Challenge-Proof (Second Try :))

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I will frist show the following:

ρ transitive $\Rightarrow \rho^n \subseteq \rho$

$$\rho^n \subseteq \rho \overset{\text{def}}{\Longleftrightarrow} \underbrace{(a,b) \in \rho^n \Rightarrow (a,b) \in \rho}_{\text{to show}}$$

Assuming ρ is transitive

I will proceed via induction:

Base case : $n = 1 \Rightarrow \rho^1 \subseteq \rho$ ✓

Induction hypothesis : $\rho^n \subseteq \rho$ for some $n \in \mathbb{N} \setminus \{0\}$

Induction step : $\rho^{n+1} \subseteq \rho$

$(a,b) \in \rho^{n+1}$

$\Rightarrow (a,c) \in \rho \wedge (c,b) \in \rho^n$ (def. ρ^n

$\Rightarrow (a,c) \in \rho \wedge (c,b) \in \rho$ (I.H., since $(x,y) \in \rho^n \Rightarrow (x,y) \in \rho$

$\Rightarrow (a,b) \in \rho$ (transitivity of ρ

So $\rho^n \subseteq \rho$, if ρ is transitive, for all $n \in \mathbb{N} \setminus \{0\}$

We have $\rho^* = \rho \cup \rho^2 \cup \ldots \cup \rho^n$

We want to show $\rho^* = \rho$, if ρ is transitive :

So we get (1) $\rho \subseteq \rho^* \wedge$ (2) $\rho^* \subseteq \rho$

(1) $\rho \subseteq \rho^*$, follows from definition of ρ^* ($\rho^* = \rho \cup \ldots$)

(2) $\rho^* \subseteq \rho$ ← This is to be shown

$\rho^* = \bigcup_{n=1}^{\infty} \rho^n$, since $\forall n \, (\rho^n \subseteq \rho)$, $n \in \mathbb{N} \setminus \{0\}$

this means, that

$\rho^* \subseteq \rho$

(1),(2) ~~concld~~ conclude the proof

So we showed $\rho^* = \rho$ if ρ is transitive