1

Let $A = \{1, 2\}, B = \{x, y\}$. List all functions from $A \to B$ that are:

(a) injections;

- $\{(1, x), (2, y)\}$
- $\{(1, y), (2, x)\}$

(b) surjections;

same as (a)

(c) bijections;

same as (a)

(d) none of the above.

- $\{(1, x), (2, x)\}$
- $\{(1, y), (2, y)\}$

2

(a) Supply a proof for Proposition 7.8.

Choose some $x, y \in A$ such that, for some $z \in C$, $g(f(x)) = g(f(y)) = z$.

Let $a = f(x)$ and $b = f(y)$. Because g is injective, $g(a) = g(b)$ implies $a = b$. Therefore $f(x) = f(y)$. Because f is injective, $f(x) = f(y)$ implies $x = y$.

Thus $g \circ f$ is injective, because for all $x, y \in A$, if $g(f(x)) = g(f(y))$ then $x = y$. $\square$

(b) Suppose $f : A \to B$ and $g : B \to C$. If $g \circ f$ is injective, is it necessarily the case that f is injective? Is it necessarily the case that g is injective? Prove or disprove your claims.

Choose some $x, y \in A$ such that $f(x) = f(y)$. We know $f(x), f(y) \in B$, so we can take g of both sides to find $g(f(x)) = g(f(y))$. As proven in part (a), since $g \circ f$ is injective, this implies $x = y$. Thus f is injective.

If we let $a = f(x)$ and $b = f(y)$, then the previous equation $(g(f(x)) = g(f(y)))$ becomes $g(a) = g(b)$. Since $x = y$, we also know $f(x) = f(y)$, or $a = b$. Note that $a, b \in f(A)$, thus g is injective over the image of A under f. $\square$

(c) Repeat part (b) but replace injective with surjective.

Choose some $z \in C$. Since $g \circ f$ is surjective, there exists an $x \in A$ such that $(g \circ f)(x) = g(f(x)) = z$. If we let $b = f(x)$, then we get $g(y) = z$. Thus g is surjective.

However f is not necessarily surjective as the image of f does not have to be all of B. For example, take $A = C = \{1\}$, $B = [2]$, and $f(x) = g(x) = 1$.

3

Find a function from $\mathbb{R}$ to $\mathbb{R}$, and supply a proof of your claim, that is:

(a) an injection but not a surjection;

$f(x) = e^x$ is an injection but not a surjection.

Injective: Choose some $a, b \in \mathbb{R}$ such that $f(a) = f(b)$. Some algebraic manipulation yields:

$$\begin{aligned} f(a) &= f(b) \\ e^a &= e^b \\ \ln e^a &= \ln e^b \\ a &= b. \end{aligned}$$

Not surjective: As an example, $-1 \in \mathbb{R}$, but there does not exist some $x \in \mathbb{R}$ such that $e^x = -1$. The range of $f(x)$ is $(0, \infty)$.

(b) a surjection but not an injection;

$f(x) = x^3 + x^2$ is a surjection but not an injection.

Surjective: We know f is continuous for $\mathbb{R}$ and $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$. Therefore, by the intermediate value theorem, for any $y \in (-\infty, \infty) = \mathbb{R}$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$.

Not Injective: As an example, if $a = 0$ and $b = -1$, we can have $f(a) = f(b) = 0$ while $a \neq b$.

(c) a bijection;

$f(x) = x$ is a bijection.

Injective: If we choose some $a, b \in \mathbb{R}$ such that $f(a) = f(b)$, it follows that $a = b$.

Surjective: Choose some $y \in \mathbb{R}$. There exists some $x = y$ such that $f(x) = y$.

(d) neither a surjection nor an injection.

$f(x) = x^2$ is neither an injection nor a surjection.

Not Injective: Choose some $a, b \in \mathbb{R}$ such that $a = -b$. We find that $f(a) = f(b) = a^2$ even though $a \neq b$.

Not Surjective: As an example, $-1 \in \mathbb{R}$, but there is no x such that $f(x) = -1$. The range of f is $[0, \infty)$.

4

Let $f : A \to B$, and let $G_1, G_2 \subseteq A$, and let $H_1, H_2 \subseteq B$.

(a) Is it true that $f^{-1}(f(G_1)) = G_1$? If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.

We know the image of G_1 under f, or $f(G_1)$, will map to a set in B, which we'll call G'. The pre-image of G', or $f^{-1}(G')$ will map back to the original values in A that map to G', which we originally said was the set G_1. Thus $f^{-1}(f(G_1)) = G_1$.

(b) Is it true that $f(f^{-1}(H_1)) = H_1$? If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.

The pre-image of H_1, or $f^{-1}(H_1)$, will map to a set of values in A (which we'll call H') that are characterized by mapping to H_1 under f. The image of H', or $f(H')$, will map to a set of values in B under the relation f. But we originally said that this set was H_1, thus $f(f^{-1}(H_1)) = H_1$.

(c) Is it true that $f(G_1 \cap G_2) = f(G_1) \cap f(G_2)$? If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.

Not true: As a counterexample, let $f : [2] \to \{1\}$, and let $G_1 = \{1\}$, $G_2 = \{2\}$. The left side evaluates to $f(G_1 \cap G_2) = f(\varnothing) = \varnothing$. However, the right side evaluates to $f(G_1) \cap f(G_2) = \{1\} \cap \{1\} = \{1\}$.

The correct relation is $f(G_1 \cap G_2) \subseteq f(G_1) \cap f(G_2)$. We know $G_1 \cap G_2 \subseteq G_1$, which implies $f(G_1 \cap G_2) \subseteq f(G_1)$. A similar argument but with G_2 shows $f(G_1 \cap G_2) \subseteq f(G_2)$. Combining these, we get $f(G_1 \cap G_2) \subseteq f(G_1) \cap f(G_2)$.

A more standard way to prove this would be to choose some $x \in f(G_1 \cap G_2)$. If we find the pre-image of both sides, we get $f^{-1}(x) \in G_1 \cap G_2$. We can split this into $f^{-1}(x) \in G_1$ and $f^{-1}(x) \in G_2$, then find the images which are $x \in f(G_1)$ and $x \in f(G_2)$ respectively. Combining these yeilds $x \in f(G_1) \cap f(G_2)$. Thus $f(G_1 \cap G_2) \subseteq f(G_1) \cap f(G_2)$.

(d) Is it true that $f^{-1}(H_1 \cap H_2) = f^{-1}(H_1) \cap f^{-1}(H_2)$? If so, prove it; if not, provide a counterexample, and provide the correct relation between the two sets, and justify your answer.

Choose some $h \in f^{-1}(H_1 \cap H_2)$. If we find the image of both sides, we get $f(h) \in H_1 \cap H_2$. We can split this into $f(h) \in H_1$ and $f(h) \in H_2$, then find the pre-images which are $h \in f^{-1}(H_1)$ and $h \in f^{-1}(H_2)$ respectively. Combining these yeilds $h \in f^{-1}(H_1) \cap f^{-1}(H_2)$. Thus $f^{-1}(H_1 \cap H_2) \subseteq f^{-1}(H_1) \cap f^{-1}(H_2)$.

Choose some $h' \in f^{-1}(H_1) \cap f^{-1}(H_2)$. We can split this into $h' \in f^{-1}(H_1)$ and $h' \in f^{-1}(H_2)$. If we find the images of both sides of each, we get $f(h') \in H_1$ and $f(h') \in H_2$. Combining these yields $f(h') \in H_1 \cap H_2$, and finding the pre-image of both sides gives $h' \in f^{-1}(H_1 \cap H_2)$. Thus $f^{-1}(H_1) \cap f^{-1}(H_2) \subseteq f^{-1}(H_1 \cap H_2)$.

Having proven subset equality in both directions, we can conclude $f^{-1}(H_1 \cap H_2) = f^{-1}(H_1) \cap f^{-1}(H_2)$.
