1

(a) Prove that for any real x, if $x^2 < 73$, then $0 < 1$.

$0 < 1$ will always be true, so the conclusion is true regardless of the premises. Thus the implication is trivial.

(b) Prove that for any integer x, if $-x^2 > 0$, then $x = 5$.

$(-x^2 > 0)$ will never be true for any x, so the premise is always false. Thus the implication is vacuous.

2

Prove the following by **direct** proof.

(a) Prove that if x is an odd integer, then $7x - 5$ is even.

Let x be an odd integer. Then there exists some integer k such that $x = 2k + 1$. We can substitute and do some basic arithmetic to find:

$$\begin{aligned} 7x - 5 &= 7(2k + 1) - 5 \\ &= 14k + 2 \\ &= 2(7k + 1). \end{aligned}$$

We can deduce that $(7k + 1) \in \mathbb{Z}$ because the multiplication and addition of integers can only yield an integer. Therefore $2(7k + 1)$ can also be expressed as $2(k')$ for some $k' \in \mathbb{Z}$. This is the definition of an even integer, so we can conclude that if x is an odd integer, then $7x - 5$ is even.

(b) Let a, b, c be integers. Prove that if a and c are odd, then $ab + bc$ is even.

Let a, b, c be integers, and let a and c be odd. Then there exist some integers k_1 and k_2 such that $a = 2k_1 + 1$ and $c = 2k_2 + 1$. We can substitute and do some basic arithmetic to find:

$$\begin{aligned} ab + bc &= (2k_1 + 1)b + b(2k_2 + 1) \\ &= b(2k_1 + 2k_2 + 2) \\ &= 2(b(k_1 + k_2 + 1)). \end{aligned}$$

We can deduce that $(b(k_1 + k_2 + 1)) \in \mathbb{Z}$ because the multiplication and addition of integers can only yield an integer. Therefore $2(b(k_1 + k_2 + 1))$ can also be expressed as $2(k')$ for some $k' \in \mathbb{Z}$. This is the definition of an even integer, so we can conclude that if a and c are odd, then $ab + bc$ is even.

(c) Prove that every odd integer is the difference of two square integers.

Let x be an odd integer. Then there exists some integer k such that $x = 2k + 1$. We can do some basic arithmetic to find:

$$\begin{aligned} x &= 2k + 1 \\ &= 2k + 1 + k^2 - k^2 \\ &= (k^2 + 2k + 1) - k^2 \\ &= (k + 1)^2 - k^2. \end{aligned}$$

We see that $k+1$ and k are integers, so we can conclude that that any odd integer x can be represented as the difference of two square integers.

3

(a) Prove that for an integer x, x is odd if and only if x^3 is odd. [Hint: You have to prove two directions for this, x is odd $\Rightarrow x^3$ is odd, and x is odd $\Leftarrow x^3$ is odd. It is customary to do this in two stages, where you label the direction you are proving with the relevant arrow. I would start the first stage by writing ($\Rightarrow$), and the second stage with a ($\Leftarrow$) to make it clear what I am doing. Also, for the second stage, the contrapositive may help you.]

($\Rightarrow$) Let x be an odd integer. Then there exists some integer k such that $x = 2k+1$. We can substitute and do some basic arithmetic to find:

$$\begin{aligned} x^3 &= (2k+1)^3 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1. \end{aligned}$$

We can deduce that $(4k^3 + 6k^2 + 3k) \in \mathbb{Z}$ because the multiplication and addition of integers can only yield an integer. Therefore $2(4k^3 + 6k^2 + 3k) + 1$ can also be expressed as $2(k') + 1$ for some $k' \in \mathbb{Z}$. This is the definition of an odd integer, so we can conclude that if x is odd, then x^3 is odd.

($\Leftarrow$) We will prove that x^3 is odd implies that x is odd by proving the contrapositive: if x is even, then x^3 is even. Let x be an even integer. This means there exists some integer k (no relation to previous section) such that $x = 2k$. We can substitute and do some basic arithmetic to find:

$$\begin{aligned} x^3 &= (2k)^3 \\ &= 8k^3 \\ &= 2(4k^3). \end{aligned}$$

We can deduce that $4k^3 \in \mathbb{Z}$ because the multiplication and addition of integers can only yield an integer. Therefore $2(4k^3)$ can also be expressed as $2(k') + 1$ for some $k' \in \mathbb{Z}$ (no relation to previous section). This is the definition of an even integer, so we can conclude that if x is even, then x^3 is even. Thus, by contrapositive, if x^3 is odd then x is also odd.

Having proven both directions of implication, we can conclude that an integer x is odd if and only if x^3 is odd.

(b) Consider the following definition:

Definition 3.19: If x **divides** y, then there is some integer k such that $y = kx$.

Prove that if four does not divide x^2 when x is an integer, then x is odd.

We will proceed by proving the contrapositive: if x is an even integer, then 4 divides x^2. Let x be an even integer. Then there exists some integer k_1 such that $x = 2k_1$. We check if 4 divides x^2 by substituting them into definition 3.19 and solving for k:

$$\begin{aligned}(x^2) &= k(4)\\ (2k_1)^2 &= 4k\\ k_1^2 &= k.\end{aligned}$$

We observe that k is an integer, so we can conclude that if x is an even integer, then 4 divides x^2 by definition 3.19. Thus by contrapositive, if 4 does not divide x^2, then x is an odd integer.

4

(a) Prove that there is no largest integer.

Assume for the sake of contradiction that there does exist some largest integer x such that it is greater than all integers. Let $y = x + 1$ be an integer. We observe that $y > x$, which contradicts our assumption that x is greater than all integers. Therefore there exists no largest integer.

(b) Prove that there is no smallest positive rational number.

Assume for the sake of contradiction that there does exist some smallest positive rational number x. Let $y = x/2$ be a rational number. We observe that $y < x$ and $y > 0$, which contradicts our assumption that x is less than all positive rational numbers. Therefore there exists no smallest positive rational number.

(c) Prove or disprove: The product of two irrational numbers is irrational.

Let x be a rational number such that $\sqrt{x}$ is irrational. We observe that $\sqrt{x} \cdot \sqrt{x} = x$, proving tht the product of two irrational numbers could be irrational.

(d) Prove or disprove: The sum of a rational and irrational number is irrational.

Let p/q be a rational number and let x be an irrational number. For the sake of contradiction, let their sum be a rational number, p'/q'. We can rearrange this statement to find:

$$\begin{aligned}\frac{p}{q} + x &= \frac{p'}{q'}\\ x &= \frac{p'}{q'} - \frac{p}{q}\\ x &= \frac{p'q - pq'}{qq'}.\end{aligned}$$

We observe that $p'q$, pq', and qq' all yield integers, so x can be expressed as a ratio of nonzero integers. But this contradicts our premise that x is an irrational number. Therefore the sum of a rational and irrational number must be irrational.

5

(a) (Triangle Inequality). Prove that for any $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$. [Hint: Proof by cases.]

Let x and y be real numbers. Then we have the following cases:

- **Case 1:** Let $|x+y| = x+y$. From this, we can simplify:

$$\begin{aligned} |x+y| &\leq |x| + |y| \\ x+y &\leq |x| + |y|, \end{aligned}$$

which is necessarily true because $x \leq |x|$ and $y \leq |y|$. Therefore the Triangle Inequality holds for the case $|x+y| = x+y$.

- **Case 2:** Let $|x+y| = -(x+y)$. From this, we can simplify:

$$\begin{aligned} |x+y| &\leq |x| + |y| \\ -x-y &\leq |x| + |y|, \end{aligned}$$

which is necessarily true because $-x \leq |x|$ and $-y \leq |y|$. Therefore the Triangle Inequality holds for the case $|x+y| = -(x+y)$.

The Triangle Inequality holds for both cases, so we can conclude that it is true.

(b) (Reverse Triangle Inequality). Prove that for any $x, y \in \mathbb{R}$, $||x| - |y|| \leq |x-y|$. [Hint: $x = x - y + y$.]

Let x and y be real numbers. Then we have the following cases:

- **Case 1:** Let $||x| - |y|| = |x| - |y|$. We can prove this by expanding $|x|$:

$$\begin{aligned} |x| &= |x + y - y| \\ &= |(x-y) + (y)|, \end{aligned}$$

at which point we can apply the Triangle Inequality to find:

$$\begin{aligned} |x| &\leq |x-y| + |y| \\ |x| - |y| &\leq |x-y|. \end{aligned}$$

Thus the Reverse Triangle Inequality holds for $||x| - |y|| = |x| - |y|$.

- **Case 2:** Let $||x| - |y|| = -(|x| - |y|) = |y| - |x|$. We repeat the same process from case 1 but replacing x and y to find $|y| - |x| \leq |x-y|$. Thus the Reverse Triangle Inequality holds for $||x| - |y|| = -(|x| - |y|)$.

The Reverse Triangle Inequality holds for both cases $\big(\big||x| - |y|\big| = \pm(|x| - |y|)\big)$, so we can conclude that it is true.