1

Consider $[5] = \{1, 2, \ldots, 5\}$. Give an example of, or show that such a request impossible, a nonempty relation on $[5]$ that is:

Let $a, b, c \in [5]$.

(a) not symmetric, but reflexive and transitive.

$a \leq b$:

- Reflexive: $a \leq a$.
- Not Symmetric: $a \leq b \not\Rightarrow b \leq a$.
- Transitive: $(a \leq b) \wedge (b \leq c) \Rightarrow a \leq c$.

Another example is "a divides b".

(b) not transitive, but reflexive and symmetric.

$|a - b| \leq 1$

- Reflexive: $|a - a| = 0 \leq 1$.
- Symmetric: $|a - b| \leq 1 \Rightarrow |b - a| \leq 1$, because the relationship essentially says that the numbers are either equal or consecutive.
- Not transitive, example: $a = 1, b = 2, c = 3$.

(c) not reflexive, but symmetric and transitive.

$(a - 3)(b - 3) > 0$

(An easier way to think of this is $ab > 0$ "shifted to the right by 3".)

- Not reflexive, example: $a = 3$.
- Symmetric: $(a - 3)(b - 3) > 0 \Rightarrow (b - 3)(a - 3) > 0$
  - Because $(a - 3)(b - 3) = (b - 3)(a - 3)$.
- Transitive: $((a - 3)(b - 3) > 0) \wedge ((b - 3)(c - 3) > 0) \Rightarrow (a - 3)(c - 3) > 0$
  - Because the premise is only true if $a, b, c \in \{1, 2\}$ or $a, b, c \in \{4, 5\}$.

(d) both symmetric and antisymmetric.

$a = b$:

- Symmetric: $a = b \Rightarrow b = a$.
- Antisymmetric: $(a = b) \wedge (b = a) \Rightarrow (a = b)$.

2

Determine if each of the following relation on a set A is an equivalence relation or not. If so, exhibit the equivalence classes. Justify each

(a) $A = \mathbb{R}^2$, $(a, b)R(c, d)$ if $a^2 + b^2 = c^2 + d^2$

- Reflexive: $a^2 + b^2 = a^2 + b^2$, so $(a, b)R(a, b)$.
- Symmetric: $\left(a^2 + b^2 = c^2 + d^2\right)$ implies $\left(c^2 + d^2 = a^2 + b^2\right)$, so $(a, b)R(c, d) \Rightarrow (c, d)R(a, b)$.
- Transitive: $\left(a^2 + b^2 = c^2 + d^2\right)$ and $\left(c^2 + d^2 = e^2 + f^2\right)$ implies $\left(a^2 + b^2 = e^2 + f^2\right)$, so $(a, b)R(c, d) \wedge (c, d)R(e, f) \Rightarrow (a, b)R(e, f)$.

Therefore R is an equivalence relation. The equivalence class of an ordered pair $(a, b) \in A$ consists of the set of all ordered pairs $(c, d) \in A$ such that $a^2 + b^2 = c^2 + d^2$.

$$\begin{aligned} [(a, b)] &= \{(c, d) \in A \mid (c, d)R(a, b)\} \\ &= \left\{(c, d) \in \mathbb{R}^2 \mid c^2 + d^2 = a^2 + b^2\right\}. \end{aligned}$$

(b) $A = \mathbb{Q}$, $R = \varnothing$ the empty relation.

Let $a, b, c \in \mathbb{Q}$.

- Not reflexive: $(a, a) \notin \varnothing$.
- Symmetric: $(a, b) \in \varnothing \Rightarrow (b, a) \in \varnothing$ is vacuously true.
- Transitive: $((a, b) \in \varnothing) \wedge ((b, c) \in \varnothing) \Rightarrow (a, c) \in \varnothing$ is vacuously true.

It is not reflexive, so $R = \varnothing$ is not an equivalence relation.

3

Prove or disprove the converse of Proposition 6.9: that is, a partition $\mathcal{A}$ on A induces an equivalence relation on A by $x \sim y$ if and only if there exists some $B \in \mathcal{A}$ such that B contains both x and y.

Let the partition on A be $\mathcal{A}$, and let $\sim$ be the relation induced by $\mathcal{A}$.

Pick some $a \in A$. The union of partitions in $\mathcal{A}$ equals A, so there exists a partition A_a such that $a \in A_a$. Therefore $a \sim a$, and the relation is reflexive.

Pick some $a, b \in A$ such that $a \sim b$. Then there exists a partition A_x such that $a, b \in A_x$. Therefore $b \sim a$, so the relation is symmetric.

Pick some $a, b, c \in A$ such that $a \sim b$ and $b \sim c$. Then there exist partitions A_x and A_y such that $a, b \in A_x$ and $b, c \in A_y$. We know $A_x \in A$ and $A_y \in A$, so either $A_x = A_y$ or $A_x \cap A_y = \varnothing$. But we know $b \in A_x$ and $b \in A_y$, so $A_x = A_y$. Therefore a and c are in the same partition, so $a \sim c$, and the relation is transitive.

The relation $\sim$ is reflexive, symmetric, and transitive, so it is an equivalence relation.

4

Consider the relation $D := \{(a,b) \in \mathbb{N} \times \mathbb{N} \mid a \text{ divides } b\}$. Show that D is a partial ordering.

Pick some $a, b, c \in \mathbb{N}$.

We know a divides a because there exists an integer k such that $a = ka$, which is $k = 1$. Therefore the relation is reflexive

If a divides b, then there exists an integer k such that $a = kb$. Similarly, if b divides a, then there exists an integer k' such that $b = k'a$. Combining these yields $a = k(k'a)$, which implies $kk' = 1$. Because $a, b \in \mathbb{N}$, we know k and k' are positive. So $k = k' = 1$. Thus $a = b$, meaning the relation is anti-symmetric.

If a divides b, then there exists an integer k such that $a = kb$. Similarly, if b divides c, then there exists an integer k' such that $b = k'c$. Combining these yields $a = k(k'c)$. The product of two integers is an integer, so there exists some integer $k'' = kk'$ such that $a = k''c$. Thus a divides c, meaning the relation is transitive.

The relation is reflexive, anti-symmetric, and transitive, so it is a partial ordering.

5

This problem will deal with what we call a total ordering.

Definition 6.16: A **total ordering**, or a **linear ordering** is a partial ordering such that any two elements are comparable; that is, for any a and b, either $a \prec b$ or $b \prec a$.

Suppose A and B are two sets, with total orderings $\prec_A$ and $\prec_B$ respectively. Define

$$\prec_L := \{((a,b),(c,d)) \mid (a \neq c \wedge a \prec_A c) \vee (a = c \wedge b \prec_B d)\},$$

and

$$\prec_P := \{((a,b),(c,d)) \mid a \prec_A c \wedge b \prec_B d\}.$$

Note 6.17: $\prec_L$ is known as the lexicographic ordering, and $\prec_P$ is known as the product ordering.

(a) Describe in your own words how $\prec_L$ and $\prec_P$ work.

$\prec_L$: We can use lexicographic ordering to sort in "increasing" order by grouping together all 2-tuples with the same first element, sorting those according to $\prec_A$ on the first elements, then sorting each of those groups by $\prec_B$ on the second elements.

For example, if $A = B = [3]$ and $\prec_A$ and $\prec_B$ both functioned as $<$, then $\prec_L$ would order:

$$(1,1),(1,2),(1,3), \quad (2,1),(2,2),(2,3), \quad (3,1),(3,2),(3,3)$$

similar to a table of contents. Note that the first and second element of each two-tuple come from different sets which can be ordered in different ways.

$\prec_P$: In an "increasing" ordering of 2-tuples, both elements must "increase" according to $\prec_A$ and $\prec_B$ respectively when compared to the previous 2-tuple.

(b) Show that $\prec_P$ is a partial ordering.

We know $\prec_A$ and $\prec_B$ are partial orderings themselves by definition.

Reflexive: $a \prec_A a$ and $b \prec_B b$ are true by reflexivity of $\prec_A$ and $\prec_B$, therefore $(a, b) \prec_P (a, b)$.

Anti-symmetric: $a \prec_A c$ implies $c \prec_A a$ by anti-symmetry of $\prec_A$. Same applies to b, d, and $\prec_B$. Therefore $((a, b) \prec_P (c, d)) \wedge ((c, d) \prec_P (a, b)) \Rightarrow (a, b) = (c, d)$.

Transitive: $a \prec_A c$ and $c \prec_A e$ implies $a \prec_A e$ by transitivity of $\prec_A$. Same applies to b, d, f, and $\prec_B$. Therefore $((a, b) \prec_P (c, d)) \wedge ((c, d) \prec_P (e, f)) \Rightarrow (a, b) \prec_P (e, f)$.

(c) Show that $\prec_P \subseteq \prec_L$.

Choose some a, b, c, d such that $(a, b) \prec_P (c, d)$. This means $a \prec_A c$ and $b \prec_B d$. Thus we know the condition for $\prec_L$ $((a \neq c \wedge a \prec_A c) \vee (a = c \wedge b \prec_B d))$ must be true, as either $a = c$ or $a \neq c$ must be true. Therefore $(a, b) \prec_P (c, d)$ necessarily implies $(a, b) \prec_L (c, d)$.

6

Let A be a nonempty set, and consider $\mathcal{P}(A)$ with $\subseteq$ being the partial ordering. Let $\mathcal{A}$ is any family of subsets of A. Find $\sup \mathcal{A}$ and $\inf \mathcal{A}$ and prove your answers.

We're given $\mathcal{A} \subseteq \mathcal{P}(A)$. By the definition of a powerset, for all $X \in \mathcal{A}$, we also know $X \in \mathcal{P}(A)$, so X is necessarily a subset of A. Thus A is an upper upper bound for $\mathcal{A}$. We also know $A \in \mathcal{P}(A)$, so it is possible for $\mathcal{A} = A$. Therefore A is the *least* upper bound for $\mathcal{A}$. Thus, $\sup \mathcal{A} = A$.

The empty set is a subset of all sets, so $\varnothing \subseteq \mathcal{A}$. We know for all $X \in \mathcal{A}$, $\varnothing \subseteq X$, so $\varnothing$ is a lower bound of $\mathcal{A}$. We also know $\varnothing \in \mathcal{P}(A)$, so it is possible for $\varnothing = \mathcal{A}$. Therefore $\varnothing$ is a *least* lower bound of $\mathcal{A}$ (this is also because there are no other lower boudns for $\mathcal{A}$). Thus, $\inf \mathcal{A} = \varnothing$.

7

Let A be a nonempty set, and consider $\mathcal{S} = \{\mathcal{A} \subseteq \mathcal{P}(A) \mid \mathcal{A} \text{ is a partition of } A\}$. Define a relation $\preceq$ on $\mathcal{S}$ by $\mathcal{B} \preceq \mathcal{C}$ if for each $C \in \mathcal{C}$, there exists some $B \in \mathcal{B}$ such that $C \subseteq B$. In this case, we say that $\mathcal{C}$ is a **refinement** of $\mathcal{B}$. Show that $\preceq$ defines a partial order on $\mathcal{S}$. What is the maximal element?

Reflexive: For all $B \in \mathcal{B}$, $B \subseteq B$ (because $B = B$), so $\mathcal{B} \preceq \mathcal{B}$.

Anti-symmetric: If $\mathcal{B} \preceq \mathcal{C}$, then either $\mathcal{B} = \mathcal{C}$ or there exists some $C \in \mathcal{C}$ such that, for the unique $B \in \mathcal{B}$ such that $C \subseteq B$, we have $C \subset B$ (note that B and C are unique because we're dealing with partitions). The latter option is ruled out by $\mathcal{C} \preceq \mathcal{B}$ because then we would also need $B \subseteq C$, so $\mathcal{B} = \mathcal{C}$.

Transitive: The relation $\mathcal{C} \preceq \mathcal{D}$ implies that for all $D \in \mathcal{D}$, there exist some $C \in \mathcal{C}$ such that $D \subseteq C$. By similar notation, $\mathcal{B} \preceq \mathcal{C}$ implies $C \subseteq B$. The subset relation "$\subseteq$" is transitive, so $D \subseteq B$. Therefore $\mathcal{B} \preceq \mathcal{D}$.

The relation is reflexive, anti-symmetric, and transitive, so it is a partial ordering.

The maximal element or "most refined" set $\mathcal{M}$ is the set of singletons containing every unique element of A. This is because for all possible partitions of A ($\mathcal{X} \in \mathcal{S}$), for all $M \in \mathcal{M}$, there must exist some $X \in \mathcal{X}$ such that $M \subseteq X$.