A more generalized problem is to look at a ruler of length r split at n points, and to find the average length of the piece that contains $\frac{r}{2}$.

Since the n points are sliced randomly, it follows that the number of points before and after $\frac{r}{2}$ is a binomial distribution.

That is, the probability that k points are in $(0,\frac{r}{2})$ and $n-k$ points are in $(\frac{r}{2},r)$ is $\frac{\binom{n}{k}}{2^n}$.

Note that j points randomly distributed between a and b create $j+1$ pieces of average length $\frac{b-a}{j+1}$.

Given that k points are in $(0,\frac{r}{2})$ and $n-k$ points are in $(\frac{r}{2},r)$, the average length from the k^{th} point to $\frac{r}{2}$ is:

$$\frac{(\frac{r}{2}-0)}{k+1}$$

$$=\frac{r}{2(k+1)}$$

Similarly, the average length from $\frac{r}{2}$ to the $(k+1)^{th}$ point is:

$$\frac{(r-\frac{r}{2})}{n-k+1}$$

$$=\frac{r}{2(n-k+1)}$$

Thus, the average length of all pieces containing the middle is:

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{2^n}\times\left(\frac{r}{2(n-k+1)}+\frac{r}{2(k+1)}\right)$$

$$=\frac{r}{2^{n+1}}\sum_{k=0}^{n}\binom{n}{k}\times\left(\frac{(k+1)+(n-k+1)}{(n-k+1)(k+1)}\right)$$

$$=\frac{r}{2^{n+1}}\sum_{k=0}^{n}\frac{n!}{(n-k)!\,k!}\times\frac{n+2}{(n-k+1)(k+1)}$$

$$=\frac{r}{2^{n+1}}\times\frac{1}{(n+1)}\sum_{k=0}^{n}\frac{n!\,(n+1)(n+2)}{(n-k)!\,(n-k+1)k!\,(k+1)}$$

$$=\frac{r}{2^{n+1}(n+1)}\sum_{k=0}^{n}\frac{(n+2)!}{(n-k+1)!\,(k+1)!}$$

Note that the 0^{th} and $(n+2)^{th}$ terms were missing from $\binom{n+2}{k}$.

$$= \frac{r}{2^{n+1}(n+1)}\left(\left(\sum_{k=0}^{n+2}\binom{n+2}{k}\right) - \binom{n+2}{0} - \binom{n+2}{n+2}\right)$$

$$= \frac{r}{2^{n+1}(n+1)}\left(2^{n+2} - 2\right)$$

$$= \boxed{\frac{r}{n+1}\left(\frac{2^{n+1}-1}{2^n}\right)}$$

For $r = 12$ and $n = 3$, the average length of the piece containing the 6-inch mark is:

$$= \boxed{\frac{45}{8} \text{ or } 5.625in}$$

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2020.08.17