PROBLEM 5.118

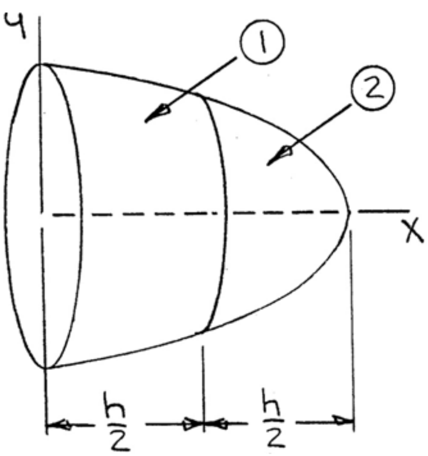
Determine by direct integration the values of $\bar{x}$ for the two volumes obtained by passing a vertical cutting plane through the given shape of Fig. 5.21. The cutting plane is parallel to the base of the given shape and divides the shape into two volumes of equal height.

A paraboloid of revolution.

SOLUTION

Choose as the element of volume a disk of radius *r* and thickness *dx*. Then

$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$



The equation of the generating curve is $x = h - \frac{h}{a^2}y^2$ so that

$r^2 = \frac{a^2}{h}(h - x)$ and then

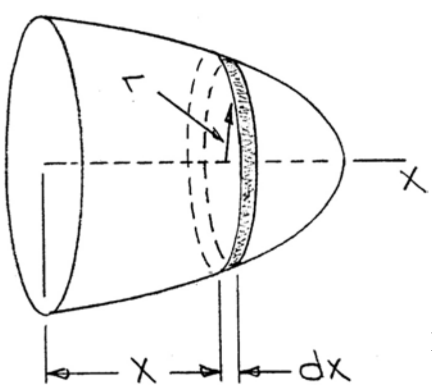
$$dV = \pi\frac{a^2}{h}(h - x)dx$$

Component 1

$$V_1 = \int_0^{h/2} \pi\frac{a^2}{h}(h - x)dx$$

$$= \pi\frac{a^2}{h}\left[hx - \frac{x^2}{2}\right]_0^{h/2}$$

$$= \frac{3}{8}\pi a^2 h$$



and $$\int_1 \bar{x}_{EL} dV = \int_0^{h/2} x\left[\pi\frac{a^2}{h}(h - x)dx\right]$$

$$= \pi\frac{a^2}{h}\left[h\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{h/2}$$

$$= \frac{1}{12}\pi a^2 h^2$$

Now $$\bar{x}_1 V_1 = \int_1 \bar{x}_{EL} dV: \quad \bar{x}_1\left(\frac{3}{8}\pi a^2 h\right) = \frac{1}{12}\pi a^2 h^2$$

or $\bar{x}_1 = \frac{2}{9}h$ ◀

PROBLEM 5.118 CONTINUED

Component 2

$$V_2 = \int_{h/2}^{h} \pi \frac{a^2}{h}(h - x)dx = \pi \frac{a^2}{h}\left[hx - \frac{x^2}{2}\right]_{h/2}^{h}$$

$$= \pi \frac{a^2}{h}\left\{\left[h(h) - \frac{(h)^2}{2}\right] - \left[h\left(\frac{h}{2}\right) - \frac{\left(\frac{h}{2}\right)^2}{2}\right]\right\}$$

$$= \frac{1}{8}\pi a^2 h$$

and $$\int_2 \overline{x}_{EL} dV = \int_{h/2}^{h} x\left[\pi \frac{a^2}{h}(h - x)dx\right] = \pi \frac{a^2}{h}\left[h\frac{x^2}{2} - \frac{x^3}{3}\right]_{h/2}^{h}$$

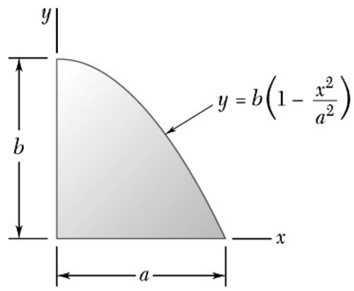
$$= \pi \frac{a^2}{h}\left\{\left[h\frac{(h)^2}{2} - \frac{(h)^3}{3}\right] - \left[h\frac{\left(\frac{h}{2}\right)^2}{2} - \frac{\left(\frac{h}{2}\right)^3}{3}\right]\right\}$$

$$= \frac{1}{12}\pi a^2 h^2$$

Now $$\overline{x}_2 V_2 = \int_2 \overline{x}_{EL} dV: \quad \overline{x}_2\left(\frac{1}{8}\pi a^2 h\right) = \frac{1}{12}\pi a^2 h^2$$

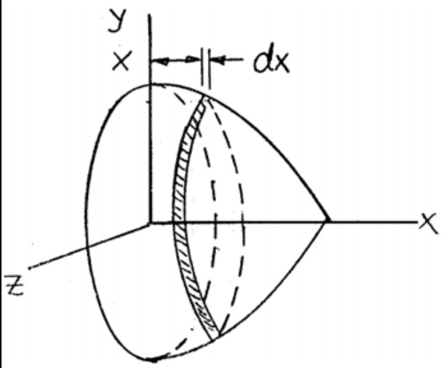
$$\text{or } \overline{x}_2 = \frac{2}{3}h \blacktriangleleft$$

PROBLEM 5.119



Locate the centroid of the volume obtained by rotating the shaded area about the x axis.

SOLUTION



First note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

Now $r = b\left(1 - \dfrac{x^2}{a^2}\right)$ so that

$$dV = \pi b^2 \left(1 - \frac{x^2}{a^2}\right)^2 dx$$

Then $$V = \int_0^a \pi b^2 \left(1 - \frac{x^2}{a^2}\right)^2 dx = \int_0^a \pi b^2 \left(1 - \frac{2x^2}{a^2} + \frac{x^4}{a^4}\right) dx$$

$$= \pi b^2 \left(x - \frac{2x^3}{3a^2} + \frac{x^5}{5a^4}\right)\Bigg|_0^a$$

$$= \pi a b^2 \left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$= \frac{8}{15}\pi a b^2$$

and $$\int \bar{x}_{EL} dV = \int_0^a \pi b^2 x \left(1 - \frac{2x^2}{a^2} + \frac{x^4}{a^4}\right) dx$$

$$= \pi b^2 \left(\frac{x^2}{2} - \frac{2x^4}{4a^2} + \frac{x^6}{6a^4}\right)\Bigg|_0^a$$

$$= \pi a^2 b^2 \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right)$$

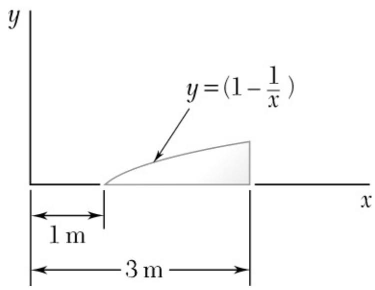
$$= \frac{1}{6}\pi a^2 b^2$$

PROBLEM 5.119 CONTINUED

Then $\quad \bar{x}V = \int x_{EL}dV: \quad \bar{x}\left(\frac{8}{15}\pi ab^2\right) = \frac{1}{16}\pi a^2b^2$

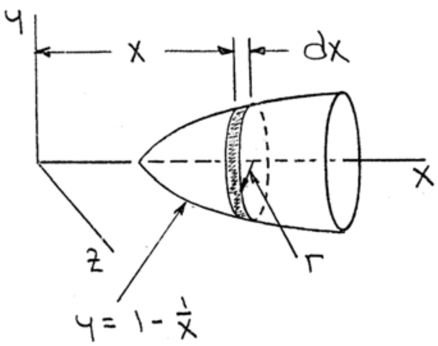
or $\bar{x} = \frac{15}{6}a$ ◀

PROBLEM 5.120



Locate the centroid of the volume obtained by rotating the shaded area about the x axis.

SOLUTION



First, note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$

Now $r = 1 - \frac{1}{x}$ so that

$$dV = \pi\left(1 - \frac{1}{x}\right)^2 dx$$

$$= \pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx$$

Then $$V = \int_1^3 \pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx = \pi\left[x - 2\ln x - \frac{1}{x}\right]_1^3$$

$$= \pi\left[\left(3 - 2\ln 3 - \frac{1}{3}\right) - \left(1 - 2\ln 1 - \frac{1}{1}\right)\right]$$

$$= (0.46944\pi)\ \text{m}^3$$

and $$\int \bar{x}_{EL} dV = \int_1^3 x\left[\pi\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)dx\right] = \pi\left[\frac{x^2}{2} - 2x + \ln x\right]_1^3$$

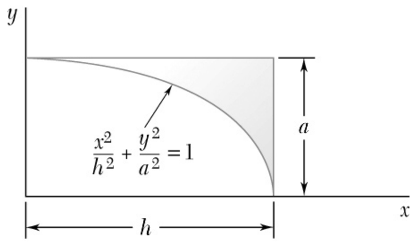
$$= \pi\left\{\left[\frac{3^2}{2} - 2(3) + \ln 3\right] - \left[\frac{1^3}{2} - 2(1) + \ln 1\right]\right\}$$

$$= (1.09861\pi)\ \text{m}$$

Now $$\bar{x}V = \int \bar{x}_{EL} dV: \quad \bar{X}(0.46944\pi\ \text{m}^3) = 1.09861\pi\ \text{m}^4$$

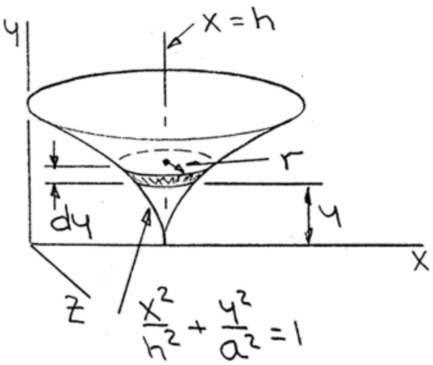
or $\bar{x} = 2.34$ m ◀

PROBLEM 5.121



Locate the centroid of the volume obtained by rotating the shaded area about the line $x = h$.

SOLUTION



First, note that symmetry implies $\bar{x} = h$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius r and thickness dx. Then

$$dV = \pi r^2 dy, \quad \bar{y}_{EL} = y$$

Now $x^2 = \frac{h^2}{a^2}\left(a^2 - y^2\right)$ so that $r = h - \frac{h}{a}\sqrt{a^2 - y^2}$

Then $$dV = \pi\frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy$$

and $$V = \int_0^a \pi\frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy$$

Let $$y = a\sin\theta \Rightarrow dy = a\cos\theta\, d\theta$$

Then
$$\begin{aligned}
V &= \pi\frac{h^2}{a^2}\int_0^{\pi/2}\left(a - \sqrt{a^2 - a^2\sin^2\theta}\right)^2 a\cos\theta\, d\theta \\
&= \pi\frac{h^2}{a^2}\int_0^{\pi/2}\left[a^2 - 2a\left(a\cos\theta\right) + \left(a^2 - a^2\sin^2\theta\right)\right]a\cos\theta\, d\theta \\
&= \pi a h^2\int_0^{\pi/2}\left(2\cos\theta - 2\cos^2\theta - \sin^2\theta\cos\theta\right)d\theta \\
&= \pi a h^2\left[2\sin\theta - 2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) - \frac{1}{3}\sin^3\theta\right]_0^{\pi/2} \\
&= \pi a h^2\left[2 - 2\left(\frac{\frac{\pi}{2}}{2}\right) - \frac{1}{3}\right] \\
&= 0.095870\pi a h^2
\end{aligned}$$

PROBLEM 5.121 CONTINUED

and $$\int \bar{y}_{EL}\,dV = \int_0^a y\left[\pi\frac{h^2}{a^2}\left(a - \sqrt{a^2 - y^2}\right)^2 dy\right]$$

$$= \pi\frac{h^2}{a^2}\int_0^a\left(2a^2y - 2ay\sqrt{a^2 - y^2} - y^3\right)dy$$

$$= \pi\frac{h^2}{a^2}\left[a^2y^2 + \frac{2}{3}a\left(a^2 - y^2\right)^{3/2} - \frac{1}{4}y^4\right]_0^a$$

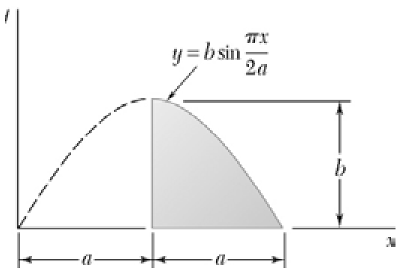
$$= \pi\frac{h^2}{a^2}\left\{\left[a^2(a)^2 - \frac{1}{4}a^4\right] - \left[\frac{2}{3}a\left(a^2\right)^{3/2}\right]\right\}$$

$$= \frac{1}{12}\pi a^2h^2$$

Now $$\bar{y}V = \int \bar{y}_{EL}\,dV: \quad \bar{y}\left(0.095870\pi ah^2\right) = \frac{1}{12}\pi a^2h^2$$

or $\bar{y} = 0.869a$ ◀

PROBLEM 5.122



Locate the centroid of the volume generated by revolving the portion of the sine curve shown about the *x* axis.

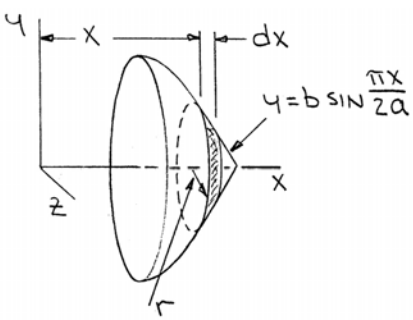
SOLUTION

First, note that symmetry implies $\bar{y} = 0$ ◀

$\bar{z} = 0$ ◀

Choose as the element of volume a disk of radius *r* and thickness *dx*. Then

$$dV = \pi r^2 dx, \quad \bar{x}_{EL} = x$$



Now $\quad r = b\sin\dfrac{\pi x}{2a}$

so that $\quad dV = \pi b^2 \sin^2\dfrac{\pi x}{2a}dx$

Then

$$V = \int_a^{2a} \pi b^2 \sin^2\frac{\pi x}{2a}dx$$

$$= \pi b^2\left[\frac{x}{2} - \frac{\sin\frac{\pi x}{2a}}{2\frac{\pi}{a}}\right]_a^{2a}$$

$$= \pi b^2\left[\left(\tfrac{2a}{2}\right) - \left(\tfrac{a}{2}\right)\right]$$

$$= \frac{1}{2}\pi ab^2$$

and $\quad \displaystyle\int \bar{x}_{EL}dV = \int_a^{2a} x\left(\pi b^2 \sin^2\frac{\pi x}{2a}dx\right)$

Use integration by parts with

$$u = x \qquad dV = \sin^2\frac{\pi x}{2a}$$

$$du = dx \qquad V = \frac{x}{2} - \frac{\sin\frac{\pi x}{a}}{\frac{2\pi}{a}}$$

PROBLEM 5.122 CONTINUED

Then $\int \overline{x}_{EL} dV = \pi b^2 \left\{ \left[x\left(\frac{x}{2} - \frac{\sin \frac{\pi x}{a}}{\frac{2\pi}{a}} \right) \right]_a^{2a} - \int_a^{2a} \left(\frac{x}{2} - \frac{\sin \frac{\pi x}{a}}{\frac{2\pi}{a}} \right) dx \right\}$

$$= \pi b^2 \left\{ \left[2a\left(\frac{2a}{2} \right) - a\left(\frac{a}{2} \right) \right] - \left[\frac{1}{4}x^2 + \frac{a^2}{2\pi^2} \cos \frac{\pi x}{a} \right]_a^{2a} \right\}$$

$$= \pi b^2 \left\{ \left(\frac{3}{2}a^2 \right) - \left[\frac{1}{4}(2a)^2 + \frac{a^2}{2\pi^2} - \frac{1}{4}(a)^2 + \frac{a^2}{2\pi^2} \right] \right\}$$

$$= \pi a^2 b^2 \left(\frac{3}{4} - \frac{1}{\pi^2} \right)$$

$$= 0.64868\pi a^2 b^2$$

Now $\overline{x}V = \int \overline{x}_{EL} dV$: $\overline{x}\left(\frac{1}{2}\pi ab^2 \right) = 0.64868\pi a^2 b^2$

or $\overline{x} = 1.297a$ ◀