O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA'LIM VAZIRLIGI

ABU RAYHON BERUNIY NOMIDAGI
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Qatorlar nazariyasi

O'quv qo'llanma

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Ushbu o'quv qo'llanma oliy matematikaning bo'limlaridan biri bo'lgan qatorlar nazariyasi oliy texnika o'quv yurtlarining bakalavriat, magistrlar va aspirantlari uchun mo'ljallab yozilgan, undan o'qituvchilar va o'z malakasini oshiruvchi muhandislar ham foydalanishlari mumkin.

O'quv qo'llanma ikki qismdan iborat bo'lib, birinchi qismi faqat nazariya, ikkinchi qismi esa masala va misollarning to'la yechimlari bilan keltirilgan. Bundan tashqari mustaqil ishlash uchun masala va misollar tavsiya etilgan.

Abu Rayxon Beruniy nomidagi Toshkent Davlat texnika universiteti ilmiy-metodik kengashi qarori asosida chop etildi.

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Kirish

Bu o'quv qo'llanmaning asosiy maqsadi talabada darajali va trigonometrik Furye qatorlari nazariyasini o'rgatishdan iboratdir. O'quv qo'llanma oliy texnika o'quv yurtlari dasturi bo'yicha oliy matematika kursini o'zlashtirgan kitobxonlarga mo'ljallangan.

Mualliflar har bir qadamda foydalanayotgan usul qonuniyatining aniq bo'lishiga intilganlar.

O'quv qo'llanma ikki qismdan iborat bo'lib, birinchi qismda qatorlar nazariyasi va bu nazariya masala va misollarning izohli yechimlari bilan keltirilgan, ikkinchi qismda esa masala va misollar to'la yechimi bilan keltirilgan.

Mualliflar tomonidan yaratilgan ushbu qo'llanma talabani mustaqil o'zlashtirishiga yordam beradi.

I QISM. QATORLAR NAZARIYASI

I BOB. SONLI QATORLAR

1-§. Sonli qatorlar

Umumiy tushunchalar. Quyidagi cheksiz sonli ketma-ketlik berilgan bo'lsin:

$$a_1, a_2, \dots, a_n, \dots$$

1-ta'rif. Ushbu

$$a_1 + a_2 + \dots + a_n + \dots \qquad (1)$$

ifodaga cheksiz sonli qator yoki qisqacha sonli qator deyiladi. $a_1, a_2, \dots, a_n, \dots$ sonlarga qatorning hadlari deyiladi.

(1) qator qisqacha qilib yig'indi belgisi orqali ushbu ko'rinishda yoziladi:

$$\sum_{n=1}^{\infty} a_n .$$

Qatorning chekli sondagi hadlari yig'indisini tuzamiz:

$$S_1 = a_1$$
$$S_2 = a_1 + a_2 \qquad (2)$$
$$\dots\dots\dots\dots\dots$$
$$S_n = a_1 + a_2 + \dots + a_n$$

(2) ga qatorning xususiy yig'indilari deyiladi. Xususiy yig'indilardan tuzilgan ushbu ketma-ketlikni ko'ramiz:

$$S_1, S_2, \dots, S_n, \dots \qquad (3)$$

2-ta'rif. Agar $\lim\limits_{n\to\infty} S_n = S$ limit mavjud bo'lib, hamda u chekli bo'lsa, (1) qator yaqinlashuvchi deyiladi va uning yig'indisi S ga teng bo'ladi.

Agar (3) ketma-ketlik limitga ega bo'lmasa yoki cheksizga teng bo'lsa, u holda qator uzoqlashuvchi deyiladi va u yig'indiga ega bo'lmaydi.

1-misol. Ushbu qatorni yaqinlashishga tekshiring:

$$1 + q + q^2 + \dots + q^n + \dots \qquad (4)$$

Yechish. Bu qator geometrik progressiya hadlaridan tuzilgan qatordir. Agar $q \neq 1$ bo'lsa,

$$S_n = 1 + q + q^2 + \ldots + q^n = \frac{q^n - 1}{q - 1}$$

bo'ladi.

Bundan

$$S_n = \frac{1}{1-q} - \frac{q^n}{1-q}$$

deb yozish mumkin.

Agar $|q| < 1$ bo'lsa,

$$\lim_{n\to\infty} S_n = \frac{1}{1-q}$$

bo'ladi. Bu holda (4) qator yaqinlashuvchi bo'ladi va uning yig'indisi $\frac{1}{1-q}$ ga teng bo'ladi.

Agar $q = 1$ bo'lsa, ushbu qatorni hosil qilamiz

$$1 + 1 + \ldots + 1 + \ldots .$$

Demak, $S_n = n$, $\lim\limits_{n\to\infty} S_n = \infty$, u holda (4) qator uzoqlashuvchi bo'ladi.

Agar $|q| > 1$ bo'lganda $n \to \infty$ da $|q^n| \to \infty$ bo'ladi. Shuning uchun $\lim\limits_{n\to\infty} \frac{1-q^n}{1-q} = \pm\infty$ bo'ladi. Shunday qilib, $|q| > 1$ bo'lganda (4) qator uzoqlashuvchidir.

Demak, $|q| < 1$ da (4) yaqinlashuvchi, $|q| \geq 1$ da esa uzoqlashuvchi ekan.

2-misol. Quyidagi qatorni yaqinlashishga tekshiring:

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)n} .$$

Yechish. $\frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}$ $(n = 2, 3, \ldots)$ ekanligi ravshandir.

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = = 1 - \frac{1}{n} .$$

Demak, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$ qator yaqinlashuvchi.

2-§. Qatorlarning asosiy xossalari.

Ushbu qator

$$a_1 + a_2 + \ldots + a_n + \ldots \tag{1}$$

Agar (1) qatordagi hadlaridan chekli sondagi m ta hadini tashlab yuborilsa, u holda ushbu qatorni hosil qilamiz:

$$a_{m+1} + a_{m+2} + \ldots + a_{m+k} + \ldots . \tag{2}$$

Bu (1) qatorning m- qoldiq hadi deyiladi va r_m deb belgilanadi.

1-xossa. (2) qator (1) qator bilan bir vaqtda yaqinlashadi yoki uzoqlashadi.

Isbot. (2) qatorni S'_k deb belgilaymiz.

$$S'_k = a_{m+1} + a_{m+2} + \ldots + a_{m+k}.$$

Bundan

$$S'_k = S_{m+k} - S_m. \tag{3}$$

Faraz qilaylik (1) qator yaqinlashuvchi bo'lsin, ya'ni

$$\lim_{n\to\infty} S_n = S$$

$$\lim_{k\to\infty} S'_k = \lim_{k\to\infty}\left(S_{m+k} - S_m\right) = \lim_{k\to\infty} S_{m+k} - S_m = \lim_{n\to\infty} S_n - S_m = S - S_m.$$

Bundan (1) qatorning m- qoldiq hadi ham yaqinlashuvchi ekanligi kelib chiqadi.

2-xossa. Qatorning chekli sondagi hadlarini tashlab yuborish uning yaqinlashishiga ta'sir etmaydi.

Isbot. Faraz qilaylik (1) qator yaqinlashuvchi bo'lsin. Bu holda 1-xossaga asosan (2) qator ham yaqinlashuvchi bo'ladi. Demak, uning yig'indisi mavjud bo'ladi.

Tashlab yuborilgan k ta hadlar yig'indisini η_k, qolgan hadlar yig'indisini σ_{n-k} bilan belgilaymiz. Bu holda quyidagi tenglik hosil bo'ladi:

$$S_n = \eta_k + \sigma_{n-k}. \tag{4}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}(\eta_k + \sigma_{n-k}) = \eta_k + \lim_{n\to\infty} \sigma_{n-k}. \qquad (5)$$

Shartga ko'ra (1) qator yaqinlashuvchi bo'lgani uchun $\lim_{n\to\infty} S_n = 0$, η_k esa n ga bog'liq emas, demak, u o'zgarmas son. Natijada (8) ning o'ng tomonida limit ham mavjud bo'ladi, ya'ni

$$\lim_{n\to\infty} \sigma_{n-k} = \sigma .$$

Demak, agar qator yaqinlashuvchi bo'lsa, tashlab yuborilgandan qolgan qator ham yaqinlashuvchi ekan.

3-xossa. Yaqinlashuvchi qatorning m-qoldiq hadining $m \to \infty$ dagi limiti nolga teng bo'ladi.

4-xossa. Agar (1) qator yaqinlashuvchi bo'lib va uning yig'indisi S ga teng bo'lsa, u holda

$$\sum_{n=1}^{\infty} ca_n \qquad (6)$$

qator yaqinlashuvchi va uning yig'indisi cS ga teng bo'ladi, bu yerda c - ixtiyoriy son.

Isbot. S_n va σ_n lar (1) va (6) qatorlarning mos ravishda xususiy yig'indilari bo'lsin. Bu holda

$$\sigma_n = ca_1 + ca_2 + \ldots + ca_n + \ldots .$$

Bundan

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} cS_n = c\lim_{n\to\infty} S_n = cS .$$

5-xossa. Agar

$$a_1 + a_2 + \ldots + a_n + \ldots \qquad (7)$$

va

$$b_1 + b_2 + \ldots + b_n + \ldots \qquad (8)$$

qatorlar yaqinlashuvchi bo'lib, ularning yig'indilari S va σ bo'lsa, u holda

$$\sum_{n=1}^{\infty} (a_n + b_n) \qquad (9)$$

qator ham yaqinlashuvchi, hamda uning yig'indisi $S + \sigma$ ga teng bo'ladi.

Isbot. Faraz qilaylik (7), (8) va (9) qatorlarning yig'indisi mos ravishda S_n, σ_n va τ_n bo'lsin. U holda

$$\tau_n = (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_n + b_n) = (a_1 + a_2 + \ldots + a_n) +$$
$$+ (b_1 + b_2 + \ldots + b_n) = S_n + \sigma_n .$$
$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} (S_k + \sigma_n) = S + \sigma .$$

1-eslatma. 5-xossaning shartiga ko'ra

$$\sum_{n=1}^{\infty} (a_n - b_n)$$

qator yaqinlashuvchi, hamda uning yig'indisi $S - \sigma$ ga teng bo'ladi.

2-eslatma. Agar (1) qator yaqinlashuvchi va uning yig'indisi S ga teng bo'lsa, u holda qatorning istalgan hadlarining o'rnini almashtirmasdan guruhlash mumkin bo'ladi, masalan

$$a_1 + (a_2 + a_3) + (a_4 + a_5) + (a_6 + a_7 + a_8) + \ldots .$$

Hosil bo'lgan yangi qator ham yaqinlashuvchi va uning yig'indisi S ga teng bo'ladi.

3-§. Qator yaqinlashishining zaruriy sharti

Teorema. Agar (1) qator yaqinlashuvchi bo'lsa, u holda $n \to \infty$ da n - hadining limiti nolga intiladi, ya'ni

$$\lim_{n\to\infty} a_n = 0 .$$

Isbot. Teoremaning shartiga ko'ra (1) qator yaqinlashuvchi, demak, $\lim_{n\to\infty} S_n = S$ va $\lim_{n\to\infty} S_{n-1} = S$ oxirgi ikki tenglikni bir-biridan ayiramiz

$$\lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0 .$$

Bundan $a_n = S_n - S_{n-1}$ bo'lgani uchun

$$\lim_{n\to\infty} a_n = 0$$

ekanligi kelib chiqadi.

Teorema isbot bo'ldi.

Eslatma. Agar (1) qatorning n- hadi $n \to \infty$ da nolga intilmasa, u holda qator uzoqlashuvchi bo'ladi.

Isbot qilingan teoremaning teskarisi har doim ham to'g'ri bo'lavermaydi, ya'ni qatorning n-hadini nolga intilishidan uning yaqinlashuvchiligi kelib chiqmaydi.

Misol tariqasida garmonik qator deb ataluvchi quyidagi qatorni ko'raylik:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots. \tag{1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$$

bo'lgani bilan, bu qator uzoqlashuvchidir. Buni isbot qilish uchun (1) qatorning ko'proq hadlarini ko'rib chiqaylik.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots+\frac{1}{n}+\ldots.$$

Bu qatorning birinchi 2^m ta hadini olib, bu hadlarni quyidagi ko'rinishda gruppalaymiz:

$$S_{2m} = 1+\frac{1}{2}+\underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{2-\text{had}}+\underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{2^2-\text{had}}+$$

$$+\underbrace{\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)}_{2^3-had}+\ldots$$

$$\ldots+\underbrace{\left(\frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}+\frac{1}{2^{m-1}+3}+\ldots+\frac{1}{2^m}\right)}_{2^{m-1}-\text{had}}+\ldots.$$

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2},$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2},$$

$$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}>$$

$$>\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{2},$$

. . . .

$$\frac{1}{2^{m-1}+1}+\frac{1}{2^{m-1}+2}+\ldots+\frac{1}{2^m}>$$

$$>\underbrace{\frac{1}{2^m}+\frac{1}{2^m}+\frac{1}{2^m}+\ldots+\frac{1}{2^m}}_{2^{m-1}-\text{had}}=\frac{2^{m-1}}{2^m}=\frac{1}{2}$$

tengsizliklar o'rinlidir.

Shunday qilib, har bir qavs ichidagi hadlar yig'indisi $\frac{1}{2}$ dan katta. Qavslarning soni birinchi ikkita hadni hisobga olmaganda $m-1$ ga teng bo'lgani uchun

$$S_{2^m}>1+\frac{1}{2}+\underbrace{\frac{1}{2}+\ldots+\frac{1}{2}}_{(m-1)-\text{had}}=1+\frac{m}{2}.$$

Agar S_{2^m} yig'indida hadlar soni $n=2^m$ katta bo'lib borsa, m ham katta bo'lib boradi. Shuning uchun $S_{2^m}\to\infty$. Demak (1) garmonik qator uzoqlashuvchidir.

4-§. Musbat hadli qatorlar

Ta'rif. Hadlari manfiy bo'lmagan qatorlarga musbat hadli qatorlar deyiladi.

Ushbu musbat hadli qator berilgan bo'lsin:

$$a_1+a_2+\ldots+a_n+\ldots. \tag{1}$$

Bu yerda $a_n\geq 0$ $(n=1, 2, \ldots)$. U holda

$$S_{n+1}=S_n+a_{n+1}\geq S_n \quad (n=1, 2, 3, \ldots)$$

ekanligi ravshandir, ya'ni $S_1, S_2, \ldots, S_n, \ldots$ ketma-ketlik kamaymaydigan. Quyidagi teoremani isbotsiz keltiramiz:

1-teorema. (1) musbat hadli qatorning yaqinlashuvchi bo'lishi uchun bu qatorning xususiy yig'indilaridan tuzilgan ketma-ketlik yuqoridan chegaralangan bo'lishi zarur va etarlidir.

Musbat hadli qatorlarni taqqoslash haqidagi teoremalarni keltiramiz,

2-teorema. Quyidagi ikkita musbat hadli qatorlar berilgan bo'lsin:

$$a_1 + a_2 + \ldots + a_n + \ldots \qquad (2)$$

$$b_1 + b_2 + \ldots + b_n + \ldots. \qquad (3)$$

(2) qatorning hadlari (3) qatorning mos hadlaridan katta bo'lmasa, ya'ni

$$a_n \le b_n \quad (n = 1, 2, \ldots) \qquad (4)$$

hamda (3) qator yaqinlashuvchi bo'lsa, u holda (2) qator ham yaqinlashuvchi bo'ladi.

Isbot. (2) va (3) qatorlarning xususiy yig'indilarini mos ravishda A_n va B_n deb belgilaymiz. (4) dan

$$A_n \le B_n$$

tengsizlikni yozish mumkin.

Agar (3) qator yaqinlashuvchi bo'lsa, u holda 1-teoremaga asosan (2) qator ham yaqinlashuvchi bo'ladi. Chunki shart bo'yicha

$$\lim_{n\to\infty} B_n = B.$$

Demak, $A_n \le B$ bo'ladi.

Teorema isbot bo'ladi.

Misol. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ qatorni yaqinlashishga tekshiring.

Yechish. Yuqorida biz

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

qatorning yaqinlashuvchiligini ko'rsatgan edik. Berilgan qatorni shu qator bilan solishtiramiz

$$\frac{1}{(n+1)^2} < \frac{1}{n(n-1)} \quad (n = 2, 3, \ldots).$$

2-teoremaga asosan berilgan qator yaqinlashuvchidir.

3-teorema. Quyidagi ikkita musbat hadli qatorlar berilgan bo'lsin:

$$a_1 + a_2 + \ldots + a_n + \ldots \qquad (2)$$

$$b_1 + b_2 + \ldots + b_n + \ldots. \qquad (3)$$

(2) qatorning hadlari (3) qatorning mos hadlaridan kichik bo'lmasa, ya'ni

$$a_n \geq b_n \quad (n=1, 2, \ldots) \tag{5}$$

hamda (3) qator uzoqlashuvchi bo'lsa, u holda (2) qator ham uzoqlashuvchi bo'ladi.

Isbot. Yana (2) va (3) qatorlarning xususiy yig'indilarini mos ravishda A_n va B_n deb belgilaymiz. (5) tengsizlikdan $A_n \geq B_n$ bo'lishi kelib chiqadi. (3) qator uzoqlashuvchi hamda uning xususiy yig'indilari o'suvchi bo'lgani uchun

$$\lim_{n\to\infty} B_n = \infty$$

bo'ladi. Natijada $\lim_{n\to\infty} A_n = \infty$ bo'ladi.

Demak, (2) qator ham uzoqlashuvchi bo'ladi.

Teorema isbot bo'ldi.

Eslatma. Qatorlarni taqqoslash teoremalaridan foydalanishda qatorlarni taqqoslash uchun yaqinlashuvchi yoki uzoqlashuvchiligi ma'lum bo'lgan qatorlarni olish kerak bo'ladi.

Misol. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n-1)}}$ qatorni yaqinlashishga tekshiring.

Yechish. $a_n = \frac{1}{\sqrt{n(n-1)}}$ berilgan qatorning umumiy hadi. Uni $b_n = \frac{1}{\sqrt{n^2}}$ bilan solishtiramiz

$$\frac{1}{\sqrt{n(n-1)}} \geq \frac{1}{\sqrt{n^2}} = \frac{1}{n}.$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ garmonik qator bo'lib, u uzoqlashuvchidir. Demak, 3-teoremaga asosan berilgan qator uzoqlashuvchi bo'ladi.

5-§. Dalamber alomati

Quyidagi musbat hadli qator berilgan bo'lsin:

$$a_1 + a_2 + \ldots + a_n + \ldots . \tag{1}$$

Teorema. Agar (1) qator $(n+1)$-hadining n - hadiga nisbati $n \to \infty$ da chekli limitga ega bo'lib, ya'ni

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$$

bo'lsa, u holda agar:

1) $l < 1$ bo'lsa (1) qator yaqinlashuvchi
2) $l > 1$ bo'lsa (1) qator uzoqlashuvchi bo'ladi.

Isbot. 1) Ketma-ketlikning ta'rifiga asosan $\forall \varepsilon > 0$ soni uchun shunday $\exists N(\varepsilon)$ sonini topish mumkin bo'lsaki, n ning $n > N$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida

$$\left| \frac{a_{n+1}}{a_n} - l \right| < \varepsilon$$

yoki

$$l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon \tag{2}$$

tengsizlik o'rinli bo'ladi.

Agar $l < 1$ bo'lsa, ε ni shunday tanlaymizki $l + \varepsilon$ birdan kichik bo'lsin. (2) tengsizlikning o'ng tomonini $q = l + \varepsilon$ deb olamiz va quyidagi tengsizlikni yozamiz:

$$\frac{a_{n+1}}{a_n} < q$$

yoki

$$a_{n+1} < a_n q \,.$$

$n = N+1,\, N+2, \ldots$ qiymatlarni oxirgi tengsizlikka qo'yib quyidagi tengsizliklarni hosil qilamiz:

$$a_{N+2} < a_{N+1} q$$

$$a_{N+3} < a_{N+2} q < a_{N+1} q^2$$

$$a_{N+4} < a_{N+3} q < a_{N+1} q^3$$

$$\ldots\ldots\ldots\ldots\ldots$$

Demak,

$$a_{N+2} + a_{N+3} + a_{N+4} + \ldots$$

qatorning hadlari yaqinlashuvchi bo'lgan

$$a_{N+1}q + a_{N+2}q^2 + a_{N+3}q^3 + a_{N+4}q^4 + \ldots$$

qatorning hadlaridan kichikdir. U holda taqqoslash xaqidagi 2-teoremaga asosan qator yaqinlashuvchidir.

2) Endi $l > 1$ bo'lsin. ε ni juda kichik qilib olamizki, bunda $l - \varepsilon > 1$ bo'lsin. Bu holda (2) ning o'ng tomoni $n > N$ bo'lganda

$$\frac{a_{n+1}}{a_n} > 1$$

yoki

$$a_{n+1} > a_n$$

bo'ladi. Shunday qilib, (1) qatorning $n = N + 1$ dan boshlab n ortishi bilan hadlari o'sib boradi. (1) ning umumiy hadi $n \to \infty$ da nolga intilmaydi. Demak, qator uzoqlashuvchidir.

Teorema isbot bo'ldi.

1-eslatma. Agar $l = 1$ bo'lsa, teorema qatorning yaqinlashuvchi yoki uzoqlashuvchiligini aniqlab bermaydi.

2-eslatma. $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$ bo'lsa, (1) qator uzoqlashuvchi bo'ladi. $n = N$ dan boshlab, $\frac{a_{n+1}}{a_n} > 1$ bo'ladi. Demak, $n \to \infty$ da a_n nolga intilmaydi.

1-misol. $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$ qatorni Dalamber alomati yordamida yaqinlashishga tekshiring.

Yechish. $a_{n+1} = \dfrac{n+1}{2^{n+1}}, \quad a_n = \dfrac{n}{2^n},$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right) = \frac{1}{2} < 1.$$

Demak, berilgan qator $l = \frac{1}{2} < 1$ bo'lgani uchun yaqinlashuvchidir.

2-misol. $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ qatorni Dalamber alomati yordamida yaqinlashishga tekshiring.

Yechish. $a_{n+1}=\frac{(n+1)^{n+1}}{(n+1)!}$, $a_n=\frac{n^n}{n!}$,

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^{n+1}\cdot n!}{(n+1)!\cdot n^n}=\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e>1\cdot$$

Demak, berilgan qator $l=e>1$ bo'lgani uchun uzoqlashuvchi ekan.

6-§. Koshi alomati

Ushbu musbat hadli qator berilgan bo'lsin.

$$a_1+a_2+\ldots+a_n+\ldots \qquad (1)$$

Teorema. Agar $n\to\infty$ da $\sqrt[n]{a_n}$ miqdorning limiti mavjud bo'lib, ya'ni

$$\lim_{n\to\infty}\sqrt[n]{a_n}=l \qquad (2)$$

bo'lsa, bu holda agar:

1) $l<1$ bo'lsa qator yaqinlashuvchi,

2) $l>1$ bo'lsa qator uzoqlashuvchi bo'ladi.

Isbot. 1) $l<1$ bo'lsin. $l<q<1$ tengsizlikni qanoatlantiruvchi q sonini qaraylik.

(2) shartga ko'ra $n=N$ qiymatdan boshlab

$$\left|\sqrt[n]{a_n}-l\right|<q-l$$

tengsizlik o'rinli bo'ladi. Bundan $\sqrt[n]{a_n}<q$ yoki $a_n<q^n$ tengsizlikni yozish mumkin.

Umumiy hadi q^n bo'lgan geometrik qator yaqinlashuvchi bo'ladi, chunki $0<q<1$. Taqqoslash haqidagi 2-teoremasiga asosan berilgan (1) qator yaqinlashuvchi bo'ladi.

2) $l>1$ bo'lsin. Bu holda n ning biror qiymatidan boshlab

$$\sqrt[n]{a_n} > 1$$

yoki

$$a_n > 1$$

o'rinli bo'ladi. (1) qator uzoqlashuvchi bo'ladi, chunki uning umumiy hadi nolga intilmaydi.

Teorema isbot bo'ldi.

Eslatma. Bu yerda ham $l = 1$ bo'lsa, Koshi alomati qatorning yaqinlashuvchi yoki uzoqlashuvchiligini aniqlamaydi.

1-misol. $\sum_{n=1}^{\infty} \frac{1}{n^n}$ qatorni Koshi alomati yordamida yaqinlashishga tekshiring.

Yechish. $a_n = \frac{1}{n^n}$.

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1.$$

Demak, $l = 0 < 1$ bo'lgani uchun berilgan qator yaqinlashuvchi.

2-misol. $\sum_{n=1}^{\infty} \left(\frac{3n+2}{n}\right)^n$ qatorni Koshi alomati yordamida yaqinlashishga tekshiring.

Yechish. $a_n = \left(\frac{3n+2}{n}\right)^n$.

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+2}{n}\right)^n} = \lim_{n\to\infty} \frac{3n+2}{n} = 3 > 1.$$

Berilgan qator $l = 3 > 1$ bo'lgani uchun uzoqlashuvchi bo'ladi.

7-§. Koshining integral alomati

Quyidagi musbat hadli qator berilgan bo'lsin:

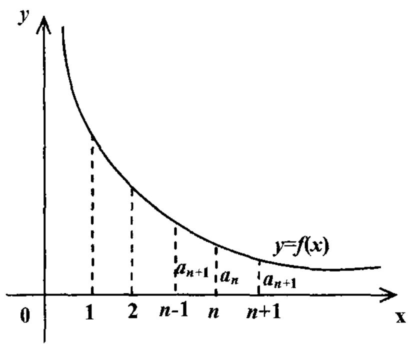
$$a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

Teorema. (1) qatorning hadlari monoton kamayuvchi bo'lib, ya'ni $a_n \geq a_{n+1}$ va

$$\int_{1}^{\infty} f(x)dx \qquad (2)$$

xosmas integral yaqinlashuvchi bo'lsa, (1) qator ham yaqinlashuvchi, (2) integral uzoqlashuvchi bo'lsa, (1) qator ham uzoqlashuvchi bo'ladi, bu yerda $x \geq 1$ bo'lganda $f(x)$ uzluksiz va musbat.

Isbot. Yuqoridan $y = f(x)$ funksiyaning grafigi va asosi Ox o'qidagi $x = 1$ dan $x = n$ gacha bo'lgan kesma bilan chegaralangan egri chiziqli trapesiyani qaraymiz (1-rasm)



1-rasm.

Asosi $[1; 2],[2; 3],...$ kesmalardan iborat bo'lgan tashqi va ichki to'g'ri to'rtburchaklarni chizamiz. Aniq integralning geometrik ma'nosini e'tiborga olib, quyidagi tengsizlikni yozamiz:

$$f(2)\cdot 1 + f(3)\cdot 1 + ... + f(n)\cdot 1 < \int_{1}^{n} f(x)dx < f(1)\cdot 1 + f(2)\cdot 1 + ... + f(n-1)\cdot 1$$

yoki

$$a_2 + a_3 + ... + a_n < \int_{1}^{n} f(x)dx < a_1 + a_2 + ... + a_{n-1}$$

yoki

$$S_n - a_1 < \int_{1}^{n} f(x)dx < S_n - a_n \cdot \qquad (3)$$

1-hol. $\int\limits_1^{\infty} f(x)dx$ xosmas integral yaqinlashuvchi bo'lsin, ya'ni $\int\limits_1^{\infty} f(x)dx = A$. $\int\limits_1^{n} f(x)dx < \int\limits_1^{+\infty} f(x)dx = A$ bo'lgani uchun, hamda (3) tengsizlikni e'tiborga olib, $S_n - a_1 < A$, ya'ni $S_n < a_n + A$ tengsizltkni yozish mumkin. Bundan xususiy yig'indilar ketma-ketligi monoton o'suvchi va yuqoridan chegaralangan bo'lgani uchun, limitning mavjudligi haqidagi teoremaga asosan limiti mavjud bo'ladi.

Demak, (1) qator yaqinlashuvchi.

2-hol. $\int\limits_1^{+\infty} f(x)dx$ xosmas integral uzoqlashuvchi bo'lsin. Bu holda $\int\limits_1^{+\infty} f(x)dx = +\infty$ bo'ladi va $n \to +\infty$ da $\int\limits_1^{n} f(x)dx$ chegaralanmagan o'suvchi bo'ladi.

$S_n > \int\limits_1^{n} f(x)dx + a_n$ e'tiborga olgan holda $n \to +\infty$ da $S_n \to +\infty$ kelib chiqadi. Demak, (1) qator uzoqlashuvchi.

Yechish. $\int\limits_1^{+\infty} f(x)dx$ integralning o'rniga $\int\limits_k^{+\infty} f(x)dx$ integralni olish mumkin, bu yerda $k \in N$, $k > 1$. (1) qatorning k ta birinchi hadlarini tashlab yuborish uning yaqinlashishi yoki uzoqlashishiga ta'sir etmaydi.

Misol. $\sum\limits_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ qatorni yaqinlashishga tekshiring.

Yechish. $f(x) = \frac{1}{x(\ln x)^2}$.

$$\int\limits_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{\alpha\to\infty} \int\limits_2^{\alpha} \frac{d(\ln x)}{(\ln x)^2} = -\lim_{\alpha\to\infty}\left(\frac{1}{\ln x}\right)\Bigg|_2^{\alpha} = -\lim_{\alpha\to\infty}\left(\frac{1}{\ln\alpha} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

.

Xosmas integral yaqinlashuvchi, demak, berilgan qator ham yaqinlashuvchidir.

8-§. Ishoralari navbatlashuvchi qatorlar

Ta'rif. Quyidagi qatorga

$$a_1 - a_2 + a_3 - \ldots + (-1)^n a_n + \ldots \qquad (1)$$

ishoralari navbatlashuvchi qator deyiladi, bu yerda $a_n > 0 \ (n = 1, 2, 3, \ldots)$.

1-teorema. (Leybnis teoremasi). Agar (1) qator hadlarining absolyut qiymatlari monoton kamayuvchi ketma-ketlik, ya'ni

$$a_{n+1} \le a_n \ (n = 1, 2, 3, \ldots) \qquad (2)$$

bo'lib va

$$\lim_{n\to\infty} a_n = 0 \qquad (3)$$

bo'lsa, u holda (1) qator yaqinlashuvchi bo'lib, uning yig'indisi musbat bo'ladi va birinchi hadidan katta bo'lmaydi.

Isbot. Qatorning S_{2m} xususiy yig'indisini quyidagi ko'rinishda yozib olaylik:

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2m-1} - a_{2m}) \qquad (4)$$

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2m-2} - a_{2m-1}) - a_{2m}. \qquad (5)$$

(4) va (5) xususiy yig'indilarning har bir qavs ichidagi ifoda (2) shartga asosan musbatdir. (4) dan $S_{2m} \ge 0$ ekanligi kelib chiqadi. $\{S_{2m}\}$ ketma-ketlik monoton o'suvchi bo'ladi. (5) dan $S_{2m} < a_1$, ya'ni $\{S_{2m}\}$ ketma-ketlik chegaralanganligi ravshandir. Demak, bu ketma-ketlik chekli limitga ega, ya'ni

$$\lim_{m\to\infty} S_{2m} = S \qquad (6)$$

bunda $0 < S < a_1$. (6) va (3) larni e'tiborga olgan holda

$$\lim_{m\to\infty} S_{2m+1} = \lim_{m\to\infty} S_{2m} + \lim_{m\to\infty} a_{2m+1} = S. \qquad (7)$$

(6) va (7) lardan

$$\lim_{n\to\infty} S_n = S$$

kelib chiqadi. Bundan (1) qatorning yaqinlashuvchiligi kelib chiqadi.

Teorema isbot bo'ldi.

1-misol. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ qatorning yaqinlashishga tekshiring.

Yechish. Berilgan qator hadlarining absolyut qiymatlari $1>\frac{1}{2}>\frac{1}{3}>\ldots$ monoton kamayuvchi ketma-ketlikni tashkil qiladi.

Shuningdek, $\lim\limits_{n\to\infty}\frac{1}{n}=0$.

Demak, Leybnis teoremasining shartlari bajarilganligi sababli berilgan qator yaqinlashuvchidir.

2-teorema. (1) ishoralari navbatlashuvchi qatorning r_n qoldig'i Leybnis teoremasining shartlarini qanoatlantirgani uchun o'zining birinchi hadining ishorasi bilan bir xil va absolyut qiymat bo'yicha birinchi hadidan kichik bo'ladi.

Isbot. Agar n juft bo'lsa, u holda

$$r_n=a_{n+1}-a_{n+2}+\ldots.$$

Bu qator Leybnis teoremasining shartlarini qanoatlantirgani uchun

$$0<S<a_1$$

bo'ladi. Bu tengsizlikdan

$$0<r_n<a_{n+1}$$

kelib chiqadi.

Agar n toq bo'lsa,

$$r_n=-a_{n+1}+a_{n+2}-\ldots.$$

Bundan

$$-r_n=a_{n+1}-a_{n+2}+\ldots$$

bo'ladi.

$$0<S<a_1.$$

Tengsizlikka asosan

$$0<-r_n<a_{n+1}$$

bo'ladi. Demak, $r_n>0$ va $|r_n|<a_{n+1}$ o'rinli bo'ladi.

Teorema isbot bo'ldi.

Misol. 0,1 aniqlik bilan $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ yaqinlashuvchi qatorning yig'indisini hisoblang.

Yechish. Qatorning xususiy yig'indisi S_n, uning taqribiy yig'indisi S bo'lsin.

$|r_n| < 0,1$, teoremaga asosan $|r_n| < \frac{1}{n+1}$. Demak, $n+1=10$, $n=9$ deb olish kerak

$$S = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}\approx 0,74.$$

9-§. O'zgaruvchan ishorali qatorlar
Absolyut va shartli yaqinlashish

Biz 8-§ paragrafda ishoralari navbatlashuvchi bo'lgan qatorlarni ko'rib chiqdik. Bu paragrafda umumiyroq bo'lgan hol o'zgaruvchan ishorali qatorlarni ko'rib chiqamiz.

Agar qatorning hadlari orasida bir nechta musbat va bir nechta manfiy hadlari bo'lsa, bunday qatorlarga o'zgaruvchan ishorali qatorlar deyiladi.

O'zgaruvchan ishorali qator yaqinlashishni tekshirishda muhim bo'lgan yetarli shartini o'rganamiz.

Teorema. O'zgaruvchi ishorali qator

$$a_1 + a_2 + \ldots + a_n + \ldots \qquad (1)$$

hadlarining absolyut qiymatlaridan tuzilgan

$$|a_1| + |a_2| + \ldots + |a_n| + \ldots \qquad (2)$$

qator yaqinlashsa, u holda (1) qator ham yaqinlashadi.

Isbot. (1) va (2) qatorning xususiy yig'indilarini mos ravishda S_n va σ_n lar bilan belgilaymiz. (1) qatorning n ta musbat hadining yig'indisini S_n', manfiylarini esa S_n'' deb belgilaymiz.

U holda $S_n = S_n' - S_n''$, $\sigma_n = S_n' + S_n''$. Lekin 2-teorema shartiga ko'ra (2) qator yaqinlashuvchi bo'lgani uchun

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} (S_n' + S_n'') = \lim_{n\to\infty} S_n' + \lim_{n\to\infty} S_n'' = S' + S''.$$

Bundan

$$\lim_{n\to\infty} S_n' = S', \quad \lim_{n\to\infty} S_n'' = S''.$$

Natijada

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_n' - S_n'') = \lim_{n\to\infty} S_n' - \lim_{n\to\infty} S_n'' = S' - S''$$

bo'ladi. Bu esa (1) qatorning yaqinlashuvchiligini ko'rsatadi.

Teorema isbot bo'ldi.

Bu teoremadan o'zgaruvchi ishorali qatorni yaqinlashishga tekshirish musbat hadli qatorni tekshirishga keladi.

1-ta'rif. Agar (2) qator yaqinlashuvchi bo'lsa, u holda (1) qatorni absolyut yaqinlashuvchi deyiladi.

2-ta'rif. (1) qator yaqinlashuvchi bo'lib, (2) qator uzoqlashuvchi bo'lsa, u holda (1) qator shartli yaqinlashuvchi deyiladi.

1-misol. $\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}$ qatorni yaqinlashishga tekshiring.

Yechish. Berilgan qatorning absolyut qiymatlarilan tuzilgan qatorni qaraylik

$$1+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}+\ldots.$$

Bu qator cheksiz kamayuvchi geometrik progressiya bo'lib, uning mahraji $q=\frac{1}{2}<1$ bo'lgani uchun yaqinlashuvchidir. Ikkinchi tomondan qatorning o'zi Leybnis teoremasining shartlarini qanoatlantirgani uchun yaqinlashuvchi. Demak, berilgan qator absolyut yaqinlashuvchi bo'ladi.

2-misol. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ qatorni yaqinlashishga tekshiring.

Yechish. Bu qator hadlarining absolyut qiymatlaridan tuzilgan qator garmonik qator bo'lgani uchun uzoqlashuvchi bo'ladi. Lekin Leybnis teoremasiga asosan bu qator yaqinlashuvchidir, ya'ni

$$1>\frac{1}{2}>\frac{1}{3}>\ldots>\frac{1}{n}>\ldots$$

va

$$\lim_{n\to\infty}\frac{1}{n}=0.$$

Demak, berilgan qator shartli yaqinlashuvchi ekan.

II BOB. FUNKSIONAL QATORLAR

1-§. Funksional qatorlar

1°. Umumiy tushunchalar. $[a;b]$ kesmada aniqlangan

$$u_1(x), u_2(x), \ldots, u_n(x), \ldots$$

ketma-ketlik berilgan bo'lsin. Bu ketma-ketlikning hadlaridan tuzilgan quyidagi qatorni qaraylik:

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots \tag{1}$$

(1) qatorning hadlari x ning funksiyasi bo'lgani uchun u funksional qator deyiladi.

$[a;b]$ kesmadagi x ning har bir qiymatida (1) qator sonli qatorlarga aylanadi. Bu sonli qatorlarning bir nechtasi yaqinlashuvchi, yana boshqa bir nechtasi uzoqlashuvchi bo'lishi mumkin. Bundan quyidagi ta'rifni yozamiz:

1-ta'rif. Funksional qator yaqinlashadigan x ning qiymatlar to'plamiga shu qatorning yaqinlashish oralig'i deyiladi

2-ta'rif. Agar $x_0 \in [a;b]$ qiymatda (1) funksional qator yaqinlashuvchi bo'lsa, bu holda x_0 nuqtaga (1) funksional qatorning yaqinlashish nuqtasi deyiladi. Bu ta'rifdan (1) funksional qatorning

$$S_n(x)=u_1(x)+u_2(x)+\ldots+u_n(x) \tag{2}$$

xususiy yig'indisi $S(x_0)$ chekli limitga ega ekanligi kelib chiqadi. Bundan, agar $\forall\varepsilon>0$ soni uchun shunday $\exists N(\varepsilon,x_0)$ soni mavjud bo'lsaki, n ning $n>N(\varepsilon,x_0)$ ni qanoatlantiruvchi barcha qiymatlarida

$$|S(x_0)-S_n(x)|<\varepsilon$$

tengsizlik o'rinli bo'lishi kelib chiqadi. Funksional qatorning $[a;b]$ kesmada yaqinlashuvchiligini ta'riflaymiz.

3-ta'rif. Agar (1) funksional qatorning xususiy yig'indisini $n\to\infty$ dagi chekli limiti $x\in[a;b]$ ning barcha qiymatlarida mavjud bo'lsa, u holda (1) qatorni $[a;b]$ kesmada yaqinlashuvchi deyiladi.

Ta'rifdan

$$\lim_{n\to\infty} S_n(x) = S(x). \quad (3)$$

Bu limitni $[a;b]$ kesmada (1) qatorning yig'indisi deyiladi. Bundan, agar ixtiyoriy $\forall\varepsilon > 0$ soni uchun shunday $\exists N(\varepsilon, x)$ soni topilsaki, n ning $n > N(\varepsilon, x)$ ni qanoatlantiruvchi barcha qiymatlarida

$$|S(x) - S_n(x)| < \varepsilon$$

tengsizlikni o'rinli bo'lishi kelib chiqadi. Agar

$$\lim_{n\to\infty} S_n(x) = S(x) \quad (4)$$

bo'lsa, u holda

$$S(x) = S_n(x) + r_n(x) \quad (5)$$

deb yozish mumkin, bu yerda

$$r_n(x) = u_{n+1}(x) + u_{n+2}(x) + \ldots \quad (6)$$

$r_n(x)$ - funksional qatorning qoldiq hadi.

(5) va (6) lardan quyidagi ifodani yozamiz:

$$\lim_{n\to\infty} r_n(x) = \lim_{n\to\infty}(S(x) - S_n(x)) = 0.$$

Demak, yaqinlashuvchi qatorning $n \to \infty$ da qoldiq hadi nolga intilar ekan.

2-§. Funksional qatorning tekis yaqinlashishi

1-ta'rif. Agar $\forall\varepsilon > 0$ soni uchun shunday x ga bog'liq bo'lmagan $\exists N(\varepsilon)$ soni topilsaki n ni $n > N(\varepsilon)$ qanoatlantiruvchi barcha qiymatlarida

$$|S(x) - S_n(x)| < \varepsilon$$

tengsizlik o'rinli bo'lsa, u holda (1) funksional qator $[a;b]$ kesmada tekis yaqinlashuvchi deyiladi.

Teorema (Veyershtrass alomati). Agar $[a;b]$ kesmada (1) funksional qatorning har bir hadi

$$c_1 + c_2 + \ldots + c_n + \ldots \quad (7)$$

yaqinlashuvchi musbat hadli qator hadlari uchun

$$|u_n(x)| < c_n \tag{8}$$

Tengsizlik o'rinli bo'lsa, u holda (1) qator $[a;b]$ kesmada tekis va absolyut yaqinlashuvchi bo'ladi.

Isbot. Ixtiyoriy $\forall\varepsilon > 0$ sonini olamiz. Shartga ko'ra (7) qator yaqinlashuvchi bo'gani uchun $\lim\limits_{n\to\infty} r_n = 0$ bo'ladi. Bu yerda r_n - musbat hadli qatorning qoldiq hadi. Shuning uchun, shunday N natural son mavjudki, $n > N$ o'rinli bo'lganda

$$\sum_{k=n+1}^{n+p} c_k < \varepsilon \tag{9}$$

o'rinli bo'ladi.

(8) tengsizlikdan foydalanib, ixtiyoriy natural p soni uchun

$$\sum_{k=n+1}^{n+p} |u_k(x)| \le \sum_{k=n+1}^{n+p} c_k \ . \tag{10}$$

$p\to\infty$ da va (9) tengsizlikni e'tiborga olib, quyidagi tengsizlikni yozish mumkin:

$$\sum_{k=n+1}^{\infty} |u_k(x)| \le \sum_{k=n+1}^{\infty} c_k < \varepsilon \quad (n > N). \tag{11}$$

(10) tengsizlikning chap tomoni yuqoridan chegaralangan bo'lgani uchun $p\to\infty$ da uning limiti mavjud bo'ladi. Bu esa (1) qatorning absolyut yaqinlashuvchiligini ko'rsatadi.

Bu holda (11) ni e'tiborga olib, hamda $n > N$ va barcha x lar uchun quyidagi tengsizlikni yozamiz:

$$|r_n(x)| = \left|\sum_{k=n+1}^{\infty} u_k(x)\right| \le \sum_{k=n+1}^{\infty} |u_k(x)| < \varepsilon \ . \tag{12}$$

Bundan (1) funksional qatorning $[a;b]$ kesmada tekis yaqinlashuvchiligi kelib chiqadi.

Agar (8) tengsizlik o'rinli bo'lsa, (1) funksional qatorni $[a;b]$ kesmada kuchaytirilgan qator deyiladi.

Veyershtrass alomatiga asosan kuchaytirilgan funksional qator tekis yaqinlashuvchi bo'ladi.

Misol. $\sum_{n=1}^{\infty}\frac{\cos nx}{2^n}$ funksional qator x ning barcha qiymatlarida kuchaytirilgan qator bo'lib, u absolyut yaqinlashuvchidir. Haqiqatan x ning barcha qiymatlari uchun

$$\left|\frac{\cos nx}{2^n}\right| \le \frac{1}{2^n}$$

tengsizlik o'rinlidir. Lekin $\sum_{n=1}^{\infty}\frac{1}{2^n}$ qator yaqinlashuvchi.

3-§. Qator yig'indisining uzluksizligi

$[a;b]$ kesmada yaqinlashuvchi bo'lgan quyidagi funksional qator berilgan bo'lsin:

$$u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots. \qquad (1)$$

(1) qatorning yig'indisi $S(x)$, uning hadlari $[a;b]$ kesmada uzluksiz bo'lsin.

Teorema. Agar $[a;b]$ kesmada tekis yaqinlashuvchi bo'lgan (1) qatorning hadlari uzluksiz bo'lsa, u holda shu oraliqda uning yig'indisi ham uzluksiz bo'ladi.

Isbot. (1) funksional qatorning $S(x)$ yig'indisi $[a;b]$ kesmaning har bir x_0 nuqtasida uzluksizligini isbot qilish kerak, ya'ni $\forall\varepsilon>0$ soni uchun shunday $\exists\delta>0$ mavjud bo'lsaki, $|x-x_0|<\delta$ bajarilganda

$$|S(x)-S(x_0)|<\varepsilon \qquad (2)$$

tengsizlik o'rinli ekanligini ko'rsatish kerak.

Ixtiyoriy qayd qilingan n uchun (1) qatorning yig'indisini ixtiyoriy $\forall x\in[a;b]$ qiymatlarda

$$S(x)=S_n(x)+r_n(x) \qquad (3)$$

ko'rinishda yozish mumkin.

Xususiy holda $x=x_0$ bo'lganda (3) tenglikni

$$S(x_0)=S_n(x_0)+r_n(x_0) \qquad (4)$$

ko'rinishda yoziladi. Bundan

$$S(x)-S(x_0)=S_n(x)-S_n(x_0)+r_n(x)-r_n(x_0) \tag{5}$$

va

$$|S(x)-S(x_0)|\le|S_n(x)-S_n(x_0)|+|r_n(x)|+|r_n(x_0)| \tag{6}$$

kelib chiqadi.

Endi $\varepsilon>0$ sonini qayd qilamiz. Teoremaning shartiga ko'ra (1) qator $[a;b]$ kesmada tekis yaqinlashuvchi bo'lgani uchun, $\frac{\varepsilon}{3}$ songa shunday $N(\varepsilon)$ son mos keladiki, bunda barcha $x\in[a;b]$ lar va $n>N(\varepsilon)$ uchun quyidagi tengsizlik o'rinli bo'ladi:

$$|r_n(x)|<\frac{\varepsilon}{3}. \tag{7}$$

Xususiy holda $x=x_0$ da

$$|r_n(x_0)|<\frac{\varepsilon}{3} \tag{8}$$

o'rinli bo'ladi.

Endi $n>N(\varepsilon)$ tengsizlikni qanoatlantiruvchi n sonini qayd qilamiz. (1) qatorning $S_n(x)$ xususiy yig'indisi, chekli sondagi uzluksiz funksiyalarning yig'indisi uzluksiz bo'lgani uchun $S_n(x)$ ham uzluksiz bo'ladi. Shuning uchun $\frac{\varepsilon}{3}$ songa shunday $\delta>0$ son mos keladiki, $|x-x_0|<\delta$ bajarilganda

$$|S_n(x)-S_n(x_0)|<\frac{\varepsilon}{3} \tag{9}$$

tengsizlik o'rinli bo'ladi.

(6) tengsizlikning o'ng tomonidagi har bir had $|x-x_0|<\delta$ bajarilganda $\frac{\varepsilon}{3}$ dan kichik, ya'ni

$$|S(x)-S(x_0)|\le|S_n(x)-S_n(x_0)|+|r_n(x)|+|r_n(x_0)|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

yoki

$$|S(x)-S(x_0)|<\varepsilon$$

bo'ladi.

Demak, bundan qatorning yig'indisi $S(x)$ x_0 nuqtada uzluksiz ekanligi ravshandir.

Teorema isbot bo'ldi.

4-§. Funksional qatorlarni hadma-had integrallash va differensiallash

1-teorema (Qatorni hadma-had integrallash haqida). $[a;b]$ kesmada hadlari uzluksiz bo'lgan tekis yaqinlashuvchi qatorni shu kesmada hadma-had integrallash mumkin.

Isbot. $a<\alpha<\beta<b$ uchun ushbu tenglik o'rinli bo'ladi:

$$\int_{\alpha}^{\beta}\left[\sum_{n=1}^{\infty}S_n(x)\right]dx=\sum_{n=1}^{\infty}\int_{\alpha}^{\beta}S_n(x)dx. \tag{1}$$

(1) formulaning o'rinli bo'lishi uchun (1) dagi qo'shish va integrallash amallar o'rnini almashtirish mumkinligini ko'rsatish qatorni hadma-had integrallashning mumkinligini isbot qilishdir.

$$\sum_{n=1}^{\infty}\int_{\alpha}^{\beta}S_n(x)dx \tag{2}$$

qatorning yaqinlashuvchiligi va uning yig'indisi

$$\int_{\alpha}^{\beta}S(x)dx \tag{3}$$

integralga teng ekanligini ko'rsatamiz.

$$S(x)=S_n(x)+r_n(x) \tag{4}$$

tenglikning ikkala qismi $(a;b)$ oraliqda uzluksizdir. Shuning uchun chekli sondagi uzluksiz funksiyalarning yig'indisi uzluksiz bo'lganligi sababli $S_n(x)$ ham uzluksiz bo'ladi. 3-§ paragrafdagi teoremaga asosan $S(x)$ ham uzluksiz bo'ladi. $r_n(x)$ funksiya ham $S(x)$ va $S_n(x)$ funksiyalarning ayirmasi bo'gani uchun bu funksiya ham uzluksiz bo'ladi.

Bu mulohazalarga asosan $S(x)$, $S_n(x)$ va $r_n(x)$ funksiyalar integrallanuvchidir.

(4) tenglikni $[\alpha\,;\beta]$ kesmada integrallab, ushbu tenglikni hosil qilamiz:

$$\int_\alpha^\beta S(x)dx = \int_\alpha^\beta \left[\sum_{k=1}^{n} S_k(x)\right]dx + \int_\alpha^\beta r_n(x)dx \cdot \tag{5}$$

Chekli yig'indida qo'shish va integrallash amallar o'rnini almashtirish mumkin. Shu sababli

$$\sigma_n = \int_\alpha^\beta \left[\sum_{k=1}^{n} S_k(x)\right]dx = \sum_{k=1}^{n}\int_\alpha^\beta S_k(x)dx \tag{6}$$

tenglik (2) qatorning xususiy yig'indisi bo'ladi.

$R_n = \int_\alpha^\beta r_n(x)dx$ belgilashni kiritib, $n \to \infty$ da $R_n \to 0$ ekanligini isbotlaymiz.

Teoremaning shartiga ko'ra $[\alpha\,;\beta]$ kesmada qator tekis yaqinlashuvchiligidan $\frac{\varepsilon}{b-a} > 0$ songa $N(\varepsilon)$ son mos keladiki barcha $n > N(\varepsilon)$ va $x \in [\alpha\,;\beta]$ lar uchun $|r_n(x)| < \frac{\varepsilon}{b-a}$ o'rinli bo'ladi. (4) formulada n ni $N(\varepsilon)$ dan katta qilib olingan. Shuning uchun

$$|R_n| \le \int_\alpha^\beta |r_n(x)|dx < \int_\alpha^\beta \frac{\varepsilon}{b-a}dx < \varepsilon$$

va bundan $\lim\limits_{n\to\infty} R_n = 0$ kelib chiqadi.

(5) tenglikni

$$\int_\alpha^\beta S(x)dx = \sigma_n + r_n \tag{7}$$

ko'rinishda yozamiz va buni har ikkala tomonidan $n \to \infty$ da limitga o'tamiz. (7) ning o'ng tomoni n ga bog'liq emas.

Natijada

$$\int_\alpha^\beta S(x)dx = \lim_{n\to\infty}\int_\alpha^\beta \left[\sum_{k=1}^{n} S_k(x)\right]dx + \lim_{n\to\infty} R_n = \sum_{k=1}^{\infty}\int_\alpha^\beta S_k(x)dx \cdot$$

bo'ladi.

Teorema isbot bo'ldi.

2-teorema (Qatorni hadma-had differensiallash haqida). Agar $[a; b]$ kesmada

1) $u_1(x)+u_2(x)+\ldots+u_n(x)+\ldots$ (8)

qator $S(x)$ funksiyaga yaqinlashsa;

2) (8) qatorning hadlari $[a; b]$ kesmada hosilalari uzluksiz;

3) $g(x)=u_1'(x)+u_2'(x)+\ldots+u_n'(x)+\ldots$ (9)

hosilalardan tuzilgan qator tekis yaqinlashuvchi bo'lsa, u holda (8) qatorni hadma-had differensiallash mumkin, ya'ni

$$\left[\sum_{n=1}^{\infty} S_n(x)\right]' = \sum_{n=1}^{\infty} S_n'(x) \tag{10}$$

Isbot. $g(x)$ funksiya, 3-§ paragrafdagi qator yig'indisining uzluksizligi haqidagi teoremaga asosan uzluksiz bo'ladi. Shuning uchun $g(x)$ funksiya $[a; b]$ kesmada integrallanuvchidir. Qatorni hadma-had integrallash haqidagi teoremaning hamma shartlarini (10) qator qanoatlantiradi. (9) qatorni a dan x gacha integrallab

$$\int_a^x g(t)dt = \int_a^x \left[\sum_{n=1}^{\infty} S_n(t)\right]' dt = \sum_{n=1}^{\infty}\int_a^x S_n'(t)dt = \sum_{n=1}^{\infty} S_n(x) - \sum_{n=1}^{\infty} S_n(a) =$$
$$= S(x) - S(a)$$

ifodani hosil qilamiz, bu yerda $a < x \le b$.

Natijada

$$\int_a^x g(t)dt = S(x) - S(a) \tag{11}$$

tenglik kelib chiqadi. (11) tenglikni yuqori chegarasi x bo'yicha hosilasini olib, $g(x)=S'(x)$ tenglikni hosil qilamiz.

Teorema isbot bo'ldi.

1-misol. $1+x+x^2+\ldots+x^n+\ldots=\dfrac{1}{1-x}$ qator $|x| \le r < 1$ oraliqda 2-teoremaning shartlarini qanoatlantiradi, shuning uchun bu qatorni hadma-had differensiallash mumkin

$$1+2x+3x^2+\ \ldots +nx^{n-1}+\ldots =\frac{1}{(1-x)^2}.$$

Bu qator Veyershtrass alomatiga asosan tekis yaqinlashuvchidir, chunki

$$\left|nx^{n-1}\right|\le nr^n.$$

$\sum\limits_{n=1}^{\infty} nr^n$ qator Dalamber alomatiga asosan yaqinlashuvchi bo'ladi.

2-misol. $\sum\limits_{n=1}^{\infty}\frac{\sin n^2 x}{n^2}$ qator ixtiyoriy kesmada tekis yaqinlashuvchi, lekin bu qatorni hadma-had differensiallab bo'lmaydi, chunki uning hosilalaridan tuzilgan

$$\sum_{n=1}^{\infty}\cos n^2 x$$

qator uzoqlashuvchidir, bu qatorning umumiy hadi nolga intilmaydi.

III BOB. DARAJALI QATORLAR

1-§. Darajali qatorlar va ularning yaqinlashish sohasi

Ta'rif. Quyidagi funksional qatorga darajali qator deyiladi:

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots , \qquad (1)$$

bu yerda $a_0, a_1, a_2, \ldots, a_n, \ldots$ - haqiqiy o'zgarmas sonlar. Bu sonlarni darajali qatorning koeffitsientlari deyiladi.

Darajali qator funksional qatorning xususiy holidir.

1. Har qanday darajali qator $x = 0$ nuqtada yaqinlashuvchi va shu nuqtada uning yig'indisi a_0 ga teng bo'ladi.

2. Faqat $x = 0$ nuqtada yaqinlashuvchi bo'lgan qatorlar mavjuddir. Masalan

$$x + 2x^2 + 3x^3 + \ldots + nx^n + \ldots . \qquad (2)$$

Haqiqatan $x_0 \neq 0$ ixtiyoriy son bo'lsin. Bu holda $n \to \infty$ da n ning biror qiymatidan boshlab (nx_0) miqdor $(nx_0) > 2$ tengsizlikni qanoatlantiradi. Demak, qatorning umumiy hadi $(nx_0)^n > 2^n$ tengsizlikni qanoatlantiradi.

$n \to \infty$ da 2^n miqdor cheksiz kattalashgani uchun $(nx_0)^n$ ham cheksiz kattalashadi. Bu holda

$$x_0 + 2x_0^{\,2} + 3x_0^{\,3} + \ldots + nx_0^{\,n} + \ldots \qquad (3)$$

qator $x_0 \neq 0$ ixtiyoriy nuqtada uzoqlashuvchi bo'ladi.

3. $(-\infty; +\infty)$ oraliqning barcha nuqtalarida yaqinlashuvchi bo'lgan qatorlar ham mavjuddir.

Misol sifatida quyidagi qatorni ko'ramiz:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^n}{n} + \ldots \qquad (4)$$

$x = x_0$ sonlar o'qining ixtiyoriy nuqtasi bo'lsin . n orta borgani uchun shunday $N > 0$ soni topiladiki barcha $n > N$ lar uchun

$\left|\frac{x_0}{n}\right| < \frac{1}{2^n}$ tengsizlik o'rinli bo'ladi. Bu holda (4) qatorni yaqinlashuvchi bo'lgan

$$\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}+\ldots \tag{5}$$

qator bilan solishtirib (4) qatorni ixtiyoriy $x = x_0$ nuqtada absolyut yaqinlashuvchiligiga ishonch hosil qilish mumkin.

4. x ning biror qiymatlarida yaqinlashuvchi va boshqa qiymatlarida uzoqlashuvchi bo'lgan darajali qatorlar ham mavjud. Masalan, ushbu qator berilgan bo'lsin

$$\frac{x}{3}+\frac{x^2}{3^2}+\ldots+\frac{x^n}{3^n}+\ldots \tag{6}$$

Bu qator $|x| < 3$ bo'lganda $\left|\frac{x}{3}\right| = |q| < 1$ bo'ladi. Bunda berilgan qatorning mahraji $|q| < 1$ bo'lgan geometrik progressiyadir.

Demak, qator absolyut yaqinlashuvchi bo'ladi, chunki geometrik progressiyaning $q = \frac{x}{3}$ mahraji $|q| < 1$ tengsizlikni qanoatlantiradi.

$|x| \geq 3$ bo'lganda berilgan qator uzoqlashuvchi bo'ladi, chunki geometrik progressiyaning $q = \frac{x}{3}$ mahraji $|q| \geq 1$ tengsizlikni qanoatlantiradi.

Shunday qilib, (6) qator $|x| < 3$ bo'lganda yaqinlashuvchi va $|x| \geq 3$ da esa uzoqlashuvchi ekan. Demak, (6) qatorning yaqinlashish sohasi $(-3; 3)$ oraliqdan iborat ekan.

Ixtiyoriy darajali qatorning yaqinlashish sohasini aniqlaydigan teoremani keltiramiz.

Abel teoremasi. 1) Agar (1) darajali qator biror $x_0 \neq 0$ nuqtada yaqinlashuvchi bo'lsa, u holda $|x| < |x_0|$ tengsizlikni qanoatlantaruvchi x ning barcha qiymatlarida (1) qator absolyut yaqinlashuvchi bo'ladi.

2) Agar (1) qator x_0 nuqtada uzoqlashuvchi bo'lsa, bu holda $|x| > |x_0|$ tengsizlikni qanoatlantiruvchi x ning barcha qiymatlarida (1) qator uzoqlashuvchi bo'ladi.

Isbot. 1) Faraz qilaylik $x = x_0 \neq 0$ nuqtada (1) qator yaqinlashuvchi bo'lsin. U holda

$$a_0 + a_1 x_0 + a_2 x_0^{\,2} + \ldots + a_n x_0^{\,n} + \ldots = \sum_{n=0}^{\infty} a_n x_0^{\,n} \tag{7}$$

sonli qator yaqinlashuvchi bo'ladi. Demak, bu qatorning umumiy hadi $n \to \infty$ da nolga intiladi, ya'ni

$$\lim_{n\to\infty} a_n x_0^n = 0.$$

Bundan $a_n x_0^n$ miqdorni chegaralanganligi kelib chiqadi, ya'ni shunday n soni topiladiki, ixtiyoriy n uchun

$$\left|a_n x_0^n\right| \leq \mu$$

tengsizlik o'rinli bo'ladi.

Berilgan (1) qatorni ushbu ko'rinishda yozamiz

$$a_0 + a_1 x_0 \left(\frac{x}{x_0}\right) + a_2 x_0^2 \left(\frac{x}{x_0}\right)^2 + \ldots + a_n x_0^n \left(\frac{x}{x_0}\right)^n + \ldots .$$

Bu qator hadlarining absolyut qiymatlaridan tuzilgan qatorni ko'raylik

$$|a_0| + |a_1 x_0| \left|\frac{x}{x_0}\right| + |a_2 x_0^2| \left|\frac{x}{x_0}\right|^2 + \ldots + |a_n x_0^n| \left|\frac{x}{x_0}\right|^n + \ldots .$$

$\left|\dfrac{x}{x_0}\right| = q$ desak, $q < 1$ bo'lganda (1) qatorning umumiy hadi ushbu tengsizlikni qanoatlantiradi:

$$\left|a_n x^n\right| = \left|a_n x_0^n\right| \cdot \left|\frac{x}{x_0}\right|^n = \left|a_n x_0^n\right| \cdot q^n \leq \mu q^n.$$

Lekin

$$\mu + \mu q + \mu q^2 + \ldots + \mu q^n + \ldots = \mu\left(1 + q + q^2 + \ldots + q^n + \ldots\right)$$

qatorning mahraji $q < 1$ bo'lganda yaqinlashuvchi geometrik progressiyadir. (1) qator bilan bu progressiyani solishtirsak, (1) qatorni

x ning $|x|<|x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida absolyut yaqinlashuvchiligi kelib chiqadi, bu yerda $x_0 \neq 0$.

2) Endi teoremaning ikkinchi qismini isbot qilamiz.

Faraz qilaylik, berilgan qator x_1 nuqtada uzoqlashuvchi bo'lsin. U holda bu qator $|x|>|x_1|$ tengsizlikni qanoatlantiruvchi barcha x nuqtalarda uzoqlashadi.

Haqiqatda, bu tengsizlikni qanoatlantiruvchi biror x nuqtada qator yaqinlashsa, u holda teoremaning isbot qilingan birinchi qismiga asosan $|x|>|x_1|$ tengsizlik o'rinli bo'lgani uchun, qator x_1 nuqtada yaqinlashuvchi bo'lishi kerak edi. Lekin bu x_1 nuqtada qator uzoqlashadi degan shartga qarama-qarshi bo'ladi. Demak, qator x_1 nuqtada uzoqlashadi.

Teorema isbot bo'ldi.

Abel teoremasidan kelib chiqadigan ayrim natijalarni ta'riflaymiz.

1-natija. Agar $x_0 \neq 0$ qator yaqinlashish nuqtasi bo'lsa, u holda $(-x_0;\, x_0)$ oraliqning barcha nuqtalarida qator yaqinlashuvchi bo'ladi.

2-natija. Har bir darajali qator uchun quyidagi xossalarga ega bo'lgan aniq $R>0$ soni mavjud bo'ladi:

a) (1) qator x ning $|x|<R$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida absolyut yaqinlashuvchi bo'ladi;

b) x ning $|x|>R$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida (1) qator uzoqlashuvchi bo'ladi.

Bu R soniga darajali qatorning yaqinlashish radiusi deyiladi. Xususiy holda, (1) qator $x_0=0$ nuqtada yaqinlashuvchi bo'lsa, yaqinlashish radiusi $R=0$ bo'ladi.

Agar qator x ning har qanday qiymatida yaqinlashuvchi bo'lsa, yaqinlashish radiusini $R=\infty$ deb olish qabul qilingan.

3-natija. (1) darajali qatorning yaqinlashish sohasi $(-R;\, R)$ oraliqdan iborat, bu yerda R - yaqinlashish radiusi. Demak, darajali qatorning yaqinlashish sohasi koordinata boshiga nisbatan simmetrik ekan. Oraliqning har bir chetki $x=-R$ va $x=R$ nuqtalarida qator yaqinlashuvchi yoki uzoqlashuvchi bo'lishi mumkin. Shuning uchun qatorni bu nuqtalardagi yaqinlashishini alohida tekshirish kerak bo'ladi.

Darajali qatorning yaqinlashish sohasini topish uchun Dalamber yoki Koshi alomatlaridan foydalaniladi.

Qator hadlarining absolyut qiymatlaridan tuzilgan qatorni yozamiz

$$|a_0|+|a_1x|+|a_2x^2|+\ \ldots+|a_nx^n|+\ \ldots\ .$$

Bu qatorning $n+1$ hadini n hadiga bo'lgan nisbatini ko'raylik

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_nx^n}\right|=|x|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|x|L\cdot$$

Bu yerda $L=\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$.

Dalamber alomatidan ma'lumki, agar $|x|L<1$ bo'lsa, qator absolyut yaqinlashuvchi va $|x|L>1$ bo'lsa uzoqlashuvchi bo'ladi.

Bundan (1) qator $|x|<\dfrac{1}{L}$ tengsizlikni qanoatlantiruvchi x ning barcha qiymatlarida yaqinlashuvchi va $|x|>\dfrac{1}{L}$ da esa uzoqlashuvchiligi kelib chiqadi. Natijada

$$\frac{1}{L}=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\cdot$$

Demak, qatorning yaqinlashish radiusini quyidagi Dalamber alomati bo'yicha hisoblash mumkin:

$$R=\frac{1}{L}=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|\cdot$$

Koshi alomatidan foydalanib qatorning yaqinlashish radiusini quyidagi formula yordamida ham hisoblash mumkin:

$$R=\frac{1}{L}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{|a_n|}}\cdot$$

1-misol. $1+\frac{x}{2}+\frac{x^2}{3}+\ldots+\frac{x^{n-1}}{n}+\ldots$ qatorni yaqinlashish radiusi va yaqinlashish sohasini toping.

Yechish. $a_n=\frac{1}{n}$, $a_{n+1}=\frac{1}{n+1}$.

$$R=\lim_{n\to\infty}\left|\frac{n+1}{n}\right|=1.$$

Demak, $L=1$. Dalamber alomatiga asosan $|x|<1$ bo'lganda qator yaqinlashuvchi va $|x|>1$ da uzoqlashuvchi bo'ladi. Yaqinlashish radiusi $R=1$, yaqinlashish sohasi $(-1;1)$ oraliqdan iborat. Endi oraliqning chetki $x=-1$ va $x=1$ nuqtalardagi yaqinlashishni tekshiramiz.

$x=1$ da quyidagi qator hosil bo'ladi:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

Bu garmonik qatordir, demak, berilgan qator $x=1$ nuqtada uzoqlashuvchi.

$x=-1$ da

$$1-\frac{1}{2}+\frac{1}{3}-\ldots+\frac{(-1)^n}{n}+\ldots$$

qator hosil bo'ladi. Bu qator Leybnis teoremasiga asosan shartli yaqinlashuvchi bo'ladi. Demak, berilgan qatorning yaqinlashish sohasi $[-1;1)$ oraliqdan iborat.

2-misol. $1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots$ qatorning yaqinlashish radiusini toping.

Yechish. $R=\frac{1}{L}=\lim_{n\to\infty}(n+1)=\infty$.

Demak, berilgan qator x ning barcha qiymatlarida yaqinlashuvchi bo'ladi.

2-§. Darajali qatorlarning xossalari

1-teorema. Har qanday

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots , \qquad (1)$$

darajali qator $(-R;\ R)$ oraliqqa tegishli $[-\rho;\ \rho]$ kesmada tekis yaqinlashuvchi bo'ladi.

Isbot. Shart bo'yicha $0 < \rho < R$. Shuning uchun

$$|a_0| + |a_1|\rho + |a_2|\rho^2 + \ldots + |a_n|\rho^n + \ldots$$

musbat hadli qator yaqinlashuvchi bo'ladi. Bunda

$$|a_n x^n| \le |a_n|\rho^n \quad (n = 0, 1, 2, \ldots)$$

tengsizlik o'rinli bo'ladi. Demak, Veyershtrass alomatiga asosan (1) darajali qator $[-\rho;\ \rho]$ kesmada tekis yaqinlashuvchi bo'ladi.

Qolgan xossalarni isbotsiz keltiramiz.

2-teorema. Agar (1) darajali qator $(-R;\ R)$ oraliqda yaqinlashuvchi bo'lsa, u holda uning $S(x)$ yig'indisi shu oraliq ichida uzluksiz bo'ladi.

3-teorema. $(-R;\ R)$ oraliqda yaqinlashuvchi bo'lgan darajali qatorni shu oraliqda hadma-had ixtiyoriy tartibli hosilasini olish mumkin, ya'ni

$$S(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

qatorning $S'(x),\ S''(x),\ \ldots,\ S^{(n)}(x)$ hosilalari mavjuddir. Bu yerda

$$S'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots ,$$

$$S''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots + n(n-1)a_n x^{n-2} + \ldots ,$$

. . . .

Qator hadlarining hosilalaridan tuzilgan qatorning yaqinlashish sohasi $(-R;\ R)$ oraliqdan iborat bo'ladi.

4-teorema. $(-R;\ R)$ oraliqda yaqinlashuvchi bo'lgan darajali qatorni shu oraliqda yotgan ixtiyoriy kesmada hadma-had integrallash mumkin.

$|x| < R$ bo'lsin. U holda

$$\int_0^x S(x)dx = \int_0^x a_0 dx + \int_0^x a_1 x dx + \int_0^x a_2 x^2 dx + \ \ldots +$$

$$+\int_0^x a_n x^n dx + \ldots = a_0 x + \frac{a_1}{2}x^2 + \frac{a_3}{3}x^3 + \ \ldots + \frac{a_n}{n+1}x^{n+1} + \ldots$$

Qatorni hadma-had integrallashdan hosil bo'lgan qatorning yaqinlashish sohasi ham $(-R;\ R)$ oraliqdan iborat bo'ladi.

3-§. $x-a$ ning darajalari bo'yicha qatorlar

Quyidagi ko'rinishdagi funksional qatorga ham darajali qator deyiladi:

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n + \ldots \quad (1)$$

Bu yerdagi $a_0, a_1, a_2, \ \ldots, a_n, \ldots$ o'zgarmas sonlarga darajali qatorning koeffitsientlari deyiladi.

(1) darajali qatorning yaqinlashish sohasini topish uchun $x - x_0 = t$ almashtirish kiritamiz. Natijada, quyidagi darajali qator hosil bo'ladi:

$$a_0 + a_1 t + a_2 t^2 + \ \ldots + a_n t^n + \ \ldots\ , \quad (2)$$

(2) qator ham darajali qatordir, u faqat t ning darajalari bo'yicha yoyilgan.

(2) qatorning yaqinlashish sohasi $(-R;\ R)$ oraliqdan iborat, ya'ni $-R < t < R$. Bundan x ning $-R < x - x_0 < R$ yoki $x_0 - R < x < x_0 + R$ tengsizlikni qanoatlantiruvchi qiymatlarida (1) qatorning yaqinlashuvchiligi kelib chiqadi.

$|t| > R$ bo'lganda (2) qator uzoqlashuvchi bo'lgani uchun, $|x - x_0| > R$ bo'lganda (1) qator ham uzoqlashuvchi bo'ladi. Demak, $|x - x_0| < R$ oraliqdan tashqarida uzoqlashuvchi, $|x - a| < R$ oraliqda esa qator yaqinlashuvchidir.

Bundan (1) qatorning yaqinlashish oralig'i markazi x_0 nuqtada bo'lgan $(x_0 - R;\ a + R)$ oraliqdan iborat bo'ladi.

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots, \tag{3}$$

darajali qatorning $(-R;\,R)$ yaqinlashish oralig'i ichida barcha xossalari (1) darajali qator uchun $(x_0-R;\,x_0+R)$ yaqinlashish oralig'i ichida saqlanadi.

Agar integrallash chegaralari $(x_0-R;\,x_0+R)$ yaqinlashish oralig'i ichida yotsa (1) darajali qatorni hadma-had integrallashdan so'ng yig'indisi berilgan (1) qatorning yig'indisidan olingan integralga teng bo'lgan qator hosil bo'ladi. x ning $(x_0-R;\,x_0+R)$ yaqinlashish oralig'i ichida yotuvchi hamma qiymatlari uchun (1) darajali qatorni hadma-had differensiallashdan, yig'indisi berilgan (1) qatorning yig'indisidan olingan hosilasiga teng bo'lgan qator hosil bo'ladi.

Misol. $(x-4)+(x-4)^2+\ldots+(x-4)^n+\ldots$ qatorning yaqinlashish sohasini toping.

Yechish. $x-4=t$ deb olamiz. U holda

$$t+t^2+\ldots+t^n+\ldots$$

qator hosil bo'ladi.

Bu qator $-1<t<1$ oraliqda yaqinlashuvchi bo'ladi. Demak, berilgan qator x ning $-1<x-4<1$, ya'ni $1<x<3$ tengsizlikni qanoatlantiruvchi barcha qiymatlarida yaqinlashadi.

4-§. Teylor va Makloren qatorlari

Agar $f(x)$ funksiya

$$a_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots+a_n(x-x_0)^n+\ldots \tag{1}$$

qatorning yig'indisi bo'lsa, u holda $f(x)$ funksiya $x-x_0$ ning darajalari bo'yicha qatorga yoyiladi deyish mumkin.

1-teorema. Agar $f(x)$ funksiyani $(a-R;\,a+R)$ oraliqda

$$f(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots+a_n(x-x_0)^n+\ldots \tag{2}$$

qatorga yoyish mumkin bo'lsa, u holda bunday yoyilma yagona bo'ladi.

Isbot. Darajali qatorni hadma-had differensiallash haqidagi teoremaga asosan quyidagi ifodalarni yozish mumkin:

$$f'(x)=a_1+2a_2(x-x_0)+3a_3(x-x_0)^2+\ldots+$$
$$+na_n(x-x_0)^{n-1}+\ldots,$$
$$f''(x)=2a_2+2\cdot 3a_3(x-x_0)+\ldots+$$
$$+n(n-1)a_n(x-x_0)^{n-2}+\ldots,$$
$$f'''(x)=1\cdot 2\cdot 3a_3+2\cdot 3\cdot 4a_4(x-x_0)+\ldots+$$
$$+n(n-1)(n-2)a_n(x-x_0)^{n-3}+\ldots$$
. . . .
$$f^{(n)}(x)=1\cdot 2\cdot 3\cdot\ldots\cdot n(n-1)(n-2)a_n+\ldots.$$

Bu tengliklarga hamda (2) ga $x=x_0$ ni qo'yib noma'lum $a_0, a_1, a_2, \ldots, a_n, \ldots$ koeffitsientlarni topamiz.

$f(x_0)=a_0$, $f'(x_0)=a_1$, $f''(x_0)=1\cdot 2\cdot a_2$, $f'''(x_0)=1\cdot 2\cdot 3\cdot a_3$, . . ., $f^{(n)}(x_0)=1\cdot 2\cdot 3\cdot\ldots\cdot(n-2)(n-1)n\cdot a_n$

yoki

$$a_0=f(x_0),\quad a_1=f'(x_0),\quad a_2=\frac{f''(x_0)}{2!},$$
$$a_3=\frac{f'''(x_0)}{3!},\ \ldots,\ a_n=\frac{f^{(n)}(x_0)}{n!}. \tag{3}$$

Hosil bo'lgan koeffitsientlarni (2) qatorga qo'yib quyidagi qatorni hosil qilamiz:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots \tag{4}$$

(4) qatorni Teylor qatori deyiladi.

$$a_0=f(x_0),\quad a_1=f'(x_0),\quad a_2=\frac{f''(x_0)}{2!},\ldots\quad a_n=\frac{f^{(n)}(x_0)}{n!}$$

o'zgarmas sonlarga $f(x)$ funksiyaning $x=x_0$ nuqtadagi Teylor qatorining koeffitsientlari deyiladi.

Shunday qilib, agar $f(x)$ funksiyani $x=x_0$ ning darajalari bo'yicha yoyish mumkin bo'lsa, u holda bu qator albatta $f(x)$ funksiyaning Teylor qatori bo'ladi.

Agar Teylor qatorida $x_0 = 0$ desak, u holda Teylor qatorining xususiy holi kelib chiqadi.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \quad (5)$$

(5) qatorga Makloren qatori deyiladi.

Yuqorida qilingan mulohazalar $f(x)$ funksiyani darajali qatorga yoyish mumkin bo'lgandagina o'rinlidir.

Faraz qilaylik, $f(x)$ funksiyani $(x_0 - R;\, x_0 + R)$ oraliqda ixtiyoriy tartibli hosilalri mavjud bo'lsin. U xolda shu oraliqqa tegishli ixtiyoriy x ning qiymatlari va ixtiyoriy n lar uchun Teylor formulasi o'rinli bo'ladi:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots +$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x) \quad (6)$$

Bu erda

$$R_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1} \quad (7)$$

bu yerda $0 < \theta < 1$.

(7) ga Teylor formulasining Lagranj ko'rinishidagi qoldiq hadi deyiladi.

Endi $f(x)$ funksiya uchun tuzilgan

$$f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \quad (8)$$

Teylor qatori $(x_0 - R;\, x_0 + R)$ oraliqda yaqinlashuvchi va uning yig'indisi $f(x)$ ga teng bo'lish shartini aniqlaymiz.

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + .\ + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

bo'lsin.

$R_n(x)$ ko'phad qandaydir ma'noda $f(x)$ funksiyaga yaqinlashishini anglaydi. Buning yordamida $f(x)$ funksiyaning aniqlik darajasini aniqlash mumkin bo'ladi.

Bu holda

$$f(x)=P_n(x)+R_n(x)$$

yoki

$$R_n(x)=f(x)-P_n(x)$$

bo'ladi. (8) Teylor qatori $(x_0-R;\ x_0+R)$ oraliqda $f(x)$ funksiyaga intilishi uchun Teylor formulasining qoldiq hadi $n\to\infty$ da nolga intilishi kerak, ya'ni

$$\lim_{n\to\infty}R_n(x)=0.$$

Endi Teylor qatorining $f(x)$ funksiyaga yaqinlashish tartibini aniqlaymiz.

2-teorema. Agar $(x_0-R;\ x_0+R)$ oraliqda $f(x)$ funksiya ixtiyoriy tartibli hosilalarga ega bo'lib, hamda

$$\left|f^{(n)}(x)\right|\le\mu \qquad (9)$$

tenglik o'rinli bo'lsa, u holda shu oraliqda (4) yoyilma o'rinli bo'ladi.

Isbot.

$$R_n(x)=\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1}$$

va (9) lardan foydalanib, quyidagi tengsizlikni yozamiz:

$$\left|R_n(x)\right|=\left|\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1}\right|\le\mu\frac{\left|x-x_0\right|^{n+1}}{(n+1)!} \qquad (10)$$

Lekin

$$\sum_{n=0}^{\infty}\frac{\left|x-x_0\right|^{n+1}}{(n+1)!}$$

qator Dalamber alomatiga asosan yaqinlashuvchi bo'lgani uchun

$$\lim_{n\to\infty}\frac{\left|x-x_0\right|^{n+1}}{(n+1)!}=0$$

bo'ladi. Bundan (10) ni e'tiborga olsak, $(x_0-R;\ x_0+R)$ oraliqda $\lim\limits_{n\to\infty}R_n(x)=0$ o'rinli ekanligi kelib chiqadi.

Teorema isbot bo'ldi.

5-§. Elementar funksiyalarni darajali qatorga yoyish

1. $f(x)=e^x$ funksiyani Makloren qatoriga yoyish. Makloren qatorining koeffitsientlari

$$a_n=\frac{f^{(n)}(0)}{n!},\ (n=0,\ 1,\ 2,\ \ldots)$$

formula orqali ifodalanadi. Berilgan funksiyadan hosilalar olib, $x_0=0$ nuqtadagi qiymatlarini topamiz.

$$\begin{array}{ll} f(x)=e^x, & f(0)=1, \\ f'(x)=e^x, & f'(0)=1, \\ f''(x)=e^x, & f''(0)=1, \\ \ldots\ldots\ldots\ldots & \\ f^{(n)}(x)=e^x, & f^{(n)}(0)=1, \\ \ldots\ldots\ldots\ldots & \end{array}$$

Demak, $a_n=\frac{1}{n!}$, $(n=0,\ 1,\ 2,\ \ldots)$. Natijada, $f(x)=e^x$ funksiya uchun quyidagi qatorni hosil qilamiz:

$$e^x=1+x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n+\ldots\ . \qquad (1)$$

Bu qatorning yaqinlashish sohasini aniqlaymiz.

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{(n+1)!}{n!}=\lim_{n\to\infty}(n+1)=\infty\,.$$

Demak, (1) qator $(-\infty;\ \infty)$ oraliqda yaqinlashuvchi ekan.

Endi x ning har qanday qiymatida qatorning yig'indisi $f(x)=e^x$ ga teng ekanligini ko'rsatamiz. Buning uchun ixtiyoriy x lar uchun

$$\lim_{n\to\infty}R_n(x)=0$$

ekanligini isbotlash yetarli bo'ladi.

4-§ paragrafdagi 2-teoremaga asosan (9) tengsizlik ixtiyoriy $|x|<r$ oraliqda o'rinli bo'ladi, chunki $f(x)=e^x<e^r$. Shuning uchun x ning barcha qiymatlari uchun (1) yoyilma o'rinli bo'ladi.

2. $f(x)=\sin x$ funksiyani Makloren qatoriga yoyish. berilgan funksiyadan hosilalarini topamiz.

$$f(x)=\sin x$$

$$f'(x)=\cos x=\sin\left(x+\frac{\pi}{2}\right),$$

$$f''(x)=-\sin x=\sin\left(x+2\cdot\frac{\pi}{2}\right),$$

$$f'''(x)=-\cos x=\sin\left(x+3\cdot\frac{\pi}{2}\right),$$

$$f^{IV}(x)=\sin x=\sin\left(x+4\cdot\frac{\pi}{2}\right),$$

. . . .

$$f^{(n)}(x)=\sin\left(x+n\cdot\frac{\pi}{2}\right).$$

$x_0=0$ da $f(0)=0$, $f'(0)=1$, $f''(0)=0$, $f'''(0)=-1$, $f^{IV}(0)=0$ va hokazo. Demak, $a_0=0$, $a_1=1$, $a_2=0$, $a_3=-\frac{1}{3!}$, $a_4=0$, . . . , $a_n=\frac{1}{n!}\sin\frac{n\pi}{2}$. Bundan $n=2k$ bo'lganda $f^{(n)}(0)=\sin\frac{n\pi}{2}=0$, $n=2k+1$ bo'lganda esa $f^{(n)}(0)=(-1)^n$ bo'lishi ravshandir. $(-\infty;\infty)$ oraliqdagi barcha x lar uchun $\left|f^{(n)}(x)\right|\le 1$ tengsizlik o'rinlidir.

4-§ paragrafdagi 2-teoremaga asosan $(-\infty;\infty)$ oraliqdagi barcha x lar uchun quyidagi qator bo'ladi:

$$\sin x=x-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots .$$

3. $f(x)=\cos x$ funksiya ham x ning $(-\infty;\infty)$ oraliqdagi barcha qiymatlari uchun quyidagi qator o'rinli bo'ladi:

$$\cos x=1-\frac{x^2}{2!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots .$$

6-§. Binomial qator

$f(x)=(1+x)^m$ bo'lsin., bu yerda m - ixtiyoriy haqiqiy son.

Teylor qatorining koeffitsientlarini aniqlash uchun funksiyaning hosilalarini topamiz.

$$f(x)=(1+x)^m$$
$$f'(x)=m(1+x)^{m-1},$$
$$f''(x)=m(m-1)(1+x)^{m-2},$$
$$f'''(x)=m(m-1)(m-2)(1+x)^{m-3},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$f^{(n)}(x)=m(m-1)(m-2)\dots(m-n+1)(1+x)^{m-n}.$$

Bu hosilalarga $x_0=0$ qiymatni qo'yib quyidagi tengliklarni hosil qilamiz:

$$f(0)=1,\ f'(0)=m,\ f''(0)=m(m-1),\ \dots,$$
$$f^{(n)}(0)=m(m-1)(m-2)\dots(m-n+1).$$

Endi Teylor qatorining koeffitsientlarini hisoblaymiz.

$$a_0=1,\ a_1=m,\ a_2=\frac{m(m-1)}{2!},\ \dots,\ a_n=\frac{m(m-1)\dots(m-n+1)}{n!}.$$

U holda Teylor qatori quyidagi ko'rinishda bo'ladi:

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\dots+\frac{m(m-1)\dots(m-n+1)}{n!}x^n+\dots \qquad (1)$$

Bu qatorning yaqinlashish sohasini aniqlaymiz.

$$R=\lim_{n\to\infty}\left|\frac{m(m-1)\dots(m-n+1)\cdot(n+1)!}{n!\,m(m-1)\dots(m-n+1)(m-n)}\right|=\lim_{n\to\infty}\left|\frac{n+1}{m-n}\right|=1.$$

Demak (1) qator $|x|<1$ oraliqda yaqinlashuvchi ekan.

Endi (1) qatorning yig'indisi $f(x)=(1+x)^m$ funksiyaga teng ekanligini ko'rsatamiz. Buning uchun uning yig'indisini $S(x)$ deb belgilaymiz.

$$S(x)=1+mx+\frac{m(m-1)}{2!}x^2+\dots+\frac{m(m-1)\dots(m-n+1)}{n!}x^n+\dots$$

qatorni hadma-had differensiallaymiz.

$$S'(x)=m+2\cdot\frac{m(m-1)}{2!}x+3\cdot\frac{m(m-1)(m-2)}{3!}x^2+\ldots+$$
$$+n\cdot\frac{m(m-1)\ldots(m-n+1)}{n!}x^{n-1}+\ldots \tag{2}$$
(2) qatorni x ga ko'paytirib, quyidagi ifodani hosil qilamiz:
$$xS'(x)=mx+2\cdot\frac{m(m-1)}{2!}x^2+3\cdot\frac{m(m-1)(m-2)}{3!}x^3+$$
$$+\ldots+n\cdot\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+\ldots \tag{3}$$
(2) bilan (3) ni hadma-had qo'shamiz.
$$(1+x)S'(x)=m+\left(m+2\cdot\frac{m(m-1)}{2!}\right)x+\left(2\cdot\frac{m(m-1)}{2!}+\right.$$
$$\left.+3\cdot\frac{m(m-1)(m-2)}{3!}\right)x^2+\ldots+\left((n-1)\cdot\frac{m(m-1)\ldots(m-n+2)}{(n-1)!}+\right.$$
$$\left.+.n\cdot\frac{m(m-1)\ldots(m-n+1)}{n!}\right)x^n+\ldots \tag{4}$$
(4) qatordagi koeffitsientlarni quyidagicha yozamiz:
$$(n-1)\cdot\frac{m(m-1)\ldots(m-n+2)}{(n-1)!}+n\cdot\frac{m(m-1)\ldots(m-n+1)}{n!}=$$
$$\frac{m(m-1)\ldots(m-n+1)}{(n-1)!}((m-n+1)+(n-1))=m\cdot\frac{m(m-1)\ldots(m-n+1)}{(n-1)!}.$$
U holda
$$S'(x)(1+x)=m\left(1+mx+\frac{m(m-1)}{2!}x^2+\ldots++\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+\ldots\right)=mS(x)$$
ifoda hosil bo'ladi.

Natijada noma'lum $S(x)$ funksiya va uning hosilasi $S'(x)$ bilan bog'langan quyidagi differensial tenglamani hosil qildik.
$$S'(x)(1+x)=mS(x) \tag{5}$$
yoki
$$\frac{dS(x)}{dx}(1+x)=m\cdot S(x).$$
Bundan

$$\frac{dS(x)}{S(x)} = m \cdot \frac{dx}{1+x}.$$

Bu tenglamani integrallab, quyidagi yyechimni hosil qilamiz:

$$\ln S(x) = m \cdot \ln(1+x) + C. \tag{6}$$

Bu yerda C topilishi kerak bo'lgan son. Buning uchun $x=0$ da $S(0)=1$ shartni qo'yamiz. U holda $\ln S(0) = \ln 1 = 0$. (6) tenglikdan $C=0$ ekanligi kelib chiqadi.

Shunday qilib,

$$\ln S(x) = m \cdot \ln(1+x)$$

tenglikni hosil qildik. Buni potensirlab, qidirilayotgan funksiyani topamiz.

$$S(x) = (1+x)^m.$$

Demak, (1) qatorning yig'indisi oldindan berilgan $f(x) = (1+m)^m$ funksiyaga teng ekan.

Bu funksiyani $|x| < 1$ oraliqda quyidagi qatorga yoyish mumkin ekan:

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + \ldots \tag{7}$$

(7) darajali qatorga binomial qator deyiladi.

Xususiy holda agar $m=-1$ bo'lsa

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ \ldots \tag{8}$$

darajali qator hosil bo'ladi.

Agar $m = -\frac{1}{2}$ bo'lsa quyidagi qatorga ega bo'lamiz:

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \ \ldots\ . \tag{9}$$

$m = \frac{1}{2}$ bo'lganda

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \ \ldots$$

bo'ladi.

Binomial qatorni boshqa funksiyalarning yoyilmasiga tatbiq qilamiz.

$f(x) = \arcsin x$ funksiyani Makloren qatoriga yoyamiz.

(9) tenglikdagi x o'rniga $-x^2$ ni qo'yamiz:

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \ldots .$$

$|x| < 1$ bo'lganda darajali qatorlarni integrallash haqidagi teoremaga asosan quyidagini hosil qilamiz:

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \arcsin x = x + \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \ldots +$$

$$+ \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot 7\cdot \ldots \cdot 2n} \cdot \frac{x^{2n+1}}{2n+1} + \ldots .$$

7-§. $\ln(1+x)$ funksiyani darajali qatorga yoyish

6-§ paragrafdagi (8) tenglikni $|x| < 1$ bo'lganda 0 dan x gacha kesmada integrallasak, quyidagi qator hosil bo'ladi:

$$\int_0^x \frac{dx}{1+t} = \int_0^x \left(1 - t + t^2 - t^3 + \ldots\right)dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\frac{x^n}{n} + ..$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots \qquad (1)$$

Bu tenglik $(-1, 1)$ oraliqda o'rinlidir.

Agar (1) formuladagi x ni $-x$ ga almashtirsak $(-1, 1)$ oraliqda yaqinlashuvchi qator hosil bo'ladi.

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \qquad (2)$$

(1) va (2) formulalar yordamida 0 bilan 2 orasidagi sonlarning logarifmlarini hisoblash mumkin. $x = 1$ bo'lganda ham (1) formula o'rinli bo'ladi .

Ixtiyoriy butun sonlarning natural logarifmlarini hisoblash uchun formula chiqaramiz.

(1) qatorni (2) dan hadma-had ayiramiz.

$$\ln|1+x|-\ln|1-x|=\ln\left|\frac{1+x}{1-x}\right|=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right) \qquad (3)$$

(3) yoyilmadan foydalanib ixtiyoriy musbat sonning logarifmini ixtiyoriy aniqlikda hisoblash mumkin. Buning uchun berilgan z sonini $z=\dfrac{1+x}{1-x}$ ko'rinishda olish yetarli bo'ladi. Bundan

$$x=\frac{z-1}{z+1}.$$

Bu qiymat $|x|<1$ tengsizlikni qanoatlantiradi. Bundan z ning qanday bo'lishidan qat'iy nazar (3) qator $(-1,1)$ oraliqda yaqinlashuvchi bo'ladi. Masalan, $x=\dfrac{9}{11}$ ni (3) qatorga qo'yib yoyilmasini yozamiz.

$$\ln 10=\ln\frac{1+\frac{9}{11}}{1-\frac{9}{11}}=2\left(\frac{9}{11}+\frac{1}{3}\left(\frac{9}{11}\right)^3+\frac{1}{5}\left(\frac{9}{11}\right)^5+\ldots\right)\approx 2{,}3026.$$

8-§. Aniq integrallarni qatorlar yordamida hisoblash

Bu paragrafda ba'zi integrallanmaydigan aniq integrallarni qaraymiz.

1. $\int_0^a e^{-x^2}dx$ integralni hisoblash talab etiladi. Bunda e^{-x^2} funksiyaning boshlang'ich funksiyasi elementar funksiya bo'lmaydi.

Bu integralni hisoblash uchun

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

yoyilmadagi x ning o'rniga $-x^2$ ni qo'yamiz. Natijada

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n \frac{x^{2n}}{n!} + \ldots$$

qator hosil bo'ladi.

Bu tenglikning har ikkala tomonini 0 dan a gacha oraliqda integrallab, quyidagi qatorni hosil qilamiz:

$$\int_0^a e^{-x^2}dx = \int_0^a \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n \frac{x^{2n}}{n!} + \ldots\right)dx =$$

$$= \left(x - \frac{x^3}{1!3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + \ldots + (-1)^n \frac{x^{2n+1}}{n!(2n+1)} + \ldots\right)\Bigg|_0^a =$$

$$= \left(a - \frac{a^3}{1!3} + \frac{a^5}{2!5} - \frac{a^7}{3!7} + \ldots + (-1)^n \frac{a^{2n+1}}{n!\,(2n+1)} + \ldots\right)$$

Bu tenglik yordamida a ning ixtiyoriy qiymatida berilgan integralni ixtiyoriy darajadagi aniqlik bilan hisoblash mumkin bo'ladi.

2. $\int_0^a \frac{\sin x}{x}dx$ integralni hisoblash talab etiladi. Integral ostidagi funksiyani qatorga yoyamiz. Buning uchun $\sin x$ funksiyaning yoyilmasidan foydalanamiz.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots .$$

Bu tenglikdan

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots$$

qatorni hosil qilamiz. Bu qatorni hadlab integrallaymiz.

$$\int_0^a\frac{\sin x}{x}dx=\int_0^a\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\ldots\right)dx=\left(x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-\frac{x^7}{7\cdot 7!}+\ldots\right)\Bigg|_0^a$$

$$=a-\frac{a^3}{3\cdot 3!}+\frac{a^5}{5\cdot 5!}-\frac{a^7}{7\cdot 7!}+\ldots.$$

Bu yerda ham a ning har qiymatida qatorning yig'indisini ixtiyoriy darajadagi aniqlik bilan hisoblash mumkin.

9-§. Differensial tenglamalarni qatorlar yordamida yechish

Differensial tenglamalarni yechish ularni integrallashga keladi, lekin ayrim hosil bo'lgan integralni integrallash mumkin bo'lmaydi. Bunday hollarda tenglamani yechish uchun taqribiy usullardan foydalanish kerak bo'ladi. Bu usullardan biri tenglamaning yyechimini Teylor qatori ko'rinishida chekli sondagi hadlarining yig'indisi taqriban izlangan xususiy yyechimga teng bo'ladi.

Berilgan

$$y''=F(x,\,y,\,y') \qquad (1)$$

ikkinchi tartibli differensial tenglamaning berilgan boshlang'ich shartni qanoatlantiruvchi yechimini topish kerak bo'lsin.

Boshlang'ich shart

$$y\big|_{x=x_0}=y_0,\qquad y'\big|_{x=x_0}=y'_0\,. \qquad (2)$$

Faraz qilaylik, $y=f(x)$ yechim mavjud va uni Teylor qatoriga quyidagi ko'rinishda yoyish mumkin bo'lsin:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots\,. \qquad (3)$$

Biz xususiy yyechimdan olingan hosilalarning $x=x_0$ bo'lgandagi qiymatlarini, ya'ni

$$f(x_0),\ f'(x_0),\ f''(x_0),\ \ldots$$

larni topishimiz kerak.

Lekin buni (1) tenglama hamda (2) boshlang'ich shartlar yordamida bajarish mumkin. (2) boshlang'ich shartlardan foydalanib quyidagi tenglikni yozamiz:

$$f''(x_0)=y''\big|_{x=x_0}=F(x_0\,,\,y_0\,,\,y'_0). \qquad (4)$$

(1) tenglamaning ikkala tomonini x bo'yicha differensiallaymiz.

$$y'''=F_x'(x\,,\,y\,,\,y')+F_y'(x\,,\,y\,,\,y')\cdot y'+F_{yy}''(x\,,\,y\,,\,y')\cdot y'' \qquad (5)$$

$x=x_0$ qiymatni (5) tenglamaning o'ng tomoniga qo'yib, quyidagi tenglikni hosil qilamiz:

$$f'''(x_0)=y'''\big|_{x=x_0}.$$

(5) munosabatni yana bir marta x bo'yicha differensiallab, quyidagi tenglikka ega bo'lamiz:

$$f^{IV}(x_0)=y^{IV}\big|_{x=x_0}.$$

Bu topilgan hosilalarni (3) tenglikka qo'yamiz. Bu hosil bo'lgan qator x ning yaqinlashadigan qiymatlarida (3) qator berilgan differensial tenglamaning yechimi bo'ladi.

Misol. $y''=x^2+y^2$ tenglamaning $y\big|_{x=1}=1,\quad y'\big|_{x=1}=0$ boshlang'ich shartlarni qanoatlantiruvchi yechimini toping.

Yechish. Yechimni quyidagi ko'rinishda izlaymiz:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots \qquad (6)$$

Bizning misolimizda $x_0=1$. Endi $f(x_0),\ f'(x_0),\ f''(x_0),\ \ldots$ larning qiymatlarini topamiz.

$f(1)=1,\qquad f'(1)=0.$

$y''=x^2+y^2,\quad y'\big|_{x=1}=f''(1)=2.$

$y'''=2x+2yy',\quad y''\big|_{x=1}=f''(1)=2.$

$y^{IV}=2+2y'^2+2yy'',\quad y^{IV}\big|_{x=1}=f^{IV}(1)=6.$

$y^{V}=6y'y''+2yy''',\quad y^{V}\big|_{x=1}=f^{V}(1)=4.$

. . . .

Bu topilgan qiymatlarni (6) ga qo'yamiz.

$$y=f(x)=1+\frac{2}{2!}(x-1)^2+\frac{2}{3!}(x-1)^3+\frac{6}{4!}(x-1)^4+\frac{4}{5!}(x-1)^5+\ldots=$$

$$=1+(x-1)^2+\frac{(x-1)^3}{3}+\frac{(x-1)^4}{4}+\frac{(x-1)^5}{30}+\ldots$$

Agar berilgan tenglama chiziqli bo'lsa, xususiy yechim koeffitsientlarini noma'lum koeffitsientlar usuli bo'yicha topish qulay bo'ladi. U holda

$$y=f(x)=a_0+a_1x+a_2x^2+\ \ldots+a_nx^n+\ldots \tag{7}$$

qatorni berilgan differensial tenglamaga qo'yib, tenglamaning turli qismlaridagi x larning bir xil darajalari oldidagi koeffitsientlarni tenglash kerak bo'ladi.

Misol. $y''=2xy'+4y$ tenglamaning $y|_{x=0}=0$, $y'|_{x=0}=1$ boshlang'ich shartlarni qanoatlantiruvchi yechimini toping.

Yechish. Faraz qilaylik

$$y=a_0+a_1x+a_2x^2+\ \ldots+a_nx^n+\ldots$$

berilgan bo'lsin.

Boshlang'ich shartlarga asosan

$$a_0=0,\ a_1=1.$$

Demak,

$$y=x+a_2x^2+\ \ldots+a_nx^n+\ldots$$

$$y'=1+2a_2x+3a_3x^2+\ \ldots+na_nx^{n-1}+\ldots$$

$$y''=2a_2+3\cdot 2a_3x+\ \ldots+n(n-1)a_nx^{n-2}+\ldots.$$

Bu ifodalarni berilgan tenglamaga qo'yib hamda x larning bir xil darajalari oldidagi koeffitsientlarni tenglab, quyidagilarni hosil qilamiz:

$2a_2=0$, bundan $a_2=0$.

$3\cdot 2\cdot a_3=2+4$, bundan $a_3=1$.

$4\cdot 3\cdot 2\cdot a_4=4a_2+4a_2$, bundan $a_4=0$.

. . . .

$n(n-1)a_n=(n-2)2a_{n-2}+4a_{n-2}$, bundan $a_n=\dfrac{2a_{n-2}}{n-1}$.

. . . .

Demak,

$$a_5=\frac{2\cdot 1}{4}=\frac{1}{2!},\ a_7=\frac{2\cdot\frac{1}{2}}{6!}=\frac{1}{3!},\ a_9=\frac{1}{4!},\ldots,\ a_{2k+1}=\frac{2\cdot\frac{1}{(k-1)!}}{2k}=\frac{1}{k!},\ldots$$

$a_2=0$, $a_4=0$, . . . , $a_{2k}=0$.

Bu qiymatlarni o'rniga qo'yib, izlanayotgan yechimni topamiz.

$$y=f(x)=x+\frac{x^3}{1!}+\frac{x^5}{2!}+\frac{x^7}{3!}+\ldots+\frac{x^{2k+1}}{k!}+\ldots.$$

Bu qator x ning barcha qiymatlarida yaqinlashuvchi bo'ladi. Bu xususiy yechimni elementar funksiyalar orqali ifodalash mumkin. Haqiqatan, agar x ni qavsdan tashqariga chiqarsak, qavs ichida e^{x^2} funksiyaning yoyilmasi hosil bo'ladi.

$$y=x\left(1+x^2+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}+\ldots+\frac{x^{2k}}{k!}+\ldots\right)=xe^{x^2}.$$

10-§. Qatorlar yordamida taqribiy hisoblash

Darajali qatorlar juda kuchli hisoblash vositasidir. Ular yordamida funksiyalarning taqribiy qiymatlarini, integrallanmaydigan integrallarni va differensial tenglamalarning taqribiy xususiy yechimlarini hisoblash mumkin.

1-misol. $e^{0,2}$ ni 0,0001 aniqlikda qiymatini hisoblang.

Yechish. Quyidagi qatordan foydalanamiz:

$$e^x=1+x+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots. \tag{1}$$

Bu qatordagi x larni o'rniga 0,2 qiymatini qo'yamiz.

$$e^{0,2}=1+0,2+\frac{(0,2)^2}{2!}+\frac{(0,2)^3}{3!}+\frac{(0,2)^4}{4!}+\ldots.$$

Bu qatorning 5-hadidan boshlab qolgan hadlarini tashlab yuboramiz. Endi qilingan xatoni baholaymiz.

$$r_n=\frac{(0,2)^4}{4!}+\frac{(0,2)^5}{5!}+\frac{(0,2)^6}{6!}+\ldots=\frac{(0,2)^4}{4!}\left(1+\frac{0,2}{5}+\frac{(0,2)^2}{5\cdot 6}+\ldots\right)<$$

$$<\frac{(0,2)^4}{4!}\left(1+\frac{0,2}{5}+\left(\frac{0,2}{5}\right)^2+\ldots\right)=\frac{0,0016}{24}\cdot\frac{1}{1-\frac{0,2}{5}}<0,0001\cdot$$

Demak,

$$e^{0,2}\approx 1+0,2+\frac{(0,2)^2}{2!}+\frac{(0,2)^3}{3!}+\frac{(0,2)^4}{4!}=1,2+\frac{0,04}{2}+\frac{0,008}{6}=1,2213\,.$$

2-misol. $\int\limits_0^{1/4} e^{-x^2}dx$ integralni 0,0001 gacha aniqlik bilan hisoblang.

Yechish. (1) qatordagi x larni o'rniga $-x^2$ ni qo'yib, quyidagi qatorni hosil qilamiz:

$$e^{-x^2}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots\,.$$

Bu qatorning har ikkala tomonini $\left[0,\frac{1}{4}\right]$ kesmada integrallaymiz.

$$\int\limits_0^{1/4} e^{-x^2}dx=\int\limits_0^{1/4}dx-\frac{1}{1!}\int\limits_0^{1/4}x^2dx+\frac{1}{2!}\int\limits_0^{1/4}x^4dx-\frac{1}{3!}\int\limits_0^{1/4}x^6dx+\ldots=$$

$$=0,25-\frac{1}{3\cdot 4^3}+\frac{1}{10\cdot 4^5}-\frac{1}{42\cdot 4^7}+\ldots\,. \qquad (2)$$

Bu qator ishoralari navbatlashuvchi qator bo'lgani uchun Leybnis teoremasining shartlarini qanoatlantiradi.

$$\frac{1}{10\cdot 4^5}=\frac{1}{10240}<0,0001$$

bo'lgani uchun faqat (2) qatorning ikkita hadini olish kifoya bo'ladi.

$$\int\limits_0^{1/4} e^{-x^2}dx=0,25-\frac{1}{3\cdot 4^3}=0,25-0,0052=0,2448\,.$$

IV BOB. FURYE QATORI

1-§. Boshlang'ich ma'lumotlar

Har xil texnik masalalarni yechishda aniq vaqt oralig'ida davri qayta-qayta takrorlanadigan hodisalarga to'g'ri kelamiz. Masalan, o'zgaruvchi tok bilan bog'langan yoki muayyan harakatda bo'lgan ichki yonar dvigatellar hodisalari misol bo'lishi mumkin.

Bunday davriy jarayonni xarakterlaydigan miqdor vaqtga bog'liq bo'lgan davriy $F(t)$ funksiya bo'ladi. Bu funksiya uchun

$$F(t+T)=F(t)$$

tenglik o'rinli bo'ladi.

Bu yerda T - vaqtning davri. Vaqt o'tishi bilan qaralayotgan miqdor o'zining boshlang'ich qiymatini oladi. O'zgarmas sonni hisobga olmaganda eng oddiy davriy funksiya sinusoidal funksiyadir. Bu funksiya quyidagi ko'rinishga ega:

$$y=A\sin(\omega t+\varphi),$$

bu yerda A - amplituda, φ - boshlang'ich faza, ω esa T bilan bog'langan ($\omega=\frac{2\pi}{T}$ yoki $T=\frac{2\pi}{\omega}$) chastota.

Agar $y_0=A_0,\ y_1=A_1\sin(\omega t+\varphi_1),\ y_2=A_2\sin(\omega t+\varphi_2),\ .\ .\ .\ .\ .\ ,\ y_k=A_k\sin(\omega t+\varphi_k),\ .\ .\ .$ davriy funksiyalarni qo'shsak, natijada

$$F(t)=A_0+A_1\sin(\omega t+\varphi_1)+A_2\sin(2\omega t+\varphi_2)+$$
$$+\ldots+A_k\sin(k\omega t+\varphi_k)+\ldots \qquad (1)$$

funksiyani hosil qilamiz.

Bularning chastotalari

$$\omega,\ 2\omega,\ 3\omega,\ \ldots,\ k\omega,\ \ldots,$$

davrlari esa mos ravishda

$$T,\ \frac{1}{2}T,\ \frac{1}{3}T,\ \ldots,\ \frac{1}{k}T,\ \ldots$$

larga teng bo'ladi. (1) funksiya ham T davrga ega bo'ladi.

Davri T bo'lgan funksiyani quyidagi trigonometrik qator ko'rinishida yozish mumkinmi degan teskari savol tug'iladi, ya'ni

$$F(t)=A_0+\sum_{k=1}^{\infty}A_k\sin(k\omega t+\varphi_k). \qquad (2)$$

Bu masalani yechish uchun avval $\omega t=x$ deb olamiz. Bundan $t=\frac{x}{\omega}$ ni topib, hamda bundan foydalanib

$$F(t)=F\left(\frac{x}{\omega}\right)=f(x)$$

tenglikni yozamiz, bu yerda $f(x)$ funksiya $T=2\pi$ davrga ega. U holda (2) tenglik quyidagi ko'rinishga ega bo'ladi:

$$f(x)=A_0+\sum_{k=1}^{\infty}A_k\sin(kx+\varphi_k). \qquad (3)$$

Endi

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

formuladan foydalanib, $A_k\sin(kx+\varphi_k)$ ni quyidagicha yozamiz:

$A_k\sin(kx+\varphi_k)=A_k\cos\varphi_k\sin kx+A_k\sin\varphi_k\cos kx$

φ_k $(k=1,2,3,...)$ - o'zgarmas bo'lgani uchun $A_k\sin\varphi_k=a_k$, $A_k\cos\varphi_k=b_k$ deb olamiz. $A_0=\frac{a_0}{2}$ deb belgilaymiz. U holda (3) tenglik quyidagi ko'rinishga ega bo'ladi:

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx). \qquad (4)$$

(3) yoki (4) munosabatga quyidagi mexanik ma'noni berish mumkin: $f(x)$ funksiya bilan ifodalanayotgan murakkab tebranish ikkita oddiy garmonik tebranishlar yig'indisi ko'rinishida ifodalanar ekan. (4) ning har bir qo'shiluvchilarini garmonikalar, funksiyaning yoyilmasini esa garmonik analiz deyiladi.

2-§. Furye koeffitsientlari

Quyidagi masalani qo'yamiz: qanday shartlar bajarilganda davri 2π bo'lgan $f(x)$ funksiyani

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos kx+b_k\sin kx\right) \tag{1}$$

qatorga yoyish mumkin.

Faraz qilaylik, $f(x)$ funksiya $[-\pi;\pi]$ kesmada uzluksiz, hamda $T=2\pi$ davrga ega bo'lsin.

a_0, a_k va b_k $(k=1,2,3,...)$ koeffitsientlarni topish talab etiladi. Buning uchun avval bir necha yordamchi munosabatlarni yozamiz.

$$\int_{-\pi}^{\pi}\sin kx\,dx=-\frac{1}{k}\cos kx\Big|_{-\pi}^{\pi}=-\frac{1}{k}\left(\cos k\pi-\cos k(-\pi)\right)=0. \tag{2}$$

Ixtiyoriy $k\neq 0$ da $\cos x$ juft bo'lgani uchun, bu munosabat o'rinli bo'ladi.

Ixtiyoriy butun $k\neq 0$ da

$$\int_{-\pi}^{\pi}\cos kx\,dx=\frac{1}{k}\sin kx\Big|_{-\pi}^{\pi}=\frac{1}{k}\left(\sin k\pi-\sin k(-\pi)\right)=0 \tag{3}$$

bo'ladi.

Ixtiyoriy k va m ning butun qiymatlarida

$$\int_{-\pi}^{\pi}\sin kx\cos mx\,dx=0$$

bo'ladi.

$$\int_{-\pi}^{\pi}\sin kx\cdot\sin mx\,dx=\begin{cases}0, & \text{агар } k\neq m\ (k,m-\text{бутун сон})\\ \pi, & \text{агар } k=m\neq 0\end{cases} \tag{4}$$

$$\int_{-\pi}^{\pi}\cos kx\cdot\cos mx\,dx==\begin{cases}0, & \text{агар } k\neq m\ (k,m-\text{бутун сон})\\ \pi, & \text{агар } k=m\neq 0\end{cases} \tag{5}$$

Endi $k=m$ bo'lsin.

$$\int_{-\pi}^{\pi}\sin kx\cos mx\,dx=\frac{1}{2}\int_{-\pi}^{\pi}\sin 2kx\,dx=0,$$

$$\int_{-\pi}^{\pi}\sin kx\cdot\sin mx\,dx=\int_{-\pi}^{\pi}\sin^2 kx\,dx=\int_{-\pi}^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2kx\right)dx=$$

$$=\frac{1}{2}\left(x-\frac{1}{2k}\sin 2kx\right)\Bigg|_{-\pi}^{\pi}=\pi\cdot$$

Qolganlari ham shunga o'xshash isbot qilinadi.

Faraz qilaylik, (1) qatorni $[-\pi;\pi]$ kesmada integrallash mumkin bo'lsin.

Integrallash quyidagini beradi:

$$\int_{-\pi}^{\pi}f(x)dx=\int_{-\pi}^{\pi}\frac{a_0}{2}dx+\sum_{k=1}^{\infty}\left(\int_{-\pi}^{\pi}a_k\cos kx\,dx+\int_{-\pi}^{\pi}b_k\sin kx\,dx\right)=$$

$$=\frac{a_0}{2}\int_{-\pi}^{\pi}dx+\sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kx\,dx+b_k\int_{-\pi}^{\pi}\sin kx\,dx\right)=\frac{a_0}{2}\cdot 2\pi=a_0\pi.$$

Bu yerda birinchi qo'shiluvchidan boshqalari nolga teng bo'ladi.

$$\int_{-\pi}^{\pi}f(x)dx=a_0\pi$$

tenglikdan

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx \qquad (6)$$

kelib chiqadi. a_k koeffitsientni topish uchun (1) qatorning har ikkala tomonini $\cos mx$ ga ko'paytirib, keyin $[-\pi;\pi]$ kesmada integrallaymiz.

$$\int_{-\pi}^{\pi}f(x)\cos mx\,dx=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx\,dx+$$

$$+\sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kx\cos mx\,dx+b_k\int_{-\pi}^{\pi}\sin kx\cos mx\,dx\right)$$

$k=m$ bo'lganda bu tenglikning o'ng tomonidagi birinchi va uchinchi qo'shiluvchilar nolga teng bo'ladi. Natijada,

$$\int_{-\pi}^{\pi}f(x)\cos kx\,dx=a_k\int_{-\pi}^{\pi}\cos kx\cos kx\,dx=a_k\pi.$$

$$\int_{-\pi}^{\pi} f(x)\cos kx\, dx = a_k \pi$$

tenglikdan

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx \tag{7}$$

hosil bo'ladi, bu yerda $k = 1, 2, 3, \ldots$. b_k koeffitsientni topish uchun (1) qatorning har ikkala tomonini $\sin mx$ ga ko'paytirib, keyin $[-\pi; \pi]$ kesmada integrallaymiz.

$$\int_{-\pi}^{\pi} f(x)\sin mx\, dx = \frac{a_0}{2}\int_{-\pi}^{\pi} \sin mx\, dx +$$

$$+\sum_{k=1}^{\infty}\left(a_k \int_{-\pi}^{\pi}\cos kx \sin mx\, dx + b_k \int_{-\pi}^{\pi}\sin kx \sin mx\, dx\right)$$

$k = m$ bo'lganda bu tenglikning o'ng tomonidagi birinchi va ikkinchi qo'shiluvchilar nolga teng bo'ladi. Natijada,

$$\int_{-\pi}^{\pi} f(x)\sin kx\, dx = b_k \pi .$$

Bundan

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx \tag{8}$$

kelib chiqadi, bu yerda $k = 1, 2, 3, \ldots$. (6), (7), va (8) formulalar yordamida topilgan a_0, a_k va b_k larni Furye koeffitsientlari deyiladi, (1) qatorni esa $f(x)$ funksiya uchun Furye qatori deyiladi.

3-§. Furye qatorining yaqinlashishi

Furye koeffitsientlarini topishda biz trigonometrik qatorni yaqinlashuvchi va uning yig'indisi berilgan $f(x)$ funksiyaga teng deb faraz qilgan edik.

Endi Furye qatoriga yoyilgan $f(x)$ funksiya qanday shartlar bajarilganda yaqinlashuvchi bo'lishi va uning yig'indisi $f(x)$ funksiyaga teng bo'lishini aniqlashimiz kerak.

Yangi tushunchalarni kiritamiz.

Agar $[a;b]$ kesmani $a_0 = a, a_1, a_2, \quad , \ldots , a_n = b$ nuqtalar bilan chekli sondagi oraliqlarga ajratish mumkin bo'lib, hamda $f(x)$ funksiya har bir oraliqda monoton o'suvchi yoki monoton kamayuvchi bo'lsa, $f(x)$ funksiyani $[a;b]$ kesmada bo'lakli monoton funksiya deyiladi.

Agar $f(x)$ funksiya:

a) $[a;b]$ kesmada uzluksiz yoki unda chekli sondagi birinchi tur uzilish nuqtalariga ega bo'lsa,

b) $[a;b]$ kesmada bo'lakli monoton bo'lsa, u holda $f(x)$ funksiya $[a;b]$ kesmada Dirixle shartlarini qanoatlantiradi deyiladi.

Quyidagi teoremani isbotsiz keltiramiz:

Dirixle teoremasi. Agar $[-\pi;\pi]$ kesmada $f(x)$ funksiya Dirixle shartlarini qanoatlantirsa, u holda $f(x)$ funksiyaning Furye qatori $[-\pi;\pi]$ kesmada yaqinlashuvchi va uning yig'indisi uzluksiz nuqtalarda $f(x)$ funksiyaga teng bo'ladi.

x_0 uzilish nuqtalarida

$$\frac{f(x_0-0)+f(x_0+0)}{2},$$

$[-\pi;\pi]$ kesmaning chetki nuqtalarida esa

$$\frac{f(\pi-0)+f(\pi+0)}{2}$$

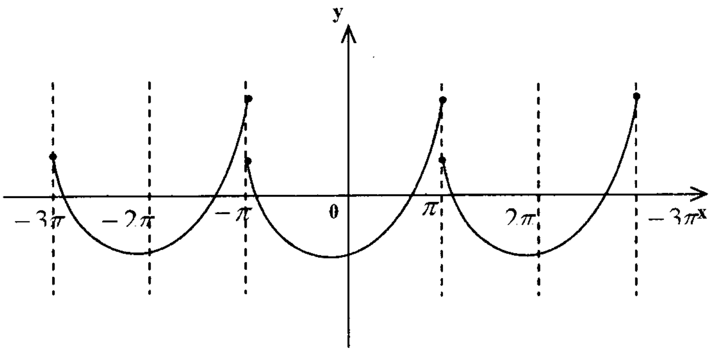
ga teng bo'ladi.

Dirixle teoremasiga oid izohni eslatib o'tamiz.

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos kx+b_k\sin kx\right) \qquad (1)$$

qatorning hadlari davri 2π ga teng bo'lgan funksiyalardir.

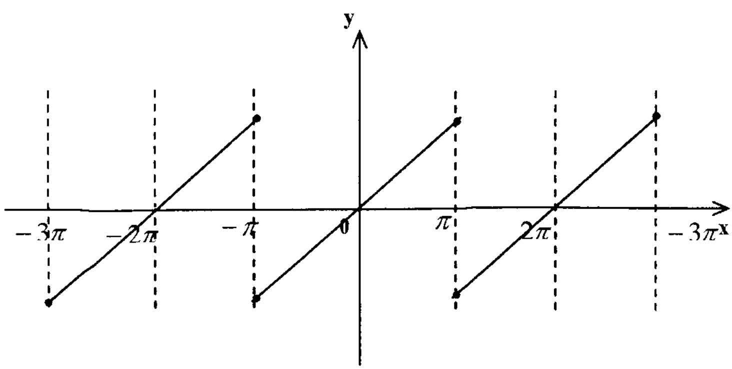
Shuning uchun, agar (1) qator $[-\pi;\pi]$ kesmada yaqinlashuvchi bo'lsa, u holda x ning barcha haqiqiy qiymatlarida yaqinlashuvchi va uning yig'indisi $[-\pi;\pi]$ kesmadagi avvalgi qiymatlari, davri 2π bo'lgan qiymatlar bilan takrorlanadi (1-rasm).



1-rasm.

Misol. Davri 2π bo'lgan $f(x)=\dfrac{x}{2}$ funksiyani $[-\pi;\pi]$ kesmada Furye qatoriga yoying.

Yechish. Bu funksiya Dirixle shartlarini qanoatlantiradi (2-rasm).



2-rasm.

Demak, Furye qatoriga yoyish mumkin. a_0, a_k va b_k koeffitsientlarni topamiz.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x}{2}dx = \frac{1}{2\pi}\cdot\frac{x^2}{2}\Bigg|_{-\pi}^{\pi} = 0.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x}{2}\cos kx\,dx = \begin{bmatrix} u = x,\ du = dx \\ dv = \cos kx\,dx \\ v = \frac{1}{k}\sin kx \end{bmatrix} =$$

$$= \frac{1}{2\pi}\left[\frac{x}{k}\sin kx\Bigg|_{-\pi}^{\pi} - \frac{1}{k}\int_{-\pi}^{\pi}\sin kx\,dx\right] = \frac{1}{2k^2\pi}\cos kx\Bigg|_{-\pi}^{\pi} =$$

$$= \frac{1}{2k^2\pi}\left(\cos k\pi - \cos k\pi\right) = 0.$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x}{2}\sin kx\,dx =$$

$$= \begin{bmatrix} u = x,\ du = dx \\ dv = \sin kx\,dx \\ v = -\frac{1}{k}\cos kx \end{bmatrix} = \frac{1}{2\pi}\left[-\frac{x}{k}\cos kx\Bigg|_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos kx\,dx\right] =$$

$$= \frac{1}{2k}\left(-\cos k\pi - \cos k\pi\right) = (-1)^{k+1}\frac{1}{k}.$$

Berilgan funksiya uchun Furye qatori quyidagi ko'rinishda bo'ladi:

$$\frac{x}{2} = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}\sin kx.$$

4-§. Juft va toq funksiyalarni Furye qatoriga yoyish

Faraz qilaylik $\varphi(x)$ funksiya juft, ya'ni $\varphi(-x)=\varphi(x)$ bo'lsin. U holda

$$\int_{-\pi}^{\pi} \varphi(x)\, dx = 2\int_{0}^{\pi} \varphi(x) dx \tag{1}$$

tenglik o'rinli bo'ladi.

Haqiqatan,

$$\int_{-\pi}^{\pi} \varphi(x)\, dx = \int_{-\pi}^{0} \varphi(x)\, dx + \int_{0}^{\pi} \varphi(x)\, dx = \int_{0}^{\pi} \varphi(-x) dx +$$

$$+\int_{0}^{\pi} \varphi(x)\, dx = \int_{0}^{\pi} \varphi(x) dx + \int_{0}^{\pi} \varphi(x)\, dx = 2\int_{0}^{\pi} \varphi(x)\, dx .$$

Endi $\varphi(x)$ funksiya toq funksiya, ya'ni $\varphi(-x)=-\varphi(x)$ o'rinli bo'lsin.

$$\int_{-\pi}^{\pi} \varphi(x)\, dx = \int_{-\pi}^{0} \varphi(x)\, dx + \int_{0}^{\pi} \varphi(x)\, dx = \int_{0}^{\pi} \varphi(-x) dx +$$

$$+\int_{0}^{\pi} \varphi(x)\, dx = -\int_{0}^{\pi} \varphi(x) dx + \int_{0}^{\pi} \varphi(x)\, dx = 0.$$

Demak, $\varphi(x)$ toq funksiya bo'lganda

$$\int_{-\pi}^{\pi} \varphi(x)\, dx = 0 \tag{2}$$

bo'lar ekan.

$f(x)$ funksiya juft bo'lganda, (1) formulaga asosan

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\, dx$$

bo'ladi.

$f(x)$ va $\cos kx$ funksiyalar juft bo'lgani uchun, $f(x)\cdot \cos kx$ ko'paytma ham juft bo'ladi. Demak, (1) formulaga asosan

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos kx\, dx$$

Formulani hosil qilamiz.

$f(x)$ juft, $\sin kx$ funksiya toq bo'lgani uchun, $f(x)\cdot \sin kx$ ko'paytma ham toq bo'lib, u holda (2) formulaga asosan quyidagi formulani yozish mumkin:

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx = 0.$$

Endi $f(x)$ funksiya toq bo'lsin. U holda (2) formulaga asosan

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = 0$$

bo'ladi.

$f(x)$ toq, $\cos kx$ funksiya esa juft bo'lgani uchun, $f(x)\cdot \cos kx$ ko'paytma ham toq bo'lib, u holda (2) formulaga asosan

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx = 0$$

tenglikka ega bo'lamiz.

$f(x)$ va $\sin kx$ funksiyalar toq bo'lgani uchun, $f(x)\cdot \sin kx$ ko'paytma juft bo'ladi, shuning uchun (1) formulaga asosan quyidagi formulani hosil qilamiz:

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin kx\, dx\,.$$

Shunday qilib, agar $f(x)$ juft bo'lsa, Furye qatori faqat kosinuslardan iborat bo'ladi, ya'ni

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx\,.$$

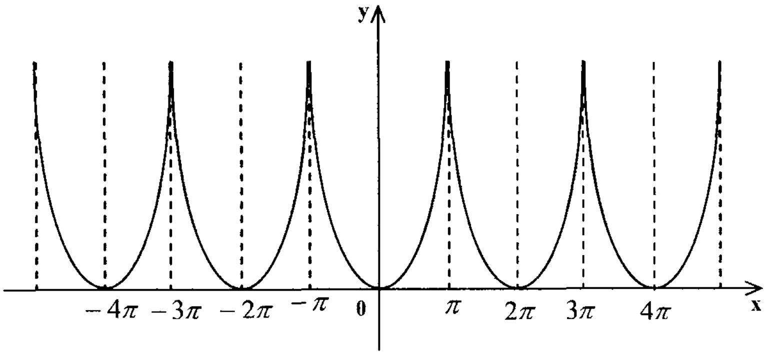
$f(x)$ funksiya toq bo'lsa, Furye qatori faqat sinuslardan iborat bo'ladi, ya'ni

$$f(x)=\sum_{k=1}^{\infty} b_k \sin kx .$$

Keltirilgan formulalar berilgan funksiyani juft yoki toqligiga qarab Furye koeffitsientlarin topishda hisoblashlarni ancha osonlashtiradi.

Misol. Davri 2π bo'lgan $f(x)=x^2$ funksiyani $-\pi<x<\pi$ oraliqda Furye qatoriga yoying.

Yechish. Berilgan funksiya Dirixle shartlarini qanoatlantiradi (3-rasm).



3-rasm.

$f(x)=x^2$ juft bo'lgani uchun a_0 va a_k koeffitsientlarni topamiz, b_k esa nolga teng bo'ladi.

$$a_0=\frac{2}{\pi}\int_0^{\pi} x^2\,dx=\frac{2}{\pi}\cdot\frac{x^3}{3}\Bigg|_0^{\pi}=\frac{2\pi^2}{3},$$

$$a_k=\frac{2}{\pi}\int_0^{\pi} x^2\cos kx\,dx=\begin{bmatrix} u=x^2,\ du=2xdx \\ dv=\cos kx\,dx,\ v=\frac{1}{k}\sin kx \end{bmatrix}=$$

$$=\frac{2}{\pi}\left[\frac{x^2}{k}\sin kx\Bigg|_0^{\pi}-\frac{2}{k}\int_0^{\pi} x\sin kx\,dx\right]=-\frac{4}{k\pi}\int_0^{\pi} x\sin kx\,dx=$$

$$= \begin{bmatrix} u = x, \; du = dx \\ dv = \sin kx dx, v = -\frac{1}{k}\cos kx \end{bmatrix} = -\frac{4}{k\pi}\left[-\frac{x}{k}\cos kx\Big|_0^{\pi} + \frac{1}{k}\int_0^{\pi}\cos kx\,dx\Big| \right] =$$

$$= \frac{4}{k^2\pi}\pi\cos k\pi + \frac{4}{k^2\pi}\cdot\frac{1}{k}\sin kx\Big|_0^{\pi} = (-1)^k\frac{4}{k^2}\cdot$$

Shunday qilib, quyidagi qatorni hosil qilamiz:

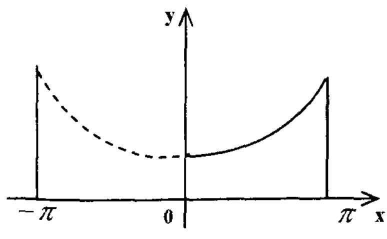
$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}(-1)^k\frac{\cos kx}{k^2}.$$

5-§. Davriy bo'lmagan funksiyalarni Furye qatoriga yoyish

Berilgan $f(x)$ funksiyani $[0, \pi]$ kesmada kosinuslar yoki sinuslar bo'yicha Furye qatoriga yoyish masalasi qo'yiladi. $f(x)$ funksiyani kosinuslar yoki sinuslar bo'yicha Furye qatoriga yoyish uchun quyidagicha mulohaza yuritiladi.

Berilgan $f(x)$ funksiyani $-\pi \le x < 0$ oraliqda davom ettirib, keyin uni butun son o'qiga 2π davr bilan davriy davom ettiriladi. Berilgan funksiyani juft yoki toq ravishda quyidagicha davom ettiriladi

$f(x)$ funksiyani kosinuslar bo'yicha Furye qatoriga yoyish uchun, funksiyani $-\pi \le x < 0$ oralikda juft, ya'ni $f(-x) = f(x)$ tarzda davom ettiriladi. Natijada juft funksiya hosil bo'ladi. U holda $f(x)$ funksiyani $[0, \pi]$ kesmadan $[-\pi, 0]$ kesmaga juft ravishda ixtiyoriy davom ettirilgan deyiladi (4-rasm).



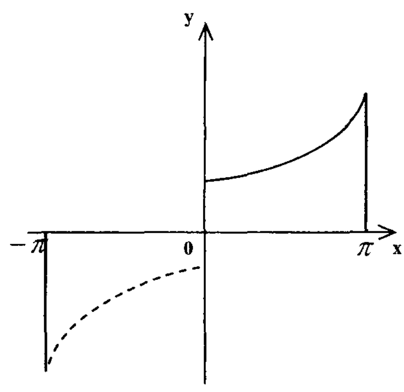
4-rasm.

Davom ettirilgan juft funksiya uchun faqat a_0 va a_k koeffitsientlar topiladi, $b_k = 0$ bo'ladi.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx\,, \quad a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos kx\, dx\ .$$

Bu formulalarda $f(x)$ funksiyaning $[0, \pi]$ dagi qiymatlari qatnashgani uchun funksiyani amalda juft tarzda davom ettirish shart emas.

Agar $f(x)$ funksiyani sinuslar bo'yicha Furye qatoriga yoyish kerak bo'lsa, funksiyani toq, ya'ni $f(-x) = -f(x)$ tarzda davom ettiriladi. Natijada toq funksiya hosil bo'ladi. U holda $f(x)$ funksiyani $[0, \pi]$ kesmadan $[-\pi, 0]$ kesmaga toq ravishda ixtiyoriy davom ettirilgan deyiladi (5-rasm).



5-rasm.

Bu holda funksiya toq bo'lgni uchun $a_0 = 0$, $a_k = 0$ bo'ladi, faqat b_k topiladi.

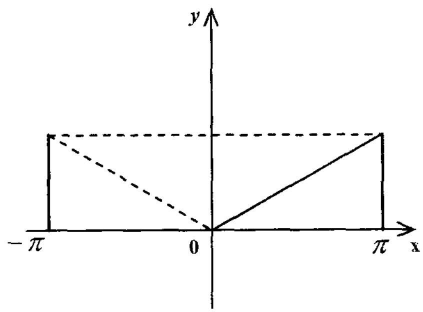
$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx\, dx\ .$$

Bu yerda ham $f(x)$ funksiyaning $[0,\pi]$ dagi qiymatlari qatnashgani uchun funksiyani $[0,\pi]$ kesmadan $[-\pi,0]$ kesmaga davom ettirmasa ham bo'ladi.

Endi quyidagi mulohazani eslatib o'tamiz: agar $f(x)$ funksiyani juft tarzda davom ettirilsa, Furye qatorida faqat kosinuslar, toq tarzda davom ettirilsa, faqat sinuslar qatnashadi.

Misol. $f(x)=\frac{x}{2}$ funksiyani $[0,\pi]$ kesmada kosinuslar bo'yicha Furye qatoriga yoying.

Yechish. Bu funksiyani juft ravishda $[-\pi,0]$ kesmaga davom ettiramiz (6-rasm).



6-rasm.

Funksiya juft tarzda davom ettirilgani uchun u juft bo'ladi. Shuning uchun, a_0 va a_k koeffitsientlarni topamiz. $b_k=0$ bo'ladi.

$$a_0=\frac{2}{\pi}\int_0^{\pi}\frac{x}{2}\,dx=\frac{1}{\pi}\cdot\frac{x^2}{2}\bigg|_0^{\pi}=\frac{\pi}{2},$$

$$a_k=\frac{2}{\pi}\int_0^{\pi}\frac{x}{2}\cos kx\,dx=\frac{1}{\pi}\int_0^{\pi}x\cos kx\,dx=\begin{bmatrix} u=x,\ du=dx \\ dv=\cos kx\,dx,\ v=\frac{1}{k}\sin kx \end{bmatrix}=$$

$$=\frac{1}{\pi}\left[\frac{x}{k}\sin kx\bigg|_0^{\pi}-\frac{1}{k}\int_0^{\pi}\sin kx\,dx\right]=-\frac{1}{k\pi}\cdot\left(-\frac{1}{k}\cos kx\right)\bigg|_0^{\pi}=$$

$$=\frac{1}{k^2\pi}(\cos k\pi-\cos 0)=(-1)^{k+1}\frac{2}{k^2\pi}.$$

$$\frac{x}{2}=\frac{\pi}{4}+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}.$$

6-§. Davri $2l$ bo'lgan funksiyalar uchun Furye qatori

Davri $2l$ bo'lgan $f(x)$ funksiya Dirixle shartlarini qanoatlantirsin, bu yerda l - ixtiyoriy musbat son.

Bu funksiyani $[-\pi,\pi]$ kesmada Furye qatoriga yoyish uchun yangi o'zgaruvchini kiritamiz, ya'ni $x=\frac{lt}{\pi}$ deb olamiz. Bunda $-\pi\le t\le\pi$.

Bu holda $\varphi(t)=f\left(\frac{lt}{\pi}\right)$ funksiya t ning funksiyasi, uning davri esa 2π ga teng bo'ladi. Haqiqatan,

$$\varphi(t+2\pi)=f\left(\frac{l(t+2\pi)}{\pi}\right)=f\left(\frac{lt}{\pi}+2l\right)=f(x+2l)=f(x)=f\left(\frac{lt}{\pi}\right)=\varphi(t)\cdot\cdot$$

Bu funksiya $[-\pi,\pi]$ kesmada Dirixle shartlarini qanoatlantiradi. Shuning uchun, $[-\pi,\pi]$ kesmada Furye qatoriga yoyish mumkin, ya'ni

$$\varphi(t)=f\left(\frac{lt}{\pi}\right)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kt+b_k\sin kt).$$

Bu yerda

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{lt}{\pi}\right)dt\ , \qquad (1)$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{lt}{\pi}\right)\cos kt\ dt \qquad (2)$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{lt}{\pi}\right)\sin kt\ dt\ . \qquad (3)$$

O'zgaruvchi t ni x orqali ifodalaymiz

$$t=\frac{\pi x}{l},\quad dt=\frac{\pi}{l}dx.$$

Bularni (1), (2) va (3) formulalarga qo'yib, quyidagi formulalarni hosil qilamiz:

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)\,dx,\quad a_k=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi}{l}x\,dx$$

$$b_k=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi}{l}x\,dx.$$

$f(x)$ funksiyaning $[-l, l]$ kesmadagi Furye qatori

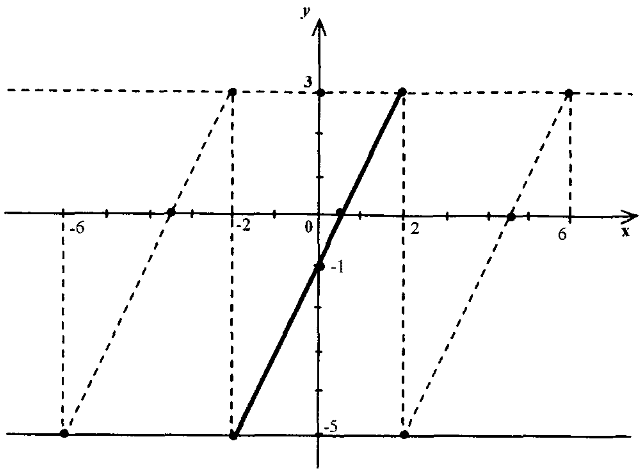
$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi}{l}x+b_k\sin\frac{k\pi}{l}x\right)$$

ko'rinishda bo'ladi.

Davri 2π bo'lgan davriy funksiyalarning Furye qatoriga yoyishdan hosil bo'lgan qatorlar uchun o'rinli bo'lgan barcha teorema va xossalar davri $2l$ bo'lgan davriy funksiyalardan hosil qilingan Furye qatorlari uchun ham o'z kuchini saqlaydi.

Misol. $f(x)=2x-1$ funksiyani $[-2, 2]$ kesmada Furye qatoriga yoying.

Yechish. Berilgan funksiya $[-2, 2]$ kesmada Dirixle shartlarini qanoatlantiradi (7-rasm).



7-rasm.

Funksiyaning davri $T = 2l = 2 \cdot 2 = 4$, demak, $l = 2$. a_0, a_k va b_k koeffitsientlarni topamiz.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2}\int_{-2}^{2}(2x-1)dx = \frac{1}{2}\left(x^2\Big|_{-2}^{2} - x\Big|_{-2}^{2}\right) =$$

$$= \frac{1}{2}(4-4-2-2) = -2,$$

$$a_k = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi}{l}x\,dx = \frac{1}{2}\int_{-2}^{2}(2x-1)\cos\frac{k\pi}{2}xdx =$$

$$= \begin{bmatrix} u = 2x-1,\ du = 2dx \\ dv = \cos\frac{k\pi}{2}xdx, \quad v = \frac{2}{k\pi}\sin\frac{k\pi}{2}x \end{bmatrix} =$$

$$= \frac{1}{2}\left[\frac{2(2x-1)}{k\pi}\sin\frac{k\pi}{2}x\Big|_{-2}^{2} - \frac{4}{k\pi}\int_{-2}^{2}\sin\frac{k\pi}{2}xdx\right] =$$

$$= -\frac{2}{k\pi}\int_{-2}^{2}\sin\frac{k\pi}{2}x\,dx = \frac{2}{k\pi}\cdot\frac{2}{k\pi}\cos\frac{k\pi}{2}x\Big|_{-2}^{2} =$$

$$= \frac{4}{k^2\pi^2}\left(\cos\left(\frac{k\pi}{2}\cdot 2\right) - \cos\left(\frac{k\pi}{2}\cdot(-2)\right)\right) = 0\cdot$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi}{l}x\,dx = \frac{1}{2}\int_{-2}^{2}(2x-1)\sin\frac{k\pi}{2}xdx =$$

$$= \left[\begin{array}{l} u = 2x-1,\ du = 2dx \\ dv = \sin\frac{k\pi}{2}xdx,\quad v = -\frac{2}{k\pi}\cos\frac{k\pi}{2}x \end{array}\right] =$$

$$= -\frac{1}{k\pi}\left[(2x-1)\cos\frac{k\pi}{2}x\Big|_{-2}^{2} - 2\int_{-2}^{2}\cos\frac{k\pi}{2}xdx\right] =$$

$$= -\frac{1}{k\pi}\left[3\cos k\pi + 5\cos k\pi\right] - \frac{4}{2k^2\pi^2}\sin\frac{k\pi}{2}x\Big|_{-2}^{2} =$$

$$= -(-1)^k\frac{4}{k\pi} = (-1)^{k+1}\frac{4}{k\pi}.$$

$f(x) = 2x-1$ funksiya uchun Furye qatori quyidagicha bo'ladi:

$$2x-1 = -1 + \frac{4}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin\frac{k\pi}{2}x}{k}.$$

II QISM. QATORLAR NAZARIYASIDAN MASHQLAR

I BOB. SONLI QATORLAR

1-§. Sonli qatorlar

1-misol. Umumiy hadi $a_n = \dfrac{1}{(2n-1)(2n+1)}$ bo'lgan qator yig'indisini toping.

Yechish. n ga ketma-ket $1, 2, 3, \ldots,$ qiymatlar berib quyidagi

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}+\ldots=\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$$

qatorni hosil qilamiz. Qatorning birinchi n ta hadlarining yig'indisi

$$S_n=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}$$

ga teng. S_n xususiy yig'indini soddaroq ko'rinishga keltirish uchun qatorning umumiy $a_n=\dfrac{1}{(2n-1)(2n+1)}$ hadini aniqmas koeffitsientlar usuli bo'yicha sodda kasrlarga ajratamiz:

$$\frac{1}{(2n-1)(2n+1)}=\frac{A}{2n-1}+\frac{B}{2n+1}.$$

Umumiy mahraj berib quyidagi ifodani hosil qilamiz

$$1=2An+A+2Bn-B \;\Rightarrow\; 1=(2A+2B)n+A-B.$$

Bundan

$$\begin{cases} 2A+2B=0 \\ A-B=1 \end{cases}$$

sistemani yozamiz.

Bu sistemadan $A=\dfrac{1}{2}$, $B=-\dfrac{1}{2}$ kelib chiqadi.

$$\frac{1}{(2n-1)(2n+1)}=\frac{1}{2(2n-1)}-\frac{1}{2(2n+1)}=\frac{1}{2}\left(\frac{1}{(2n-1)}-\frac{1}{(2n+1)}\right).$$

Qatorning har bir hadini ikki qo'shiluvchi yig'indisi ko'rinishida ifodalasak, S_n xususiy yig'indi quyidagi ko'rinishga keladi:

$$S_n = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) +$$

$$+ \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right).$$

Qator yig'indisi S ni quyidagicha topamiz

$$S = \lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}.$$

Demak, berilgan qator yaqinlashuvchi bo'lib, qatorning yig'indisi $\frac{1}{2}$ ga teng bo'ladi.

2-misol. $\frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \ldots$ qator yaqinlashishning zaruriy shart bajarilishini tekshiring.

Yechish. Berilgan qatorning umumiy hadi $\frac{2n}{2n+1}$ ko'rinishda bo'ladi. U holda

$$\frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \ldots + \frac{2n}{2n+1} + \ldots, \qquad a_n = \frac{2n}{2n+1}.$$

Ma'lumki, qator yaqinlashuvchi bo'lsa, uning umumiy hadi $n \to \infty$ da nolga intiladi. Buni e'tiborga olgan holda

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n}{2n+1} = \lim_{n\to\infty}\frac{2}{2+\frac{1}{n}} = 1$$

bo'ladi.

Qator yaqinlashishining zaruriy sharti bajarilmadi, shuning uchun berilgan qator uzoqlashuvchi.

Mustaqil yechish uchun mashqlar

Quyidagi qatorlarinng yig'indilarini toping.

1. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{3\cdot 2^{n-1}} + \ldots$ Javob: $\frac{2}{3}$

2. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$ Javob: 1

3. $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\ldots+$

$+\frac{1}{n(n+1)(n+2)}+\ldots$ Javob: $\frac{1}{4}$

4. $\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\ldots+\frac{2n+1}{n^2(n+1)^2}+\ldots$ Javob: 1

5. $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\ldots$ Javob: $\frac{3}{4}$

6. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots$ Javob: $\frac{1}{2}$

Quyidagi qatorlarning yaqinlashishini zaruriy sharti yordamida tekshiring.

1. $\frac{2}{1}+\frac{5}{8}+\frac{10}{27}+\ldots+\frac{n^2+1}{n^2}+\ldots$ Javob: uzoqlashuvchi

2. $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\ldots+\frac{2n-1}{2n}+\ldots$ Javob: uzoqlashuvchi

3. $\cos 1+\cos\frac{1}{2}+\cos\frac{1}{3}+\ldots+\cos\frac{1}{n}+\ldots$ Javob: uzoqlashuvchi

4. $\ln\frac{2}{1}+\ln\frac{3}{2}+\ln\frac{4}{3}+\ldots$ Javob: uzoqlashuvchi

5. $\frac{2}{3}+\frac{3}{5}+\frac{4}{7}+\ldots+\frac{n+1}{2n+1}+\ldots$ Javob: uzoqlashuvchi

6. $(1+1)+\left(1+\frac{1}{2}\right)^2+\left(1+\frac{1}{3}\right)^3+\ldots$ Javob: uzoqlashuvchi

2-§. Musbat hadli qatorlar

Quyidagi qatorlarning yaqinlashish yoki uzoqlashishini tekshiring.

1-misol. $\sum_{n=1}^{\infty}\frac{1}{n\cdot 3^{n-1}}$.

Yechish. Berilgan qatorning $a_n = \dfrac{1}{n \cdot 3^{n-1}}$ umumiy hadi yaqinlashuvchi bo'lgan quyidagi $\sum\limits_{n=1}^{\infty} \dfrac{1}{3^{n-1}}$ geometrik $\left(q = \dfrac{1}{3} < 1\right)$ qatorning umumiy hadidan kichik, ya'ni $a_n = \dfrac{1}{n \cdot 3^{n-1}} \le \dfrac{1}{3^{n-1}} = b_n$ solishtirish alomatiga asosan, berilgan qator yaqinlashuvchi bo'ladi.

2-misol. $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n-1}$.

Yechish. Umumiy hadi $a_n = \dfrac{1}{2n-1}$ bo'lgan berilgan qatorni umumiy hadi $b_n = \dfrac{1}{n}$ bo'lgan uzoqlashuvchi $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ garmonik qator bilan solishtiramiz.

Ikkinchi solishtirish alomatiga asosan

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1}{2n-1}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0$$

bo'lgani uchun berilgan qator ham uzoqlashuvchi bo'ladi.

3-misol. $\sum\limits_{n=1}^{\infty} \dfrac{n}{3^n}$.

Yechish. Bu qatorning umumiy hadi $a_n = \dfrac{n}{3^n}$, keyingi hadi $a_{n+1} = \dfrac{n+1}{3^{n+1}}$. Dalamber alomatiga asosan

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{n+1}{3^{n+1}}}{\dfrac{n}{3^n}} = \lim_{n\to\infty} \frac{(n+1)\cdot 3^n}{n \cdot 3^{n+1}} = \frac{1}{3} \lim_{n\to\infty} \frac{(n+1)}{n} = \frac{1}{3} < 1$$

bo'lgani uchun berilgan qator yaqinlashuvchi bo'ladi.

4-misol. $\sum\limits_{n=1}^{\infty} \dfrac{3^n \cdot n!}{n^n}$.

Yechish. Bu yerda $a_n=\dfrac{3^n\cdot n!}{n^n}$, $a_{n+1}=\dfrac{3^{n+1}\cdot(n+1)!}{(n+1)^{n+1}}$.

Dalamber alomatiga asosan

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}:\frac{3^n\cdot n!}{n^n}=\lim_{n\to\infty}\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{3^n\cdot n!}=$$

$$=\lim_{n\to\infty}\frac{3n^n}{(n+1)^n}=3\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=3\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{3}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n}=\frac{3}{e}>1$$

bo'lgani uchun berilgan qator uzoqlashuvchi.

5-misol. $\sum\limits_{n=1}^{\infty}\left(\dfrac{n}{n+1}\right)^{n^2}$.

Yechish. Koshining radikal alomatiga ko'ra

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1$$

bo'lgani uchun qator yaqinlashuvchi.

6-misol. $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^{\alpha}}$ (Dirixle qatori) qatorni yaqinlashishga tekshiring.

Yechish. Qatorni yaqinlashishga tekshirish uchun Koshining integral alomatidan foydalanamiz.

$f(x)=\dfrac{1}{x^{\alpha}}$ deb olib, ushbu $\int\limits_1^{\infty}\dfrac{dx}{x^{\alpha}}$ xosmas integralni tekshiramiz.

$$\int_1^{\infty}\frac{dx}{x^{\alpha}}=\lim_{b\to\infty}\int_1^b x^{-\alpha}dx=\lim_{b\to\infty}\begin{cases}\dfrac{1}{1-\alpha}\left(b^{-\alpha}-1\right), & \alpha\neq 1\\ \ln b, & \alpha=1\end{cases}=$$

$$
=\begin{cases}
\dfrac{1}{\alpha-1}, & \text{agar } \alpha>1 \text{ bo'lsa,} \quad \text{yaqinlashuvchi} \\
\infty, & \text{agar } \alpha<1 \text{ bo'lsa,} \quad \text{uzoqlashuvchi} \\
\infty, & \text{agar } \alpha=1 \text{ bo'lsa,} \quad \text{uzoqlashuvchi}
\end{cases}
$$

Demak, xosmas integral $\alpha>1$ da yaqinlashuvchi, $\alpha\le 1$ da uzoqlashuvchi . Koshining integral alomatiga ko'ra $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ qator $\alpha>1$ da yaqinlashuvchi, $\alpha\le 1$ da uzoqlashuvchi bo'ladi. $\alpha=1$ bo'lganda $\sum_{n=1}^{\infty}\frac{1}{n}$ garmonik qator hosil bo'ladi.

Mustaqil yechish uchun mashqlar

Solishtirish alomatlaridan foydalanib quyidagi qatorlarning yaqinlashishi yoki uzoqlashishini ko'rsating.

1. $\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^{n}}$ Javob: yaqinlashuvchi

2. $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ Javob: uzoqlashuvchi

3. $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2(1+2^{2})}}+\frac{1}{\sqrt{3(1+3^{2})}}+\ldots+$
$+\frac{1}{\sqrt{n(1+n^{2})}}+\ldots$ Javob: yaqinlashuvchi

4. $\sin\frac{\pi}{3}+\sin\frac{\pi}{9}+\ldots+\sin\frac{\pi}{3^{n}}+\ldots$ Javob: yaqinlashuvchi

5. $\frac{1}{\sqrt{1\cdot 2}}+\frac{1}{\sqrt{2\cdot 3}}+\frac{1}{\sqrt{3\cdot 4}}+\ldots+\frac{1}{\sqrt{n(n+1)}}+\ldots$ Javob: uzoqlashuvchi

6. $1+\frac{1+2}{1+2^{2}}+\frac{1+3}{1+3^{2}}+\ldots+\frac{1+n}{1+n^{2}}+\ldots$ Javob: uzoqlashuvchi

7. $\sin 1+\sin\frac{1}{2}+\ldots+\sin\frac{1}{n}+\ldots$ Javob: yaqinlashuvchi

8. $\ln\frac{3}{2}+\ln\frac{2^2+2}{2^2+1}+\ln\frac{3^2+2}{3^2+1}+\ldots+\ln\frac{n^2+2}{n^2+1}+\ldots$ Javob: yaqinlashuvchi

9. $2\sin\frac{\pi}{3}+2^2\sin\frac{\pi}{3^2}+\ldots+2^n\sin\frac{\pi}{3^n}+\ldots$ Javob: uzoqlashuvchi

Dalamber va Koshi alomatlaridan foydalanib quyidagi qatorlarning yaqinlashish yoki uzoqlashishini tekshiring.

1. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\ldots+\frac{n}{2^n}+\ldots$ Javob: yaqinlashuvchi

2. $\frac{5}{1!}+\frac{5^2}{2!}+\frac{5^3}{3!}+\ldots+\frac{5^n}{n!}+\ldots$ Javob: uzoqlashuvchi

3. $1+\frac{2^2}{2!}+\frac{3^3}{3!}+\ldots+\frac{n^n}{n!}+\ldots$ Javob: uzoqlashuvchi

4. $2+\left(\frac{2+1}{2\cdot 2-1}\right)^2+\left(\frac{3+1}{2\cdot 3-1}\right)^3+\ldots+\left(\frac{n+1}{2n-1}\right)^n+\ldots$ Javob: yaqinlashuvchi

5. $\frac{2}{3}+\frac{\left(\frac{3}{2}\right)^4}{9}+\ldots+\frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}+\ldots$ Javob: yaqinlashuvchi

6. $2+\frac{2\cdot 5}{1\cdot 5}+\frac{2\cdot 5\cdot 8}{1\cdot 5\cdot 9}+\ .\ .\ .\ +$

$+\frac{2\cdot 5\cdot 8\cdot\ .\ .\ .\ \cdot(3n-1)}{1\cdot 5\cdot 9\cdot\ .\ .\ .\ \cdot(4n-3)}+\ .\ .\ .$ Javob: yaqinlashuvchi

7. $1+\frac{1\cdot 4}{3!!}+\frac{1\cdot 4\cdot 7}{5!!}+\ .\ .\ .\ +$

$+\frac{1\cdot 4\cdot 7\cdot\ .\ .\ .\ \cdot(3n-2)}{(2n-1)!!}+\ .\ .\ .$ Javob: uzoqlashuvchi

8. $\sin\frac{\pi}{2}+\sin^2\frac{\pi}{4}+\ldots+\sin^n\frac{\pi}{2n}+\ldots$ Javob: yaqinlashuvchi

9. $\frac{3}{2^{-1}}+\frac{3^3}{2^2}+\frac{3^5}{2^5}+\ldots+\frac{3^{2n+1}}{2^{3n-1}}+\ldots$ Javob: uzoqlashuvchi

10. $\frac{1}{\sqrt{3}}+\frac{5}{\sqrt{2\cdot 3^2}}+\frac{9}{\sqrt{3\cdot 3^3}}+\ldots+\frac{4n-3}{\sqrt{n\cdot 3^n}}+\ldots$ Javob: yaqinlashuvchi

Koshining integral alomatidan foydalanib, quyidagi qatorlarning yaqinlashishini tekshiring.

1. $\frac{1}{2(\ln 2)^2}+\frac{1}{3(\ln 3)^2}+\ \dots\ +\frac{1}{n(\ln n)^2}+\ \dots$ Javob: yaqinlashuvchi

2. $1+\frac{1}{\sqrt{5}}+\frac{1}{\sqrt{9}}+\dots+\frac{1}{\sqrt{4n+1}}+\dots$ Javob: uzoqlashuvchi

3. $\frac{2}{3+1^2}+\frac{2}{3+2^2}+\frac{2}{3+3^2}+\dots+\frac{2}{3+n^2}+\dots$ Javob: yaqinlashuvchi

4. $\left(\frac{1+1}{1+1^2}\right)^2+\left(\frac{1+2}{1+2^2}\right)^2+\dots+\left(\frac{1+n}{1+n^2}\right)^2+\dots$ Javob: yaqinlashuvchi

5. $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ Javob: yaqinlashuvchi

6. $\frac{e^{-\sqrt{1}}}{\sqrt{1}}+\frac{e^{-\sqrt{2}}}{\sqrt{2}}+\frac{e^{-\sqrt{3}}}{\sqrt{3}}+\ \dots\ +\frac{e^{-\sqrt{n}}}{\sqrt{n}}+\ \dots$ Javob: yaqinlashuvchi

Quyidagi qatorlarni yaqinlashishga tekshiring.

1. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}+\dots$ Javob: uzoqlashuvchi

2. $1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}+\dots$ Javob: yaqinlashuvchi

3. $\frac{1}{2^3}+\frac{2}{3^3}+\frac{3}{4^3}+\dots+\frac{n}{(n+1)^3}+\dots$ Javob: yaqinlashuvchi

4. $\sum_{n=2}^{\infty}\frac{1}{\sqrt{n(n-1)}}$ Javob: uzoqlashuvchi

5. $\sum_{n=1}^{\infty}\frac{n}{2n-1}$ Javob: uzoqlashuvchi

6. $\sum_{n=1}^{\infty}\frac{n!}{3^{n-1}}$ Javob: uzoqlashuvchi

7. $\sum_{n=1}^{\infty}\frac{2n-1}{\left(\sqrt{3}\right)^n}$ Javob: yaqinlashuvchi

8. $\sum_{n=1}^{\infty}\frac{n^2}{2n^2+1}$ Javob: uzoqlashuvchi

9. $\sum_{n=1}^{\infty}\frac{n^3}{e^n}$ Javob: yaqinlashuvchi

10. $\sum_{n=1}^{\infty}\left(\frac{1+n}{1+n^2}\right)^2$ Javob: yaqinlashuvchi

11. $\sum_{n=0}^{\infty}\frac{1}{\sqrt{4n+1}}$ Javob: uzoqlashuvchi

12. $\sum_{n=3}^{\infty}\frac{n}{n^4-9}$ Javob: yaqinlashuvchi

13. $\sum_{n=1}^{\infty}\frac{2}{2^n}\left(\frac{n+1}{n}\right)^{n^2}$ Javob: uzoqlashuvchi

3-§. Ishoralari o'zgaruvchi qatorlar

Quyidagi misollarda ishoralari o'zgaruvchi qatorlarning absolyut, shartli yaqinlashishi yoki uzoqlashishini tekshiring.

1-misol. $\sum_{n=1}^{\infty}\frac{\sin n\alpha}{3^n}$ $(\alpha$ – ихтиёрий сон$)$.

Yechish. Berilgan qator hadlarining absolyut qiymatlaridan yangi qator tuzamiz

$$\sum_{n=1}^{\infty}\frac{|\sin n\alpha|}{3^n}$$

va bu qatorni $\sum_{n=1}^{\infty}\frac{1}{3^n}$ qator bilan taqqoslaymiz. Ma'lumki $\frac{|\sin n\alpha|}{3^n}\leq\frac{1}{3^n}$. Geometrik qator $\left(q=\frac{1}{3}<1\right)$ $\sum_{n=1}^{\infty}\frac{1}{3^n}$ yaqinlashuvchi bo'lgani uchun, musbat hadli qatorlarni solishtirish alomatiga ko'ra qator yaqinlashuvchi bo'ladi. Demak, berilgan qator absolyut yaqinlashuvchidir.

2-misol. $\sum_{n=1}^{\infty} \cos\frac{n\pi}{4}$.

Yechish. Berilgan ishoralari o'zgaruvchi qator uchun qator yaqinlashishining zaruriy sharti bajarilmaydi. Haqiqatan,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos\frac{n\pi}{4}$$

limit mavjud emas. Demak, berilgan qator uzoqlashuvchi.

3-misol. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 2^n}$.

Yechish. Ishoralari almashinuvchi berilgan qatorning hadlari absolyut qiymat bo'yicha monoton kamayadi, ya'ni

$$\frac{1}{2} > \frac{1}{2\cdot 2^2} > \frac{1}{3\cdot 2^3} > \ldots$$

va

$$\lim_{n\to\infty} \frac{1}{n\cdot 2^n} = 0.$$

Leybnis teoremasiga asosan berilgan qator yaqinlashuvchi. Qatorning absolyut yoki shartli yaqinlashishini tekshirish uchun berilgan qator hadlarining absolyut qiymatlaridan $\sum_{n=1}^{\infty} \frac{1}{n\cdot 2^n}$ qator tuzamiz. Bu qatorning yaqinlashishini Dalamber alomatiga ko'ra tekshiramiz.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)\cdot 2^{n+1}}}{\frac{1}{n\cdot 2^n}} = \lim_{n\to\infty} \frac{n\cdot 2^n}{(n+1)\cdot 2^{n+1}} = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2} < 1.$$

Demak, musbat hadli qator yaqinlashuvchi. Shuning uchun, berilgan qator absolyut yaqinlashuvchi bo'ladi.

4-misol. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$.

Yechish. Berilgan qator Leybnis teoremasi shartlarini qanoatlantiradi, ya'ni

$$1 > \frac{1}{3} > \frac{1}{5} > \ldots$$

va

$$\lim_{n\to\infty}\frac{1}{2n-1}=0.$$

Demak, berilgan qator yaqinlashuvchi. Berilgan qatorning absolyut yoki shartli yaqinlashishini tekshiramiz. Buning uchun qator hadlarining absolyut qiymatlaridan tuzilgan $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ qatorni tekshiramiz. Koshining integral alomatidan foydalanamiz.

$$\int_1^{\infty}\frac{dx}{2x-1}=\frac{1}{2}\lim_{n\to\infty}\int_1^n\frac{d(2x-1)}{2x-1}=\frac{1}{2}\lim_{n\to\infty}\ln(2x-1)\Big|_1^n=\frac{1}{2}\lim_{n\to\infty}\ln(2n-1)=+\infty$$

bo'lgani uchun xosmas integral uzoqlashuvchi. Demak, musbat hadli $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ qator uzoqlashuvchi. U holda berilgan qator shartli yaqinlashuvchi.

Mustaqil yechish uchun mashqlar

1. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ Javob: shartli yaqinlashuvchi

2. $\frac{1}{\ln 2}-\frac{1}{\ln 3}+\frac{1}{\ln 4}-\ldots$ Javob: shartli yaqinlashuvchi

3. $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\ldots+(-1)^{n-1}\frac{1}{\sqrt{n}}+\ldots$ Javob: shartli yaqinlashuvchi

4. $\frac{\sin\alpha}{1}+\frac{\sin 2\alpha}{2^2}+\frac{\sin 3\alpha}{3^2}+\ldots$ Javob: absolyut yaqinlashuvchi

5. $-\frac{3}{4}+\left(\frac{5}{7}\right)^2-\left(\frac{7}{10}\right)^3+\ldots+$
$+(-1)^n\left(\frac{2n+1}{3n+1}\right)^n+\ldots$ Javob: absolyut yaqinlashuvchi

6. $\sum_{n=1}^{\infty}\frac{\cos n\alpha}{(\ln 10)^n}$ Javob: absolyut yaqinlashuvchi

7. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2n+1}{n(n+1)}$ Javob: absolyut yaqinlashuvchi

8. $\sum_{n=1}^{\infty}(-1)^{n-1}\cos\frac{\pi}{3n}$ Javob: uzoqlashuvchi

9. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n+1)a^{2n}}$ Javob: $\begin{cases} |a|>1 \text{da abs. yaqinlash.} \\ |a|<1 \text{da uzoqlashuvchi} \end{cases}$

10. $1-\frac{1}{3}+\frac{1}{2}-\frac{1}{3^3}+\frac{1}{2^2}-\frac{1}{3^5}+\ldots$ Javob: absolyut yaqinlashuvchi

11. $\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)=1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\ldots$ tenglikni isbotlang.

II BOB. FUNKSIONAL VA DARAJALI QATORLAR

1-§. Funksional qatorlar

1-misol. Quyidagi funksional qatorning yaqinlashish sohasini toping:

$$\sum_{n=1}^{\infty} \frac{1}{n(x+3)^n}.$$

Yechish. Dalamber alomatidan foydalanamiz.

$$u_n(x) = \frac{1}{n(x+3)^n}; \quad u_{n+1}(x) = \frac{1}{(n+1)(x+3)^{n+1}},$$

$$l = \lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{n(x+3)^n}{(n+1)(x+3)^{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{(n+1)(x+3)}\right| =$$

$$= \frac{1}{|x+3|}\lim_{n\to\infty}\frac{n}{n+1} = \frac{1}{|x+3|}.$$

Dalamber alomatiga ko'ra qator yaqinlashishi uchun $l<1$ bo'lishi kerak. Bu holda

$$\frac{1}{|x+3|} < 1 \Rightarrow |x+3| > 1 \Rightarrow x+3 > 1$$

va

$$x+3 < -1 \Rightarrow x > -2 \text{ ва } x < -4.$$

Demak, qatorning yaqinlashish sohasi $(-\infty; -4)\cup(-2; +\infty)$ dan iborat bo'ladi. Berilgan qatorni topilgan interval chegaralarida tekshiramiz.

$x=-4$ bo'lganda $\sum\limits_{n=1}^{\infty}\frac{1}{n(-1)^n}$ ishoralari almashinuvchi qator hosil bo'lib, bu qator Leybnis alomatiga ko'ra yaqinlashuvchi bo'ladi. $x=-2$ bo'lganda $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ uzoqlashuvchi bo'lgan garmonik qator hosil bo'ladi.

Shunday qilib, berilgan qatorning yaqinlashish sohasi $(-\infty; -4]\cup(-2; +\infty)$ dan iborat.

2-misol. Quyidagi

$$\sum_{n=1}^{\infty}\left(8-x^2\right)^n$$

funksional qatorning yaqinlashish sohasini toping.

Yechish. Berilgan funksional qatorni mahraji $q=8-x^2$ bo'lgan geometrik qator deb qarash mumkin. $|q|=\left|8-x^2\right|<1$ bo'lganda ma'lumki qator yaqinlashuvchi bo'ladi. $\left|8-x^2\right|<1$ tengsizlikni echamiz. $-1<8-x^2<1$ yoki $7<x^2<9$, bundan $\sqrt{7}<|x|<3$. Qatorning yaqinlashish sohasi $\left(-3;-\sqrt{7}\right)\cup\left(\sqrt{7};3\right)$ dan iborat. Endi qatorning yaqinlashishini oraliq chegaralarida tekshirib ko'ramiz: $x=-3$, $x=-\sqrt{7}$, $x=\sqrt{7}$ va $x=3$ bo'lganda $\sum_{n=1}^{\infty}(-1)^n$ va $\sum_{n=1}^{\infty}1^n$ uzoqlashuvchi qatorlar hosil bo'ladi.

Demak, berilgan qatorning yaqinlashish sohasi $\left(-3;-\sqrt{7}\right)\cup\left(\sqrt{7};3\right)$ dan iborat.

3-misol. Funksional qatorning yaqinlashish sohasini toping:

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{|x|}{n}\right)^n,\ x\neq 0.$$

Yechish. Funksional qatorning hadlari sonlar o'qining $x=0$ dan boshqa barcha nuqtalarda aniqlangan. Agar $x<0$ bo'lsa $\frac{|x|}{x}=\frac{-x}{x}=-1$ bo'ladi. U holda ishoralari almashinuvchi $\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{|x|}{n}\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ qator Leybnis alomatiga ko'ra yaqinlashuvchi bo'ladi. Agar $x>0$ bo'lsa $\frac{|x|}{x}=\frac{x}{x}=1$ bo'ladi. U holda $\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{|x|}{n}\right)^n=\sum_{n=1}^{\infty}\frac{1}{n}$ uzoqlashuvchi garmonik qator hosil bo'ladi.

Demak, berilgan qatorning yaqinlashish sohasi $(-\infty;0)$ dan iborat bo'ladi.

4-misol. Quyidagi funksional qatorning yaqinlashish sohasini toping:

$$\sum_{n=1}^{\infty} 2^{n-1} \cdot x^{2n-2}.$$

Yechish. Funksional qatorda x ning barcha natural (x ning toq darajalari qatnashmaydi) darajalari qatnashmaganligi uchun qatorning yaqinlashish sohasini topishda Dalamber alomatidan foydalanamiz.

$$u_n(x) = 2^{n-1} \cdot x^{2n-2}\ ;\ u_{n+1}(x) = 2^n \cdot x^{2n},$$

$$l = \lim_{n\to\infty} \frac{u_{n+1}(x)}{u_n(x)} = \lim_{n\to\infty} \frac{2^n \cdot x^{2n}}{2^{n-1} \cdot x^{2n-2}} = \lim_{n\to\infty} 2x^2 = 2x^2 < 1 \cdot$$

Bundan $x^2 < \frac{1}{2}$ yoki $|x| < \frac{1}{\sqrt{2}}$ bo'ladi. Qatorning yaqinlashish intervali $\left(-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$ bo'ladi.

Endi qator yaqinlashishini intervalning chegaralarida tekshiramiz. $x = \pm\frac{1}{\sqrt{2}}$ bo'lganda

$$1 + 1 + 1 + \ldots + 1 + \ldots$$

uzoqlashuvchi qator hosil bo'ladi. Shunday qilib, berilgan darajali qator $\left(-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$ intervalda yaqinlashuvchi bo'ladi.

5-misol. Quyidagi funksional qatorning yaqinlashish sohasini toping:

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + x^2}.$$

Yechish. x ning barcha qiymatlari uchun

$$\frac{1}{n^4 + x^2} \le \frac{1}{n^4}$$

tengsizlik o'rinli. Ma'lumki, $\sum_{n=1}^{\infty} \frac{1}{n^4}$ qator ($\alpha = 4 > 1$ bo'lganligi uchun, umumlashgan garmonik qatorni eslang) yaqinlashuvchi.

$\frac{1}{n^4+x^2} \le \frac{1}{n^4}$ tengsizlik o'rinli bo'lgani uchun Veyershtrass alomatiga ko'ra, berilgan qator x ning barcha qiymatlarida tekis yaqinlashuvchi bo'ladi.

6-misol. $\sum_{n=1}^{\infty} \frac{\sin 3^n \pi x}{3^n}$ funksional qatorning tekis yaqinlashishini ko'rsating.

Yechish. $|\sin kx| \le 1$ bo'lgani uchun

$$\left|\frac{\sin 3^n \pi x}{3^n}\right| \le \frac{1}{3^n} \qquad (*)$$

tengsizlik o'rinli. n - hadi $\frac{1}{3^n}$ bo'lgan $\sum_{n=1}^{\infty} \frac{1}{3^n}$ qator geometrik qator bo'lib, $\left(q = \frac{1}{3} < 1\right)$ bo'lganligi uchun yaqinlashuvchi. Yuqoridagi (*) tengsizlik o'rinli bo'lgani uchun, Veyershtrass alomatiga ko'ra, berilgan qator x ning barcha qiymatlarida tekis yaqinlashuvchi bo'ladi.

7-misol. $\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{1+(2n+1)x}}$ $(0 \le x < \infty)$ funksional qatorning tekis yaqinlashishini ko'rsating. Qanday n lar uchun $|\varphi_n(x)| < 0{,}01$ tengsizlik bajariladi?

Yechish. Qaralayotgan oraliqda quyidagi

$$\frac{1}{2^n \sqrt{1+(2n+1)x}} \le \frac{1}{2^n} \qquad (**)$$

tengsizlik o'rinli. Mahraji $q = \frac{1}{2} < 1$ bo'lgan $\sum_{n=1}^{\infty} \frac{1}{2^n}$ geometrik qator yaqinlashuvchi bo'lganligi uchun (**) Veyershrass teoremasiga ko'ra, berilgan qator qaralayotgan oraliqda tekis yaqinlashuvchi bo'ladi. Funksional qatorning qoldig'i $r_n(x) = S(x) - S_n(x)$ ni baholash uchun $\sum_{n=1}^{\infty}\frac{1}{2^n}$ sonli qatorning qoldig'i $r_n = S - S_n$ ni

baholaymiz. Ma'lumki, $S_n = \dfrac{\frac{1}{2}-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$,

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{\frac{1}{2}-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 1$$

va

$$r_n = S - S_n = \frac{1}{2^n}.$$

Berilgan funksional qator $\varphi_n(x)$ qoldig'i $\sum_{n=1}^{\infty}\frac{1}{2^n}$ sonli qatorning r_n qoldig'idan katta bo'la olmaydi. Shuning uchun $\varphi_n(x) \le \frac{1}{2^n}$ bo'ladi. $\varphi_n(x) < 0{,}01$ tengsizlik o'rinli bo'lishi uchun $\frac{1}{2^n} < 0{,}01$ tengsizlikni echamiz. U holda $2^n > 100$ bo'lib, $n \ge 7$ bo'ladi.

Mustaqil yechish uchun mashqlar

Quyidagi funksional qatorlarning yaqinlashish sohasini toping.

1. $\sum_{n=1}^{\infty} e^{-nx}$ Javob: $(0\,;\,\infty)$
2. $\sum_{n=1}^{\infty}\frac{1}{x^n}$ Javob: $(-\infty\,;\,-1)\cup(1\,;\,\infty)$
3. $\sum_{n=1}^{\infty}\frac{n!}{x^n}$ Javob: butun sonlar o'qida uzoqlashuvchi

4. $\sum_{n=1}^{\infty}\left(2-x^2\right)^n$ Javob: $\left(-\sqrt{3}\,;-1\right)\cup\left(1\,;\sqrt{3}\right)$

5. $\sum_{n=1}^{\infty}2^n\sin\frac{x}{2^n}$ Javob: $(-\infty\,;+\infty)$

6. $\sum_{n=1}^{\infty}\ln^n x$ Javob: $\left(\frac{1}{e}\,;e\right)$

7. $\sum_{n=1}^{\infty}n\sqrt[3]{\sin^n x}$ Javob: $x_k\neq\frac{\pi}{2}+k\pi\ \ (k=0,\pm1,\pm2,\ldots)$

8. $\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ Javob: $(-\infty\,;+\infty)$

9. $\sum_{n=1}^{\infty}\frac{n(x+3)}{3^{n+1}}$ Javob: $(-6\,;0)$

10. $\sum_{n=1}^{\infty}(3x)^{n^2}$ Javob: $\left(-\frac{1}{3}\,;\frac{1}{3}\right)$

11. $\sum_{n=1}^{\infty}\frac{\cos nx}{n^n}$ Javob: $(-\infty\,;+\infty)$

12. $\sum_{n=1}^{\infty}\frac{\cos 2^n x}{n^2}$ Javob: $(-\infty\,;+\infty)$

13. $\sum_{n=1}^{\infty}\frac{1}{n^2+\cos^2 x}$ Javob: $(-\infty\,;+\infty)$

14. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{x^{2n}+n}$ Javob: $(-\infty\,;-1]\cup[1\,;\infty)$

2-§. Darajali qatorlar

Quyidagi qatorlarning yaqinlashish radiuslari va yaqinlashish intervallarini toping. Yaqinlashish intervalining chegaralarida qatorning yaqinlashishini tekshiring.

1-misol. $\sum_{n=1}^{\infty}\frac{2^n x^n}{\sqrt{3^n}}$.

Yechish. Bu yerda $a_n = \frac{2^n x^n}{\sqrt{3^n}}$. Qatorning yaqinlashish radiusini quyidagi Koshi alomati yordamida topamiz.

$$R = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{a_n}} = \frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{\frac{2^n}{\sqrt{3^n}}}} = \frac{1}{\frac{2}{\sqrt{3}}} = \frac{\sqrt{3}}{2}, \quad R = \frac{\sqrt{3}}{2}.$$

Demak, yaqinlashish intervali $\left(-\frac{\sqrt{3}}{2}; \frac{\sqrt{3}}{2}\right)$ bo'ladi. Intervalning chegaralarida qatorning yaqinlashishini alohida tekshiramiz.

$x = -\frac{\sqrt{3}}{2}$ bo'lsa $\sum\limits_{n=1}^{\infty}(-1)^n$ uzoqlashuvchi qator,

$x = \frac{\sqrt{3}}{2}$ bo'lsa $\sum\limits_{n=1}^{\infty}1^n = 1+1+1+ \ldots +1+ \ldots$ uzoqlashuvchi qator hosil bo'ladi. Demak berilgan qatorning yaqinlashish intervali $\left(-\frac{\sqrt{3}}{2}; \frac{\sqrt{3}}{2}\right)$ dan iborat.

2-misol. $\sum\limits_{n=1}^{\infty}\frac{x^n}{n\cdot 2^n}$.

Yechish. n - hadining koeffitsienti $a_n = \frac{1}{n\cdot 2^n}$. Qatorning yaqinlashish radiusini topish formulaga asosan topamiz:

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)\cdot 2^{n+1}}{n\cdot 2^n} = \lim_{n\to\infty}\frac{2n+2}{n} = 2,$$

$R = 2$ va yaqinlashish intervali $(-2; 2)$ bo'ladi. Interval chegaralarida, ya'ni $x=-2$ va $x=2$ nuqtalarda tekshiramiz.

$x=-2$ bo'lganda $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}$ qator hosil bo'ladi va u Leybnis teoremasi shartlarini qanoatlantiradi. Demak, qator $x=-2$ nuqtada yaqinlashadi.

$x=2$ bo'lganda $\sum_{n=1}^{\infty}\frac{1}{n}$ uzoqlashuvchi garmonik qator hosil bo'ladi. Shunday qilib, berilgan darajali qator $[-2\,;2)$ yarim oraliqda yaqinlashuvchi.

3-misol. $\sum_{n=1}^{\infty}\frac{4^n}{n^3(x^2-4x+7)^n}$.

Yechish. Qatorning n-hadi va $(n+1)$- hadini yozamiz:

$$u_n(x)=\frac{4^n}{n^3(x^2-4x+7)^n}\,;\; u_{n+1}(x)=\frac{4^{n+1}}{(n+1)^3(x^2-4x+7)^{n+1}}.$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{4^{n+1}}{(n+1)^3(x^2-4x+7)^{n+1}}\cdot\frac{n^3(x^2-4x+7)^n}{4^n}\right|=$$

$$=\lim_{n\to\infty}\left|\frac{4n^3}{(n+1)^3(x^2-4x+7)}\right|=\frac{4}{x^2-4x+7}<1$$

Bu tengsizlikni yechib yaqinlashish oralig'ini topamiz.

$$x\in(-\infty,\,1)\cup(3,\,\infty)$$

Endi qatorni chegaradagi nuqtalarda yaqinlashishini tekshiramiz. $x=1$ da

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

qator hosil bo'ladi. Bu garmonik qatorning umumlashgan $p=3$ bo'lgani uchun yaqinlashuvchidir $x=3$ da

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

hosil bo'ladi. Bu qator ham yaqinlashuvchi. Demak yaqinlashish oralig'i $(-\infty,\,1]\cup[3,\,\infty)$.

Mustaqil yechish uchun mashqlar

Quyidagi qatorlarning yaqinlashish radiuslari va yaqinlashish intervallarini toping. Yaqinlashish intervalining chegaralarida qatorning yaqinlashishini tekshiring.

1. $\sum_{n=1}^{\infty}x^n$ Javob: $R=1,\ (-1;\,1)$

2. $\sum_{n=1}^{\infty}\frac{(x-4)^n}{n}$ Javob: $R=1,\ [3;\ 5]$

3. $\sum_{n=1}^{\infty} n!\cdot x^n$ Javob: $R=0,\ x=0$

4. $\sum_{n=1}^{\infty}(-2)^n x^{2n}$ Javob: $R=\frac{1}{\sqrt{2}},\ \left(-\frac{1}{\sqrt{2}};\ \frac{1}{\sqrt{2}}\right)$

5. $\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}$ Javob: $R=1,\ [-1;\ 1]$

6. $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ Javob: $R=1,\ [-1;\ 1]$

7. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^{2n}}{n\cdot 4^n}$ Javob: $R=2,\ [0;\ 4]$

8. $\sum_{n=1}^{\infty}\frac{n}{n+1}\left(\frac{x}{2}\right)^n$ Javob: $R=2,\ (-2;\ 2)$

9. $\sum_{n=1}^{\infty}\frac{n^5}{(n+1)!}(x+5)^{2n+1}$ Javob: $R=\infty, (-\infty;\ \infty)$

10. $\sum_{n=1}^{\infty}\frac{10^n x^n}{\sqrt{n}}$ Javob: $R=\frac{1}{10},\ (-0{,}1;0{,}1)$

11. $\sum_{n=1}^{\infty}\frac{(2x+1)^n}{3n-2}$ Javob: $R=1, [-1;\ 0)$

12. $\sum_{n=1}^{\infty}\frac{(-x)^n}{3^{n-1}\sqrt{n}}$ Javob: $R=3,\ (-3;\ 3]$

13. $\sum_{n=1}^{\infty}\frac{2^n\cdot n!}{(2n)!}x^{2n}$ Javob: $R=\infty,\ (-\infty;\ \infty)$

14. $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}x^n$ Javob: $R=\frac{1}{e},\ \left(-\frac{1}{e};\ \frac{1}{e}\right)$

3-§. Teylor va Makloren qatorlari

1-misol. $f(x)=\sin\frac{\pi x}{4}$ funksiyani $x=2$ nuqta atrofida Teylor qatoriga yoying.

Yechish. Berilgan funksiyaning hosilalarini va $x=2$ nuqtadagi qiymatlarini topamiz.

$$f(x)=\sin\frac{\pi x}{4},\quad f(2)=\sin\frac{2\pi}{4}=1,$$

$$f'(x)=\frac{\pi}{4}\cos\frac{\pi x}{4}=\frac{\pi}{4}\sin\left(\frac{\pi x}{4}+\frac{\pi}{2}\right),\quad f'(2)=\sin\left(\frac{2\pi}{4}+\frac{\pi}{2}\right)=0,$$

$$f''(x)=-\frac{\pi^2}{4^2}\sin\frac{\pi x}{4}=\frac{\pi^2}{4^2}\sin\left(\frac{\pi x}{4}+2\cdot\frac{\pi}{2}\right),$$

$$f''(2)=\frac{\pi^2}{4^2}\sin\left(\frac{2\pi}{4}+\pi\right)=-\frac{\pi^2}{4^2},$$

$$f'''(x)=-\frac{\pi^3}{4^3}\cos\frac{\pi x}{4}=\frac{\pi^3}{4^3}\sin\left(\frac{\pi x}{4}+3\cdot\frac{\pi}{2}\right),$$

$$f'''(2)=\frac{\pi^3}{4^3}\sin\left(\frac{2\pi}{4}+\frac{3\pi}{2}\right)=0,$$

$$f^{IV}(x)=\frac{\pi^4}{4^4}\sin\frac{\pi x}{4}=\frac{\pi^4}{4^4}\sin\left(\frac{\pi x}{4}+4\cdot\frac{\pi}{2}\right),$$

$$f^{IV}(2)=\frac{\pi^4}{4^4}\sin\left(\frac{2\pi}{4}+2\pi\right)=\frac{\pi^4}{4^4},$$

. . . .

$$f^{(2k)}(x)=\frac{\pi^{2k}}{4^{2k}}\sin\left(\frac{\pi x}{4}+2k\cdot\frac{\pi}{2}\right),$$

$$f^{(2k)}(2)=\frac{\pi^{2k}}{4^{2k}}\sin\left(\frac{2\pi}{4}+k\pi\right)=(-1)^k\frac{\pi^{(2k)}}{4^{(2k)}},$$

$$f^{(2k+1)}(x)=\frac{\pi^{2k+1}}{4^{2k+1}}\sin\left(\frac{\pi x}{4}+(2k+1)\cdot\frac{\pi}{2}\right),$$

$$f^{(2k+1)}(2)=\frac{\pi^{2k+1}}{4^{2k+1}}\sin\left(\frac{2\pi}{4}+\frac{(2k+1)\pi}{2}\right)=0.$$

Topilganlarni Teylor qatoriga qo'ysak, quyidagiga ega bo'lamiz

$$\sin\frac{\pi x}{4}=1-\frac{\pi^2}{4^2}\frac{(x-2)^2}{2!}+\frac{\pi^4}{4^4}\frac{(x-2)^4}{4!}-\ldots+$$

$$+\ldots+(-1)^k\frac{\pi^{2k}}{4^{2k}}\frac{(x-2)^{2k}}{(2k)!}+\ldots.$$

Bu darajali qatorning yaqinlashish intervalini Dalamber alomati yordamida topamiz.

$$\lim_{k\to\infty}\left|\frac{u_{2k+2}(x)}{u_{2k}(x)}\right|=\lim_{k\to\infty}\left|\frac{\pi^{2k+2}}{4^{2k+2}}\frac{(x-2)^{2k+2}}{(2k+2)!}\cdot\frac{4^{2k}}{\pi^{2k}}\frac{(2k)!}{(x-2)^{2k}}\right|=$$

$$=\frac{\pi^2}{4^2}(x-2)^2\lim_{k\to\infty}\frac{1}{(2k+1)(2k+2)}=0<1$$

bo'lgani uchun x ning barcha qiymatlarida yuqorida berilgan qator yaqinlashuvchi, ya'ni qatorning yaqinlashish intervali $(-\infty\,;\infty)$ dan iborat. Endi Teylor formulasidan

$$R_{k+1}=\frac{\pi^{2k+1}}{4^{2k+1}}\frac{(x-2)^{2k+1}}{(2k+1)!}\sin\left(\frac{\pi\xi}{4}+(2k+1)\frac{\pi}{2}\right)$$

qoldiq hadni tekshiramiz. Har qanday k va ξ uchun $\left|\sin\left(\frac{\pi\xi}{4}+(2k+1)\frac{\pi}{2}\right)\right|\le 1$ $\lim\limits_{k\to\infty}\left(\frac{\pi}{4}\right)^{2k+1}=0$ o'rinli. Har qanday chekli x uchun

$$\lim_{k\to\infty}\frac{(x-2)^{2k+1}}{(2k+1)!}=0\cdot$$

U holda $\lim\limits_{k\to\infty}R_{2k+1}(x)=0$. Demak, berilgan $f(x)=\sin\frac{\pi x}{4}$ funksiya ixtiyoriy tartibdagi hosilaga ega va $\lim\limits_{k\to\infty}R_{2k+1}(x)=0$ bo'lgani uchun berilgan funksiya $(x-2)$ ning darajalari bo'yicha Teylor qatoriga yoyiladi.

$$\sin\frac{\pi x}{4}=1-\frac{\pi^2}{4^2}\frac{(x-2)^2}{2!}+\frac{\pi^4}{4^4}\frac{(x-2)^4}{4!}-\ldots+(-1)^k\frac{\pi^{2k}}{4^{2k}}\frac{(x-2)^{2k}}{(2k)!}+\ldots.$$

2-misol. $f(x)=\dfrac{1}{x}$ funksiyani $x=-3$ nuqta atrofida Teylor qatoriga yoying.

Yechish. Berilgan $f(x)=\dfrac{1}{x}$ funksiya hosilalarini va ularning $x=-3$ nuqtadagi qiymatlarini topamiz.

$$f(x)=\frac{1}{x}=x^{-1}, \qquad f(-3)=-\frac{1}{3},$$

$$f'(x)=-1\cdot x^{-2}=-\frac{1}{x^2}, \qquad f'(-3)=-\frac{1}{3^2},$$

$$f''(x)=1\cdot 2\cdot x^{-3}=\frac{2!}{x^3}, \qquad f''(-3)=-\frac{2!}{3^3},$$

$$f'''(x)=-1\cdot 2\cdot 3\cdot x^{-4}=-\frac{3!}{x^4}, \qquad f'''(-3)=-\frac{3!}{3^4},$$

. . . .

$$f^{(n)}(x)=(-1)^n\frac{n!}{x^{n+1}}, \qquad f^{(n)}(-3)=-\frac{n!}{3^{n+1}},$$

. . . .

Topilganlarni Teylor qatoriga qo'ysak, $f(x)=\dfrac{1}{x}$ funksiya uchun Teylor qatori

$$-\frac{1}{3}-\frac{1!}{3^2}\frac{x+3}{1!}-\frac{2!}{3^4}\frac{(x+3)^2}{2!}-\ldots-\frac{n!}{3^{n+1}}\frac{(x+3)^n}{n!}-\ldots=$$

$$=-\frac{1}{3}\left(1+\frac{x+3}{3}+\frac{(x+3)^2}{3^2}+\ldots+\frac{(x+3)^n}{3^n}+\ldots\right).$$

ko'rinishda bo'ladi. Bu qatorning yaqinlashish intervalini Dalamber alomati bo'yicha topamiz.

$$u_n(x)=\frac{(x+3)^n}{3^n}\ ;\ u_{n+1}(x)=\frac{(x+3)^{n+1}}{3^{n+1}}.$$

$$l=\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right|=\lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{3^{n+1}}\cdot\frac{3^n}{(x+3)^n}\right|==\lim_{n\to\infty}\frac{|x+3|}{3}=\frac{|x+3|}{3}<1.$$

Bundan $|x+3|<3$, $-3<x+3<3$, $-6<x<0$. Intervalning chegaralarida, ya'ni $x=-6$, $x=0$ da mos ravishda

$$1-1+1-1+\ldots,\qquad 1+1+1+1+\ldots$$

uzoqlashuvchi qatorlar hosil bo'ladi. Demak, qatorning yaqinlashish intervali $(-6;\,0)$ bo'ladi. Hosil bo'lgan qatorda

$$\lim_{n\to\infty}R_n(x)=0$$

ekanligini oson isbotlash mumkin. Demak, quyidagi tenglik o'rinli:

$$\frac{1}{x}=-\frac{1}{3}\left(1+\frac{x+3}{3}+\frac{(x+3)^2}{3^2}+\ldots+\frac{(x+3)^n}{3^n}+\ldots\right).$$

3-misol. $f(x)=\dfrac{1}{x^2-3x+2}$ funksiyani x ning darajalari bo'yicha Makloren qatoriga yoying.

Yechish. Berilgan funksiyani sodda kasrlarga ajratamiz

$$\frac{1}{x^2-3x+2}=\frac{1}{x-2}-\frac{1}{x-1}.$$

Bu misolni yechish uchun quyidagi binomial qator yoyilmasidan foydalanamiz

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\ldots+\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+\ldots,$$

$(-1<x<1)$.

Demak,

$$\frac{1}{x-2}=\frac{-1}{2\left(1-\frac{x}{2}\right)}=-\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{2^n}+\ldots\right)=$$

$$=-\frac{1}{2}-\frac{x}{2^2}-\frac{x^2}{2^3}-\ldots-\frac{x^n}{2^{n+1}}-\ldots,\quad(-2<x<2)$$

$$\frac{1}{x-1}=-\frac{1}{1-x}=-\left(1+x+x^2+\ldots+x^n+\ldots\right)=$$

$$=-1-x-x^2-\ldots-x^n-\ldots,\quad(-1<x<1).$$

Qatorlarni hadma-had ayirish natijasida quyidagiga ega bo'lamiz

$$\frac{1}{x^2-3x+2}=\frac{1}{x-2}-\frac{1}{x-1}=\frac{1}{2}+\frac{3}{4}x+\ldots+\frac{2^{n+1}-1}{2^{n+1}}x^n+\ldots$$

$(-1<x<1)$.

4-misol. $f(x)=e^{-x}\sin x$ funksiyani Makloren qatoriga yoying.

Echish. Ma'lumki

$$e^x=1+x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n+\ldots,\quad (-\infty<x<\infty),$$

$$\sin x=x-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots,(-\infty<x<\infty).$$

U holda

$$e^{-x}=1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\ldots+(-1)^n\frac{1}{n!}x^n+\ldots .$$

Ko'paytmani aniqlaymiz

$$f(x)=e^{-x}\sin x=\left(1-x+\frac{1}{2!}x^2-\frac{1}{3!}x^3+\ldots+(-1)^n\frac{1}{n!}x^n+\ldots\right)\times$$

$$\times\left(x-\frac{x^3}{3!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\ldots\right)=x-x^2+\frac{1}{3}x^3-\frac{3}{40}x^5+\ldots,$$

$(-\infty<x<\infty)$.

5-misol. $f(x)=\ln\sqrt{\frac{1+3x}{1-x}}$ funksiyani Makloren qatoriga yoying.

Yechish. Ma'lumki

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n},\qquad (-1<x<1).$$

U holda

$$\ln(1+3x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3^n\cdot x^n}{n},\qquad \left(-\frac{1}{3}<x<\frac{1}{3}\right),$$

$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n},\qquad (-1<x<1),$$

$$\ln\sqrt{\frac{1+3x}{1-x}}=\frac{1}{2}(\ln(1+3x)-\ln(1-x))=$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left((-1)^{n+1}3^n+1\right)\frac{x^n}{n}, \qquad \left(-\frac{1}{3}<x<\frac{1}{3}\right).$$

Mustaqil yechish uchun mashqlar

Quyidagi funksiyalarni $(x-x_0)$ ning darajalari bo'yicha Teylor qatoriga yoying.

1. $f(x)=\frac{1}{x}$; $x_0=3$

Javob: $f(x)=\frac{1}{3}-\frac{x-3}{3^2}+...+(-1)^{n+1}\frac{(x-3)^{n-1}}{3^n}+...,0<x<6$.

2. $f(x)=x^2-3x$; $x_0=1$

Javob: $f(x)=-2+3(x-1)^2+(x-1)^3$.

3. $f(x)=x^4-3x^2$; $x_0=-2$

Javob: $f(x)=-16(x+2)+20(x+2)^2-8(x+2)^3+(x+2)^4$.

4. $f(x)=\ln(x+2)$; $x_0=1$

Javob: $f(x)=\ln 3+\frac{1}{3}(x-1)-\frac{1}{3^2}\frac{(x-1)}{2}+\frac{1}{3^3}\frac{(x-1)}{3}+...+$

$+\frac{(-1)^{n-1}}{3^n}\frac{(x-1)^{n-1}}{n}+...$, $-2\le x\le 4$.

5. $f(x)=\cos x$; $x_0=\frac{\pi}{4}$

Javob: $f(x)=\frac{\sqrt{2}}{2}\left(1-\frac{x-\frac{\pi}{4}}{1!}-\frac{\left(x-\frac{\pi}{4}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{4}\right)^3}{3!}+...\right)-\infty\le x\le+\infty$.

6. $f(x)=e^x$; $x_0=-2$

Javob: $f(x)=e^{-2}\left(1+\frac{x+2}{1!}-\frac{(x+2)^2}{2!}+...+\frac{(x+2)^n}{n!}+...\right)-\infty\le x\le+\infty$.

Quyidagi funksiyalarni x ning darajalari bo'yicha Makloren qatoriga yoying.

1. $f(x)=5^x$

Javob: $f(x)=5^x=1+\sum_{n=1}^{\infty}\frac{x^n \ln^n 5}{n!}$; $(-\infty<x<\infty)$.

2. $f(x)=\frac{e^x-1}{x}$

Javob: $f(x)=\frac{e^x-1}{x}=1+\frac{x}{2!}+...+\frac{x^{n-1}}{n!}+...,(-\infty<x<\infty)$.

3. $f(x)=\ln(10+x)$

Javob: $f(x)=\ln(10+x)=\ln 10+\frac{x}{10}-\frac{x^2}{2\cdot 10^2}+\frac{x^3}{3\cdot 10^3}-...+$

$+(-1)^{n-1}\frac{x^n}{n\cdot 10^n}+...$, $(-\infty<x<\infty)$.

4. $f(x)=\sin^2 x$

Javob: $f(x)=\sin^2 x=\frac{2x^2}{2!}-\frac{8x^4}{4!}+...+(-1)^{n-1}\frac{2^{2n-1}x^{2n}}{(2n)!}+...$

$(-\infty<x<\infty)$.

5. $f(x)=\sin^2 x\cos^2 x$

Javob: $f(x)=\sin^2 x\cos^2 x=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^{4n-3}x^{2n}}{(2n)!}$,

$\sin^2 x\cos^2 x=\frac{1}{4}\sin^2 2x$, $(-\infty<x<\infty)$.

6. $f(x)=\frac{3}{(1-x)(1+2x)}$

Javob: $f(x)=\frac{3}{(1-x)(1+2x)}=\sum_{n=0}^{\infty}\left(1+(-1)^n 2^{n+1}\right)x^n$,

$\frac{3}{(1-x)(1+2x)}=\frac{1}{1-x}+\frac{2}{1+2x}$, $\frac{1}{1-x}=1+x+x^2+...=\sum_{n=0}^{\infty}x^n$,

$\frac{1}{1+2x}=1-2x+(2x)^2-...+(-1)^n 2^n\cdot x^n+...=\sum_{n=0}^{\infty}(-1)^n 2^n x^n$,

$$f(x)=\sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty}(-1)^n 2^n x^n = \sum_{n=0}^{\infty}\left(1+(-1)^n 2^{n+1}\right)x^n .$$

7. $f(x)=(1+x)e^{-x}$

Javob: $f(x)=(1+x)e^{-x}=1+\sum_{n=2}^{\infty}(-1)^{n-1}\frac{n-1}{n!}x^n, (-\infty<x<\infty)$.

8. $f(x)=\ln\frac{1+x}{1-x}$

Javob: $f(x)=\ln\frac{1+x}{1-x}=2x+\frac{2x^3}{3}+\ldots+\frac{2x^{2n+1}}{2n+1}+\ldots,(-1<x<1)$

9. $f(x)=\frac{x^2}{\sqrt{1-x^2}}$

Javob: $f(x)=\frac{x^2}{\sqrt{1-x^2}}=x^2+\frac{1}{2}x^2+\ldots+\frac{(2n-1)!!}{(2n)!!}x^{2n-2}$,

$(-1<x<1))$

10. $f(x)=\frac{\ln(1+x)}{1+x}$

Javob: $f(x)=\frac{\ln(1+x)}{1+x}=x-\left(1+\frac{1}{2}\right)x^2+\left(1+\frac{1}{2}+\frac{1}{3}\right)x^3-$

$-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)x^4+\ldots,\ (-1<x<1)$

11. $f(x)=\ln\left(1+3x+2x^2\right)$

Javob: $f(x)=\ln\left(1+3x+2x^2\right)=(-1)^{n-1}\left(1+2^n\right)\frac{x^n}{n}, \left(-\frac{1}{2}<x\le\frac{1}{2}\right)$

12. Binomial qator yordami bilan $|x|<1$ bo'lganda

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\ldots+$$

$$+\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}x^{2n}+\ldots$$

ekanini ko'rsating va qatorni hadma-had integrallab, $\arcsin x$ uchun qatorga yoying.

Javob: $\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \ldots + \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\frac{x^{2n+1}}{2n+1} + \ldots$.

13. Binomial qator yordami bilan $|x|<1$ bo'lganda

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{1}{2}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}x^4 - \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^6 + \ldots$$

ekanligini ko'rsating va qatorni hadma-had integrallab, $\ln\left(x+\sqrt{1+x^2}\right)$ funksiya uchun qatorga yoying.

4-§. Qatorlarning taqribiy hisoblashlarga tatbiqi

1-misol. $\ln 1{,}1$ ни 10^{-4} gacha aniqlikda hisoblang.

Yechish. Ma'lumki

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n+1}\frac{x^n}{n} + \ldots$$

$(-1 < x \le 1)$ da $x = 0{,}1$ deb olsak

$$\ln(1{,}1) = 0{,}1 - \frac{(0{,}1)^2}{2} + \frac{(0{,}1)^3}{3} - \frac{(0{,}1)^4}{4} + \ldots.$$

Bu ishoralari almashinuvchi Leybnis qatori. Qator to'rtinchi hadining absolyut qiymati 10^{-4} dan kichik bo'lgani sababli, $\ln 1{,}1$ ни 10^{-4} gacha aniqlikda hisoblash uchun qatorning uchta $(n=3)$ hadini olish yetarlidir. Demak,

$$\ln(1{,}1) \approx 0{,}1 - \frac{0{,}01}{2} + \frac{0{,}001}{3} \approx 0{,}0953 \cdot$$

$$\ln(1{,}1) \approx 0{,}0953 .$$

2-misol. $\ln 2$ ни 10^{-5} gacha aniqlikda hisoblang.

Yechish. Logarifmlarni taqribiy hisoblashda

$$\ln(N+1) = \ln N + 2\left(\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \right.$$

$$\left. + \frac{1}{5(2N+1)^5} + \ldots\right)$$

dan foydalanamiz. Bu yerda $N=1$ bo'ladi.

$$\ln 2 = 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \ldots + \frac{1}{(2n+1)\cdot 3^{2n+1}} + \ldots\right).$$

R_{n+1} qoldiq hadni baholaymiz.

$$R_{n+1} = 2\left(\frac{1}{2n+3}\cdot\frac{1}{3^{2n+1}} + \frac{1}{2n+5}\cdot\frac{1}{3^{2n+5}} + \ldots\right) <$$

$$< \frac{2}{(2n+3)\cdot 3^{2n+3}}\left(1 + \frac{1}{3^2} + \frac{1}{3^n}\right) = \frac{2\cdot 9}{(2n+3)\cdot 3^{2n+3}\cdot 8} =$$

$$= \frac{1}{4(2n+3)\cdot 3^{2n+1}} < 10^{-5}.$$

Bundan $4(2n+3)\cdot 3^{2n+1} > 10^5$ va bu tengsizlik $n=4$ bo'lganda o'rinli bo'ladi. Demak,

$$\ln 2 = 2\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \frac{1}{9\cdot 3^9}\right) \approx 0{,}69314 \text{ yoki } \ln 2 \approx 0{,}69314.$$

3-misol. $e^{0,1}$ ни 10^{-3} gacha aniqlikda hisoblang.

Yechish. e^x ning yoyilmasidan foydalanamiz.

$$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots + \frac{1}{(n-1)!}x^{n-1} + R_n,$$

$R_n = \frac{1}{n!}x^{\xi}$, $0 < \xi < x$. $x = 0{,}1$ desak, $e^{\xi} < e^{0,1} < e < 3$ bo'ladi. U holda $R_n < \frac{3}{10^n\cdot n!} < 0{,}001$ tengsizlik $n=3$ bo'lganda o'rinli bo'ladi. Demak, $e^{0,1}$ ни 10^{-3} gacha aniqlikda hisoblash uchun qatorning uchta hadini hisoblash kifoya, ya'ni

$$e^{0,1} \approx 1 + \frac{1}{10} + \frac{1}{200} \approx 1{,}105$$

yoki $e^{0,1} \approx 1{,}105$.

4-misol. $\sqrt[3]{130}$ ни $0{,}0001$ gacha aniqlikda taqribiy hisoblang.

Yechish. Ma'lumki

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 +$$

$$+ \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + \ldots, \ -1 < x < 1$$

$\sqrt[3]{130}$ ni quyidagi ko'rinishda yozamiz

$$\sqrt[3]{130} = \sqrt[3]{125+5} = \sqrt[3]{125\left(1+\frac{1}{25}\right)} = 5\left(1+\frac{1}{25}\right)^{\frac{1}{3}}. \quad m = \frac{1}{3} \text{ bo'lsa}$$

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3 + \ldots =$$

$$= 1 + \frac{1}{3}x - \frac{1\cdot 2}{3^2\cdot 2!}x^2 + \frac{1\cdot 2\cdot 5}{3^3\cdot 3!}x^3 - \frac{1\cdot 2\cdot 5\cdot 8}{3^4\cdot 4!}x^4 + \ldots.$$

Endi hosil bo'lgan qatorda x ning o'rniga $\frac{1}{25}$ ni qo'yamiz.

$$\left(1+\frac{1}{25}\right)^{\frac{1}{3}} = 1 + \frac{1}{3\cdot 5^2} - \frac{1\cdot 2}{3^2\cdot 2!\cdot 5^4} + \frac{1\cdot 2\cdot 5}{3^3\cdot 3!\cdot 5^6} - \frac{1\cdot 2\cdot 5\cdot 8}{3^4\cdot 4!\cdot 5^8} + \ldots.$$

Ishoralari almashinuvchi qator hosil bo'ldi. Ildizning qiymatini 0,0001 gacha aniqlikda taqribiy hisoblash uchun qatorning 3 ta hadini olish kifoya, chunki to'rtinchi hadi

$$\frac{5\cdot 1\cdot 2\cdot 5}{3^3\cdot 3!\cdot 5^6} = \frac{1\cdot 2}{3^3\cdot 6\cdot 5^4} = \frac{1}{81\cdot 625} < 0{,}0001$$

tengsizlik o'rinli. Demak,

$$\sqrt[3]{130} \approx 5\left(1 + \frac{1}{3\cdot 5^2} - \frac{1\cdot 2}{3^2\cdot 2!\cdot 5^4}\right) \approx 5{,}0000 + 0{,}0667 - 0{,}0009 \approx 5{,}0658$$

$\sqrt[3]{130} \approx 5{,}0658$.

5-misol. $\sin 5^0$ ni 10^{-6} gacha aniqlikda taqribiy hisoblang.

Yechish. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Foydalanib, 5^0 radian hisobida $\frac{\pi}{36}$ bo'lganligi uchun $\left(\frac{2\pi}{360}\cdot 5=\frac{\pi}{36}\right)$.

$$\sin\frac{\pi}{36}=\frac{\pi}{36}-\frac{\pi^3}{36^3\cdot 3!}+\frac{\pi^5}{36^5\cdot 5!}-\frac{\pi^7}{36^7\cdot 7!}+\dots$$

bo'ladi. Qatorning uchinchi hadini baholaymiz.

$$\frac{1}{5!}\left(\frac{\pi}{36}\right)^5<\frac{1}{5!}(0{,}1)^5=\frac{1}{120}\cdot 10^{-5}=\frac{5}{6}\cdot 10^{-7}.$$

Qatorning ikkita hadi bilan chegaralansak ham bo'lar ekan, chunki hisoblashda qilingan xatolik $\frac{5}{6}\cdot 10^{-7}$ dan kichik bo'ladi. Shuning uchun

$$\sin\frac{\pi}{36}\approx\frac{\pi}{36}-\frac{\pi^3}{36^3\cdot 3!}\approx 0{,}0872665-0{,}0001107=0{,}0871558$$

$\sin\frac{\pi}{36}\approx 0{,}0871558$.

6-misol. $\int_0^1\cos\sqrt{x}\,dx$ integralni 0,0001 gacha aniqlikda taqribiy hisoblang.

Yechish. Ma'lumki

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots+(-1)^n\frac{x^{2n}}{(2n)!}+\dots.$$

Agar x ni $\sqrt{x}$ bilan almashtirsak,

$$\cos\sqrt{x}=1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\dots+(-1)^n\frac{x^n}{(2n)!}+\dots,\quad (x\ge 0)$$

hosil bo'ladi. Bu tenglikning ikkala tomonini 0 dan 1 gacha chegaralarda integrallab quyidagini topamiz:

$$\int_0^1\cos\sqrt{x}\,dx=\left(x-\frac{x^2}{2\cdot 2!}+\frac{x^3}{3\cdot 4!}-\frac{x^4}{4\cdot 6!}+\dots\right)\Bigg|_0^1=1-\frac{1}{2\cdot 2!}+\frac{1}{3\cdot 4!}-\frac{1}{4\cdot 6!}+\dots.$$

hosil bo'lgan ishoralari almashinuvchi qator 5-hadining absolyut qiymati 0,0001 dan kichik bo'lganligi sababli qatorning birinchi 4 ta hadini olish kifoya. Demak,

$$\int_0^1 \cos\sqrt{x}\,dx \approx 1-\frac{1}{4}+\frac{1}{72}-\frac{1}{2880} \approx 0{,}7635$$

$$\int_0^1 \cos\sqrt{x}\,dx \approx 0{,}7635 .$$

Mustaqil yechish uchun mashqlar

Quyidagi ifodalarni berilgan ε aniqlikda taqribiy hisoblang.

1. $\ln 5$; $\varepsilon = 0{,}001$ Javob: 1,609
2. $\lg 101$; $\varepsilon = 0{,}0001$ Javob: 2,0043
3. $\ln 17$; $\varepsilon = 0{,}001$ Javob: 2,833
4. $\sqrt[3]{30}$; $\varepsilon = 0{,}001$ Javob: 3,1072
5. $\sqrt[3]{10}$; $\varepsilon = 0{,}001$ Javob: 2,154
6. $\sqrt[5]{250}$; $\varepsilon = 0{,}001$ Javob: 3,617
7. $\cos 10^0$; $\varepsilon = 0{,}0001$ Javob: 0,9948
8. $\sin 0{,}4$; $\varepsilon = 0{,}001$ Javob: 0,3894
9. $\sin 18^0$; $\varepsilon = 0{,}001$ Javob: 0,309
10. $\operatorname{arctg} 0{,}2$; $\varepsilon = 0{,}0001$ Javob: 0,1973
11. $\sqrt{e}$; $\varepsilon = 0{,}001$ Javob: 1,649
12. $\frac{1}{\sqrt[4]{e}}$; $\varepsilon = 0{,}001$ Javob: 0,779

Quyidagi integrallarni 0,001 aniqlikda taqribiy hisoblang.

1. $\int_0^1 \sqrt[3]{x}\cos x\,dx$ Javob: 0,608
2. $\int_0^1 \sin x^2\,dx$ Javob: 0,310

3. $\int\limits_0^{\pi/4} \frac{\sin x}{x} dx$ Javob: 0,758

4. $\int\limits_0^1 e^{-x^2} dx$ Javob: 0,747

5. $\int\limits_0^{1/2} \frac{\operatorname{arctg} x}{x} dx$ Javob: 0,487

6. $\int\limits_0^{1/2} \cos \frac{x^2}{4} dx$ Javob: 0,500

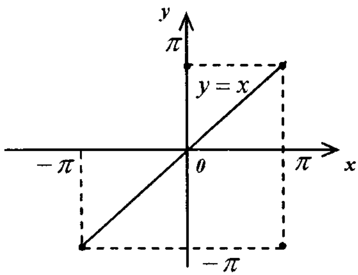
7. $\int\limits_0^{1/4} \ln\left(1+\sqrt{x}\right) dx$ Javob: 0,072

8. $\int\limits_0^{1/2} \sqrt{1+x^3}\, dx$ Javob: 0,508

III BOB. FURYE QATORI

1-§. Davri 2π bo'lgan funksiyalarni Furye qatoriga yoyish

1-misol. Davri $T=2\pi$ bo'lgan $f(x)=x$ funksiyani $[-\pi;\pi]$ kesmada Furye qatoriga yoying (8-rasm).



8-rasm.

Yechish. $f(x)=x$ funksiya $[-\pi;\pi]$ da Dirixle shartlarini qanoatlantiradi. Shuning uchun Furye qatoriga yoyish mumkin va a_0, a_k, b_k koeffitsientlarini hammasini quyidagi formulalar orqali topamiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\,dx=\frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi}=\frac{1}{2\pi}\left(\pi^2-\pi^2\right)=0,$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos kx\,dx=\begin{bmatrix}u=x, & dv=\cos kx\,dx\\ du=dx, & v=\frac{1}{k}\sin kx\end{bmatrix}=$$

$$=\frac{1}{\pi}\left[\frac{x}{k}\sin kx\Big|_{-\pi}^{\pi}-\frac{1}{k}\int_{-\pi}^{\pi}\sin kx\,dx\right]=\frac{1}{\pi}\left[0+\frac{1}{k^2}\cos kx\Big|_{-\pi}^{\pi}\right]=$$

$$=\frac{1}{k^3\pi}\left(\cos k\pi-\cos k\pi\right)=0,$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin kx\,dx=\begin{bmatrix}u=x, & dv=\sin kx\,dx\\ du=dx, & v=-\frac{1}{k}\cos kx\end{bmatrix}=$$

$$= \frac{1}{\pi}\left[-\frac{x}{k}\cos kx\bigg|_{-\pi}^{\pi} + \frac{1}{k}\int_{-\pi}^{\pi}\cos kx\,dx\bigg|\right] = \frac{1}{\pi}\left[\left(-\frac{\pi}{k}-\frac{\pi}{k}\right)\cos k\pi + 0\bigg|\right] =$$

$$= -\frac{2}{k}\cos k\pi = (-1)^{k+1}\frac{2}{k}.$$

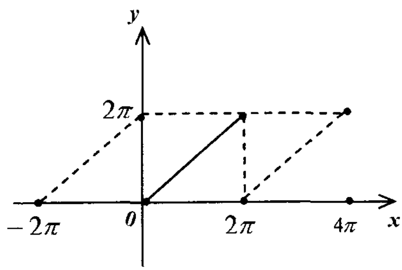
$$b_k = (-1)^{k+1}\frac{2}{k}.$$

Javob: $f(x) = x = 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$

yoki

$$x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots + (-1)^{k+1}\frac{\sin kx}{k} + \ldots\right).$$

2-misol. Davri $T = 2\pi$ bo'lgan $f(x) = x$ funksiyani $[0;\, 2\pi]$ kesmada Furye qatoriga yoying (9-rasm).



9-rasm.

Yechish. Davri $T = 2\pi$ bo'lgan $f(x) = x$ funksiya uchun quyidagi tenglik o'rinlidir:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{\lambda}^{\lambda+2\pi} f(x)dx$$

va $\lambda = 0$ bo'lsa

$$\int_{-\pi}^{\pi} f(x)dx = \int_{0}^{2\pi} f(x)dx. \qquad (1)$$

Bu formulalardan foydalanib, a_0, a_k, b_k koeffitsientlarni topamiz.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}x\,dx=\frac{1}{\pi}\frac{x^2}{2}\Bigg|_0^{2\pi}=\frac{1}{2\pi}\left(4\pi^2-0\right)=2\pi,$$

$$a_k=\frac{1}{\pi}\int_0^{2\pi}x\cos kx\,dx=\begin{bmatrix}u=x, & dv=\cos kx\,dx\\ du=dx, & v=\frac{1}{k}\sin kx\end{bmatrix}=\frac{1}{\pi}\left[\frac{x}{k}\sin kx\Big|_0^{2\pi}-\right.$$

$$\left.-\frac{1}{k}\int_0^{2\pi}\sin kx\,dx\right]=\frac{1}{\pi}\left[0+\frac{1}{k^2}\cos kx\Big|_0^{2\pi}\right]=\frac{1}{k^2\pi}\left(\cos 2k\pi-\cos 0\right)=0,$$

$$b_k=\frac{1}{\pi}\int_0^{2\pi}x\sin kx\,dx=\begin{bmatrix}u=x, & dv=\sin kx\,dx\\ du=dx, & v=-\frac{1}{k}\cos kx\end{bmatrix}=\frac{1}{\pi}\left[-\frac{x}{k}\cos kx\Big|_0^{2\pi}+\right.$$

$$\left.+\frac{1}{k}\int_0^{2\pi}\cos kx\,dx\right]=\frac{1}{\pi}\left[-\frac{2\pi}{k}+0-\frac{1}{k^2}\sin kx\Big|_0^{2\pi}\right]=\frac{1}{\pi}\left(-\frac{2\pi}{k}\right)=-\frac{2}{k}.$$

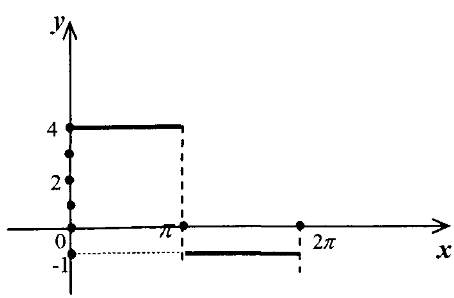
Javob: $x=\pi-2\sum_{k=1}^{\infty}\frac{\sin kx}{k}$

yoki

$$x=\pi-2\left(\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\ldots+\frac{\sin kx}{k}+\ldots\right).$$

3-misol. Davri $T=2\pi$ bo'lgan quyidagi funksiyani $[0;\,2\pi]$ kesmada Furye qatoriga yoying: (10-rasm).

$$f(x)=\begin{cases}4, & \text{agar } 0\le x<\pi\\ -1, & \text{agar } \pi\le x\le 2\pi\end{cases}$$



10-rasm.

Yechish. 2-misoldagi (1) tenglikdan foydalanib, a_0, a_k, b_k koeffitsientlarni topamiz.

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\left(\int_0^{\pi}4\,dx-\int_{\pi}^{2\pi}dx\right)=\frac{1}{\pi}\left(4x\Big|_0^{\pi}-x\Big|_{\pi}^{2\pi}\right)=3,$$

$$a_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos kx\,dx=\frac{1}{\pi}\left[\int_0^{\pi}4\cos kx\,dx-\int_{\pi}^{2\pi}\cos kx\,dx\right]=$$

$$=\frac{1}{\pi}\left[\frac{4}{k}\sin kx\Big|_0^{\pi}-\frac{1}{k}\sin kx\Big|_{\pi}^{2\pi}\right]=\frac{1}{\pi}(0-0)=0,$$

$$b_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin kx\,dx=\frac{1}{\pi}\left[\int_0^{\pi}4\sin kx\,dx-\int_{\pi}^{2\pi}\sin kx\,dx\right]=$$

$$=\frac{1}{\pi}\left[-\frac{4}{k}\cos kx\Big|_0^{\pi}+\frac{1}{k}\cos kx\Big|_{\pi}^{2\pi}\right]=\frac{1}{\pi}\left(-\frac{4}{k}(\cos k\pi-1)+\frac{1}{k}-\frac{1}{k}\cos k\pi\right)=$$

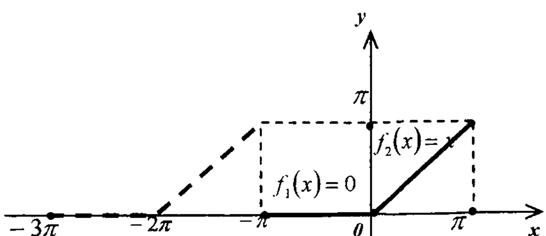
$$=(-1)^{k+1}\frac{5}{k\pi}+\frac{5}{k\pi}=\frac{5}{k\pi}\left[1-(-1)^k\right]=\begin{cases}0 & \text{, agar } k=2n\\ \dfrac{10}{k\pi} & \text{, agar } k=2n+1\end{cases}.$$

Javob: $f(x)=\dfrac{3}{2}+\dfrac{10}{\pi}\sum\limits_{k=0}^{\infty}\dfrac{\sin(2k+1)x}{2k+1}$.

4-misol. Davri $T=2\pi$ bo'lgan quyidagi funksiyani:

$$f(x)=\begin{cases}0, & \text{agar } -\pi\le x<0\\ x, & \text{agar } 0\le x\le\pi\end{cases}$$

$[-\pi;\pi]$ kesmada Furye qatoriga yoying. (11-rasm)



11-rasm.

Yechish. Berilgan funksiya $[-\pi;\pi]$ da Dirixle shartlarini qanoatlantiradi. a_0, a_k, b_k koeffitsientlarini topamiz.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{0}f_1(x)dx+\frac{1}{\pi}\int_{0}^{\pi}f_2(x)dx=$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cdot dx+\int_{0}^{\pi}x\,dx\right]=\frac{1}{\pi}\cdot\left.\frac{x^2}{2}\right|_0^{\pi}=\frac{\pi^2}{2\pi}=\frac{\pi}{2},$$

$$a_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos kx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}f_1(x)\cos kx\,dx+\frac{1}{\pi}\int_{0}^{\pi}f_2(x)\cos kx\,dx=$$

$$=\begin{bmatrix}u=x, & dv=\cos kx\,dx\\ du=dx, & v=\dfrac{1}{k}\sin kx\end{bmatrix}=\frac{1}{\pi}\left[\left.\frac{x}{k}\sin kx\right|_0^{\pi}-\frac{1}{k}\int_0^{\pi}\sin kx\,dx\right]=$$

$$=\frac{1}{\pi}\left[0+\left.\frac{1}{k^2}\cos kx\right|_0^{\pi}\right]=\frac{1}{k^2\pi}(\cos k\pi-\cos 0)=\frac{1}{k^2\pi}\left[(-1)^k-1\right]=$$

$$=\begin{cases}-\dfrac{2}{k^2\pi}, & \text{agar } k=2n+1,\\ 0, & \text{agar } k=2n\end{cases}$$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin kx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}0\cdot\sin kx\,dx+\frac{1}{\pi}\int_{0}^{\pi}x\sin kx\,dx=$$

$$=\frac{1}{\pi}\int_0^{\pi}x\sin kx\,dx=\begin{bmatrix}u=x, & dv=\sin kx\,dx\\ du=dx, & v=-\dfrac{1}{k}\cos kx\end{bmatrix}=\frac{1}{\pi}\left[-\left.\frac{x}{k}\cos kx\right|_0^{\pi}+\right.$$

$$\left.+\frac{1}{k}\int_0^{\pi}\cos kx\,dx\right]=-(-1)^k\frac{1}{\pi}\cdot\frac{\pi}{k}+0=(-1)^{k+1}\frac{1}{k}.$$

Javob: $f(x)=\dfrac{\pi}{4}-\dfrac{2}{\pi}\sum\limits_{k=1}^{\infty}\dfrac{\cos(2k+1)x}{(2k+1)^2}+\sum\limits_{k=1}^{\infty}\dfrac{(-1)^{k+1}}{k}\sin kx\cdot$

Mustaqil yechish uchun mashqlar

1. Davri $T=2\pi$ bo'lgan $f(x)=x^2$ funksiyani $[0;\,2\pi]$ kesmada Furye qatoriga yoying.

Javob: $f(x)=\dfrac{4\pi}{3}+4\sum\limits_{k=1}^{\infty}\left(\dfrac{\cos kx}{k^2}-\pi\dfrac{\sin kx}{k}\right).$

2. Davri $T=2\pi$ bo'lgan $f(x)=e^x$ funksiyani $[-\pi;\,\pi]$ kesmada Furye qatoriga yoying.

Javob: $f(x)=\dfrac{2}{\pi}\operatorname{sh}\pi\left[\dfrac{1}{2}+4\sum\limits_{k=1}^{\infty}\dfrac{(-1)^k}{k^2+1}(\cos kx-k\sin kx)\right].$

3. Davri $T=2\pi$ bo'lgan $f(x)=x-2$ funksiyani $[-\pi;\,\pi]$ kesmada Furye qatoriga yoying.

Javob:

$$f(x)=\frac{4+\pi^2}{2\pi}+\frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{\left((-1)^k 2k-\sin 2k\right)\sin kx}{k^2}+\frac{\left((-1)^k-\cos 2k\right)\cos kx}{k^2}\right).$$

4. Davri $T=2\pi$ bo'lgan quyidagi:

$$f(x)=\begin{cases}0, & \text{agar } -\pi\le x<0\\ x^2, & \text{agar } 0\le x\le\pi\end{cases}$$

funksiyani $[-\pi;\,\pi]$ kesmada Furye qatoriga yoying.

Javob:

$$f(x)=\frac{\pi^2}{6}+2\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kx+\pi\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin kx}{k^3}.$$

5. Davri $T=2\pi$ bo'lgan quyidagi:

$$f(x)=\begin{cases}-2x, & \text{agar } -\pi\le x<0\\ 3x, & \text{agar } 0\le x\le\pi\end{cases}$$

funksiyani $[-\pi;\,\pi]$ kesmada Furye qatoriga yoying.

Javob: $f(x)=\dfrac{5\pi}{4}-\dfrac{10}{\pi}\sum\limits_{k=0}^{\infty}\dfrac{\cos(2k+1)x}{(2k+1)^2}+\sum\limits_{k=1}^{\infty}\dfrac{\sin kx}{k}.$

6. Davri $T=2\pi$ bo'lgan $f(x)=\pi-x$ funksiyani $[0;\,2\pi]$ kesmada Furye qatoriga yoying.

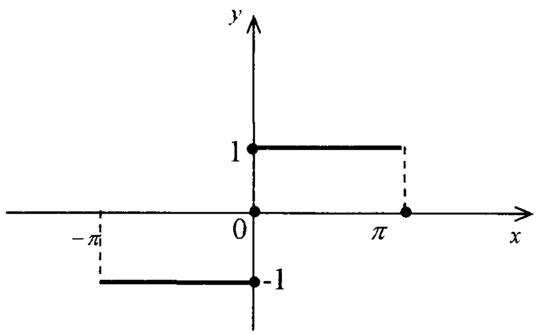
Javob: $f(x) = 2\sum_{k=1}^{\infty} \frac{\sin kx}{k}$.

2-§. Juft va toq funksiyalarni Furye qatoriga yoyish

1-misol. Davri $T = 2\pi$ bo'lgan quyidagi:

$$f(x) = \begin{cases} -1, & \text{agar } -\pi \le x < 0 \\ 1, & \text{agar } 0 \le x \le \pi \end{cases}$$

funksiyani $[-\pi; \pi]$ kesmada Furye qatoriga yoying. (12-rasm)



12-rasm.

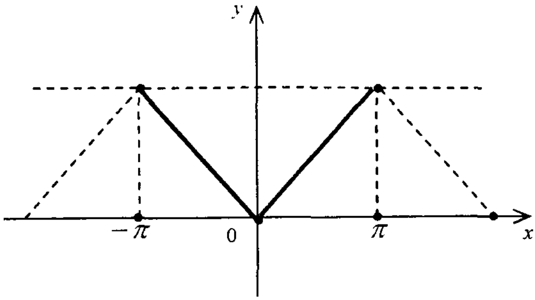
Yechish. Bu funksiya toq funksiya. U holda $a_0 = 0$, $a_k = 0$. b_k koeffitsientni topamiz.

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin kx\,dx = -\frac{2}{k\pi}\cos kx\Big|_0^{\pi} =$$

$$= -\frac{2}{k\pi}(\cos k\pi - \cos 0) = -\frac{2}{k\pi}\left[(-1)^k - 1\right] = \begin{cases} 0, & \text{agar } k = 2n \\ \dfrac{4}{k\pi}, & \text{agar } k = 2n+1 \end{cases}.$$

Javob: $f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$.

2-misol. Davri $T = 2\pi$ bo'lgan $f(x)=|x|$ funksiyani $[-\pi;\pi]$ kesmada Furye qatoriga yoying (13-rasm).



13-rasm.

Yechish. Bu funksiya juft funksiya. U holda $b_k = 0$. a_0, a_k koeffitsientlarni topamiz.

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^\pi = \pi,$$

$$a_k = \frac{2}{\pi}\int_0^\pi x\cos kx\,dx = \begin{bmatrix} u = x, & dv = \cos kx\,dx \\ du = dx, & v = \frac{1}{k}\sin kx \end{bmatrix} = \frac{2}{\pi}\left[\frac{x}{k}\sin kx\Big|_0^\pi - \right.$$

$$\left. -\frac{1}{k}\int_0^\pi \sin kx\,dx\right] = \frac{2}{k^2\pi}\cos kx\Big|_0^\pi = \frac{2}{k^2\pi}(\cos k\pi - \cos 0) =$$

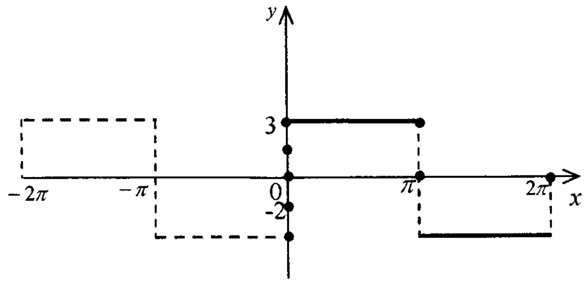
$$= \frac{2}{k^2\pi}\left[(-1)^k - 1\right] = \begin{cases} 0\;, & \text{agar } k = 2n \\ -\dfrac{4}{k^2\pi}, & \text{agar } k = 2n+1 \end{cases}.$$

Javob: $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi}\sum\limits_{k=0}^{\infty}\dfrac{\cos(2k+1)x}{(2k+1)^2}$.

3-misol. Davri $T = 2\pi$ bo'lgan quyidagi

$$f(x) = \begin{cases} 3, & \text{agar } 0 \le x < \pi \\ -2, & \text{agar } \pi \le x \le 2\pi \end{cases}$$

funksiyani $[0;\,2\pi]$ kesmada Furye qatoriga yoying. (14-rasm)



14-rasm.

Yechish. Bu funksiya 2π davrli bo'lib, u toq ham, juft ham emas. Uning Furye koeffitsientlarini topish uchun ko'rilgan lemma shartidan foydalanamiz.

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\left(\int_0^{\pi}3\,dx-\int_{\pi}^{2\pi}2dx\right)=\frac{1}{\pi}\left(3x\big|_0^{\pi}-2x\big|_{\pi}^{2\pi}\right)=1,$$

$$a_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos kx\,dx=\frac{1}{\pi}\left[\int_0^{\pi}3\cos kx\,dx-\int_{\pi}^{2\pi}2\cos kx\,dx\right]=$$

$$=\frac{1}{\pi}\left[\frac{3}{k}\sin kx\bigg|_0^{\pi}-\frac{2}{k}\sin kx\bigg|_{\pi}^{2\pi}\right]=\frac{1}{\pi}(0-0)=0,$$

$$b_k=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin kx\,dx=\frac{1}{\pi}\left[\int_0^{\pi}3\sin kx\,dx-\int_{\pi}^{2\pi}2\sin kx\,dx\right]=$$

$$=\frac{1}{\pi}\left[-\frac{3}{k}\cos kx\bigg|_0^{\pi}+\frac{2}{k}\cos kx\bigg|_{\pi}^{2\pi}\right]=\frac{1}{\pi}\left(-\frac{3}{k}\left[(-1)^k-1\right]+\frac{2}{k}\left[1-(-1)^k\right]\right)=$$

$$=\frac{5}{k\pi}\left[1-(-1)^k\right]=\begin{cases}0\quad, \text{агар}\quad k=2n\\ \dfrac{10}{k\pi},\ \text{агар}\quad k=2n+1\end{cases}.$$

Javob: $f(x)=\dfrac{1}{2}+\dfrac{10}{\pi}\sum\limits_{k=0}^{\infty}\dfrac{\sin(2k+1)x}{2k+1}$.

Mustaqil yechish uchun mashqlar

1. Davri $T = 2\pi$ bo'lgan $f(x) = x^3$ funksiyani $[-\pi; \pi]$ kesmada Furye qatoriga yoying.

Javob: $f(x) = \sum_{k=1}^{\infty}(-1)^k\left(\frac{12}{k^3} - \frac{2\pi^2}{k}\right)\sin kx$.

2. Davri $T = 2\pi$ bo'lgan quyidagi

$$f(x) = \begin{cases} -2, & \text{agar } -\pi \le x < 0 \\ 2, & \text{agar } 0 \le x \le \pi \end{cases}$$

funksiyani $[-\pi; \pi]$ kesmada Furye qatoriga yoying.

Javob: $f(x) = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}$.

3. Davri $T = 2\pi$ bo'lgan $f(x) = x^2$ funksiyani $[0; 2\pi]$ kesmada Furye qatoriga yoying.

Javob: $f(x) = \frac{4\pi}{3} + 4\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\cos kx - \frac{\pi}{k}\sin kx\right)$.

4. Davri $T = 2\pi$ bo'lgan $f(x) = x$ funksiyani $[-\pi; \pi]$ kesmada Furye qatoriga yoying.

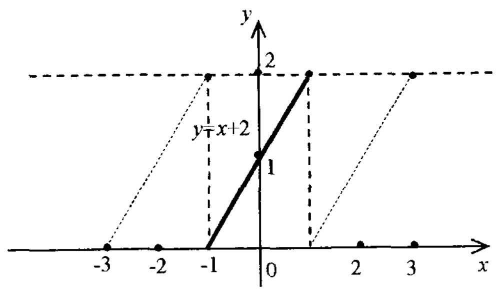
Javob: $f(x) = 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}$.

5. Davri $T = 2\pi$ bo'lgan $f(x) = \sin x$ funksiyani $[-\pi; \pi]$ kesmada Furye qatoriga yoying.

Javob: $f(x) = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2kx}{4k^2 - 1}$.

3-§. Davri $2l$ bo'lgan funksiyalarni Furye qatoriga yoyish

1-misol. Davri $T=2$ bo'lgan $f(x)=x+1$ funksiyani $[-1;\,1]$ kesmada Furye qatoriga yoying (15-rasm).



15-rasm.

Yechish. $f(x)=x+1$ funksiya toq ham, juft ham emas. Shuning uchun a_0, a_k, b_k koeffitsientlarni topamiz. Bunda $l=1$.

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx=\int_{-1}^{1}(x+1)dx=\left(\frac{x^2}{2}+x\right)\Bigg|_{-1}^{1}=2,$$

$$a_k=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{k\pi}{l}x\,dx=\int_{-1}^{1}(x+1)\cos k\pi x\,dx=$$

$$=\begin{bmatrix} u=x+1, & dv=\cos k\pi x\,dx \\ du=dx, & v=\dfrac{1}{k\pi}\sin k\pi x \end{bmatrix}=\frac{x+1}{k\pi}\sin k\pi x\Bigg|_{-1}^{1}-\frac{1}{k\pi}\int_{-1}^{1}\sin k\pi x\,dx=$$

$$=\frac{2}{k\pi}\sin k\pi-0+\frac{1}{k^2\pi^2}\cos k\pi x\Bigg|_{-1}^{1}=\frac{2}{k^2\pi^2}(\cos k\pi-\cos k\pi)=0,$$

$$b_k = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{k\pi}{l}x\,dx = \int_{-1}^{1}(x+1)\sin k\pi x\,dx =$$

$$= \begin{bmatrix} u = x+1, & dv = \sin k\pi x\,dx \\ du = dx, & v = -\dfrac{1}{k\pi}\cos k\pi x \end{bmatrix} = -\frac{x+1}{k\pi}\cos k\pi x\Big|_{-1}^{1} +$$

$$+\frac{1}{k\pi}\int_{-1}^{1}\cos k\pi x\,dx = -\frac{2}{k\pi}\cos k\pi - 0 + \frac{1}{k^2\pi^2}\sin k\pi x\Big|_{-1}^{1} =$$

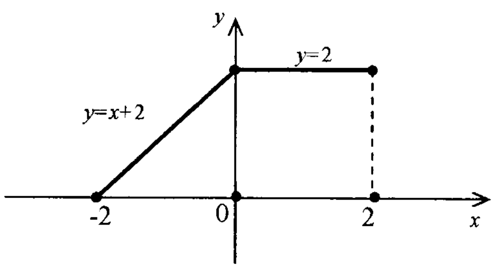
$$= -\frac{2}{k\pi}(-1)^k + \frac{1}{k^2\pi^2}(\sin k\pi - \sin k\pi) = -\frac{2}{k\pi}(-1)^k = (-1)^{k+1}\frac{2}{k\pi}.$$

Javob: $f(x) = 1 + \frac{2}{\pi}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k}.$

2-misol. Davri $T = 4$ bo'lgan quyidagi:

$$f(x) = \begin{cases} 2+x, & \text{agar } -2 \le x < 0 \\ 2, & \text{agar } 0 \le x \le 2 \end{cases}$$

funksiyani $[-2; 2]$ kesmada Furye qatoriga yoying. (16-rasm)



16-rasm.

Yechish. Bu funksiya juft ham, toq ham emas, $l =$
Shuning uchun a_0, a_k va b_k koeffitsientlarning topamiz.

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{1}{2}\int_{-2}^{0}(x+2)dx + \frac{1}{2}\int_{0}^{2}2\,dx = \frac{1}{2}\left(\frac{x^2}{2} + 2x\right)\Bigg|_{-2}^{0} + \frac{1}{2}\cdot 2x\Big|_{0}^{2} =$$

$$a_k=\frac{1}{2}\int_{-2}^{0}(x+2)\cos\frac{k\pi}{2}x\,dx+\frac{1}{2}\int_{0}^{2}2\cos\frac{k\pi}{2}x\,dx=$$

$$=\begin{bmatrix} u=x+2, & dv=\cos\frac{k\pi}{2}x\,dx \\ du=dx, & v=\frac{2}{k\pi}\sin\frac{k\pi}{2}x \end{bmatrix}=\frac{1}{2}\left[\frac{2(x+2)}{k\pi}\sin\frac{k\pi}{2}x\Big|_{-2}^{0}-\right.$$

$$\left.-\frac{2}{k\pi}\int_{-2}^{0}\sin\frac{k\pi}{2}x\,dx+\frac{2}{k\pi}\sin\frac{k\pi}{2}x\Big|_{0}^{2}\right]=\frac{2}{k^2\pi^2}\cos\frac{k\pi}{2}x\Big|_{-2}^{0}=$$

$$=\frac{2}{k^2\pi^2}\left(1-(-1)^k\right)=\begin{cases}\frac{4}{k^2\pi^2}, & k=2n+1\\ 0, & k=2n\end{cases},$$

$$b_k=\frac{1}{2}\int_{-2}^{0}(x+2)\sin\frac{k\pi}{2}x\,dx+\frac{1}{2}\int_{0}^{2}2\sin\frac{k\pi}{2}x\,dx=$$

$$=\begin{bmatrix} u=x+2, & dv=\sin\frac{k\pi}{2}x\,dx \\ du=dx, & v=-\frac{2}{k\pi}\cos\frac{k\pi}{2}x \end{bmatrix}=\frac{1}{2}\left[-\frac{2(x+2)}{k\pi}\cos\frac{k\pi}{2}x\Big|_{-2}^{0}+\right.$$

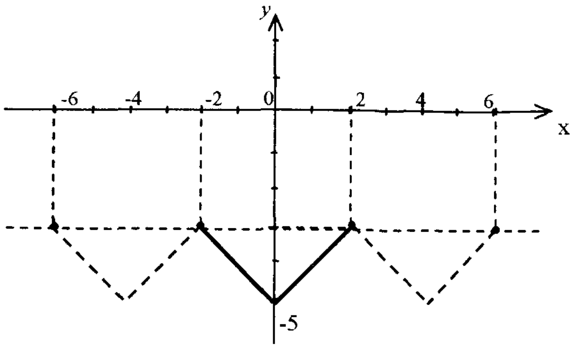
$$\left.+\frac{2}{k\pi}\int_{-2}^{0}\cos\frac{k\pi}{2}x\,dx-\frac{2}{k\pi}\cos\frac{k\pi}{2}x\Big|_{0}^{2}\right]=-\frac{2}{k\pi}+\frac{2}{k^2\pi^2}\sin\frac{k\pi}{2}x\Big|_{-2}^{0}-$$

$$-\frac{2}{k\pi}\left((-1)^k-1\right)=(-1)^{k+1}\frac{2}{k\pi}.$$

Javob:

$$f(x)=\frac{3}{2}+\frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{2k+1}\cos\frac{(2k+1)\pi}{2}x+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin\frac{k\pi}{2}x.$$

3-misol. Davri $T=4$ bo'lgan $f(x)=|x|-5$ funksiyani $[-2,\ 2]$ kesmada Furye qatoriga yoying (17-rasm).



17-rasm.

Yechish. Berilgan funksiya juft bo'lgani uchun faqat a_0 va a_n koeffitsientlarni topamiz, $b_k = 0$ bo'ladi. $l = 2$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{2}\int_0^2 (x-5)dx = \frac{x^2}{2}\Big|_0^2 = 2 - 10 = -8$$

$$a_k = \frac{2}{l}\int_0^l f(x)\cos\frac{k\pi}{l}xdx = \frac{2}{2}\int_0^2 (x-5)\cos\frac{k\pi}{2}xdx =$$

$$= \begin{bmatrix} u = x-5 \Rightarrow du = dx \\ dv = \cos\frac{k\pi}{2}xdx \Rightarrow v = \frac{2}{k\pi}\sin\frac{k\pi}{2}x \end{bmatrix} =$$

$$= \frac{2(x-5)}{k\pi}\sin\frac{k\pi}{2}x\Big|_0^2 - \frac{2}{k\pi}\int_0^2 \sin\frac{k\pi}{2}xdx =$$

$$= \frac{4}{k^2\pi^2}\cos\frac{k\pi}{2}x\Big|_0^2 = \frac{4}{k^2\pi^2}(\cos k\pi - \cos 0) =$$

$$\frac{4}{k^2\pi^2}\left((-1)^k - 1\right) = \begin{cases} 0, & \text{agar } k = 2n \\ -\frac{8}{k^2\pi^2}, & \text{agar } k = 2n+1 \end{cases}$$

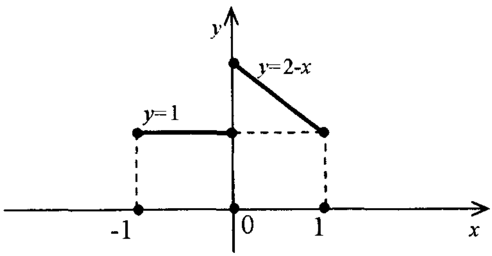
Javob:

$$|x| - 5 = -4 - \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2n+1)^2}\cos\frac{(2n+1)\pi}{2}x$$

4-misol. Davri $T=2$ bo'lgan quyidagi

$$f(x)=\begin{cases} 1, & \text{agar } -1\le x<0 \\ 2-x, & \text{agar } \ 0\le x\le 1 \end{cases}$$

funksiyani $[-1;1]$ kesmada Furye qatoriga yoying. (18-rasm)



18-rasm.

Yechish. Berilgan funksiya juft ham, toq ham emas, $l=1$. Demak, a_0, a_k va b_k koeffitsientlarning hammasini topamiz.

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx=\int_{-1}^{0}dx+\int_{0}^{1}(2-x)dx=x\Big|_{-1}^{0}+\left(2x-\frac{x^2}{2}\right)\Big|_{0}^{1}=\frac{5}{2},$$

$$a_k=\int_{-1}^{0}\cos k\pi x\,dx+\int_{0}^{1}(2-x)\cos k\pi x\,dx=\begin{bmatrix} u=2-x, & dv=\cos k\pi x\,dx \\ du=-dx, & v=\dfrac{1}{k\pi}\sin k\pi x \end{bmatrix}=$$

$$=\frac{1}{k\pi}\sin k\pi x\Big|_{-1}^{0}+\frac{2-x}{k\pi}\sin k\pi x\Big|_{0}^{1}+\frac{1}{k\pi}\int_{0}^{1}\sin k\pi x\,dx=$$

$$=-\frac{1}{k^2\pi^2}\cos k\pi x\Big|_{0}^{1}=\frac{1}{k^2\pi^2}\left[1-(-1)^k\right]=\begin{cases} \dfrac{2}{k^2\pi^2}, & \text{agar } \ k=2n+1, \\ 0 \ , & \text{agar } \ k=2n \end{cases}$$

$$b_k=\int_{-1}^{0}\sin k\pi x\,dx+\int_{0}^{1}(2-x)\sin k\pi x\,dx=\begin{bmatrix} u=2-x, & dv=\sin k\pi x\,dx \\ du=-dx, & v=-\dfrac{1}{k\pi}\cos k\pi x \end{bmatrix}=$$

$$= -\frac{1}{k\pi}\cos k\pi x\Big|_{-1}^{0} - \frac{2-x}{k\pi}\cos k\pi x\Big|_{0}^{1} - \frac{1}{k\pi}\int_0^1 \cos k\pi x\,dx =$$

$$= \frac{1}{k\pi}\left[(-1)^k - 1\right] - \frac{1}{k\pi}\cos k\pi + \frac{2}{k\pi} - \frac{2}{k^2\pi^2}\sin k\pi x\Big|_0^1 = \frac{1}{k\pi}.$$

Javob: $f(x) = \frac{5}{4} + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{2k+1} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\pi x}{k}$.

Mustaqil yechish uchun mashqlar

1. Davri $T = 2$ bo'lgan quyidagi:

$$f(x) = \begin{cases} -\dfrac{3x+1}{2}, & \text{agar } -1 \le x < 0 \\ 1, & \text{agar } \quad 0 \le x \le 1 \end{cases}$$

funksiyani $[-1;1]$ kesmada Furye qatoriga yoying.

Javob: $f(x) = \frac{5}{8} + 3\sum_{k=1}^{\infty}\left(\frac{\cos k\pi x}{k^2\pi^2} + \frac{\sin k\pi x}{2k\pi}\right)$.

2. Davri $T = 2$ bo'lgan $f(x) = x^2$ funksiyani $[-1;1]$ kesmada Furye qatoriga yoying.

Javob: $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos k\pi x$.

3. Davri $T = 2$ bo'lgan $f(x) = |x|$ funksiyani $[-1;1]$ kesmada Furye qatoriga yoying.

Javob: $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}$.

4. Davri $T = 2$ bo'lgan $f(x) = x - [x]$ funksiyani $[-1;1]$ kesmada Furye qatoriga yoying.

Javob: $f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\pi x}{k}$.

5. Davri $T = 4$ bo'lgan $f(x) = |x|$ funksiyani $[-2;2]$ kesmada Furye qatoriga yoying.

Javob: $f(x)=1-\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\frac{(2k+1)\pi x}{2}$.

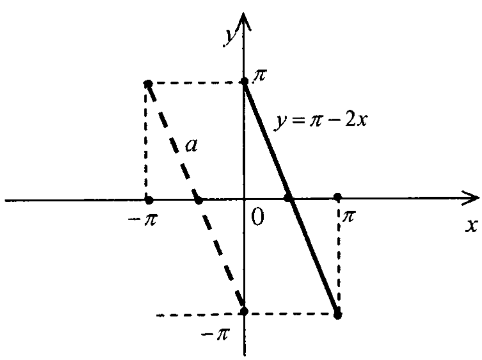
4-§. Davriy bo'lmagan funksiyalarni Furye qatoriga yoyish

Biror $[0;l]$ kesmada uzluksiz va bo'lakli monoton $f(x)$ funksiyani $[-l;0]$ kesmada davom ettirib, bu funksiyani $[-l;l]$ kesmada toq yoki juft holdagi funksiyaga to'ldirib uni Furye qatoriga yoyish mumkin.

Davom ettirilgan toq funksiya uchun Furye qatori faqat sinuslarni, juft funksiya uchun Furye qatori faqat kosinuslarni o'z ichiga oladi.

1-misol. $f(x)=\pi-2x$ funksiyani $[0;\pi]$ kesmada toq holda davom ettirib, $[-\pi;\pi]$ kesmada Furye qatoriga yoying. (19-rasm).

Yechish. Bu funksiya $T=2\pi$ davrli bo'lib, $[-\pi;\pi]$kesmada toq funksiya bo'ladi. U holda $a_0=0$, $a_k=0$, $b_k\neq 0$ bo'lib, $b_k$ni umumiy formuladan topish kifoya. Bu yerda $l=\pi$.



19-rasm.

$$b_k=\frac{2}{\pi}\int_0^{\pi}(\pi-2x)\sin kx\,dx=\begin{bmatrix} u=\pi-2x, & dv=\sin kx\,dx \\ du=-2dx, & v=-\frac{1}{k}\cos kx \end{bmatrix}=$$

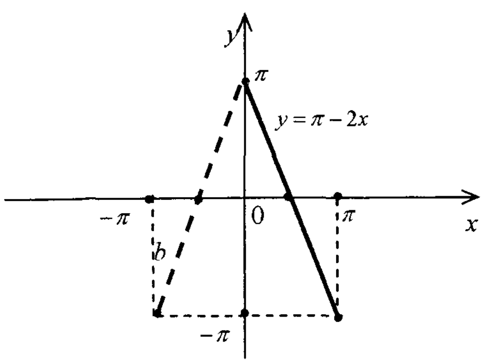
$$=\frac{2}{\pi}\left[-\frac{\pi-2x}{k}\cos kx\Big|_0^{\pi}-\frac{2}{k}\int_0^{\pi}\cos kx\,dx\Big|\right]=\frac{2}{\pi}\left[\frac{\pi}{k}\cos k\pi+\frac{\pi}{k}\cos 0-0\Big|\right]=$$

$$=\frac{2}{\pi}\cdot\frac{\pi}{k}\left[(-1)^k+1\right]=\frac{2}{k}\left[(-1)^k+1\right]=\begin{cases}\frac{4}{k}, & \text{agar } k=2n \\ 0, & \text{agar } k=2n+1\end{cases}.$$

Javob: $f(x)=4\sum_{k=1}^{\infty}\frac{\sin 2kx}{2k}$.

2-misol. $f(x)=\pi-2x$ funksiyani $[0;\pi]$ kesmada juft holda davom ettirib, $[-\pi;\pi]$ kesmada Furye qatoriga yoying. (20-rasm).

Yechish. Bu davriy bo'lmagan $f(x)=\pi-2x$ funksiyani juft holda davom ettirib, $[-\pi;\pi]$ kesmada $T=2\pi$ davrli juft funksiyaga to'ldirsak bo'ladi. U holda $b_k=0$ bo'lib, a_0, a_k larni umumiy formulalardan topish kifoya. Bu yerda $l=\pi$.



20-rasm.

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)dx = \frac{2}{\pi}\left(\pi x - 2\cdot\frac{x^2}{2}\right)\Bigg|_0^{\pi} = \frac{2}{\pi}\left(\pi^2 - \pi^2\right) = 0,$$

$$a_k = \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\cos kx\,dx = \begin{bmatrix} u = \pi - 2x, & dv = \cos kx\,dx \\ du = -2dx, & v = \dfrac{1}{k}\sin kx \end{bmatrix} =$$

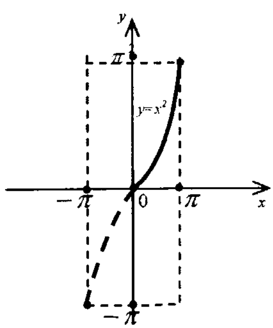
$$= \frac{2}{\pi}\left[\frac{\pi - 2x}{k}\sin kx\Bigg|_0^{\pi} + \frac{2}{k}\int_0^{\pi}\sin kx\,dx\right] = \frac{2}{\pi}\left[-\frac{2}{k}\cdot\frac{1}{k}\cos kx\Bigg|_0^{\pi}\right] =$$

$$= -\frac{4}{k^2\pi}(\cos k\pi - 1) = \frac{4}{k^2\pi}\left[1 - (-1)^k\right] = \begin{cases} \dfrac{8}{k^2\pi}, & \text{agar } k = 2n+1 \\ 0, & \text{agar } k = 2n \end{cases}.$$

Javob: $f(x) = \pi - 2x = \dfrac{8}{\pi}\sum\limits_{k=0}^{\infty}\dfrac{\cos(2k+1)x}{2k+1}$.

3-misol. $f(x) = x^2$ funksiyani $[0;\pi]$ kesmada toq holda davom ettirib, $[-\pi;\pi]$ kesmada Furye qatoriga yoying. (21-rasm).

Yechish. Agar $f(x) = x^2$ funksiyani toq holda davom ettirsak u $[-\pi;\pi]$ kesmada $T = 2\pi$ davrli toq funksiyaga aylanadi. Demak, uni $[-\pi;\pi]$ kesmada toq funksiya deb qarab, uning Furye koeffitsientlarini topamiz. Bunda $a_0 = 0$, $a_k = 0$, $b_k \neq 0$, $l = \pi$.



21-rasm.

$$b_k = \frac{2}{\pi}\int_0^{\pi} x^2 \sin kx\,dx = \begin{bmatrix} u = x^2, & dv = \sin kx\,dx \\ du = 2xdx, & v = -\frac{1}{k}\cos kx \end{bmatrix} =$$

$$= \frac{2}{\pi}\left[-\frac{x^2}{k}\cos kx\Big|_0^{\pi} + \frac{2}{k}\int_0^{\pi} x\cos kx\,dx \right] = \begin{bmatrix} u = x, & dv = \cos kx\,dx \\ du = dx, & v = \frac{1}{k}\sin kx \end{bmatrix} =$$

$$= \frac{2}{\pi}\left[-\frac{\pi^2}{k}\cos k\pi + 0 \right] + \frac{4}{k\pi}\left[\frac{x}{k}\sin kx\Big|_0^{\pi} - \frac{1}{k}\int_0^{\pi}\sin kx\,dx \right] = -\frac{2\pi}{k}\cos k\pi +$$

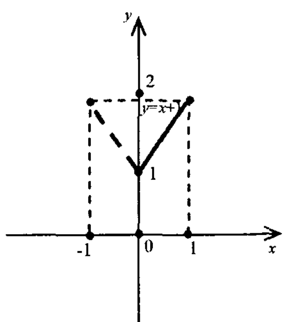
$$+ \frac{4}{k\pi}\left[0 + \frac{1}{k^2}\cos kx\Big|_0^{\pi} \right] = (-1)^{k+1}\frac{2\pi}{k} + \frac{4}{k^3\pi}\left[\cos k\pi - \cos 0\right] =$$

$$= (-1)^{k+1}\frac{2\pi}{k} + \frac{4}{k^3\pi}\left[(-1)^k - 1\right] = (-1)^{k+1}\frac{2\pi}{k} - \begin{cases} \dfrac{8}{k^3\pi}, & \text{agar } k = 2n+1 \\ 0, & \text{agar } k = 2n \end{cases}.$$

Javob: $f(x) = 2\pi\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin kx}{k} - \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{(2k+1)^3}$.

4-misol. $f(x) = x + 1$ funksiyani $[0;1]$ kesmada juft holda davom ettirib, Furye qatoriga yoying. (22-rasm).

Yechish. Agar $f(x) = x + 1$ funksiyani juft holda davom ettirsak u $[-1;1]$ kesmada $T = 2$ davrli juft funksiyaga aylanadi. Uning Furye koeffitsientlarini topamiz. Bunda $b_k = 0$, $l = 1$.



22-rasm.

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx=2\int_0^1(x+1)dx=2\left(\frac{x^2}{2}+x\right)\Bigg|_0^1=2\left(\frac{1}{2}+1\right)=3,$$

$$a_k=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{k\pi}{l}x\,dx=2\int_0^1(x+1)\cos k\pi x\,dx=$$

$$=\begin{bmatrix}u=x+1, & dv=\cos k\pi x\,dx\\ du=dx, & v=\frac{1}{k\pi}\sin k\pi x\end{bmatrix}=2\left[\frac{x+1}{k\pi}\sin k\pi x\Bigg|_0^1-\frac{1}{k\pi}\int_0^1\sin k\pi x\,dx\right]$$

$$=0+\frac{2}{k^2\pi^2}\cos k\pi x\Bigg|_0^1=\frac{2}{k^2\pi^2}(\cos k\pi-\cos 0)=\frac{2}{k^2\pi^2}\left[(-1)^k-1\right]=$$

$$=\begin{cases}-\frac{4}{k^2\pi^2}, & \text{агар}\quad k=2n+1\\ 0, & \text{агар}\quad k=2n\end{cases}$$

Javob: $f(x)=\frac{3}{2}-\frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}$.

Mustaqil yechish uchun mashqlar.

1. Davriy bo'lmagan $f(x)=\cos 2x$ funksiyani $[0;\pi]$ kesmada toq holda davom ettirib (sinuslar buyicha), $[-\pi;\pi]$ da Furye qatoriga yoying.

Javob: $f(x)=-\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(2k+1)\sin(2k+1)x}{4-(2k+1)^2}$.

2. Davriy bo'lmagan quyidagi:

$$f(x)=\begin{cases}x, & \text{agar}\quad 0\le x\le 1\\ 2-x, & \text{agar}\quad 1<x\le 2\end{cases}$$

funksiyani $[0;2]$ kesmada juft holda davom ettirib, $[-2;2]$ da Furye qatoriga yoying.

Javob: $f(x)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}$.

3. Davriy bo'lmagan $f(x)=x-\frac{1}{2}x^2$ funksiyani $[0;\,2]$ kesmada juft holda davom ettirib, $[-2;\,2]$ kesmada Furye qatoriga yoying.

Javob: $f(x)=\frac{1}{3}-\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1+(-1)^k}{k^2}\cos\frac{k\pi}{2}x$.

4. Davriy bo'lmagan $f(x)=x-\frac{1}{2}x^2$ funksiyani $[0;\,2]$ kesmada toq holda davom ettirib, $[-2;\,2]$ kesmada Furye qatoriga yoying.

Javob: $f(x)=\frac{8}{\pi^3}\sum_{k=1}^{\infty}\frac{1-(-1)^k}{k^3}\sin\frac{k\pi}{2}x$.

5. Davriy bo'lmagan $f(x)=\frac{\pi}{4}-\frac{x}{2}$ funksiyani $[0;\,\pi]$ kesmada toq holda davom ettirib, $[-\pi;\,\pi]$ da Furye qatoriga yoying.

Javob: $f(x)=\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$.

Adabiyotlar

1. Б.М.Будак, С.В.Фомин. Кратные интрегралы и ряды.
- Москва: Наука, 1967.
2. Н.Н.Воробьев. Теория рядов. – Москва: Наука, 1986.
3. В.А.Ильин, А.В.Куркина. Высшая математка. – Москва: Проспект, 2006.
4. Н.С.Пискунов. Дифференциал ва интеграл ҳисоб. 2-қисм. – Тошкент: Ўқитувчи, 1974.
5. Г.П.Толстов. Ряды Фурье. – Москва: Наука, 1980.
6. Ю.М.Данилов и другие. Математика. – Москва: Инфра-М, 2006.
7. П.Е.Данко, А.Г.Попов, Т.Я.Кожевникова. Высшая математика в упражнениях и задачах. II часть. – Москва: Высшая школа, 1985.
8. Ё.У. Соатов. Олий математика. 3-қисм. – Тошкент: Ўқитувчи, 1985.
9. Г.Н.Берман. Сборник задач по курсу математического анализа. – Москва: Наука, 1977.
10. Ю.И.Клименко. Высшая математика для экономистов в примерах и задачах. – Москва: Экзамен, 2006.
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www.gov.uz.

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