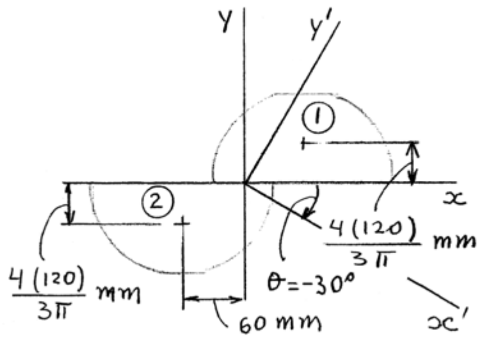
PROBLEM 9.81

Determine the moments of inertia and the product of inertia of the area of Problem 9.73 with respect to new centroidal axes obtained by rotating the x and y axes through 30° clockwise.

SOLUTION



From Problem 9.73,

$$\bar{I}_{xy} = 138.24 \times 10^6 \text{ mm}^4$$

$$\bar{I}_x = (I_x)_1 + (I_x)_2 \qquad (I_x)_1 = (I_x)_2$$

$$= 2\left[\frac{\pi}{8}(120 \text{ mm})^4\right]$$

$$= 51.84\pi \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = (I_y)_1 + (I_y)_2 \qquad (I_y)_1 = (I_y)_2$$

$$= 2\left[\frac{\pi}{8}(120 \text{ mm})^4 + \frac{\pi}{2}(120 \text{ mm})^2(60 \text{ mm})^2\right]$$

$$= 103.68\pi \times 10^6 \text{ mm}^4$$

Have

$$\bar{I}_x = 2\left(25.92\pi \times 10^6\right) = 51.84\pi \times 10^6 \text{ mm}^4$$

and

$$\bar{I}_y = 2\left(51.84\pi \times 10^6\right) = 103.68\pi \times 10^6 \text{ mm}^4$$

Then

$$\frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) = 77.76\pi \times 10^6 \text{ mm}^4$$

and

$$\frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right) = -25.92\pi \times 10^6 \text{ mm}^4$$

PROBLEM 9.81 CONTINUED

Now, from Equations 9.18, 9.19, and 9.20

Equation 9.18: $$\bar{I}_{x'} = \frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) + \frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right)\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$

$$= \left[77.76\pi \times 10^6 - 25.92\pi \times 10^6 \cos\left(-60°\right) - 138.24 \times 10^6 \sin\left(-60°\right)\right]\text{mm}^4$$

$$= 323.29 \times 10^6 \text{ mm}^4$$

or $\bar{I}_x = 323 \times 10^6 \text{ mm}^4$ ◀

Equation 9.19: $$\bar{I}_{y'} = \frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) - \frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right)\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= \left[77.76\pi \times 10^6 + 25.92\pi \times 10^6 \cos\left(-60°\right) + 138.24 \times 10^6 \sin\left(-60°\right)\right]\text{mm}^4$$

$$= 165.29 \times 10^6 \text{ mm}^4$$

or $\bar{I}_{y'} = 165.29 \times 10^6 \text{ mm}^4$ ◀

Equation 9.20: $$\bar{I}_{x'y'} = \frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right)\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$

$$= \left[-25.92\pi \times 10^6 \sin\left(-60°\right) + 138.24 \times 10^6 \cos\left(-60°\right)\right]\text{mm}^4$$

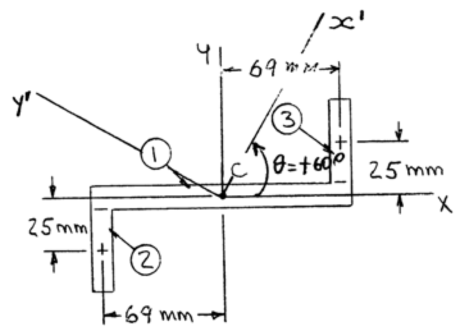
$$= 139.64 \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'y'} = 139.6 \times 10^4 \text{ mm}^4$ ◀

PROBLEM 9.82

Determine the moments of inertia and the product of inertia of the area of Problem 9.75 with respect to new centroidal axes obtained by rotating the x and y axes through 60° counterclockwise.

SOLUTION



From Problem 9.75 $\quad \bar{I}_{xy} = 1.5732 \times 10^6 \text{ mm}^4$

Now $\quad \bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3$

where $\quad (I_x)_1 = \frac{1}{12}(150 \text{ mm})(12 \text{ mm})^3 = 21\,600 \text{ mm}^4$

and $\quad (I_x)_2 = (I_x)_3 = \frac{1}{12}(12 \text{ mm})(38 \text{ mm})^3 + [(12 \text{ mm})(38 \text{ mm})](25 \text{ mm})^2$

$$= 339\,872 \text{ mm}^4$$

Then $\quad \bar{I}_x = [21\,600 + 2(339\,872)]\text{mm}^4 = 701\,344 \text{ mm}^4 = 0.70134 \times 10^6 \text{ mm}^4$

Also $\quad \bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3$

where $\quad (I_y)_1 = \frac{1}{12}(12 \text{ mm})(150 \text{ mm})^3 = 3.375 \times 10^6 \text{ mm}^4$

and $\quad (I_y)_2 = (I_y)_3 = \frac{1}{12}(38 \text{ mm})(12 \text{ mm})^3 + [(12 \text{ mm})(38 \text{ mm})](69 \text{ mm})^2$

$$= 2.1765 \times 10^6 \text{ mm}^4$$

Then $\quad \bar{I}_y = [(3.375 + 2(2.1765)] \times 10^6 \text{ mm}^4 = 7.728 \times 10^6 \text{ mm}^4$

Now $\quad \frac{1}{2}(\bar{I}_x + \bar{I}_y) = 4.2146 \times 10^6 \text{ mm}^4$

and $\quad \frac{1}{2}(\bar{I}_x - \bar{I}_y) = -3.5133 \times 10^6 \text{ mm}^4$

PROBLEM 9.82 CONTINUED

Using Equations 9.18, 9.19, and 9.20

From Equation 9.18:

$$\bar{I}_{x'} = \frac{\bar{I}_x + \bar{I}_y}{2} + \frac{\bar{I}_x - \bar{I}_y}{2}\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$

$$= \left[4.2147\times10^6 + \left(-3.5133\times10^6\right)\cos\left(120°\right) - 1.5732\times10^6\sin\left(120°\right)\right]\text{mm}^4$$

$$= 4.6089\times10^6\ \text{mm}^4$$

or $\bar{I}_{x'} = 4.61\times10^6\ \text{mm}^4$ ◀

From Equation 9.19:

$$\bar{I}_{y'} = \frac{\bar{I}_x + \bar{I}_y}{2} - \frac{\bar{I}_x - \bar{I}_y}{2}\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= \left[4.2147\times10^6 - \left(-3.5133\times10^6\right)\cos\left(120°\right) + 1.5732\times10^6\sin\left(120°\right)\right]\text{mm}^4$$

$$= 3.8205\times10^6\ \text{mm}^4$$

or $\bar{I}_{y'} = 3.82\times10^6\ \text{mm}^4$ ◀

From Equation 9.20:

$$\bar{I}_{x'y'} = \frac{\bar{I}_x - \bar{I}_y}{2}\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$

$$= \left[-3.5133\times10^6\sin\left(120°\right) + 1.5732\times10^6\cos\left(120°\right)\right]\text{mm}^4$$

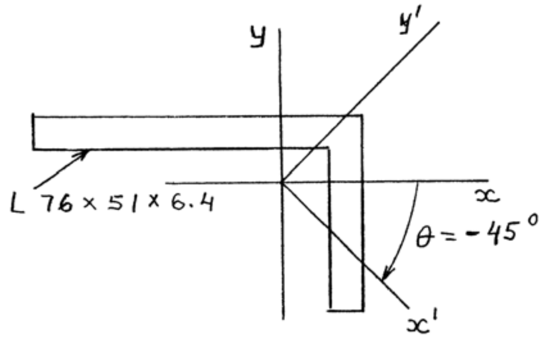
$$= -3.8292\times10^6\ \text{mm}^4$$

or $\bar{I}_{x'y'} = -3.83\times10^6\ \text{mm}^4$ ◀

PROBLEM 9.83

Determine the moments of inertia and the product of inertia of the L76 × 51 × 6.4-mm angle cross section of Problem 9.74 with respect to new centroidal axes obtained by rotating the *x* and *y* axes through 45° clockwise.

SOLUTION



From Problem 9.74

$$\bar{I}_{xy} = -0.1596 \times 10^6 \text{ mm}^4$$

From Figure 9.13

$$\bar{I}_x = 0.166 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.453 \times 10^6 \text{ mm}^4$$

Now

$$\frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) = 0.3095 \times 10^6 \text{ mm}^4$$

$$\frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right) = -0.1453 \times 10^6 \text{ mm}^4$$

Using Equations (9.18), (9.19), and (9.20)

Equation (9.18):

$$\bar{I}_{x'} = \frac{\bar{I}_x + \bar{I}_y}{2} + \frac{\bar{I}_x - \bar{I}_y}{2}\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$

$$= \left[0.3095 \times 10^6 + \left(-0.1435 \times 10^6\right)\cos\left(-90^{\circ}\right) - \left(-0.1596 \times 10^6\right)\sin\left(-90^{\circ}\right)\right]\text{mm}^4$$

$$= 0.1499 \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'} = 0.1499 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.83 CONTINUED

Equation (9.19):

$$\bar{I}_{y'} = \frac{\bar{I}_x + \bar{I}_y}{2} - \frac{\bar{I}_x - \bar{I}_y}{2}\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= \left[0.3095 \times 10^6 - \left(-0.1435 \times 10^6\right)\cos\left(-90°\right) + \left(-0.1596 \times 10^6\right)\sin\left(-90°\right)\right]\text{mm}^4$$

$$= 0.4691 \times 10^6 \text{ mm}^4$$

or $\bar{I}_{y'} = 0.469 \times 10^6 \text{ mm}^4$ ◀

Equation (9.20):

$$\bar{I}_{x'y'} = \frac{\bar{I}_x - \bar{I}_y}{2}\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$

$$= \left[-0.1435 \times 10^6 \sin\left(-90°\right) + 0.1596 \times 10^6 \cos\left(-90°\right)\right]\text{mm}^4$$

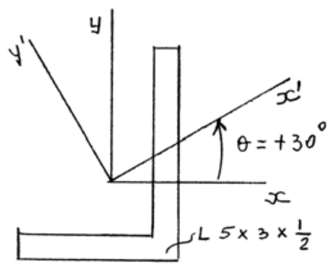
$$= 0.1435 \times 10^6 \text{ mm}^4$$

or $\bar{I}_{x'y'} = 0.1435 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.84

Determine the moments of inertia and the product of inertia of the L5 × 3 × $\frac{1}{2}$-in. angle cross section of Problem 9.78 with respect to new centroidal axes obtained by rotating the x and y axes through 30° counterclockwise.

SOLUTION



From Problem 9.78

$$\bar{I}_{xy} = 2.8125 \text{ in}^4$$

From Figure 9.13

$$\bar{I}_x = 9.45 \text{ in}^4, \qquad \bar{I}_y = 2.58 \text{ in}^4$$

Now

$$\frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) = 6.015 \text{ in}^4$$

$$\frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right) = 3.435 \text{ in}^4$$

Using Equations (9.18), (9.19), and (9.20)

Equation (9.18):

$$\bar{I}_{x'} = \frac{\bar{I}_x + \bar{I}_y}{2} + \frac{\bar{I}_x - \bar{I}_y}{2}\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$

$$= \left[6.015 + 3.435\cos(60^{\circ}) - 2.8125\sin(60^{\circ})\right]\text{in}^4 = 5.2968 \text{ in}^4$$

or $\bar{I}_{x'} = 5.30 \text{ in}^4$ ◀

Equation (9.19):

$$\bar{I}_{y'} = \frac{\bar{I}_x + \bar{I}_y}{2} - \frac{\bar{I}_x - \bar{I}_y}{2}\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= \left[6.015 - 3.435\cos(60^{\circ}) + 2.8125\sin(60^{\circ})\right]\text{in}^4 = 6.7332 \text{ in}^4$$

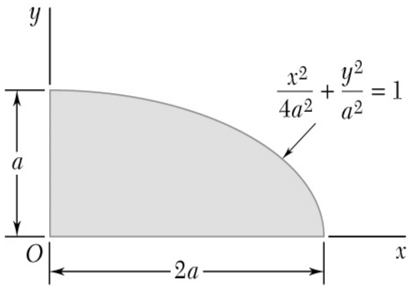
or $\bar{I}_{y'} = 6.73 \text{ in}^4$ ◀

Equation (9.20):

$$\bar{I}_{x'y'} = \frac{\bar{I}_x - \bar{I}_y}{2}\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$

$$= \left[3.435\sin(60^{\circ}) + 2.8125\cos(60^{\circ})\right]\text{in}^4 = 4.3810 \text{ in}^4$$

or $\bar{I}_{x'y'} = 4.38 \text{ in}^4$ ◀



PROBLEM 9.85

For the quarter ellipse of Problem 9.67, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

SOLUTION

From Problem 9.79: $I_x = \frac{\pi}{8}a^4 \qquad I_y = \frac{\pi}{2}a^4$

Problem 9.67: $I_{xy} = \frac{1}{2}a^4$

Now, Equation (9.25):
$$\tan 2\theta_m = -\frac{2I_{xy}}{I_x - I_y} = -\frac{2\left(\frac{1}{2}a^4\right)}{\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4}$$

$$= \frac{8}{3\pi} = 0.84883$$

Then $\quad 2\theta_m = 40.326° \quad$ and $\quad 220.326°$

or $\theta_m = 20.2°$ and $110.2°$ ◀

Also, Equation (9.27):
$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

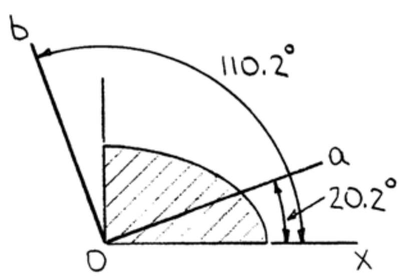
$$= \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right)$$

$$\pm \sqrt{\left[\frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right)\right]^2 + \left(\frac{1}{2}a^4\right)^2}$$

$$= (0.981748 \pm 0.772644)a^4$$

or $I_{\max} = 1.754a^4$ ◀

and $I_{\min} = 0.209a^4$ ◀



By inspection, the a axis corresponds to $I_{\min}$ and the b axis corresponds to $I_{\max}$.

PROBLEM 9.86

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.72

SOLUTION

From the solutions to Problem 9.72 and 9.80

$$\bar{I}_{xy} = 501.1875 \text{ in}^4 \qquad \frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) = 2812.21875 \text{ in}^4$$

$$\frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right) = -1946.53125 \text{ in}^4$$

Then Equation (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{501.1875}{-1946.53125} = 0.257477$$

or

$$2\theta_m = 14.4387° \qquad \text{and} \qquad 194.4387°$$

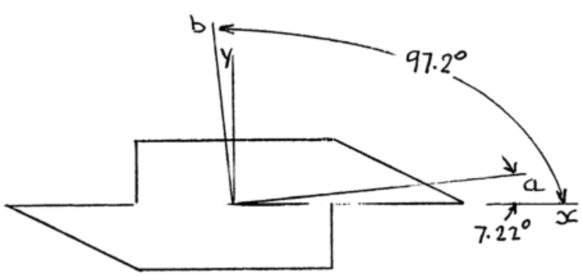
or $\theta_m = 7.22°$ and $97.2°$ ◀

Equation (9.27):

$$\bar{I}_{\text{max, min}} = \frac{\bar{I}_x + \bar{I}_y}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

$$= 2812.21875 \pm \sqrt{(-1946.53125)^2 + (501.1875)^2}$$

$$= (2812.21875 \pm 2010.0181)\text{in}^4$$



or $\bar{I}_{\text{max}} = 4.82 \times 10^3 \text{ in}^4$ ◀

and $\bar{I}_{\text{min}} = 802 \text{ in}^4$ ◀

By inspection, the *a* axis corresponds to $\bar{I}_{\text{min}}$ and the *b* axis corresponds to $\bar{I}_{\text{max}}$.

PROBLEM 9.87

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.73

SOLUTION

From Problems 9.73 and 9.81

$$\bar{I}_x = 51.84\pi \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 103.68\pi \times 10^6 \text{ mm}^4$$

$$\bar{I}_{xy} = 138.24 \times 10^6 \text{ mm}^4$$

Equation (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2\left(138.24 \times 10^6\right)}{51.84\pi \times 10^6 - 103.68\pi \times 10^6}$$

$$= 1.69765$$

$$2\theta_m = 59.500° \qquad \text{and} \qquad 239.500°$$

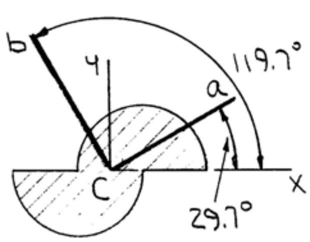
$$\text{or} \quad \theta_m = 29.7° \text{ and } 119.7° \blacktriangleleft$$

Then

$$\bar{I}_{\max,\min} = \frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

$$= \frac{(51.84 + 103.68)\pi \times 10^6}{2} \pm \sqrt{\left[\frac{(51.84 - 103.68)\pi \times 10^6}{2}\right]^2 + \left(138.24 \times 10^6\right)^2}$$

$$= (244.29 \pm 160.44) \times 10^6 \text{ mm}^4$$



$$\text{or} \quad \bar{I}_{\max} = 405 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\text{and} \quad \bar{I}_{\min} = 83.9 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

Note: By inspection the *a* axis corresponds to $\bar{I}_{\min}$ and the *b* axis corresponds to $I_{\max}$.

PROBLEM 9.88

For the area indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

Area of Problem 9.75

SOLUTION

From Problems 9.75 and 9.82

$$\overline{I}_x = 0.70134 \times 10^6 \text{ mm}^4$$

$$\overline{I}_y = 7.728 \times 10^6 \text{ mm}^4$$

$$\overline{I}_{xy} = 1.5732 \times 10^6 \text{ mm}^4$$

Then
$$\frac{1}{2}\left(\overline{I}_x + \overline{I}_y\right) = 4.2147 \times 10^6 \text{ mm}^4$$

$$\frac{1}{2}\left(\overline{I}_x - \overline{I}_y\right) = -3.5133 \times 10^6 \text{ mm}^4$$

Equation (9.25):
$$\tan 2\theta = -\frac{2\overline{I}_{xy}}{\overline{I}_x - \overline{I}_y} = -\frac{2\left(1.5732 \times 10^6\right)}{0.70134 \times 10^6 - 7.728 \times 10^6}$$

$$= 0.44778$$

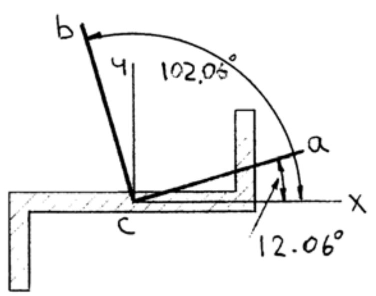
Then
$$2\theta_m = 24.12° \quad \text{and} \quad 204.12°$$

or $\theta_m = 12.06°$ and $102.1°$ ◀

Also, Equation (9.27):
$$\overline{I}_{\text{max, min}} = \frac{\overline{I}_x + \overline{I}_y}{2} \pm \sqrt{\left(\frac{\overline{I}_x - \overline{I}_y}{2}\right)^2 \overline{I}_{xy}^2}$$

$$= 4.2147 \times 10^6 \pm \sqrt{\left(-3.5133 \times 10^6\right)^2 + \left(1.5732 \times 10^6\right)^2}$$

$$= \left(4.2147 \pm 3.8494\right) \times 10^6 \text{ mm}^4$$



or $\overline{I}_{\text{max}} = 8.06 \times 10^6 \text{ mm}^4$ ◀

and $\overline{I}_{\text{min}} = 0.365 \times 10^6 \text{ mm}^4$ ◀

By inspection, the *a* axis corresponds to $\overline{I}_{\text{min}}$ and the *b* axis corresponds to $\overline{I}_{\text{max}}$.

PROBLEM 9.89

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The L76 × 51 × 6.4-mm angle cross section of Problem 9.74

SOLUTION

From Problems 9.74 and 9.83

$$\bar{I}_x = 0.166 \times 10^6 \text{ mm}^4$$

$$\bar{I}_y = 0.453 \times 10^6 \text{ mm}^4$$

$$\bar{I}_{xy} = -0.1596 \times 10^6 \text{ mm}^4$$

Then
$$\frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) = 0.3095 \times 10^6 \text{ mm}^4$$

$$\frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right) = -0.1435 \times 10^6 \text{ mm}^4$$

Equation (9.25):
$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{I_x - I_y} = -\frac{2\left(-0.1596 \times 10^6\right)}{(0.166 - 0.453) \times 10^6} = -1.1122$$

Then
$$2\theta_m = -48.041° \quad \text{and} \quad 131.96°$$

or
$$\theta_m = -24.0° \text{ and } 66.0° \blacktriangleleft$$

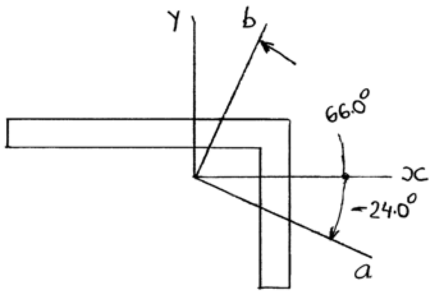
Also, Equation (9.27):
$$\bar{I}_{\max,\min} = \frac{\left(\bar{I}_x + \bar{I}_y\right)}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + \bar{I}_{xy}^2}$$

$$= 0.3095 \times 10^6 \pm \sqrt{\left(-0.1435 \times 10^6\right)^2 + \left(-0.1596 \times 10^6\right)^2}$$

$$= (0.3095 \pm 0.21463) \times 10^6 \text{ mm}^4$$

or
$$\bar{I}_{\max} = 0.524 \times 10^6 \text{ mm}^4 \blacktriangleleft$$

$$\bar{I}_{\min} = 0.0949 \times 10^6 \text{ mm}^4 \blacktriangleleft$$



By inspection, the *a* axis corresponds to $\bar{I}_{\min}$ and the *b* axis corresponds to $\bar{I}_{\max}$.

PROBLEM 9.90

For the angle cross section indicated, determine the orientation of the principal axes at the origin and the corresponding values of the moments of inertia.

The $L5 \times 3 \times \frac{1}{2}$-in. angle cross section of Problem 9.78

SOLUTION

From Problems 9.78 and 9.84

$$\bar{I}_{xy} = 2.81 \text{ in}^4$$

$$\bar{I}_x = 9.45 \text{ in}^4$$

$$\bar{I}_y = 2.58 \text{ in}^4$$

Then

$$\frac{1}{2}\left(\bar{I}_x + \bar{I}_y\right) = 6.015 \text{ in}^4$$

$$\frac{1}{2}\left(\bar{I}_x - \bar{I}_y\right) = 3.435 \text{ in}^4$$

Equation (9.25):

$$\tan 2\theta_m = -\frac{2\bar{I}_{xy}}{\bar{I}_x - \bar{I}_y} = -\frac{2(2.81)}{9.45 - 2.58} = -0.8180$$

Then

$$2\theta_m = -39.2849 \quad \text{and} \quad 140.7151$$

or $\theta_m = -19.64$ and 70.36 ◀

Also, Equation (9.27):

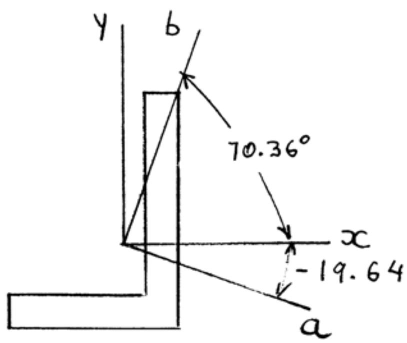
$$\bar{I}_{\max,\min} = \frac{\left(\bar{I}_x + \bar{I}_y\right)}{2} \pm \sqrt{\left(\frac{\bar{I}_x - \bar{I}_y}{2}\right)^2 + I_{xy}^2}$$

$$= 6.015 \pm \sqrt{3.435^2 - 2.81^2}$$

$$= \left(6.015 \pm 4.438\right)\text{in}^4$$

or $\bar{I}_{\max} = 10.45 \text{ in}^4$ ◀

and $\bar{I}_{\min} = 1.577 \text{ in}^4$ ◀



Note: By inspection, the a axis corresponds to $I_{\max}$ and the b axis corresponds to $I_{\min}$.