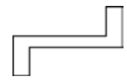
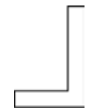
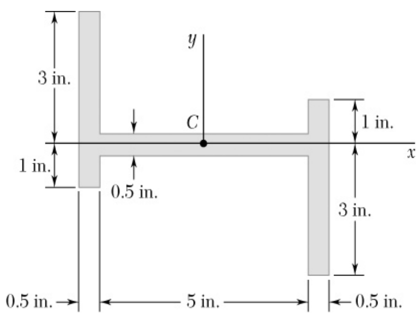
The following table is provided for the convenience of the instructor, as many problems in this and the next lesson are related.

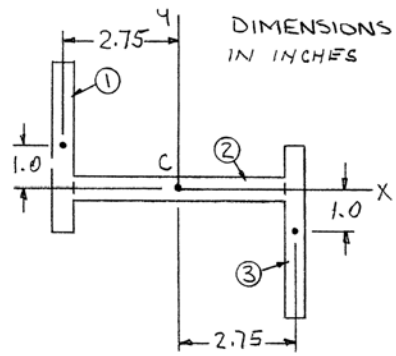
Type of Problem						
Compute I_x and I_y	Fig. 9.12			Fig. 9.13B		Fig. 9.13A
Compute I_{xy}	9.67	9.72	9.73	9.74	9.75	9.78
$I_{x'}$, $I_{y'}$, $I_{x'y'}$ by equations	*9.79*	9.80	*9.81*	9.83	9.82	9.84
Principal axes by equations	*9.85*	9.86	*9.87*	9.89	9.88	9.90
$I_{x'}$, $I_{y'}$, $I_{x'y'}$ by Mohr's circle	*9.91*	9.92	*9.93*	9.95	9.94	9.96
Principal axes by Mohr’s circle	9.97	9.98	9.100	9.101	9.103	9.106

PROBLEM 9.71



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



Have $$\bar{I}_{xy} = (I_{xy})_1 + (\bar{I}_{xy})_2 + (I_{xy})_3$$

Symmetry implies $$(I_{xy})_2 = 0$$

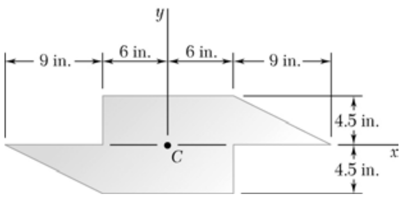
For the other rectangles $$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

Where symmetry implies $$I_{x'y'} = 0$$

	A in^2	$\bar{x}$, in.	$\bar{y}$, in.	$A\bar{x}\,\bar{y}$ in^4
1	$4(0.5) = 2$	–2.75	1.0	–5.5
3	$4(0.5) = 2$	2.75	–1.0	–5.5
Σ				–11.00

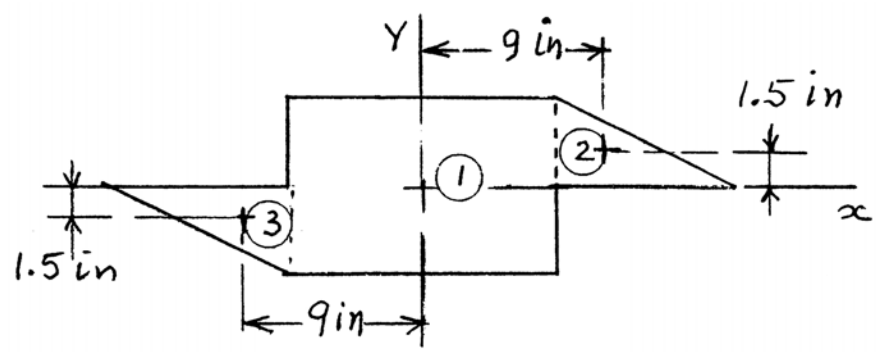
or $\bar{I}_{xy} = -11.00\ \text{in}^4$ ◀

PROBLEM 9.72



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



Note: Orientation of A_3 corresponding to a 180° rotation of the axes. Equation 9.20 then yields

$$I_{x'y'} = I_{xy}$$

Symmetry implies $\quad (I_{xy})_1 = 0$

Using Sample Problem 9.6 $\quad (\bar{I}_{x'y'})_2 = -\frac{1}{72}(9\text{ in.})^2(4.5\text{ in.})^2 = -22.78125\text{ in}^4$

and $\quad \bar{X}_2 = 9\text{ in.} \qquad \bar{Y}_2 = 1.5\text{ in.} \qquad A_2 = \frac{1}{2}(9\text{ in.})(4.5\text{ in.}) = 20.25\text{ in}^2$

Similarly, $\quad (\bar{I}_{x'y'})_3 = -\frac{1}{72}(9\text{ in.})^2(4.5\text{ in.})^2 = -22.78125\text{ in}^4$

and $\quad \bar{X}_3 = -9\text{ in.} \qquad \bar{Y}_2 = -1.5\text{ in.} \qquad A_3 = \frac{1}{2}(9\text{ in.})(4.5\text{ in.}) = 20.25\text{ in}^2$

Then $\quad \bar{I}_{xy} = \cancelto{0}{(I_{xy})_1} + (I_{xy})_2 + (I_{xy})_3 \qquad \text{with} \qquad (I_{xy})_2 = (I_{xy})_3$

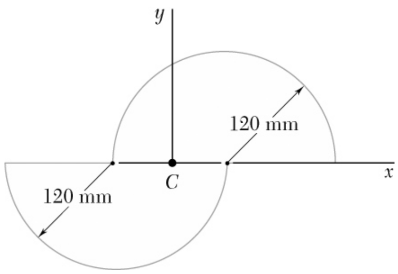
and $\quad I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}\,A$

Therefore, $\quad \bar{I}_{xy} = 2\left[-22.78125 + (9)(1.5)(20.25)\right]\text{in}^4$

$$= 501.1875\text{ in}^4$$

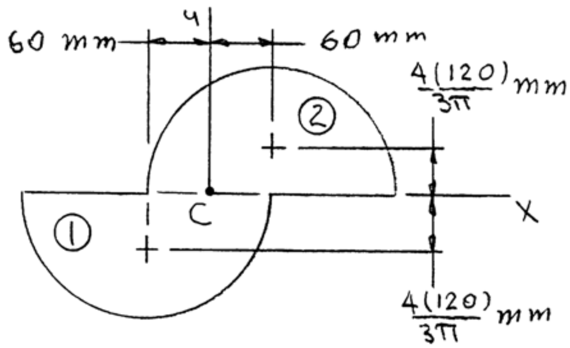
or $\bar{I}_{xy} = 501\text{ in}^4$ ◀

PROBLEM 9.73



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal *x* and *y* axes.

SOLUTION



Have $\bar{I}_{xy} = \left(I_{xy}\right)_1 + \left(I_{xy}\right)_2$

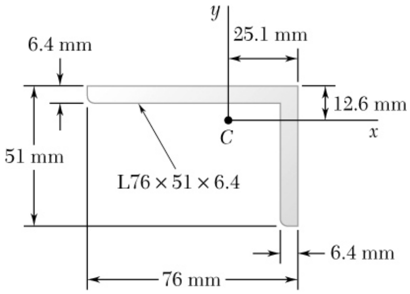
For each semicircle $I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}\,A$ and $\bar{I}_{x'y'} = 0$ (symmetry)

Thus $\bar{I}_{xy} = \Sigma \bar{x}\,\bar{y}\,A$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$A\bar{x}\,\bar{y}$, mm^4
1	$\frac{\pi}{2}(120)^2 = 7200\pi$	-60	$-\frac{160}{\pi}$	69.12×10^6
2	$\frac{\pi}{2}(120)^2 = 7200\pi$	60	$\frac{160}{\pi}$	69.12×10^6
Σ				138.24×10^6

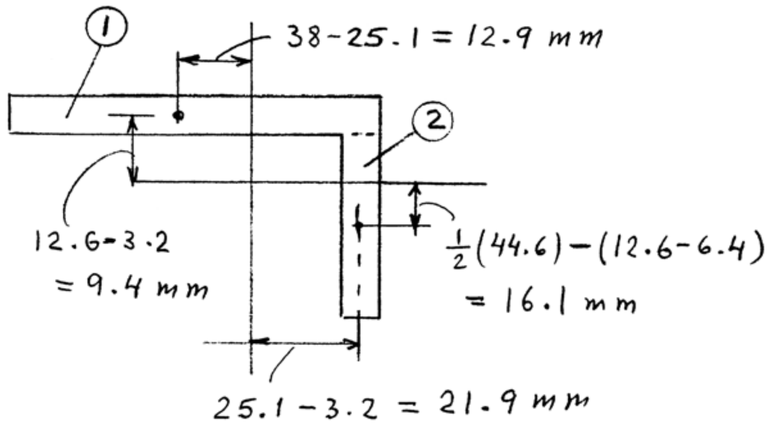
or $\bar{I}_{xy} = 138.2 \times 10^6$ mm^4 ◀

PROBLEM 9.74



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



Have $$I_{xy} = (I_{xy})_1 + (I_{xy})_2$$

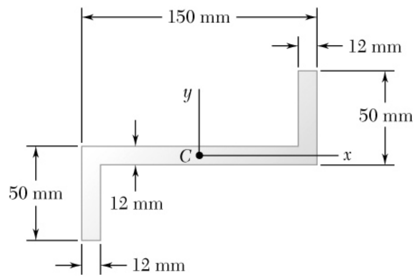
For each rectangle $$I_{xy} = \bar{I}_{x'y'} + A\bar{x}\,\bar{y} \qquad \text{and} \qquad \bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

Thus $$\bar{I}_{xy} = \Sigma \bar{x}\,\bar{y}\,A$$

	A, mm^2	$\bar{x}$, mm	$\bar{y}$, mm	$A\bar{x}\,\bar{y}$, mm^4
1	$76(6.4) = 486.4$	–12.9	9.4	–58 980.86
2	$44.6(6.4) = 285.44$	21.9	–16.1	–100 643.29
Σ				–159 624.15

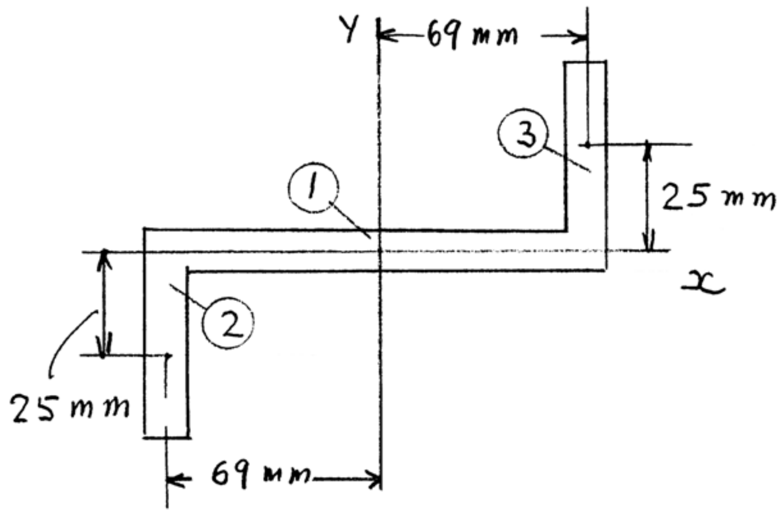
or $\bar{I}_{xy} = -0.1596 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.75



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



Have $$\bar{I}_{xy} = \left(\bar{I}_{xy}\right)_1 + \left(I_{xy}\right)_2 + \left(I_{xy}\right)_3$$

Now symmetry implies $$\left(\bar{I}_{xy}\right)_1 = 0$$

and for the other rectangles $$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A \qquad \text{where} \qquad \bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

Thus $$\bar{I}_{xy} = \left(\bar{x}\,\bar{y}A\right)_2 + \left(\bar{x}\,\bar{y}\right)A_3$$

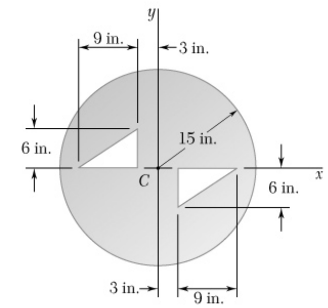
$$= (-69 \text{ mm})(-25 \text{ mm})\left[(12 \text{ mm})(38 \text{ mm})\right]$$

$$+ (69 \text{ mm})(25 \text{ mm})\left[(12 \text{ mm})(38 \text{ mm})\right]$$

$$= (786\,600 + 786\,600) \text{ mm}^4 = 1\,573\,200 \text{ mm}^4$$

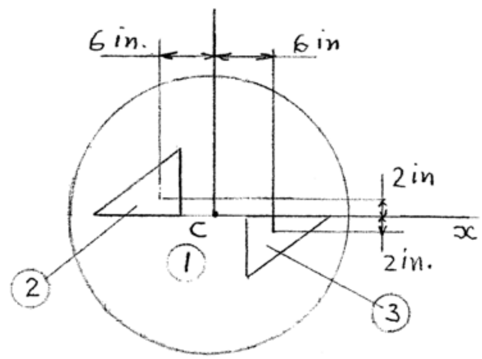
or $\bar{I}_{xy} = 1.573 \times 10^6 \text{ mm}^4$ ◀

PROBLEM 9.76



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



Symmetry implies $$(I_{xy})_1 = 0$$

Using Sample Problem 9.6 and Equation 9.20, note that the orientation of A_2 corresponds to a 90° rotation of the axes; thus $$(\bar{I}_{x'y'})_2 = \frac{1}{72}b^2h^2$$

Also, the orientation of A_3 corresponds to a 270° rotation of the axes; thus $$(\bar{I}_{x'y'})_3 = \frac{1}{72}b^2h^2$$

Then $$(\bar{I}_{x'y'})_2 = \frac{1}{72}(9\text{ in.})^2(6\text{ in.})^2 = 40.5\text{ in}^4$$

and $$\bar{x}_2 = 6\text{ in.},\qquad \bar{y}_2 = -2\text{ in.},\qquad A_2 = \frac{1}{2}(9\text{ in.})(6\text{ in.}) = 27\text{ in}^2$$

Also $$(\bar{I}_{x'y'})_3 = (\bar{I}_{x'y'})_2 = 40.5\text{ in}^4$$

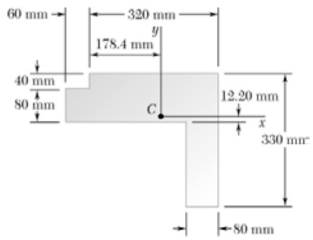
and $$\bar{x}_3 = -6\text{ in.},\qquad \bar{y}_3 = 2\text{ in.},\qquad A_3 = A_2 = 27\text{ in}^2$$

Now $$\bar{I}_{xy} = \cancelto{0}{(I_{xy})_1} - (I_{xy})_2 - (I_{xy})_3 \qquad \text{and} \qquad I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A \qquad (I_{xy})_2 = (I_{xy})_3$$

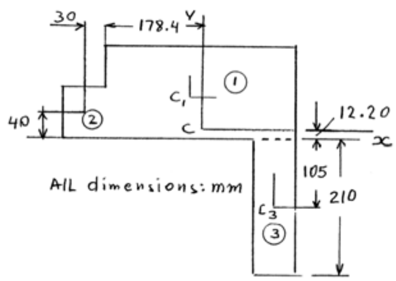
Then $$\bar{I}_{xy} = -2\left[40.5\text{ in}^4 + (6\text{ in.})(-2\text{ in.})(27\text{ in}^2)\right]$$

$$= -2(40.5 - 324)\text{ in}^4$$

or $\bar{I}_{xy} = 567\text{ in}^4$ ◀

	**PROBLEM 9.77** Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal x and y axes.

SOLUTION



Have $$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A$$

Where $\bar{I}_{x'y'} = 0$ for each rectangle

Then $$\bar{I}_{xy} = \left(I_{xy}\right)_1 + \left(I_{xy}\right)_2 + \left(I_{xy}\right)_3$$
$$= \Sigma\,\bar{x}\,\bar{y}A$$

Now $$\bar{x}_1 = -(178.4\text{ mm} - 160\text{ mm}) = -18.4\text{ mm}$$

$$\bar{y}_1 = 60\text{ mm} - 12.2\text{ mm} = 47.8\text{ mm}$$

$$A_1 = 320\text{ mm} \times 120\text{ mm} = 38400\text{ mm}^2$$

and $$\bar{x}_2 = -(178.4\text{ mm} + 30\text{ mm}) = -208.4\text{ mm}$$

$$\bar{y}_2 = 40\text{ mm} - 12.2\text{ mm} = 27.8\text{ mm}$$

$$A_2 = 60\text{ mm} \times 80\text{ mm} = 4800\text{ mm}^2$$

and $$\bar{x}_3 = (320\text{ mm} - 178.4\text{ mm}) - 40\text{ mm} = 101.6\text{ mm}$$

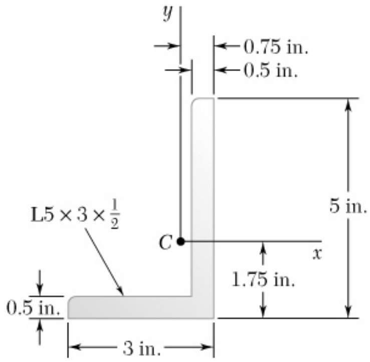
$$\bar{y}_3 = -(12.2\text{ mm} + 105\text{ mm}) = -117.2\text{ mm}$$

$$A_3 = (80\text{ mm} \times 210\text{ mm}) = 16800\text{ mm}^2$$

Then $$\bar{I}_{xy} = \left[(-18.4\text{ mm})(47.8\text{ mm})\left(38400\text{ mm}^2\right)\right] + \left[(-208.4\text{ mm})(27.8\text{ mm})\left(4800\text{ mm}^2\right)\right]$$
$$+ \left[(101.6\text{ mm})(-117.2\text{ mm})\left(16800\text{ mm}^2\right)\right]$$
$$= -(33.7736 + 27.8089 + 200.0463) \times 10^6\text{ mm}^4$$
$$= -261.6288 \times 10^6\text{ mm}^4$$

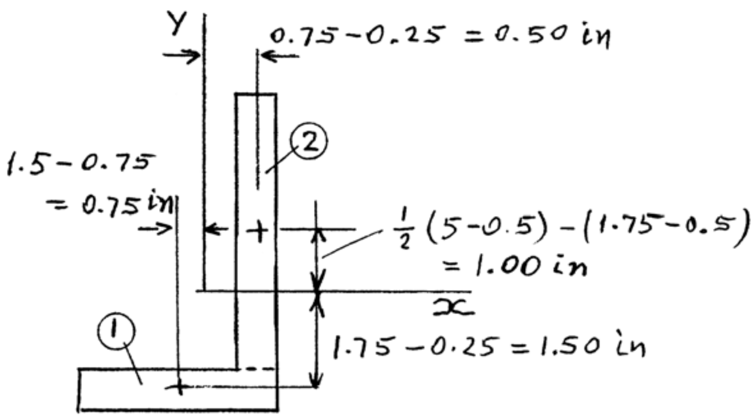
or $\bar{I}_{xy} = -262 \times 10^6\text{ mm}^4$ ◀

PROBLEM 9.78



Using the parallel-axis theorem, determine the product of inertia of the area shown with respect to the centroidal *x* and *y* axes.

SOLUTION



Have $$\bar{I}_{xy} = \left(I_{xy}\right)_1 + \left(I_{xy}\right)_2$$

For each rectangle

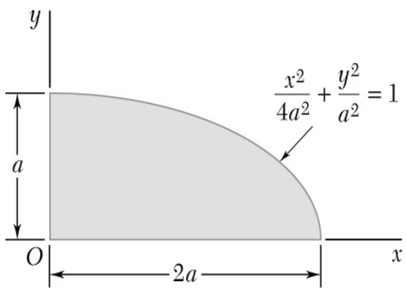
$$I_{xy} = \bar{I}_{x'y'} + \bar{x}\,\bar{y}A \qquad \text{and} \qquad \bar{I}_{x'y'} = 0 \text{ (symmetry)}$$

Then $$\bar{I}_{xy} = \Sigma\bar{x}\,\bar{y}A = (-0.75\text{ in.})(-1.5\text{ in.})\left[(3\text{ in.})(0.5\text{ in.})\right]$$

$$+\,(0.5\text{ in.})(1.00\text{ in.})\left[(4.5\text{ in.})(0.5\text{ in.})\right]$$

$$= (1.6875 + 1.125)\text{in}^4 = 2.8125\text{ in}^4$$

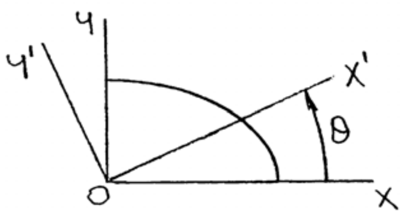
or $\bar{I}_{xy} = 2.81\text{ in}^4$ ◀



PROBLEM 9.79

Determine for the quarter ellipse of Problem 9.67 the moments of inertia and the product of inertia with respect to new axes obtained by rotating the x and y axes about O (a) through 45° counterclockwise, (b) through 30° clockwise.

SOLUTION



From Figure 9.12:
$$I_x = \frac{\pi}{16}(2a)(a)^3$$
$$= \frac{\pi}{8}a^4$$
$$I_y = \frac{\pi}{16}(2a)^3(a)$$
$$= \frac{\pi}{2}a^4$$

From Problem 9.67:
$$I_{xy} = \frac{1}{2}a^4$$

First note
$$\frac{1}{2}(I_x + I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 + \frac{\pi}{2}a^4\right) = \frac{5}{16}\pi a^4$$
$$\frac{1}{2}(I_x - I_y) = \frac{1}{2}\left(\frac{\pi}{8}a^4 - \frac{\pi}{2}a^4\right) = -\frac{3}{16}\pi a^4$$

Now use Equations (9.18), (9.19), and (9.20).

Equation (9.18):
$$I_{x'} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\cos 2\theta - I_{xy}\sin 2\theta$$
$$= \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4\cos 2\theta - \frac{1}{2}a^4\sin 2\theta$$

Equation (9.19):
$$I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta + I_{xy}\sin 2\theta$$
$$= \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4\cos 2\theta + \frac{1}{2}a^4\sin 2\theta$$

Equation (9.20):
$$I_{x'y'} = \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta$$
$$= -\frac{3}{16}\pi a^4\sin 2\theta + \frac{1}{2}a^4\cos 2\theta$$

PROBLEM 9.79 CONTINUED

(*a*) $\theta = +45°$: $I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos 90° - \frac{1}{2}a^4 \sin 90°$

or $I_{x'} = 0.482a^4$ ◀

$$I_{y'} = \frac{5}{16}\pi + \frac{3}{16}\pi a^4 \cos 90° + \frac{1}{2}a^4$$

or $I_{y'} = 1.482a^4$ ◀

$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin 90° + \frac{1}{2}a^4 \cos 90°$$

or $I_{x'y'} = -0.589a^4$ ◀

(*b*) $\theta = -30°$:

$$I_{x'} = \frac{5}{16}\pi a^4 - \frac{3}{16}\pi a^4 \cos(-60°) - \frac{1}{2}a^4 \sin(-60°)$$

or $I_{x'} = 1.120a^4$ ◀

$$I_{y'} = \frac{5}{16}\pi a^4 + \frac{3}{16}\pi a^4 \cos(-60°) + \frac{1}{2}a^4 \sin(-60°)$$

or $I_{y'} = 0.843a^4$ ◀

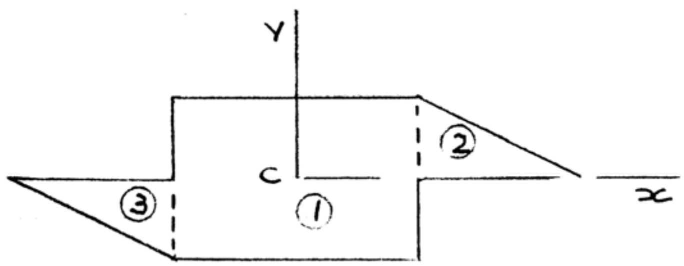
$$I_{x'y'} = -\frac{3}{16}\pi a^4 \sin(-60°) + \frac{1}{2}a^4 \cos(-60°)$$

or $I_{x'y'} = 0.760a^4$ ◀

PROBLEM 9.80

Determine the moments of inertia and the product of inertia of the area of Problem 9.72 with respect to new centroidal axes obtained by rotating the x and y axes 45° clockwise.

SOLUTION



From the solution to Problem 9.72

$$\bar{I}_{xy} = 501.1875 \text{ in}^4$$

$$A_2 = A_3 = 20.25 \text{ in}^2$$

First compute the moment of inertia

$$\bar{I}_x = (I_x)_1 + (I_x)_2 + (I_x)_3 \qquad \text{with} \qquad (I_x)_2 = (I_x)_3$$

$$= \left[\frac{1}{12}(12 \text{ in.})(9 \text{ in.})^3\right] + 2\left[\frac{1}{12}(9 \text{ in.})(4.5\text{in.})^3\right]$$

$$= (729 + 136.6875) \text{ in}^4 = 865.6875 \text{ in}^4$$

and
$$\bar{I}_y = (I_y)_1 + (I_y)_2 + (I_y)_3 \qquad \text{with} \qquad (I_y)_2 = (I_y)_3$$

$$= \left[\frac{1}{12}(9 \text{ in.})(12 \text{ in.})^3\right] + 2\left[\frac{1}{36}(4.5 \text{ in.})(9 \text{ in.})^3 + \left(20.25 \text{ in}^2\right)(9 \text{ in.})^2\right]$$

$$= (1296 + 182.25 + 3280.5)\text{in}^4 = 4758.75 \text{ in}^4$$

From Equation 9.18
$$\bar{I}_{x'} = \frac{\bar{I}_x + \bar{I}_y}{2} + \frac{\bar{I}_x - \bar{I}_y}{2}\cos 2\theta - \bar{I}_{xy}\sin 2\theta$$

$$= \frac{865.6875 \text{ in}^4 + 4758.75 \text{ in}^4}{2} + \frac{865.6875 \text{ in}^4 - 4758.75 \text{ in}^4}{2}\cos\left[2(-45°)\right]$$

$$-501.1875 \text{ in}^4 \sin\left[2(-45°)\right]$$

$$= (2812.21875 + 501.1875)\text{in}^4 = 3313.4063 \text{ in}^4$$

or $\bar{I}_{x'} = 3.31 \times 10^3 \text{ in}^4$ ◀

PROBLEM 9.80 CONTINUED

Similarly $$\bar{I}_{y'} = \frac{\bar{I}_x + \bar{I}_y}{2} - \frac{\bar{I}_x - \bar{I}_y}{2}\cos 2\theta + \bar{I}_{xy}\sin 2\theta$$

$$= (2812.21875 - 501.1875)\text{in}^4 = 2311.0313 \text{ in}^4$$

or $\bar{I}_{y'} = 2.31 \times 10^3 \text{ in}^4$ ◀

and $$\bar{I}_{x'y'} = \frac{\bar{I}_x - \bar{I}_y}{2}\sin 2\theta + \bar{I}_{xy}\cos 2\theta$$

$$= \frac{865.6875 \text{ in}^4 - 4758.75 \text{ in}^4}{2}\sin\left[2(-45°)\right]$$

$$+ 501.1875\cos\left[2(-45°)\right]$$

$$= (-1946.53125)(-1)\text{in}^4 = 1946.53125 \text{ in}^4$$

or $\bar{I}_{x'y'} = 1.947 \times 10^3 \text{ in}^4$ ◀