1 Standard Form for Linear Equations

Given $F(t, y, y') = 0$, an equation where y and y' are linear, we can separate coefficients to find the standard form:

$$\boxed{y' + p(t)y = g(t)}. \tag{1}$$

1.1 Separable Case

If $g(t) = 0$, note that our linear equation also becomes **separable** ($y' + p(t)y = 0$), which we can easily solve to yield the general solution:

$$\boxed{y_{gen} = Ce^{-\int p(t)\ dt}}. \tag{2}$$

1.2 Non-Separable Case

Consider an equation of form eq (1). We choose some "integrating factor" $\mu(t) \neq 0$ (shortened to μ) such that $\mu' = p(t)\mu$. This property allows us to modify eq (1) as follows:

$$\begin{aligned}
y' + p(t)y &= g(t) \\
y'\mu + p(t)\mu y &= g(t)\mu \\
y'\mu + \mu' y& \\
(y\mu)'& \\
y\mu &= \int \mu(t)g(t)\ dt.
\end{aligned}$$

This gives us the general equation:

$$\boxed{y_{gen} = \frac{1}{\mu(t)} \int \mu(t)g(t)\ dt.} \tag{3}$$

Knowing $\mu' = p(t)\mu$, we can readily find:

$$\boxed{\mu(t) = e^{\int p(t)\ dt}.} \tag{4}$$

1.3 Examples

2lazy2type lol, just look at the red boxes for things to note:

$$\begin{cases} t^3 y' + 4t^2 y = e^{-t} \\ y(-1) = 0, \quad t < 0 \end{cases}$$

eq. in standard form : $y' + \underbrace{\frac{4}{t}}_{p(t)} y = \underbrace{t^{-3} e^{-t}}_{g(t)}$

$$\mu = e^{\int \frac{4}{t}\,dt} = e^{4\ln|t|} = \boxed{e^{\ln t^4} = t^4}$$

$$\int \mu g\,dt = \int t^4 t^{-3} e^{-t}\,dt = \int \underbrace{t}_{u} \underbrace{e^{-t}\,dt}_{dv} \qquad \begin{aligned} du &= dt \\ v &= -e^{-t} \end{aligned}$$

$$= -te^{-t} + \int e^{-t}\,dt = -te^{-t} - e^{-t}$$

$$y_{gen} = \frac{1}{\mu}\left[\int \mu g\,dt + c\right] = \frac{1}{t^4}\left(-te^{-t} - e^{-t} + c\right)$$

at $t = -1$: $\cancel{e} - \cancel{e} + c = 0 \Rightarrow c = 0$

Sol. : $y = -e^{-t}(t+1)\frac{1}{t^4} = -e^{-t}\left(t^{-3} + t^{-4}\right), \quad \boxed{t < 0.}$