get ready for the  iest proofs of your lifetime. i am very sorry

1

There is an error in Lemma 8.7, similar to the last time I pointed out my own error. Find it, fix the statement of the lemma, and amend the proof to adapt to this. Discuss in Discord.

If $\{b\}$ were an "infinite sequence" of empty sequences it would just be empty, so we'd run into the problem where the "then there would be no elements from b_{N+1} onwards" condition for contradiction isn't satisfied.

So we must specify at the beginning that "...let B be the set of **non-empty** finite sequences on A."

2

(Pigeonhole Principle).

(a) Show that there is no injection from $[n+1]$ to $[n]$, for any nonnegative integer.

We know $|[n+1]| > |[n]|$, so $f : [n+1] \to [n]$ can't be injective. $\square$

(b) † Suppose I give you 100 naturals. Prove that it is always possible to pick a subset of size 15 such that the difference of any two is a multiple of 7.

no 

3

Exhibit a bijection from $\mathbb{Z}$ to $\mathbb{Q}$.

From proposition 8.8, we know that $|\mathbb{N}| = |\mathbb{Q}^+|$. We can apply the same reasoning from the proposition to negative numbers to find $|-\mathbb{N}| = |\mathbb{Q}^-|$. Because all sets so far are countably infinite, we can add a single element (0) to both without changing cardinality*. Therefore $|\mathbb{N}| = |\mathbb{N} \cup \{0\}| = |\mathbb{Q}^+| = |\mathbb{Q}^+ \cup \{0\}|$. Knowing that $\mathbb{N} \cup (-\mathbb{N}) \cup \{0\} = \mathbb{Z}$ and $\mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\} = \mathbb{Q}$, we can finally reason that $|\mathbb{Z}| = |\mathbb{Q}|$, and thus there is a bijection from $\mathbb{Z}$ to $\mathbb{Q}$.

(*Not sure if we have to prove this but here:)

Claim: If we add/remove a finite number of elements from a countably infinite set, it is still countably infinite.

Proof: Let $\mathcal{A}$ be a countably infinite set and choose some $A \subseteq \mathcal{A}$ where A must be a finite set. Let $B = \mathcal{A} \setminus A$. Suppose for the sake of contradiction that B is a finite set. Then $B \cup A = \mathcal{A}$, which would imply A is also a finite set which is a contradiction. Thus B is a countably infinite set.

4

Show that if a set A has a bijection to a subset of $\mathbb{N}$, then it is countable. Conclude that if $|A| \leq |\mathbb{N}|$, then it is countable.

Let $N \subseteq \mathbb{N}$. A bijection between A and N implies $|A| = |N|$.

- Case 1: If $N \subset \mathbb{N}$, then $|A| < |\mathbb{N}|$, meaning A is a finite set and therefore countable.
- Case 2: If $N = \mathbb{N}$, then $|A| = |\mathbb{N}|$, meaning A is equinumerous with the naturals (countably infinite) and therefore countable.

Therefore, A is countable for $|A| \leq |\mathbb{N}|$.

5

(a) Suppose A_1, A_2 are countable sets. Show that there is an injection $f : A_1 \times A_2 \to \mathbb{N}$. [Hint: Fundamental theorem of Arithmetic.]

Let $f(x, y) = 2^x \cdot 3^y$. By the Fundamental theorem of Arithmetic, for all $n \in \mathbb{N}$, there exist unique $x_0, y_0 \in \mathbb{N}$ such that $n = f(x_0, y_0)$.

Now choose some $a, c \in A_1$ and $b, d \in A_2$ such that $f(a, b) = f(c, d)$. Some algebraic manipulation reveals

$$\begin{aligned} f(a,b) &= f(c,d) \\ 2^a \cdot 3^b &= 2^c \cdot 3^d \\ 2^{a-c} \cdot 3^{b-d} &= 1, \end{aligned}$$

which implies $a = c$ and $b = d$. This maps to a unique natural number, so f is injective.

(b) Generalize to any countable $A_1 \times A_2 \times \cdots \times A_n$.

Let P be a sequence of the first n prime numbers indexed by $[n]$ $(P_1 = 2,\ P_2 = 3, \dots)$. Let $\mathbf{x} \in A_1 \times A_2 \times \cdots \times A_n$ where $\mathbf{x}$ is also indexed by $[n]$ $(x_1 \in A_1, \dots)$. By the Fundamental theorem of Arithmetic, for all $n \in \mathbb{N}$, there exist unique $\mathbf{x}$ such that $n = f(\mathbf{x}) = \prod_{i=1}^{n} (P_i)^{x_i}$. This is injective by similar reasoning as part (a).

(c) Use parts (a) and/or (b) to get another proof that $\mathbb{Q}$ is countable.

Allowing negative exponents in the function in part (b) allows f to represent any positive real number. This is because if any natural number $n = f(\mathbf{x}) = \prod_{i=1}^{n} (P_i)^{x_i}$, then any rational number in $(0, 1]$ can be represented with $\prod_{i=1}^{n} (P_i)^{-x_i} = \prod_{i=1}^{n} \frac{1}{(P_i)^{x_i}}$. We denote the function that can map onto any positive real number f'.

Thus $f' : \mathbb{N}^n \to \mathbb{Q}^+$. As shown in part (a), for all $n \in \mathbb{N}$, $|\mathbb{N}^n| = |\mathbb{N}|$. As shown in question (3), $|Q^+| = |Q^+ \cup Q^- \cup \{0\}| = |Q|$. Thus $|\mathbb{N}| = |\mathbb{Q}|$, so $\mathbb{Q}$ is countable.

6

(a) Suppose that $\mathcal{A}$ is a countable collection of countably infinite sets. That is, there are countably many sets contained in $\mathcal{A}$. Show that $\bigcup_{A\in\mathcal{A}} A$ is countable.

Denote the countably infinite sets in $\mathcal{A}$ as $\mathcal{A} = (A_1, A_2, A_3, \cdots)$, where the elements of A_i are indexed as $(A_{i,1}, A_{i,1}, \ldots)$. Put each element of $\mathcal{A}$ in a row of a 2×2 grid like such:

$$\begin{matrix} A_{1,1} & A_{1,2} & A_{1,3} & \cdots \\ A_{2,1} & A_{2,2} & A_{2,3} & \cdots \\ A_{3,1} & A_{3,2} & A_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

We end up with an $\mathbb{N} \times N$ grid, where $N \subseteq \mathbb{N}$. We know that there exists an injection f such that $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Thus, we can conclude $\left|\bigcup_{A\in\mathcal{A}} A\right| \leq |\mathbb{N}|$, so the joint union is countable.

(b) Deduce that the collection of all finite subsets of $\mathbb{N}$ is countable.

We will proceed by induction. Let $S(n)$ be the set of subsets of $\mathbb{N}$ such that each subset has cardinality less than or equal to n. We know $|S(0)| = |\{\varnothing\}| = 1$, so it is countable. Similarly, $|S(1)| = |S(0)| + |\mathbb{N}^1|$, which is countable as proven in question 3.

Suppose $S(k)$ is countable for some $k \in \mathbb{N}$. Then $|S(k+1)| = |S(k)| + |\mathbb{N}^{k+1}|$. Both terms are countably finite (there is an injection $\mathbb{N}^{k+1} \to \mathbb{N}$ which implies $|\mathbb{N}^{k+1}| \leq |\mathbb{N}|$ as prove in 5b). The sum of countable infinities is a countable infinity (as proven in 3), so $S(k+1)$ is countable.