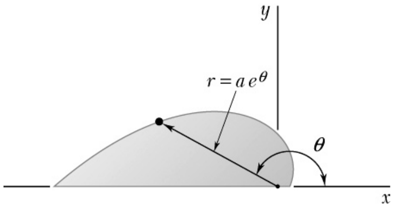
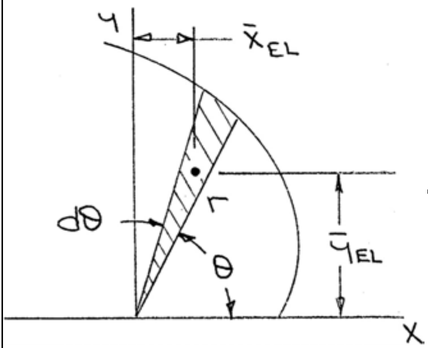
PROBLEM 5.45



Determine by direct integration the centroid of the area shown.

SOLUTION



Have $\quad \bar{x}_{EL} = \dfrac{2}{3}r\cos\theta = \dfrac{2}{3}ae^{\theta}\cos\theta$

$$\bar{y}_{EL} = \frac{2}{3}r\sin\theta = \frac{2}{3}ae^{\theta}\sin\theta$$

and $\quad dA = \dfrac{1}{2}(r)(rd\theta) = \dfrac{1}{2}a^2e^{2\theta}d\theta$

Then

$$A = \int dA = \int_0^{\pi} \frac{1}{2}a^2e^{2\theta}d\theta = \frac{1}{2}a^2\left[\frac{1}{2}e^{2\theta}\right]_0^{\pi} = \frac{1}{4}a^2\left(e^{2\pi} - 1\right) = 133.623a^2$$

and $\quad \int \bar{x}_{EL}dA = \int_0^{\pi} \dfrac{2}{3}ae^{\theta}\cos\theta\left(\dfrac{1}{2}a^2e^{2\theta}d\theta\right) = \dfrac{1}{3}a^3\int_0^{\pi} e^{3\theta}\cos\theta\, d\theta$

To proceed, use integration by parts, with

$$u = e^{3\theta} \quad \text{and} \quad du = 3e^{3\theta}d\theta$$

$$dv = \cos\theta d\theta \quad \text{and} \quad v = \sin\theta$$

Then $\quad \int e^{3\theta}\cos\theta d\theta = e^{3\theta}\sin\theta - \int \sin\theta\left(3e^{3\theta}d\theta\right)$

Now let $\quad u = e^{3\theta} \quad$ then $\quad du = 3e^{3\theta}d\theta$

$$dv = \sin\theta d\theta, \quad \text{then} \quad v = -\cos\theta$$

Then $\quad \int e^{3\theta}\sin\theta d\theta = e^{3\theta}\sin\theta - 3\left[-e^{-3\theta}\cos\theta - \int(-\cos\theta)\left(3e^{3\theta}d\theta\right)\right]$

So that $\quad \int e^{3\theta}\cos\theta d\theta = \dfrac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)$

$$\therefore \quad \int \bar{x}_{EL}dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}(\sin\theta + 3\cos\theta)\right]_0^{\pi} = \frac{a^3}{30}\left(-3e^{3\pi} - 3\right) = -1239.26a^3$$

Also $\quad \int \bar{y}_{EL}dA = \int_0^{\pi} \dfrac{2}{3}ae^{\theta}\sin\theta\left(\dfrac{1}{2}a^2e^{2\theta}d\theta\right) = \dfrac{1}{3}a^3\int_0^{\pi} e^{3\theta}\sin\theta\, d\theta$

PROBLEM 5.45 CONTINUED

Using integration by parts, as above, with

$$u = e^{3\theta} \qquad \text{and} \qquad du = 3e^{3\theta}d\theta$$

$$dv = \int \sin\theta d\theta \qquad \text{and} \qquad v = -\cos\theta$$

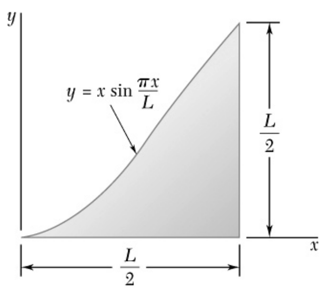
Then $\int e^{3\theta}\sin\theta d\theta = -e^{3\theta}\cos\theta - \int\left(-\cos\theta\right)\left(3e^{3\theta}d\theta\right)$

So that $\int e^{3\theta}\sin\theta d\theta = \dfrac{e^{3\theta}}{10}\left(-\cos\theta + 3\sin\theta\right)$

$$\therefore \quad \int \overline{y}_{EL}dA = \frac{1}{3}a^3\left[\frac{e^{3\theta}}{10}\left(-\cos\theta + 3\sin\theta\right)\right]_0^\pi = \frac{a^3}{30}\left(e^{3\pi} + 1\right) = 413.09a^3$$

Hence $\overline{x}A = \int \overline{x}_{EL}dA$: $\quad \overline{x}\left(133.623a^2\right) = -1239.26a^3 \qquad$ or $\overline{x} = -9.27a$ ◀

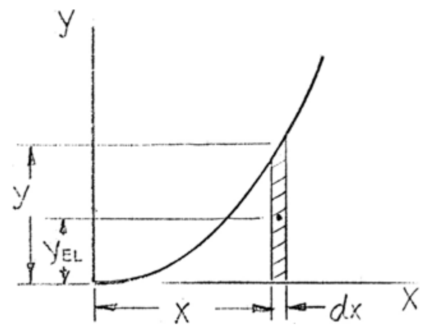
$\overline{y}A = \int \overline{y}_{EL}dA$: $\quad \overline{y}\left(133.623a^2\right) = 413.09a^3 \qquad$ or $\overline{y} = 3.09a$ ◀



PROBLEM 5.46

Determine by direct integration the centroid of the area shown.

SOLUTION



Have $$\overline{x}_{EL} = x, \quad \overline{y}_{EL} = \frac{1}{2} x \sin \frac{\pi x}{L}$$

and $$dA = y dx$$

$$A = \int dA = \int_0^{L/2} x \sin \frac{\pi x}{L} dx = \left[\frac{L^2}{\pi^2} \sin \frac{\pi x}{L} - \frac{L}{\pi} x \cos \frac{\pi x}{L} \right]_0^{L/2} = \frac{L^2}{\pi^2}$$

and $$\overline{x} = \int \overline{x}_{EL} dA = \int_0^{L/2} x \left(x \sin \frac{\pi x}{L} dx \right)$$

$$= \left[\frac{2L^2}{\pi^2} x \sin\left(\frac{\pi x}{L}\right) + \frac{2L^3}{\pi^3} \cos\left(\frac{\pi x}{L}\right) - \frac{L}{\pi} x^2 \sin\left(\frac{\pi x}{L}\right) \right]_0^{L/2} = \frac{L^3}{\pi^2} - 2\frac{L^3}{\pi^3}$$

Also $$\overline{y} = \int \overline{y}_{EL} dA = \int_0^{L/2} \frac{1}{2} x \sin \frac{\pi x}{L} \left(x \sin \frac{\pi x}{L} dx \right)$$

$$= \frac{1}{2} \left[\frac{2L^2}{\pi^2} x \sin \frac{\pi x}{L} - \left(\frac{L}{\pi} x - \frac{2L^3}{\pi^3} \right) \cos \frac{\pi x}{L} \right]_0^{L/2}$$

$$= \frac{1}{2} \left[\frac{1}{6} \left(\frac{L^3}{8} \right) - \frac{L^2}{4\pi^2} \left(\frac{L}{2} \right) (-1) \right] = \frac{L^3}{96\pi^2} \left(6 + \pi^2 \right)$$

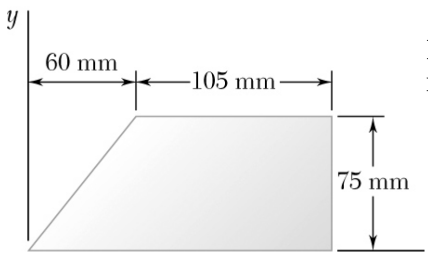
PROBLEM 5.46 CONTINUED

Hence $$\bar{x}A = \int \bar{x}_{EL}dA: \quad \bar{x}\left(\frac{L^2}{\pi^2}\right) = L^3\left(\frac{1}{\pi^2} - \frac{z}{\pi^3}\right)$$

or $\bar{x} = 0.363L$ ◀

$$\bar{y}A = \int \bar{y}_{EL}dA: \quad \bar{y}\left(\frac{L^2}{\pi^2}\right) = \frac{L^3}{96\pi^2}\left(\frac{1}{\pi^2} - \frac{2}{\pi^3}\right)$$

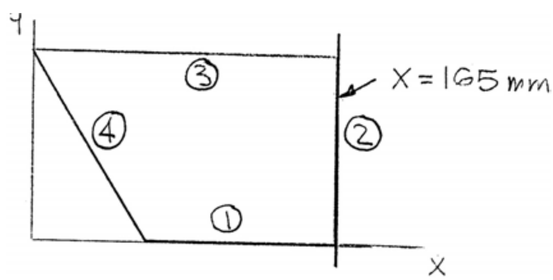
or $\bar{y} = 0.1653L$ ◀



PROBLEM 5.47

Determine the volume and the surface area of the solid obtained by rotating the area of Problem. 5.2 about (a) the x axis, (b) the line $x = 165$ mm.

SOLUTION



From the solution to Problem 5.2:

$$A = 10\,125\text{ mm}^2, \bar{X}_{\text{area}} = 96.4\text{ mm}, \bar{Y}_{\text{area}} = 34.7\text{ mm} \qquad (\text{Area})$$

From the solution to Problem 5.22:

$$L = 441.05\text{ mm } \bar{X}_{\text{line}} = 92.2\text{ mm}, \ \bar{Y}_{\text{line}} = 32.4\text{ mm} \qquad (\text{Line})$$

Applying the theorems of Pappus-Guldinus, we have

(a) Rotation about the x axis:

$$\text{Area} = 2\pi\,\bar{Y}_{\text{line}}\,L = 2\pi(32.4\text{ mm})(441.05\text{ mm}) = 89.786\times10^3\text{ mm}^2$$

$$A = 89.8\times10^3\text{ mm}^2 \blacktriangleleft$$

$$\text{Volume} = 2\pi\bar{Y}_{\text{area}}A = 2\pi(34.7\text{ mm})(10\,125\text{ mm}) = 2.2075\times10^6\text{ mm}^3$$

$$V = 2.21\times10^6\text{ mm}^3 \blacktriangleleft$$

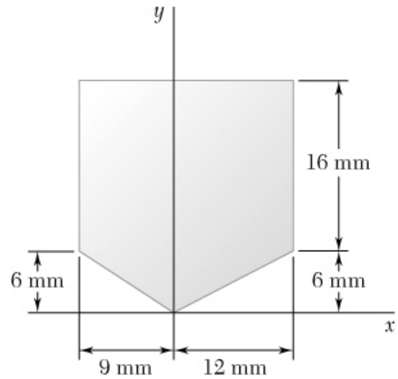
(b) Rotation about $x = 165$ mm:

$$\text{Area} = 2\pi(165 - \bar{X}_{\text{line}})L = 2\pi[(165 - 92.2)\text{ mm}](441.05\text{ mm}) = 2.01774\times10^5\text{ mm}^2$$

$$A = 0.202\times10^6\text{ mm}^2 \blacktriangleleft$$

$$\text{Volume} = 2\pi(165 - \bar{X}_{\text{area}})A = 2\pi[(165 - 96.4)\text{ mm}](10\,125\text{ mm}) = 4.3641\times10^6\text{ mm}^3$$

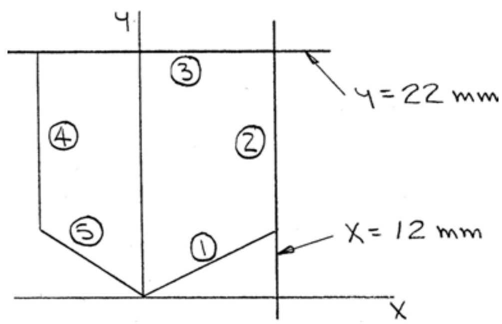
$$V = 4.36\times10^6\text{ mm}^3 \blacktriangleleft$$



PROBLEM 5.48

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.4 about (*a*) the line $y = 22$ mm, (*b*) the line $x = 12$ mm.

SOLUTION



From the solution to Problem 5.4:

$$A = 399\text{ mm}^2,\ \bar{X}_{\text{area}} = 1.421\text{ mm},\ \bar{Y}_{\text{area}} = 12.42\text{ mm} \qquad \text{(Area)}$$

From the solution to Problem 5.23:

$$L = 77.233\text{ mm},\ \bar{X}_{\text{line}} = 1.441\text{ mm},\ \bar{Y}_{\text{line}} = 12.72\text{ mm} \qquad \text{(Line)}$$

Applying the theorems of Pappus-Guldinus, we have

(*a*) Rotation about the line $y = 22$ mm:

$$\text{Area} = 2\pi\left(22 - \bar{Y}_{\text{line}}\right)L = 2\pi\left[(22 - 12.72)\text{ mm}\right](77.233\text{ mm}) = 4503\text{ mm}^2$$

$$A = 4.50 \times 10^3\text{ mm}^2 \blacktriangleleft$$

$$\text{Volume} = 2\pi\left(22 - \bar{Y}_{\text{area}}\right)A = 2\pi\left[(22 - 12.42)\text{ mm}\right]\left(399\text{ mm}^2\right) = 24\,016.97\text{ mm}^3$$

$$V = 24.0 \times 10^3\text{ mm}^3 \blacktriangleleft$$

(*b*) Rotation about line $x = 12$ mm:

$$\text{Area} = 2\pi\left(12 - \bar{X}_{\text{line}}\right)L = 2\pi\left[(12 - 1.441)\text{ mm}\right](77.233\text{ mm}) = 5124.45\text{ mm}^2$$

$$A = 5.12 \times 10^3\text{ mm}^2 \blacktriangleleft$$

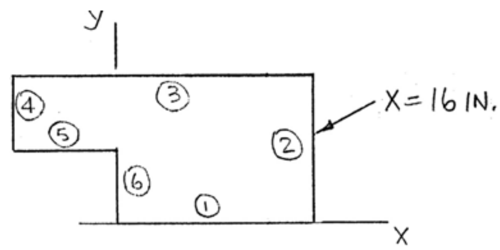
$$\text{Volume} = 2\pi(12 - 1.421)A = 2\pi\left[(12 - 1.421)\text{ mm}\right]\left(399\text{ mm}^2\right) = 26\,521.46\text{ mm}^3$$

$$V = 26.5 \times 10^3\text{ mm}^3 \blacktriangleleft$$

PROBLEM 5.49

Determine the volume and the surface area of the solid obtained by rotating the area of Problem 5.1 about (*a*) the x axis, (*b*) the line $x = 16$ in.

SOLUTION



From the solution to Problem 5.1:

$$A = 240 \text{ in}^2, \bar{X}_{\text{area}} = 5.60 \text{ in.}, \bar{Y}_{\text{area}} = 6.60 \text{ in.} \qquad \text{(Area)}$$

From the solution to Problem 5.21:

$$L = 72 \text{ in.}, \bar{X}_{\text{line}} = 4.67 \text{ in.}, \bar{Y}_{\text{line}} = 6.67 \text{ in.}$$

Applying the theorems of Pappus-Guldinus, we have

(*a*) Rotation about the x axis:

$$A_x = 2\pi Y_{\text{line}} L = 2\pi(6.67 \text{ in.})(72 \text{ in.}) = 3017.4 \text{ in}^2$$

$$A = 3020 \text{ in}^2 \blacktriangleleft$$

$$V_x = 2\pi Y_{\text{area}} A = 2\pi(6.60 \text{ in.})(240 \text{ in}^2) = 9952.6 \text{ in}^3$$

$$V = 9950 \text{ in}^3 \blacktriangleleft$$

(*b*) Rotation about $x = 16$ in.:

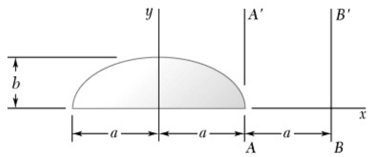
$$A_{x=16} = 2\pi(16 - \bar{X}_{\text{line}})L = 2\pi[(16 - 4.67) \text{ in.}](72 \text{ in.}) = 5125.6 \text{ in}^2$$

$$A_{x=16} = 5130 \text{ in}^2 \blacktriangleleft$$

$$V_{x=16} = 2\pi(16 - \bar{X}_{\text{area}})A = 2\pi[(16 - 5.60) \text{ in.}](240 \text{ in}^2) = 15\ 682.8 \text{ in}^3$$

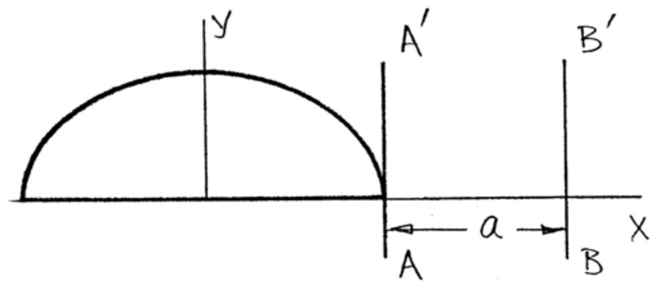
$$V_{x=16} = 15.68 \times 10^3 \text{ in}^3 \blacktriangleleft$$

PROBLEM 5.50



Determine the volume of the solid generated by rotating the semielliptical area shown about (*a*) the axis AA', (*b*) the axis BB', (*c*) the *y* axis.

SOLUTION



Applying the second theorem of Pappus-Guldinus, we have

(*a*) Rotation about axis AA':

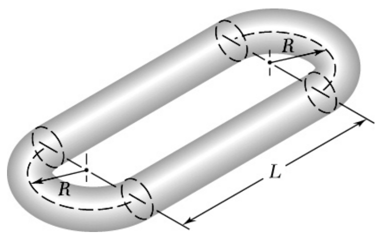
$$\text{Volume} = 2\pi\overline{y}A = 2\pi(a)\left(\frac{\pi ab}{2}\right) = \pi^2a^2b \qquad V = \pi^2a^2b \blacktriangleleft$$

(*b*) Rotation about axis BB':

$$\text{Volume} = 2\pi\overline{y}A = 2\pi(2a)\left(\frac{\pi ab}{2}\right) = 2\pi^2a^2b \qquad V = 2\pi^2a^2b \blacktriangleleft$$

(*c*) Rotation about *y*-axis:

$$\text{Volume} = 2\pi\overline{y}A = 2\pi\left(\frac{4a}{3\pi}\right)\left(\frac{\pi ab}{2}\right) = \frac{2}{3}\pi a^2b \qquad V = \frac{2}{3}\pi a^2b \blacktriangleleft$$



PROBLEM 5.51

Determine the volume and the surface area of the chain link shown, which is made from a 2-in.-diameter bar, if $R = 3$ in. and $L = 10$ in.

SOLUTION

First note that the area A and the circumference C of the cross section of the bar are

$$A = \frac{\pi}{4} d^2 \qquad \text{and} \qquad C = \pi d$$

Observe that the semicircular ends of the link can be obtained by rotating the cross section through a horizontal semicircular arc of radius R. Then, applying the theorems of Pappus-Guldinus, we have

$$\text{Volume} = 2(V_{\text{side}}) + 2(V_{\text{end}}) = 2(AL) + 2(\pi RA) = 2(L + \pi R)A$$

$$= 2\left[10\text{ in.} + \pi(3\text{ in.})\right]\left[\frac{\pi}{4}(2\text{ in.})^2\right]$$

$$= 122.049\text{ in}^3$$

$$V = 122.0\text{ in}^3 \blacktriangleleft$$

$$\text{Area} = 2(A_{\text{side}}) + 2(A_{\text{end}}) = 2(CL) + 2(\pi RC) = 2(L + \pi R)C$$

$$= 2\left[10\text{ in.} + \pi(3\text{ in.})\right]\left[\pi(4\text{ in.})\right]$$

$$= 488.198\text{ in}^2$$

$$A = 488\text{ in}^2 \blacktriangleleft$$