

For a pipeline with exactly one execution unit per stage, prove that

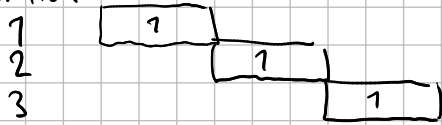
pipeline balanced \Leftrightarrow no stage longer than the first

\Leftarrow : Prove for any such pipeline, i.e. for such a pipeline of any length

$n=1$:



iteration

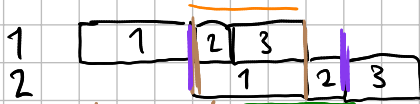
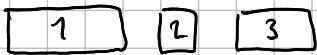


\rightarrow balanced

I.H.: Assume statement holds for some n , i.e. for any pipeline of length n where no stage is longer than stage 1, the pipeline is balanced.

$n \rightarrow n+1$:

Proof idea:



does stage 2 have to wait? $d_2 < d_1 \Rightarrow$ no

does stage 3 have to wait? $d_2 + d_3 < d_1 + d_2 \Rightarrow$ no

Consider a pipeline of length $n+1$:

To show: all iterations have the same latency.

We look at iter. 1 and 2.






when does iteration 2 finish stage n ?



By I.H. we know that the pipeline $1, \dots, n$ is balanced ∇

\Rightarrow duration of  is $d_1 + d_2 + \dots + d_n$ (d_i := duration of stage i)

duration of  is $d_2 + d_3 + \dots + d_{n+1}$

\Rightarrow  -  = $d_1 - d_n \geq 0$

\Rightarrow stage n ends after $n+1$ in the prev. iteration

\Rightarrow duration of iteration 2 is $d_1 + \dots + d_n + d_{n+1}$ = duration of iter. 1

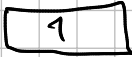

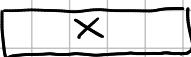
Then the same argument can be applied for all other iterations

(could do this with induction to be more precise)

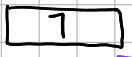

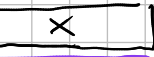
\Rightarrow : We prove it indirectly:

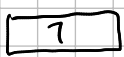


there exists a stage longer than the first \Rightarrow pipeline unbalanced



Let x be the first stage longer than the first

 \dots   \dots


$d_1 < d_x$,


1.  \dots  



2. a)  \dots   \rightarrow unbalanced

2. b)   \rightarrow ok

Since x is the first longer stage and $d_1 \geq d_i \forall i \in \{2, \dots, x\}$,
the pipeline $1, \dots, x-1$ is balanced ∇

\Rightarrow  = $d_1 + d_2 + \dots + d_{x-1}$

 = $d_2 + d_3 + \dots + d_x$

\Rightarrow  -  = $d_x - d_1 > 0$

\Rightarrow stage x will have to wait in iter. 2

\Rightarrow iter 2. longer than iter. 1

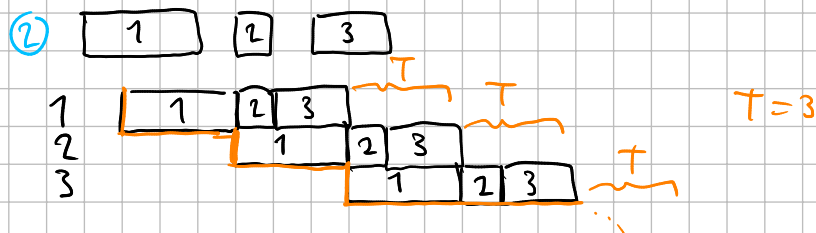
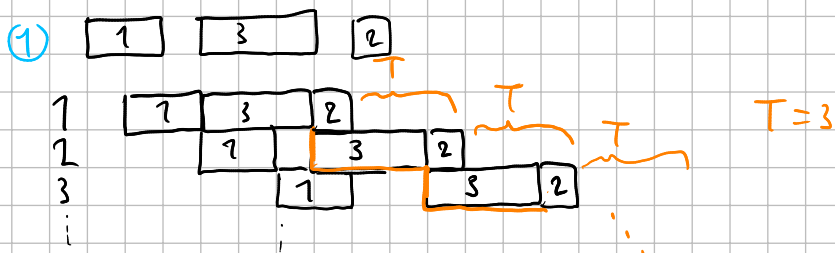
\Rightarrow pipeline unbalanced

□

Prove that (in a pipeline with 1 execution unit per stage)

$$\text{throughput} = \frac{1}{\text{length of longest stage}}$$

Examples:



Let x the longest stage.

• We see that x in iteration $k+1$ starts right when iter. k finishes with x (\dagger)

Assuming (\dagger) were always true:

If we look at the execution from stages x to n , then this is the execution of the pipeline with stages x, \dots, n . Since $d_x > d_i \forall i$, the pipeline x, \dots, n is balanced ∇



$\boxed{x} = \boxed{n}$ because balanced

$$\Rightarrow T = d_x + \underline{d_{x+1} + \dots + d_n} - (\underline{d_{x+1} + \dots + d_n})$$

$$= \underline{d_x}$$

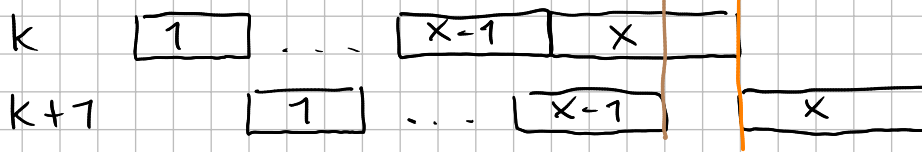
and T same for every iteration

Now we prove (†):

The earliest x can start in iter. $k+1$ is when iter. k finishes x

\Rightarrow we need to show that iter. $k+1$ always finishes stages

$1, \dots, x-1$ before (\leq) iter. k finishes x .



Trick \forall Consider the pipeline $0, 1, \dots, x, \dots, n$ with $d_0 = d_x$.

This pipeline is balanced, since $d_0 = d_x \geq d_i \forall i \forall$



pipeline balanced \Rightarrow duration is sum of stages;

$$\begin{aligned}
 \text{duration} &= d_0 + d_1 + \dots + d_{x-1} \\
 &= d_x + d_1 + \dots + d_{x-1} \\
 &= \text{duration of iter. k}
 \end{aligned}$$

\Rightarrow We finish $x-1$ in iter. $k+1$ exactly when we finish x in iter. k

Since $d_0 \geq d_1$, adding stage 0 only delayed the execution of stages $1, \dots, x-1$ by $d_0 - d_1$ in iter. $k+1$ compared to the original pipeline.

\Rightarrow stages $1, \dots, x-1$ in iter. $k+1$ can finish before iter. k finishes x .

\Rightarrow (†) proven

□