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TASHKENT STATE TECHNICAL UNIVERSITY NAMED AFTER ABU RAIKHAN BERUNI

These lectures contain materials of mathematics for Bachelors of natural science, engineer work. This course is intended for the group of talented and intelligent students in Tashkent state technical university.

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Key words.

Matrix - матрица
Entry - элемент
Multiple- сомножитель
transpose operation – операция транспонирования
symmetrical and skew-symmetrical – симметричный и кососимметричный

Determinant - детерминант
Minor - минор
Cofactor - кофактор
Factor - фастор

Inverse - обратный
matrix equation – матричное уравнение
a reduced matrix – редуцированная матрица
an adjoint - ассоцированный

n-dimensional vector – *n*-мерный вектор
vector space – векторное пространство
linearly independent set – линейно – независимое множество
basis - базис

Linear - линейный
inconsistent - несовместный
Gaussian elimination – Гауссовый метод исключения

Rank of a matrix – ранг матрицы
Homogeneous linear systems – однородная линейная система

Geometric vector – геометрический вектор
Initial point – начальная точка
Terminal point – конечная точка
Right-handed system – правосторонняя система
Left-handed system – левосторонняя система

Translation - перенос
norm of a vector – норма вектора
length of a vector – длина вектора
Dot product – скалярное произвединие

Orthogonal projections – ортогональная проекция
Cross product – векторное произведение
Standard unit vectors – единичный вектор

Area of the parallelogram – площадь параллелограмма
scalar triple product – тройное скалярное произведение
volume of the parallelelpepid - объем параллелепипеда

PREFACE

These lectures are designed to equip students with essential mathematical knowledge and skills for solving business and management, engineer work problems. This book contains 10 lectures and has following structure: matrix algebra, determinants, linear vector spaces, linear equations, geometrical vectors.

LECTURE # 1

Matrix Algebra

1. Type of Matrix.
2. Equality of Matrices.
3. Matrix addition
4. Scalar Multiplication.
5. Matrix Multiplication.
6. Transpose of a Matrix.

Definition 1. A rectangular array of numbers consisting of m horizontal rows and n vertical columns,

$$\begin{bmatrix} a_{11} & a_{12} & \bullet & \bullet & a_{1n} \\ a_{21} & a_{22} & \bullet & \bullet & a_{2n} \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ a_{m1} & a_{m2} & \bullet & \bullet & a_{mn} \end{bmatrix}$$

is called an *mxn* matrix or a matrix of the order *mxn*. The numbers a_{ij} , $i = 1,2,\ldots.m$, $j = 1,2,\ldots.n$ in a matrix are called its entries or elements. For the entry a_{ij}, we call i the row subscript and j the column subscript.

Example.

a. The matrix $\begin{bmatrix} 1 & -6 \\ 5 & 1 \\ 9 & 4 \end{bmatrix}$ has order 3x2.

b. The matrix [7] has order 1x1.

An *mxn* matrix whose entries are all 0 is called the mxn zero matrix and is denoted by O. Thus the 2x3 zero matrix is

$$O=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

A matrix having the same number of columns as rows, for example n rows and n columns, is called a square matrix of order n. That is, an mxn matrix is square if and only if $m = n$.
A square matrix A is called a diagonal matrix if all the entries that are off the main diagonal are zero; that is, $a_{ij}=0$ for $i \neq j$. For example, 2x2 identity matrix $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and 3x3 matrix $\begin{bmatrix}3 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 9\end{bmatrix}$ are diagonal matrices.
We now define equality of matrices.

<u>*Definition 2*</u>. Matrices $A=[a_{ij}]$ and $B = [b_{ij}]$ are equal if and only if they have the same order and $a_{ij} = b_{ij}$ for each i and j (that is, corresponding entries are equal).
For example, by the definition of equality,

$$\begin{bmatrix}1+1 & 1\\ 2\cdot 3 & 0\end{bmatrix}=\begin{bmatrix}2 & 1\\ 6 & 0\end{bmatrix},$$

but $[1 \;\; 1] \neq \begin{bmatrix}1\\ 1\end{bmatrix}$ and $[1 \;\; 1]\neq[1 \;\; 1 \;\; 1]$.

Matrix addition.

<u>*Definition 3.*</u> If $A=[a_{ij}]$ and $B = [b_{ij}]$ are both mxn matrices, then the sum $A+B$ is the *mxn* matrix obtained by adding corresponding entries of A and B; that is $A+B=[a_{ij}+b_{ij}]$.

For example, let

$$A=\begin{bmatrix} 3 & 0 & -2 \\ 2 & -1 & 4 \end{bmatrix} \quad \text{and } B=\begin{bmatrix} 5 & -3 & 6 \\ 1 & 2 & -5 \end{bmatrix}$$

since A and B are of the same order (2x3), their sum is defined. We have

$$A+B=\begin{bmatrix} 3+5 & 0+(-3) & -2+6 \\ 2+1 & -1+2 & 4+(-5) \end{bmatrix} = \begin{bmatrix} 8 & -3 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

If $A.B,C$ and O have the same order, then the following properties hold for matrix addition:

1. $A+B = B+A$ (commutative property)
2. $A+(B+C) = (A+B)+C$ (associated property)
3. $A+O = O+A = A$ (identity property)

Scalar multiplication.

<u>*Definition 4*</u>. If A is an mxn matrix and k is a real number (also called a scalar), then by kA we denote the mxn matrix obtained by multiplying each entry in A by k. This operation is called scalar multiplication, and kA is called a scalar multiple of A.

The nxn identity matrix, denoted I_n, is the diagonal matrix whose main diagonal entries are 1's.

For example,

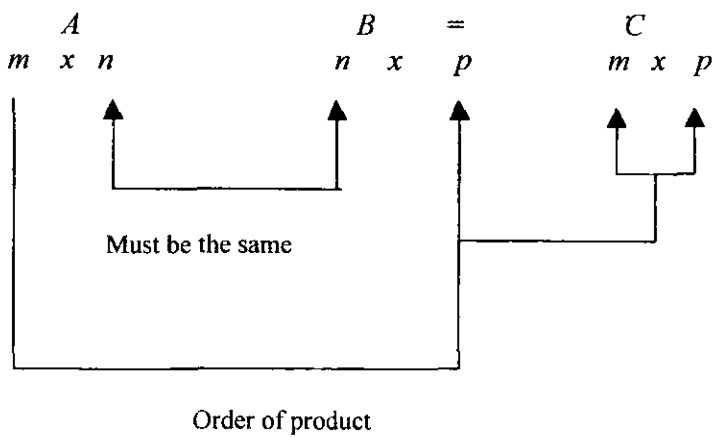
$$-3\begin{bmatrix} 1 & 0 & -2 \\ 2 & -1 & 4 \end{bmatrix} = \begin{bmatrix} -3(1) & -3(0) & -3(-2) \\ -3(2) & -3(-1) & -3(4) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 6 \\ -6 & 3 & -12 \end{bmatrix}$$

If A,B and O are the same order, then for any scalars, k, k_1, k_2 we have the following properties of scalar multiplication:

1. $k(A+B) = kA+kB.$
2. $(k_1 + k_2)A = k_1A + k_2A.$
3. $k_1(k_2A) = (k_1k_2)A.$
4. $0*A = O$
5. $k*O = O$

Matrix Multiplication.

<u>*Definition 5*</u>. Let A be an *mxn* marix and B be an *nxp* matrix. Then the product AB is the *mxp* matrix C whose entry c_{ij} in row i and column j is obtained as follows: sum the products formed by multiplying, in order, each entry (that is, first, second, etc.) in row i of A by the "corresponding" entry (that is, first, second, etc.) in column j of B, so
$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$,
$i = 1, 2, ...,m;\ \ j = 1, 2,.....,n.$



Example.

$$A*B=\begin{bmatrix}1 & 2\\ 0 & -1\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 4 & 0\\ 3 & -1 & 1\end{bmatrix}=\begin{bmatrix}1+6 & 4-2 & 0+2\\ 0-3 & 0+1 & 0-1\\ 3+3 & 12-1 & 0+1\end{bmatrix}=$$

$$=\begin{bmatrix}7 & 2 & 2\\ -3 & 1 & -1\\ 6 & 11 & 1\end{bmatrix}=C$$

Matrix multiplication satisfies the following properties if it assumed that all sums and products are defined:

1. $A*(B*C) = (A*B)*C$ (associative property).
2. $A*(B+C) = A*B+A*C$
   $(A+B)*C = A*C+B*C$ (distributive properties).

If A is a square matrix and both A and I have the same order, then $A*I = I*A = A$, where I is identity matrix.

If A is a square matrix, and p is a positive integer, then the p^{th} power of A, written A^p, is the product of p factors of A:

$$A^p = \underbrace{A*A*....*A}_{\text{p factors}}.$$

If A is nxn, we defined $A^0 = I$.

Transpose of a Matrix.

If A is a matrix, the matrix formed from A by interchanging its rows with its columns is called the transpose of A.

Definition 6. The transpose of an mxn matrix A, denoted A^T, is the nxm matrix whose i-th row is the i-th column of A.

Example. If $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \end{bmatrix}$, *then* $\begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 4 & 1 \end{bmatrix}$ *is* A^T.

Properties for transpose operation:

1. $(A^T)^T = A$.
2. $(A+B)^T = A^T + B^T$.
3. $(kA)^T = k\,A^T$.
4. $(A*B)^T = B^T * A^T$.

If A is a square matrix and $A^T = A$, then A is called the symmetrical matrix;

$A^T = (-1)*A$, then A is called the skew-symmetrical matrix.

Theorem. If A is any square matrix, then $A = B+C$, where B is a symmetrical matrix and C – skew – symmetrical matrix.

LECTURE # 2

DETERMINANTS.

1. Definition of determinant.
2. Minor and Cofactor of a matrix.
3. Determinant of a square Matrix.
4. Properties of Determinants

We now introduce a new function, the determinant function. If A is a square matrix, then the determinant function associates with A exactly one real number called the determinant of A. Denoting the determinant of A by |A| or detA.

Definition. If $A = [a_{11}]$ is a square matrix of order 1, then $|A| = a_{11}$.
Hence if A = [6], then |A| = 6.

Definition. If $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is a square matrix of order 2,

then $|A| = a_{11}a_{22} - a_{12}a_{21}$

Example. $\begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = (2)(-4) - (1)(3) = -8 - 3$

Minor and Cofactor of a Matrix.

Given the following matrix; $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

the minor of a_{21} is $\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

and the cofactor of a_{21} is $c_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

Determinant of a square matrix.

To find the determinant of any square matrix A of order n (n>2), select any row (or column) of A and multiply each entry in the row

(column) by its cofactor. The sum of these products is defined to be the determinant of A and is called a determinant of order n.
For example, we shall find the determinant of

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{bmatrix}$$

If we had expanded along the second column, then

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{bmatrix} = (-1)\,(-1)^{1+2} \begin{vmatrix} 3 & -5 \\ 2 & 1 \end{vmatrix} + 0 + (1)\,(-1)^{3+2} \begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix} = 32.$$

Properties of Determinants.

1. If each of the entries in a row (or column) of A is 0, then $|A| = 0$, where A denotes a square matrix.
2. If two rows (or columns) of a square matrix A are identical, $|A| = 0$.
3. If A is upper (or lower) triangular, then $|A|$ is equal to the product of the main diagonal entries.

Thus $$\begin{vmatrix} 2 & 6 & 1 \\ 0 & 5 & 7 \\ 0 & 0 & -2 \end{vmatrix} = (2)(5)(-2) = -20.$$

4. If B is the matrix obtained by adding a multiple of one row (or column) of A to another row (or column), then |B| = |A|.

Thus if $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 5 \\ 1 & 2 & 1 \end{bmatrix}$ and B is the matrix obtained from A

by adding -2 times row 3 to row 1, then

$$|A| = \begin{vmatrix} 2 & 4 & 2 \\ 1 & 3 & 5 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 5 \\ 1 & 2 & 1 \end{vmatrix} = |B|$$

By property 1, |B| = 0 and hence |A| = 0.

5. If B is the matrix obtained by interchanging two rows (or columns) of A, then |B| = - |A|.
For example, by interchanging rows2 and 4 we have

$$\begin{vmatrix} 2 & 2 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -3 & 4 \end{vmatrix} = -\begin{vmatrix} 2 & 2 & 1 & 6 \\ 0 & 1 & -3 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -(2)(1)(2)(1) = -4,$$

by property 3.

6. If B is the matrix obtained by multiplying each entry of a row (or column) of A by the same number k, then |B| = k |A|.

Thus $\begin{vmatrix} 6 & 10 & 14 \\ 5 & 2 & 1 \\ 9 & 15 & 21 \end{vmatrix} = 2 \begin{vmatrix} 3 & 5 & 7 \\ 5 & 2 & 1 \\ 9 & 15 & 21 \end{vmatrix}$

7. If k is a constant and A has order n, then $|kA| = k^n |A|$. This follows from property 6, since each of the n rows of kA has a common factor of k. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \text{ then } |A| = -2, \text{ so}$$

$$|4A| = 4^2 |A| = 16\,(-2) = -32.$$

8. The determinant of the product of two matrices of order n is the product of their determinants. That is, $|AB| = |A||B|$.
9. The determinant of a square matrix and its transpose are equal. $|A| = |A^T|$.

LECTURE # 3.
Inverses.

1. Inverse of matrix.
2. Using the inverse to solve a system.
3. Elementary Row operations.
4. Reduced matrix.
5. Method to find the inverse of a matrix.
6. Inverses using adjoint.

Inverse of a matrix.

Definition 1. If A is a square matrix and there exists a matrix C such that CA=I, then C is a called an inverse of A, and A is said to be invertible.

Let A=$\begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}$ and C=$\begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix}$. Since CA =$\begin{bmatrix}7 & -2\\-3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 7\end{bmatrix}$

=$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$=I

Matrix C is an inverse of A. We will denote it by the symbol A^{-1}. Thus $A^{-1}A$=I and $A^{-1}A= A A^{-1}$=I. Systems of linear equations can be represented by using matrix multiplication. For example, consider the matrix equation

$$\begin{bmatrix}1 & 4 & -2\\2 & -3 & 1\end{bmatrix}\times\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}4\\-3\end{bmatrix}(1).$$

The product on the left side has order 2x1 and hence is a column matrix:

$$\begin{bmatrix}x_1 & + & 4x_2 & - & 2x_3\\2x_1 & - & 3x_2 & + & x_3\end{bmatrix}=\begin{bmatrix}4\\-3\end{bmatrix}$$

By equality of matrices, corresponding entries must be equal so we obtain the system

$$\begin{cases}x_1 + 4x_2 - 2x_3 = 4\\2x_1 - 3x_2 + x_3 = -3\end{cases}$$

Hence this system of linear equations can be defined by matrix equation (1). We *usually describe* Eg. (1) *by saying that it has the form.*

$$AX=B,$$

where A is the matrix obtained from the coefficients of the variables, X is a column matrix obtained from the variables, and B is a column matrix obtained from the constants. Matrix A is called the coefficient matrix for the system using the inverse to solve a system.

Returning to the matrix equation AX=B, we can now state the following:

If A is an invertible matrix, then the matrix equations AX=B has the unique solution $X=A^{-1}B$.

For example, solve the system

$$\begin{cases} x_1 + 2x_2 = 5 \\ 3x_1 + 7x_2 = 18 \end{cases}$$

In matrix form we have AX=B, where

A= $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$, X= $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, B= $\begin{bmatrix} 5 \\ 18 \end{bmatrix}$

In example 1 we showed that $A^{-1}=\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$.

Thus $X=A^{-1}B=\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix}=\begin{bmatrix} -1 \\ 3 \end{bmatrix}$,

So $x_{1=-1}$ and x_2=3.

Elementary row operations.

1. Interchanging two rows of a matrix;
2. Adding a multiple of one row of a matrix to a different row of that matrix;
3. Multiplying a row of a matrix by a nonzero scalar.

These elementary row operations correspond to the three elementary operations used in the algebraic method of elimination. Whenever a matrix can be obtained from another by one or more elementary row operations, we say that the matrices are equivalent. We describing particular elementary row operations ,for convenience we will use the following notation:

$R_i \leftrightarrow R_j$ Interchange rows R_i and R_j

kR_i Multiply row R_i by the constant k

kR_i+R_j Add k times row R_i to row R_j (but row R_i remains the same).

We are now ready to describe a matrix procedure for solving a system of linear equations . First, form the augmented coefficient matrix of he system; then, by means of elementary row operations, determine an equivalent matrix hat clearly indicates the solution. It is a matrix, called a reduced matrix, defined as follows :

A matrix is said to be a reduced matrix provided all of the following are true:

1. If a row does not consist entirely of zeros , then the first nonzero entry in the row, called the leading entry, is 1,whereas all entries in the column in which the 1 appears are zeros.
2. The first nonzero entry in each row is to the right of the first nonzero entry of each row about it.
3. Any rows that consist entirely of zeros are at the bottom of the matrix .

In other words, to solve the system we must find the reduced matrix such that the augmented coefficient matrix is equivalent to it .

For example, solve the system $\begin{cases} 3x - y = 1 \\ x + 2y = 5 \end{cases}$

We have A=$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 1 & 2 & 5 \end{array}\right]$,then

$$\left[\begin{array}{cc|c} 3 & -1 & 1 \\ 1 & 2 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & -1 & 1 \end{array}\right] \xrightarrow{(-3)R_1 + R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -7 & -14 \end{array}\right] \xrightarrow{\left(-\frac{1}{7}\right)R_2} \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{(-2)R_2 + R_1}$$

$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$ = E and E is a reduced matrix . Therefore we get the equivalent system

$$\begin{cases} x + 0y = 1 \\ 0x + y = 2 \end{cases}$$

Where x=1 and y=2, so the original system is solved.

Method to find the inverse of a Matrix.

If M is an invertible n*n matrix, form the n*(2n) matrix $[M / I]$.
Then perform elementary row operations until the first n columns form a reduced matrix equal to I. The last n columns will be M^{-1}.

$$[M / I] \rightarrow [I / M^{-1}]$$

If a matrix M does not reduce to I, then M^{-1}does not exists.

This procedure can be extended to find the inverse of any invertible matrix .For example, determine M^{-1} if M is invertible:

$$M=\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

Finding M^{-1} by the above procedure can be done conveniently by using the following format. First, we write the matrix

$$[M / I]=\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{(-2)R_1+R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right] = [I / M^{-1}]$$

Note that the first two columns of $[I / M^{-1}]$ form a reduced matrix.

Inverses using the Adjoint

Determinants and cofactors can be used to find the inverse of a matrix, if it exists. To begin we need the idea of the adjoint of a matrix.
Definition 2. The adjoint of a square matrix A, denoted adj A, is the transpose of the matrix obtained by replacing each entry a_{ij} in A by its cofactor c_{ij}. That is, it the transpose of the cofactor matrix $[c_{ij}]$.

If $A=\begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{bmatrix}$,find adj A.

We first find the cofactor c_{ij} of each entry a_{ij} in A.

$$C_{11}=(-1)^{1+1}\begin{vmatrix}0 & -5\\1 & 1\end{vmatrix}=(1)(5)=5.$$

$$C_{12}=(-1)^{1+2}\begin{vmatrix}3 & -5\\2 & 1\end{vmatrix}=(-1)(13)=-13$$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}3 & 0\\2 & 1\end{vmatrix}=(1)(3)=3$$

$$C_{21}=(-1)^{2+1}\begin{vmatrix}-1 & 3\\1 & 1\end{vmatrix}=(-1)(-4)=4$$

$$C_{22}=(-1)^{2+2}\begin{vmatrix}2 & 3\\2 & 1\end{vmatrix}=(1)(-4)=-4$$

$$C_{23}=(-1)^{2+3}\begin{vmatrix}2 & -1\\2 & 1\end{vmatrix}=(-1)(4)=-4$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}-1 & 3\\0 & -5\end{vmatrix}=(1)(5)=5$$

$$C_{32}=(-1)^{3+2}\begin{vmatrix}2 & 3\\3 & -5\end{vmatrix}=(-1)(-19)=19$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}2 & -1\\3 & 0\end{vmatrix}=(1)(3)=3$$

The cofactor matrix $\left[c_{ij}\right]$ is thus

$$\left[C_{ij}\right]=\begin{bmatrix}5 & -13 & 3\\4 & -4 & -4\\5 & 19 & 13\end{bmatrix}$$

The adjoint is adj $A=\left[Cij\right]^{T}=\begin{bmatrix}5 & 4 & 5\\-13 & -4 & 19\\3 & -4 & 3\end{bmatrix}$

Using the adjoint we can now state a formula for the inverse of an invertible matrix. It can be shown that if $|A| \neq 0$, then A^{-1} exists and

$A^{-1} = \frac{1}{|A|}$ adj A (2). we find that $|A|=32 \neq 0$. Thus A^{-1} exists. Hence, from Eg. (2) we obtained.

$$A^{-1}=\frac{1}{32}\begin{bmatrix} 5 & 4 & 5 \\ -13 & -4 & 19 \\ 3 & -4 & 3 \end{bmatrix}=\begin{bmatrix} \frac{5}{32} & \frac{1}{8} & \frac{5}{32} \\ -\frac{13}{32} & -\frac{1}{8} & \frac{19}{32} \\ \frac{3}{32} & -\frac{1}{8} & \frac{3}{32} \end{bmatrix}$$

You should verify that $A^{-1}A=I$

LECTURE 4.

Arithmetic Vector Spaces

1. Vectors in n-space.
2. Vector Space
3. Linearly independent set.
4. Basis for a vector space.
5. Properties of linearly dependent vectors.

Vectors in n-Space

Definition 1. If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(a_1,a_2,\ldots,a_n)$.

The set of all ordered n-tuples is called n-space and is denoted by R^n.

We shall describe the n-tuple $(a_1,a_2,\ldots,a_n)$either as a point in R^n or a vector in R^n, where a_i, i=1,2,…n, is called the i-th component of $(a_1,a_2,\ldots,a_n)$.

Definition 2. Two vectors $a=(a_1,a_2,...,a_n)$ either as a point in R^n, where a_i, $i=1,2,\dots,n$, is called i-th component of $(a_1,a_2,a_3,\dots,a_n)$ and $b=(b_1,b_2,\dots,b_n)$ in R^n are called equal if

$$a_1=b_1, a_2=b_2, \dots, a_n=b_n$$

The sum a+b is defined by $a+b=(a_1+b_1,a_2+b_2,\dots,a_n+b_n)$ and if R is any scalar ,the scalar multiple R_a is defined by

$Ka=(ka_1, ka_2,...,ka_n)$

The operations of addition and scalar multiplication in this definition are called the standart operations on R^n.

The zero vector in R^n is denoted by 0 and is defined to be the vector

$$0=(0,0\dots,0)$$

If $a=(a_1,a_2,\dots,a_n)$ is any vector in R^n, then the negative of a is denoted by –a and is defined by $-a=(-a_1,-a_2,...,-a_n)$. The difference of vectors in R^n is defined by

$$a\text{-}b=a+(\text{-}b)$$

or in terms of components

$a\text{-}b=(a_1\text{-} b_1,a_2\text{-}b_2,\dots,a\text{-}b)$

The most important arithmetic properties of addition and scalar multiplication of vectors in R^n are listed in the following theorem.

Theorem 3 If $a=(a_1,a_2,\dots,a_n)$, $b=(b_1,b_2,\dots,b_n)$, and $c=(c_1,c_2,\dots,c_n)$ are vectors in

R^n and k and L are scalars, then :

(i) a+(b+c)=(a+b)+c Associative law of vector addition.

(ii) a+b=b+a Commutative law of vector addition.

(iii) a+0=0+a=a existence of 0as an additive identity.

(iv) a+(-a)=0 Existence of additive inverses.

(v) k(a+b)= ka + kb Scalars distribute over vector addition.

(vi) (k + l) a = ka + la Vectors distribute over scalar addition.

(vii) (kl) a = k (la) Associative law for multiplication by scalars.

(viii) 1*a = a Number 1 is a multiplicative identity.

Arithmetic Vector space.

If components a_1, a_2, ..., a_n are real numbers, then set of all n-dimensional vectors is called real arithmetic vector space R^n; if

components are complex numbers, then set of all n-dimensional vectors is complex vector space C^n .
All of our vector components will be real numbers.
Definition 4 *a* vector *a* is called a linear combination of the vectors $v_1, v_2, \ldots, v_k$ if it can be expressed in the form.

$a = k_1 \ v_1 + k_2 \ v_2 + \ldots + k_k \ v_k$

Where $k_1 \ k_2, \ldots, k_k$ are scalars.

Example. Every vector $v = (\alpha, \beta, \gamma)$ in R^3 is expressible as a linear combination of the standard basic vectors.
i=(1, 0, 0); j = (0, 1, 0); k = (0, 0, 1)
Since
$v = (\alpha, \beta, \gamma,) = \alpha \ (1, 0, 0) + \beta \ (0, 1, 0) + \gamma (0, 0, 1) = \alpha i + \beta j + \gamma k.$
Linearly independent set.
Definition 5. If $S = \{v_1, v_2, \ldots, v_m\}$ is a nonempty set of vectors, then the vector equation

$k_1 v_1 + k_2 \ v_2 + \ldots + k_m \ v_m = 0$

has at least one solution, namely

$k_1 = 0; k_2 = 0; \ldots; k_m = 0$

I this is the only solution, then S is called a linearly independent set. If these are other solutions, then S is called a linearly dependent set.
Basis for a vector space.
Definition 6. If V is any vector space and $S = \{e_1, e_2, \ldots, e_n\}$ is a set ofvectors in V, then S is called a basis for V if the following two conditions hold:

(a) S is linearly independent.
(b) Every vector *v* in V can be expressed in the form
$v = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n$ in exactly one way.

Example. Let $e_1 = (1, 1, 1, 1, 1,)$, $e_2 = (0,1, 1, 1, 1)$, $e_3 = (0, 0, 1, 1, 1)$,
$e_4 = (0, 0, 0, 1, 1)$ and $e_5 = (0, 0 ,0, 0, 1)$. Show that the set
$S = \{ e_1, e_2, e_3, e_4, e_5\}$ is a basis for R^5 .

Solution. To prove that S is linearly independent, we must show that the only solution of

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 = 0$$

is $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$

If $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 = (\lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) = 0$

Hence

$\lambda_1 = 0$, $\lambda_1 + \lambda_2 = 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0$ and we obtain $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$.

Now, we must show that an arbitrary vector $b = (b_1, b_2, b_3, b_4, b_5)$ can be expressed as a linear combination.

$b = b'_1 e_1 + b'_2 e_2 + b'_3 e_3 + b'_4 e_4 + b'_5 e_5$

of the vectors in S.

We have

$b = (b_1, b_2, b_3, b_4, b_5) = (b_1\ b_1, b_1, b_1, b_1,) + (0, b_2 - b_1, b_2 - b_1, b_2 - b_1, b_2 - b_1) + (0,0, b_3 - b_2, b_3 - b_2, b_3 - b_2) + (0,0,0, b_4 - b_3, b_4 - b_3) + (0,0,0,0, b_5 - b_4) = b_1(1,1,1,1,1) + (b_2 - b_1)(0,1,1,1,1) + (b_3 - b_2)(0,0,1,1,1) + (b_4 - b_3)(0,0,0,1,1) + (b_5 - b_4)(0,0,0,0,1) = b_1 e_1 + (b_2 - b_1) e_2 + (b_3 - b_2) e_3 + (b_4 - b_3) e_4 + (b_5 - b_4) e_5$.

If $b = (b_1, b_2, b_3, b_4, b_5) \neq 0$, then coefficients $b_1, b_2 - b_1, b_3 - b_2, b_4 - b_3, b_5 - b_4$ are not all zero. Hence $S = \{e_1, e_2, e_3, e_4, e_5\}$ is a basic for R^5.

Properties of linearly dependent vectors.

Theorem 7. A set S with two are more vectors is:

(a) Linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.
(b) Linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.

We shall prove part (a) and leave the proof of part (b) as an exercise.

Proof (a). Let S={ v_1, v_2, ..., v_r } be a set with two or more vectors. If we assume that S is linearly dependent, then there are scalars k_1, k_2, ..., k_r, not all zero, such that

$k_1 v_1 + k_2 v_1 + \ldots + k_2 v_r = 0$ (1)

To be specific, suppose that $k_1 \neq 0$. then (1) can be rewritten as

$$v_1 = (-\frac{k_2}{k_1})v_2 + \ldots + (-\frac{k_r}{k_1})v_r$$

which expresses v_1 as a linear combination of the other vectors in S. Similarly, if $k_j \neq 0$ in (1) for some j=2,3,...r, then v_j is expressible as a linear combination of the other vectors in S.

Conversely, let us assume that at least one of the vectors in S is expressible as a linear combination of the other vectors. To be specific, suppose that

$$v_1 = c_2 v_2 + c_3 v_3 + \ldots + c_r v_r$$

so

$v_1 - c_2 v_2 - c_3 v_3 - \ldots - c_r v_r = 0$

It follows that S is linearly dependent since the equation

$k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$

is satisfied by

$$k_1 = 1, k_2 = -c_2, \ldots, k_r = -c_r$$

which are not all zero. The proof in the case where some vector other than v_1 is expressible as a linear combination of the other vectors in S is similar.

Theorem 8.

(a) A finite set of vectors that contains the zero vector is linearly dependent .

(b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

We shall prove part (a) and leave the proof of part (b) as an exercise.

Proof (a). For any vectors $v_{1,}\ v_2, \dots v_r$, the set S={ $v_{1,}\ v_2, \dots, v_r\ 0$} is linearly dependent since the equation.

$0 \cdot v_1 + 0 \cdot v_2 + \dots + 0\ v_r + 1\ (0) = 0$

expresses 0 as a linear combination of the vectors in S with coefficients that are not all zero.

The next Theorem shows that a linearly independent set in R ^{n} can contain at most n vectors.

Theorem 9. Let S= { $v_{1,}\ v_2, \dots v_r$ } be a set of vectors in R ^{n}. If r>n, then S is linearly dependent.

Definition 10. A vector space V has dimension n if and only if V has a basis containing n vectors.

To signify that a vector space in n-dimensional, we write

$$\dim V = n$$

Theorem 11. Every set of n linearly independent vectors { $v_{1,}\ v_2, \dots v_n$ } of an n-dimensional vector space V is a basis of V.

Proof. Each of the vectors $v_{1,}\ v_2, \dots v_n$ is a linear combination of these n vectors because every vector is obviously equal to itself. For any vector v of V that is not one of the vectors $v_{1,}\ v_2, \dots v_n$, the set { $v_{1,}\ v_2$, $\dots v_n$, v} is linearly dependent because each vector of this set is linear combination of any n basis vectors. Constants $c_1, c_2, \dots, c_n, c$ therefore exist, at least one of which is not zero, such that $c_1\ v_{1,} + c_2\ v_2 + \dots + c_n\ v_n +$ c v=0. Clearly, c $\neq 0$, since $v_{1,}\ v_2, \dots, v_n$ are linearly independent. The preceding equation can thus be solved for V. This proves that every vector of V is a linear combination of $v_{1,}\ v_2$, $\dots v_n$ and they are a basis of V.

LECTURE # 5

Introduction to systems of linear equations.

1. Linear equations.
2.Liner systems.
3. Augmented matrices.
4. Row -echelon form.
5. Gaussian elimination.

The study of systems of linear equation and their solutions is one of the major topics
in linear algebra .In this section we shall introduce some basic terminology and discuss a method for solving such systems.

Linear equations

A line in the xy-plane can be represented algebraically by an equation of the form

$$a_1x+a_2y=b$$

An equation of this kind is called a linear equation in the variables x and y. More generally, we define a linear equation in the n variables $x_1, x_2,.., x_n$ to be one that can be expressed in the form

$$a_1x_1+a_2x_2+...+a_nx_n=b$$

where $a_1,a_2,...,a_n$, and b are real constants. The variables in linear equation are sometimes called the unknowns. A solution of a linear equation $a_1x_1+a_2x_2+...+a_nx_n=b$ is a sequense of n numbers s_1, $s_2,..,s_n$ such that the equation is satisfied when we substitute $x_1=s_1$, $x_2=s_2,...,x_n=s_n$.

The set of all solutions of the equation is called its solution set or sometimes the general solution of the equation.

Linear systems.

A finite set of linear equations in the variables $x_1, x_2,.., x_n$ is called a system of linear equations or a linear system. A sequence of numbers $s_1, s_2,..,s_n$ is called a solution of the system if $x_1=s_1, x_2=s_2,...,x_n=s_n$ is a solution of every equation in the system.

For example, the system

$$\begin{cases} 4x_1 - x_2 + 3x_3 = -1 \\ 3x_1 + x_2 + 9x_3 = -4 \end{cases}$$

has the solution $x_1=1$ $x_2=2$ $x_3=-1$ since these values satisfy both equations. However

$x_1=1$ $x_2=8$ $x_3=1$ is not a solution since these values satisfy only the first of the two equations in the system.

Not all systems of linear equations have solutions. For example, if we multipy the second equation of the system

$$\begin{cases} x + y = 4 \\ 2x + 2y = 6 \end{cases}$$

by $\frac{1}{2}$ it becomes evident that there are no solutions since the resulting equivalent system

$$\begin{cases} x + y = 4 \\ x + y = 3 \end{cases}$$

has contradictory equations.

A system of equations that has no solutions is said to be inconsistent, if there is at least one solution of the system ,it is called consistent. Every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.

An arbitrary system of m linear equations in n unknowns can be written as

$a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n=b_1$

$a_{21}x_2+a_{22}x_2+\ldots+a_{2n}x_n=b_2$

. . .

. . .

. . .

$a_{m1}x_1+a_{m2}x_2+\ldots+a_{m}x_n=b_m$

Where $x_1, x_2,.., x_n$ are the unknowns and the subscripted a's and b's denote constants.

Augmented Matrices.

We consider following matrix:

$$\begin{bmatrix} a_{11}a_{12}\cdots a_{1n}b_1 \\ a_{21}a_{22}\cdots a_{2n}b_2 \\ . \\ . \\ . \\ a_{m1}a_{m2}\cdots a_{mn}b_m \end{bmatrix}$$

This is called the augmented matrix for the system .

Row -echelon form.

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a one .(we call this a leading 1)
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else.

A matrix having properties 1,2 and 3 (but not necessarily 4 is said to be in row -echelon form.)

Gaussian elimination

Now we shall give a step -by-step procedure that can be used to reduce any matrix to reduced row-echelon form. As we state each step in the procedure, we shall illustrate
the idea by reducing the following matrix to reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1 Locate the leftmost column that does not consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.
The first and second rows in the preceding matrix were interchanged.

$$\begin{bmatrix} 2 & 0 & -2 & 0 & 7 & 12 \\ 0 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 3. If the entry that is now at the top of the column found in step *1* is a, multiply the first row by $1/a$ in order to introduce a leading 1.
The first row of the preceding matrix was multiplied by 1/2.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

-2 times the first row of the preceding matrix was added to the third row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & 29 \end{bmatrix}$$

Step 5. Now cover the top row in the matrix and begin again with step 1 applied to the sub matrix that remains. Continue in this way until the entire matrix is in row –echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

leftmost nonzero column in the submatrix

The first row in the submatrix was multiplied by $-\frac{1}{2}$ to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-5 times the first row of the submatrix was added to he second row of the submatrix to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was cowered, and we returned again to step 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Leftmost nonzero column in the new submatrix

The first (and only)row in the new submatrix was multiplied by 2 to introduce a leading 1.

The entire matrix is now in row-echelon form. To find the reduced row -echelon form we need the following additional step.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros , above the leading 1's.

$\frac{7}{2}$ times the third row of the preceding matrix was added to the second row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

-6 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the second row was added to the first row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The last matrix is in reduced row- echelon form. The above procedure for reducing a matrix to reduced row -echelon form is called Gauss-Jordan elimination. If we use only the first five steps, the procedure produces a row-echelon form and is called Gaussian elimination.

LECTURE # 6.

Rank and homogeneous linear systems.

1. Rank of a matrix.
2. Homogeneous linear systems.

Rank of a matrix.

One of the most important characteristics of a matrix with real or complex elements is its rank.

Definition 1. The rank of a zero matrix is 0. The (row) rank of a matrix A with scalar elements is the maximum number of linearly independent row vektors of A.

Still another definition of the rank of a matrix is the following. The rank of a nonzero matrix A is the largest integer r for which there exists an r-th-order minor of A whose value is not zero.

We have the following results.
Theorem 2. If A and B are n*n matrices of rank n, then both AB and BA are of rank n.
In theory of matrices, we introduced three types of elementary operations for manipulating the rows of a matrix. For some purposes, it is necessary to consider the effect of performing elementary column operations on a matrix. These defined analogously to be the following.
1.Interchanging two columns.
2. Multiplying a column by a nonzero number.
3. Adding a constant times the elements of one columns to the corresponding elements of another column.
For convenience, we shall speak of elementary row and column operations collectively as elementary operations . One of their most important properties is the following.
Theorem 3.The rank of a matrix is not altered by sequence of elementary operations.
Theorem 4. If A and B are conformable matrices of rank and p, respectively, the rank of the product AB is equal to or less than the smaller of r and p.
Theorem 5.If A is an m*n matrix of rank r and if B is an n*p matrix of rank p, the rank of the product AB is equal to or greater than r+p-n.
Theorem 6. Every complete solution of a consistent system of linear algebraic equations.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad & \qquad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

in n unknowns contains n-r arbitrary constants, where r is the rank of A.

Homogeneous linear systems.

A system of linear equations is said to be homogeneous of the constant terms are all zero; that is, the system has the form

$$
\begin{aligned}
a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n&=0\\
a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n&=0\\
\vdots\qquad\quad \vdots\qquad\qquad\quad &\vdots\\
a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n&=0
\end{aligned}
$$

Every homogeneous system of linear equations is consistent, since all such systems have $x_1=0,\ x_2=0,\ \ldots,x_n=0$ as a solution. This solution is called the trivial solution ; if there are other solutions, they are called nontrivial solutions .

Because a homogeneous linear system always has the trivial solution , there are only two possibilities for its solutions:

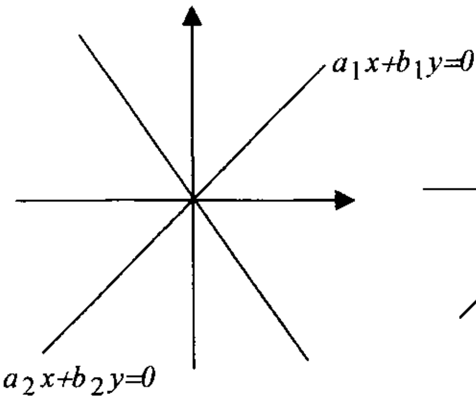
- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

In the special case of a homogeneous linear system of two equations in two unknowns, say

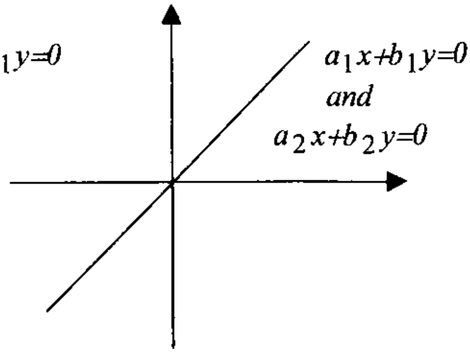
$a_1x+b_1y=0$ $\qquad$ (a_1,b_1 not both zero)

$a_2x+b_2y=0$ $\qquad$ (a_1,b_1 not both zero)

the graphs of the equations are lines through the origin, and trivial solution corresponds to the point of intersection at the origin (Figure)



Only the trivial solution



Infinitely many solutions

There is one case in which a homogeneous system is assured of having nontrivial solutions , namely, whenever the system involves more unknowns than equations. To see why , consider the following example of four equations in five unknowns ,

Example. Solve the following homogeneous system of linear equations by Gauss-Jordan elimination.

$$
\begin{array}{ll}
2x_1 +2x_2-x_3+x_5=0 & \\
-x_1- x_2-2x_3-3x_4+x_5=0 & (1) \\
x_1+x_2-2x_3-x_5=0 & \\
x_3+x_4+x_5=0 &
\end{array}
$$

Solution. The augmented matrix for the system is

$$
\begin{bmatrix}
2 & 2 & -1 & 0 & 1 & 0 \\
-1 & -1 & 2 & -3 & 1 & 0 \\
1 & 1 & -2 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0
\end{bmatrix}
$$

Reducing this matrix to reduced row-echelon form, we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} &x_1+x_2+x_5=0 \\ &x_3+x_5=0 \qquad (2) \\ &x_4=0 \end{aligned}$$

Solving for the leading variables yields

$$\begin{aligned} &x_1=-x_2-x_5 \\ &x_3=-x_5 \\ &x_4=0 \end{aligned}$$

Thus, the general solution is

$x_1=-s-t$, $x_2=s$, $x_3=-t$, $x_5=t$, $x_4=0$.

Note that the trivial solution is obtained when $s=t=0$.

Example, illustrates two important points about solving homogeneous systems of linear equations. First, none of the three elementary row operations alters the final column of zeros in the augmented matrix , so that the system of equations corresponding to the reduced row- echelon form of the augmented matrix must also be a homogeneous system [see system (2)]. Second, depending on whether the reduced row-echelon form of the augmented matrix has any zero rows, the number of equations in the reduced system is the same as or less than the number of equations in the original system [compare system (1)and (2)].Thus , if the given homogeneous system has m equations in n unknowns with $m<n$, and if there are r nonzero rows in the reduced row -echelon form of the augmented matrix, we will have $r<n$. It follows that the system of equation corresponding to the reduced row- echelon form of the augmented matrix will have the form

$$\begin{aligned} \ldots x_{k_1} \quad &+\sum() = 0 \\ \ldots x_{k_2} \quad &+\sum() = 0 \qquad (3) \\ &\vdots \\ x_{k_r} &+\sum() = 0 \end{aligned}$$

where $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$ are the leading variables and $\sum()$ denotes sums

(possibly all different) that involve the n-r free variables [compare system (3) with system (2) above]. Solving for the leading variables gives

$$x_{k_1} = -\sum()$$
$$x_{k_2} = -\sum()$$
$$x_{k_r} = -\sum()$$

As in example, we can assign arbitrary values to the free variables on the right- hand side and thus obtain infinitely many solutions to the system.

In summary, we have the following important theorem.

Theorem. A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

LECTURE # 7

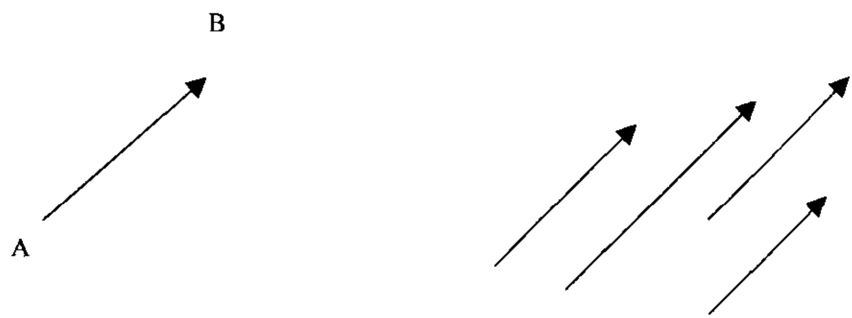
Introduction to geometric vectors.

1.Geometric vectors.
2. Vectors in coordinate systems.
3. Vectors in 3-space.

Geometric vectors.

Vectors can be represented geometrically as directed line segments or arrows in 2-space or 3-space. The tail of the arrow is called the initial point of the vector, and the tip of the arrow the terminal point. We shall denote vectors in lowercase boldface type (for instance, a, k, v, w, and x).

If, as in figure 1(a), the initial point of a vector v is A and the terminal point is B, we write $v = \vec{AB}$



(a) (b)

Figure 1.

Vectors with the same length and same direction, such as those in Figure 1(b), are called equivalent. Since we want a vector to be determined solely by its length and direction, equivalent vectors are regarded as equal even though they may be located in different positions.

If v and w are equivalent, we write

$$v = w$$

Definition 1. If v and w are any two vectors, then the sum $v + w$ is the vector determined as follows: Position the vector w so that its initial point coincides with the terminal point of v. The vector $v + w$ is represented by the arrow from the initial point of v to the terminal point of w (Figure 2a).

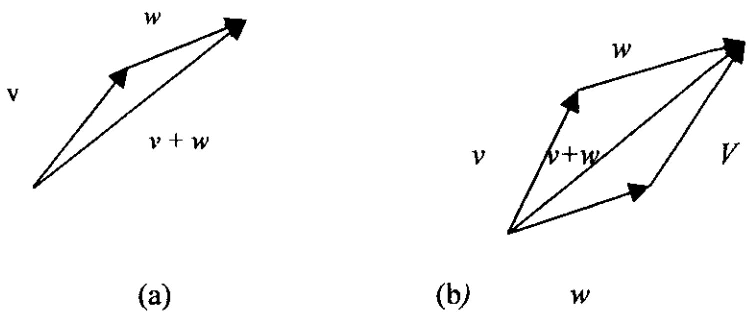


Figure 2.

In Figure 2b we have constructed two sums, $v + w$ and $w + v$. It is evident that

$$v + w = w + v$$

and that the sum coincides with the diagonal of the parallelogram determined by v and w when these vectors are positioned so they have the same initial point.

The vector of length zero is called the zero vector and is denoted by 0. We define

$$0 + v = v + 0 = v$$

for every vector v. Since there is no natural direction for the zero vector, we shall agree that it can be assigned any direction that is convenient for the problem being considered.

If v is any nonzero vector, then $-v$, the negative of v, is defined to be the vector having the same magnitude as v, but oppositely directed (Figure 3).

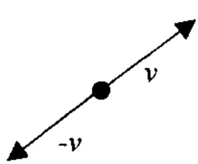


Figure 3.

The negative of v has the same length as v, but is oppositely directed. This vector has the property

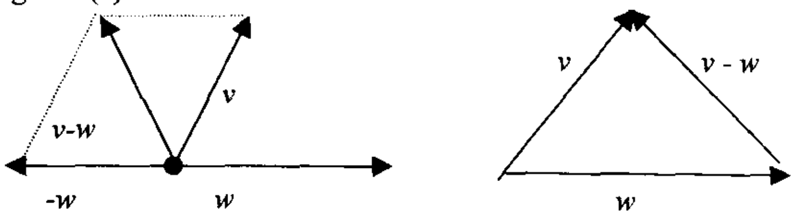
$$v + (-v) = 0$$

In addition, we define $-0 = 0$. Subtraction of vectors is defined as follows.

Definition 2. If v and w are any two vectors, then difference of w from v is defined by

$$v - w = v + (-w)$$

Figure 4(a)



(a) (b)

To obtain the difference $v - w$ without constructing $-w$, position v and w so their initial points coincide; the vector from the terminal point of w to the terminal point of v is then the vector $v - w$ (Figure 4b).

Definition 3. If v is a nonzero vector and k is a nonzero real number (scalar), then the product kv is defined to be the vector whose length is $|k|$ times the length of v and whose direction is the same as that of v if $k>0$ and opposite to that of v if $k<0$. We define $kv = 0$ if $k = 0$ or $v = 0$. A vector of the form kv is called a scalar multiple of v.

Vectors in coordinate systems.

Let v be any vector in the plane, and assume, as in Figure 5, that v has been positioned so its initial point is at the origin of a rectangular coordinate system.

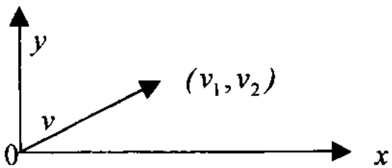


Figure 5.

The coordinates (v_1, v_2) of the terminal point of *v* are called the components of *v*, and we write

$$v = (v_1, v_2)$$

Two vectors

$$v = (v_1, v_2) \text{ and } w = (w_1, w_2)$$

are equivalent if and only if

$$v_1 = w_1, \ v_2 = w_2$$

The operations of vector addition and multiplication by scalars are easy to carry out in terms of components.

If $v = (v_1, v_2)$ and $w = (w_1, w_2)$ then

$$v + w = (v_1 + w_1, v_2 + w_2) \quad (1)$$

If $v = (v_1, v_2)$ and *k* is any scalar, then by using a geometric argument involving similar triangles, it can be shown that

$$kv = (kv_1, kv_2) \qquad (2)$$

Thus, for example, *if* $v = (1, -2)$ *and* $w = (7, 6)$, *then*

$$v + w = (1, -2) + (7,6) = (1 + 7, -2 + 6) = (8, 4)$$

and $$4v = 4\,(1, -2) = (4(1), 4(-2)) = (4, -8)$$

Since $v - w = v + (-1)w$, it follows from Formulas (1) and (2) that

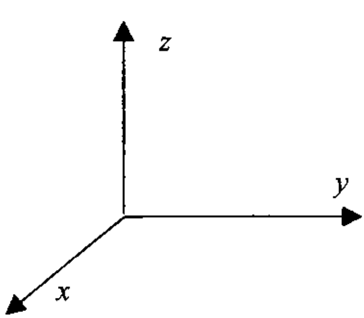
$$v - w = (v_1 - w_1, v_2 - w_2)$$

Vectors in 3 – space.

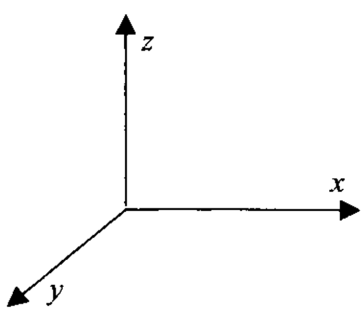
Just as vectors in the plane can be described by pairs of real numbers, vectors in 3 – space can be described by triples of real numbers by introducing a rectangular coordinate system. To construct such a coordinate system, select a point O, called the origin, and choose three mutually perpendicular lines, called coordinate axes, passing through the origin. Label these axes *x, y,* and *z,* and select a positive direction for each coordinate axis as well as a unit of length for measuring distances. Each pair of coordinate axes determines a plane called a coordinate plane.

These are referred to as the *xy* - plane , the *xz* – plane, and the *yz* – plane. To each point P in 3 – space we assign a triple of numbers (*x, y, z*), called the coordinates of P, as follows: Pass three planes through P parallel to the coordinate planes, and denote the points of intersections of these planes with the three coordinate axes X, Y, and Z.

Rectangular coordinate systems in 3-space fall into two categories, left-handed and right-handed. A right handed system has the property that an ordinary screw pointed in the positive direction on the *z* – axis would be advanced if the positive *x*-axis is rotated 90^0 toward the positive *y* – axis; the system is left-handed if the screw would be retracted.



Right-handed



Left – handed

A vector v in 3-space in positioned so its initial point is at the origin of a rectangular coordinate system, then the coordinates of the terminal point are called the components of v, and we write

$$v = (v_1, v_2, v_3)$$

If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are two vectors in 3-space, then arguments similar to those used for vectors in a plane can be used to establish the following results.

v and w are equivalent if and only if $v_1 = w_1$, $v_2 = w_2$, and $v_3 = w_3$

$v + w = (v_1 + w_1,\ v_2 + w_2, v_3 + w_3)$

$kv = (k v_1, k v_2, k v_3)$, where k is any scalar.

Example. If $v = (1, -3, 2)$ and $w = (4, 2, 1)$, then $v + w = (5, -1, 3)$, $2v = (2, -6, 4)$, $-w = (-4, -2, -1)$

$v - w = v + (-w) = (-3, -5, 1)$.

LECTURE # 8.

Norm of a vector. Length of a vector.

1. Translation of axes.
2. Norm of a vector.
3. Dot product.
4. Orthogonal vectors.

Sometimes a vector is positioned so that its initial point is not at the origin. If the vector $\overrightarrow{P_1P_2}$ has initial point $P_1(x_1,y_1,z_1)$ and terminal point $P_2(x_2,y_2,z_2)$, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

In 2-space the vector with initial point $P_1(x_1,y_1)$ and terminal point $P_2(x_2,y_2)$ is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

That is, the components of $\overrightarrow{P_1P_2}$ are obtained by subtracting the coordinates of the initial point from the coordinates of the terminal point.

Translation of axes.

The solutions to many problems can be simplified by translating the coordinate axes to obtain new axes parallel to the original ones.
In figure 1(a) we have translated the axes of an xy-coordinate system to obtain an *x'y'*-coordinate system whose origin o' is at the point $(x,y)=(k,l)$. A point P in spacr now he has both (x,y) coordinates and (x',y') coordinates.

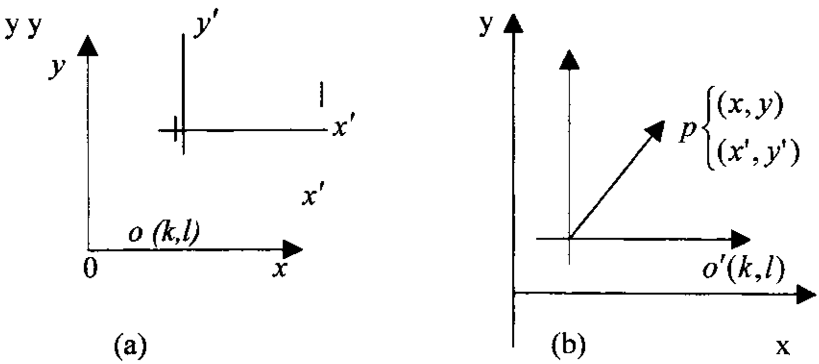


Figure 1

To see how the two are related, consider the vector $\overrightarrow{O'P}$ (figure 1(b). In the xy-system it initial point is at (k,l) and its terminal point is at (x',y'), so $\overrightarrow{O'P}=(x-k,y-l)$. In the $x\,y$ –system its initial point is at (0,0) and its terminal point is at (x',y'). So $O'P = (x'y')$. Therefore,

$$x' = x - k \qquad y' = y - l$$

there formulas are called the translation equations. In 3-space

$$x' = x - k \qquad x' = y - l \qquad z' = z - m$$

Where (k,l,m) are the xyz- coordinations of the $x'y'z'$-origin.

Norm of a vector.

The length of a vector u is often called the norm of u and is denoted by $\|u\|$. It follows from the theorem of Pythagorean that the norm of a vector $u=(u_1,u_2)$ in 2-space is

$$\|u\| = \sqrt{u_1^2 + u_2^2} \qquad (1)$$

Let $u=(u_1,u_2,u_3)$ be a vector in 3-space. Thus,

$$\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} \qquad (2)$$

A vector of norm 1 is called a unit vector.

If $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ are two points in of 3-space, then the distance d

between them is the norm of the vector $\overrightarrow{P_1P_2}$ (figure 2). Since

$$\overrightarrow{P_1P_2}=(x_2-x_1,y_2-y_1,z_2-z_1)$$

it follows from (2) that

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \quad (3)$$

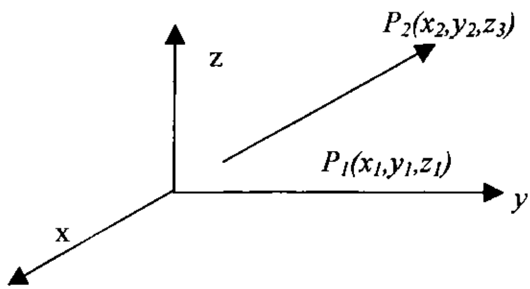


Figure 2.

Similarly, if (x_1,y_1) and $P_2(x_2,y_2)$ are points in 2-space, then the distance between then is given by

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \qquad (4)$$

From the definition of the product *ku*, the length of the vector *ku* is |k| times the length of *u*. Expressed as an equation, this statement says that

$$\|ku\|=|k|\|u\| \qquad (5)$$

This useful formula is applicable in both 2-space and 3-space.

Dot product.

Let *u* and *v* be two nonzero vectors in 2-space or 3-space, and assume these vectors have been positioned so their initial points coincide. By the angle between *u* and *v*, we shall mean the angle θ determined by u and v that satisfies $0 \le \theta \le \pi$.

Definition 1. If u and V are vectors in 2-space or 3-space and θ is the angle between u and v, then the dot product or Euclidean inner product *uv* is defined

$$uv=\begin{cases}\|u\|\|v\|\cos\theta & \text{if } u\neq 0 \text{and } v\neq o\\ 0 & \text{If } u=0 \text{ or } v=0\end{cases} \qquad (1)$$

Let $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ be two nonzero vectors, then Formula (1) can be written as

$$uv=u_1v_1+u_2v_2+u_3v_3 \qquad (2)$$

If *u* and *v* are nonzero vectors, then Formula (1) can be written as

$$\cos\theta=\frac{uv}{\|u\|\|v\|} \qquad (3)$$

The following theorem shows how the dot product can be used to obtain information about the angle between two vectors; It also establishes an important relationship between the norm and the dot product.

Theorem 2. Let *u* and *v* be vectors in 2-or 3-space.

(a) $vv=\|v^2\|$; that is, $\|v\|=(vv)^{\frac{1}{2}}$

(b) If the vectors *u* and *v* are nonzero and θ is the angle between them, then

θ is acute if and only if $uv>0$

θ is obtuse if and only if uv<0

$\theta=\frac{\pi}{2}$ if and only if uv=0

Proof (a). Since the angle θ between v and v is 0, we have

$$vv=\|v\|\|v\|\cos\theta=\|v\|^2\cos 0=\|v\|^2$$

Proof (b). θ Since satisfies $\theta\leq\theta\leq\pi$ <, it follows that:

θ is acute if and only if $\cos\theta$; θ is obtuse if and only if $\cos\theta$<0; and $\theta=\frac{\pi}{2}$ if and only if $\cos\theta$=0 . But $\cos\theta$ has the same sign as uv since $uv=\|v\|\|v\|\cos\theta=\|v\|^2$, $\|u\|$>0, and $\|v\|$>0.thus, the result follows.

Therefore, u and v make obtuse angle, v and w make an acute angle , and u and w are perpendicular.

Orthogonal vectors.

Perpendicular vectors are also called orthogonal vectors. In light of theorem 2 (b), two nonzero vectors are orthogonal if and only if their dot product is zero. If we agree to consider u and v to be perpendicular when either or both of these vectors is 0, then we can state without exception that two vectors u and v are orthogonal (perpendicular) if only if $u\cdot v=0$. vectors. To indicate that u and v are orthogonal vectors we write $u\perp v$.

The following theorem lists the most important properties of the dot product . They are useful in calculations involving vectors.

Theorem 3. If u,v and w are vectors in 2-or 3-space and k is a scalar, then:

(a) $uv=vu$

(b) $u(v+w)=uv+uw$

(c) $k(uv)=(ku)v=u(kv)$

(d) $vv>0$ if $v\neq 0$, and $vv=0$ if $v=0$

Proof. We shall prove (c) for vectors in 3-space and leave the remaining proofs as exercises. Let $u=(u_1,u_2,u_3)$; then

$$k(uv)=k(u_1v_1+u_2v_2+u_3v_3)=(ku_1)v_1+(ku_2)v_2+(ku_3v_3)=(ku)v$$

Similarly $\quad k(uv)=u(kv)$

LECTURE 9.

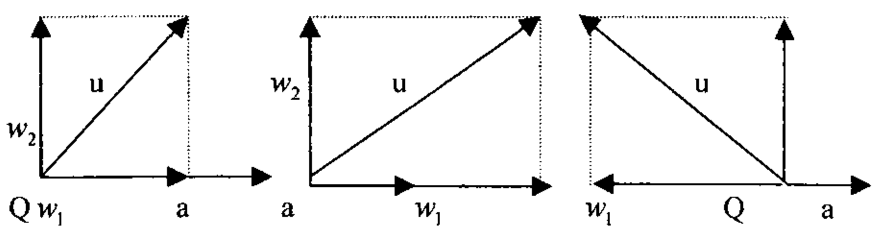
Cross Product.

1. Orthogonal projections.
2. Cross product of vectors.
3. Determinant formula for cross product.

As indicated in Figure 1, the vector w_1 is parallel to a, the vector w_2 is perpendicular to a, and

$$w_1 \quad + w_2 = (w_1 + (u - w_1) = u$$

Figure 1.



The vector w_1 is called the orthogonal projection of u on a or sometimes the vector component of u along a. It is denoted by

$$proj_a u \qquad (1)$$

The vector w_2 is called the vector component of u orthogonal to a. Since we have

$w_2 = u - w_1$, this vector can be written in notation (1) as

$$w_2 = u - proj_a u$$

The following theorem gives formulas for calculating the vectors $proj_a u$ and $u - proj_a u$.

Theorem 1. If u and a are vectors in 2-space or 3-space and if $a \neq 0$, then

$$proj_a u = \frac{u \cdot a}{\| a \|^2} a \text{ (vector component of } u \text{ along } a\text{)}$$

$$u - proj_a u = u - \frac{u \cdot a}{\| a \|^2} a$$ (vector component of u orthogonal to a).

Proof. Let $w_1 = proj_a u$ and $w_2 = u - proj_a u$. Since w_1 is parallel to a, it must be a scalar multiple of a, so it can be written in the form $w_1 = ka$. Thus

$$u = w_1 + w_2 = ka + w_2 \qquad (2)$$

Taking the dot product of both sides of (2) with a and using properties of dot product yields

$$u \cdot a = (ka + w_2) \cdot a = k \| a \|^2 + w_2 a \qquad (3)$$

But $w_2 \cdot a = 0$ since w_2 is perpendicular to a; so (3) yields

$$k = \frac{u \cdot a}{\| a \|^2}$$

Since $proj_a u = w_1 = ka$, we obtain

$$proj_a u = \frac{u \cdot a}{\| a \|^2} a$$

Example. Let $u = (2, -1, 3)$ and $a = (4, -1, 2)$. Find the vector component of u along a and the vector component of u orthogonal to a.

Solution.

$u \cdot a = (2)(4) + (-1)(-1) + (3)(2) = 15.$

$\|a\|^2 = 4^2 + (-1)^2 + 2^2 = 21.$

Thus, the vector component of u along a is

$$proj_a u = \frac{u \cdot a}{\| a \|^2} a = \frac{15}{21}(4,-1,2) = (\frac{20}{7};-\frac{5}{7};\frac{10}{7})$$

and the vector component of u orthogonal to a is

$$u - proj_a u = (2, -1, 3) - (\frac{20}{7};-\frac{5}{7};\frac{10}{7}) = (-\frac{6}{7};-\frac{2}{7};\frac{11}{7})$$

As a check, the reader may wish to verify that the vectors u - $proj_a u$ and a are perpendicular by showing that their dot product is zero.

A formula for the length of the vector component of u along a can be obtained by writing

$$\| proj_a u \| = \left\| \frac{u \cdot a}{\| a \|^2} a \right\| = \left| \frac{u \cdot a}{\| a \|^2} \right| \| a \| = \frac{|u \cdot a|}{\| a \|^2} \| a \|$$

which yields

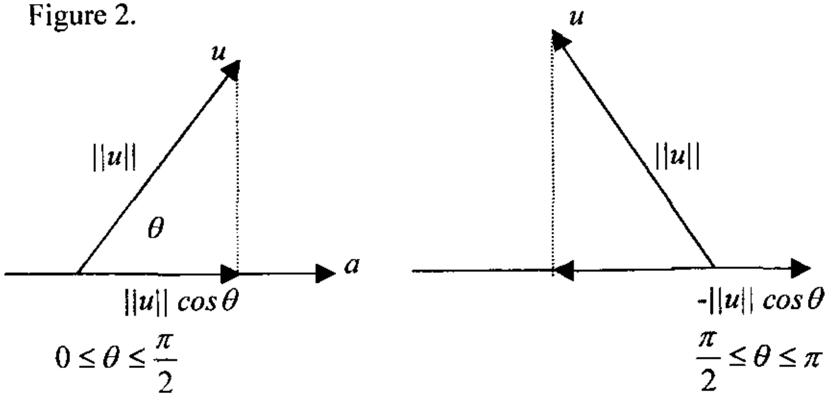
$$\| proj_a u \| = \frac{|u \cdot a|}{\| a \|} \qquad (4)$$

If θ denotes the angle between u and a, then $u \cdot a = \| u \| \| a \| \cos\theta$, so that (4) can also be written as

$$\| proj_a u \| = \|u\| \, |\cos \theta|$$

A geometric interpretation of this result is given in Figure 2.

Figure 2.



Cross product of vectors.

In many applications of vectors to problems in geometry, physics, and engineering, it is of interest to construct a vector in 3-space that is perpendicular to two given vectors.

In this section we shall introduce a type of vector multiplication that produces such a vector.

Definition. If $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ are vectors in 3-space, then the cross product $u\times v$ is the vector defined by

$u\times v=(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$ or in determinant notation

$$u\times v=\left(\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix},-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix},\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right)\quad (*)$$

There is an important difference between the dot product and cross product of two vectors – the dot product is a scalar and the cross product is a vector. The following theorem gives some important relationship between the dot product and cross product and also shows that $u \times v$ is orthogonal to both u and v.

Theorem 2. If u, v, and w are vectors in 3-space, then:

a) $u \cdot (u \times v) = 0$ ($u \times v$ is orthogonal to u).

b) $v \cdot (u \times v) = 0$ ($u \times v$ is orthogonal to v).

c) $\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u\ v)^2$ (Lagrange's identity).

d) $u \times (v \times w) = (u \cdot w) \cdot v - (u \cdot v) \cdot w$ (Relationship between cross and dot products)

e) $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$ (relationship between cross and dot products)

Proof (a). Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. Then

$$u \cdot (u \times v) = (u_1, u_2, u_3) \cdot (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) =$$
$$= u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) = 0$$

Proof (b). Similar to (a).

Proof (c). Since

$$\|u \times v\|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 \qquad (5)$$

and

$$\|u\|^2 \|v\|^2 - (u \cdot v)^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3) \qquad (6)$$

The proof can be completed by "multiplying out" the right sides of (5) and (6) and verifying their equality.

Proof (d) and (e). The proofs of these parts are left as exercises.

The main arithmetic properties of the cross product are listed in the next theorem.

Theorem 3. If u, v, and w are any vectors in 3-space and k is any scalar, then:

a) $u \times v = -(v \times u)$

b) $u \times (v + w) = (u \times v) + (u \times w)$

c) $(u + v) \times w = (u \times w) + (v \times w)$

d) $k(u \times v) = (ku) \times v = u \times (kv)$

e) $u \times 0 = 0 \times u = 0$

f) $u \times u = 0$

The proofs follow immediately from Formula (*) and properties of determinants; for example, (a) can be proved as follows:
Proof (a). Interchanging u and v in (*) interchanges the rows of the three determinants on the right side of (*) and hence changes the sign of each components in the cross product. Thus,

$u \times v = -(v \times u)$.

The proofs of the remaining parts are left as exercises.

Determinant formula for cross product.

Consider the vectors

$\vec{i} = (1, 0, 0)$ $\quad \vec{j} = (0, 1, 0)$ $\quad \vec{k} = (0, 0, 1)$

These vectors each have length 1 and lie along the coordinate axes. They are called the standard unit vectors in 3-space. Every vector $v = (v_1, v_2, v_3)$ in 3-space is expressible in terms of $\vec{i}, \vec{j}, and, \vec{k}$ since we can write $v = (v_1, v_2, v_3) =$

$v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

From (*) we obtain

$$\vec{i} \times \vec{j} = \left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) = (0,0,1) = \vec{k}$$

The reader should have no trouble obtaining the following results:

$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$

$$\vec{i} \times \vec{j} = \vec{k}$$
$$\vec{j} \times \vec{i} = -\vec{k}$$
$$\vec{j} \times \vec{k} = \vec{i}$$
$$\vec{k} \times \vec{j} = -\vec{i}$$
$$\vec{k} \times \vec{i} = \vec{j}$$
$$\vec{i} \times \vec{k} = -\vec{j}$$

It is also worth noting that a cross product can be represented symbolically in the form of a 3×3 determinant:

$$u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

LECTURE 10

Geometric interpretation of the cross product

1. Lagrange's identity.
2. Scalar triple product.

Lagrange's identity

If ***u*** and ***v*** are vectors in 3 – space , then the norm of $u * v$ has a useful geometric interpretation. Lagrange's identity, states that.

$$|| u * v ||^2 = || u ||^2 \quad || v ||^2 - (u\, v)^2 \quad (1)$$

If θ denotes the angle between u and v , then $u - v = || u || \, || v ||$ Cos θ, so that (1) can be rewritten as $|| u * v || = || u ||^2 \quad || v ||^2 - || u ||^2 \, || v ||^2 \; Cos^2 \; \theta = || u||^2 * || v ||^2 \; (1- Cos^2 \; \theta) = || u ||^2 \; || v ||^2$ Sin ^{2} θ

Since $O \leq \theta \leq \pi$, it follows that $\sin\,\theta \geq O$, so this can be rewritten as,

$$\| u * v\| = \| u \| \| v \| \, Sin\ \theta \quad (2)$$

But $\| v \| \, Sin\ \theta$ is the altitude of the parallelogram determined by u and v (Figure 1)

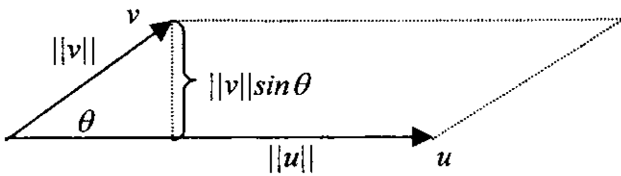


Figure 1.

Thus from, (2), the area A of this parallelogram is given by

A= (base) (Altitude) $=\|u\| \, \|v\| \, Sin\ \theta = \|u \times v\|$

This result is even correct if u and v are collinear, since the parallelogram determined by u and v has zero area and from (2) we have $u \times v=0$ because $\theta=0$ in this case.

Thus we have the following theorem.

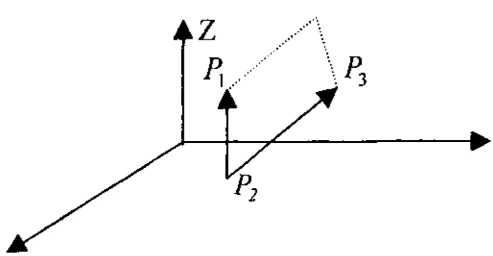
Theorem1. If u and v are vectors in 3-space, then $\|u \times v\|$ is equal to the area of the parallelogram determined by u and v

Example 2. Find the area of the triangle determined by the points $P_1(2,2,0)$,

$P_2(-1,0,2)$, and $P_3\ (0,4,3)$.

Solution. The area A of the triangle is $\frac{1}{2}$ the area of the parallelogram determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$

(Figure 2).



Using the method discussed in section of cross product, we obtain that

$$P_1 P_2 \times P_1 P_3 = (-10,5,-10)$$

Where $P_1 P_2$ =(-3,-2,2) and $P_1 P_3$=(-2,2,3)

Hence

$$A = \frac{1}{2}\|P_1P_2 \times P_1P_3\| = \frac{1}{2}(15) = \frac{15}{2}$$

Scalar Triple Product.

Definition 3. If $u \cdot (v \times w)$ are vectors in 3-space, then
is called the scalar triple product of u, v, and w. The scalar triple product of $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3),$ and $w = (w_1, w_{2,} w_3)$ can be calculated from the formula

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \qquad (3)$$

This follows from Formula

$$u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

Since

$$u \cdot (v \times w) = u \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k} \right) = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_1 -$$

$$- \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} u_3 = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example 4. Calculate the scalar triple product $u \cdot (v \times w)$ of the vectors

$$u = 3\vec{i} - 2\vec{j} - 5\vec{k}, v = \vec{i} + 4\vec{j} - 4\vec{k}, w = 3\vec{j} + 2\vec{k}$$

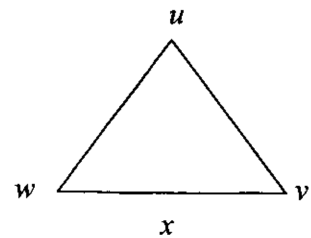
Solution. From (3)

$$u \cdot (v \times w) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} +$$

$$+ (-5)\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} = 60 + 4 - 15 = 49$$

Remark 5. The symbol $u \cdot (v \times w)$ makes no sense since we cannot form the cross product of a scalar and a vector. Thus, no ambiguity arises if we write $u \cdot v \times w$ rather then $u \cdot (v \times w)$. However, for clarity we shall usually keep the parentheses.
At follows from (3) that
$u \cdot (v \times w) = w \cdot (u \times v) = v \cdot (w \cdot u)$

Since the 3x3 determinants that represent these products can be obtained from one another by two row interchanges. These relationships can be remembered by moving the vectors u, v, and w clockwise around the vertices of the triangle in Figure 3.



Geometric interpretation of determinants.

The next theorem provides a useful geometric interpretation of 2x2 and 3x3 determinants.

Theorem 6.

(a) The absolute value of the determinant

$$\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$$

is equal to the area of the parallelogram in 2-space determined by the vectors $u = (u_1, u_2)$ and $v = (v_1, v_2)$

(b) The absolute value of the determinant

$$\det\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

is equal to the volume of the parallelepiped in 3-space determined by the vectors $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$, and $w = (w_1, w_{2,} w_3)$.

Proof (a). The key to the proof is to use theorem 1. However, that theorem applies to vectors in 3-space, whereas $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are vectors in 2-space. To circumvent this "dimension problem", we shall view u and v as vectors in the xy-plane of an xyz-

coordinate system, in which case these vectors are expressed as $u = (u_1, u_2, 0)$ and $v = (v_1, v_2, 0)$.

Thus,

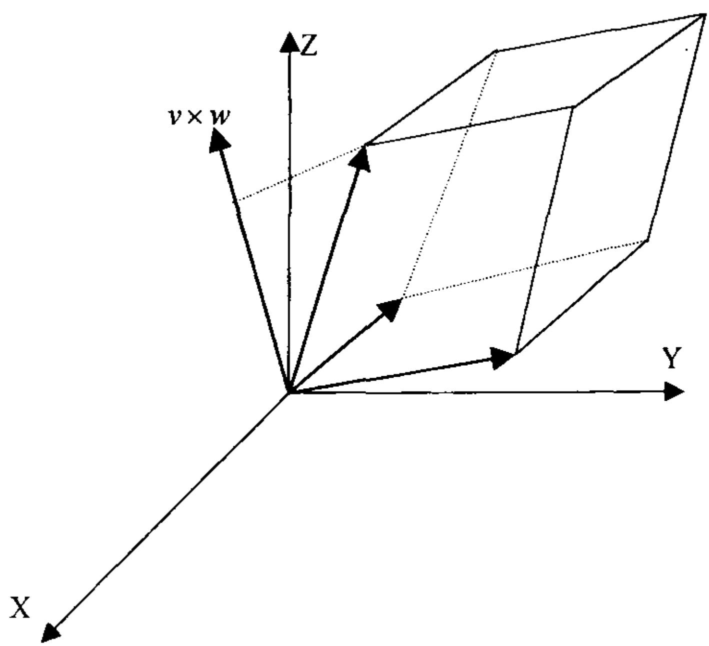
$$u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k} = \det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \vec{k}$$

At now follows Theorem 1 and the fact that $||k||=1$, that the area A of the parallelogram determined by u and v is

$$A = \| u \times v \| = \left\| \det\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k} \right\| = \left| \det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right| \| \vec{k} \| = \left| \det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$$

which completes the proof.

Proof (b). As shown in Figure 4, take the base of the parallelepiped determined by *u, v,* and *w* to be the parallelogram determined by *v* and *w*.



It follows from Theorem 1 that the area of the base is $\| v\times w \|$ and, as illustrated in Figure 4, the height h of the parallelepiped is the length of the orthogonal projection of u and $v\times w$. Therefore, by formula

$$\| proj_a u \| = \frac{|u\cdot a|}{\| a \|}$$

we have

$$h = \| proj_{v\times w} u \| = \frac{|u\cdot (v\times w)|}{\| v\times w \|}$$

It follows that the volume V of the parallelepiped is

$$V = (area\ of\ base)height = \| v\times w \| \frac{|u\cdot (v\times w)|}{\| v\times w \|} = |u\cdot (v\times w)|$$

So from (3)

$$V = \left| \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right|$$

which completes the proof.

Remark 7. If V denotes the volume of the parallelepiped determined by vectors u, v, and w, then it follows from Theorem 6 and formula (3) that

$$\mathrm{V} = \left\{ \begin{matrix} \text{Volume of parallellepiped} \\ \\ \text{Determined by u, v, and w} \end{matrix} \right\} = |u\cdot (v\times w)| \qquad (4)$$

From this and properties of dot product, we can conclude that

$$u \cdot (v \times w) = \pm V$$

where the + or – results depending on whether u makes an acute or obtuse angle with $(v \times w)$.

Formula (4) leads to a useful test for ascertaining whether three given vectors lie in the same plane. Since three vectors not in the same plane determine a parallelepiped of positive volume, it follows from (4) that $|u \cdot (v \times w)| = 0$ if and only if the vectors *u, v,* and *w* lie in the same plane. Thus, we have the following result.

Theorem 8. If the vectors $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3)$, and $w = (w_1, w_{2,} w_3)$ have the same initial point, then they lie in the same plane if and only if

$$u \cdot (v \times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

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