1

See dynalist notes, transfer eventually (just main results, derivations probably aren't necessary for anything ever at all)

2 Energy of EM Waves: The Poynting Vector

Recall that the "energy density" of an electromagnetic field is the potential energy per volume:

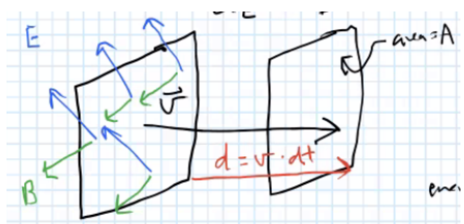
$$u = \frac{U}{V} = \tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2\mu_0} B^2, \tag{1}$$

which when combined with $E = cB$ from the previous lecture, yields:

$$\begin{aligned} u &= \tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2\mu_0}\tfrac{E^2}{c^2}, \\ &\left[use\ c = \tfrac{1}{\sqrt{\epsilon_0\mu_0}}\right] \\ &= \tfrac{1}{2}\epsilon_0 E^2 + \tfrac{1}{2\mu_0}\epsilon_0\mu_0 E^2 \\ &= \epsilon_0 E^2. \end{aligned}$$

So E and B have the same energy density at any given moment in time ($u_E = u_B$). Neat!

Recall that for the plane wave, $\mathbf{E} \times \mathbf{B} = \mathbf{v}$. So we consider an EM plane wave that travels some distnace:



By taking the derivative of equation (1), we find:

$$\begin{aligned} dU &= u \cdot dV \\ &= \left(\epsilon_0 E^2\right)(A \cdot c\ dt) \\ \Rightarrow \frac{1}{A} \cdot \frac{dU}{dt} &= \epsilon_0 c E^2 \\ &[use\ E = cB] \\ &= \epsilon_0 c E(cB) \\ &= \epsilon_0 c^2 EB \\ &\left[use\ c = \tfrac{1}{\sqrt{\epsilon_0\mu_0}}\right] \\ &= \tfrac{1}{\mu_0} EB. \end{aligned}$$

If we define $S := \frac{1}{A} \cdot \frac{dU}{dt}$, we can promote S to a vector:

$$\boxed{\mathbf{S} = \tfrac{1}{\mu_0}\ \mathbf{E} \times \mathbf{B},} \tag{2}$$

which we call the Poynting Vector, representing the energy flowing through EM waves.

Remark: This better describes how energy in an AC circuit flows in one direction, in spite of the electron flow / E-field constantly reversing directions. This Veritasium video gives a great visualization of how the it's the fields transfer energy, not the electrons.

Now consider some EM plane wave moving in the $\hat{x}$ *direction*, given by $\begin{cases} \mathbf{E} = E_0 \cos(kx - \omega t)\hat{y} \\ \mathbf{B} = B_0 \cos(kx - \omega t)\hat{z} \end{cases}$.

We "reverse" the wave to move in $-\hat{x}$ *direction* by flipping $\mathbf{B}$ and changing the phase such that as t increases, the cos wave shifts to the left ($-x$ direction): $\begin{cases} \mathbf{E} = E_0 \cos(kx + \omega t)\hat{y} \\ \mathbf{B} = B_0 \cos(kx + \omega t)(-\hat{z}) \end{cases}$.

We plug these $\mathbf{E}$ and $\mathbf{B}$ into the Poynting Vector from (2) to get

$$\mathbf{S} = \tfrac{1}{\mu_0} E_0 B_0 \cos^2(kx - \omega t)\hat{x}.$$

Note that this is positive for all t, which solidifies the notion of energy transfer (if it were oscillating back and forth, it would cancel out).

The average of $\cos^2$ is a half, so the average of the Poynting vector becomes:

$$\boxed{\langle S \rangle = \tfrac{1}{2\mu_0} E_0 B_0.}$$
