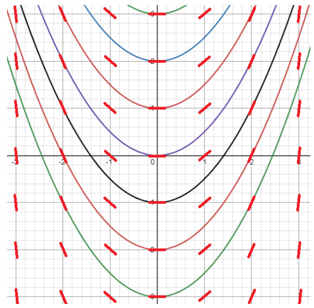
1 Slope Fields

Given a family of functions representing the solutions to an ODE, we can visualize the differential equation itself as a slope field that describes the "flow" (tangent vectors) in $\mathbb{R}^2$.

For example, consider the first order ODE $y' = t$ with the solution $y = \frac{1}{2}t^2 + C$:



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2 Separable ODEs

Recall that the explicit form of first-order (linear) ODEs is:

$$y' = f(t, y).$$

Similarly, separable ODEs take the explicit form:

$$\boxed{y' = g(t)\ h(y)}. \tag{1}$$

Note that the obvious implication of this is that not all first-order ODEs are separable.

2.1 Nonlinear Separable ODEs (Procedure to Solve)

Note that if $h(y_0) = 0$, then y_0 is a solution of (1): the RHS is forced to be 0, and the LHS is 0 because the derivative of a constant (y_0) is 0.

To find **non-constant** solutions, we take a similar approach to section 2.1 in Lecture Notes 1, and rewrite (1):

$$\frac{dy}{dt} = g(t)\ h(y)$$
$$\int \frac{dy}{h(y)} = \int g(t)\ dt$$
$$a(y) = b(t). \tag{2}$$

This is called the implicit solution, which effectively eliminates the derivative in favor of arbitrary functions in terms of y and t. Finally, we can solve for some $y = y(t)$.

Note 2.1.A: If we're given some IVP (initial value proposition $y(t_0) = y_0$), we can fully define our general solution (ie find a value of C),,,

Be sure to define an interval for the domain of t to avoid indeterminate solutions (eg div by 0). The interval **must** be "connected" to the IVP (see 2.1.2). [WHY THE FUCK??? ASK IN RECITATION]

Example 2.1:

Problem: Solve $y' = t^2 y^2$.

Solution:

y' is *separable*, with $\begin{cases} g(t) = t^2 \\ h(y) = y^2 \end{cases}$.

To find *constant solutions*, let $h(y) = y^2 = 0$. Thus $y = 0$.

For *general solutions* ($y \neq 0$), find the implicit form:

$$\int \frac{dy}{y^2} = \int t^2 \ dt$$
$$\frac{-1}{y} = \frac{t^3}{3} + C_1$$
$$\implies y = \frac{-3}{t^3 + C}.$$

Thus, $y_{gen} = \left\{0,\ \frac{-3}{t^3+C}\right\}$.

Example 2.2:

Problem: $\begin{cases} y' = (1-2x)y^2 \\ y(0) = \frac{-1}{6} \end{cases}$

Solution:

For $y \neq 0$,

$$\int \tfrac{dy}{y^2} = \int (1-2x) \ dx$$
$$y = \frac{-1}{x - x^2 + C}.$$

Our IVP implies $C = 6$. Thus $y(x) = \frac{1}{(x-3)(x+2)}$.

Since $x \neq 3$ and $x \neq -2$, we may naively conclude that the domain of y is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$. However, with Note 2.1.A in mind, we know the interval must be "connected" to $x_0 = 0$, as defined by our IVP. Thus, the domain of y is $(-2, 3)$.