,\bar{z}\right) = m_3\bar{y}_3\bar{z}_3 + m_4\bar{y}_4\bar{z}_4$$

where $m_3 = m_4$, $\bar{y}_3 = \bar{y}_4$, $\bar{z}_4 = -\bar{z}_3$, so that $I_{yz} = 0$

$$I_{zx} = \Sigma\left(I_{z'x'} + m\bar{z}\,\bar{x}\right) = m_1\bar{z}_1\bar{x}_1 + m_2\bar{z}_2\bar{x}_2 = 0$$

PROBLEM 9.173 CONTINUED

From the solution to Problem 9.149

$$I_x = 23.170 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_y = 21.444 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$I_z = 17.992 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

Now $\qquad \boldsymbol{\lambda}_{OL} = \frac{1}{7}(-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

Have

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - \cancelto{0}{2I_{yz}\lambda_y}\lambda_z - \cancelto{0}{2I_{zx}}\lambda_z\lambda_x \quad [\text{Eq. } (9.46)]$$

$$= \left[23.170\left(-\frac{3}{7}\right)^2 + 21.444\left(-\frac{6}{7}\right)^2 + 17.992\left(\frac{2}{7}\right)^2\right.$$

$$\left. - 2(-5.1777)\left(-\frac{3}{2}\right)\left(-\frac{6}{7}\right)\right] \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$= (4.2557 + 15.755 + 1.4687 + 3.8040) \times 10^{-3}\ \text{kg}\cdot\text{m}$$

$$= 25.283 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

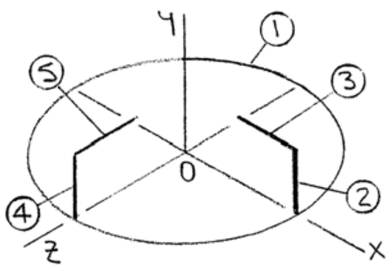
or $\quad I_{OL} = 25.3 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

PROBLEM 9.174

For the wire figure of the problem indicated, determine the mass moment of inertia of the figure with respect to the axis through the origin characterized by the unit vector $\boldsymbol{\lambda} = \left(-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}\right)/7$.

Problem 9.148

SOLUTION



First compute the mass of each component.

Have $$m = \left(m/L\right)L$$

$$= \left(0.049 \text{ kg/m}\right)L$$

Then $$m_1 = \left(0.049 \text{ kg/m}\right)\left(2\pi \times 0.32 \text{ m}\right)$$

$$= 0.09852 \text{ kg}$$

$$m_2 = m_3 = m_4 = m_5 = \left(0.049 \text{ kg}\right)\left(0.160 \text{ m}\right)$$

$$= 0.00784 \text{ kg}$$

Now observe that $\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{z'x'} = 0$ for each component.

Also, $\bar{x}_1 = \bar{x}_4 = \bar{x}_5 = 0, \quad \bar{y}_1 = 0, \quad \bar{z}_1 = \bar{z}_2 = \bar{z}_3 = 0$

Then

$$I_{xy} = \Sigma\left(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}\right) = m_2\bar{x}_2\bar{y}_2 + m_3\bar{x}_3\bar{y}_3$$

$$= \left(0.00784 \text{ kg}\right)\left[\left(0.32 \text{ m}\right)\left(0.08 \text{ m}\right) + \left(0.24 \text{ m}\right)\left(0.16 \text{ m}\right)\right]$$

$$= 0.50176 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

By symmetry $$I_{yz} = I_{xy} = 0.50176 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

Now $$I_{zx} = \Sigma\left(I_{z'x'} + m\bar{z}\,\bar{x}\right) = 0$$

From the solution to Problem 9.148

$$I_x = I_z = 6.8505 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

$$I_y = 12.630 \times 10^{-3} \text{ kg} \cdot \text{m}^2$$

PROBLEM 9.174 CONTINUED

Now $$\boldsymbol{\lambda}_{OL} = \frac{1}{7}\left(-3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}\right)$$

Have

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2\cancelto{0}{I_{zx}}\lambda_z\lambda_x \quad \left[\text{Eq.}\left(9.46\right)\right]$$

$$= \left\{\left(6.8505\right)\left[\left(-\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2\right]\right\} \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$- \left\{\left(12.63\right)\left(\frac{2}{7}\right)^2\right] \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

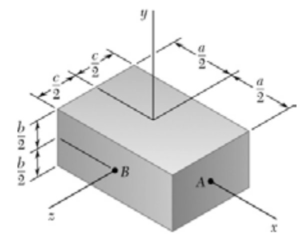
$$-2\left(0.50176\right)\left[\left(-\frac{3}{7}\right)\left(-\frac{6}{7}\right) + \left(-\frac{6}{3}\right)\left(\frac{2}{7}\right)\right]\right\} \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$= \left(6.29128 + 1.03102 - 0.12288\right) \times 10^{-3}\ \text{kg}\cdot\text{m}^2$$

$$= 0.719942\ \text{kg}\cdot\text{m}^2$$

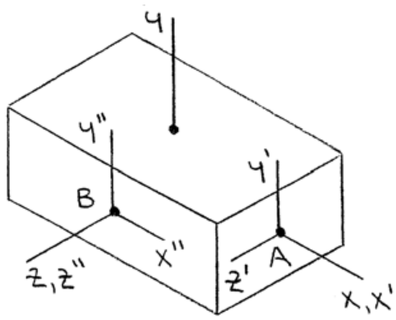
or $I_{OL} = 7.2 \times 10^{-3}\ \text{kg}\cdot\text{m}^2$ ◀

PROBLEM 9.175



For the rectangular prism shown, determine the values of the ratios b/a and c/a so that the ellipsoid of inertia of the prism is a sphere when computed (*a*) at point *A*, (*b*) at point *B*.

SOLUTION



(*a*) Using Figure 9.28 and the parallel-axis theorem have at point *A*..

$$I_{x'} = \frac{1}{12}m\left(b^2 + c^2\right)$$

$$I_{y'} = \frac{1}{12}m\left(a^2 + c^2\right) + m\left(\frac{a}{2}\right)^2 = \frac{1}{12}m\left(4a^2 + c^2\right)$$

$$I_{z'} = \frac{1}{12}m\left(a^2 + b^2\right) + m\left(\frac{a}{2}\right)^2 = \frac{1}{12}m\left(4a^2 + b^2\right)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

Using Equation (9.48), the equation of the ellipsoid of inertia is then

$$I_{x'}x^2 + I_{y'}y^2 + I_{z'}z^2 = 1$$

or $$\frac{1}{12}m\left(b^2 + c^2\right)x^2 + \frac{1}{12}m\left(4a^2 + c^2\right)y^2 + \frac{1}{12}m\left(4a^2 + b^2\right)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore,

$$\frac{1}{12}m\left(b^2 + c^2\right) = \frac{1}{12}m\left(4a^2 + c^2\right) = \frac{1}{12}m\left(4a^2 + b^2\right)$$

Then $$b^2 + c^2 = 4a^2 + c^2 \qquad \text{or} \quad \frac{b}{a} = 2 \blacktriangleleft$$

and $$b^2 + c^2 = 4a^2 + b^2 \qquad \text{or} \quad \frac{c}{a} = 2 \blacktriangleleft$$

(*b*) Using Figure 9.28 and the parallel-axis theorem, we have at point *B*

$$I_{x''} = \frac{1}{12}m\left(b^2 + c^2\right) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m\left(b^2 + 4c^2\right)$$

$$I_{y''} = \frac{1}{12}m\left(a^2 + c^2\right) + m\left(\frac{c}{2}\right)^2 = \frac{1}{12}m\left(a^2 + 4c^2\right)$$

$$I_{z''} = \frac{1}{12}m\left(a^2 + b^2\right)$$

PROBLEM 9.175 CONTINUED

Now observe that symmetry implies

$$I_{x''y''} = I_{y''z''} = I_{z''x''} = 0$$

From part *a* it then immediately follows that

$$\frac{1}{12}m\left(b^2 + 4c^2\right) = \frac{1}{12}m\left(a^2 + 4c^2\right) = \frac{1}{12}m\left(a^2 + b^2\right)$$

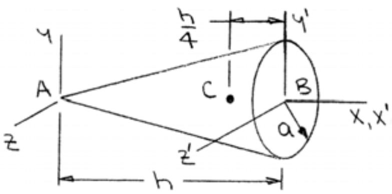
Then $\qquad b^2 + 4c^2 = a^2 + 4c^2 \qquad$ or $\quad \dfrac{b}{a} = 1$ ◀

and $\qquad b^2 + 4c^2 = a^2 + b^2 \qquad$ or $\quad \dfrac{c}{a} = \dfrac{1}{2}$ ◀

PROBLEM 9.176

For the right circular cone of Sample Prob. 9.11, determine the value of the ratio a/h for which the ellipsoid of inertia of the cone is a sphere when computed (*a*) at the apex of the cone, (*b*) at the center of the base of the cone.

SOLUTION



(*a*) From sample Problem 9.11, we have at the apex *A*

$$I_x = \frac{3}{10}ma^2$$

$$I_y = I_z = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)$$

Now observe that symmetry implies $I_{xy} = I_{yz} = I_{zx} = 0$

Using Equation (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1$$

or $$\frac{3}{10}ma^2x^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)y^2 + \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)z^2 = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore,

$$\frac{3}{10}ma^2 = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right) \qquad \text{or } \frac{a}{h} = 2 \blacktriangleleft$$

(*b*) From Sample Problem 9.11, we have

$$I_{x'} = \frac{3}{10}ma^2$$

and at the centroid *C* $$\bar{I}_{y''} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right)$$

Then $$I_{y'} = I_{z'} = \frac{3}{20}m\left(a^2 + \frac{1}{4}h^2\right) + m\left(\frac{h}{4}\right)^2$$

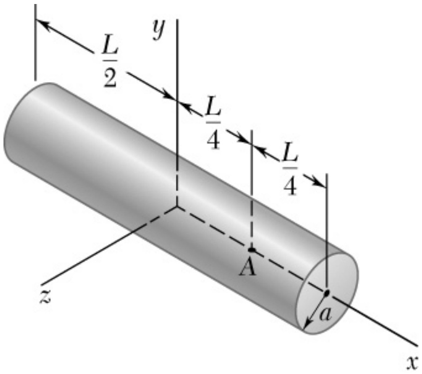
$$= \frac{1}{20}m\left(3a^2 + 2h^2\right)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From part *a* it then immediately follows that

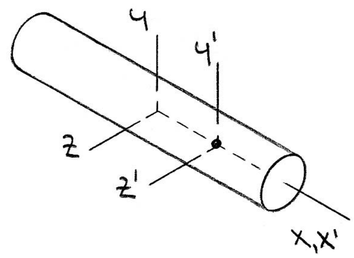
$$\frac{3}{10}ma^2 = \frac{1}{20}m\left(3a^2 + 2h^2\right) \qquad \text{or } \frac{a}{h} = \sqrt{\frac{2}{3}} \blacktriangleleft$$



PROBLEM 9.177

For the homogeneous circular cylinder shown, of radius *a* and length *L*, determine the value of the ration *a*/*L* for which the ellipsoid of inertia of the cylinder is a sphere when computed (*a*) at the centroid of the cylinder, (*b*) at point *A*.

SOLUTION



(*a*) From Figure 9.28

$$\bar{I}_x = \frac{1}{2}ma^2 \qquad \bar{I}_y = \bar{I}_z = \frac{1}{12}m\left(3a^2 + L^2\right)$$

Now observe that symmetry implies

$$I_{xy} = I_{yz} = I_{zx} = 0$$

Using Equation (9.48), the equation of the ellipsoid of inertia is then

$$I_x x^2 + I_y y^2 + I_z z^2 = 1: \ \frac{1}{2}ma^2x^2 + \frac{1}{12}m\left(3a^2 + L^2\right)y^2 + \frac{1}{12}m\left(3a^2 + L^2\right) = 1$$

For the ellipsoid to be a sphere, the coefficients must be equal. Therefore,

$$\frac{1}{2}ma^2 = \frac{1}{12}m\left(3a^2 + L^2\right) \qquad \text{or } \frac{a}{L} = \frac{1}{\sqrt{3}} \blacktriangleleft$$

(*b*) Using Fig. 9.28 and the parallel-axis theorem

Have $I_{x'} = \frac{1}{2}ma^2$

$$I_{y'} = I_{z'} = \frac{1}{12}m\left(3a^2 + L^2\right) + m\left(\frac{L}{4}\right)^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right)$$

Now observe that symmetry implies

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

From Part a it then immediately follows that

$$\frac{1}{2}ma^2 = m\left(\frac{1}{4}a^2 + \frac{7}{48}L^2\right) \qquad \text{or } \frac{a}{L} = \sqrt{\frac{7}{12}} \blacktriangleleft$$

PROBLEM 9.178

Given an arbitrary body and three rectangular axes *x*, *y*, and *z*, prove that the moment of inertia of the body with respect to any one of the three axes cannot be larger than the sum of the moments of inertia of the body with respect to the other two axes. That is, prove that the inequality $I_x \le I_y + I_z$ and the two similar inequalities are satisfied. Further, prove that $I_y \ge \frac{1}{2} I_x$ if the body is a homogeneous solid of revolution, where *x* is the axis of revolution and *y* is a transverse axis.

SOLUTION

(i) To prove $$I_y + I_z \ge I_x$$

By definition $$I_y = \int \left(z^2 + x^2\right) dm \qquad I_z = \int \left(x^2 + y^2\right) dm$$

Then $$I_y + I_z = \int \left(z^2 + x^2\right) dm + \int \left(x^2 + y^2\right) dm$$

$$= \int \left(y^2 + z^2\right) dm + 2\int x^2 dm$$

Now.. $$\int \left(y^2 + z^2\right) dm = I_x \qquad \text{and} \qquad \int x^2 dm \ge 0$$

$$\therefore \quad I_y + I_z \ge I_x \qquad \text{Q.E.D.}$$

The proofs of the other two inequalities follow similar steps.

(ii) If the *x* axis is the axis of revolution, then

$$I_y = I_z$$

and from part (i) $$I_y + I_z \ge I_x$$

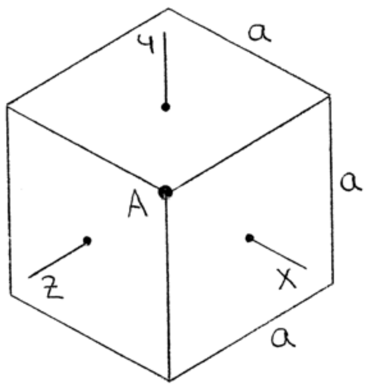
or $$2I_y \ge I_x$$

or $$I_y \ge \frac{1}{2} I_x \qquad \text{Q.E.D.}$$

PROBLEM 9.179

Consider a cube of mass m and side a. (a) Show that the ellipsoid of inertia at the center of the cube is a sphere, and use this property to determine the moment of inertia of the cube with respect to one of its diagonals. (b) Show that the ellipsoid of inertia at one of the corners of the cube is an ellipsoid of revolution, and determine the principal moments of inertia of the cube at that point.

SOLUTION



(a) At the center of the cube have (using Figure 9.28)

$$I_x = I_y = I_z = \frac{1}{12}m\left(a^2 + a^2\right) = \frac{1}{6}ma^2$$

Now observe that symmetry implies $\qquad I_{xy} = I_{yz} = I_{zx} = 0$

Using Equation (9.48), the equation of the ellipsoid of inertia is

$$\left(\frac{1}{6}ma^2\right)x^2 + \left(\frac{1}{6}ma^2\right)y^2 + \left(\frac{1}{6}ma^2\right)z^2 = 1$$

or $$x^2 + y^2 + z^2 = \frac{6}{ma^2} \quad \left(= R^2\right) \blacktriangleleft$$

which is the equation of a sphere.

Since the ellipsoid of inertia is a sphere, the moment of inertia with respect to any axis OL through the center O of the cube must always be the same $\left(R = \frac{1}{\sqrt{I_{OL}}}\right)$.

$$\therefore \quad I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

(b) The above sketch of the cube is the view seen if the line of sight is along the diagonal that passes through corner A. For a rectangular coordinate system at A and with one of the coordinate axes aligned with the diagonal, an ellipsoid of inertia at A could be constructed. If the cube is then rotated 120° about the diagonal, the mass distribution will remain unchanged. Thus, the ellipsoid will also remain unchanged after it is rotated. As noted at the end of section 9.17, this is possible only if the ellipsoid is an ellipsoid of revolution, where the diagonal is both the axis of revolution and a principal axis.

It then follows that $$I_{x'} = I_{OL} = \frac{1}{6}ma^2 \blacktriangleleft$$

PROBLEM 9.179 CONTINUED

In addition, for an ellipsoid of revolution, the two transverse principal moments of inertia are equal and any axis perpendicular to the axis of revolution is a principal axis. Then, applying the parallel-axis theorem between the center of the cube and corner *A* for any perpendicular axis

$$I_{y'} = I_{z'} = \frac{1}{6}ma^2 + m\left(\frac{\sqrt{3}}{2}a\right)^2$$

$$\text{or } I_{y'} = I_{z'} = \frac{11}{12}ma^2 \blacktriangleleft$$

Note: Part *b* can also be solved using the method of Section 9.18. First note that at corner *A*

$$I_x = I_y = I_z = \frac{2}{3}ma^2 \qquad I_{xy} = I_{yz} = I_{zx} = \frac{1}{4}ma^2$$

Substituting into Equation (9.56) yields

$$k^3 - 2ma^2k^2 + \frac{55}{48}m^2a^6k - \frac{121}{864}m^3a^9 = 0$$

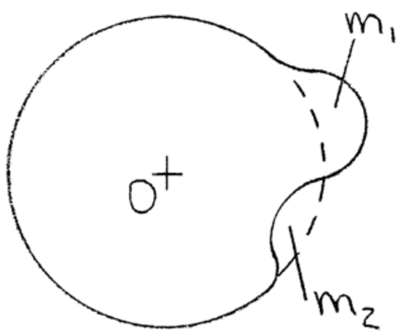
For which the roots are

$$k_1 = \frac{1}{6}ma^2 \qquad k_2 = k_3 = \frac{11}{12}ma^2$$

PROBLEM 9.180

Given a homogeneous body of mass m and of arbitrary shape and three rectangular axes x, y, and z with origin at O, prove that the sum $I_x + I_y + I_z$ of the moments of inertia of the body cannot be smaller than the similar sum computed for a sphere of the same mass and the same material centered at O. Further, using the results of Prob. 9.178, prove that if the body is a solid of revolution, where x is the axis of revolution, its moment of inertia I_y about a transverse axis y cannot be smaller than $\frac{3ma^2}{10}$, where a is the radius of the sphere of the same mass and the same material.

SOLUTION



(i) Using Equation (9.30), we have

$$I_x + I_y + I_z = \int\left(y^2 + z^2\right)dm + \int\left(z^2 + x^2\right)dm + \int\left(x^2 + y^2\right)dm$$

$$= 2\int\left(x^2 + y^2 + z^2\right)dm$$

$$= 2\int r^2 dm$$

where r is the distance from the origin O to the element of mass dm. Now assume that the given body can be formed by adding and subtracting appropriate volumes V_1 and V_2 from a sphere of mass m and radius a which is centered at O; it then follows that $m_1 = m_2 \left(m_{\text{body}} = m_{\text{sphere}} = m\right)$.

Then

$$\left(I_x + I_y + I_z\right)_{\text{body}} = \left(I_x + I_y + I_z\right)_{\text{sphere}} + \left(I_x + I_y + I_z\right)_{V_1} - \left(I_x + I_y + I_z\right)_{V_2}$$

or

$$\left(I_x + I_y + I_z\right)_{\text{body}} = \left(I_x + I_y + I_z\right)_{\text{sphere}} + 2\int_{m_1} r^2 dm - 2\int_{m_2} r^2 dm$$

Now, $m_1 = m_2$ and $r_1 \geq r_2$ for all elements of mass dm in volumes 1 and 2.

$$\therefore \quad \int_{m_1} r^2 dm - \int_{m_2} r^2 dm \geq 0$$

so that $\left(I_x + I_y + I_z\right)_{\text{body}} \geq \left(I_x + I_y + I_z\right)_{\text{sphere}}$ Q.E.D.

PROBLEM 9.180 CONTINUED

(ii) First note from Figure 9.28 that for a sphere

$$I_x = I_y = I_z = \frac{2}{5}ma^2$$

Thus, $$\left(I_x + I_y + I_z\right)_{\text{sphere}} = \frac{6}{5}ma^2$$

For a solid of revolution, where the *x* axis is the axis of revolution, have

$$I_y = I_z$$

Then, using the results of part i

$$\left(I_x + 2I_y\right)_{\text{body}} \geq \frac{6}{5}ma^2$$

From Problem 9.178 have $$I_y \geq \frac{1}{2}I_x$$

or $$\left(2I_y - I_x\right)_{\text{body}} \geq 0$$

Adding the last two inequalities yields

$$\left(4I_y\right)_{\text{body}} \geq \frac{6}{5}ma^2$$

or $$\left(I_y\right)_{\text{body}} \geq \frac{3}{10}ma^2 \quad \text{Q.E.D.}$$