1

Find the error in the proof of Proposition 5.14 (it is not true!). Respond in the homework channel on Discord rather than here, and discuss potential solutions.

The proof wrongfully assumes that s exists. This is not the case in a tree with a single vertex ($G = (\{v_1\}, \{\})$), where v_1 is not connected to any vertices.

This can be fixed by specifying that V has atleast 2 elements ($|V| \geq 2$) at the beginning of the proof.

2

Prove that the sum of degrees of vertices in a simple graph is even.

Let $P(n)$ be the statement that the sum of degrees of vertices in a simple graph with n vertices is even.

$P(0)$ is true, as a graph with 0 vertices has 0 connections. $P(1)$ is also true, as a graph with 1 vertex has 0 connections.

Now suppose $P(k)$ is true for some integer $k \geq 1$. Let G be a simple graph with $k+1$ vertices. Choose some vertex v in G, and let V' be the set of vertices connected to v. Removing v from G will decrease the sum of degrees of vertices by $2 \cdot |V'|$, as v has a degree of $|V'|$, and the degree of all vertices in V' will decrease by one.

Note that $2 \cdot |V'|$ is even because $2 \cdot |V'| = 2k$' for some integer $k' = |V'| \in \mathbb{Z}$.

G is now a simple graph with k vertices, so the sum of their degrees is even. If we re-add v (and its original connections) to G, the sum of degrees of vertices will increase by $2 \cdot |V'|$. The sum of two even numbers is even, so $P(k+1)$ follows. Thus by induction, $P(n)$ is true for all integers $n \geq 0$. $\square$

3

(a) Prove that a connected, simple graph with $|V| - 1$ edges is acyclic.

Let $P(n)$ be the statement that at least $n-1$ edges are required to connect a simple graph with n vertices. $P(1)$ is true, as a simple graph with one vertex has zero edges and is connected.

Now suppose $P(k)$ for some natural number $k \geq 1$. Take a simple graph with $k+1$ vertices and zero edges. Removing an arbitrary vertex, v, allows us to form a connected graph with $k-1$ edges, which is the minimum. If we now re-add v, we need to add a minimum of one edge to ensure connectedness. Thus we have a graph with $k+1$ vertices and the minimum number of edges, $(k-1)+1 = k$, so $P(k+1)$ follows. Thus by induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Now suppose for the sake of contradiction that there exists a connected simple graph with $|V|-1$ edges that is also cyclic. Because it's cyclic, we can remove at least one edge while remaining connected. However, this leads to a connected graph with less than $|V|-1$ edges, which contradicts $P(|V|)$. Therefore, a connected simple graph with $|V|-1$ edges must be acyclic. $\square$

(b) Prove that an acyclic, simple graph with $|V| - 1$ edges is connected.

Let $P(n)$ be the statement that an acyclic, simple graph with n vertices and $n-1$ edges is connected. $P(1)$ is true because there is only one configuration (a single vertex), and it is both acyclic and connected.

Now supposed $P(k)$ for some natural number $k \geq 1$. Let G be a simple, acyclic graph with $k+1$ vertices and k edges. Because G is acyclic and has atleast one edge, we can find a subgraph of G containing a set of connected, acyclic vertices. This subgraph is a tree, so it must have at least two leaves by Proposition 5.14 (note that we escape the error in problem 1 because there are at least two vertices). Thus G has at least two leaves. Removing one of these leaves will remove a single edge, resulting in a new graph, G', with k vertices and $k-1$ edges. G' is still acyclic as we are not adding any new edges, so $P(k)$ implies that G' is connected. If we re-add the leaf (and edge) we removed, we have a simple, acyclic graph with $k+1$ vertices and k edges that is connected, so $P(k+1)$ follows. Thus by induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

4

Prove that a simple graph is bipartite iff it contains no odd-length cycles.

($\Rightarrow$) Divide the vertices of a bipartite graph into two disjoint and independent sets, U and V such that every edge connects a vertex in U to one in V. Suppose for the sake of contradiction that the bipartite graph contains an odd-length cycle, represented by the path $v_1, v_2, \ldots, v_n, v_1$ where n must be odd. Let v_1 be an element of one of the sets, say U. Then $v_i \in U$ if i is odd (there exists some integer k such that $i = 2k+1$) as every 2 "steps" bring us back to an element of U. Thus $v_n \in U$, but this leads to a contradiction as v_n and v_1 share an edge but are both elements of U. Therefore if a graph is bipartite, then it contains no odd-length cycles.

($\Leftarrow$) Let G be a connected graph with no odd-length cycles. Choose some vertex v. Let U be the set of vertices with an odd-length path from v, and let V be the set of vertices with an even-length path from v. Thus U and V are disjoint, and together they comprise all vertices in G.

Suppose for the sake of contradiction that there exist two unique vertices, v_a and v_b, such that they belong to the same set ($v_a, v_b \in U$ or $v_a, v_b \in V$) and are connected by an edge.

- Case 1: If $v_a = v$, then the distance between v and v_a is zero, which is even. Vertex v_b is in the same set as v_a, so the distance between v_b and v (equivalently, the distance between v_b and v_a) must also be even. But this leads to a contradicion, as v_a and v_b being connected by an edge would imply the distance between them is one, which is odd.
- Case 2: Assume $v_a \neq v_b \neq v$. Let A be the shortest path from v to v_a, and let B be the shortest path from v to v_b. Vertices v_a and v_b are both in the same set, so the lengths of A and B must be of the same parity. Thus, traveling from v_a to v_b by traversing path A in the reverse direction and then path B in the forward direction will have an even-length, as the sum of two numbers with the same parity is even. This means there exists some integer k such that the path length equals $2k$. If we then travel from v_b to v_a via the edge that connects them, the path length becomes $2k+1$. However this leads to a contradiction as we now have an odd-length cycle.

Thus, every edge connects a vertex in U to one in V and vice versa, so G is bipartite. $\square$

5

A simple graph is said to be k-colorable if there is a function $f : V \to [k]$ such that for any edge (a, b), $f(a) \neq f(b)$.

(a) Show that a simple graph is 2-colorable iff it is bipartite.

($\Rightarrow$) Let G be a 2-colorable graph, meaning all vertices get one of two colors and no vertices that share an edge have the same color. Let U be the set of vertices with the first color, and let V be the set of vertices with the second color. No vertices are in both sets, both sets comprise all vertices, and every edge connects vertices of different colors. Thus G is bipartite.

($\Leftarrow$) Let G be a bipartite graph. Divide the vertices into two disjoint and independent sets, U and V, such that every edge connects a vertex in U to one in V. Coloring the vertices in U the first color and V the second color ensures vertices that share an edge have different colors, and thus is a valid 2-coloring. □

(b) † Show that a simple graph whose vertices all have degree at most k is $(k + 1)$-colorable.

Let $P(n)$ be the statement that a simple graph whose vertices all have degree at most n is $(n+1)$-colorable. $P(1)$ is true, as a single vertex must have degree zero and is 1-colorable.

Suppose $P(k)$ is true for some natural number $k \geq 1$. Take a simple graph with $k + 2$ vertices and degree at most $k + 1$ (eg star graph S_{k+1}). If we remove one vertex, it is still a simple graph but the degree is now at most k, thus $P(k)$ implies it is $(k + 1)$-colorable. We can now re-add the vertex (and edges) back to the graph and color the vertex differently from all other verticies to ensure a valid coloring. This yields a simple graph with degree at most $k+1$ that is $(k+2)$-colorable, so $P(k+1)$ follows. Thus by induction, $P(n)$ holds for all $n \in \mathbb{N}$. □
