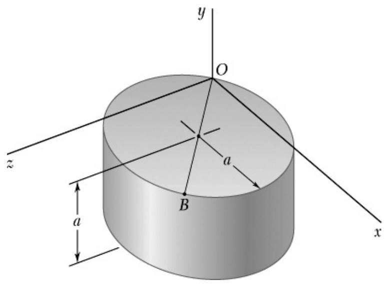
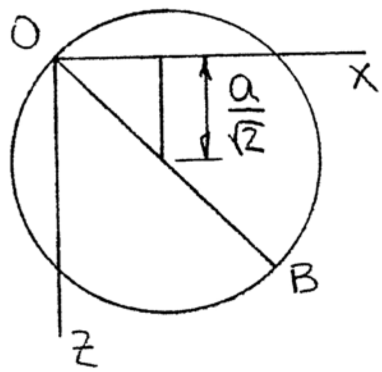
PROBLEM 9.181



The homogeneous circular cylinder shown has a mass m, and the diameter OB of its top surface forms 45° angles with the x and z axes. (a) Determine the principal moments of inertia of the cylinder at the origin O. (b) Compute the angles that the principal axes of inertia at O form with the coordinate axes. (c) Sketch the cylinder, and show the orientation of the principal axes of inertia relative to the x, y, and z axes.

SOLUTION



(a) First compute the moments of inertia using Figure 9.28 and the parallel-axis theorem.

$$I_x = I_z = \frac{1}{12}m\left(3a^2 + a^2\right) + m\left[\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{2}\right)^2\right] = \frac{13}{12}ma^2$$

$$I_y = \frac{1}{2}ma^2 + m\left(a\right)^2 = \frac{3}{2}ma^2$$

Next observe that the centroidal products of inertia are zero because of symmetry. Then

$$I_{xy} = \cancelto{0}{\bar{I}_{x'y'}} + m\bar{x}\,\bar{y} = m\left(\frac{a}{\sqrt{2}}\right)\left(-\frac{a}{2}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{yz} = \cancelto{0}{\bar{I}_{y'z'}} + m\bar{y}\,\bar{z} = m\left(-\frac{a}{2}\right)\left(\frac{a}{\sqrt{2}}\right) = -\frac{1}{2\sqrt{2}}ma^2$$

$$I_{zx} = \cancelto{0}{\bar{I}_{z'x'}} + m\bar{z}\,\bar{x} = m\left(\frac{a}{\sqrt{2}}\right)\left(\frac{a}{\sqrt{2}}\right) = \frac{1}{2}ma^2$$

Substituting into Equation (9.56)

$$K^3 - \left(\frac{13}{12} + \frac{3}{2} + \frac{13}{12}\right)ma^2K^2$$

$$+\left[\left(\frac{13}{12}\times\frac{3}{2}\right) + \left(\frac{3}{2}\times\frac{13}{12}\right) + \left(\frac{13}{12}\times\frac{13}{12}\right) - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2\right]\left(ma^2\right)^2 K$$

$$-\left[\left(\frac{13}{12}\times\frac{3}{2}\times\frac{13}{12}\right) - \left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2 - \left(\frac{13}{12}\right)\left(-\frac{1}{2\sqrt{2}}\right)^2 - 2\left(-\frac{1}{2\sqrt{2}}\right)\left(-\frac{1}{2\sqrt{2}}\right)\left(\frac{1}{2}\right)\right]\left(ma^2\right)^3 = 0$$

Simplifying and letting $K = ma^2\zeta$ yields

$$\zeta^3 - \frac{11}{3}\zeta^2 + \frac{565}{144}\zeta - \frac{95}{96} = 0$$

PROBLEM 9.181 CONTINUED

Solving yields

$$\zeta_1 = 0.363383 \qquad \zeta_2 = \frac{19}{12} \qquad \zeta_3 = 1.71995$$

The principal moments of inertia are then

$$K_1 = 0.363ma^2 \blacktriangleleft$$

$$K_2 = 1.583ma^2 \blacktriangleleft$$

$$K_3 = 1.720ma^2 \blacktriangleleft$$

(*b*) To determine the direction cosines $\lambda_x,\ \lambda_y,\ \lambda_z$ of each principal axis, we use two of the equations of Equations (9.54) and Equation (9.57). Thus

$$(I_x - K)\lambda_x - I_{xy}\lambda_y - I_{zx}\lambda_z = 0 \qquad (9.54a)$$

$$-I_{zx}\lambda_x - I_{yz}\lambda_y + (I_z - K)\lambda_z = 0 \qquad (9.54c)$$

$$\lambda_x^2 + \lambda_y^2 + \lambda_z^2 = 1 \qquad (9.57)$$

Note: Since $I_{xy} = I_{yz}$, Equations (9.54a) and (9.54c) were chosen to simplify the "elimination" of λ_y during the solution process.

Substituting for the moments and products of inertia in Equations (9.54a) and (9.54c)

$$\left(\frac{13}{12}ma^2 - K\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y - \left(\frac{1}{2}ma^2\right)\lambda_z = 0$$

$$-\left(\frac{1}{2}ma^2\right)\lambda_x - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_y + \left(\frac{13}{12}ma^2 - K\right)\lambda_z = 0$$

or

$$\left(\frac{13}{12} - \zeta\right)\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y - \frac{1}{2}\lambda_z = 0 \qquad (i)$$

and

$$-\frac{1}{2}\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y + \left(\frac{13}{12} - \zeta\right)\lambda_z = 0 \qquad (ii)$$

Observe that these equations will be identical, so that one will need to be replaced, if

$$\frac{13}{12} - \zeta = -\frac{1}{2} \qquad \text{or} \qquad \zeta = \frac{19}{12}$$

Thus, a third independent equation will be needed when the direction cosines associated with K_2 are determined. Then for K_1 and K_3

PROBLEM 9.181 CONTINUED

Eq.(i) – Eq.(ii) $\left[\frac{13}{12} - \zeta - \left(-\frac{1}{2}\right)\right]\lambda_x + \left[-\frac{1}{2} - \left(\frac{13}{12} - \zeta\right)\right]\lambda_z = 0$

or $\lambda_z = \lambda_x$

Substituting into Eq.(i) $\left(\frac{13}{12} - \zeta\right)\lambda_x + \frac{1}{2\sqrt{2}}\lambda_y - \frac{1}{2}\lambda_x = 0$

or $\lambda_y = 2\sqrt{2}\left(\zeta - \frac{7}{12}\right)\lambda_x$

Substituting into Equation (9.57)

$$\lambda_x^2 + \left[2\sqrt{2}\left(\zeta - \frac{7}{12}\right)\lambda_x\right]^2 + \left(\lambda_x\right)^2 = 1$$

or $$\left[2 + 8\left(\zeta - \frac{7}{12}\right)^2\right]\lambda_x^2 = 1 \qquad \text{(iii)}$$

K$_1$: Substituting the value of ζ_1 into Eq.(iii)

$$\left[2 + 8\left(0.363383 - \frac{7}{12}\right)^2\right]\left(\lambda_x\right)_1^2 = 1$$

or $\left(\lambda_x\right)_1 = \left(\lambda_z\right)_1 = 0.647249$

and then $\left(\lambda_y\right)_1 = 2\sqrt{2}\left(0.363383 - \frac{7}{12}\right)(0.647249)$

$= -0.402662$

$\therefore \quad \left(\theta_x\right)_1 = \left(\theta_z\right)_1 = 49.7° \quad \left(\theta_y\right)_1 = 113.7°$ ◀

K$_3$: Substituting the value of ζ_3 into Eq.(iii)

$$\left[2 + 8\left(1.71995 - \frac{7}{12}\right)^2\right]\left(\lambda_x\right)_3^2 = 1$$

or $\left(\lambda_x\right)_3 = \left(\lambda_z\right)_3 = 0.284726$

and then $\left(\lambda_y\right)_3 = 2\sqrt{2}\left(1.71995 - \frac{7}{12}\right)(0.284726)$

$= 0.915348$

$\therefore \quad \left(\theta_x\right)_3 = \left(\theta_z\right)_3 = 73.5° \quad \left(\theta_y\right)_3 = 23.7°$ ◀

PROBLEM 9.181 CONTINUED

$\mathbf{K}_2$: For this case, the set of equations to be solved consists of Equations (9.54a), (9.54b), and (9.57). Now

$$-I_{xy}\lambda_x + \left(I_y - K\right)\lambda_y - I_{yz}\lambda_z = 0 \qquad (9.54b)$$

Substituting for the moments and products of inertia.

$$-\left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_x + \left(\frac{3}{2}ma^2 - K\right)\lambda_y - \left(-\frac{1}{2\sqrt{2}}ma^2\right)\lambda_z = 0$$

or
$$\frac{1}{2\sqrt{2}}\lambda_x + \left(\frac{3}{2} - \zeta\right)\lambda_y + \frac{1}{2\sqrt{2}}\lambda_z = 0 \qquad (iv)$$

Substituting the value of ζ_2 into Eqs.(i) and (iv)

$$\left(\frac{13}{12} - \frac{19}{12}\right)(\lambda_x)_2 + \frac{1}{2\sqrt{2}}\left(\lambda_y\right)_2 - \frac{1}{2}(\lambda_z)_2 = 0$$

$$\frac{1}{2\sqrt{2}}(\lambda_x)_2 + \left(\frac{3}{2} - \frac{19}{12}\right)\left(\lambda_y\right)_2 + \frac{1}{2\sqrt{2}}(\lambda_z)_2 = 0$$

or
$$-(\lambda_x)_2 + \frac{1}{\sqrt{2}}\left(\lambda_y\right)_2 - (\lambda_z)_2 = 0$$

and
$$(\lambda_x)_2 - \frac{\sqrt{2}}{6}\left(\lambda_y\right)_2 + (\lambda_z)_2 = 0$$

Adding yields
$$\left(\lambda_y\right)_2 = 0$$

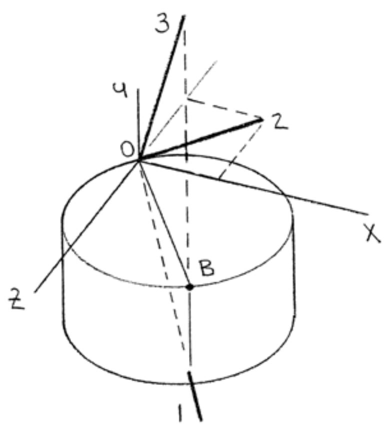
and then
$$(\lambda_z)_2 = -(\lambda_x)_2$$

Substituting into Equation (9.57)

$$(\lambda_x)_2^2 + \cancel{\left(\lambda_y\right)_2^2}^{\,0} + (-\lambda_x)_2^2 = 1$$

or
$$(\lambda_x)_2 = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad (\lambda_z)_2 = -\frac{1}{\sqrt{2}}$$

$$\therefore \quad (\theta_x)_2 = 45.0° \quad \left(\theta_y\right)_2 = 90.0° \quad (\theta_z)_2 = 135.0° \blacktriangleleft$$



(*c*) Principal axes 1 and 3 lie in the vertical plane of symmetry passing through points *O* and *B*. Principal axis 2 lies in the *xz* plane.

PROBLEM 9.182

Prob. 9.167
For the component described in the problem indicated, determine (*a*) the principal moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

From the solution to Problem 9.143 and 9.167

$$I_x = 34.106 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$
$$I_y = 50.125 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$
$$I_z = 34.876 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$
$$I_{xy} = 0.96211 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$
$$I_{yz} = I_{zx} = 0$$

(*a*) From Equation 9.55

$$\begin{vmatrix} I_x - K & I_{xy} & 0 \\ I_{xy} & I_y - K & 0 \\ 0 & 0 & I_z - K \end{vmatrix} = 0$$

or $$(I_x - K)(I_y - K)(I_z - K) - (I_z - K)I_{xy}^2 = 0$$

or $$(I_z - K)\left[(I_x - K)(I_y - K) - I_{xy}^2\right] = 0$$

Then $$I_z - K = 0 \text{ and } I_xI_y - (I_x + I_y)K + K^2 - I_{xy}^2 = 0$$

Now $$K_1 = I_z = 34.876 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

or $K_1 = 34.9 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

and

$$(34.876 \times 10^{-3})(50.125 \times 10^{-3}) - (34.106 \times 10^{-3} + 50.125 \times 10^{-3})K$$
$$+ K^2 - (0.96211 \times 10^{-3})^2 = 0$$

or $$1.70864 \times 10^{-3} - 84.231 \times 10^{-3}K + K^2 = 0$$

Solving yields $$K_2 = 34.0486 \times 10^{-3} \quad K_3 = 50.1824 \times 10^{3}$$

or $K_2 = 34.0 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

and $K_3 = 50.2 \times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$ ◀

PROBLEM 9.182 CONTINUED

(b) To determine the directions cosines λ_x, λ_y and λ_z of each principal axis use two of the Equations 9.54 and Equation 9.57

$\mathbf{K}_1$: Using Equation 9.54(a) and Equation 9.54(b) with $I_{yz} = I_{zx} = 0$, we have

$$\left(I_x - K_1\right)\left(\lambda_x\right)_1 - I_{xy}\left(\lambda_y\right)_1 = 0$$

$$-I_{xy}\left(\lambda_x\right)_1 + \left(I_y - K_1\right)\left(\lambda_y\right)_1 = 0$$

Substituting

$$\left(34.106 \times 10^{-3} - 34.876 \times 10^{-3}\right)\left(\lambda_x\right)_1 - 0.96211 \times 10^{-3}\left(\lambda_y\right)_1 = 0$$

$$-0.96211 \times 10^{-3}\left(\lambda_x\right)_1 + \left(50.125 \times 10^{-3} - 34.876 \times 10^{-3}\right)\left(\lambda_y\right)_1 = 0$$

or

$$-0.770 \times 10^{-3}\left(\lambda_x\right)_1 - 0.96211 \times 10^{-3}\left(\lambda_y\right)_1 = 0$$

$$-0.96211 \times 10^{-3}\left(\lambda_x\right)_1 + 15.249 \times 10^{-3}\left(\lambda_y\right)_1 = 0$$

Solving yields $\quad \left(\lambda_x\right)_1 = \left(\lambda_y\right)_1 = 0$

From Equation 9.57 $\quad \cancelto{0}{\left(\lambda_x\right)_1^2} + \cancelto{0}{\left(\lambda_y\right)_1^2} + \left(\lambda_z\right)_1^2 = 1 \quad \text{or} \quad \left(\lambda_z\right)_1 = 1$

and $\quad \left(\theta_x\right)_1 = 90.0°, \quad \left(\theta_y\right)_1 = 90.0°, \quad \left(\theta_z\right)_1 = 0°$ ◀

$\mathbf{K}_2$: Using Equation 9.54(b) and Equation 9.54(c) with $I_{yz} = I_{zx} = 0$

$$-I_{xz}\left(\lambda_x\right)_2 + \left(I_y - k_2\right)\left(\lambda_y\right)_2 = 0$$

$$\left(I_z - K_2\right)\left(\lambda_z\right)_2 = 0$$

Now $\quad I_z \neq K_2 \Rightarrow \left(\lambda_z\right)_2 = 0$

Substituting

$$-0.96211 \times 10^{-3}\left(\lambda_x\right)_2 + \left(50.125 \times 10^{-3} - 34.0486 \times 10^{-3}\right)\left(\lambda_y\right)_2 = 0$$

or $\quad \left(\lambda_y\right)_2 = 0.05985\left(\lambda_x\right)_2$

PROBLEM 9.182 CONTINUED

Then $$(\lambda_x)_2^2 + \left[0.05985(\lambda_x)_2\right]^2 + \cancelto{0}{(\lambda_z)_2} = 1$$

$$(\lambda_x)_2 = 0.99821$$

$$(\lambda_y)_2 = 0.05974$$

and $$(\theta_x)_2 = 3.43°,\quad (\theta_y)_2 = 86.6°,\quad (\theta_z)_2 = 90.0° \blacktriangleleft$$

$\mathbf{K}_3$: $$-I_{xy}(\lambda_x)_3 + (I_y - K_3)(\lambda_y)_3 = 0$$

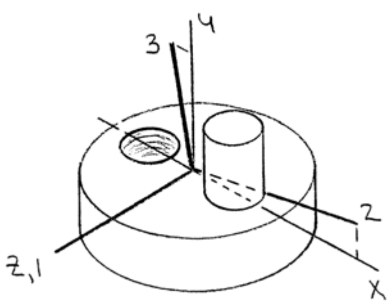
$$(I_z - K_3)(\lambda_z)_3 = 0$$

Now $$I_z \neq K_3 \Rightarrow (\lambda_z)_3 = 0$$

Substituting

$$-0.96211\times10^{-3}(\lambda_x)_3 + \left(50.125\times10^{-3} - 50.1824\times10^{-3}\right)(\lambda_y)_3 = 0$$

$$-0.96211\times10^{-3}(\lambda_x)_3 - 0.0574\times10^{-3}(\lambda_y)_3 = 0$$



or $$(\lambda_y)_3 = -16.7615(\lambda_x)_3$$

Have $$(\lambda_x)_3^2 + \left[-16.7615(\lambda_x)_3\right]^2 + \cancelto{0}{(\lambda_z)_3^2} = 1$$

yields $(\lambda_x)_3 = -0.059555$ and $(\lambda_y)_3 = 0.998231$

and $$(\theta_x)_3 = 93.4°,\quad (\theta_y)_3 = 3.41°,\quad \theta_z = 90.0° \blacktriangleleft$$

(c) Principal axis 1 coincides with the z axis, while the principal axes 2 and 3 lie in the xy plane

PROBLEM 9.183

Prob. 9.147 and 9.151
For the component described in the problem indicated, determine (*a*) the principal moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the x, y, and z axes.

SOLUTION

From Problem 9.147:

$$I_x = 9.8821\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_y = 11.5344\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_z = 2.1878\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

From Problem 9.151:

$$I_{xy} = 0.48776\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_{yz} = 1.18391\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

$$I_{zx} = 2.6951\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

(*a*) From Equation 9.56

$$K^3 - \left(I_x + I_y + I_z\right)K^2 + \left(I_xI_y + I_yI_z + I_zI_x - I_{xy}^2 - I_{yz}^2 - I_{zx}^2\right)K$$

$$-\left(I_xI_yI_z - I_xI_{yz}^2 - I_yI_{zx}^2 - I_zI_{xy}^2 - 2I_{xy}I_{yz}I_{zx}\right) = 0$$

Substituting

$$K^3 - \left[(9.8821 + 11.5344 + 2.1878)\times 10^{-3}\right]K^2 + \left\{\left[(9.8821)(11.5344) + (11.5344)(2.1878)\right.\right.$$

$$\left.\left. + (2.1878)(9.8821) - (0.48776)^2 - (1.18391)^2 - (2.6951)^2\right]\times 10^{-6}\right\}K$$

$$-\left[(9.8821)(11.5344)(2.1878) - (9.8821)(1.18391)^2 - (11.5344)(2.6951)^2\right.$$

$$\left. - (2.1878)(0.48776)^2 - 2(0.48776)(1.18391)(2.6951)\right]\times 10^{-9} = 0$$

or

$$K^3 - \left(23.6043\times 10^{-3}\right)K^2 + \left(151.9360\times 10^{-6}\right)K - 148.1092\times 10^{-9} = 0$$

Solving numerically

$$K_1 = 1.180481\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2 \quad \text{or} \quad K_1 = 1.180\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

$$K_2 = 10.72017\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2 \quad \text{or} \quad K_2 = 10.72\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

$$K_3 = 11.70365\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2 \quad \text{or} \quad K_3 = 11.70\times 10^{-3} \text{ lb}\cdot\text{ft}\cdot\text{s}^2 \blacktriangleleft$$

PROBLEM 9.183 CONTINUED

Solving numerically

$$K_1 = 1.180481 \times 10^{-3} \text{ lb·ft·s}^2 \qquad \text{or} \qquad K_1 = 1.180 \times 10^{-3} \text{ lb·ft·s}^2 \blacktriangleleft$$

$$K_2 = 10.72017 \times 10^{-3} \text{ lb·ft·s}^2 \qquad \text{or} \qquad K_2 = 10.72 \times 10^{-3} \text{ lb·ft·s}^2 \blacktriangleleft$$

$$K_3 = 11.70365 \times 10^{-3} \text{ lb·ft·s}^2 \qquad \text{or} \qquad K_3 = 11.70 \times 10^{-3} \text{ lb·ft·s}^2 \blacktriangleleft$$

(*b*) From Equations 9.54(a) and 9.54(b)

$$\left(I_x - K\right)\left(\lambda_x\right) - I_{xy}\left(\lambda_y\right) - I_{zx}\left(\lambda_z\right) = 0$$

$$-I_{xy}\left(\lambda_x\right) + \left(I_y - K_1\right)\left(\lambda_y\right) - I_{yz}\left(\lambda_z\right) = 0$$

$\mathbf{K}_1$: Substitute K_1 and solve for λ to get $\left(\lambda_x\right)$, $\left(\lambda_{xy}\right)$ and $\left(\lambda_y\right)_3$.

$$\left[\left(9.8821 - 1.180481\right)\left(\lambda_x\right)_1 - 0.48776\left(\lambda_y\right)_1 - 2.6951\left(\lambda_z\right)_1\right] \times 10^{-3} = 0$$

$$\left[-0.48776\left(\lambda_x\right)_1 + \left(11.5344 - 1.180481\right)\left(\lambda_y\right)_1 - 1.18391\left(\lambda_z\right)_1\right] \times 10^{-3} = 0$$

or

$$17.83996\left(\lambda_x\right)_1 - \left(\lambda_y\right)_1 - 5.52546\left(\lambda_z\right)_1 = 0$$

$$-0.0471\left(\lambda_x\right)_1 + \left(\lambda_{y1}\right)_1 - 0.11434\left(\lambda_z\right)_1 = 0$$

Then

$$\left(\lambda_z\right)_1 = 3.1549\left(\lambda_x\right)_1$$

and

$$\left(\lambda_y\right)_1 = 0.40769\left(\lambda_x\right)_1$$

Equation 9.57:

$$\left(\lambda_x\right)_1^2 + \left(\lambda_y\right)_1^2 + \left(\lambda_z\right)_1^2 = 1$$

Substituting

$$\left(\lambda_x\right)_1^2 + \left[0.40769\left(\lambda_x\right)_1\right]^2 + \left[3.1549\left(\lambda_x\right)_1\right]^2 = 1$$

$$\text{or} \quad \left(\lambda_x\right)_1 = 0.29989 \qquad \text{then} \qquad \left(\theta_x\right)_1 = 72.5° \blacktriangleleft$$

$$\text{and} \quad \left(\lambda_y\right)_1 = 0.122262 \qquad \text{then} \qquad \left(\theta_y\right)_1 = 83.0° \blacktriangleleft$$

$$\left(\lambda_z\right)_1 = 0.94612 \qquad \text{then} \qquad \left(\theta_z\right)_1 = 18.89° \blacktriangleleft$$

$\mathbf{K}_2$: Substitute K_2 and solve for λ.

$$\left[\left(9.8821 - 10.72017\right)\left(\lambda_x\right)_2 - 0.48776\left(\lambda_y\right)_2 - 2.6951\left(\lambda_z\right)_2\right] \times 10^{-3} = 0$$

$$\left[-0.48776\left(\lambda_x\right)_2 + \left(11.5344 - 10.72017\right)\left(\lambda_y\right)_2 - 1.18391\left(\lambda_z\right)_2\right] \times 10^{-3} = 0$$

PROBLEM 9.183 CONTINUED

or $$-1.718202(\lambda_x)_2 - (\lambda_y)_2 - 5.52546(\lambda_z)_2 = 0$$

$$-0.599045(\lambda_x)_2 + (\lambda_y)_2 - 1.45402(\lambda_z)_2 = 0$$

Then $$(\lambda_z)_2 = -0.33201(\lambda_x)_2$$

and $$(\lambda_y)_2 = 0.116306(\lambda_x)_2$$

Then $$(\lambda_x)_2^2 + \left[0.116306(\lambda_x)_2\right]^2 + \left[-0.33201(\lambda_x)_2\right]^2 = 1$$

or $(\lambda_x)_2 = 0.94333$ then $(\theta_x)_2 = 19.38°$ ◀

And $(\lambda_y)_2 = 0.109715$ then $(\theta_y)_2 = 83.7°$ ◀

$(\lambda_z)_2 = -0.31320$ then $(\theta_z)_2 = 108.3°$ ◀

K$_3$: Substitute K_3 and solve for λ.

$$\left[(9.8821 - 11.70365)(\lambda_x)_3 - 0.48776(\lambda_y)_3 - 2.6951(\lambda_z)_3\right] \times 10^{-3} = 0$$

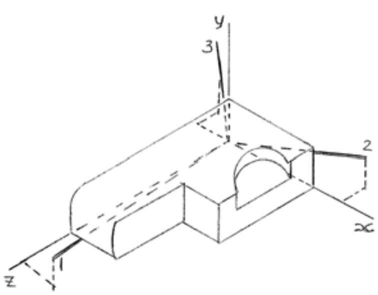
$$\left[-0.48776(\lambda_x)_3 + (11.5344 - 11.70365)(\lambda_y)_3 - 1.18391(\lambda_z)_3\right] \times 10^{-3} = 0$$

or $$-3.73452(\lambda_x)_3 - (\lambda_y)_3 - 5.52546(\lambda_z)_3 = 0$$

$$2.88189(\lambda_x)_3 + (\lambda_y)_3 + 6.99504(\lambda_z)_3 = 0$$

Then $$(\lambda_z)_3 = 0.58019(\lambda_x)_3$$

and $$(\lambda_y)_3 = -6.9403(\lambda_y)_3$$



$$(\lambda_x)_3^2 + \left[-0.69403(\lambda_x)_3\right]^2 + \left[0.58019(\lambda_x)_3\right]^2 = 1$$

or $(\lambda_x)_3 = -0.142128^*$ then $(\theta_x)_3 = 98.2°$ ◀

$(\lambda_y)_3 = 0.98641$ then $(\theta_y)_3 = 9.46°$ ◀

$(\lambda_z)_3 = -0.082461$ then $(\theta_z)_3 = 94.7°$ ◀

*Note: the negative root of $(\lambda_x)_3$ is taken so that axes 1, 2, 3 form a right-handed set.

PROBLEM 9.184

Prob. 9.169

For the component described in the problem indicated, determine (*a*) the principal moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION

(*a*) From the solution of Problem 9.169 have

$$I_x = \frac{1}{2}\frac{W}{g}a^2 \qquad I_{xy} = \frac{1}{4}\frac{W}{g}a^2$$

$$I_y = \frac{W}{g}a^2 \qquad I_{yz} = \frac{1}{8}\frac{W}{g}a^2$$

$$I_z = \frac{5}{6}\frac{W}{g}a^2 \qquad I_{zx} = \frac{3}{8}\frac{W}{g}a^2$$

Substituting into Equation (9.56)

$$K^3 - \left[\left(\frac{1}{2} + 1 + \frac{5}{6}\right)\left(\frac{W}{g}a^2\right)\right]K^2 + \left[\left(\frac{1}{2}\right)(1) + (1)\left(\frac{5}{6}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{4}\right)^2 - \left(\frac{1}{8}\right)^2 - \left(\frac{3}{8}\right)^2\right]\left(\frac{W}{g}a^2\right)^2 K$$

$$-\left[\left(\frac{1}{2}\right)(1)\left(\frac{5}{6}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{8}\right)^2 - (1)\left(\frac{3}{8}\right)^2 - \left(\frac{5}{6}\right)\left(\frac{1}{4}\right)^2 - 2\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)\left(\frac{3}{8}\right)\right]\left(\frac{W}{g}a^2\right)^3 = 0$$

Simplifying and letting $K = \frac{W}{g}a^2\zeta$ yields

$$\zeta^3 - 2.33333\zeta^2 + 1.53125\zeta - 0.192708 = 0$$

Solving yields

$$\zeta_1 = 0.163917 \qquad \zeta_2 = 1.05402 \qquad \zeta_3 = 1.11539$$

The principal moments of inertia are then

$$K_1 = 0.1639\frac{W}{g}a^2 \blacktriangleleft$$

$$K_2 = 1.054\frac{W}{g}a^2 \blacktriangleleft$$

$$K_3 = 1.115\frac{W}{g}a^2 \blacktriangleleft$$

PROBLEM 9.184 CONTINUED

(*b*) To determine the direction cosines $\lambda_x,\ \lambda_y,\ \lambda_z$ of each principal axis, use two of the equations of Equations (9.54) and (9.57). Then

$\mathbf{K}_1$: Begin with Equations (9.54a) and (9.54b).

$$(I_x - K_1)(\lambda_x)_1 - I_{xy}(\lambda_y)_1 - I_{zx}(\lambda_z)_1 = 0$$

$$-I_{xy}(\lambda_x)_1 + (I_y - K_2)(\lambda_y)_1 - I_{yz}(\lambda_z)_1 = 0$$

Substituting

$$\left[\left(\frac{1}{2} - 0.163917\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_1 - \left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_1 - \left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_1 = 0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_1 + \left[(1 - 0.163917)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_1 - \left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_1 = 0$$

Simplifying yields

$$1.34433(\lambda_x)_1 - (\lambda_y)_1 - 1.5(\lambda_z)_1 = 0$$

$$-0.299013(\lambda_x)_1 + (\lambda_y)_1 - 0.149507(\lambda_z)_1 = 0$$

Adding and solving for $(\lambda_z)_1$

$$(\lambda_z)_1 = 0.633715(\lambda_x)_1$$

and then

$$(\lambda_y)_1 = [1.34433 - 1.5(0.633715)](\lambda_x)_1$$

$$= 0.393758(\lambda_x)_1$$

Now substitute into Equation (9.57)

$$(\lambda_x)_1^2 + [0.393758(\lambda_x)_1]^2 + [0.633715(\lambda_x)_1]^2 = 1$$

or

$$(\lambda_x)_1 = 0.801504$$

and

$$(\lambda_y)_1 = 0.315599 \qquad (\lambda_z)_1 = 0.507925$$

$$\therefore\quad (\theta_x)_1 = 36.7° \qquad (\theta_y)_1 = 71.6° \qquad (\theta_z)_1 = 59.5° \blacktriangleleft$$

$\mathbf{K}_2$: Begin with Equations (9.54a) and (9.54b).

$$(I_x - K_2)(\lambda_x)_2 - I_{xy}(\lambda_y)_2 - I_{zx}(\lambda_z)_2 = 0$$

$$-I_{xy}(\lambda_x)_2 + (I_y - K_2)(\lambda_y)_2 - I_{yz}(\lambda_z)_2 = 0$$

PROBLEM 9.184 CONTINUED

Substituting

$$\left[\left(\frac{1}{2}-1.05402\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_2-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_2-\left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_2=0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_2+\left[(1-1.05402)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_2-\left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_2=0$$

Simplifying yields

$$-2.21608(\lambda_x)_2-(\lambda_y)_2-1.5(\lambda_z)_2=0$$

$$4.62792(\lambda_x)_2+(\lambda_y)_2+2.31396(\lambda_z)_2=0$$

Adding and solving for $(\lambda_z)_2$

$$(\lambda_z)_2=-2.96309(\lambda_x)_2$$

and then

$$(\lambda_y)_2=[-2.21608-1.5(-2.96309)](\lambda_x)_2$$

$$=2.22856(\lambda_x)_2$$

Now substitute into Equation (9.57)

$$(\lambda_x)_2^2+\left[2.22856(\lambda_x)_2\right]^2+\left[-2.96309(\lambda_x)_2\right]^2=1$$

or

$$(\lambda_x)_2=0.260410$$

and

$$(\lambda_y)_2=0.580339 \qquad (\lambda_z)_2=-0.771618$$

$$\therefore \quad (\theta_x)_2=74.9° \qquad (\theta_y)_2=54.5° \qquad (\theta_z)_2=140.5° \blacktriangleleft$$

$\mathbf{K}_3$: Begin with Equations (9.54a) and (9.54b).

$$(I_x-K_3)(\lambda_x)_3-I_{xy}(\lambda_y)_3-I_{zx}(\lambda_z)_3=0$$

$$-I_{xy}(\lambda_x)_3+(I_y-K_3)(\lambda_y)_3-I_{yz}(\lambda_z)_3=0$$

Substituting

$$\left[\left(\frac{1}{2}-1.11539\right)\left(\frac{W}{g}a^2\right)\right](\lambda_x)_3-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_y)_3-\left(\frac{3}{8}\frac{W}{g}a^2\right)(\lambda_z)_3=0$$

$$-\left(\frac{1}{4}\frac{W}{g}a^2\right)(\lambda_x)_3+\left[(1-1.11539)\left(\frac{W}{g}a^2\right)\right](\lambda_y)_3-\left(\frac{1}{8}\frac{W}{g}a^2\right)(\lambda_z)_3=0$$

PROBLEM 9.184 CONTINUED

Simplifying yields

$$-2.46156(\lambda_x)_3 - (\lambda_y)_3 - 1.5(\lambda_z)_3 = 0$$

$$2.16657(\lambda_x)_3 + (\lambda_y)_3 + 1.08328(\lambda_z)_3 = 0$$

Adding and solving for $(\lambda_z)_3$

$$(\lambda_z)_3 = -0.707885(\lambda_x)_3$$

and then $$(\lambda_y)_3 = [-2.46156 - 1.5(-0.707885)](\lambda_x)_3$$

$$= -1.39973(\lambda_x)_3$$

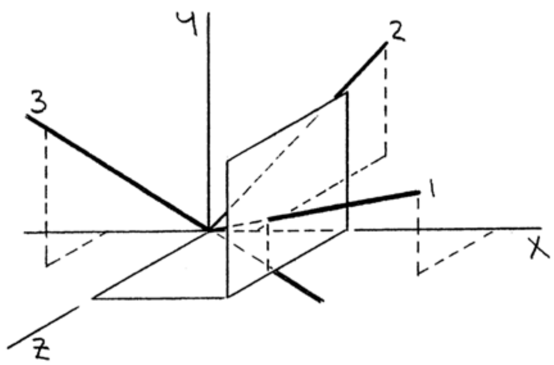
Now substitute into Equation (9.57)

$$(\lambda_x)_3^2 + [-1.39973(\lambda_x)_3]^2 + [-0.707885(\lambda_x)_3]^2 = 1 \qquad \text{(i)}$$

or $$(\lambda_x)_3 = 0.537577$$

and $$(\lambda_y)_3 = -0.752463 \qquad (\lambda_z)_3 = -0.380543$$

$$\therefore \quad (\theta_x)_3 = 57.5° \qquad (\theta_y)_3 = 138.8° \qquad (\theta_z)_3 = 112.4° \blacktriangleleft$$



(*c*) Note: Principal axis 3 has been labeled so that the principal axes form a right-handed set. To obtain the direction cosines corresponding to the labeled axis, the negative root of Equation (i) must be chosen; that is, $(\lambda_x)_3 = -0.537577$

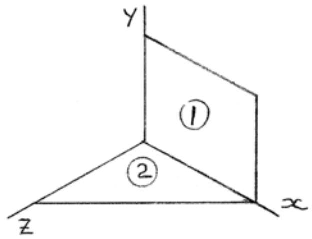
Then $$(\theta_x)_3 = 122.5° \qquad (\theta_y)_3 = 41.2° \qquad (\theta_z)_3 = 67.6°$$

PROBLEM 9.185

Prob. 9.170

For the component described in the problem indicated, determine (*a*) the principal moments of inertia at the origin, (*b*) the principal axes of inertia at the origin. Sketch the body and show the orientation of the principal axes of inertia relative to the *x*, *y*, and *z* axes.

SOLUTION



From Problem 9.170 $$m_1 = \rho t a^2 \qquad m_2 = \frac{1}{2}\rho t a^2$$

Now $$I_x = (I_x)_1 + (I_x)_2 = \frac{1}{3}(\rho t a^2)a^2 + \frac{1}{6}\left(\frac{1}{2}\rho t a^2\right)a^2 = \frac{5}{12}\rho t a^4$$

$$I_y = (I_y)_1 + (I_y)_2 = \frac{1}{3}(\rho t a^2)a^2 + \frac{1}{6}\left(\frac{1}{2}\rho t a^2\right)(a^2 + a^2) = \frac{1}{2}\rho t a^4$$

$$I_z = (I_z)_1 + (I_z)_2 = \frac{1}{3}(\rho t a^2)(a^2 + a^2) + \frac{1}{6}\left(\frac{1}{2}\rho t a^2\right)a^2 = \frac{3}{4}\rho t a^4$$

Now note that symmetry implies $$(\bar{I}_{x'y'})_1 = (\bar{I}_{y'z'})_1 = (\bar{I}_{z'x'})_1 = 0$$

$$(\bar{I}_{x'y'})_2 = (\bar{I}_{y'z'})_2 = 0$$

Have $$I_{uv} = \bar{I}_{u'v'} + m\bar{u}\,\bar{v}$$

PROBLEM 9.185 CONTINUED

Then
$$I_{xy} = m_1\bar{x}_1\bar{y}_1 + m_2\bar{x}_2\cancelto{0}{\bar{y}_2} = \rho ta^2\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) = \frac{1}{4}\rho ta^4$$

$$I_{yx} = m_1\bar{y}_1\cancelto{0}{\bar{z}_1} + m_2\cancelto{0}{\bar{y}_2}\bar{z}_2$$

$$I_{zx} = m_1\cancelto{0}{\bar{z}_1}\bar{x}_1 + \left[\left(\bar{I}_{z'x'}\right)_2 + m_2\bar{z}_2\bar{x}_2\right]$$

From Problem 9.170
$$\left(\bar{I}_{z'x'}\right)_2 = -\frac{1}{72}\rho ta^4$$

Then
$$I_{zx} = -\frac{1}{72}\rho ta^4 + \left(\frac{1}{2}\rho ta^2\right)\left(\frac{1}{3}a\right)\left(\frac{1}{3}a\right) = \frac{1}{24}\rho ta^4$$

(*a*) Equation 9.56

$$K^3 - \left(I_x + I_y + I_z\right)K^2 + \left(I_xI_y + I_yI_z + I_zI_x\right) - I_{xy}^2 - I_{yz}^2 - I_{zx}^2\Big)K$$

$$-\left(I_xI_yI_z - I_xI_{yz}^2 - I_yI_{zx}^2 - I_zI_{xy}^2 - 2I_{xy}I_{yz}I_{zx} = 0\right.$$

Substituting

$$K^3 - \left[\left(\frac{5}{12} + \frac{1}{2} + \frac{3}{4}\right)\rho ta^4\right]K^2 + \left\{\left[\left(\frac{5}{12}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)\left(\frac{5}{12}\right) - \left(\frac{1}{4}\right)^2 - 0 - \left(\frac{1}{24}\right)^2\right]\left(\rho ta^4\right)^2\right\}K$$

$$-\left\{\left[\left(\frac{5}{12}\right)\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) - 0 - \left(\frac{1}{2}\right)\left(\frac{1}{24}\right)^2 - \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 - 0\right]\left(\rho ta^4\right)^3\right\} = 0$$

Simplifying and letting $\qquad K = \rho ta^4\zeta$

yields
$$\zeta^3 = \frac{5}{3}\zeta^2 + \frac{479}{576}\zeta - \frac{125}{1152} = 0$$

Solving numerically...

$\zeta_1 = 0.203032 \quad$ or $\quad K_1 = 0.203\rho ta^4$ ◀

$\zeta_2 = 0.698281 \quad$ or $\quad K_2 = 0.698\rho ta^4$ ◀

$\zeta_3 = 0.765354 \quad$ or $\quad K_3 = 0.765\rho ta^4$ ◀

(*b*) Equations 9.54a and 9.54b

$$\left(I_x - K\right)\left(\lambda_x\right) - I_{xy}\left(\lambda_y\right) - I_{zx}\left(\lambda_z\right) = 0$$

$$-I_{xy}\left(\lambda_x\right) + \left(I_y - K\right)\left(\lambda_y\right) - I_{yz}\left(\lambda_z\right) = 0$$

PROBLEM 9.185 CONTINUED

Substituting K_1

$$\left[\left(\frac{5}{12} - 0.203032\right)(\lambda_x)_1 - \frac{1}{4}(\lambda_y)_1 - \frac{1}{24}(\lambda_z)_1\right]\rho t a^4 = 0$$

$$\left[-\frac{1}{4}(\lambda_x)_1 + \left(\frac{1}{2} - 0.203032\right)(\lambda_y)_1 - 0\right]\rho t a^4 = 0$$

or $$(\lambda_y)_1 = 0.841842(\lambda_x)_1$$

and $$(\lambda_z)_1 = 0.0761800(\lambda_x)_1$$

Equation 9.57 $$(\lambda_x)_1^2 + (\lambda_y)_1^2 + (\lambda_z)_1^2 = 1$$

Substituting $$(\lambda_x)_1^2 + \left[0.841842(\lambda_x)_1\right]^2 + \left[0.0761800(\lambda_x)_1\right]^2 = 1$$

or $(\lambda_x)_1 = 0.763715$ then $(\theta_x)_1 = 40.2°$ ◀

$(\lambda_y)_1 = 0.642927$ then $(\theta_y)_1 = 50.0°$ ◀

$(\lambda_z)_1 = 0.0581798$ then $(\theta_z)_1 = 86.7°$ ◀

Substituting K_2

$$\left[\left(\frac{5}{12} - 0.698281\right)(\lambda_x)_2 - \frac{1}{4}(\lambda_y)_2 - \frac{1}{24}(\lambda_z)_2\right]\rho t a^4 = 0$$

$$\left[-\frac{1}{4}(\lambda_x)_2 + \left(\frac{1}{2} - 0.698281\right)(\lambda_y)_2 - 0\right]\rho t a^4 = 0$$

or $$(\lambda_y)_2 = -1.260837(\lambda_x)_2$$

and $$(\lambda_z)_2 = 0.806278(\lambda_x)_2$$

Then $$(\lambda_x)_2^2 + \left[-1.260837(\lambda_x)_2\right]^2 + \left[0.806278(\lambda_x)_2\right]^2 = 1$$

or $(\lambda_x)_2 = 0.555573$ then $(\theta_x)_2 = 56.2°$ ◀

$(\lambda_y)_2 = -0.700487$ then $(\theta_y)_2 = 134.5°$ ◀

$(\lambda_z)_2 = 0.447946$ then $(\theta_z)_2 = 63.4°$ ◀

PROBLEM 9.185 CONTINUED

Substituting K_3

$$\left[\left(\frac{5}{12} - 0.765354\right)(\lambda_x)_3 - \frac{1}{4}(\lambda_y)_3 - \frac{1}{24}(\lambda_z)_3\right]\rho ta^4 = 0$$

$$\left[-\frac{1}{4}(\lambda_x)_3 + \left(\frac{1}{2} - 0.765354\right)(\lambda_y)_3 - 0\right]\rho ta^4 = 0$$

or $$(\lambda_y)_3 = -0.942138(\lambda_x)_3$$

And $$(\lambda_z)_3 = -2.71567(\lambda_x)_3$$

Then $$(\lambda_x)_3^2 + \left[-0.942138(\lambda_x)_3\right]^2 + \left[-2.71567(\lambda_x)_3\right]^2 = 1$$

or $(\lambda_x)_3 = 0.328576$ then $(\theta_x)_3 = 70.8°$ ◀

$(\lambda_y)_3 = -0.309564$ then $(\theta_y)_3 = 108.0°$ ◀

$(\lambda_z)_3 = -0.892304$ then $(\theta_z)_3 = 153.2°$ ◀

(*c*)