# Beweisaufgabe mit Polynomen

Let $F$ be a finite field. Show that there exists a non-constant polyomial $p(x) \in F[x]$ with no roots.

# Proof systems (HS21)

($\star$) Let $\Pi_1 = (\mathcal{S}_1, \mathcal{P}_1, \tau_1, \phi_1)$ and $\Pi_2 = (\mathcal{S}_2, \mathcal{P}_2, \tau_2, \phi_2)$ be two proof systems. We combine $\Pi_1$ and $\Pi_2$ into a third proof system

$$\Pi_3 = (\mathcal{S}_1 \times \mathcal{S}_2, \mathcal{P}_1 \times \mathcal{P}_2, \tau_3, \phi_3),$$

where

$$\tau_3(s_1, s_2) = 1 \quad \overset{\text{def}}{\iff} \quad \tau_1(s_1) = 1 \text{ or } \tau_2(s_2) = 1,$$

and

$$\phi_3((s_1, s_2), (p_1, p_2)) = 1 \quad \overset{\text{def}}{\iff} \quad \phi_1(s_1, p_1) = 1 \text{ or } \phi_2(s_2, p_2) = 1.$$

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(a) Prove or disprove: If $\Pi$ complete, then $\Pi_1$ complete or $\Pi_2$ complete.

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(a) Prove or disprove: If $\Pi$ complete, then $\Pi_1$ complete or $\Pi_2$ complete.

(b) Prove or disprove: If $\Pi_1$ sound or $\Pi_2$ sound, then $\Pi$ sound.

# Kalküle (FS22)

Consider the calculus consisting of the following four derivation rules:

$$\emptyset \vdash_{R_1} F \to F$$
$$\{F\} \vdash_{R_2} F \lor F$$
$$\{\neg F \lor \neg F\} \vdash_{R_3} F \to (\neg F \lor \neg F)$$
$$\{F \to (G \lor H), G \to H\} \vdash_{R_4} F \to H$$

Formally derive $A \to \neg A$ from $\{\neg A\}$.