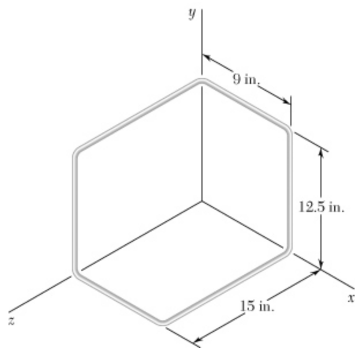
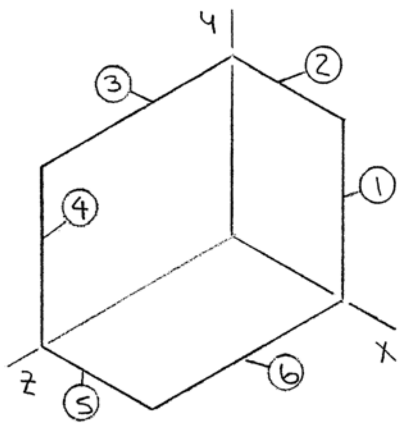
PROBLEM 9.161



The figure shown is formed of 0.075-in.-diameter aluminum wire. Knowing that the specific weight of aluminum is 0.10 lb/in^3, determine the products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION



First note $$m = \rho V = \frac{\gamma}{g} V = \frac{\gamma}{g} AL$$

Specific weight of aluminium $= 0.10 \text{ lb/in}^3 = 172.8 \text{ lb/ft}^3$

Then $$m = \frac{172.8 \text{ lb/ft}^3}{32.2 \text{ ft/s}^2}\left[\frac{\pi}{4}\frac{(0.075 \text{ ft})^2}{12}\right]L$$

$$= \left(0.16464 \times 10^{-3}\right) L \text{ lb}\cdot\text{s}^2/\text{ft}^2$$

Now $$L_1 = L_4 = 12.5 \text{ in.} = 1.04167 \text{ ft}$$

$$m_1 = m_4 = 0.17150 \times 10^{-3} \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$L_2 = L_5 = 9 \text{ in.} = 0.75 \text{ ft}$$

$$m_2 = m_5 = 0.12348 \times 10^{-3} \text{ lb}\cdot\text{s}^2/\text{ft}$$

$$L_3 = L_6 = 15 \text{ in.} = 1.25 \text{ ft}$$

$$m_3 = m_6 = 0.20580 \times 10^{-3} \text{ lb}\cdot\text{s}^2/\text{ft}$$

and $$\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{z'x'} = 0$$

PROBLEM 9.161 CONTINUED

	m, lb·s^2/ft	$\bar{x}$, ft	$\bar{y}$, ft	$\bar{z}$, ft	$m\bar{x}\,\bar{y}$, lb·ft·s^2	$m\bar{y}\,\bar{z}$, lb·ft·s^2	$m\bar{z}\,\bar{x}$, lb·ft·s^2
1	0.17150×10^{-3}	0.75	0.5208	0	0.06699×10^{-3}	0	
2	0.12348×10^{-3}	0.375	1.04167	0	0.04823×10^{-3}	0	
3	0.20580×10^{-3}	0	1.04167	0.625	0	0.13398×10^{-3}	
4	0.17150×10^{-3}	0	0.5208	1.25	0	0.111646×10^{-3}	
5	0.12348×10^{-3}	0.375	0	1.25	0	0	0.05788×10^{-3}
6	0.20580×10^{-3}	0.75	0	0.625	0	0	0.09647×10^{-3}
Σ					0.11522×10^{-3}	0.24563×10^{-3}	0.15435×10^{-3}

$$I_{xy} = \Sigma\left(\cancelto{0}{I_{x'y'}} + m\bar{x}\,\bar{y}\right) = 0.115222 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_{xy} = 0.1152 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

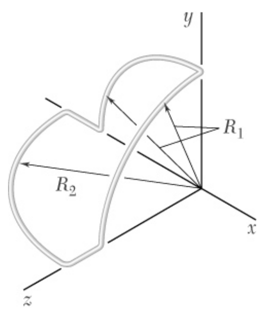
$$I_{yz} = \Sigma\left(\cancelto{0}{I_{y'z'}} + m\bar{y}\,\bar{z}\right) = 0.24563 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_{yz} = 0.246 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

$$I_{zx} = \Sigma\left(\cancelto{0}{I_{z'x'}} + m\bar{z}\,\bar{x}\right) = 0.15435 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

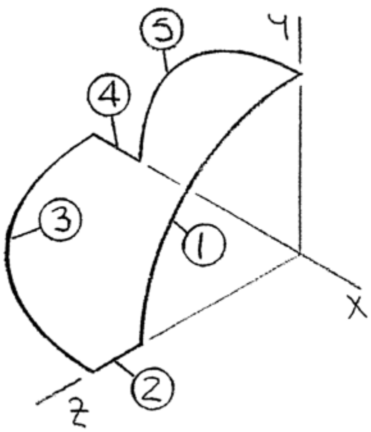
or $I_{zx} = 0.1543 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.162



Thin aluminum wire of uniform diameter is used to form the figure shown. Denoting by m' the mass per unit length of wire, determine the products of inertia I_{xy}, I_{yz}, and I_{zx} of the wire figure.

SOLUTION



First compute the mass of each component. Have

$$m = \left(\frac{m}{L}\right)L = m'L$$

Then $$m_1 = m_5 = m'\left(\frac{\pi}{2}R_1\right) = \frac{\pi}{2}m'R_1$$

$$m_2 = m_4 = m'(R_2 - R_1)$$

$$m_3 = m'\left(\frac{\pi}{2}R_2\right) = \frac{\pi}{2}m'R_2$$

Now observe that because of symmetry the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'},$ and $\bar{I}_{z'x'},$ of components 2 and 4 are zero and

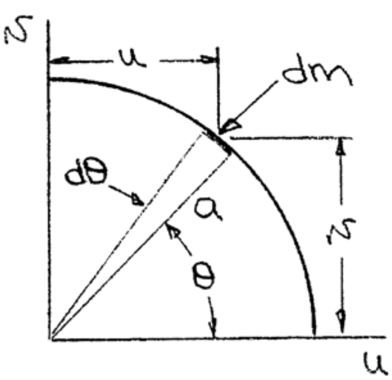
$$\left(\bar{I}_{x'y'}\right)_1 = \left(\bar{I}_{z'x'}\right)_1 = 0 \qquad \left(\bar{I}_{x'y'}\right)_3 = \left(\bar{I}_{y'z'}\right)_3 = 0$$

$$\left(\bar{I}_{y'z'}\right)_5 = \left(\bar{I}_{z'x'}\right)_5 = 0$$

Also $\quad \bar{x}_1 = \bar{x}_2 = 0 \qquad \bar{y}_2 = \bar{y}_3 = \bar{y}_4 = 0 \qquad \bar{z}_4 = \bar{z}_5 = 0$

Using the parallel-axis theorem [Equations (9.47)], it follows that

$I_{xy} = I_{yz} = I_{zx}$ for components 2 and 4.



To determine I_{uv} for one quarter of a circular arc have $\qquad dI_{uv} = uvdm$

where $$u = a\cos\theta \qquad v = a\sin\theta$$

and $$dm = \rho dV = \rho\left[A(ad\theta)\right]$$

where A is the cross-sectional area of the wire. Now

$$m = m'\left(\frac{\pi}{2}a\right) = \rho A\left(\frac{\pi}{2}a\right)$$

so that $$dm = m'ad\theta$$

and $$dI_{uv} = (a\cos\theta)(a\sin\theta)(m'ad\theta)$$

$$= m'a^3\sin\theta\cos\theta d\theta$$

PROBLEM 9.162 CONTINUED

Then $$I_{uv} = \int dI_{uv} = \int_0^{\frac{\pi}{2}} m'a^3 \sin\theta \cos\theta d\theta$$

$$= m'a^3 \left[\frac{1}{2} \sin^2\theta \right]_0^{\frac{\pi}{2}} = \frac{1}{2} m'a^3$$

Thus, $$\left(I_{yz}\right)_1 = \frac{1}{2} m'R_1^3$$

and $$\left(I_{zx}\right)_3 = -\frac{1}{2} m'R_2^3 \qquad \left(I_{xy}\right)_5 = -\frac{1}{2} m'R_1^3$$

Because of 90° rotations of the coordinate axes. Finally,

$$I_{xy} = \Sigma\left(I_{xy}\right) = \left[\cancelto{0}{\left(\bar{I}_{x'y'}\right)_1} + m_1\bar{x}_1\bar{y}_1\right] + \left[\cancelto{0}{\left(\bar{I}_{x'y'}\right)_3} + m_3\bar{x}_3\bar{y}_3\right] + \left(I_{xy}\right)_5$$

or $I_{xy} = -\frac{1}{2} m'R_1^3$ ◀

$$I_{yz} = \Sigma\left(I_{yz}\right) = \left(I_{yz}\right)_1 + \left[\cancelto{0}{\left(\bar{I}_{y'z'}\right)_3} + m_3\bar{y}_3\bar{z}_3\right] + \left[\cancelto{0}{\left(\bar{I}_{y'z'}\right)_5} + m_5\bar{y}_5\bar{z}_5\right]$$

or $I_{yz} = \frac{1}{2} m'R_1^3$ ◀

$$I_{zx} = \Sigma\left(I_{zx}\right) = \left[\cancelto{0}{\left(\bar{I}_{z'x'}\right)_1} + m_1\bar{z}_1\bar{x}_1\right] + \left(I_{zx}\right)_3 + \left[\cancelto{0}{\left(\bar{I}_{z'x'}\right)_5} + m_5\bar{z}_5\bar{x}_5\right]$$

or $I_{zx} = -\frac{1}{2} m'R_2^3$ ◀

PROBLEM 9.163

Complete the derivation of Eqs. (9.47), which express the parallel-axis theorem for mass products of inertia.

SOLUTION

Have

$$I_{xy} = \int xy\,dm \qquad I_{yz} = \int yz\,dm \qquad I_{zx} = \int zx\,dm \tag{9.45}$$

and

$$x = x' + \overline{x} \qquad y = y' + \overline{y} \qquad z = z' + \overline{z} \tag{9.31}$$

Consider

$$I_{xy} = \int xy\,dm$$

Substituting for x and for y

$$I_{xy} = \int (x' + \overline{x})(y' + \overline{y})\,dm$$

$$= \int x'y'dm + \overline{y}\int x'dm + \overline{x}\int y'dm + \overline{x}\,\overline{y}\int dm$$

By definition

$$\overline{I}_{x'y'} = \int x'y'dm$$

and

$$\int x'dm = m\overline{x}\,'$$

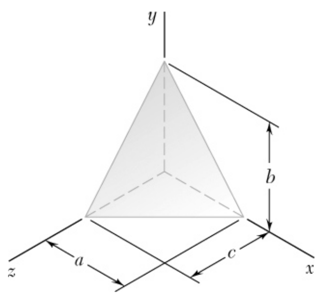
$$\int y'dm = m\overline{y}\,'$$

However, the origin of the primed coordinate system coincides with the mass center G, so that

$$\overline{x}\,' = \overline{y}\,' = 0$$

$$\therefore\ I_{xy} = \overline{I}_{x'y'} + m\overline{x}\,\overline{y} \quad \text{Q.E.D.} \blacktriangleleft$$

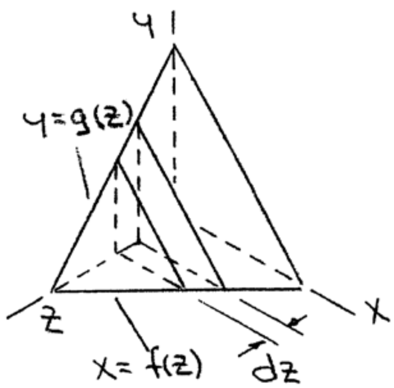
The expressions for I_{yz} and I_{zx} are obtained in a similar manner.



PROBLEM 9.164

For the homogeneous tetrahedron of mass m shown, (a) determine by direct integration the product of inertia I_{zx}, (b) deduce I_{yz} and I_{xy} from the results obtained in part a.

SOLUTION



(a) First divide the tetrahedron into a series of thin vertical slices of thickness dz as shown.

Now $$x = -\frac{a}{c}z + a = a\left(1 - \frac{z}{c}\right)$$

and $$y = -\frac{b}{c}z + b = b\left(1 - \frac{z}{c}\right)$$

The mass dm of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2}xydz\right) = \frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz$$

Then $$m = \int dm = \int_0^c \frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz = \frac{1}{2}\rho ab\left[\left(-\frac{c}{3}\right)\left(1 - \frac{z}{c}\right)^3\right]_0^c$$

$$= \frac{1}{6}\rho abc$$

Now $$dI_{zx} = d\bar{I}_{z'x'} + \bar{z}_{EL}\bar{x}_{EL}dm$$

where $$d\bar{I}_{z'x'} = 0 \qquad \text{(symmetry)}$$

and $$\bar{z}_{EL} = z \qquad \bar{x}_{EL} = \frac{1}{3}x = \frac{1}{3}a\left(1 - \frac{z}{c}\right)$$

Then $$I_{zx} = \int dI_{zx} = \int_0^c z\left[\frac{1}{3}a\left(1 - \frac{z}{c}\right)\right]\left[\frac{1}{2}\rho ab\left(1 - \frac{z}{c}\right)^2 dz\right]$$

$$= \frac{1}{6}\rho a^2 b\int_0^c\left(z - 3\frac{z^2}{c} + 3\frac{z^3}{c^2} - \frac{z^4}{c^3}\right)dz$$

$$= \frac{m}{c}a\left[\frac{1}{2}z^2 - \frac{z^3}{c} + \frac{3}{4}\frac{z^4}{c^2} - \frac{1}{5}\frac{z^5}{c^3}\right]_0^c$$

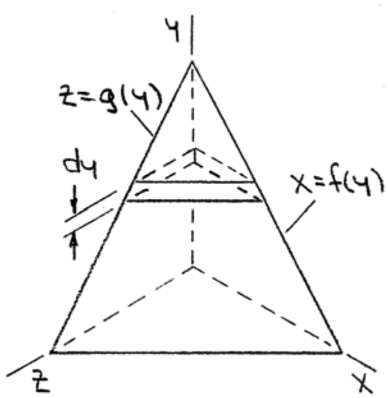
or $$I_{zx} = \frac{1}{20}mac \blacktriangleleft$$

PROBLEM 9.164 CONTINUED

(*b*) Because of the symmetry of the body, I_{xy} and I_{yz} can be deduced by considering the circular permutation of (x, y, z) and (a, b, c). Thus

$$I_{xy} = \frac{1}{20}mab \blacktriangleleft$$

$$I_{yz} = \frac{1}{20}mbc \blacktriangleleft$$



Alternative solution for part *a*
First divide the tetrahedron into a series of thin horizontal slices of thickness dy as shown.

Now $$x = -\frac{a}{b}y + a = a\left(1 - \frac{y}{b}\right)$$

and $$z = -\frac{c}{b}y + c = c\left(1 - \frac{y}{b}\right)$$

The mass dm of the slab is

$$dm = \rho dV = \rho\left(\frac{1}{2}xzdy\right) = \frac{1}{2}\rho ac\left(1 - \frac{y}{b}\right)^2 dy$$

Now $$dI_{zx} = \rho t dI_{zx,\text{ Area}}$$

where $$t = dy$$

and $dI_{zx,\text{ Area}} = \dfrac{1}{24}x^2z^2$ from the results of Sample Problem 9.6

Then $$dIzx = \rho(dy)\left\{\frac{1}{24}\left[a\left(1 - \frac{y}{b}\right)^2\right]\left[c\left(1 - \frac{y}{b}\right)\right]^2\right\}$$

$$= \frac{1}{24}\rho a^2c^2\left(1 - \frac{y}{b}\right)^4 dy = \frac{1}{4}\frac{m}{b}ac\left(1 - \frac{y}{b}\right)^4 dy$$

Finally $$I_{zx} = \int dI_{zx} = \int_0^b \frac{1}{4}\frac{m}{b}ac\left(1 - \frac{y}{b}\right)^4 dy$$

$$= \frac{1}{4}\frac{m}{b}ac\left[\left(-\frac{b}{5}\right)\left(1 - \frac{y}{b}\right)^5\right]_0^b$$

or $$I_{zx} = \frac{1}{20}mac \blacktriangleleft$$

PROBLEM 9.164 CONTINUED

Alternative solution for part a

The equation of the included face of the tetrahedron is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

so that
$$y = b\left(1 - \frac{x}{a} - \frac{z}{c}\right)$$

For an infinitesimal element of sides $dx, dy,$ and dz

$$dm = \rho dV = \rho dydxdz$$

From part a
$$x = a\left(1 - \frac{z}{c}\right)$$

Now

$$I_{zx} = \int zxdm = \int_0^c \int_0^{a\left(1-\frac{z}{c}\right)} \int_0^{b\left(1-\frac{x}{a}-\frac{z}{c}\right)} zx\left(\rho dydxdz\right)$$

$$= \rho \int_0^c \int_0^{a\left(1-\frac{z}{c}\right)} zx\left[b\left(1 - \tfrac{x}{a} - \tfrac{z}{c}\right)\right]dxdz$$

$$= \rho b \int_0^c z\left[\frac{1}{2}x^2 - \frac{1}{3}\frac{x^3}{a} - \frac{1}{2}\frac{z}{c}x^2\right]_0^{a\left(1-\frac{z}{c}\right)} dz$$

$$= \rho b \int_0^c z\left[\frac{1}{2}a^2\left(1 - \frac{z}{c}\right)^2 - \frac{1}{3a}a^3\left(1 - \frac{z}{c}\right)^3 - \frac{1}{2}\frac{z}{c}a^2\left(1 - \frac{z}{c}\right)^2\right]dz$$

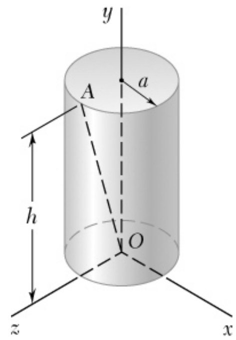
$$= \rho b \int_0^c \frac{1}{6}a^2 z\left(1 - \frac{z}{c}\right)^3 dz$$

$$= \frac{1}{6}\rho a^2 b \int_0^c \left(z - 3\frac{z^2}{c} + 3\frac{z^3}{c^2} - \frac{z^4}{c^3}\right)dz$$

$$= \frac{m}{c}a\left[\frac{1}{2}z^2 - \frac{z^3}{c} + \frac{3}{4}\frac{z^4}{c^2} - \frac{1}{5}\frac{z^5}{c^3}\right]_0^c$$

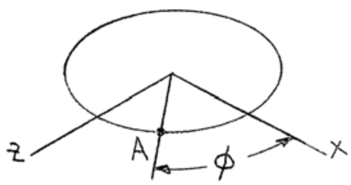
or $I_{zx} = \dfrac{1}{20}mac$ ◀

PROBLEM 9.165



The homogeneous circular cylinder shown has a mass m. Determine the moment of inertia of the cylinder with respect to the line joining the origin O and point A which is located on the perimeter of the top surface of the cylinder.

SOLUTION



From Figure 9.28 $\qquad I_y = \frac{1}{2}ma^2$

and using the parallel-axis theorem

$$I_x = I_z = \frac{1}{12}m\left(3a^2 + h^2\right) + m\left(\frac{h}{2}\right)^2 = \frac{1}{12}m\left(3a^2 + 4h^2\right)$$

Symmetry implies $\qquad I_{xy} = I_{yz} = I_{zx} = 0$

For convenience, let point A lie in the yz plane. Then

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}\left(h\mathbf{j} + a\mathbf{k}\right)$$

With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$I_{OA} = I_x\underbrace{\cancel{\lambda_x^2}}_{0} + I_y\lambda_y^2 + I_z\lambda_z^2$$

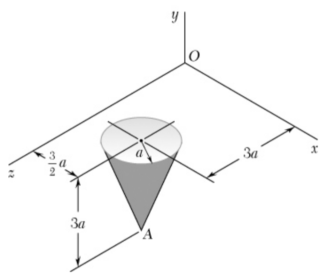
$$= \frac{1}{2}ma^2\left(\frac{h}{\sqrt{h^2 + a^2}}\right)^2 + \frac{1}{12}m\left(3a^2 + 4h^2\right)\left(\frac{a}{\sqrt{h^2 + a^2}}\right)^2$$

or $\quad I_{OA} = \frac{1}{12}ma^2\frac{10h^2 + 3a^2}{h^2 + a^2}$ ◀

Note: For point A located at an arbitrary point on the perimeter of the top surface, λ_{OA} is given by

$$\lambda_{OA} = \frac{1}{\sqrt{h^2 + a^2}}\left(a\cos\phi\mathbf{i} + h\mathbf{j} + a\sin\phi\mathbf{k}\right)$$

which results in the same expression for I_{OA}



PROBLEM 9.166

The homogeneous circular cone shown has a mass m. Determine the moment of inertia of the cone with respect to the line joining the origin O and point A.

SOLUTION

First note that
$$d_{OA} = \sqrt{\left(\frac{3}{2}a\right)^2 + (-3a)^2 + (3a)^2} = \frac{9}{2}a$$

Then
$$\lambda_{OA} = \frac{1}{\frac{9}{2}a}\left(\frac{3}{2}a\mathbf{i} - 3a\mathbf{j} + 3a\mathbf{k}\right) = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

For a rectangular coordinate system with origin at point A and axes aligned with the given x, y, z axes, have (using Figure 9.28)

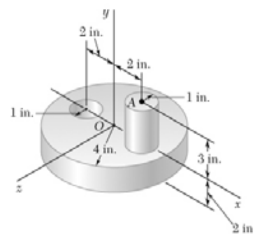
$$I_x = I_z = \frac{3}{5}m\left[\frac{1}{4}a^2 + (3a)^2\right] \qquad I_y = \frac{3}{10}ma^2$$
$$= \frac{111}{20}ma^2$$

Also, symmetry implies
$$I_{xy} = I_{yz} = I_{zx} = 0$$

With the mass products of inertia equal to zero, Equation (9.46) reduces to

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2$$
$$= \frac{111}{20}ma^2\left(\frac{1}{3}\right)^2 + \frac{3}{10}ma^2\left(-\frac{2}{3}\right)^2 + \frac{111}{20}ma^2\left(\frac{2}{3}\right)^2$$
$$= \frac{193}{60}ma^2$$

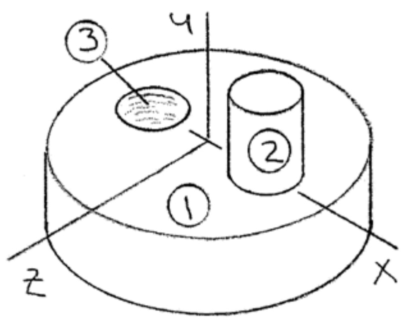
or $I_{OA} = 3.22ma^2$ ◀



PROBLEM 9.167

Shown is the machine element of Problem 9.143. Determine its moment of inertia with respect to the line joining the origin *O* and point *A*.

SOLUTION



First compute the mass of each component

Have $$m = \rho_{sT} V = \frac{0.284\ \text{lb/in}^3}{32.2\ \text{ft/s}^2} V = \left(0.008819\ \text{lb}\cdot\text{s}^2/\text{ft}\cdot\text{in}^3\right) V$$

Then

$$m_1 = 0.008819\ \text{lb}\cdot\text{s}^2/\text{ft}\cdot\text{in}^3 \left[\pi (4\ \text{in.})^2 (2\ \text{in.})\right] = 0.88667\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$m_2 = 0.008819\ \text{lb}\cdot\text{s}^2/\text{ft}\cdot\text{in}^3 \left[\pi (1\ \text{in.})^2 (3\ \text{in.})\right] = 0.083125\ \text{lb}\cdot\text{s}^2/\text{ft}$$

$$m_3 = 0.008819\ \text{lb}\cdot\text{s}^2/\text{ft}\cdot\text{in}^3 \left[\pi (1\ \text{in.})^2 (2\ \text{in.})\right] = 0.055417\ \text{lb}\cdot\text{s}^2/\text{ft}$$

Symmetry implies $$I_{yz} = I_{zx} = 0 \quad \left(I_{xy}\right)_1 = 0$$

and $$\left(\bar{I}_{x'y'}\right)_2 = \left(\bar{I}_{x'y'}\right)_3 = 0$$

Now

$$I_{xy} = \Sigma\left(\bar{I}_{x'y'} + m\bar{x}\,\bar{y}\right) = m_2\bar{x}_2\bar{y}_2 - m_3\bar{x}_3\bar{y}_3$$

$$= \left(0.083125\ \text{lb}\cdot\text{s}^2/\text{ft}\right)\left[(2\ \text{in.})(1.5\ \text{in.})\right] \times \left(\frac{1\ \text{ft}^2}{144\ \text{in}^2}\right)$$

$$-\left(0.055417\ \text{lb}\cdot\text{s}^2/\text{ft}\right)\left[(-2\ \text{in.})(-1\ \text{in.})\right] \times \left(\frac{1\ \text{ft}^2}{144\ \text{in}^2}\right)$$

$$= 0.96209 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

From the solution to Problem 9.143:

$$I_x = 34.106 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_y = 50.125 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_z = 34.876 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

PROBLEM 9.167 CONTINUED

By observation $\boldsymbol{\lambda}_{OA} = \frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{j})$

Then

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + \cancelto{0}{I_z\lambda_z^2} - 2I_{xy}\lambda_x\lambda_y - \cancelto{0}{2I_{yz}\lambda_y\lambda_z} - \cancelto{0}{2I_{zx}\lambda_z\lambda_x}$$

$$= \left(34.106 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2\right)\left(\frac{2}{\sqrt{13}}\right)^2$$

$$+ \left(50.125 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2\right)\left(\frac{3}{\sqrt{13}}\right)^2$$

$$- 2\left(0.96209 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2\right)\left(\frac{2}{\sqrt{13}}\right)\left(\frac{3}{\sqrt{13}}\right)$$

$$= (10.4942 + 34.7019 - 0.8881) \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 44.308 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

or $I_{OA} = 44.3 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.168

Determine the moment of inertia of the steel machine element of Problems 9.147 and 9.151 with respect to the axis through the origin which forms equal angles with the *x*, *y*, and *z* axes.

SOLUTION

From Problem 9.147:

$$I_x = 9.8821 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_y = 11.5344 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_z = 2.1878 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

Problem 9.151:

$$I_{xy} = 0.48776 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_{yz} = 1.18391 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_{zx} = 2.6951 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

Now

$$\lambda_x = \lambda_y = \lambda_z$$

and

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1$$

Therefore,

$$3\lambda_x^2 = 1$$

or

$$\lambda_x = \lambda_y = \lambda_z = \frac{1}{\sqrt{3}}$$

Equation 9.46

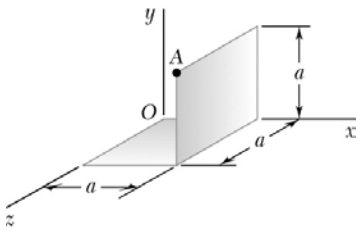
$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

$$= \left[9.8821\left(\frac{1}{\sqrt{3}}\right)^2 + 11.5344\left(\frac{1}{\sqrt{3}}\right)^2 + 2.1878\left(\frac{1}{\sqrt{3}}\right)^2 - 2(0.48776)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\right.$$

$$\left. - 2(1.18391)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) - 2(2.6951)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\right] \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$= 4.95692 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

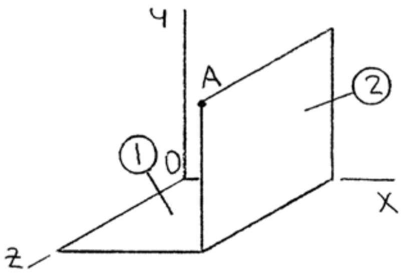
or $I_{OL} = 4.96 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀



PROBLEM 9.169

The thin bent plate shown is of uniform density and weight W. Determine its mass moment of inertia with respect to the line joining the origin O and point A.

SOLUTION



First note that $$m_1 = m_2 = \frac{1}{2}\frac{W}{g}$$

And that $$\lambda_{OA} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Using Figure 9.28 and the parallel-axis theorem have

$$I_x = (I_x)_1 + (I_x)_2$$

$$= \left[\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)^2\right]$$

$$+\left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)(a^2 + a^2) + \frac{1}{2}\frac{W}{g}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{12} + \frac{1}{4}\right)a^2 + \left(\frac{1}{6} + \frac{1}{2}\right)a^2\right] = \frac{1}{2}\frac{W}{g}a^2$$

$$I_y = (I_y)_1 + (I_y)_2$$

$$= \left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)(a^2 + a^2) + \frac{1}{2}\frac{W}{g}\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right]\right\}$$

$$+\left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left[(a)^2 + \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{6} + \frac{1}{2}\right)a^2 + \left(\frac{1}{12} + \frac{5}{4}\right)a^2\right] = \frac{W}{g}a^2$$

$$I_z = (I_z)_1 + (I_z)_2$$

$$= \left[\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)^2\right]$$

$$+\left\{\frac{1}{12}\left(\frac{1}{2}\frac{W}{g}\right)a^2 + \frac{1}{2}\frac{W}{g}\left[(a)^2 + \left(\frac{a}{2}\right)^2\right]\right\}$$

$$= \frac{1}{2}\frac{W}{g}\left[\left(\frac{1}{12} + \frac{1}{4}\right)a^2 + \left(\frac{1}{12} + \frac{5}{4}\right)a^2\right] = \frac{5}{6}\frac{W}{g}a^2$$

Now observe that the centroidal products of inertia, $\bar{I}_{x'y'}, \bar{I}_{y'z'},$ and $\bar{I}_{z'x'}$, of both components are zero because of symmetry. Also, $\bar{y}_1 = 0$

Then $$I_{xy} = \Sigma\left(\cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y}\right) = m_2\bar{x}_2\bar{y}_2 = \frac{1}{2}\frac{W}{g}(a)\left(\frac{a}{2}\right) = \frac{1}{4}\frac{W}{g}a^2$$

$$I_{yz} = \Sigma\left(\cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z}\right) = m_2\bar{y}_2\bar{z}_2 = \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = \frac{1}{8}\frac{W}{g}a^2$$

$$I_{zx} = \Sigma\left(\cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x}\right) = m_1\bar{z}_1\bar{x}_1 + m_2\bar{z}_2\bar{x}_2$$

$$= \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + \frac{1}{2}\frac{W}{g}\left(\frac{a}{2}\right)(a) = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Equation (9.46)

$$I_{OA} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Noting that

$$\lambda_x^2 = \lambda_y^2 = \lambda_z^2 = \lambda_x\lambda_y = \lambda_y\lambda_z = \lambda_z\lambda_x = \frac{1}{3}$$

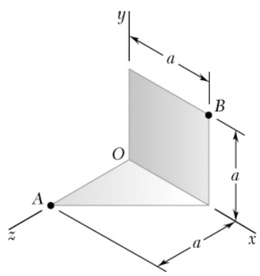
Have

$$I_{OA} = \frac{1}{3}\left[\frac{1}{2}\frac{W}{g}a^2 + \frac{W}{g}a^2 + \frac{5}{6}\frac{W}{g}a^2\right.$$

$$\left. - 2\left(\frac{1}{4}\frac{W}{g}a^2 + \frac{1}{8}\frac{W}{g}a^2 + \frac{3}{8}\frac{W}{g}a^2\right)\right]$$

$$= \frac{1}{3}\left[\frac{14}{6} - 2\left(\frac{3}{4}\right)\right]\frac{W}{g}a^2$$

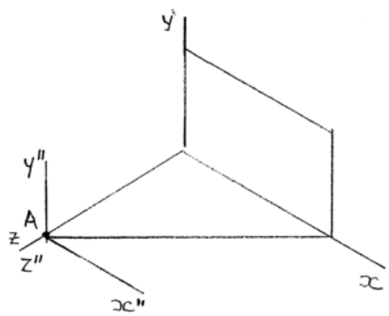
or $I_{OA} = \dfrac{5}{18}\dfrac{W}{g}a^2$ ◀



PROBLEM 9.170

A piece of sheet metal of thickness t and density ρ is cut and bent into the shape shown. Determine its mass moment of inertia with respect to a line joining points A and B.

SOLUTION



Have $\qquad m = \rho V = \rho t A$

Then $\qquad m_1 = \rho t a^2 \qquad m_2 = \dfrac{1}{2}\rho t a^2$

Compute moments and moments of inertia with respect to point A

Now

$$I_{x''} = (I_{x''})_1 + (I_{x''})_2$$

$$= \rho t a^2\left\{\left(\frac{1}{2}a^2\right) + \left[\left(\frac{a}{2}\right)^2 + (a)^2\right]\right\} + \frac{1}{2}\rho t a^2\left\{\left[\frac{1}{18}a^2\right] + \left[\left(\frac{2}{3}a\right)^2\right]\right\}$$

$$= \frac{19}{12}\rho t a^4$$

$$I_{y''} = (I_{y''})_1 + (I_{y''})_2$$

$$= \rho t a^2\left\{\left[\frac{1}{12}a^2\right] + \left[\left(\frac{a}{2}\right)^2 + (a)^2\right]\right\}$$

$$+\frac{1}{2}\rho t a^2\left\{\frac{1}{18}\left[a^2 + a^2\right] + \left[\left(\frac{b}{3}\right)^2 + \left(\frac{2^2}{3}\right)^2\right]\right\}$$

$$= \frac{5}{3}\rho t a^4$$

$$I_{z''} = (I_{z''})_1 + (I_{z''})_2$$

$$= \rho t a^2\left[\frac{1}{3}\left(a^2 + a^2\right)\right] + \frac{1}{2}\rho t a^2\left(\frac{1}{6}a^2\right)$$

$$= \frac{3}{4}\rho t a^4$$

Now note symmetry implies $\qquad \left(\bar{I}_{x'y'}\right)_1 = \left(\bar{I}_{y'z'}\right)_1 = \left(\bar{I}_{z'x'}\right)_1 = 0$

$$\left(\bar{I}_{x'y}\right)_2 = \left(\bar{I}_{y'z'}\right)_2 = 0$$

PROBLEM 9.170 CONTINUED

Now $$I_{uv} = I_{u'v'} + m\bar{u}\,\bar{v}$$

Therefore $$I_{x''y''} = m_1\bar{x}_1''\bar{y}_1'' + \cancelto{0}{m_2\bar{x}_2''\bar{y}_2''} = \rho t a^2\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = \frac{1}{4}\rho t a^4$$

$$I_{y''z''} = m_1\bar{y}_1''\,\bar{z}_1'' + \cancelto{0}{m_2\bar{y}_2''}\bar{z}_2'' = \rho t a^2\left(\frac{a}{2}\right)(-a) = -\frac{1}{2}\rho t a^4$$

$$I_{z''x''} = m_1\bar{z}_1''\,\bar{x}_1'' + \left[\left(\bar{I}_{z'x'}\right)_2 + m_2\bar{z}_2''\,\bar{x}_2''\right]$$

From Sample Problem 9.6 $$\left[\left(I_{z'x'}\right)_2\right]_{\text{area}} = -\frac{1}{72}a^4$$

Then $$\left(\bar{I}_{z''x''}\right)_2 = \rho t\left[\left(\bar{I}_{z'x'}\right)_2\right]_{\text{area}} = -\frac{1}{72}\rho t a^4$$

Then $$I_{z'x'} = \rho\frac{1}{2}a^2(-a)\left(\frac{a}{2}\right)$$

$$+\left[-\frac{1}{72}\rho\frac{1}{2}ta^4 + \frac{1}{2}\rho t a^2\left(-\frac{2}{3}a\right)\left(\frac{1}{3}a\right)\right]$$

$$= -\frac{5}{8}\rho t a^4$$

By observation $$\boldsymbol{\lambda}_{AB} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} - \mathbf{k})$$

Now, Equation 9.46

$$I_{AB} = I_{x''}\lambda_x^2 + I_{y''}\lambda_y^2 + I_{z''}\lambda_z^2 - 2I_{x''y''}\lambda_{x''y''} - 2I_{y''z''}\lambda_{y''}\lambda_{z''} - 2I_{z''x''}\lambda_{z''}\lambda_{x''}$$

$$= \rho t a^4\left[\frac{19}{12}\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{5}{3}\left(\frac{1}{\sqrt{3}}\right)^2 + \frac{3}{4}\left(-\frac{1}{\sqrt{3}}\right)^2\right.$$

$$-2\left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) - 2\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right)$$

$$\left.-2\left(-\frac{5}{8}\right)\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\right]$$

or $$I_{AB} = \frac{5}{12}\rho t a^4 \blacktriangleleft$$