INTEGRATION AND EVALUATION OF LINE INTEGRALS

A. QUESTIONS

1. Find $I = \int_0^2 [3x^2 + 2x + 1]\, dx$
2. Find $I = \int sin^6 x cosx\ \ dx$
3. Integrate $\frac{8x}{(3+2x)^2}$ with respect to x
4. Evaluate $\int_{-2}^{2} \frac{x^2}{64 + x^6}\, dx$
5. Evaluate the line integrate $I = \int_0^2 (x^2 + 2y)\, dy + (x + y^2)\, dy.$ From the point A(0,1) to the other point B(2,3) along the curve C defined by $y = x + 1$
6. Evaluate the line integral $I = \int_0^1 (x + y)\, dx$ from A(0,1) to B(0,-1) along the semicircle $y = \sqrt{1 - x^2}$
7. Show that $\int_0^{2\pi} cosmxcosnxdx = 0\,,\ \ m \neq n$
8. Use the substitution $x = acos^2\theta + bsin^2\theta$ to evaluate $\int \frac{dx}{(x-a)(x-b)}$
9. Show that $\int e^{ax} cosbxdx = \frac{e^{ax}}{a^2+b^2}(bsinbx + acosbx)$
10. Use the substitution $u = t - \frac{1}{t}$ to show that $\int \frac{t^2+1}{t^4+1} dt = \int \frac{du}{u^2+2}$ and use the substitution $v = t + \frac{1}{t}$ to show that $\int \frac{t^2-1}{t^4+1} dt = \int \frac{dv}{v^2-2}$. Hence, evaluate $\int \frac{dt}{t^4+1}$ in terms of u and v

B. SOLUTION

1. $$I = \int_0^2 [3x^2 + 2x + 1]\, dx$$

$$= \left[\frac{3x^3}{3} + \frac{2x^2}{2} + x\right]_0^2$$

$$= [x^3 + x^2 + x\,]_0^2$$

$$I \quad = \ 14$$

2. $$I = \int sin^6 x cosx \ dx$$

Let$u \ = \ sinx, \frac{du}{dx} = cosx \therefore dx = \frac{du}{cosx}$

$$I = \int u^6 . cosx \ \frac{du}{cosx}$$

$$= \int u^6 \ du$$

$$= \frac{u^7}{7} + \ c$$

$$I \ = \frac{sin^7 x}{7} + \ c$$

3. Let $I = \int \frac{8x}{(3+2x)^2} dx$

Let $u = 3 + 2x$

$$\frac{du}{dx} = 2$$

$$dx = \frac{du}{2}$$

But $u = 3 + 2x$

$$x = \frac{u-3}{2}$$

Substitute for x and dx in I

$$I = \int \frac{8\left(\frac{u-3}{2}\right)}{\left[3+2\left(\frac{u-3}{2}\right)\right]^2} . \frac{du}{2}$$

$$= \int \frac{4(u-3)}{[3+(u-3)]^2} . \frac{du}{2}$$

$$= \int \frac{4(u-3)}{u^2} . \frac{du}{2}$$

$$= \int \frac{2(u-3)}{u^2} du$$

$$= \int \frac{2u}{u^2} du - \int \frac{6}{u^2} du$$

$$= \int \frac{2}{u} du - \int \frac{6}{u^2} du$$

$$= \int \frac{2}{u} du - \int 6u^{-2}\, du$$

$$= \int 2 \ln u\, du - 6u^{-1} + c$$

But $u = 3 + 2x$

$$\therefore I = \int 2 \ln(3 + 2x)\, du - 6(3 + 2x)^{-1} + c$$

4. To evaluate $\int_{-2}^{2} \frac{x^2}{64 + x^6}\, dx$

Let $u = x^3,\ x = u^{\frac{1}{3}}$

$$I = \int_{-2}^{2} \frac{x^2}{64 + u^2}\, dx$$

$$= \int_{-2}^{2} \frac{(u^{1/3})^2}{64 + u^2}\, dx$$

$$= \int_{-2}^{2} \frac{u^{2/3}}{64 + u^2}\, dx$$

But $\frac{du}{dx} = 3x^2$

$$dx = \frac{du}{3x^2} = \frac{du}{3(u^{1/3})^2} = \frac{du}{3u^{2/3}}$$

$$I = \int_{-2}^{2} \frac{u^{2/3}}{64 + u^2} \frac{du}{3u^{2/3}}$$

$$= \frac{1}{3} \int_{-2}^{2} \frac{1}{64 + u^2}\, du$$

$$= \frac{1}{3} \left[\int_{-2}^{2} \frac{1}{8^2 + u^2}\, du \right]$$

$$= \frac{1}{3} \left[\frac{1}{8} \tan^{-1} \frac{u}{8} \right]_{-2}^{2}$$

$$= \frac{1}{3} \left[\frac{1}{8} tan^{-1} \frac{2}{8} - \frac{1}{8} tan^{-1} \left(\frac{-2}{8} \right) \right]$$

$$= \frac{1}{3} \left[\frac{1}{8} tan^{-1} \frac{1}{4} - \frac{1}{8} tan^{-1} \left(\frac{-1}{4} \right) \right]$$

$$= \frac{1}{3} \left[\frac{1}{8} tan^{-1} \frac{1}{4} + \frac{1}{8} tan^{-1} \left(\frac{1}{4} \right) \right]$$

$$= \frac{1}{3}\left[\frac{2}{8} tan^{-1}\frac{1}{4}\right]$$

$$= \frac{2}{24} tan^{-1}\frac{1}{4} = \frac{1}{12} tan^{-1}\frac{1}{4}$$

5. If we write the integral in terms of y alone, we have $x = y - 1$ and $dx = dy$

Hence

$$I = \int_1^3 \left[(y-1)^2 + 2y\right] dy + \left[(y-1) + y^2\right] dy$$

Since $x = y - 1$ $then$ $\frac{dx}{dy} = 1, dx = dy$

$$I = \int_1^3 \left[y^2 - 2y + 1 + 2y + y - 1 + y^2\right] dy$$

$$= \int_1^3 \left[2y^2 + y\right] dy$$

$$= \left[\frac{2y^3}{3} + \frac{y^2}{2}\right]_1^3$$

$$= \left(\frac{54}{3} + \frac{9}{2}\right) - \left(\frac{2}{3} + \frac{1}{2}\right)$$

$$I = \frac{64}{3}$$

6. $y = \sqrt{1 - x^2}$

$$y^2 = 1 - x^2 \quad \rightarrow \quad x^2 = 1 - y^2 \quad \rightarrow \quad x = \sqrt{1 - y^2}$$

Since $x = \sqrt{1 - y^2}$, differentiate wrt y

$$\frac{dx}{dy} = \frac{d\left(\sqrt{1 - y^2}\right)}{dy} = \frac{1}{2\left(\sqrt{1 - y^2}\right)}$$

Substitute for x and dx in I

$$I = \int_{-1}^1 \left(\sqrt{1 - y^2} + y\right) . \frac{dy}{2\left(\sqrt{1 - y^2}\right)}$$

$$I = \int_{-1}^1 \left(\frac{1}{2} + \frac{y}{2\left(\sqrt{1 - y^2}\right)}\right) dy$$

$$I = \int_{-1}^{1}\left(\frac{1}{2}\right)dy + \int_{-1}^{1}\left(\frac{y}{2\left(\sqrt{1-y^2}\right)}\right)dy$$

$$= \left[\frac{y}{2}\right]_{-1}^{1} + \frac{1}{2}\left[\int_{-1}^{1}\left(\frac{y}{\sqrt{1-y^2}}\cdot\frac{du}{2y}\right)\right]$$

$$= \left[\frac{y}{2}\right]_{-1}^{1} + \frac{1}{4}\left[\int_{-1}^{1}(1-u)^{-1/2}du\right]$$

$$= \left[\frac{y}{2}\right]_{-1}^{1} + \frac{1}{4}\left[\frac{(1-u)^{(-1/2+1)}}{(-1/2+1)}\right]_{-1}^{1}$$

$$= 1 + \frac{1}{2}\left(-2^{1/2}\right) = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$$

7. To show that $\int_{0}^{2\pi} cosmxcosnxdx = 0\,,\ \ m \neq n$

Solving by 'Integration by parts' and neglecting the boundaries

$$\int cosmxcosnxdx = cosmx\int cosnxdx - \int\left[\frac{d(cosmx)}{dx}\int cosnxdx\right]dx$$

$$\int cosmxcosnxdx = cosmx\frac{1}{n}(\sin nx) - \int\left[-msimx.\frac{1}{n}(\sin nx)\right]dx$$

$$\int cosmxcosnxdx = \frac{1}{n}\sin nx\, cosmx - \int\left[-msimx.\frac{1}{n}(\sin nx)\right]dx$$

$$\int cosmxcosnxdx = \frac{1}{n}\sin nx\, cosmx + \frac{m}{n}\int[\sin mx \sin nx]\,dx$$

$$\int cosmxcosnxdx = \frac{1}{n}\sin nx\, cosmx + \frac{m}{n}\left[\sin mx\int \sin nx\, dx - \int\left[\frac{d(\sin mx)}{dx}\int \sin nx\, dx\right]dx\right]$$

$$\int cosmxcosnxdx = \frac{1}{n}\sin nx\, cosmx + \frac{m}{n}\left[\sin mx\left(-\frac{1}{n}\cos nx\right) - \int\left[(mcosmx)\left(-\frac{1}{n}\cos nx\right)\right]dx\right]$$

$$\int cosmxcosnxdx = \frac{1}{n}\sin nx\, cosmx + \frac{m}{n}\left[-\frac{1}{n}\sin mx\cos nx + \frac{m}{n}\int[cosmx\cos nx]\,dx\right]$$

$$\int cosmxcosnxdx = \frac{1}{n}\sin nx\, cosmx - \frac{m}{n^2}\sin mx\cos nx + \frac{m^2}{n^2}\int cosmx\cos nx\, dx$$

Collect like terms

$$\int cosmxcosnxdx - \frac{m^2}{n^2}\int cosmx\cos nx\, dx = \frac{1}{n}\sin nx\, cosmx - \frac{m}{n^2}\sin mx\cos nx$$

$$\left(1-\frac{m^2}{n^2}\right)\int cosmx\cos nx\,dx = \frac{1}{n}\sin nx\,cosmx - \frac{m}{n^2}\sin mx\cos nx$$

$$\left(\frac{n^2-m^2}{n^2}\right)\int cosmx\cos nx\,dx = \frac{1}{n}\sin nx\,cosmx - \frac{m}{n^2}\sin mx\cos nx$$

$$\int cosmx\cos nx\,dx = \left(\frac{n^2}{n^2-m^2}\right)\left[\frac{1}{n}\sin nx\,cosmx - \frac{m}{n^2}\sin mx\cos nx\right]$$

Including the boundaries, we have $0 \le x \le 2\pi$

$$\int_0^{2\pi} cosmx\cos nx\,dx = \left(\frac{1}{n^2-m^2}\right)[n\,sin\,nx\;cosmx - \;m\,sin\,mx\;\;cos\,nx\;]_0^{2\pi}$$

$$= \left(\frac{1}{n^2-m^2}\right)[0-0]$$

$$= 0$$

$$\int_0^{2\pi} cosmx\cos nx\,dx = 0,\quad m \ne n$$

8. To evaluate $\int \frac{dx}{(x-a)(x-b)}$

First solve $\frac{1}{(x-a)(x-b)}$ by partial fraction

$$\frac{1}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} \qquad \alpha$$

$1 = A(x-b) + B(x-a)$
$1 = Ax - Ab + Bx - Ba$
$1 = (A+B)x - (Ab+Ba)$
Comparing the coefficients
$A + B = 0 \qquad \beta$
$-(Ab+Ba) = 1 \qquad \gamma$
From (β) $B = -A.$ $Substitute\ for\ this\ in\ (\gamma)$
$-(Ab+Ba) = 1$
$-(Ab+(-A)a) = 1$
$-Ab + Aa = 1$
$A(a-b) = 1$
$A = \frac{1}{(a-b)} \qquad \delta$

Substitute for A in β

$$A + B = 0$$
$$B = -A$$
$$= -\frac{1}{(a-b)}$$

Substitute for A and B in α

$$\frac{1}{(x-a)(x-b)} = \frac{1}{(a-b)(x-a)} - \frac{1}{(a-b)(x-b)}$$
$$= \frac{1}{(a-b)}\left[\frac{1}{(x-a)} - \frac{1}{(x-b)}\right]$$

Integrating both sides $wrt\ x$

$$\int \frac{dx}{(x-a)(x-b)} = \int \frac{1}{(a-b)}\left[\frac{1}{(x-a)} - \frac{1}{(x-b)}\right] dx$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\int \left[\frac{1}{(x-a)} - \frac{1}{(x-b)}\right] dx$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}[\ln(x-a) - \ln(x-b)]$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\left[\ln\left(\frac{x-a}{x-b}\right)\right]$$

But $x = acos^2\theta + bsin^2\theta$

Hence,

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\left[\ln\left(\frac{acos^2\theta + bsin^2\theta - a}{acos^2\theta + bsin^2\theta - b}\right)\right]$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\left[\ln\left(\frac{a(cos^2\theta - 1) + bsin^2\theta}{acos^2\theta + b(sin^2\theta - 1)}\right)\right]$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\left[\ln\left(\frac{-a(1-cos^2\theta) + bsin^2\theta}{acos^2\theta - b(1-sin^2\theta)}\right)\right]$$

Since $sin^2\theta = 1 - cos^2\theta$ and $cos^2\theta = 1 - sin^2\theta$

$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\left[\ln\left(\frac{-asin^2\theta + bsin^2\theta}{acos^2\theta - bcos^2\theta}\right)\right]$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\left[\ln\left(\frac{(b-a)sin^2\theta}{(a-b)cos^2\theta}\right)\right]$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\left[\ln\left(\frac{-(a-b)sin^2\theta}{(a-b)cos^2\theta}\right)\right]$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(a-b)}\left[\ln\left(\frac{-sin^2\theta}{cos^2\theta}\right)\right]$$
$$\int \frac{dx}{(x-a)(x-b)} = -\frac{1}{(a-b)}\left[\ln\left(\frac{sin^2\theta}{cos^2\theta}\right)\right]$$
$$\int \frac{dx}{(x-a)(x-b)} = -\frac{1}{(a-b)}[\ln(tan^2\theta)]$$
$$\int \frac{dx}{(x-a)(x-b)} = \frac{1}{(b-a)}[\ln(tan^2\theta)]$$

9. To show that $\int e^{ax}cosbxdx = \frac{e^{ax}}{a^2+b^2}(bsinbx + acosbx)$

Method: Integration by parts

$$\int e^{ax}cosbxdx = e^{ax}\int cosbxdx - \int\left[\frac{d(e^{ax})}{dx}\int cosbxdx\right]dx$$
$$\int e^{ax}cosbxdx = e^{ax}.\frac{1}{b}sinbx - \int\left[ae^{ax}.\frac{1}{b}sinbx\right]dx$$
$$\int e^{ax}cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\int e^{ax}sinbx\,dx$$

$$\int e^{ax}cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\left[e^{ax}\int sinbxdx - \int\left[\frac{d(e^{ax})}{dx}\int sinbxdx\right]dx\right]$$
$$\int e^{ax}cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\left[e^{ax}\left(-\frac{1}{b}cosbx\right) - \int\left[ae^{ax}\left(-\frac{1}{b}cosbx\right)\right]dx\right]$$
$$\int e^{ax}cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}cosbx + \frac{a}{b}\int e^{ax}cosbx\,dx\right]$$
$$\int e^{ax}cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}cosbx + \frac{a}{b}\int e^{ax}cosbx\,dx\right]$$
$$\int e^{ax}cosbxdx = \frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx - \frac{a^2}{b^2}\int e^{ax}cosbx\,dx$$
Collect like terms
$$\int e^{ax}cosbxdx + \frac{a^2}{b^2}\int e^{ax}cosbx\,dx = \frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx$$
$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}cosbx\,dx = \frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx$$
$$\left(\frac{b^2+a^2}{b^2}\right)\int e^{ax}cosbx\,dx = \frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx$$
$$\int e^{ax}cosbx\,dx = \left(\frac{b^2}{a^2+b^2}\right)\left[\frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx\right]$$
$$\int e^{ax}cosbx\,dx = \left(\frac{1}{a^2+b^2}\right)\left[be^{ax}sinbx + ae^{ax}cosbx\right]$$
$$\int e^{ax}cosbx\,dx = \frac{e^{ax}}{a^2+b^2}\left[bsinbx + acosbx\right] \qquad \textit{Proved}$$

10. To show that $\int \frac{t^2+1}{t^4+1}dt = \int \frac{du}{u^2+2}$, $if\ u = t - \frac{1}{t}$

$\int \frac{t^2+1}{t^4+1}dt$ $\qquad \vartheta$

Let $u = t - \frac{1}{t} = \frac{t^2-1}{t}$

$$\frac{du}{dt} = \frac{t\frac{d(t^2-1)}{dt} - (t^2-1)\frac{dt}{dt}}{t^2}$$

$$= \frac{t(2t) - (t^2-1)}{t^2}$$
$$\frac{du}{dt} = \frac{t^2+1}{t^2}$$
Then
$$dt = \frac{t^2}{t^2+1}du$$
Substitute for dt in equation ϑ
$$\int \frac{t^2+1}{t^4+1}.\frac{t^2}{t^2+1}du = \int \frac{t^2}{t^4+1}du$$
μ

But $u = \frac{t^2-1}{t}, to\ get\ t^4, square\ both\ sides$

$u^2 = \left(\frac{t^2-1}{t}\right)^2 = \frac{t^4-2t^2+1}{t^2}$

$\therefore\ u^2 = \frac{t^4-2t^2+1}{t^2}$

$u^2t^2 = t^4 - 2t^2 + 1$

$\therefore\ u^2t^2 + 2t^2 - 1 = t^4$

Sub for t^4 $in\ equation\ \mu$

$$\therefore \int \frac{t^2}{t^4+1} du = \int \frac{t^2}{[u^2t^2+2t^2-1]+1} du$$

$$= \int \frac{t^2}{u^2t^2+2t^2} du$$

$$= \int \frac{t^2}{t^2(u^2+2)} du$$

$$= \int \frac{du}{u^2+2}$$

(ii) $If\ v = t + \frac{1}{t}$, we are to show that $\int \frac{t^2-1}{t^4+1} dt = \int \frac{dv}{v^2-2}$.

Let $v = t + \frac{1}{t} = \frac{t^2+1}{t}$

$$\frac{dv}{dt} = \frac{t\frac{d(t^2+1)}{dt} - (t^2+1)\frac{dt}{dt}}{t^2}$$

$$= \frac{t(2t) - (t^2+1)}{t^2}$$

$$\frac{dv}{dt} = \frac{t^2+1}{t^2}$$

Then

$$dt = \frac{t^2}{t^2-1} dv$$

Substitute for dt in the equation

$$\int \frac{t^2-1}{t^4-1} . \frac{t^2}{t^2-1} dv = \int \frac{t^2}{t^4+1} dv$$

ρ

But $v = \frac{t^2+1}{t}$, $to\ get\ t^4, square\ both\ sides$

$$v^2 = \left(\frac{t^2+1}{t}\right)^2 = \frac{t^4+2t^2+1}{t^2}$$

$$\therefore\ v^2 = \frac{t^4+2t^2+1}{t^2}$$

$$v^2t^2 = t^4 + 2t^2 + 1$$

$$\therefore\ v^2t^2 - 2t^2 - 1 = t^4$$

Sub for t^4 $in\ equation\ \rho$

$$\therefore\ \int \frac{t^2}{t^4+1} dv = \int \frac{t^2}{[v^2t^2-2t^2-1]+1} dv$$

$$= \int \frac{t^2}{v^2t^2-2t^2} dv$$

$$= \int \frac{t^2}{t^2(v^2-2)} dv$$

$$= \int \frac{du}{v^2-2}$$

PARTIAL DERIVATIVES

A. QUESTIONS

1. Find the partial derivatives of z with respect to the independent variables x and y

(a) $x^2 + y^2 + z^2 = 25$

(b) $x^2(2y + 3z) + y^2(3x - 4z) + z^2(x - 2y) = xyz$

2. If $z = \frac{1}{x^2+y^2-1}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = -2z(1 + z)$

3. Considering x and y as independent variables. Find $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial \theta}{\partial y}$ when $x = e^{2r} cos\theta$,

$y = e^{3r} \sin \theta$

4. Find all the second partial derivatives to z if $z = x^2 + 3xy + y^2$

5. Prove that, if $v = \ln(x^2 + y^2)$, then $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

6. If $z = \sqrt{x^2 + y^2}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

7. If $z = f\left(\frac{x}{y}\right)$, show that $\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$

8. If $z = (x + y). f\left(\frac{x}{y}\right)$, where f is an arbitrary function, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

9. If $u = \frac{x+y+z}{(x^2+y^2+z^2)^{1/2}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

10. If $z = \mathrm{e^x}(\mathrm{xcosy} - \mathrm{ysin}\, y)$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

11. If $z = \sin(3x - 2y)$,verify that $3\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 z}{\partial x^2} \neq 0$

12. Show that the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ is satisfied by $z = \ln(x^2 + y^2) + \frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right)$

13. If $z = (ax + by)^2 + e^{ax+by} + \sin(ax + by)$, show that $a\frac{\partial z}{\partial y} = b\frac{\partial z}{\partial x}$

B. SOLUTION

$(1a.)\ (i) \quad x^2 + y^2 + z^2 = 25$

$$\frac{\partial(x^2+y^2+z^2=25)}{\partial x} = \frac{\partial(25)}{\partial x}$$

$$2x + 2z\frac{\partial z}{\partial x} = 0$$

$$2z\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2z} = \frac{-x}{z}$$

(1a.) (ii) $\quad x^2 + y^2 + z^2 = 25$

$$\frac{\partial(x^2+y^2+z^2=25)}{\partial y} = \frac{\partial(25)}{\partial y}$$

$$2y + 2z\frac{\partial z}{\partial y} = 0$$

$$2z\frac{\partial z}{\partial x} = -2y$$

$$\frac{\partial z}{\partial x} = \frac{-2y}{2z} = \frac{-y}{z}$$

(1b)(i) $\quad x^2(2y+3z) + y^2(3x-4z) + z^2(x-2y) = xyz$

$$2x^2y + 3x^2z + 3xy^2 - 4y^2z + xz^2 - 2yz^2 = xyz$$

$$\frac{\partial(\ 2x^2y+3x^2z+3xy^2-4y^2z+xz^2-2yz^2)}{\partial x} = \frac{\partial(xyz)}{\partial x}$$

$$2x^2\frac{\partial y}{\partial x} + 2y\frac{\partial(x^2)}{\partial x} + 3x^2\frac{\partial z}{\partial x} + 3z\frac{\partial x^2}{\partial x} + 3x\frac{\partial(y^2)}{\partial x} + 3y^2\frac{\partial x}{\partial x} - \left[4y^2\frac{\partial z}{\partial x} + 4z\frac{\partial(y^2)}{\partial x}\right]$$

$$+x\frac{\partial(z^2)}{\partial x}+z^2\frac{\partial(x)}{\partial x}-\left[2y\frac{\partial(z^2)}{\partial x}+2z^2\frac{\partial(y)}{\partial x}\right]=xy\frac{\partial z}{\partial x}+xz\frac{\partial y}{\partial x}+yz\frac{\partial x}{\partial x}$$

$$2y(2x)+3x^2\frac{\partial z}{\partial x}+3z(2x)+3y^2-4y^2\frac{\partial z}{\partial x}+x\left(2z\frac{\partial z}{\partial x}\right)$$
$$+z^2-2y\left(2z\frac{\partial z}{\partial x}\right)\ =xy\frac{\partial z}{\partial x}+yz$$

Collecting like terms

$$\frac{\partial z}{\partial x}=\frac{yz-4xy-6xz-3y^2-z^2}{3x^2-4y^2+2xz-4yz-xy}$$

(1b)(ii) With respect to y

$$x^2(2y+3z)+y^2(3x-4z)+z^2(x-2y)=xyz$$

$$2x^2y+3x^2z+3xy^2-4y^2z+xz^2-2yz^2=xyz$$

$$\frac{\partial(\ 2x^2y+3x^2z+3xy^2-4y^2z+xz^2-2yz^2)}{\partial x}=\ \frac{\partial(xyz)}{\partial x}$$

$$\frac{\partial z}{\partial x}(3x^2-4y^2+2xz-4yz-xy)=\ xz-2x^2-6xy+8yz+2z^2$$

$$\frac{\partial z}{\partial x}=\ \frac{xz-2x^2-6xy+8yz+2z^2}{3x^2-4y^2+2xz-4yz-xy}$$

(2) $$z=\ \frac{1}{x^2+y^2-1} \qquad \alpha$$

$$z(x^2+y^2-1)=1$$

$$zx^2+zy^2-z=1$$

Differentiating implicitly and finding $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

$$\frac{\partial(zx^2+zy^2-z)}{\partial x}=\ \frac{\partial(1)}{\partial x}$$

$$x^2\frac{\partial z}{\partial x}+z\frac{\partial(x^2)}{\partial x}+y^2\frac{\partial z}{\partial x}+z\frac{\partial(y^2)}{\partial x}-\frac{\partial z}{\partial x}=0$$

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial x}-\frac{\partial z}{\partial x}+2xz=0$$

$$\frac{\partial z}{\partial x}(x^2 + y^2 - 1) + 2xz = 0$$

$$\frac{\partial z}{\partial x}(x^2 + y^2 - 1) = -2xz$$

$$\frac{\partial z}{\partial x} = \frac{-2xz}{x^2+y^2-1}$$

But $z = \frac{1}{x^2+y^2-1}$

$$\therefore \quad \frac{\partial z}{\partial x} = -2xz^2$$

Also,

$$\frac{\partial(zx^2+zy^2-z)}{\partial y} = \frac{\partial(1)}{\partial y}$$

$$x^2\frac{\partial z}{\partial y} + z\frac{\partial(x^2)}{\partial y} + y^2\frac{\partial z}{\partial y} + z\frac{\partial(y^2)}{\partial y} - \frac{\partial z}{\partial y} = 0$$

$$x^2\frac{\partial z}{\partial y} + y^2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial x} + 2yz = 0$$

$$\frac{\partial z}{\partial x}(x^2 + y^2 - 1) + 2yz = 0$$

$$\frac{\partial z}{\partial x}(x^2 + y^2 - 1) = -2yz$$

$$\frac{\partial z}{\partial x} = \frac{-2xz}{x^2+y^2-1}$$

But $z = \frac{1}{x^2+y^2-1}$

$$\therefore \quad \frac{\partial z}{\partial x} = -2yz^2$$

PROOF

To show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = -2z(1+z)$

From the L.H.S. $x(-2xz^2) + y(-2yz^2) = -2x^2z^2 + -2y^2z^2$

$$= -2z^2(x^2+y^2) \qquad \beta$$

Recall from equation α

$$z = \frac{1}{x^2+y^2-1}$$

$$x^2+y^2-1 = \frac{1}{z}$$

$$x^2+y^2 = \frac{1}{z}+1$$

Substitute into equation β

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = -2z^2(\frac{1}{z}+1) = -2z(1+z)$$

(3.) We are to find $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$,$\frac{\partial \theta}{\partial y}$ where $x = e^{2r}cos\theta,\ y = e^{3r}\sin\ \theta$

$$\frac{\partial x}{\partial x} = \frac{\partial(e^{2r}cos\theta)}{\partial x} = e^{2r}\frac{\partial(cos\theta)}{\partial x} + cos\theta\frac{\partial(e^{2r})}{\partial x}$$

$$1 = -e^{2r}sin\theta\frac{\partial \theta}{\partial x} + 2e^{2r}cos\theta\frac{\partial r}{\partial x} \qquad \vartheta$$

Also, $\frac{\partial y}{\partial x} = \frac{\partial(e^{3r}\sin\ \theta)}{\partial x} = e^{3r}\frac{\partial(\sin\ \theta)}{\partial x} + \sin\ \theta\frac{\partial(e^{3r})}{\partial x}$

$$0 = e^{3r}\cos\ \theta\frac{\partial \theta}{\partial x} + 3\sin\ \theta e^{3r}\frac{\partial r}{\partial x} \qquad \mu$$

From μ

$$e^{3r}\cos\ \theta\frac{\partial \theta}{\partial x} = -3e^{3r}\sin\ \theta\frac{\partial r}{\partial x}$$

$$\frac{\partial \theta}{\partial x} = -3\frac{\sin\theta}{\cos\ \theta}\frac{\partial r}{\partial x} \qquad \gamma$$

Sub. Into ϑ

$$1 = -e^{2r}sin\theta\left(-3\frac{\sin\theta}{\cos\ \theta}\frac{\partial r}{\partial x}\right) + 2e^{2r}cos\theta\frac{\partial r}{\partial x}$$

$$1 = \frac{3e^{2r}sin^2\theta\frac{\partial r}{\partial x} + 2e^{2r}cos^2\theta\frac{\partial r}{\partial x}}{cos\theta}$$

$$cos\theta = 3e^{2r}sin^2\theta\frac{\partial r}{\partial x} + 2e^{2r}cos^2\theta\frac{\partial r}{\partial x}$$

$$= \frac{\partial r}{\partial x}e^{2r}(3sin^2\theta + 2cos^2\theta)$$

$$= \frac{\partial r}{\partial x}e^{2r}(3sin^2\theta + 2(1 - sin^2\theta))$$

$$= \frac{\partial r}{\partial x}e^{2r}(3sin^2\theta + 2 - 2sin^2\theta)$$

$$cos\theta = \frac{\partial r}{\partial x}e^{2r}(2 + sin^2\theta)$$

$$\frac{\partial r}{\partial x} = \frac{cos\theta}{e^{2r}(2+sin^2\theta)}$$

From μ

$$0 = e^{3r}\cos\ \theta\frac{\partial \theta}{\partial x} + 3\sin\ \theta e^{3r}\frac{\partial r}{\partial x}$$

$$3\sin\ \theta e^{3r}\frac{\partial r}{\partial x} = -e^{3r}\cos\ \theta\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x} = -\frac{e^{3r}}{3e^{3r}}\frac{\cos\ \theta}{\sin\ \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial r}{\partial x} = -\frac{1}{3}\frac{\cos\ \theta}{\sin\ \theta}\frac{\partial \theta}{\partial x} \qquad \beta$$

Sub. Into ϑ

$$1 = -e^{2r}sin\theta\frac{\partial \theta}{\partial x} + 2e^{2r}cos\theta\left(-\frac{1}{3}\frac{\cos\ \theta}{\sin\ \theta}\frac{\partial \theta}{\partial x}\right)$$

$$1 = \frac{-e^{2r}sin^2\theta\frac{\partial \theta}{\partial x} - \frac{2}{3}e^{2r}cos^2\theta\frac{\partial \theta}{\partial x}}{sin\theta}$$

$$sin\theta = -e^{2r}sin^2\theta\frac{\partial \theta}{\partial x} - \frac{2}{3}e^{2r}cos^2\theta\frac{\partial \theta}{\partial x}$$

$$= -\frac{\partial\theta}{\partial x}e^{2r}(sin^2\theta + \frac{2}{3}cos^2\theta)$$

$$= -\frac{\partial\theta}{\partial x}e^{2r}(sin^2\theta + \frac{2}{3}(1 - sin^2\theta))$$

$$= -\frac{\partial\theta}{\partial x}e^{2r}(3sin^2\theta + \frac{2}{3} - \frac{2}{3}sin^2\theta)$$

$$sin\theta = -\frac{\partial\theta}{\partial x}e^{2r}(\frac{1}{3}sin^2\theta + \frac{2}{3})$$

$$sin\theta = -\frac{1}{3}\frac{\partial\theta}{\partial x}e^{2r}(sin^2\theta + 2)$$

$$\frac{\partial\theta}{\partial x} = \frac{-3cos\theta}{e^{2r}(2+sin^2\theta)}$$

Differentiate wrt y

$$\frac{\partial x}{\partial y} = \frac{\partial(e^{2r}cos\theta)}{\partial y} = e^{2r}\frac{\partial(cos\theta)}{\partial y} + cos\theta\frac{\partial(e^{2r})}{\partial y}$$

$$0 = -e^{2r}sin\theta\frac{\partial\theta}{\partial y} + 2e^{2r}cos\theta\frac{\partial r}{\partial y} \qquad \sigma$$

Also, $\frac{\partial y}{\partial y} = \frac{\partial(e^{3r}\sin\theta)}{\partial y} = e^{3r}\frac{\partial(\sin\theta)}{\partial y} + \sin\theta\frac{\partial(e^{3r})}{\partial y}$

$$1 = e^{3r}\cos\theta\frac{\partial\theta}{\partial y} + 3\sin\theta e^{3r}\frac{\partial r}{\partial y} \qquad \tau$$

$$e^{3r}\cos\theta\frac{\partial\theta}{\partial y} = 1 - 3e^{3r}\sin\theta\frac{\partial r}{\partial y}$$

From σ

$$0 = -e^{2r}sin\theta\frac{\partial\theta}{\partial y} + 2e^{2r}cos\theta\frac{\partial r}{\partial y}$$

$$e^{2r}sin\theta\frac{\partial\theta}{\partial y} = 2e^{2r}cos\theta\frac{\partial r}{\partial y}$$

$$\frac{\partial\theta}{\partial y} = 2\frac{e^{2r}}{e^{2r}}\frac{cos\theta}{sin\theta}\frac{\partial r}{\partial y}$$

$$\frac{\partial\theta}{\partial y} = 2\frac{cos\theta}{sin\theta}\frac{\partial r}{\partial y}$$

Sub. Into τ

$$1 = e^{3r}cos\theta\left(2\frac{cos\theta}{sin\theta}\frac{\partial r}{\partial y}\right) + 3e^{3r}sin\theta\frac{\partial r}{\partial y}$$

$$1 = \frac{2e^{3r}cos^2\theta\frac{\partial r}{\partial y} + 3e^{3r}sin^2\theta\frac{\partial r}{\partial y}}{sin\theta}$$

$$sin\theta = 2e^{3r}cos^2\theta\frac{\partial r}{\partial y} + 3e^{3r}sin^2\theta\frac{\partial r}{\partial y}$$

$$= \frac{\partial r}{\partial y}e^{3r}(2cos^2\theta + 3sin^2\theta)$$

$$= \frac{\partial r}{\partial y}e^{3r}(2(1 - sin^2\theta) + 3sin^2\theta)$$

$$= \frac{\partial r}{\partial y}e^{3r}(2-2sin^2\theta + 3sin^2\theta)$$

$$sin\theta = \frac{\partial r}{\partial y}e^{3r}(2+sin^2\theta)$$

$$\frac{\partial r}{\partial y} = \frac{sin\theta}{e^{3r}(2+sin^2\theta)}$$

From σ

$$0 = -e^{2r}sin\theta\frac{\partial\theta}{\partial y} + 2e^{2r}cos\theta\frac{\partial r}{\partial y}$$

$$e^{2r}sin\theta\frac{\partial\theta}{\partial y} = 2e^{2r}cos\theta\frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}\frac{e^{2r}}{e^{2r}}\frac{sin\theta}{cos\theta}\frac{\partial\theta}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}\frac{sin\theta}{cos\theta}\frac{\partial\theta}{\partial y}$$

$$1 = e^{3r}cos\theta\frac{\partial\theta}{\partial y} + 3e^{3r}sin\theta\left(\frac{1}{2}\frac{sin\theta}{cos\theta}\frac{\partial\theta}{\partial y}\right)$$

$$1 = \frac{e^{3r}cos^2\theta\frac{\partial\theta}{\partial y} + \frac{3}{2}e^{3r}sin^2\theta\frac{\partial\theta}{\partial y}}{cos\theta}$$

$$cos\theta = e^{3r}cos^2\theta\frac{\partial\theta}{\partial y} + \frac{3}{2}e^{3r}sin^2\theta\frac{\partial\theta}{\partial y}$$

$$= \frac{\partial\theta}{\partial y} e^{3r}(cos^2\theta + \tfrac{3}{2} sin^2\theta)$$

$$= \frac{\partial\theta}{\partial y} e^{3r}(1 - sin^2\theta + \tfrac{3}{2} sin^2\theta)$$

$$= \frac{\partial\theta}{\partial y} e^{3r}(1 + \tfrac{1}{2} sin^2\theta)$$

$$cos\theta \quad = \tfrac{1}{2} \frac{\partial\theta}{\partial y} e^{3r}(2+sin^2\theta)$$

$$\frac{\partial\theta}{\partial y} = \frac{2cos\theta}{e^{3r}(2+sin^2\theta)}$$

4. $z = x^2 + 3xy + y^2$

$$\frac{\partial z}{\partial x} = 2x + 3y, \quad \frac{\partial z}{\partial y} = 3x + 2y$$

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 3$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3, \quad \frac{\partial^2 z}{\partial y \partial x} = 3$$

5. To prove that, if $v = \ln(x^2 + y^2)$, then $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$$v = \ln(x^2 + y^2)$$

Let $u = x^2 + y^2$

$$v = \ln u$$

$$\frac{\partial v}{\partial u} = \frac{1}{u}$$

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u} . \frac{\partial u}{\partial x}$$

$$= \frac{1}{u}.2x$$

$$= \frac{2x}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial u}.\frac{\partial u}{\partial y}$$

$$= \frac{1}{u}.2y$$

$$= \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)\frac{\partial(2x)}{\partial x} - 2x\frac{\partial(x^2+y^2)}{\partial x}}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$= \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)\frac{\partial(2y)}{\partial y} - 2y\frac{\partial(x^2+y^2)}{\partial y}}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(2) - 2y(2y)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$$= \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2y^2}{(x^2+y^2)^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2}$$

$$= 0$$

6. To prove that if $z = \sqrt{x^2 + y^2}$, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

$$z = \sqrt{x^2 + y^2}$$

$$z^2 = x^2 + y^2$$

Differentiate wrt x

$$\frac{\partial}{\partial x}(z^2) = \frac{\partial}{\partial x}(x^2 + y^2)$$

$$2z\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial x} = \frac{2x}{2z}$$

$$\frac{\partial z}{\partial x} = \frac{x}{z}$$

Differentiate wrt y

$$\frac{\partial}{\partial y}(z^2) = \frac{\partial}{\partial y}(x^2 + y^2)$$

$$2z\frac{\partial z}{\partial y} = 2y$$

$$\frac{\partial z}{\partial y} = \frac{2y}{2z}$$

$$\frac{\partial z}{\partial y} = \frac{y}{z}$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2}{z} + \frac{y^2}{z}$$

$$= \frac{x^2+y^2}{z}$$

Recall that $z^2 = x^2 + y^2$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2}{z} + \frac{y^2}{z}$$

$$= \frac{z^2}{z}$$

$$= z \qquad\qquad\qquad\qquad Proved$$

7. To prove if $z = f\left(\frac{x}{y}\right)$, then $\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[f\left(\frac{x}{y}\right)\right] = f'\left(\frac{x}{y}\right) \times \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= f'\left(\frac{x}{y}\right) \times \frac{y\frac{\partial x}{\partial x} - x\frac{\partial y}{\partial x}}{y^2}$$

$$= f'\left(\frac{x}{y}\right) \times \frac{(y-0)}{y^2}$$

$$= f'\left(\frac{x}{y}\right) \times \frac{1}{y}$$

$$\frac{\partial z}{\partial x} = \frac{1}{y}f'\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{1}{y}f'\left(\frac{x}{y}\right)\right]$$

$$= f''\left(\frac{x}{y}\right) \times \frac{\partial}{\partial x}\left[\frac{1}{y}\right]$$

$$= f''\left(\frac{x}{y}\right) \times \frac{y\frac{\partial (1)}{\partial x} - (1)\frac{\partial y}{\partial x}}{y^2}$$

$$= f''\left(\frac{x}{y}\right) \times \frac{(0-0)}{y^2}$$

$$= 0$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[f\left(\frac{x}{y}\right)\right] = f'\left(\frac{x}{y}\right) \times \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= f'\left(\frac{x}{y}\right) \times \frac{y\frac{\partial x}{\partial y} - x\frac{\partial y}{\partial y}}{y^2}$$

$$= f'\left(\frac{x}{y}\right) \times \frac{(0-x)}{y^2}$$

$$\frac{\partial z}{\partial x} \quad = -\frac{x}{y^2} f'\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[-\frac{x}{y^2} f'\left(\frac{x}{y}\right)\right]$$

$$= f''\left(\frac{x}{y}\right) \times \frac{\partial}{\partial y}\left[-\frac{x}{y^2}\right]$$

$$= f''\left(\frac{x}{y}\right) \times \frac{(y^2)\frac{\partial(-x)}{\partial y} - (-x)\frac{\partial(y^2)}{\partial y}}{(y^2)^2}$$

$$= f''\left(\frac{x}{y}\right) \times \left[\frac{0 + x(2y)}{y^4}\right]$$

$$= f''\left(\frac{x}{y}\right) \times \left[\frac{2x}{y^3}\right]$$

$$= \frac{2x}{y^3} f''\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left[-\frac{x}{y^2} f'\left(\frac{x}{y}\right)\right]$$

$$= f''\left(\frac{x}{y}\right) \times \frac{\partial}{\partial y}\left[-\frac{x}{y^2}\right]$$

$$= f''\left(\frac{x}{y}\right) \times \frac{(y^2)\frac{\partial(-x)}{\partial x} - (-x)\frac{\partial(y^2)}{\partial x}}{(y^2)^2}$$

$$= f''\left(\frac{x}{y}\right) \times \left[\frac{-y^2}{y^4}\right]$$

$$= f''\left(\frac{x}{y}\right) \times \left[-\frac{1}{y^2}\right]$$

$$= -\frac{1}{y^2} f''\left(\frac{x}{y}\right)$$

Proof : $\quad \frac{\partial^2 z}{\partial x^2} = 0$

$$2xy\frac{\partial^2 z}{\partial x \partial y} = 2xy\left[-\frac{1}{y^2} f''\left(\frac{x}{y}\right)\right] = -\frac{2x}{y} f''\left(\frac{x}{y}\right)$$

$$y^2\frac{\partial^2 z}{\partial y^2} = y^2\left[\frac{2x}{y^3} f''\left(\frac{x}{y}\right)\right] = \frac{2x}{y} f''\left(\frac{x}{y}\right)$$

Now, $\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0 - \frac{2x}{y}f''\left(\frac{x}{y}\right) + \frac{2x}{y}f''\left(\frac{x}{y}\right) = 0$

8. If $z = (x + y).f\left(\frac{x}{y}\right)$, where f is an arbitrary function, we are to show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$

$z = (x + y).f\left(\frac{x}{y}\right)$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[(x + y).f\left(\frac{x}{y}\right)\right]$

$= (x + y)\left[f'\left(\frac{x}{y}\right) \times \frac{\partial}{\partial x}\left(\frac{x}{y}\right)\right] + f\left(\frac{x}{y}\right)\left[\frac{\partial}{\partial x}(x + y)\right]$

$= (x + y)\left[f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)\right] + f\left(\frac{x}{y}\right)$

$= \frac{(x+y)}{y}\left[f'\left(\frac{x}{y}\right)\right] + f\left(\frac{x}{y}\right)$

Now,

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[(x + y).f\left(\frac{x}{y}\right)\right]$

$= (x + y)\left[f'\left(\frac{x}{y}\right) \times \frac{\partial}{\partial y}\left(\frac{x}{y}\right)\right] + f\left(\frac{x}{y}\right)\left[\frac{\partial}{\partial y}(x + y)\right]$

$= (x + y)\left[f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)\right] + f\left(\frac{x}{y}\right)$

$= \frac{-x(x+y)}{y}\left[f'\left(\frac{x}{y}\right)\right] + f\left(\frac{x}{y}\right)$

9. If $u = \frac{x+y+z}{(x^2+y^2+z^2)^{1/2}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

$u(x^2 + y^2 + z^2)^{1/2} = x + y + z$

Square both sides

$u^2(x^2 + y^2 + z^2) = (x + y + z)^2$

$$u^2x^2 + u^2y^2 + u^2z^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

Differentiate implicitly wrt x

$$\frac{\partial}{\partial x}(u^2x^2 + u^2y^2 + u^2z^2) = \frac{\partial}{\partial x}(x^2 + y^2 + z^2 + 2xy + 2xz + 2yz)$$

$$u^2(2x) + (x^2 + y^2 + z^2).2u\frac{\partial u}{\partial x} = 2x + 2y + 2z$$

$$\frac{\partial u}{\partial x} = \frac{2x+2y+2z-2u^2x}{2u(x^2+y^2+z^2)}$$

$$= \frac{x+y+z-u^2x}{ux^2+uy^2+uz^2}$$

Differentiate implicitly wrt y

$$\frac{\partial}{\partial y}(u^2x^2 + u^2y^2 + u^2z^2) = \frac{\partial}{\partial y}(x^2 + y^2 + z^2 + 2xy + 2xz + 2yz)$$

$$u^2(2y) + (x^2 + y^2 + z^2).2u\frac{\partial u}{\partial y} = 2x + 2y + 2z$$

$$\frac{\partial u}{\partial y} = \frac{2x+2y+2z-2u^2y}{2u(x^2+y^2+z^2)}$$

$$= \frac{x+y+z-u^2y}{ux^2+uy^2+uz^2}$$

Differentiate implicitly wrt z

$$\frac{\partial}{\partial z}(u^2x^2 + u^2y^2 + u^2z^2) = \frac{\partial}{\partial z}(x^2 + y^2 + z^2 + 2xy + 2xz + 2yz)$$

$$u^2(2z) + (x^2 + y^2 + z^2).2u\frac{\partial u}{\partial z} = 2x + 2y + 2z$$

$$\frac{\partial u}{\partial z} = \frac{2x+2y+2z-2u^2z}{2u(x^2+y^2+z^2)}$$

$$= \frac{x+y+z-u^2z}{ux^2+uy^2+uz^2}$$

Proof : $x\frac{\partial u}{\partial x} = x\left[\frac{x+y+z-u^2x}{u(x^2+y^2+z^2)}\right]$

$$y\frac{\partial u}{\partial y} = y\left[\frac{x+y+z-u^2y}{u(x^2+y^2+z^2)}\right]$$

$$z\frac{\partial u}{\partial z} = z\left[\frac{x+y+z-u^2z}{u(x^2+y^2+z^2)}\right]$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left[\frac{x+y+z-u^2x}{u(x^2+y^2+z^2)}\right] + y\left[\frac{x+y+z-u^2y}{u(x^2+y^2+z^2)}\right] + z\left[\frac{x+y+z-u^2z}{u(x^2+y^2+z^2)}\right]$$

$$= \frac{u^2(x^2+y^2+z^2)-u^2(x^2+y^2+z^2)}{u(x^2+y^2+z^2)} = 0$$

10. Prove that if $z = \mathrm{e^x}(\mathrm{xcosy} - \mathrm{ysin}\, y)$, then $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

$$z = \mathrm{e^x}(\mathrm{xcosy} - \mathrm{ysin}\, y)$$

$$= \mathrm{xe^xcosy} - \mathrm{ye^xsin}\, y$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\mathrm{xe^xcosy} - \mathrm{ye^xsin}\, y)$$

$$= \mathrm{cosy}(\mathrm{xe^x} + \mathrm{e^x}) - \mathrm{ye^xsin}\, y$$

$$= \mathrm{xe^xcosy} + \mathrm{e^xcosy} - \mathrm{ye^xsin}\, y$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(\mathrm{xe^xcosy} + \mathrm{e^xcosy} - \mathrm{ye^xsin}\, y)$$

$$= \mathrm{cosy}\left[x\frac{\partial}{\partial x}(\mathrm{e^x}) + \mathrm{e^x}\frac{\partial}{\partial x}(x)\right] + \mathrm{e^xcosy} - \mathrm{ye^xsin}\, y$$

$$= \mathrm{cosy}[x\mathrm{e^x} + \mathrm{e^x}] + \mathrm{e^xcosy} - \mathrm{ye^xsin}\, y$$

$$= x\mathrm{e^xcosy} + 2\mathrm{e^xcosy} - \mathrm{ye^xsin}\, y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\mathrm{xe^xcosy} - \mathrm{ye^xsin}\, y)$$

$$= -\mathrm{xe^xsiny} - \mathrm{e^xycosy} - \mathrm{ye^xsin}\, y$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(-\mathrm{xe^x siny} - \mathrm{e^x ycosy} - \mathrm{ye^x}\sin y)$$

$$= -x\mathrm{e^x cosy} - 2\mathrm{e^x cosy} + \mathrm{ye^x}\sin y$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x\mathrm{e^x cosy} + 2\mathrm{e^x cosy} - \mathrm{ye^x}\sin y - x\mathrm{e^x cosy} - 2\mathrm{e^x cosy} + \mathrm{ye^x}\sin y = 0$$

11. To prove that if $z = \sin(3x - 2y)$,then $3\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 z}{\partial x^2} \neq 0$

$z = \sin(3x - 2y)$

Let $u = 3x + 2y$

$$\frac{\partial u}{\partial x} = 3$$

$$\frac{\partial u}{\partial y} = 2$$

Then $z = \sin u$

$$\frac{\partial z}{\partial u} = \cos u$$

But $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}$

$$= \cos u \,.\, 3$$

$$= 3\cos(3x + 2y)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}$$

$$= 2\cos u$$

$$= 2\cos(3x + 2y)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}[3\cos(3x + 2y)]$$

$$= 3\cos'(3x+2y) \times \frac{\partial}{\partial x}[3x+2y]$$

$$= -3\sin(3x+2y)(3)$$

$$= -9\sin(3x+2y)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial x}[2\cos(3x+2y)]$$

$$= 2\cos'(3x+2y) \times \frac{\partial}{\partial x}[3x+2y]$$

$$= -2\sin(3x+2y)(3)$$

$$= -4\sin(3x+2y)$$

$$\text{Proof}: \ 3\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 z}{\partial x^2} = 3[-4\sin(3x+2y)] - 2[-9\sin(3x+2y)]$$

$$= 6\sin(3x+2y)$$

$$= 6z$$

$$\therefore 3\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 z}{\partial x^2} \neq 0. Infact\ 3\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2 z}{\partial x^2} = 6z$$

12. To show that the equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ is satisfied by $z = \ln(x^2+y^2) + \frac{1}{2}\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = In'(x^2+y^2) \times \frac{\partial}{\partial x}(x^2+y^2) + \frac{1}{2}\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= \frac{1}{x^2+y^2}(2x) + \frac{1}{2} \times \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{2x}{x^2+y^2} + \frac{1}{2} \times \frac{1}{\left(\frac{x^2+y^2}{x^2}\right)} \times \left(-\frac{y}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} + \frac{1}{2} \times \left(\frac{x^2}{x^2+y^2}\right) \times \left(-\frac{y}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{2(x^2+y^2)}$$

$$= \frac{4x-y}{2(x^2+y^2)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2(x^2+y^2)\frac{\partial}{\partial x}(4x-y)-(4x-y)\frac{\partial}{\partial x}[2(x^2+y^2)]}{[2(x^2+y^2)]^2}$$

$$= \frac{2(x^2+y^2)(4)-(4x-y)(4x)}{[2(x^2+y^2)]^2}$$

$$= \frac{8y^2-8x^2+4xy}{4(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y} = \text{In}'(x^2+y^2) \times \frac{\partial}{\partial y}(x^2+y^2) + \frac{1}{2}\frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= \frac{1}{x^2+y^2}(2y) + \frac{1}{2} \times \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{\partial}{\partial y}\left(\frac{y}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{1}{2} \times \frac{1}{\left(\frac{x^2+y^2}{x^2}\right)} \times \left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{1}{2} \times \left(\frac{x^2}{x^2+y^2}\right) \times \left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{x}{2(x^2+y^2)}$$

$$= \frac{4y+x}{2(x^2+y^2)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2(x^2+y^2)\frac{\partial}{\partial y}(4x+y)-(4x+y)\frac{\partial}{\partial y}[2(x^2+y^2)]}{[2(x^2+y^2)]^2}$$

$$= \frac{8x^2-8y^2-4xy}{4(x^2+y^2)^2}$$

Proof : $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{8y^2-8x^2+4xy}{4(x^2+y^2)^2} + \frac{8x^2-8y^2-4xy}{4(x^2+y^2)^2}$$

$$= \frac{8y^2-8x^2+4xy+8x^2-8y^2-4xy}{4(x^2+y^2)^2}$$

13. $z = (ax+by)^2 + e^{ax+by} + \sin(ax+by)$

$$\frac{\partial z}{\partial x} = 2a(ax+by) + ae^{ax+by} + \text{a}\cos(ax+by)$$

$$\frac{\partial z}{\partial y} = 2b(ax+by) + be^{ax+by} + \text{b}\cos(ax+by)$$

$$b\frac{\partial z}{\partial x} = 2ab(ax+by) + abe^{ax+by} + \text{ab}\cos(ax+by) \qquad \delta$$

$$a\frac{\partial z}{\partial y} = 2ab(ax+by) + abe^{ax+by} + \text{ab}\cos(ax+by) \qquad \varphi$$

$\therefore$ *From* δ *and* φ

$$a\frac{\partial z}{\partial y} = b\frac{\partial z}{\partial x} \quad (Proved)$$

CHAIN RULE

A. QUESTIONS

1. Find $\frac{dz}{dt}$, given that $z = x^2 + 3xy + 5y^2, x = \sin t\,, y = \cos t$
2. Find $\frac{dz}{dx}$, given that $z = f(x,y) = x^2 + 3xy + 4y^2, y = e^{ax}$
3. Find $\frac{dz}{dt}$, given that $z = \ln(x^2 + y^2)\,; x = e^{-t}, y = e^{t}$
4. Given that $z = f(x,y) = xy^2 + x^2y,\ y = \ln x,\ find\ (a)\frac{dz}{dx},\ (b)\frac{dz}{dy}$. Here, x is the independent variable.
5. The altitude of a right circular cone is 15cm and is increasing at $0.2cm/s$. The radius of the base is 10cm and is decreasing at $0.3cm/s$. How fast is the volume changing?
6. Find $\frac{du}{dx}$, given $u = f(x,y,z) = xy + yz + zx,\ y = \frac{1}{x}, z = x^2$.
7. Find $\frac{du}{dt}$, given $u = f(x,y,z) = xy + yz + zx,\ x = e^{t},\ y = e^{-t}, z = e^{t} + e^{-t}$
8. If $u = f(x,y)$ and $x = r\cos\theta, y = r\sin\theta$, show that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$

9. If $u = f(x+ay) + g(x-ay)$, show that $\frac{\partial^2 z}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 z}{\partial y^2}$

10. If $z = x^n f\left(\frac{y}{x}\right)$, show that $x\frac{dz}{dx} + y\frac{dz}{dy} = nz$

11. Find $\frac{dz}{dr}$ and $\frac{dz}{ds}$ given $x^2 + xy + y^2, x = 2r + s, y = r - 2s$

12. If $u = f(x,y), u = e^x cosy$ and $v = e^{-x} siny$, find $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$

B. SOLUTION

1. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + 5y^2)$$

$$= 2x + 3y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + 5y^2)$$

$$= 3x + 10y$$

$$\frac{dx}{dt} = \frac{d}{dt}(sint)$$

$$= cost$$

$$\frac{dy}{dt} = \frac{d}{dt}(cost)$$

$$= -sint$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2x + 3y)(cost) + (3x + 10y)(-sint)$$

$$= (2sint + 3cost)(cost) + (3sint + 10cost)(-sint)$$

Since $x = sint, y = cost$

$$= 2sintcost + 3cos^2t - 3sin^2t - 10sintcost$$

$$= 3cos^2t - 3sin^2t - 8sintcost$$

$$= 3(cos^2t - sin^2t) - 8sintcost$$

$$= 3[cos^2t - (1 - cos^2t)] - 8sintcost$$

$$= 3[2cos^2t - 1] - 8sintcost$$

$$= 6cos^2t - 8sintcost - 3$$

2. $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$

$$= \frac{\partial}{\partial x}(x^2 + 3xy + 4y^2) + \frac{\partial}{\partial y}(x^2 + 3xy + 4y^2).\frac{d}{dx}(e^{ax})$$

$$= (2x + 3y) + (3x + 8y)(ae^{ax}) \qquad \text{Since } y = e^{ax}$$

$$= (2x + 3e^{ax}) + a(3x + 8e^{ax})e^{ax}$$

3. $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[\ln(x^2 + y^2)]$$

$$= \text{In}'(x^2 + y^2) \times \frac{\partial}{\partial x}(x^2 + y^2)$$

$$= \frac{1}{x^2+y^2} \times 2x$$

$$= \frac{2x}{x^2+y^2}$$

$$\frac{dx}{dt} = -e^{-t}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[\ln(x^2 + y^2)]$$

$$= \text{In}'(x^2 + y^2) \times \frac{\partial}{\partial y}(x^2 + y^2)$$

$$= \frac{1}{x^2+y^2} \times 2y$$

$$= \frac{2y}{x^2+y^2}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{2x}{x^2+y^2}(-e^{-t}) + \frac{2y}{x^2+y^2}(e^{-t})$$

$$= \frac{-2xe^{-t}+2ye^{t}}{x^2+y^2}$$

$$= \frac{2(ye^{t}-xe^{-t})}{x^2+y^2}$$

4. $z = f(x, y) = xy^2 + x^2y, \; y = \ln x$

(a.) $\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xy^2 + x^2y)$$

$$= y^2 + 2xy$$

$$= y(2x + y)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xy^2 + x^2y)$$

$$= x^2 + 2xy$$

$$= x(2x + y)$$

$$\frac{dy}{dx} = \frac{d}{dx}(\ln x)$$

$$= \frac{1}{x}$$

$$\therefore \frac{dz}{dx} = y(2x + y) + x(2x + y) \times \frac{1}{x}$$

$$= 2xy + y^2 + 2y + x$$

$$= x(2y+1) + y(y+2)$$

5. Let $x = radius$ and $y = altitude\ of\ cone$

From $v = \frac{1}{3}\pi x^2 y$. Considering x and y as a function of t

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}$$

$$= \frac{1}{3}(2xy\frac{dx}{dt} + x^2\frac{dy}{dt})$$

$$= \frac{1}{3}\pi[(-2 \times 10 \times 15 \times 0.3) + 10^2 \times 0.2]$$

$$= -\frac{70\pi}{3}cm^2/s$$

$\therefore There\ is\ a\ negative\ change\ in\ the\ volume, that\ is, it\ is\ decreasing$

6. To find $\frac{du}{dx}$, given that $u = f(x,y,z) = xy + yz + zx,\ y = \frac{1}{x}, z = x^2$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial z}\frac{dz}{dx}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(xy + yz + zx) = y + z$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(xy + yz + zx) = x + z$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(xy + yz + zx) = y + x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{dz}{dx} = \frac{d}{dx}(x^2) = 2x$$

$$\frac{du}{dx} = (y+z) + (x+z)\left(-\frac{1}{x^2}\right) + (y+z)(2x)$$

$$= y(2x+1) + z(2x+1) - (x+z)\left(\frac{1}{x^2}\right)$$

Since $y = \frac{1}{x}$, $z = x^2$

$$= \frac{1}{x}(2x+1) + x^2(2x+1) - (x+x^2)\left(\frac{1}{x^2}\right)$$

7. $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(xy + yz + zx) = y + z$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(xy + yz + zx) = x + z$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(xy + yz + zx) = y + x$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^t) = e^t$$

$$\frac{dy}{dt} = \frac{d}{dt}(e^{-t}) = -e^{-t}$$

$$\frac{dz}{dt} = \frac{d}{dt}(e^t + e^{-t}) = e^t - e^{-t}$$

$$\frac{du}{dt} = (y+z)e^t + (x+z)(-e^{-t}) + (y+x)(e^t - e^{-t})$$

$$= ye^t + ze^t - xe^{-t} - ze^{-t} + ye^t + xe^t - ye^{-t} - xe^{-t}$$

$$= -2xe^{-t} + xe^t + 2ye^t - ye^{-t} + ze^t - ze^{-t}$$

8. If $u = f(x, y)$

$\frac{\partial x}{\partial \theta} = -rsin\theta$

$\frac{\partial y}{\partial \theta} = rcos\theta$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-rsin\theta) + \frac{\partial u}{\partial y}(rcos\theta)$$

Squaring the equation above

$$\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 r^2 sin^2\theta - 2r^2 sin\theta cos\theta \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 r^2 cos^2\theta$$

$$\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 sin^2\theta - 2sin\theta cos\theta \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 cos^2\theta$$

$$\therefore \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(rcos\theta) = cos\theta$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(rsin\theta) = sin\theta$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}cos\theta + \frac{\partial u}{\partial y}sin\theta \qquad \mu$$

Squaring μ gives

$$\left(\frac{\partial u}{\partial r}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 cos^2\theta + 2r^2 sin\theta cos\theta \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 sin^2\theta$$

$(i) + (ii)$ gives

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 sin^2\theta - 2sin\theta cos\theta \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 cos^2\theta$$

$$+\left(\frac{\partial u}{\partial x}\right)^2 cos^2\theta + 2r^2 sin\theta cos\theta \frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 sin^2\theta$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 (sin^2\theta + cos^2\theta) + \left(\frac{\partial u}{\partial y}\right)^2 (sin^2\theta + cos^2\theta)$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \qquad Proved$$

9. $u = f(x + ay) + g(x - ay)$, show that $\frac{\partial^2 z}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial^2 z}{\partial y^2}$

$$\frac{\partial u}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x + ay) + g(x - ay)]$$

$$= \frac{\partial}{\partial x}[f(x + ay)] + \frac{\partial}{\partial x}[g(x - ay)]$$

$$= f'(x + ay) \times \frac{\partial}{\partial x}(x + ay) + g'(x - ay) \times \frac{\partial}{\partial x}(x - ay)$$

$$= f'(x + ay) + g'(x - ay)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}[f'(x + ay) + g'(x - ay)]$$

$$= f''(x + ay) \times \frac{\partial}{\partial x}(x + ay) + g''(x - ay) \times \frac{\partial}{\partial x}(x - ay)$$

$$= f''(x + ay) + g''(x - ay) \qquad \tau$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x + ay) + g(x - ay)]$$

$$= \frac{\partial}{\partial y}[f(x + ay)] + \frac{\partial}{\partial y}[g(x - ay)]$$

$$= f'(x + ay) \times \frac{\partial}{\partial y}(x + ay) + g'(x - ay) \times \frac{\partial}{\partial y}(x - ay)$$

$$= f'(x + ay)(a) + g'(x - ay)(-a)$$

$$= af'(x + ay) - ag'(x - ay)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}[af'(x + ay) - ag'(x - ay)]$$

$$= \frac{\partial}{\partial y}[af'(x + ay)] - \frac{\partial}{\partial y}[ag'(x - ay)]$$

$$= af''(x + ay) \times \frac{\partial}{\partial y}(x + ay) - ag''(x - ay) \times \frac{\partial}{\partial y}(x - ay)$$

$$= af''(x + ay)(a) - ag''(x - ay)(-a)$$

$$= a^2[f''(x+ay)(a)+g''(x-ay)]$$

$$\therefore \frac{1}{a^2}\frac{\partial^2 z}{\partial y^2} = [f''(x+ay)(a)+g''(x-ay)] \qquad \varphi$$

Since τ and φ are equal, then it is true.

10. We are to show that if $z = x^n f\left(\frac{y}{x}\right)$, then $x\frac{dz}{dx} + y\frac{dz}{dy} = nz$

$$z = x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial x} = x^n \frac{\partial}{\partial x}\left[f\left(\frac{y}{x}\right)\right] + f\left(\frac{y}{x}\right)\frac{\partial}{\partial x}(x^n)$$

$$= x^n\left[f'\left(\frac{y}{x}\right)\times\frac{\partial}{\partial x}\left(\frac{y}{x}\right)\right] + f\left(\frac{y}{x}\right)nx^{n-1}$$

$$= x^n\left[f'\left(\frac{y}{x}\right)\times\left(-\frac{y}{x^2}\right)\right] + f\left(\frac{y}{x}\right)nx^{n-1}$$

$$= \left(-\frac{x^n}{x^2}\right)yf'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right)nx^{n-1}$$

$$= -x^{n-1}\left(\frac{y}{x}\right)f'\left(\frac{y}{x}\right) + nf\left(\frac{y}{x}\right)x^{n-1}$$

$$x\frac{\partial z}{\partial x} = x\left[-x^{n-1}\left(\frac{y}{x}\right)f'\left(\frac{y}{x}\right) + nf\left(\frac{y}{x}\right)x^{n-1}\right]$$

$$= -x^{n-1}yf'\left(\frac{y}{x}\right) + nf\left(\frac{y}{x}\right)x^n$$

ε

$$\frac{\partial z}{\partial y} = x^n\frac{\partial}{\partial y}\left[f\left(\frac{y}{x}\right)\right] + f\left(\frac{y}{x}\right)\frac{\partial}{\partial y}(x^n)$$

$$= x^n\left[f'\left(\frac{y}{x}\right)\times\frac{\partial}{\partial y}\left(\frac{y}{x}\right)\right] + 0$$

$$= x^n\left[f'\left(\frac{y}{x}\right)\times\left(\frac{1}{x}\right)\right]$$

$$= x^{n-1}f'\left(\frac{y}{x}\right)$$

$$y\frac{\partial z}{\partial y} = x^{n-1}yf'\left(\frac{y}{x}\right) \qquad \omega$$

Equation $\varepsilon + \omega$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -x^{n-1}yf'\left(\frac{y}{x}\right) + nf\left(\frac{y}{x}\right)x^n + x^{n-1}yf'\left(\frac{y}{x}\right)$$

$$= nf\left(\frac{y}{x}\right)x^n \qquad \text{Since } z = f\left(\frac{y}{x}\right)x^n$$

$$= nz \quad \blacksquare$$

11. To find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ given that $x^2 + xy + y^2, x = 2r + s, y = r - 2s$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial r}(x^2 + xy + y^2)$$

$$= 2x + y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy + y^2)$$

$$= x + 2y$$

$$\frac{\partial x}{\partial r} = 2$$

$$\frac{\partial y}{\partial r} = 1$$

$$\frac{\partial y}{\partial s} = -2$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= 5x + 4y$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2x + y)(-1) + (x + 2y)(-2)$$

$$= -3y$$

13. If $u = f(x,y), u = e^{x}cosy$ and $v = e^{-x}siny$, we are to find $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$

$$\delta u = \frac{\partial u}{\partial x} . \ \delta x + \frac{\partial u}{\partial y} . \delta y \qquad א$$

$$\delta v = \frac{\partial v}{\partial x} . \delta x + \frac{\partial v}{\partial y} . \delta y \qquad ב$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^{x}cosy)$$

$$= e^{x} \frac{\partial}{\partial x}(cosy) + cosy\frac{\partial}{\partial x}(e^{x})$$

$$= e^{x}cosy$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{x}cosy)$$

$$= e^{x} \frac{\partial}{\partial y}(cosy) + cosy\frac{\partial}{\partial y}(e^{x})$$

$$= -e^{x}siny$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(e^{-x}siny)$$

$$= e^{-x} \frac{\partial}{\partial x}(siny) + siny\frac{\partial}{\partial x}(e^{-x})$$

$$= -e^{-x}siny$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(e^{-x}siny)$$

$$= e^{-x} \frac{\partial}{\partial x}(siny) + siny\frac{\partial}{\partial x}(e^{-x})$$

$$= e^{-x}cosy$$

Substitute for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ in equations א and ב

$$e^{-x}cosy \times [\delta u = e^{x}cosy\delta x - e^{x}siny\delta y] \qquad \alpha$$

$$e^x siny \times [\delta v = -e^{-x} siny\delta x + e^{-x} cosy\delta y] \qquad \beta$$

We have

$$e^{-x} cosy\delta u = (e^{-x})(e^x)cos^2 y\delta x - (e^{-x})(e^x)sinycosy\delta y$$

$$e^x siny\delta v = -(e^x)(e^{-x})sin^2 y\delta x + (e^x)(e^{-x})sinycosy\delta y$$

This becomes

$$e^{-x} cosy\delta u = cos^2 y\delta x - sinycosy\delta y \qquad \gamma$$

$$e^x siny\delta v = -sin^2 y\delta x + sinycosy\delta y \qquad \delta$$

Adding equations $\gamma + \delta$

$$e^{-x} cosy\delta u + e^x siny\delta v = (cos^2 y - sin^2 y)\delta x$$

Recall $cos2y = cos^2 y - sin^2 y$

$$e^{-x} cosy\delta u + e^x siny\delta v = cos2y\delta x$$

$$\delta x = \frac{e^{-x} cosy}{cos2y}\delta u + \frac{e^x siny}{cos2y}\delta v$$

Comparing with

$$\delta x = \frac{\partial x}{\partial u} .\ \delta u + \frac{\partial x}{\partial v} .\delta v$$

$$\therefore\ \frac{\partial x}{\partial u} = \frac{e^{-x} cosy}{cos2y}$$

$$\frac{\partial x}{\partial v} = \frac{e^x siny}{cos2y}$$

From $\delta u = \frac{\partial u}{\partial x} .\ \delta x + \frac{\partial u}{\partial y} .\delta y$

$$= e^x cosy\delta x - e^{-x} siny\delta y$$

$$\delta v = \frac{\partial v}{\partial x} .\delta x + \frac{\partial v}{\partial y} .\delta y$$

$$= -e^{-x}siny\delta x + e^{-x}cosy\delta y$$

Now,

$$e^{-x}siny \times [\delta u = e^{x}cosy\delta x - e^{-x}siny\delta y]$$

$$e^{x}cosy \times [\delta v = -e^{-x}siny\delta x + e^{-x}cosy\delta y]$$

$$e^{-x}siny\delta u = (e^{-x}siny)(e^{x}cosy)\delta x - (e^{x})(e^{-x})sin^2y\delta y \qquad \omega$$

$$e^{x}cosy\delta v = -(e^{-x}siny)(e^{x}cosy)\delta x + (e^{-x})(e^{x})cos^2y\delta y \qquad \varphi$$

Equation $\omega + \varphi$

$$e^{-x}siny\delta u + e^{x}cosy\delta v = cos^2y\delta y - sin^2y\delta y$$

$$e^{-x}siny\delta u + e^{x}cosy\delta v = (cos^2y - sin^2y)\delta y$$

Recall $cos2y = cos^2y - sin^2y$

$$e^{-x}siny\delta u + e^{x}cosy\delta v = cos2y\delta y$$

$$\delta y = \frac{e^{-x}siny}{cos2y}\delta u + \frac{e^{x}cosy}{cos2y}\delta v$$

Comparing with

$$\delta y = \frac{\partial y}{\partial u} \,.\, \delta u + \frac{\partial y}{\partial v} \,.\, \delta v$$

$$\therefore \frac{\partial y}{\partial u} = \frac{e^{-x}siny}{cos2y} \;, \frac{\partial y}{\partial v} = \frac{e^{x}cosy}{cos2y}$$

STATIONARY VALUES OF FUNCTIONS WITH TWO VARIABLES

A. QUESTIONS

1. Investigate stationary values of the function $z = x^2 + xy + y^2 + 5x - 5y + 3$
2. Find the values of x and y for the stationary points of $z = 5xy - 6x^2 - y^2 + 7x - 2y$
3. Determine the position and nature of the stationary points of the function $z = 2x^2y^2 - 4y^3 + 4xy^2 + 16y + 5$
4. Determine the stationary values of the function $z = x^3 - 6xy + y^3$
5. Locate the stationary points of the following functions. Determine the nature of the points and calculate the critical functions' values
   (a.) $z = y^2 + xy + x^2 + 4y - 4x + 5$
   (b.) $z = y^2 + xy + 2x + 3y + 6$
6. Determine the stationary values of $z = x^3 - 3x + xy^2$ and their nature.

B. SOLUTION

1. $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + xy + y^2 + 5x - 5y + 3)$

$= 2x + y + 5$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2 + xy + y^2 + 5x - 5y + 3)$$

$$= x + 2y - 5$$

At stationary points $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

Solving these two simultaneously, we have

$2x + y + 5 = 0$
$x + 2y - 5 = 0$

$1 \times (2x + y = -5)$
$-2 \times (x + 2y = 5)$

$2x + y = -5$ $\in$
$-2x - 4y = -10$ $\ni$

Adding equations $\in$ and $\ni$, we have
$y = 5$

Substituting y in equation $\in$, we have
$x = -5$

Hence, stationary value exists at $(-5, 5)$
If $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, a stationary value occurs.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(2x + y + 5) = 2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(x + 2y - 5) = 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(x + 2y - 5) = 1$$

$$\therefore \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (2)(2) - (1)^2 = 4 - 1 = 3$$

Hence, it occurs at (-5, 5)

Since $\frac{\partial^2 z}{\partial x^2} = 2 > 0$

$\frac{\partial^2 z}{\partial y^2} = 2 > 0$

$\therefore z$ is maximum.

2. $z = 5xy - 6x^2 - y^2 + 7x - 2y$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(5xy - 6x^2 - y^2 + 7x - 2y)$

$= 5y - 12x + 7$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(5xy - 6x^2 - y^2 + 7x - 2y)$

$= 5x - 2y - 2$

At stationary points $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

Solving these two simultaneously, we have

$5y - 12x + 7 = 0$

$5x - 2y - 2 = 0$

$5 \times (5y - 12x + 7 = 0)$

$-12 \times (5x - 2y - 2 = 0\ )$

$25y - 60x + 35 = 0$ φ

$-60x + 24y + 24 = 0$ ω

Adding equations φ and ω, we have

$y = -11$

Substituting y in $5y - 12x + 7 = 0$, we have

$x = -4$

Hence, stationary value exists at $(-4, -11)$

3. $z = 2x^2y^2 - 4y^3 + 4xy^2 + 16y + 5$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(2x^2y^2 - 4y^3 + 4xy^2 + 16y + 5)$$

$$= 4xy^2 + 4y^2$$

$$= 4y^2(x + 1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(2x^2y^2 - 4y^3 + 4xy^2 + 16y + 5)$$

$$= 4x^2y - 12y^2 + 8xy + 16$$

At stationary points $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

Solving these two simultaneously, we have

$$4y^2(x + 1) = 0 \qquad \alpha$$
$$4x^2y - 12y^2 + 8xy + 16 = 0 \qquad \beta$$

From equation α

$4y^2(x + 1) = 0$

$(x + 1) = 0$

$x = -1$

Substitute for x in β

$4(-1)^2y - 12y^2 + 8(-1)y + 16 = 0$

$4y - 12y^2 - 8y + 16 = 0$

$-12y^2 - 4y + 16 = 0$

$-4(3y^2 - y - 4) = 0$

$3y^2 - y - 4 = 0$

By General formula

$a = 3, b = -1, c = -4$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{1^2 - 4(3)(-4)}}{2(3)}$$

$$= \frac{-3 \pm \sqrt{1 + 48}}{2(3)}$$

$$= \frac{-3 \pm \sqrt{49}}{2(3)}$$

$$= \frac{-3 \pm 7}{6}$$

$$y = \frac{1}{6}(-3 \pm 7)$$

Hence, the stationary values occur at $\left[-1, \frac{1}{6}(-3 \pm 7)\right]$

4. $z = x^3 - 6xy + y^3$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3 - 6xy + y^3)$$

$$= 3x^2 - 6y$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3 - 6xy + y^3)$$

$$= -6x + 3y^2$$

At stationary points $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

Solving these two simultaneously, we have

$$3x^2 - 6y = 0$$
$$-6x + 3y^2 = 0$$

$$x^2 - 2y = 0 \qquad \rho$$
$$-2x + y^2 = 0 \qquad \sigma$$

From equation ρ, we have

$$x^2 = 2y$$
$$y = \frac{x^2}{2} \qquad \vartheta$$

Substitute in equation σ

$$-2x + y^2 = 0$$
$$-2x + \left(\frac{x^2}{2}\right)^2 = 0$$
$$-2x + \frac{x^4}{4} = 0$$
$$x^4 = 8x$$
$$x^3 = 8$$
$$x = \sqrt[3]{8} = 2$$

Substituting x in equation ϑ, we have

$$y = \frac{x^2}{2}$$
$$y = \frac{2^2}{2} = 2$$

Hence, stationary value exists at (2,2)

If $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, a stationary value occurs.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 - 6y) = 6x$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(-6x + 3y^2) = 6y$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-6x + 3y^2) = -6$$

At $x = 0$, $\frac{\partial^2 z}{\partial x^2} = 0$

At $y = 0$, $\frac{\partial^2 z}{\partial y^2} = 0$

$$\therefore \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (0)(0) - (-6)^2 = -36 < 0$$

Hence, at (0, 0), there is neither maximum nor minimum

At $(2, 2)$

$$\therefore \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = [6(2)][6(2)] - (-6)^2 = 108 > 0$$

Since $\frac{\partial^2 z}{\partial x^2} > 0$

and $\frac{\partial^2 z}{\partial y^2} > 0$ at $(2, 2)$

$\therefore$ *The stationary value at* $(2, 2)$ is a minimum.

The minimum value of z is $z = x^3 - 6xy + y^3 = 2^3 - 6(2)(2) + 2^3$

$$= 40 \text{ at } (2,2)$$

5. $z = y^2 + xy + x^2 + 4y - 4x + 5$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(y^2 + xy + x^2 + 4y - 4x + 5)$$

$$= y + 2x - 4$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(y^2 + xy + x^2 + 4y - 4x + 5)$$

$$= 2y + x + 4$$

At stationary points $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

Solving these two simultaneously, we have

$y + 2x - 4 = 0$
$2y + x + 4 = 0$

$-2 \times (y + 2x = 4)$
$1 \times (2y + x = -4)$

$-2y - 4x = -8$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in$

$2y + x = -4$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \ni$

Adding equations $\in$ and $\ni$, we have

$x = 4$

Substituting x in equation $\in$, we have

$y = -5$

Hence, stationary value exists at $(4, -4)$

If $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, a stationary value occurs.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(y + 2x - 4) = 2$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(2y + x + 4) = 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(2y + x + 4) = 1$$

$$\therefore \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (2)(2) - (1)^2 = 4 - 1 = 3 > 0$$

It is neither minimum nor maximum yet a stationary value exists

Since $\frac{\partial^2 z}{\partial x^2} = 2 > 0$

$\frac{\partial^2 z}{\partial y^2} = 2 > 0$

$\therefore z$ is minimum at $(4, -4)$.

The value of z occurs $(4, -4)$

$z = y^2 + xy + x^2 + 4y - 4x + 5$

$= (-4)^2 + 4(-4) + 4^2 + 4(-4) - 4(4) + 5$

$= -21$

6. $z = x^3 - 3x + xy^2$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3x + xy^2)$$

$$= 3x^2 - 3 + y^2$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3x + xy^2)$$

$$= 2xy$$

At stationary points $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

Solving these two simultaneously, we have

$$3x^2 - 3 + y^2 = 0 \qquad \in$$
$$2xy = 0 \qquad \ni$$

From equation ∋
$x = 0$ and $y = 0$

If $x = 0$ then $y^2 = 3 \quad \rightarrow \quad y = \pm\sqrt{3}$
We have a stationary value $(0, \pm\sqrt{3})$

If $y = 0$ then $3x^2 - 3 = 0 \quad \rightarrow \quad x = \pm 1$
We have a stationary value $(\pm 1, 0)$

If $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$, a stationary value occurs.

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 - 3 + y^2) = 6x$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(x^3 - 3x + xy^2) = 2x$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(x^3 - 3x + xy^2) = 2y$$

$$\therefore \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (6x)(2x) - (2y)^2 = 12x^2 - 4y^2$$

When $x = 0, \; y = \sqrt{3}$
$12(0) - 4(3) = -12 < 0, i.e.,$ Saddle point

When $x = 0, \; y = -\sqrt{3}$
$12(0) - 4(3) = -12 < 0, i.e.,$ Saddle point

When $x = 1, \; y = 0$
$6(1) - 0 > 0, i.e.,$ minimum point

When $x = -1, \; y = 0$
$6(-1) - 0 < 0, i.e.,$ maximum point

LAGRANGE OF UNDETERMINED MULTIPLIERS OF FUNCTIONS WITH TWO VARIABLES

A. QUESTIONS

1. Find the stationary points of the function $u = x^2 + y^2$ subject to the constraints $x^2 + y^2 + 2x - 2y + 1 = 0$
2. Find the stationary points of the function $u = x^2 + 2y^2 + z$ subject to the constraints $\emptyset(x, y, z) = x^2 - z^2 - 2 = 0$
3. Use Lagrange's method of undetermined multipliers to obtain the stationary values of the following functions u, subject in each case to the constraints $\emptyset$
   a. $u = x^2y^2z^2, \quad \emptyset = x^2 + y^2 + z^2 - 4 = 0$
   b. $u = x^2 + y^2, \quad \emptyset = 4x^2 + 4y^2 + 6xy = 9$
   c. $u = x^2 + y^2 + z^2, \quad \emptyset = 3x - 2y + z - 4 = 0$
4. A hot water storage tank is a vertical cylinder surmounted by a hemisphere top of the same diameter. The tank is designed to hold $400m^3$ of liquid. Determine the total height and the diameter of the tank if the surface heat loss is to be a minimum.
5. If $z = \frac{xy}{x-y}$, show that
   i. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$
   ii. $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = 0$
   iii. $z\frac{\partial^2 z}{\partial x \partial y} = 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$
6. If $z = f(x, y)$ and $u = e^{-x}siny$. Find the derivatives $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, $\frac{\partial y}{\partial v}$
7. If $z = 2x^2 + 3xy + 4y^2$ and $u = x^2 + y^2$ and $v = x + 2y$, determine
   a. $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, $\frac{\partial y}{\partial v}$
   b. $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

B. SOLUTION

1. $u = x^2 + y^2$

$\emptyset = x^2 + y^2 + 2x - 2y + 1 = 0$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$

$\frac{\partial \emptyset}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + 2x - 2y + 1) = 2x + 2$

$\frac{\partial \emptyset}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + 2x - 2y + 1) = 2y - 2$

At stationary points

$\frac{\partial u}{\partial x} + \lambda \frac{\partial \emptyset}{\partial x} = 0 \qquad \alpha$

$\frac{\partial u}{\partial y} + \lambda \frac{\partial \emptyset}{\partial y} = 0 \qquad \beta$

$\rightarrow \quad 2x + \lambda(2x + 2) = 0 \qquad \gamma$

$\quad 2x + \lambda(2y - 2) = 0 \qquad \delta$

From γ

$2x + \lambda(2x + 2) = 0$

$\lambda = \frac{-2x}{2x+2} = \frac{-x}{x+1} \qquad \theta$

From δ

$\lambda = \frac{-2y}{2y-2} = \frac{-y}{y-1} \qquad \vartheta$

Dividing θ by ϑ

$\frac{\lambda}{\lambda} = \left(\frac{-x}{x+1}\right) \Big/ \left(\frac{-y}{y-1}\right) = \frac{x(y-1)}{y(x+1)}$

$\therefore 1 = \frac{x(y-1)}{y(x+1)}$

$\rightarrow \quad y(x + 1) = x(y - 1)$

$\therefore y = -x \qquad \mu$

Sub. for $y = -x$ in $\emptyset$

$\emptyset(x, y) \rightarrow \; x^2 + y^2 + 2x - 2y + 1 = 0$

$\rightarrow \; x^2 + (-x)^2 + 2x - 2(-x) + 1 = 0$

$\therefore \emptyset(x, y) \rightarrow 2x^2 + 4x + 1 = 0$

Solving by General formula

$a = 2, \; b = 4 \; and \; c = 1$

$\therefore x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{4^2-4(2)(1)}}{2(2)}$

$= \frac{-4 \pm \sqrt{16-8}}{4} \quad = -1 \pm \frac{\sqrt{2}}{2}$

Sub for $x = -1 \pm \frac{\sqrt{2}}{2}$ in equation μ

$\therefore y = -x$

$= -\left[-1 \pm \frac{\sqrt{2}}{2}\right]$

$= 1 \pm \frac{\sqrt{2}}{2}$

To find λ, we have equation θ

$\lambda = \frac{-x}{x+1} = -\frac{\left(-1 \pm \frac{\sqrt{2}}{2}\right)}{\left(-1 \pm \frac{\sqrt{2}}{2}+1\right)}$

$= \sqrt{2} \pm 1$

2. $u = x^2 + 2y^2 + z \, , \emptyset(x, y, z) = x^2 - z^2 - 2 = 0$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2y^2 + z) = 2x$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2y^2 + z) = 4y$

$\frac{\partial u}{\partial z} = \frac{\partial}{\partial y}(x^2 + 2y^2 + z) = 1$

$\frac{\partial \emptyset}{\partial x} = \frac{\partial}{\partial x}(x^2 - z^2 - 2) = 2x$

$\frac{\partial \emptyset}{\partial y} = \frac{\partial}{\partial y}(x^2 - z^2 - 2) = 0$

$\frac{\partial \emptyset}{\partial y} = \frac{\partial}{\partial y}(x^2 - z^2 - 2) = -2z$

At stationary points

$\frac{\partial u}{\partial x} + \lambda \frac{\partial \emptyset}{\partial x} = 0 \;\; \rightarrow \;\; 2x + 2\lambda x = 0 \;\; \rightarrow \;\; \lambda = -1$

$\frac{\partial u}{\partial y} + \lambda \frac{\partial \emptyset}{\partial y} = 0 \;\; \rightarrow \;\; 4y - \lambda(0) = 0 \;\; \rightarrow \;\; y = 0$

$\frac{\partial u}{\partial z} + \lambda \frac{\partial \emptyset}{\partial z} = 0 \;\; \rightarrow \;\; 1 - 2\lambda z = 0 \;\; \rightarrow \;\; z = \frac{1}{2\lambda} = -\frac{1}{2}$

$0 = x^2 - z^2 - 2 \;\; \rightarrow \;\; x^2 - \left(-\frac{1}{2}\right)^2 - 2 = 0 \;\; \rightarrow \;\; x^2 - \frac{9}{4} = 0$

$\rightarrow x = \pm \frac{3}{2}$

$\therefore \textit{The stationary points are at } \left(\frac{3}{2}, \; 0, \; -\frac{1}{2}\right) \textit{ and } \left(-\frac{3}{2}, \; 0, \; -\frac{1}{2}\right)$

3a. $u = x^2 y^2 z^2, \quad \emptyset = x^2 + y^2 + z^2 - 4 = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2y^2z^2) = 2xy^2z^2$$
$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2y^2z^2) = 2x^2yz^2$$
$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial y}(x^2y^2z^2) = 2x^2y^2z$$
$$\frac{\partial \emptyset}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2 - 4) = 2x$$
$$\frac{\partial \emptyset}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + z^2 - 4) = 2y$$
$$\frac{\partial \emptyset}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + z^2 - 4) = 2z$$

At stationary points

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \emptyset}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \emptyset}{\partial y} = 0$$
$$\frac{\partial u}{\partial z} + \lambda\frac{\partial \emptyset}{\partial z} = 0$$

$$\rightarrow \quad 2xy^2z^2 + \lambda(2x) = 0 \qquad \gamma$$
$$2x^2yz^2 + \lambda(2y) = 0 \qquad \delta$$
$$2x^2y^2z + \lambda(2z) = 0 \qquad \beta$$

From γ

$$2xy^2z^2 + \lambda(2x) = 0$$
$$\lambda(2x) = -2xy^2z^2$$
$$\lambda = \frac{-2xy^2z^2}{2x}$$
$$\lambda = -y^2z^2 \qquad \theta$$

From δ

$$2x^2yz^2 + \lambda(2y) = 0$$
$$\lambda = \frac{-2x^2yz^2}{2y}$$
$$\lambda = -x^2z^2 \qquad \sigma$$

From β

$$2x^2y^2z + \lambda(2z) = 0$$
$$\lambda = \frac{-2x^2y^2z}{2z}$$
$$\lambda = -x^2y^2 \qquad \varphi$$

Divide θ by σ

$$\frac{\lambda}{\lambda} = \frac{-y^2z^2}{-x^2z^2}$$
$$1 = \frac{y^2}{x^2}$$

$$x^2 = y^2$$
$$\rightarrow x = y$$

Divide σ by φ

$$\frac{\lambda}{\lambda} = \frac{-x^2z^2}{-x^2y^2}$$
$$1 = \frac{z^2}{y^2}$$
$$y^2 = z^2$$
$$\rightarrow y = z$$

Hence, $x = y = z$

Replace, y and z by x in $\emptyset$

$\emptyset \rightarrow x^2 + x^2 + x^2 - 4 = 0$

$$3x^2 - 4 = 0$$
$$3x^2 = 4$$
$$x = \pm\frac{2}{\sqrt{3}}$$

Rationalizing by the conjugate of the interval

$$x = \pm\frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \pm\frac{2\sqrt{3}}{3}$$

Since $x = y = z$ $\therefore y = \pm\frac{2\sqrt{3}}{3}$ and $z = \pm\frac{2\sqrt{3}}{3}$

But $\lambda = -x^2z^2$

$$= -\left[\pm\frac{2\sqrt{3}}{3}\right]\left[\pm\frac{2\sqrt{3}}{3}\right]$$
$$= -\left[\frac{4}{9} \times 3\right]$$
$$= -\frac{4}{9} = -1\,{}^{1}\!/_{3}$$

$\therefore$ The stationary points are $\left[3\left(\frac{2\sqrt{3}}{3}\right)\right]$ and $\left[3\left(-\frac{2\sqrt{3}}{3}\right)\right]$

MULTIPLE INTEGRALS

A. QUESTIONS

1. Evaluate $\int_0^1 dx \int_0^x e^{\frac{y}{x}} dy$

<u>Solution</u>

$$\int_0^1 dx \int_0^x e^{\frac{y}{x}} dy = \int_0^1 \left[\frac{1}{1/x} e^{\frac{y}{x}}\right]_0^x dx$$

$$= \int_0^1 \left[x e^{\frac{y}{x}}\right]_0^x dx$$

$$= \int_0^1 [xe - x] dx$$

$$= \int_0^1 x[e-1] dx$$

$$= [e-1] \int_0^1 x dx$$

$$= [e-1] \left[\frac{x^2}{2}\right]_0^1$$

$$= [e-1] \left[\frac{1}{2}\right]$$

$$= \frac{1}{2}(e-1)$$

2. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2}$

<u>Solution</u>

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{(1+x^2)+y^2}$$

$$= \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}}$$

$$= \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{0}{\sqrt{1+x^2}}\right]$$

$$= \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}(1)\right]$$

$$= \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4} \ln[x + \sqrt{1+x^2}]_0^1$$

$$= \frac{\pi}{4} \left[\ln[1+\sqrt{2}] - \ln[0+\sqrt{1}]\right]$$

$$= \frac{\pi}{4} \left[\ln[1+\sqrt{2}] - 0\right]$$

$$= \frac{\pi}{4} \ln[1+\sqrt{2}]$$

3. Evaluate $\iint_A xydxdy$ over the region of the positive quadrant for which $x + y \leq 1$

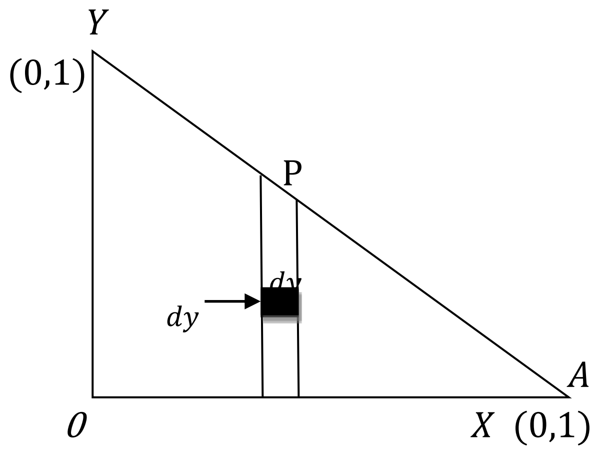
Solution

$x + y = 1$ represents a line AB in the figure

The limits of y are $1 - x$ and 0

Required integral $= \int_0^1 xdx \int_0^{1-x} ydy$

$$= \int_0^1 xdx \left[\frac{y^2}{2}\right]_0^{1-x}$$



Required integral $= \frac{1}{2}\int_0^1 xdx\,(1-x)^2$

$$= \frac{1}{2}\int_0^1 x\,(1-x)^2 dx$$

$$= \frac{1}{2}\int_0^1 [x - 2x^2 + x^3]\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

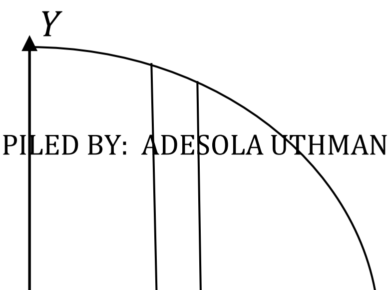
$$= \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

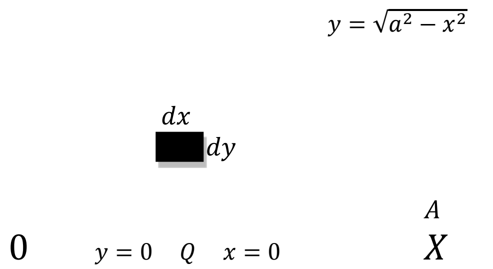
$$= \frac{1}{24}$$

4. Evaluate $\iint_A xydxdy$ where R is the quadrant of the circle $x^2 + y^2 = a^2$, where $x \geq 0$, $y \geq 0$

Solution

Let the region of integration be the first quadrant of the circle OAB





First, we integrate with respect to y and with respect to x

The limits of y are 0 and $\sqrt{a^2 - x^2}$, for $x = 0$ to a

$$\int_0^a xdx \int_0^{\sqrt{a^2-x^2}} ydy = \int_0^a xdx \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= \int_0^a xdx \left[\frac{(\sqrt{a^2-x^2})^2}{2} - 0\right]$$

$$= \frac{1}{2}\int_0^a x\,(a^2 - x^2)dx$$

$$= \frac{1}{2}\int_0^a (xa^2 - x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{a^4}{8}$$

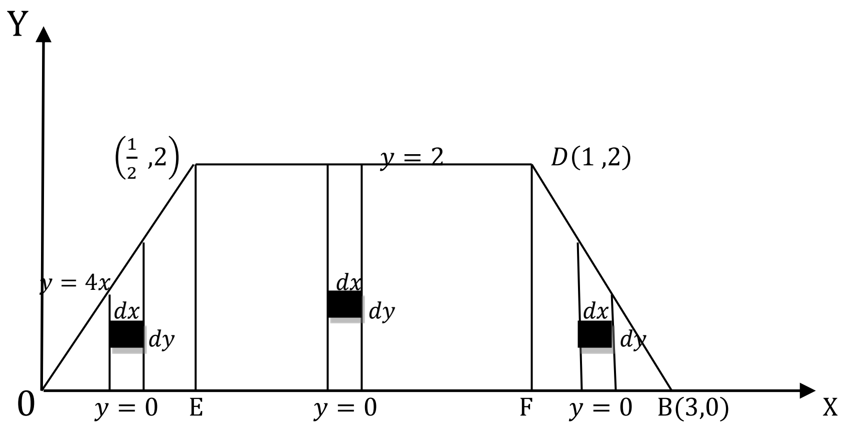
5. Evaluate $\iint_A (x^2 + y^2)dxdy$ throughout the area enclosed by the Curve $y = 4x$, $x + y = 3$, $y = 0$ and $y = 2$

<u>Solution</u>

Let OC represent $= 4x$; $BD,\ x + y = 3$; OB , $y = 0$ and $CD,\ y = 2$

The given integral is to evaluate the area A of the trapezium $OCDB$

Area $OCDB$ consists of area OCE , area $ECDF$ and area FDB



The co-ordinates of C, D and B are $\left(\frac{1}{2},2\right)$, $(1,2)$ $and (3,0)$ respectively.

$$\therefore \iint_A (x^2+y^2)dxdy = \iint_{OCE}(x^2+y^2)dxdy + \iint_{ECDF}(x^2+y^2)dxdy + \iint_{FDB}(x^2+y^2)dxdy$$

$$= \int_0^{\frac{1}{2}} dx \int_0^{4x}(x^2+y^2)dy + \int_{\frac{1}{2}}^{1} dx \int_0^{2}(x^2+y^2)dy + \int_1^3 dx \int_0^{3-x}(x^2+y^2)dy$$

Now,

$$I_1 = \int_0^{\frac{1}{2}} dx \int_0^{4x}(x^2+y^2)dy = \int_0^{\frac{1}{2}}\left[x^2y - \frac{y^3}{3}\right]_0^{4x} dx$$
$$= \int_0^{\frac{1}{2}} \frac{76}{3}x^3\,dx$$
$$= \frac{19}{48}$$

$$I_2 = \int_{\frac{1}{2}}^{1} dx \int_0^{2}(x^2+y^2)dy = \int_{\frac{1}{2}}^{1}\left[x^2y - \frac{y^3}{3}\right]_0^{2} dx$$
$$= \int_{\frac{1}{2}}^{1}\left(2x^2 + \frac{8}{3}\right)dx$$
$$= \left[\frac{2x^3}{3} + \frac{8x}{3}\right]_{\frac{1}{2}}^{1}$$
$$= \frac{23}{12}$$

$$I_3 = \int_1^3 dx \int_0^{3-x}(x^2+y^2)dy = \int_1^3\left[x^2y - \frac{y^3}{3}\right]_0^{3-x} dx$$
$$= \int_1^3\left(x^2(3-x) + \frac{(3-x)^3}{3}\right)dx$$
$$= \int_1^3\left(3x^2 - x^3 + \frac{(3-x)^3}{3}\right)dx$$
$$= \left[\frac{3x^3}{3} - \frac{x^4}{4} - \frac{(3-x)^4}{3.4}\right]_1^3$$
$$= \frac{22}{3}$$

$$\iint_A (x^2+y^2)dxdy = I_1 + I_2 + I_3 = \frac{19}{48} + \frac{23}{12} + \frac{22}{3} = \frac{463}{48} = 9\tfrac{31}{48}$$

6. Evaluate $\iiint_R (x+y+z)dxdydz$, where $R : 0 \le x \le 1$, $1 \le y \le 2$ $2 \le z \le 3$

Solution

$$\int_0^1 dx \int_1^2 dy \int_2^3 (x+y+z)dz = \int_0^1 dx \int_1^2 dy \left[\frac{(x+y+z)^2}{2}\right]_2^3$$
$$= \frac{1}{2}\int_0^1 dx \int_1^2 dy\,[(x+y+3)^2 - (x+y+2)^2]$$
$$= \frac{1}{2}\int_0^1 dx \int_1^2 [(2x+2y+5)]dy$$
$$= \frac{1}{2}\int_0^1 \left[\frac{(2x+2y+5)^2}{4}\right]_1^2 dx$$

$$= \frac{1}{8}\int_0^1 [(2x + 4 + 5)^2 - (2x + 2 + 5)^2]\, dx$$

$$= \frac{1}{8}\int_0^1 [4x + 16] \times 2dx$$

$$= \left[\frac{x^2}{2} + 4x\right]_0^1$$

$$= \frac{9}{2}$$

2012/2013 TEST

1a. Does the equation $x^2 + 2y^3 = 3$ determine y as a single-valued Function of x?

Solution

$x^2 + 2y^3 = 3$

$\frac{d}{dx}(x^2 + 2y^3) = \frac{d(3)}{dx}$

$2x + 6y\frac{dy}{dx} = 0$

$6y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{6y}$

$\frac{dy}{dx} = \frac{-x}{3y}$

Hence, $x^2 + 2y^3 = 3$ determines y as a single-valued function of x

b. If $f(x) = \frac{x+1}{x-1}$ and $g(y) = \frac{2y+5}{y-3}$, find $f[g(x)]$ and $g[f(x)]$

Solution

$f(x) = \frac{x+1}{x-1}$ and $g(y) = \frac{2y+5}{y-3}$

$g(x) = \frac{2x+5}{x-3}$

$$f[g(x)] = \frac{\left(\frac{2x+5}{x-3}\right)+1}{\left(\frac{2x+5}{x-3}\right)-1}$$

$$= \frac{2x+5+(1)(x-3)}{x-3} \bigg/ \frac{2x+5-(1)(x-3)}{x-3}$$

$$= \frac{3x+2}{x-3} \times \frac{x-3}{x+8}$$

$$= \frac{3x+2}{x+8}$$

1c. Find the nth derivatives of $\frac{1}{\sqrt{ax+b}}$

Solution

Let $y = \frac{1}{\sqrt{ax+b}}$

$y = (ax+b)^{-\frac{1}{2}}$

Find the first derivative up the nth term

$\frac{dy}{dx} = \frac{d}{dx}(ax+b)^{-\frac{1}{2}}$

$= -\frac{1}{2}(ax+b)^{-\frac{1}{2}} \times \frac{d}{dx}(ax+b)$

$= -\frac{1}{2}(ax+b)^{-\frac{1}{2}-1} \times a$

$$= \ -\frac{1}{2}a(ax+b)^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-\frac{1}{2}a(ax+b)^{-\frac{3}{2}}\right]$$

$$= \ \frac{3}{4}a^2(ax+b)^{-\frac{5}{2}}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{3}{4}a^2(ax+b)^{-\frac{5}{2}}\right]$$

$$= \ -\frac{15}{8}a^3(ax+b)^{-\frac{7}{2}}$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left[-\frac{15}{8}a^3(ax+b)^{-\frac{7}{2}}\right]$$

$$= \ \frac{105}{16}a^4(ax+b)^{-\frac{9}{2}}$$

$$\frac{d^5y}{dx^5} = \frac{d}{dx}\left[\frac{105}{16}a^4(ax+b)^{-\frac{9}{2}}\right]$$

$$= \ -\frac{945}{32}a^5(ax+b)^{-\frac{11}{2}}$$

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$$\frac{d^ny}{dx^n} = \frac{(-1)^n(a)^n(2n)!(ax+b)^{-(n+\frac{1}{2})}}{n!2^{2n}}$$

2a. Use differentials to approximate $\sin 60^{\circ}1^{'}$

Solution

Let $y = \sin 60^{\circ}1^{'}$

Let $x = \sin 60^{\circ}$ and $\Delta x = 1^{'}$

$y = \sin x = \sin 60^{\circ} = \frac{\sqrt{3}}{2} = 0.86603$

$\frac{dy}{dx} = \cos x = \cos 60 = \frac{1}{2}$

$\delta x = cos\delta y$

$= \ cos60(0.003)$

$= 0.00015$

Convert $1^{'}$ to radians

$60^{'} = 1^{\circ}$

But $\Delta x = 1^{'}$

$\Delta x = \left(\frac{1}{60}\right)^{\circ}$

But π rad $= 180^{\circ}$

Δx rad $= \left(\pi \times \frac{1}{60}\right) \Big/ 180$

$= 0.0003$ rad

Then, $\sin 60^{\circ} 1' = y + \mathrm{d}y$

$= 0.86603 + 0.00015$

$= 0.86618$

2b. Find the approximate change in the volume of a cube x cm caused by increasing the side by 1%.

Solution

The volume V is given by

$V = x^3 cm^3$

$\frac{dV}{dx} = 3x^2$

Where $\delta x = 1\%$ of $x = 0.01x$

$\delta V = 3x^2 \delta x$

$= 3x^2 (0.01)x$

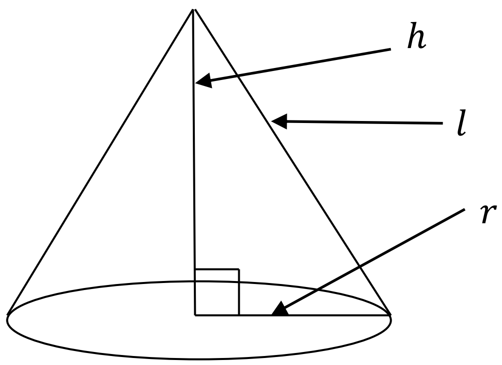
$= 0.03x^3 cm^3$

2c. Find the change in total surface area of a right circular cone , when

(i) The radius remains constant while the attitude changes by a small amount.

(ii) The altitude remains constant while the radius changes by a small amount.

Solution



The total surface area of a right circular cone, T is given by

$T = \pi r^2 + \pi r l$

Where $l^2 = h^2 + r^2$

$l = \sqrt{h^2 + r^2}$

$T = \pi r^2 + \pi r \sqrt{h^2 + r^2}$ where $r = radius$

$h = altitude$

(i.) $\left(\frac{\partial T}{\partial h}\right)_r = \frac{\partial}{\partial h}\left[\pi r^2 + \pi r\sqrt{h^2 + r^2}\right]$

$$= \frac{\partial}{\partial h}[\pi r^2] + \frac{\partial}{\partial h}\left[\pi r\sqrt{h^2 + r^2}\right]$$

$$= \frac{\pi r h}{\sqrt{h^2+r^2}}$$

(ii.) $\left(\frac{\partial T}{\partial r}\right)_h = \frac{\partial}{\partial r}\left[\pi r^2 + \pi r\sqrt{h^2 + r^2}\right]$

$$= \frac{\partial}{\partial r}[\pi r^2] + \frac{\partial}{\partial r}\left[\pi r\sqrt{h^2 + r^2}\right]$$

$$= \frac{2\pi r\sqrt{h^2+r^2} + 2\pi r^2 + \pi h^2}{\sqrt{h^2+r^2}}$$

3. Show that $\int e^{ax} cosbxdx = \frac{e^{ax}}{a^2+b^2}(bsinbx + acosbx)$

Proof

To show that $\int e^{ax} cosbxdx = \frac{e^{ax}}{a^2+b^2}(bsinbx + acosbx)$

Method: Integration by parts

$$\int e^{ax} cosbxdx = e^{ax}\int cosbxdx - \int\left[\frac{d(e^{ax})}{dx}\int cosbxdx\right]dx$$

$$\int e^{ax} cosbxdx = e^{ax}.\frac{1}{b}sinbx - \int\left[ae^{ax}.\frac{1}{b}sinbx\right]dx$$

$$\int e^{ax} cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\int e^{ax}sinbx\,dx$$

$$\int e^{ax} cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\left[e^{ax}\int sinbxdx - \int\left[\frac{d(e^{ax})}{dx}\int sinbxdx\right]dx\right]$$

$$\int e^{ax} cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\left[e^{ax}\left(-\frac{1}{b}cosbx\right) - \int\left[ae^{ax}\left(-\frac{1}{b}cosbx\right)\right]dx\right]$$

$$\int e^{ax} cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}cosbx + \frac{a}{b}\int e^{ax}cosbx\,dx\right]$$

$$\int e^{ax} cosbxdx = \frac{1}{b}e^{ax}sinbx - \frac{a}{b}\left[-\frac{1}{b}e^{ax}cosbx + \frac{a}{b}\int e^{ax}cosbx\,dx\right]$$

$$\int e^{ax} cosbxdx = \frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx - \frac{a^2}{b^2}\int e^{ax}cosbx\,dx$$

Collect like terms

$$\int e^{ax} cosbxdx + \frac{a^2}{b^2}\int e^{ax}cosbx\,dx = \frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx$$

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}cosbx\,dx = \frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx$$

$$\left(\frac{b^2+a^2}{b^2}\right)\int e^{ax}cosbx\,dx = \frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx$$

$$\int e^{ax}cosbx\,dx = \left(\frac{b^2}{a^2+b^2}\right)\left[\frac{1}{b}e^{ax}sinbx + \frac{a}{b^2}e^{ax}cosbx\right]$$

$$\int e^{ax}cosbx\,dx = \left(\frac{1}{a^2+b^2}\right)[be^{ax}sinbx + ae^{ax}cosbx]$$

$$\int e^{ax}cosbx\,dx = \frac{e^{ax}}{a^2+b^2}[bsinbx + acosbx] \qquad \textit{Proved}$$