1

(a) What set (that we know) is $\bigcup_{x\in\mathbb{Z}}\{x, x+1, x+2\}$?

This is equal to the set of all integers, $\mathbb{Z}$.

This is because $\bigcup\limits_{x\in\mathbb{Z}}\{x\} = \bigcup\limits_{x\in\mathbb{Z}}\{x+1\} = \bigcup\limits_{x\in\mathbb{Z}}\{x+2\} = \mathbb{Z}$. Redundancies are removed in sets.

(b) What set (that we know) is $\bigcup_{n\in}(-n, n)$?

This is equal to the set of all real numbers, $\mathbb{R}$.

(c) What set is $\bigcap_{n\in}(-n, n)$? Use formal set builder notation to describe it.

$\{x \in \mathbb{R} \mid (x > -1) \wedge (x < 1)\} = (-1, 1)$.

(d) What set is $\bigcup_{n=2}^{\infty}[0, 1-1/n)$ (`\bigcup_{n = 2} ^\infty [0, 1 - 1/n)`)?

$\{x \in \mathbb{R} \mid (x \geq 0) \wedge (x < 1)\} = [0, 1)$.

(e) What set is $\bigcup_{x\in\mathbb{Z}}(\bigcup_{n=2}^{\infty}[x, x+1-1/n))$? Express your answer as simply as possible.

This is equal to $\mathbb{R}$.

The problem simplifies to $\bigcup\limits_{x\in\mathbb{Z}}\{[x, x+1)\} = (-\infty, \infty)$.

2

For the sets in problem 1, find the complement of each set with respect to $\mathbb{R}$. (Recall that the complement of A with respect to B is every element in B that is not in A)

(a) $\mathbb{Z}^C$.

$\{x \in \mathbb{R} \mid x \notin \mathbb{Z}\} = \bigcup\limits_{x\in\mathbb{Z}}(x, x+1)$.

Question from student: of the two above, which method is better?

(b) $\mathbb{R}^C$

$\varnothing$.

(c) $(-1, 1)^C$

$\{x \in \mathbb{R} \mid (x \leq -1) \vee (x \geq 1)\} = (-\infty, -1] \cup [1, \infty)$.

(d) $[0, 1)^C$.

$\{x \in \mathbb{R} \mid (x < 0) \vee (x \geq 1)\} = (-\infty, 0) \cup [1, \infty)$.

(e) $\mathbb{R}^C$.

$\varnothing$.

3

(a) Prove that the cardinality of $|A \times B|$ is $|A| \cdot |B|$, assuming that A and B are finite. It does not have to be particularly rigorous.

The definition of the cartesian product is $A \times B = \{(a, b) \mid a \in A, b \in B\}$. We have $|A|$ unique options for a, and for each of those a, we have $|B|$ unique options for b. Therefore, there are $|A| \cdot |B|$ unique elements in $A \times B$ — in other words, $|A \times B| = |A| \cdot |B|$.

(b) What is the cardinality of $\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))))$? What about if I apply the powerset function to the empty set n times in general?

$|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))))| = 16$.

Recall that for an arbitrary finite set A, $|\mathcal{P}(A)| = 2^{(|A|)}$. If we were to apply the powerset function again, we'd get $|\mathcal{P}(\mathcal{P}(A))| = 2^{(|\mathcal{P}(A)|)} = 2^{(2^{(|A|)})}$. In our case, $A = \varnothing$, so $|A| = 0$, which makes $2^{(2^{(|A|)})} = 2$.

Generalizing this, applying the powerset function n times to the empty set yields a set with cardinality $2^{\cdot^{\cdot^{\cdot^{2}}}}$, where 2 appears $n-1$ times (or in "tetration" notation, ${}^{(n-1)}2$).

This is clear by manually calculating each nested powerset:

- $|\varnothing| = 0$,
- $|\mathcal{P}(\varnothing)| = 2^0 = 1$,
- $|\mathcal{P}(\mathcal{P}(\varnothing))| = 2^{2^0} = 2$,
- $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing)))| = 2^{2^{2^0}} = 4$,
- $|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathcal{P}(\varnothing))))| = 2^{2^{2^{2^0}}} = 16$.

4

(a) Prove that $(A \setminus B) \subseteq A$.

By Definition 2.17, $A \setminus B := \{x \in A \mid x \notin B\}$. Thus it trivially follows that x is an element of A. Because an arbitrary element of $A \setminus B$ is necessarily an element of A, we can conclude $A \setminus B \subseteq A$.

(b) Prove the first DeMorgan's law for sets that I listed above:

$$(A \cup B)^C = A^C \cap B^C.$$

Let $x \in (A \cup B)^C$.

Then $x \notin (A \cup B)$,

$(x \notin A) \wedge (x \notin B)$,

$(x \in A^C) \wedge (x \in B^C)$,

$x \in (A^C \cap B^C)$.

Therefore, $(A \cup B)^C \subseteq (A^C \cap B^C)$.

Let $y \in (A^C \cap B^C)$.

Then $(y \in A^C) \wedge (y \in B^C)$,

$(y \notin A) \wedge (y \notin B)$,

$y \notin (A \cup B)$,

$y \in (A \cup B)^C$.

Therefore $(A^C \cap B^C) \subseteq (A \cup B)^C$.

$(A \cup B)^C \subseteq (A^C \cap B^C)$ and $(A^C \cap B^C) \subseteq (A \cup B)^C$, so $(A \cup B)^C = A^C \cap B^C$.

5

Define the **symmetric difference** of two sets A, B as follows:

$$A \triangle B := (A \setminus B) \cup (B \setminus A).$$

Prove that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

- We begin with the definition of the symmetric difference:
  $(A \setminus B) \cup (B \setminus A)$.
- By the definition of set minus:
  $(A \cap B^C) \cup (B \cap A^C)$.
- By distribution [of form $P \cup (Q \cap R)$, where $P = (A \cap B^C)$, $Q = B$, and $R = A^C$]:
  $((A \cap B^C) \cup B) \cap ((A \cap B^C) \cup A^C)$.
- By distribution [of form $(P \cap Q) \cup R$]:
  $((A \cup B) \cap (B^C \cup B)) \cap ((A \cup A^C) \cap (B^C \cup A^C))$.

- By law of excluded middle and definition of truth:
  $(A \cup B) \cap (B^C \cup A^C)$.
- By commutivity of union:
  $(A \cup B) \cap (A^C \cup B^C)$.
- By definition of set minus:
  $(A \cup B) \setminus (A^C \cup B^C)^C$.
- By DeMorgan's first law for sets:
  $(A \cup B) \setminus (A \cap B)$.

Therefore, we find that $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$. $\square$

6

We can perform unions and intersections without an indexing set, as well. For example, we can write

$$\bigcup_{X \in \mathcal{P}(A)} X = A.$$

(You can prove this to yourself if you'd like). Now let's define the following set:

$$\mathcal{T}_A := \{X \subseteq A \mid |X| = 2\}.$$

We say this set is *parameterized* by a set A. Now, for which A is the union

$$\bigcup_{X \in \mathcal{T}_A} X = A?$$

$\bigcup_{X \in \mathcal{T}_A} X = A$ for all A with a cardinality greater than or equal to 2.