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**OLIY MATEMATIKA**  
**(Biologiya-tuproqshunoslik va kimyo  
fakulteti bakalavr talabalari uchun)**

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Mazkur o'quv qo'llanma Biologiya – tuproqshunoslik va kimyo fakulteti talabalari uchun mo'ljallangan bo'lib, unda oliv algebra, analitik geometriya, matematik – analiz va differentsial tenglamalar kurslarining muhim tushunchalari bayon qilingan.

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## **SO'Z BOSHI**

Hozirgi vaqtida matematik metodlar fan, texnika va xalq xo'jaligining ko'p sohalarida turli – tuman masalalarni hal qilishda keng qo'llanilmoqda. Ayniqsa, xalq xo'jaligining barcha sohalarida kompyutrlarning yalpi qo'llanilishi munosabati bilan matematik usullarning ahamiyati yanada ortdi.

No matematik mutaxassisliklarda matematikani o'qitishdan maqsad – talabaning mantiqiy fikrleshini o'stirish va tatbiqiy masalalarni matematik usullar bilan tekshirish malakalarini hosil qilishdan iboratdir.

Zamon talabi o'qitilayotgan fanning mos mutaxassisliklar bo'yicha amaliy tatbiqlarini hisobga olishni talab etmoqda.

Ushbu o'quv qo'llanma biologiya – tuproqshunoslik va ximiya mutaxassisliklari talabalari uchun muljallangan o'quv reja va o'tiladigan o'quv dasturidan kelib chiqqan holda yozilgan. O'quv qo'llanmada oliv algebra, analitik geometriya, matematik – analiz va differentsial tenglamalar bo'limining elementlari, hamda mustaqil yechish uchun misol va masalalar keltirilgan.

Bundan tashqari ba'zi biologiya va ximiyada uchraydigan ayrim masalalarning matematik modellari keltirilgan.

**I-BOB.**  
**DASTLABKI TUSHUNCHALAR.**

**1-§. To'plamlar va ular ustida amallar.**

**1<sup>0</sup>.** Matematikada yozuvni qisqartirib ifodalash, shuningdek iboralarni ixcham aytish maqsadida maxsus belgilar (mantiqiy belgilar) dan foydalaniadi:

- 1)  $\Rightarrow$  belgi «agar ... bo'lsa, u holda ... bo'ladi» iborasi o'rnida;
- 2)  $\forall$  belgi, «har qanday», «ixtiyoriy», «barchasi uchun» so'zлari va so'z birikmalari o'rnida;
- 3)  $\exists$  belgi, «mavjudki», «topiladiki» so'zлari o'rnida;
- 4)  $\Leftrightarrow$  belgi, ikki ekvivalent tasdiqni ifodalashda;
- 5)  $:=$  belgi, «ta'rifga ko'ra teng» so'z birikmalari o'rnida;
- 6)  $\in$  belgi, «tegishli»,  $\notin$  yoki  $\in \notin$  belgilari esa «tegishli emas» so'zлari o'rnida ishlataladi.

To'plam matematikaning boshlang'ich tushunchalaridan bo'lib, uni ob'ektlarning ma'lum belgilar bo'yicha birlashmasi (majmuasi) sifatida tushuniladi.

Bir qancha ob'ektlarni birlashtirishga yangi ob'ekt deb qarash mumkin. Bu yangi ob'ektga **to'plam** deyladi.

*Bitta ham elementga ega bo'lmagan to'plam bo'sh to'plam deyiladi va  $\emptyset$  kabi belgilanadi.*

Odatda, to'plamlar bosh harflar, uni tashkil etuvchilari (elementlari) kichik harflar bilan belgilanadi.

To'plamning  $P$  xususiyatli  $x$  elementlaridan tashkil topgan to'plam quyidagicha  $\{x | P\}$  belgilanadi.

**2<sup>0</sup>.** Ikki A va B to'plamlar berilgan bo'lsin. A to'plam B ning qismi ( $A \subset B$ ), A va B to'plamlar tengligi ( $A = B$ ), A va B to'plamlar yig'indisi ( $A \cup B$ ), A va B to'plamlar ko'paytmasi ( $A \cap B$ ), A to'plamdan B to'plamning ayirmasi ( $A \setminus B$ ) quyidagicha ta'riflanadi:

$$\begin{aligned} A \subset B : & a \in A \Rightarrow a \in B; \\ A = B : & A \subset B, B \subset A; \\ A \cup B : & = \{x | x \in A \text{ ekr } x \in B\}; \\ A \cap B : & = \{x | x \in A, x \in B\}; \\ A \setminus B : & = \{x | x \in A, x \notin B\}. \end{aligned}$$

**1-misol.**  $A = \{x \in N \mid -3 < x \leq 5\}$  va  
 $B = \{x \in N \mid x^2 + 2x - 3 = 0\}$  to'plamlar uchun  $A \cup B$ ,  $A \setminus B$ ,  $B \setminus A$ ,  $A \cap B$  lar topilsin.

♦  $A = \{x \in N \mid -3 < x \leq 5\} = \{1, 2, 3, 4, 5\}$  va  
 $B = \{x \in N \mid x^2 + 2x - 3 = 0\} = \{1\}$  ekanligidan quyidagilarni hosil qilamiz:

$$\begin{aligned}A \cup B &= \{1, 2, 3, 4, 5\}, \\A \setminus B &= \{2, 3, 4, 5\}, \\B \setminus A &= \emptyset, \\A \cap B &= \{1\}\end{aligned}$$

**2 – misol.** Agar ikki A va B to'plamlari uchun  
 $A \setminus B = B \setminus A$

bo'lsa, A va B lar haqida nima deyish mumkin?

♦ Ravshanki,

$$\begin{aligned}x \in A \setminus B \Rightarrow x \in A, x \notin B. \\x \in B \setminus A \Rightarrow x \in B, x \notin A.\end{aligned} \Rightarrow x \in A \cap B.$$

Demak, bu to'plamlar kesishmaydi ►

**3 – misol.**  $A = \{x \mid x^2 - 8x + 15 = 0\}$  to'plamning barcha qism to'plamlarini yozing.

♦  $A = \{3; 5\}$  ekanligini ko'rish qiyin emas. Bu to'plamning barcha qism to'plamlari quyidagi to'plamlardan iborat bo'ladi:

$$\emptyset, \{3\}, \{5\}, \{3; 5\} \quad \blacktriangleright$$

**4 – misol.** Ushbu

- 1)  $\{1; 2\} \in \{\{1\}, \{2\}, \{1; 3\}, \{1; 2; 3\}\}$
- 2)  $\{1; 2\} \subset \{\{1\}, \{2\}, \{1; 3\}, \{1; 2; 3\}\}$

munosabatlar to'g'rimi?

♦ Agar  $A = \{1; 2\}$  va  $B = \{\{1\}, \{2\}, \{1; 3\}, \{1; 2; 3\}\}$  deb belgilasak, B to'plam elementlari orasida A to'plamga teng bo'lgan element bo'lmasligi sababli  $A \in B \Rightarrow 1$  – munosabat to'g'ri emas.

$A \subset B$  esa bajariladi, chunki  $1 \in B$  va  $2 \in B$  Demak, 2) – munosabat to'g'ri. ►

**5 – misol.** A, B, S to'plamlari uchun ushbu

$$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$$

tenglikning o'rinli bo'lishi ko'rsatilsin.

► Aytylik,  $x \in (A \setminus B) \cap C$  bo'lsin. Yuqorida keltirilgan to'plamlar ustida amallar ta'riflaridan foydalanib topamiz:

$$x \in (A \setminus B) \cap C \Rightarrow x \in A \setminus B, x \in C \Rightarrow$$

$$\Rightarrow x \in A, x \notin B, x \in C \Rightarrow x \in A \cap C, x \notin B \cap C \Rightarrow \\ \Rightarrow x \in (A \cap C) \setminus (B \cap C).$$

Demak,

$$(A \setminus B) \cap C \subset (A \cap C) \setminus (B \cap C). \quad (1)$$

Endi  $x \in (A \cap C) \setminus (B \cap C)$  bo'lsin. Unda

$$x \in A \cap C, x \notin B \cap C \Rightarrow x \in A, x \in C, \\ x \notin B \Rightarrow x \in A \setminus B, x \in C \Rightarrow x \in (A \setminus B) \cap C.$$

Demak,

$$(A \cap C) \setminus (B \cap C) \subset (A \setminus B) \cap C. \quad (2)$$

(1) va (2) munosabatlardan

$$(A \setminus B) \cap C = (A \cap C) \setminus (B \cap C)$$

bo'lishi kelib chiqadi. ►

**6 – misol.** A, B, S to'plamlari uchun ushbu

$$(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$$

tenglikning bajarilishi ko'rsatilsin.

► Faraz qilaylik,  $x \in (A \setminus B) \setminus C$  bo'lsin. Unda

$$x \in A \setminus B, x \notin C \Rightarrow x \in A, x \notin B, \\ x \notin C \Rightarrow x \in A \setminus C, x \notin B \setminus C \Rightarrow x \in (A \setminus C) \setminus (B \setminus C),$$

Demak,

$$(A \setminus B) \setminus C \subset (A \setminus C) \setminus (B \setminus C) \quad (3)$$

Endi  $x \in (A \setminus C) \setminus (B \setminus C)$  bo'lsin. Unda

$$\begin{aligned} x \in A \setminus C, x \notin B \setminus C &\Rightarrow x \in A, x \notin C, \\ x \in A \setminus B &\Rightarrow x \in A \setminus B, x \notin C \Rightarrow x \in (A \setminus B) \setminus C, \end{aligned}$$

Demak,

$$(A \setminus C) \setminus (B \setminus C) \subset (A \setminus B) \setminus C. \quad (4)$$

(3) va (4) munosabatlardan

$$(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$$

bo'lishi kelib chiqadi. ▶

### Mustaqil yechish uchun misol va masalalar.

Berilgan A va B to'plamlar uchun

$$A \cup B, A \setminus B, B \setminus A, A \cap B$$

to'plamlar topilsin.

$$1. A = \{x \in N \mid -2 < x \leq 3\}, \quad B = \{x \mid x^2 - x - 2 = 0\}$$

$$J: A \cup B = \{-1; 1, 3\}, \quad A \setminus B = \{1; 3\}, \quad B \setminus A = \{-1\}, \quad A \cap B = \{2\}$$

$$2. A = \{x \in N \mid \sin x = 0, x \in [-\pi; \pi]\}, \quad B = \left\{x \in N \mid \cos \frac{\pi x}{2} = 0, x \in [0; 2\pi]\right\}.$$

$$J: A \cup B = \{-1; 2; 3; 5\}, \quad A \setminus B = \{2\}, \quad B \setminus A = \{5\}, \quad A \cap B = \{1; 3\}$$

$$3. A = \{1; 2; 4\}, \quad B = \{2; 3\}$$

$$J: A \cup B = \{1; 2; 3; 4\}, \quad A \setminus B = \{1; 4\}, \quad B \setminus A = \{3\}, \quad A \cap B = \{2\}.$$

$$4. A = \{x \mid x^2 - 7x + 6 = 0\}, \quad B = \{1; 6\}$$

$$J: A \cup B = \{1; 6\}, \quad A \setminus B = \emptyset, \quad B \setminus A = \emptyset, \quad A \cap B = \{1; 6\}.$$

$$5. \text{ Agar } A = \{x \in N \mid 2 < x \leq 6\}, \quad B = \{x \in N \mid 1 < x < 4\} \text{ va}$$

$$C = \{x \in N \mid x^2 - 4 = 0\} \text{ bo'lsa,}$$

1)  $B \cup C$ , 2)  $A \cap B \cap C$ , 3)  $A \cup B \cup C$ , 4)  $(A \cap B) \cup (B \cup C)$  to'plamlar topilsin.

J: 1)  $\{2;3\}$ , 2)  $\emptyset$ , 3)  $\{2; 3; 4; 5; 6\}$ , 4)  $\{2; 3\}$ .

6.  $A = \{x \in N \mid 1 < x \leq 4\}$ ,  $B = \{x \mid x^2 - 9 = 0\}$

bo'lsa,  $A \cup B$  to'plamning barcha qism to'plamlari topilsin.

J:  $\emptyset$ ,  $\{-3\}$ ,  $\{2\}$ ,  $\{3\}$ ,  $\{4\}$ ,  $\{-3; 2\}$ ,  $\{-3; 3\}$ ,  $\{-3; 4\}$ ,  $\{2; 3\}$ ,  $\{2; 4\}$ ,  $\{3; 4\}$ ,  $\{-3; 2; 3\}$ ,  $\{-3; 2; 4\}$ ,  $\{-3; 3; 4\}$ ,  $\{2; 3; 4\}$ ,  $\{-3; 2; 3\}$ .

7. Agar  $A = \{x \in N \mid 0 \leq x < 3\}$ ,  $B = \{x \in N \mid -1 < x < 4\}$  va

$C = \{x \in N \mid x^2 - 9 = 0\}$  bo'lsa, 1)  $B \cup C$ , 2)  $A \cap B \cap C$ , 3)

$A \cup B \cup C$ , 4)  $(A \cap B) \cup (B \cup C)$  to'plamalr topilsin.

J: 1)  $\{2;3\}$ , 2)  $\emptyset$ , 3)  $\{1;2;3\}$ , 4)  $\{2;3\}$ .

8.  $A = \{x \in N \mid x^2 - 16 = 0\}$   $B = \{x \in N \mid -4 \leq x \leq 2\}$  bo'lsa,

$A \cup B$  to'plamning barcha qism to'plamlari topilsin.

J:  $\emptyset, \{1\}, \{2\}, \{4\}, \{1;2\}, \{1;4\}, \{2;4\}, \{1;2;4\}$ .

9. A to'plam 2 ga bo'linuvchi barcha natural sonlar to'plami, B esa 3 ga bo'linuvchi barcha natural sonlar to'plami bo'lsa,  $A \cap B$  to'plam qanday bo'ladi?

J: 6 ga bo'linuvchi barcha natural sonlar to'plami.

10. Agar  $B \subset A$  bo'lib, ushbu

$$A \cap X = B$$

tenglik bajarilsa, unda X to'plam qanday bo'ladi?

J:  $X = B$

11. Agar  $X = A \setminus (B \cup C)$ ,  $Y = (A \setminus B) \cup (A \setminus C)$  bo'lsa, unda X va Y to'plamlar qanday munosabatda bo'ladi?

J:  $X \subset Y$

A, B va S to'plamlar uchun quyidagi munosabatlarning bajarilishi ko'rsatilsin.

12.  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

13.  $A \setminus (B \cup C) = (A \setminus B) \setminus C$ .

14. Agar

$$X = A \cup (B \setminus C), Y = (A \cup B) \setminus (A \cup C)$$

bo'lsa, X va Y to'plamlar qanday munosabatda bo'ladi?

J:  $X \supset Y$

15. A to'plam 3 ga bo'linuvchi barcha natural sonlar to'plami, B esa 5 ga bo'linuvchi barcha natural sonlar to'plami bo'lsa,  $A \cap B$  to'plam qanday bo'ladi?

J: 15 ga bo'linuvchi barcha natural sonlar to'plami

16. Ushbu

$$(A \setminus B) \cup B = A$$

tenglik faqat  $B \subset A$  bo'lgandagina bajarilishi ko'rsatilsin.

17. Ixtiyoriy A, B va S to'plamlari uchun ushbu

$$(A \cup B) \setminus B \subset (A \setminus B) \cup C$$

munosabatning bajarilishi ko'rsatilsin.

Ixtiyoriy A, B va S to'plamlar uchun quyidagi munosabatlarning bajarilishi ko'rsatilsin.

$$18. A \cap (A \cup B) = A.$$

$$19. A \setminus (A \setminus B) = A \cap B.$$

$$20. (A \setminus B) \cup (B \setminus A) \cup (A \cap B) = A \cup B.$$

$$21. (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

$$22. A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

23. Agar  $X = (A \cap B) \setminus C$ ,  $Y = (A \cap C) \cap (B \setminus C)$  bo'lsa, X va Y to'plamlar qanday munosabatda bo'ladi?

$$J: X = Y.$$

24. Agar A to'plam 4 ta elementdan iborat bo'lsa, uning qismiy to'plamlaridan tashkil topgan to'plamning elementlari soni 16 ta bo'lishi isbotlansin.

25. Agar A to'plamning elementlari soni m ta, B niki n ta bo'lib, P va q to'plamlarning elementlari soni mos ravishda r va q ta bo'lsa

$$p = m + n - q$$

bo'lishi isbotlansin.

## 2-§. Ҳақиқий sonlar.

1. Ҳақиқий sonlar. Sanash jarayonida ishlataladigan 1, 2, 3, ... sonlar **natural sonlar** deyilib, natural sonlar to'plami N bilan belgilanadi:

$$N = \{1, 2, 3, \dots, n, \dots\}$$

Barcha natural sonlar, nol soni va barcha manfiy ishora bilan olingan natural sonlardan tashkil topgan to'plam **butun sonlar** to'plami Z ni hosil qiladi:

$$Z = \{..., -3, -2, -1, 0, 1, 2, ...\}.$$

$$n! = 1 \bullet 2 \bullet 3 \bullet \dots \bullet (n-1) \bullet n.$$

Bu " $n$  faktorial" deb o'qiladi.

$0! = 1$ , deb qabul qilingan.

$C_n^k$  binomial koeffitsentni quyidagi ko'rinishda aniqlasymiz:

$$C_n^k = \frac{n!}{k!(n-k)!}, \quad n, k \in N, \quad k \leq n.$$

Barcha  $n \in N$  lar uchun  $C_n^0 = 1$  deb hisoblaymiz.

Quyidagi xossalarni keltiramiz,

$$1) \quad C_n^k = C_n^{n-k},$$

$$2) \quad C_{n+1}^{k+1} = C_n^k + C_n^{k+1}.$$

Bu xossalardan binomial koeffitsent natural son ekanligi kelib chiqadi.

$C_n^k$  dan foydalanib Nyuton binomi formulatasini keltiramiz:

$$(x+a)^n = x^n + C_n^1 x^{n-1} a + C_n^2 x^{n-2} a^2 + \dots + C_n^{n-1} x a^{n-1} + a^n = \sum_{k=0}^n C_n^k x^{n-k} a^k, n \in N.$$

Nyuton binomi formulasi xususiy hollarda quyidagi ko'rinishlarda bo'ladi:

$$(x+a)^2 = x^2 + 2xa + a^2,$$

$$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3,$$

$$(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4.$$

Ma'lumki, qisqarmaydigan  $\frac{p}{n}$  ( $p \in Z, n \in N$ ) kasr ko'rinishida ifodalananadigan son ratsional son deyiladi va barcha ratsional sonlar to'plami  $Q$  deb belgilanadi:

$$Q = \left\{ r = \frac{p}{n} \mid p \in Z, n \in N, (p, n) = 1 \right\}.$$

Agar  $\frac{p}{n}$  kasrning maxraji  $10^{k*}$  ko'rinishda bo'lsa, uni **o'nli kasr** deyiladi. O'nli kasrlar uch xil bo'ladi:

- 1) chekli o'nli kasrlar ( $1,5; 0,17; -0,01$ );
- 2) cheksiz davriy o'nli kasrlar ( $0,888\dots = 0,(8)$ ;  
 $0,11777\dots = 0,11(7)$ );
- 3) cheksiz davriy bo'limgan o'nli kasrlar ( $3,14\dots; 2,71\dots$ ).

*Har qanday ratsional son chekli o'nli kasr yoki cheksiz davriy o'nli kasr ko'rinishida ifodalanadi va aksincha.*

Cheksiz davriy bo'limgan o'nli kasr **irratsional son** deyiladi.

Ratsional hamda irratsional sonlar umumiy nom bilan haqiqiy son deb ataladi va barcha haqiqiy sonlar to'plami  $R$  deb belgilanadi.

**2. Sonlar o'qi.** Haqiqiy sonlar to'plami  $R$  bilan to'g'ri chiziq nuqtalari to'plami o'zaro bir qiymatli moslikda bo'ladi. Shu sababli haqiqiy sonlar to'plami sonlar **o'qi yoki** haqiqiy o'q, haqiqiy sonning o'zi esa **sonlar o'qining nuqtasi** ham deyiladi.

$x \in R$  son uchun

$$|x| = \begin{cases} x, & x > 0 \\ 0, & x = 0 \\ -x, & x < 0 \end{cases}$$

mi qdor  $x$  ning **absolyut qiymati** deyiladi.

$x \in R$  sonning o'zidan katta bo'limgan butun qismi  $[x]$  kabi yoziladi,  $\{x\} = x - [x]$  esa shu sonning kasr qismi bo'ladi.

Aytaylik,,  $a \in R$  va  $b \in R$  bo'lib,  $a < b$  bo'lsin.

Unda

$[a, b] = \{x \in R   a \leq x \leq b\}$	kesma (segment,)
$(a, b) = \{x \in R   a < x < b\}$	interval,
$[a, b) = \{x \in R   a \leq x < b\}$	yarim interval,
$(a, b] = \{x \in R   a < x \leq b\}$	yarim interval deyilib, a va b lar

ularning chetki nuqtalari deyiladi. Shuningdek,

$$[a, +\infty) = \{x \in R | x \geq a\},$$

$$(-\infty, a) = \{x \in R | x < a\},$$

$$(-\infty, +\infty) = R$$

deb qaraymiz.

**3. To'plamning chegaralari.** Biror  $E$  ( $E \subset R$ ) to'plam berilgan bo'lsin.

Arap

$$\exists b_0 \in E, \forall x \in E \Rightarrow x \leq b_0$$

bo'lsa,  $b_0$  soni  $E$  to'plamning eng katta elementi deyiladi va

$$b_0 = \max E = \max \{x | x \in E\}$$

kabi belgilanadi.

Agar

$$\exists a_0 \in E, \forall x \in E \Rightarrow x \geq a_0$$

bo'lsa  $a_0$  soni  $E$  to'plamning eng kichik elementi deyiladi va

$$a_0 = \min E = \min \{x | x \in E\}$$

kabi belgilanadi.

Agar

$$\exists M \in R \quad \forall x \in E \Rightarrow x \leq M \quad (\exists m \in R \quad \forall x \in E \Rightarrow m \leq x)$$

bo'lsa,  $E$  to'plam yuqoridan (quyidan) chegaralangan,  $M(m)$  soni esa  $E$  to'plamning yuqori (guyi) chegarasi deyiladi.

Agar  $a^* (a_*)$  soni

- 1)  $E$  to'plamning yuqori (quyi) chegarasi bo'lib,
  - 2)  $E$  to'plamning ixtiyoriy yuqori (quyu) chegarasi  $d$  uchun  $a^* \leq d$  ( $a_* \geq d$ ) bo'lsa,
- $a^*$  soni  $E$  to'plamning aniq yuqori (aniq guyi) chegarasi deyiladi va

$$a^* = \sup E \quad (a_* = \inf E)$$

kabi belgilanadi.

**1-teorema.** a' soni  $E$  to'plamning aniq yuqori chegarasi bo'lishi uchun ushbu

- 1) a' soni  $E$  to'plamning yuqori chegarasi,
- 2)  $\forall d < a \cdot \exists x \in E \Rightarrow x > d$  shartlarning bajarilishi zarur va etarli.

a. soni  $E$  to'plamning aniq quyisi chegarasi bo'lishi uchun ushbu

- 1') a. soni  $E$  to'plamning quyisi chegarasi,
- 2')  $\forall d > a \cdot \exists x \in E \Rightarrow x < d$  shartlarning bajarilishi zarur va etarli.

**2-teorema.** Faraz qilaylik,  $E \subset R$  bo'lib,  $E \neq \emptyset$  bo'lsin. Agar  $E$  to'plam yuqoridan chegaralangan bo'lsa, u aniq yoqori chegaraga ega. Agar  $E$  quyidan chegaralangan bo'lsa, u aniq quyisi chegaraga ega.

Agar  $E$  yuqoridan chegaralanmagan bo'lsa  $\sup E = +\infty$ , quyidan chegaralanmagan bo'lsa  $\inf E = -\infty$  deb qaratadi.

### 1-misol. Ushbu

$$\alpha = \sqrt{3}, \quad \beta = \log_6 6$$

sonlarning irratsional ekanligi ko'rsatilsin.

Faraz qilaylik,  $\alpha$  ratsional son bo'lsin. Unda  $\sqrt{3} = \frac{p}{q}$  bo'lib,

$p$  va  $q$  lar o'zaro tubdir:  $(p,q)=1$ .

Ravshanki,  $p^2 = 3q^2$  bo'ladi. Demak,  $p^2$  son 3 ga bo'linadi. Haqiqatdan ham, agar r son 3 ga bo'linmaydigan bo'lsa, unda  $p = 3k+1$  yoki  $p = 3k+2$  ( $k \in \mathbb{Z}$ ) bo'ladi.

Agar  $p = 3k+1$  bo'lsa,  $p^2 = 9k^2 + 6k + 1$ , yani  $p^2 - 3(3k^2 + 2k) = 1$  bo'ladi. Ayni paytda  $p^2$  son 3 ga bo'linganligi sababli,  $p^2 - 3(3k^2 + 2k)$  ham 3 ga bo'linishi lozim. Biroq, bu ayirma 1 ga teng bo'lishi ziddiyatga olib keladi.

Xuddi shunga o'xshash  $p = 3k+2$  bo'lganda ham ziddiyat keliib chiqadi. Demak,  $p$  soni 3 ga bo'linmaydi:  $p = 3m$ .

Shuni etiborga olib,  $q^2 = 3m^2$  bo'lishini va yuqoridagi mulohazaga ko'ra  $q$  ning ham 3 ga bo'linishini topamiz:  $q = 3n$ .

Demak,  $p$  va  $q$  sonlarning har biri 3 ga bo'linadi. Bu esa  $p$  va  $q$  larning o'zaro tub ekanligiga ziddir. Ziddiyat,  $\sqrt{3} = \frac{p}{q}$  deb olinishi oqibatida sodir bo'ldi. Demak,  $\sqrt{3}$  ni  $\frac{p}{q}$  ko'rinishida ifodalab bo'lmaydi.  $\sqrt{3}$  – irratsional son.

$\beta = \log_7 6$  sonini ham ratsional, yani  $\log_7 6 = \frac{p}{q}$  deb faraz qilamiz. Unda  $7^{\frac{p}{q}} = 6 \Rightarrow 7^p = 6^q$  bo'ladi. Bu tenglik hech qachon o'rinli bo'lmaydi, chunki  $6^q$  – juft,  $7^p$  – toq sondir.

Demak,  $\log_7 6$  ni  $\frac{p}{q}$  ko'rinishda ifodalab bo'lmaydi, u irratsional son.

**2-misol.** Agar  $r$  – ratsional son va  $\alpha$  – irratsional son bo'lsa,  $\alpha + r$  irratsional son bo'lishi isbotlansin.

$\alpha + r = \beta$  deb belgilab,  $\beta$  – ratsional son bo'lsin deb faraz qilamiz.  $\Rightarrow \alpha = \beta - r$  – ratsional son. Ziddiyat hosil bo'ldi. Demak,  $\alpha + r = \beta$  – irratsional son.

**3-misol.** Agar  $\alpha$  va  $\beta$  lar irratsional sonlar bo'lsa, unda

$$1) \alpha + \beta,$$

$$2) \alpha^\beta \text{ lar ratsional son bo'lishi mumkinmi?}$$

$\alpha + \beta$  ratsional son bo'lishi mumkin. Masalan,  $\alpha = \sqrt{3}$  va  $\beta = 2 - \sqrt{3}$  lar irratsional, lekin  $\alpha + \beta = 2$  – ratsional.

$\alpha + \beta = \alpha^\beta$  ham ratsional son bo'lishi mumkin.

Masalan,  $\alpha = \sqrt{3}$  va  $\beta = 2 \log_2 2$  larning irratsional ekanini ko'rish qiyin emas. Lekin  $\alpha^\beta = \sqrt{3}^{2 \log_2 2} = 2$  – ratsional son.

**4-misol.**  $\alpha = \sqrt{3} + \sqrt{5}$  sonining irratsional son ekanligi isbotlansin.

Teskarisini faraz qilamiz, yani  $\sqrt{3} + \sqrt{5} = r$  ratsional son bo'lsin. Unda  $\sqrt{3} + \sqrt{5} = r \Rightarrow 3 + 2\sqrt{3} \cdot \sqrt{5} + 5 = r^2 \Rightarrow \sqrt{15} = r^2 - \frac{8}{2}$ . Bu tenglikning o'ng tomonida ratsional son turibdi, chapdag'i  $\sqrt{15}$  ning irratsional ekanini ko'rish qiyin emas. Ziddiyat  $\alpha = \sqrt{3} + \sqrt{5}$  ning irratsional son ekanini isbotlaydi.

**5-misol.** Ushbu

$$\left| \frac{x}{x+1} \right| > \frac{x}{x+1}$$

tengsizlik yechilsin.

$|x| > x$  tengsizlik  $x < 0$  bo'lgandagina bajarilgani uchun berilgan tengsizlikdan

$$\frac{x}{x+1} < 0$$

ekanligini topamiz. Intervallar usulidan foydalansak —

$$-1 < x < 0$$

ekanligini ko'rish qiyin emas.

**6-misol.**  $E \subset R, F \subset R$  to'plamlar yuqoridan chegaralangan va  $E + F = \{x + y | x \in E, y \in F\}$  bo'lisin. Unda

$$\sup(E + F) = \sup E + \sup F$$

bo'lishi isbotlansin.

Aytaylik,  $\sup E = a$  va  $\sup F = b$  bo'lisin. Unda 1-teoremaga ko'ra

$$\sup E \Leftrightarrow \begin{cases} 1) \forall x \in E, \quad x \leq a \\ 2) \forall d_1 < a \exists x_0 \in E \Rightarrow x_0 > d_1, \end{cases}$$

$$\sup F \Leftrightarrow \begin{cases} 1) \forall y \in F, \quad y \leq b \\ 2) \forall d_2 < b \exists y_0 \in F \Rightarrow y_0 > d_2, \end{cases}$$

bo'ladi. Ravshanki,

$$\begin{aligned} \forall x \in F, x \leq a; \forall y \in F, y \leq b \Rightarrow x + y \leq a + b \\ \forall d_1 < a \exists x_0 \in E \Rightarrow x_0 > d_1, \forall d_2 < b \exists y_0 \in F \Rightarrow y_0 > d_2 \end{aligned}$$

bo'lishidan

$\forall d < a + b$  olib  $\exists x_1 + y_1 \in E + F : x_1 + y_1 > d$  bo'lisin.  
Yana 1-teoremaga ko'ra

$$\sup(E + F) = a + b$$

bo'ladi. Demak,

$$\sup(E + F) = \sup E + \sup F.$$

### Mustaqil yechish uchun misol va masalalar.

1. Ushbu  $\alpha = \sqrt{5}$  va  $\beta = \lg 2$  sonlarining irratsional ekanligi isbotlansin.
2. Ushbu  $\alpha = \sqrt{2}$  va  $\beta = \log_2 3$  sonlarining irratsional ekanligi isbotlansin.
3. Ushbu 1)  $2^x = 11$ , 2)  $x^2 - 7x + 1 = 0$  tenglamani qanoatlantiruvchi ratsional son mavjud emasligi ko'rsatilsin.
4. Agar  $a, b$  va  $\sqrt{a} + \sqrt{b}$  ratsional sonlar bulsa,  $\sqrt{a}$  va  $\sqrt{b}$  sonlar to'g'risida nima deyish mumkin?
5. Agar  $\alpha$  va  $\beta$  – irratsional sonlar,  $r$  – ratsional son bo'lsa, ushbu 1)  $\alpha + r$ , 2)  $\sqrt{r}$ , 3)  $\alpha \cdot r$ , 4)  $\sqrt{\alpha + \sqrt{\beta}}$ , 5)  $\sqrt{\alpha + \sqrt{r}}$  sonlarning qaysilari ratsional bo'lishi mumkin?  
J:  $\sqrt{r}, \sqrt{\alpha + \sqrt{\beta}}$ .
6. Ushbu 1)  $x^3 = 7$ , 2)  $x^2 + 3x + 1 = 0$  tenglamani qanoatlantiruvchi ratsional son mavjud emasligi isbotlansin.
7.  $\sqrt{2}$  va  $\sqrt{3}$  sonlar orasida joylashgan ratsional son topilsin.  
J: bunday sonlar cheksiz ko'p, masalan  $\frac{3}{2}$ .
8. Agar  $\alpha$  va  $\beta$  – irratsional sonlar,  $r$  – ratsional bo'lsa, ushbu 1)  $\alpha + \beta$ , 2)  $\sqrt{\alpha}$ , 3)  $\alpha \cdot \beta$ , 4)  $\sqrt{\alpha + r}$  va 5)  $\sqrt{\alpha} + \sqrt{r}$  sonlarning qaysilari ratsional bo'lishi mumkin?  
J:  $\alpha + \beta, \alpha \cdot \beta, \sqrt{\alpha + \sqrt{r}}$ .
9.  $\alpha = \sqrt{2} + \sqrt{3}$  sonining irratsional ekanligi isbotlansin.
10. Agar  $\alpha$  va  $\beta$  – irratsional sonlar bo'lib,  $\alpha + \beta$  – ratsional bo'lsa, unda  $\alpha - \beta$  va  $\alpha + 2\beta$  larning irratsional son bo'lishi ko'rsatilsin.

**Quyidagi sonlar irratsional ekanligi isbotlansin.**

11.  $\alpha = \frac{1}{\sqrt{2} + \sqrt{5}}$ .

12.  $\sqrt{5} - \sqrt{2}$ .

**Tengsizliklar yechilsin.**

13.  $|x+3| > 2$  J:  $x \in (-\infty; -5) \cup (-1; +\infty)$ .

14.  $|x-4| + |x+4| \leq 10$ . J:  $-5 \leq x \leq 5$ .

15.  $|x| < x+1$ . J:  $x < -\frac{1}{2}$ .

16.  $|x-2| < 3$ . J:  $-1 < x < 5$ .

17.  $|x+2| + |x-2| \geq 12$  J:  $x \in (-\infty; -6] \cup [6; +\infty)$

18.  $|x^2 - 2x| > x^2 - |2x|$  J:  $x < 2 \quad (x \neq 0)$

19. Agar  $a > \frac{1}{2}$ , bo'lsa,  $\frac{1}{a}$  sonning butun qismi topilsin.

J: 1, agar  $\frac{1}{2} < a \leq 1$  bo'lsa, 0 agar  $a > 1$ , bo'lsa.

20. Ushbu  $E = \left\{ \frac{n}{n+1} \mid n \geq 1 \right\}$  to'plamning chegaralanganligi ko'rsatilsin.

21. Ushbu  $A = \{r \in Q \mid r^2 < 2\}$  to'plamning aniq yuqori va quyi chegaralarini topilsin.

J: J:  $\sup A = \sqrt{2}$ ,  $\inf A = -\sqrt{2}$

22. Ushbu  $\sup \left\{ \frac{1}{\sqrt{n}} \mid n \in N \right\} = 1$  tenglik isbotlansin.

23. Aytaylik,  $E \subset R$  va  $-E = \{x \mid -x \in E\}$  bo'lsin. Unda  $\sup(-E) = -\inf E$ ,  $\inf(-E) = -\sup E$  bo'lishi isbotlansin.

24. Ushbu  $E = \left\{ \frac{n^2}{n^2 + 4} \mid n \in N \right\}$  to'plam uchun  $\sup E = 1$ ,  $\inf E = \frac{1}{5}$  bo'lishi ko'rsatilsin.

### 3-§. Matematik induktsiya usuli.

Ma'lumki, fikr yuritish asosan ikki — deduktiv va induktiv formalarda olib boriladi. *Deduktsiya* — fikrlashning umumiy tasdiqlaridan xususiy tasdiqlarga o'tish, *induktsiya* esa fikrlashning xususiy tasdiqlaridan umumiy tasdiqlarga o'tish formasidir.

Aytaylik,  $A(n)$  biror tasdiqni (fikr, mulohazani) ifodalasini ( $n \in N$ ). Matematik induktsiya usulida:

- 1)  $n = n_0$  bo'lganda  $A(n)$  tasdiqning to'g'riliqi (mulohazanining rostligi) tekshiriladi;
- 2)  $n = k$  bo'lganda ( $k > n_0$ )  $A(n)$  tasdiqni to'g'ri deb faraz qilib,  $n = k + 1$  uchun  $A(n)$  ning to'g'riliqi isbotlanadi ( $A(k) \Rightarrow A(k+1)$ );
- 3) yuqoridaagi 1) va 2) bandlar ijobiy hal bo'lgan taqdirda  $A(n)$  tasdiq barcha  $n \geq n_0$  lar uchun to'g'ri deb xulosa chiqariladi.

**1-misol.** Dastlabki n ta toq natural sonlar yig'indisi uchun

$$S_n = 1 + 3 + 5 + \dots + (2n - 1) = n^2 \quad (1)$$

bo'lishi isbotlansin.

› Bu ayniyatni matematik induktsiya usuli yordamida isbotlaymiz.

- 1)  $n = 1$  bo'lganda  $S_1 = 1 = 1^2$  bo'lib,  $A(1)$  to'g'ri.
- 2)  $n = k$  da  $A(k)$  to'g'ri bo'lsin:

$$S_k = 1 + 3 + 5 + \dots + (2k - 1) = k^2.$$

Unda

$$\begin{aligned} S_{k+1} &= 1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) = S_k + 2k + 1 = \\ &= k^2 + 2k + 1 = (k + 1)^2 \end{aligned}$$

bo'ladi. Demak,

$$A(k) \Rightarrow A(k+1).$$

Matematik induktsiya usuliga binoan (1) tenglik ixtiyoriy natural n soni uchun to'g'ri bo'ladi ›

**2-misol.**  $\forall n \in N$  uchun  $n(n^2 + 5)$  ning 6 ga bo'linishi ko'rsatilsin.

1)  $n = 1$  bo'lganda  $1 \cdot (1^2 + 5) = 6$  bo'lib, A(1) tasdiq to'g'ri.

2)  $n = k$  da A(k) to'g'ri bo'lsin, ya'ni  $k \cdot (k^2 + 5) \geq 6$  ga bo'linsin.

Unda

$$(k+1) \cdot [(k+1)^2 + 5] = (k+1) \cdot (k^2 + 5 + 2k + 1) =$$

$$= k \cdot (k^2 + 5) + 2k^2 + k + k^2 + 5 + 2k + 1 = k(k^2 + 5) + 3k(k+1) + 6$$

bo'lib,  $k(k+1)$  har doim juft son bo'lganligidan  $3k(k+1)$  soni 6 ga bo'linadi. Undan

$$A(k) \Rightarrow A(k+1)$$

ni topamiz. Demak,  $\forall n \in N$  da  $n(n^2 + 5)$  son 6 ga bo'linadi ▶

**3-misol.**  $\forall n \in N$  va  $\forall x > -1$  uchun ushbu

$$(1+x)^n \geq 1 + nx \quad (\text{Bernulli tengsizligi}) \quad (2)$$

tengsizlik isbotlansin.

1)  $n = 1$  bo'lganda  $(1+x)^1 = 1 + 1 \cdot x$  bo'lib, A(1) to'g'ri.

2)  $n = k$  da A(k) to'g'ri bo'lsin:

$$(1+x)^k \geq 1 + kx \quad (3)$$

Ravshanki,  $x > -1 \Rightarrow 1+x > 0$ . (3) tengsizlikning har ikki tomonini  $1+x$  ga ko'paytirib topamiz:

$$(1+x)^{k+1} \geq (1+kx) \cdot (1+x) = 1 + kx + x + kx^2 \geq 1 + (k+1)x$$

Demak,

$$A(k) \Rightarrow A(k+1)$$

Binobarin,  $\forall n \in N$  va  $\forall x > -1$  uchun

$$(1+x)^n \geq 1 + nx$$

bo'ladi ▶

**4-misol.**  $\forall n \in N$  uchun Ushbu

$$\cos \alpha \cdot \cos 2\alpha \cdot \cos 4\alpha \cdots \cos 2^n \alpha = \frac{\sin 2^{n+1} \alpha}{2^{n+1} \cdot \sin \alpha}, \alpha \neq 0 \quad (4)$$

tenglikning o'rinli bo'lishi ko'rsatilsin.

1)  $n=1$  da (4) tenglik o'rinli:

$$\cos \alpha \cdot \cos 2\alpha = \frac{1}{4 \sin \alpha} \cdot 4 \sin \alpha \cdot \cos \alpha \cdot \cos 2\alpha =$$

$$= \frac{1}{4 \sin \alpha} \cdot 2 \sin 2\alpha \cdot \cos 2\alpha = \frac{\sin 4\alpha}{4 \sin \alpha} = \frac{\sin 2^2 \alpha}{2^2 \cdot \sin \alpha}$$

2)  $n=k$  da  $A(k)$  to'g'ri, ya'ni

$$\cos \alpha \cdot \cos 2\alpha \cdot \cos 4\alpha \cdots \cos 2^k \alpha = \frac{\sin 2^{k+1} \alpha}{2^{k+1} \cdot \sin \alpha}$$

deb,  $n=k+1$  da  $A(k+1)$  ning to'g'ri bo'lishini ko'rsatamiz:

$$\begin{aligned} & \cos \alpha \cdot \cos 2\alpha \cdot \cos 4\alpha \cdots \cos 2^k \alpha \cdot \cos 2^{k+1} \alpha = \frac{\sin 2^{k+1} \alpha}{2^{k+1} \cdot \sin \alpha} \cdot \cos 2^{k+1} \alpha = \\ & = \frac{2 \cdot \sin 2^{k+1} \alpha \cdot \cos 2^{k+1} \alpha}{2 \cdot 2^{k+1} \cdot \sin \alpha} = \frac{\sin(2 \cdot 2^{k+1} \alpha)}{2^{k+2} \cdot \sin \alpha} = \frac{\sin 2^{k+2} \alpha}{2^{k+2} \cdot \sin \alpha} \end{aligned}$$

Demak, (4) tenglik  $\forall n \in N$  uchun o'rinli.

**5-misol.** Ixtiyoriy natural  $n > 1$  uchun

$$(2n)! < 2^{2n} \cdot (n!)^2 \quad (5)$$

tengsizlik isbotlansin.

1)  $n=2$  da  $(2 \cdot 2)! = 4! = 24$  va  $2^{2 \cdot 2} \cdot (2!)^2 = 16 \cdot 4 = 64$  bo'lib,  $A(1)$  to'g'ri.

2)  $n=k$  da  $A(k)$  to'g'ri bo'linsin, ya'ni  $(2k)! < 2^{2k} \cdot (k!)^2$  tengsizlik bajarilsin. Unda

$$\begin{aligned}
 (2k+2)! &= (2k)!(2k+1) \cdot (2k+2) < 2^{2k} \cdot (k!)^2 \cdot (2k+1) \cdot (2k+2) = \\
 &= 2^{2k+2} \cdot [(k+1)!]^2 \cdot \frac{2^{2k} \cdot (k!)^2 \cdot (2k+1) \cdot (2k+2)}{2^{2k+2} \cdot [(k+1)!]^2} = \\
 &= 2^{2k+2} \cdot [(k+1)!]^2 \cdot \frac{(2k+1)(2k+2)}{2^2 \cdot (k+1)^2} = 2^{2k+2} \cdot [(k+1)!]^2 \cdot \frac{2k^2 + 3k + 1}{2k^2 + 4k + 2} < \\
 &< 2^{2k+1} \cdot [(k+1)!]^2
 \end{aligned}$$

bo'ladi. Demak, (5) tengsizlik ixtiyoriy natural  $n > 1$  uchun o'rinli. ▶

### Mustaqil yechish uchun misol va masalalar.

Matematik induktsiya usulini qo'llab,  $\forall n \in N$  uchun quyidagi munosabatlarning to'g'riligi ko'rsatilsin.

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2.$$

**Ko'rsatma:** Avval 1 – misoldan foydalaniib, isbotlanishi kerak bo'lgan tenglikni ushbu

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

tenglik ko'rinishida yozib oling.

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(4n^2 - 1)}{3}.$$

$$4. \quad \left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{9}\right) \cdot \dots \cdot \left[1 - \frac{1}{(n+1)^2}\right] = \frac{n+2}{2n+2}.$$

$$5. \quad 1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = n^2 \cdot (2n^2 - 1).$$

$$6. \quad \arctg \frac{1}{2} + \arctg \frac{1}{8} + \dots + \arctg \frac{1}{2n^2} = \arctg \frac{n}{n+1}.$$

**Ko'rsatma:**

$$\arctg \alpha + \arctg \beta = \arctg \frac{\alpha + \beta}{1 - \alpha \beta}$$

tenglikdan foydalaning.

7.  $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
8.  $1 + 3 + 6 + 10 + \dots + \frac{(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{1}{6}n(n+1)(n+2)$ .
9.  $1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1$ .
10.  $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + (n-1)n = \frac{(n-1) \cdot n \cdot (n+1)}{3}$ .
11.  $\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \dots + \frac{n-1}{n!} = 1 - \frac{1}{n!}$ .
12.  $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n! = (n+1)! - 1$ .
13.  $2^2 + 6^2 + \dots + (4n-2)^2 = \frac{4n \cdot (2n-1) \cdot (2n+2)}{3}$ .
14.  $\cos \frac{\alpha}{2} \cdot \cos \frac{\alpha}{4} \cdot \cos \frac{\alpha}{8} \cdot \dots \cdot \cos \frac{\alpha}{2^n} = \frac{\sin \alpha}{2^n \cdot \sin \frac{\alpha}{2^n}}$ .

15 – 20 misollarda ko'rsatilgan ifodalarni  $\forall n \in N$  uchun qavs ichida ko'rsatilgan songa bo'linishi isbotlansin.

15.  $n^5 - n$  (5 ga).
16.  $6^{2n-2} + 3^{n+1} + 3^{n-1}$  (11 ga).  $n^3 + 11n$  (6 ga).
17.  $11^{n+1} + 12^{2n-1}$  (133 ga).
18.  $7 \cdot 4^{2n} + 56$  (3 ga).
19.  $2^{n+5} \cdot 3^{4n} + 5^{3n+1}$  (37 ga).
20. Ixtiyoriy  $a \in R, b \in R$  va  $\forall n \in N$  lar uchun ushbu

$$(a+b)^n = \sum_{k=0}^n C_n^k a^{n-k} b^k = a^n + n \cdot a^{n-1} b + \dots + \frac{n \cdot (n-1)}{1 \cdot 2} a^{n-2} b^2 + \dots + n a \cdot b^{n-1} + b^n \quad (6)$$

tenglik (Nyuton binomi) isbotlansin. Bu yerda

$$C_n^k := \frac{n!}{k!(n-k)!}, \quad 0!=1.$$

**J. Ko'rsatma:** Matematik induktsiya usulidan foydalanamiz.

$$1) \quad A(1) \text{ to'g'ri. } 2) \quad A(k) \text{ to'g'ri, ya'ni } (a+b)^k = \sum_{m=0}^k C_k^m a^{k-m} b^m$$

tenglik bajariladi deb faraz qilamiz. Unda

$$\begin{aligned} (a+b)^{k+1} &= (a+b) \cdot (a+b)^{k+1} = (a+b) \cdot \sum_{m=0}^k C_k^m a^{k-m} b^m = \sum_{m=0}^k C_k^m a^{k+1-m} b^m + \sum_{m=0}^k C_k^m a^{k-m} b^{m+1} = \\ &= \sum_{m=0}^k C_k^m a^{k+1-m} b^m + \sum_{m=0}^{k+1} C_k^{m-1} a^{k+1-m} b^m = a^{k+1} + \sum_{m=1}^k (C_k^m + C_k^{m-1}) a^{k+1-m} b^m + b^{k+1}. \end{aligned}$$

Agar  $C_k^m + C_k^{m-1} = C_{k+1}^m$  va  $C_{k+1}^0 = C_{k+1}^{k+1} = 1$  ekanligini e'tiborga olsak, oxirgi tenglikdan

$$(a+b)^{k+1} = a^{k+1} + \sum_{m=1}^k C_{k+1}^m a^{k+1-m} b^m + b^{k+1} = \sum_{m=1}^{k+1} C_{k+1}^m a^{k+1-m} b^m$$

tenglikni hosil qilamiz. Demak,  $A(k) \Rightarrow A(k+1) \Rightarrow \forall n \in N$  uchun Nyuton binomi o'rinni.

21.  $\forall n \in N$  uchun ushbu

$$2 \leq \left(1 + \frac{1}{n}\right)^n < 3$$

tengsizlik isbotlansin.

**Ko'rsatma:** Nyuton binomidan foydalaning.

22. Ixtiyoriy natural  $n > 1$  uchun ushbu

$$\frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 1$$

tengsizlik isbotlansin.

23. Agar  $a > b > 0$  bo'lsa,  $\forall n \in N$  uchun ushbu

$$a^n > b^n$$

tengsizlik isbotlansin.

24. Ixtiyoriy natural  $n \in N$  uchun

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \dots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$$

tengsizlik isbotlansin.

25. Agar  $x_1, x_2, \dots, x_n$  lar ixtiyoriy musbat sonlar bo'lib,

$$x_1 \cdot x_2 \cdot \dots \cdot x_n = 1 \quad (7)$$

munosabat bajarilsa, u holda

$$x_1 + x_2 + \dots + x_n \geq n \quad (8)$$

tengsizlikning bajarilishi isbotlansin.

**J. Ko'rsatma:** 1)  $n=1$  bo'lganda, (7) shartga ko'ra  $x_1=1$  bo'lib, A(1) o'rini.

2)  $n=k$ , bo'lganda A(k) o'rini deb faraz qilamiz va  $n=k+1$  da A(k+1) ning o'rini bo'lishini ko'rsatamiz.

Aytaylik,  $x_1, x_2, \dots, x_k, x_{k+1}$  lar  $x_1 \cdot x_2 \cdots x_k \cdot x_{k+1} = 1$  munosabatni qanoatlaniruvchi ixtiyoriy musbat sonlar bo'lsin.

(7) munosabat bajarilayotganda quyidagi ikki hol bo'lishi mumkin: yoki (7) munosabatdagi barcha sonlar 1 ga teng bo'lib, A(k+1) bajariladi; yoki bu sonlar ichida kamida bitta birdan farqli son bor. Hech bo'limganda yana bitta birdan farqli son bo'lib, agar ularning biri 1 dan kichik bo'lsa, ikkinchisi albatta 1 dan katta bo'ladi. Umumiylikka ziyon keltirmagan holda  $x_k > 1$  va  $x_{k+1} < 1$  deb faraz qilish mumkin.

Endi ushbu k ta  $x_1, x_2, \dots, x_{k-1}, (x_k \cdot x_{k+1})$  sonni ko'ramiz. Induktiv farazga ko'ra  $x_1 + x_2 + \dots + x_{k-1} + x_k \cdot x_{k+1} \geq k$  tengsizlik o'rini, chunki bu k ta sonning ko'paytmasi 1 ga teng. Bu tengsizlikning ikkala tomoniga  $x_k + x_{k+1}$  ni qo'shib,  $x_k \cdot x_{k+1}$  ni o'ng tomonga o'tkazamiz:

$$\begin{aligned} x_1 + x_2 + \dots + x_{k-1} + x_k + x_{k+1} &\geq k - x_k \cdot x_{k+1} + x_k + x_{k+1} = k + 1 + x_k(1 - x_{k+1}) + x_{k+1} - 1 = \\ &= k + 1 + x_k \cdot (1 - x_{k+1}) - (1 - x_{k+1}) = k + 1 + (1 - x_{k+1}) \cdot (x_k - 1) \geq k + 1 \end{aligned}$$

Demak,  $A(k) \Rightarrow A(k+1)$ . Shunday qilib, (8) – tengsizlik  $\forall n \in N$  uchun isbotlanadi.

Isbotdan shu narsa ko'rinish turibdiki, (8) – munosabat faqat  $x_1 = x_2 = \dots = x_n = 1$  bo'lgandagina tenglikka aylanadi.

26. Agar  $a_1, a_2, \dots, a_n$  lar ixtiyoriy musbat sonlar bo'lsa, ushbu

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \cdot a_2 \cdots a_n} \quad (9)$$

tengsizlikning bajarilishi isbotlansin.

**Ko'rsatma.**

$$x_k = \frac{a_k}{\sqrt[n]{a_1 \cdot a_2 \cdots a_n}} \quad (k = 1, 2, \dots, n)$$

deb olib, 25 – misoldan foydalaning.

Matematik induktsiya usulidan foydalanib quyidagi munosabatlarning o'rini ekanligi isbotlansin

$$27. n! < \binom{n+1}{2}^n \quad (n > 1).$$

**Ko'rsatma:**  $a_k = k$  ( $k = 1, n$ ) deb olib, 26 – misoldan foydalaning.

$$28. \sqrt[n]{x_1 \cdot x_2 \cdot \dots \cdot x_n} \geq \frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \quad (n \in N) \text{ bu yerda}$$

$$x_k > 0, k = 1, n.$$

**Ko'rsatma:** (9) tengsizlikni  $\frac{1}{x_1}, \frac{1}{x_2}, \dots, \frac{1}{x_n}$  sonlar uchun qo'llang.

$$29. \left( \sum_{i=1}^n x_i y_i \right)^2 \leq \left( \sum_{i=1}^n x_i^2 \right) \cdot \left( \sum_{i=1}^n y_i^2 \right) - \text{Koshi} - \text{Bunyakovskiy}$$

tengsizligi. Bu yerda  $x_i, y_i (i = 1, n)$  lar ixtiyoriy haqiqiy sonlar.

**J. Ko'rsatma:**  $\forall t \in R$  uchun  $\sum_{i=1}^n (x_i t + y_i)^2 \geq 0$  tengsizlik o'rini bo'lishidan foydalaning.

$$30. (n!)^2 < \left[ \frac{(n+1)(2n+1)}{6} \right]^n \quad (n > 1).$$

**Ko'rsatma:**  $a_k = k^2$  deb olamiz, 26 – misoldan foydalaning.

$$31. n^{n+1} > (n+1)^n \quad (n \geq 3).$$

$$32. (2n)! > \frac{4n}{n+1} \cdot (n!)^2 \quad (n > 1).$$

**II—BOB.  
OLIY ALGEBRA ELEMENTLARI.**

**1—§. 2—va 3—tartibli determinantlar, ularning xossalari. Chiziqli tenglamalar sistemasini Kramer usuli bilan yechish.**

Aytaylik,  $a_1, a_2, b_1, b_2$  sonlar berilgan bo'lsin.

Ushbu  $a_1b_2 - a_2b_1$  songa **ikkinci tartibli determinant** deb ataladi va u,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

kabi yoziladi.

$a_1, a_2, b_1, b_2$  sonlarga determinantning elementlari deyiladi.

**Misol.**  $\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = 2 \cdot (-4) - 5 \cdot 3 = -23.$

**Xossalari.**

**1<sup>0</sup>.** Agar determinantning mos qatorini mos ustuni bilan almashtirilsa, uning qiymati o'zgarmaydi, ya'ni

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - \text{tenglik o'rinni.}$$

**2<sup>0</sup>.** Agar determinantning yo'llari (yoki ustunlari) almashtirilsa, unda determinant ishorasini o'zgartiradi:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = - \begin{vmatrix} a_2 & a_1 \\ b_2 & b_1 \end{vmatrix}.$$

**3<sup>0</sup>.**  $\begin{vmatrix} ka_1 & b_1 \\ ka_2 & b_2 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$

**4<sup>0</sup>.** Birorta yo'lning (yoki ustunning) elementlari nollardan iborat bo'lsa, unda determinantning qiymati nolga teng bo'ladi.

**5<sup>0</sup>.** Agar ikkita yo'ldagi (yoki ustundagi) elementlar bir-biriga teng bo'lsa, unda determinant nolga teng bo'ladi.

**Masalan,**

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = a_1 a_2 - a_1 a_2 = 0$$

**6<sup>0</sup>.** Agar birorta yo'ldagi (yoki ustundagi) elementlar o'zgarmas songa ko'paytirilib boshqasiga qo'shilsa, unda determinantning qiymati o'zgarmaydi.

**Masalan,**

$$\begin{vmatrix} a_1 \pm ka_2 & a_2 \\ b_1 \pm kb_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Endi 3-tartibli determinantning ta'rifini beramiz.

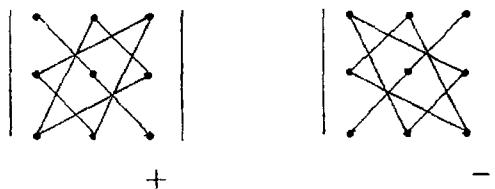
Aytaylik,  $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$  sonlar berilgan bo'lisin.

**Ushbu**

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{31}. \quad (1)$$

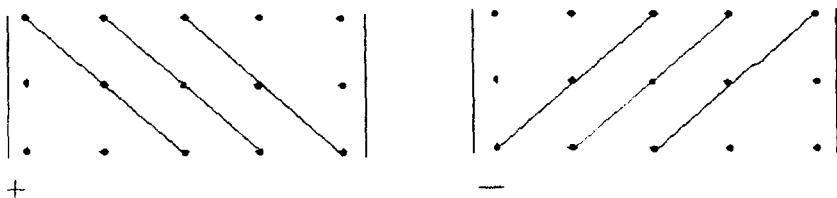
songa 3-tartibli determinant deyiladi.

$\Delta$  ning qiymatini hisoblashning (1)-formulasiga uchiburchak usuli deyiladi. Uni eslab qolishda quyidagi sxema yordam beradi:



Δ ning qiymatini **Sarryus usuli** bilan ham hisoblash mumkin;

$$\begin{aligned} & a_{11}a_{12}a_{13}a_{11}a_{12} \\ & a_{21}a_{22}a_{23}a_{21}a_{22} \\ & a_{31}a_{32}a_{33}a_{31}a_{32} \end{aligned}$$



Uchinchi tartibli determinantning qiymatini determinantning tartibini pasaytirib, minorlar yordamida ham hisoblash mumkin. Masalan, birinchi yo'lning elementlari bo'yicha minorlarga yoyish yordamida quyidagicha hisoblənadi:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (2)$$

$a_{ij}$  ( $i, j = 1, 2, 3$ ) elementlarga ko'paytirilgan ikkinchi tartibli determinantlar bu elementlarning **algebraik to'ldiruvchisi** deyiladi va  $A_{ij}$  kabi belgilanadi.

Agar  $a_{ij}$  ( $i, j = 1, 2, 3$ ) element turgan satr hamda ustunning nomerlari yig'indisi juft bo'lsa, u holda  $A_{ij} = M_{ij}$  bo'ladi. Agar yig'ndi toq bo'lsa, u holda  $A_{ij} = -M_{ij}$  bo'ladi, ya'ni

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Uchinchi tartibli determinantlar ham **1<sup>0</sup>-6<sup>0</sup>** xossalarga ega.  $n$ -tartibli determinantlar ham yuqoridagi kabi aniqlana di, faqat uning qiymati minorlar yordamida hisoblənadi.

Determinantlarning qo'llanish doirasasi juda keng. Xususan, ular yordamida chiziqli tenglamalar sistemasi osongina yechiladi.

## Aytaylik, ushbu

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \quad (3)$$

uchta noma'lum chiziqli tenglamalar sistemasi berilgan bo'lib, uni yechish talab qilinsin. Uning uchun ushbu determinantlarni hisoblaymiz:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - \text{asosiy determinant.}$$

$$\Delta_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}, \quad \Delta_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

$\Delta_x, \Delta_y, \Delta_z$  — yordamchi determinantlar.

**Teorema (Kramer).** Agar  $\Delta \neq 0$  bo'lsa, unda (3)-tenglamalar sistemasi yagona yechimga ega bo'lib, uning yechimi ushbu

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta} \quad (4)$$

formulalar yordamida topiladi.

(4) – formulaga **Kramer formulasi** deyiladi va tenglamalar sistemasini yechishning bu usuliga esa **Kramer usuli** deb ataladi.

**Izoh.** Agar  $\Delta = 0$  bo'lib,  $\Delta_x, \Delta_y, \Delta_z$  lardan birortasi  $\neq 0$  bo'lsa, unda (3)-sistema yechimga ega emas. Agar  $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$  bo'lsa, (3)-sistema cheksiz ko'p yechimga ega bo'ladi.

n ta noma'lumli chiziqli tenglamalar sistemasi uchun ham Kramer teoremasi o'rinli.

## Nazorat savollari.

- 1) 2 – tartibli determinant deb nimaga aytildi? Misollar keltiring.
- 2) 3 – tartibli determinant deb nimaga aytildi? Misollar keltiring.
- 3) Determinant xossalari. Misollar.
- 4) Chiziqli tenglamalar sistemasini yechishning Kramer usuli. Misollar.

### 2-§. Matritsalar va ular ustida amallar. Gauss usuli.

Ushbu

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \circ & \circ & \circ & \circ \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = A$$

jadvalga  $(n \times m)$  o'lchovli matritsa deb ataladi.  $n = m$  bo'lsa unga kvadrat matritsa deb ataladi.

Endi,

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \circ & \circ & \circ & \circ \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix}$$

matritsa berilgan bo'lsin.

Agar  $a_{ij} = b_{ij}$  ( $i = 1, n, j = 1, m$ ) bo'lsa, unda  $A = B$  deyiladi.

$$A + B := \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1m} \\ \circ & \circ & \cdots & \circ \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nm} + b_{nm} \end{pmatrix},$$

$$\lambda \cdot A := \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1m} \\ \circ & \cdots & \circ \\ \lambda a_{n1} & \cdots & \lambda a_{nm} \end{pmatrix}, \quad 0 := \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$$

deb qabul qilamiz.

### Xossalari:

$$1^0. A + B = B + A$$

$$2^0. (A + B) + C = A + (B + C)$$

$$3^0. A\lambda = \lambda A$$

Endi  $n = m = 2$  bo'lganda, matritsani ko'paytirishni o'rjanamiz.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ va } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$A \cdot B := \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix} \quad (5)$$

(5) – tenglikdan ko'riniib turibdiki,  $\forall A$  va  $B$  matritsalar berilganda,  $A \cdot B$  aniqlanishi uchun  $A$  matritsaning ustunlar soni  $B$  matritsaning yo'llar soniga teng bo'lishi kerak.

### Misollar.

$$1) \begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 1 \cdot 2 + 0 \cdot 2 & 2 \cdot 2 + 1 \cdot 1 + 0 \cdot 2 \\ 3 \cdot 1 + 1 \cdot 2 + 1 \cdot 2 & 3 \cdot 2 + 1 \cdot 1 + 1 \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 7 & 9 \end{pmatrix}.$$

$$2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$

$$4^0. A \cdot (B \cdot C) = (A \cdot B) \cdot C$$

$$5^0. (A + B) \cdot C = A \cdot C + B \cdot C$$

$$6^0. A \cdot E = E \cdot A = A, \text{ bu yerda } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ – birlik matritsa.}$$

$A$  – kvadrat matritsa bo'lib, uning determinanti  $|A| \neq 0$  bo'lсин.

Unda  $A^{-1}$  deb belgilanadigan shunday matritsa topiladiki,  $A \cdot A^{-1} = A^{-1} \cdot A = E$  tenglik bajariladi, hamda  $A^{-1}$ ga teskari matritsa deb ataladi.

Agar,  $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$  bo'lsa,

$$A^{-1} = \begin{pmatrix} \frac{A_{11}}{|A|} & \frac{A_{21}}{|A|} & \frac{A_{31}}{|A|} \\ \frac{A_{12}}{|A|} & \frac{A_{22}}{|A|} & \frac{A_{32}}{|A|} \\ \frac{A_{13}}{|A|} & \frac{A_{23}}{|A|} & \frac{A_{33}}{|A|} \end{pmatrix} \quad (6)$$

formula yordamida topiladi. Bu yerda  $a_{ii} - a_{ii}$  elementining algebraik to'ldiruvchisi;

$$\text{Masalan, } A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \dots$$

Matritsalar yordamida chiziqli tenglamalar sistemasi osongina yechiladi.

Aytaylik,

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases} \quad (7)$$

(7) – tenglamalar sistemasi berilgan bo'lsin. Agar

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{va} \quad C = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{deb belgilasak,}$$

(7) ni  $A \cdot X = C$  (8) matritsa ko'rinishida yozish mumkin.

$$X = A^{-1} \cdot C \quad (9)$$

**1-misol.**  $\begin{cases} x_1 + 2x_2 = 10 \\ 3x_1 + 2x_2 + x_3 = 23 \\ x_2 + 2x_3 = 13 \end{cases}$  tenglamalar sistemasi matritsalar usuli bilan yechilsin.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad C = \begin{pmatrix} 10 \\ 23 \\ 13 \end{pmatrix} \Rightarrow A^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{4}{9} & -\frac{2}{9} \\ \frac{2}{3} & -\frac{2}{9} & \frac{1}{9} \\ -\frac{1}{3} & \frac{1}{9} & \frac{4}{9} \end{pmatrix} \Rightarrow$$

$$\Rightarrow X = A^{-1} \cdot C = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \Rightarrow x_1 = 4, x_2 = 3, x_3 = 5.$$

(7) tenglamalar sistemasini noma'lumlarni ketma-ket yo'qotish usuli, ya'ni Gauss usuli bilan ham yechish mumkin.

Uni misolda tushuntiramiz.

**2-misol.**  $\begin{cases} x_1 + 0,5x_2 - 0,5x_3 = 0,5 \\ 3x_1 + 2x_2 - 2x_3 = 1 \\ x_1 - x_2 + 2x_3 = 5 \end{cases}$

Avval 1-qatorni 3 ga ko'paytirib, keyin ikkinchi qatordan ayiramiz, so'ng 1-qatorni uchinchi qatordan ayirib qo'yidagi sistemani hosil qilamiz (bunda biz determinantlarning 6<sup>0</sup>-xossasidan foydalanamiz):

$$\left. \begin{array}{l} \begin{array}{l} x_1 + 0,5x_2 - 0,5x_3 = 0,5 \\ 0,5x_2 - 0,5x_3 = -0,5 \\ -1,5x_2 + 2,5x_3 = 45 \end{array} \Rightarrow \begin{array}{l} x_1 + 0,5x_2 - 0,5x_3 = 0,5 \\ x_2 - x_3 = -1 \\ -1,5x_2 + 2,5x_3 = 45 \end{array} \Rightarrow \begin{array}{l} x_1 + 0,5x_2 - 0,5x_3 = 0,5 \\ x_2 - x_3 = -1 \\ x_3 = 3 \end{array} \end{array} \right\} \Rightarrow$$

$$\Rightarrow x_1 = 3 \Rightarrow x_2 = -1 + x_3 = 2 \Rightarrow x_1 = 1.$$

Shunday qilib,  $x_1 = 1, x_2 = 2, x_3 = 3$ .

### Nazorat savollari.

- 1.Matritsa ta'rifi. Misollar.
- 2.Matritsalarni qo'shish va ko'paytirish. Misollar.
- 3.Teskari matritsa. Misollar.
- 4.Tenglamalar sistemasini yechishning matritsalar usuli. Misollar.
- 5.Gauss usuli. Misollar.

### 3-§. Chiziqli (vektor) fazo va chiziqli almashtirishlar.

Vektor tushunchasi, ular ustida amallar, ularning skalyar ko'paytmasi mablag'i kursidan ma'lum bo'lganligi sababli biz ularga to'xtalib o'tirmaymiz.

$n$  ta  $x_1, x_2, \dots, x_n$  tartiblangan sonlar to'plamini  $n$  o'lchovli vektor deb ataymiz va  $x$  vektorni  $x = \{x_1, x_2, \dots, x_n\}$  ko'rinishda yozamiz. Agar  $A(a_1, a_2, \dots, a_n)$  va  $B(b_1, b_2, \dots, b_n)$  nuqtalarning koordinatalari uchun  $x_1 = b_1 - a_1, x_2 = b_2 - a_2, \dots, x_n = b_n - a_n$  munosabatlar o'rinni bo'lsa  $A$  nuqta  $x$  vektorning boshi,  $B$  nuqta  $x$  vektorning oxiri ekanligini bildiradi. Vektorlar quyidagicha ham belgilanadi  $x, \bar{x}$  va  $\overline{AB}$ .

$x$  vektorni tashkil qiluvchi  $x_1, x_2, \dots, x_n$  sonlar uning komponentlari (koordinatalari) deyiladi.

Barcha komponentlari nolga teng bo'lgan vektor nol vektor deyiladi.

Vektorlarni qo'shish va vektorlarni haqiqiy songa ko'paytirish vektorlar ustida chiziqli amallar bajarish deyiladi.

Ushbu

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \dots + \lambda_n e_n$$

(bu yerda  $e_1, e_2, e_3, \dots, e_n$  — turli vektorlar,  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  — sonlar) ko'rinishda tasvirlangan  $x$  vektor  $e_1, e_2, e_3, \dots, e_n$  vektorlarning chiziqli kombinatsiyasidan iborat deyiladi.

Agar hammasi bir vaqtida nolga teng bo'lmasagan  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  ( $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \dots + \lambda_n^2 > 0$ ) shunday sonlar mavjud bo'lib,

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \dots + \lambda_n e_n = 0 \quad (*)$$

tenglik o'rinni bo'lsa,  $e_1, e_2, e_3, \dots, e_n$  vektorlar chiziqli bog'liq vektorlar deyiladi. Agar bunday  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  — sonlar mavjud bo'lmasa, ya'ni (\*) tenglik faqat  $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$  bo'lgandagina bajarilsa, unda  $e_1, e_2, e_3, \dots, e_n$  vektorlar chiziqli erkli vektorlar deyiladi.

Ikkita vektor o'zaro kollinear bo'lganda ular chiziqli bog'lig bo'ladilar. Agar ucta vektor o'zaro komplanar bo'lsa, ular chiziqli bog'ligli bo'ladilar.

$\nu$ -fazoning istalgan vektori chiziqli ifodalananadigan chiziqli erkli vektorlar to'plami shu fazoning bazisi deyiladi.  $\nu$ -fazoning bazisini tashkil etuvchi vektorlar bazis vektorlar deyiladi.

$\nu$ -fazoning chiziqli erkli vektorlarining eng katta soni shu fazoning o'lchovi deyiladi.

Shunga muvofiq ravishda, to'g'ri chiziqni bir o'lchovli  $\nu_1$  fazo deyiladi, birorta to'g'ri chiziqqa joylashgan vektorlar to'plamining bazisi bitta vektordan iborat. Tekislikni ikki o'lchovli  $\nu_1$  fazo deyiladi, birorta tekislikda joylashgan vektorlar to'plamining bazislari ikkita vektordan iborat. Odatdag'i fazoni uch o'lchovli  $\nu_1$ ,  $\nu_2$ ,  $\nu_3$  fazo deyiladi, uning bazislari ucta nokomplanar vektordan iborat.

Quyidagi teorema o'rinci:  $\nu_1$ , fazoning istalgan  $x$  vektori shu fazoning ucta chiziqli erkli  $e_1, e_2, e_3$  vektorlarining chiziqli kombinatsiyasi yordamida bir qiymatli aniqlanadi, ya'ni

$$x = x_1 e_1 + x_2 e_2 + x_3 e_3.$$

Bu ifoda  $x$  vektorning  $e_1, e_2, e_3$  bazisi bo'yicha yoyilmasi deyiladi. Vektoring bazis bo'yicha yoyilmasidagi  $x_1, x_2, x_3$  koeffitsientlar vektorning shu bazisdagi koordinatalari deyiladi.

O'zaro perpendikulyar birlik vektorlardan iborat bo'lgan bazis ortonormallangan bazis deyiladi va quyidagicha yoziladi.

$$(e_i, e_j) = \begin{cases} 1, & \text{агар } i = j \text{ бўлса} \\ 0, & \text{агар } i \neq j \text{ бўлса} \end{cases}$$

$$(i, j = 1, 2, \dots, n).$$

$\nu_1$  fazoda biror qoida bo'yicha har bir vektor  $x$  ga shu fazoning  $y$  vektori mos qo'yilgan bo'lsin, ya'ni vektor argument  $x$  ning  $y = A(x)$  vektor qiymatli funktsiyasi berilgan bo'lsa, u holda bu fazoda vektorlarni almashtirish berilgan deyiladi.

Agar

$$A(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 A(x_1) + \lambda_2 A(x_2) \quad (1)$$

shart bajarilsa  $A$  almashtirish **chiziqli almashtirish** deyiladi, bu yerda  $x_1$  va  $x_2$  — qaralayotgan fazoning ixtiyoriy vektorlari,  $\lambda_1$  va  $\lambda_2$  esa istalgan sonlar.

$V_3$  fazoda biror  $e_1, e_2, e_3$  bazis tanlab olamiz va bazisning har bir vektori uchun  $A$  almashtirishni qo'llaymiz:

$$\begin{aligned} A(e_1) &= a_{11}e_1 + a_{21}e_2 + a_{31}e_3 \\ A(e_2) &= a_{12}e_1 + a_{22}e_2 + a_{32}e_3 \\ A(e_3) &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3 \end{aligned} \quad (2)$$

Ixtiyoriy  $x \in V_3$  vektorni olamiz va uni  $x = x_1e_1 + x_2e_2 + x_3e_3$  bazis bo'yicha yoyamiz. U holda

$$y = A(x) = A(x_1e_1 + x_2e_2 + x_3e_3) = x_1A(e_1) + x_2A(e_2) + x_3A(e_3)$$

(2) formuladan foydalaniib va o'xshash hadlarni ixchamlab quyidagini hosil qilamiz

$$\begin{aligned} y &= x_1(a_{11}e_1 + a_{21}e_2 + a_{31}e_3) + x_2(a_{12}e_1 + a_{22}e_2 + a_{32}e_3) + x_3(a_{13}e_1 + a_{23}e_2 + a_{33}e_3) = \\ &= (a_{11}x_1 + a_{12}x_2 + a_{13}x_3)e_1 + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3)e_2 + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3)e_3 \end{aligned}$$

$y$  vektoring  $e_1, e_2, e_3$  bazisdagi koordinatalarini  $y_1, y_2, y_3$  orqali belgilab,  $V_3$  fazodagi har qanday vektoring koordinatalarini chiziqli almashtirishini aniqlovchi formulani hosil qilamiz.

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \quad (3)$$

(3)ni matritsa shaklida yozamiz:

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

ya'ni

$$Y = AX$$

Vektorlarning har bir  $y = Ax$  chiziqli almashtirishga shu vektorlar koordinatalarining  $Y = AX$  chiziqli almashtirishi mos keladi va aksincha.

**Misol.** Ikkita chiziqli

$$\begin{cases} x_1' = 2x_1 - x_2 + 5x_3 \\ x_2' = x_1 + 4x_2 - x_3 \\ x_3' = 3x_1 - 5x_2 + 2x_3 \end{cases} \quad \text{va} \quad \begin{cases} x_1'' = x_1' + 4x_2' + 3x_3' \\ x_2'' = 5x_1' - x_2' - x_3' \\ x_3'' = 3x_1' + 6x_2' + 7x_3' \end{cases}$$

almashtirishlar berilgan.  $x_1'', x_2'', x_3''$  ni  $x_1, x_2, x_3$  orqali ifodalovchi almashtirishni toping.

**Yechilishi:** Koordinatalarning birinchi chiziqli almashtirish matritsasini  $A$  bilan ikkinchisini  $B$  bilan belgilaymiz.

$$A = \begin{pmatrix} 2 & -1 & 5 \\ 1 & 4 & -1 \\ 3 & -5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 3 \\ 5 & -1 & -1 \\ 3 & 6 & 7 \end{pmatrix}$$

$x, x', x''$  vektorlarning koordinatalaridan olingan vektor ustunlarini mos ravishda  $X, X', X''$  bilan belgilaymiz, ya'ni

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad X' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \quad X'' = \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix}.$$

U holda koordinatalarning berilgan chiziqli almashtirishlarini matritsa shaklida yozamiz:

$X' = AX$ ,  $X'' = BX'$  ularga mos vektorlarning chiziqli almashtirishlari esa  $x' = Ax$ ,  $x'' = Bx'$  ko'rinishda yoziladi.

Hosil qilingan formulalar quyidagini ifodalaydi:  $A$  chiziqli almashtirish  $x$  vektorni  $x'$  vektorga o'tkazadi, hosil qilingan  $x'$

vektor  $B$  almashtirish bilan  $x''$  vektorga o'tkaziladi.  $x$  vektorni  $y$  vektorga o'tkazuvchi chiziqli almashtirish bunday yoziladi:

$$x'' = B(Ax)$$

koordinatalarning unga mos chiziqli almashtirishi esa quyidagicha yoziladi:

$$X'' = BAX \quad (4)$$

$B$  va  $A$  matritsalarni ko'paytiramiz:

$$BA = \begin{pmatrix} 1 & 4 & 3 \\ 5 & -1 & -1 \\ 3 & 6 & 7 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 & 5 \\ 1 & 4 & -1 \\ 3 & -5 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 0 & 7 \\ 6 & -4 & 24 \\ 33 & -14 & 23 \end{pmatrix}$$

Olingan  $BA$  matritsani (4) tenglikka qo'yamiz:

$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} 15 & 0 & 7 \\ 6 & -4 & 24 \\ 33 & -14 & 23 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

bundan

$$x_1'' = 15x_1 + 0 \cdot x_2 + 7x_3$$

$$x_2'' = 6x_1 + 4x_2 + 24x_3$$

$$x_3'' = 33x_1 - 14x_2 + 23x_3$$

shuni topish talab qilingan edi.

### Nazorat savollari.

- 1) Chiziqli bog'liqli va chiziqli erkli vektorlar.
- 2) Bazis tshunchasi. Misollar.
- 3) Chiziqli almashtirishlar.

## Mustaqil yechish uchun misollar.

### Chiziqli tenglamalar sistemasini uchta usul:

- 1) Kramer usuli.
- 2) Matritsa usuli .
- 3) Gauss usuli bilan yeching.

<p>1. <math>\begin{cases} 3x_1 + 4x_2 + 2x_3 = 8 \\ 2x_1 - 4x_2 - 3x_3 = -1 \\ x_1 + 5x_2 + x_3 = 0 \end{cases}</math></p> <p>3. <math>\begin{cases} 3x_1^* + x_2 + x_3 = 5 \\ x_1 - 4x_2 - 2x_3 = -3 \\ -3x_1 + 5x_2 + 6x_3 = 7 \end{cases}</math></p> <p>5. <math>\begin{cases} x_1 + x_2 - x_3 = -2 \\ 4x_1 - 3x_2 + x_3 = 1 \\ 2x_1 + x_2 - 5 = 1 \end{cases}</math></p> <p>7. <math>\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ 3x_1 - 5x_2 + 3x_3 = 1 \\ 2x_1 + 7x_2 - x_3 = 8 \end{cases}</math></p> <p>9. <math>\begin{cases} x_1 - 2x_2 + 3x_3 = 6 \\ 2x_1 + 3x_2 - 4x_3 = 20 \\ 3x_1 - 2x_2 - 5x_3 = 6 \end{cases}</math></p> <p>11. <math>\begin{cases} 3x_1 + 2x_2 + x_3 = 5 \\ 2x_1 + 3x_2 + x_3 = 1 \\ 2x_1 + x_2 + 3x_3 = 11 \end{cases}</math></p> <p>13. <math>\begin{cases} 4x_1 - 3x_2 + 2x_3 = 9 \\ 2x_1 + 5x_2 - 3x_3 = 4 \\ 5x_1 + 6x_2 - 2x_3 = 18 \end{cases}</math></p> <p>15. <math>\begin{cases} 2x_1 - x_2 - x_3 = 4 \\ 3x_1 + 4x_2 - 2x_3 = 1 \\ 3x_1 - 2x_2 + 4x_3 = 11 \end{cases}</math></p> <p>17. <math>\begin{cases} x_1 + x_2 - x_3 = 1 \\ 8x_1 + 3x_2 - 6x_3 = 2 \\ 4x_1 + x_2 - 3x_3 = 3 \end{cases}</math></p>	<p>2. <math>\begin{cases} 5x_1 + 8x_2 - x_3 = 7 \\ 2x_1 - 3x_2 + 2x_3 = 9 \\ x_1 + 2x_2 + 3x_3 = 1 \end{cases}</math></p> <p>4. <math>\begin{cases} 2x_1 - x_2 + 5x_3 = 4 \\ 5x_1 + 2x_2 + 13x_3 = -23 \\ 3x_1 - x_3 + 5 = 0 \end{cases}</math></p> <p>6. <math>\begin{cases} 7x_1 - 5x_2 = 31 \\ 4x_1 - 11x_3 = -43 \\ 2x_1 + 3x_2 + 4x_3 = -20 \end{cases}</math></p> <p>8. <math>\begin{cases} x_1 + x_2 - x_3 = 1 \\ 8x_1 + 3x_2 - 6x_3 = 2 \\ -4x_1 - x_2 + 3x_3 = -3 \end{cases}</math></p> <p>10. <math>\begin{cases} 4x_1 - 3x_2 + 2x_3 = 9 \\ 2x_1 + 5x_2 - 3x_3 = 14 \\ 5x_1 + 6x_2 - 2x_3 = 18 \end{cases}</math></p> <p>12. <math>\begin{cases} x_1 - 2x_2 + 3x_3 = 6 \\ 2x_1 + 3x_2 - 4x_3 = 20 \\ 3x_1 - 2x_2 - 5x_3 = 6 \end{cases}</math></p> <p>14. <math>\begin{cases} x_1 + x_2 + 2x_3 = -1 \\ 2x_1 - x_2 + 2x_3 = -4 \\ 4x_1 + x_2 + 4x_3 = -2 \end{cases}</math></p> <p>16. <math>\begin{cases} 3x_1 + 4x_2 + 2x_3 = 8 \\ 2x_1 - x_2 - 3x_3 = -4 \\ x_1 + 5x_2 + x_3 = 0 \end{cases}</math></p> <p>18. <math>\begin{cases} x_1 - 4x_2 - 2x_3 = -3 \\ 3x_1 + x_2 + x_3 = 5 \\ 3x_1 - 5x_2 - 6x_3 = -9 \end{cases}</math></p>
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$$19. \begin{cases} 7x_1 - 5x_2 = 31 \\ 4x_1 - 11x_3 = -43 \\ 2x_1 + 3x_2 + 4x_3 = -20 \end{cases}$$

$$20. \begin{cases} x_1 + 2x_2 + 4x_3 = 31 \\ 5x_1 + x_2 + 2x_3 = 20 \\ 3x_1 - x_2 + x_3 = 9 \end{cases}$$

**Ikkita chiziqli almashtirish berilgan  $x_1'', x_2'', x_3''$  ni  $x_1, x_2, x_3$  orqali ifodalovchi almashtirishni toping.**

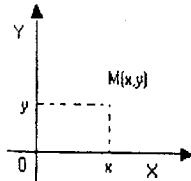
1.	$\begin{cases} x_1 = 5x_1 - x_2 + 3x_3 \\ x_2 = x_1 - 2x_2 \\ x_3 = 7x_2 - x_3 \end{cases}$	va	$\begin{cases} x_1 = 2x_1 + x_3 \\ x_2 = x_2 - 5x_3 \\ x_3 = 2x_1 \end{cases}$
2.	$\begin{cases} x_1 = x_1 + 2x_2 + 2x_3 \\ x_2 = -3x_2 + x_3 \\ x_3 = 2x_1 + 3x_3 \end{cases}$	va	$\begin{cases} x_1 = 3x_1 + x_2 \\ x_2 = x_1 - 2x_2 - x_3 \\ x_3 = 3x_2 + 2x_3 \end{cases}$
3.	$\begin{cases} x_1 = x_1 - 3x_2 + 4x_3 \\ x_2 = 2x_1 + 2x_2 - 5x_3 \\ x_3 = -3x_1 + 5x_2 + x_3 \end{cases}$	va	$\begin{cases} x_1 = 4x_1 + 5x_2 - 3x_3 \\ x_2 = x_1 - x_2 - x_3 \\ x_3 = 7x_1 + 4x_3 \end{cases}$
4.	$\begin{cases} x_1 = 4x_1 + 3x_2 + 5x_3 \\ x_2 = 6x_1 + 7x_2 + x_3 \\ x_3 = 9x_1 + x_2 + 8x_3 \end{cases}$	va	$\begin{cases} x_1 = -x_1 + 5x_2 - 3x_3 \\ x_2 = x_1 - x_2 - x_3 \\ x_3 = 7x_1 + 4x_3 \end{cases}$
5.	$\begin{cases} x_1 = -x_1 - x_2 - x_3 \\ x_2 = -x_1 + 4x_2 + 7x_3 \\ x_3 = 8x_1 + x_2 - x_3 \end{cases}$	va	$\begin{cases} x_1 = 9x_1 + 3x_2 + 5x_3 \\ x_2 = 2x_1 + 3x_3 \\ x_3 = x_2 - x_3 \end{cases}$
6.	$\begin{cases} x_1 = 4x_1 + 3x_2 + 2x_3 \\ x_2 = -2x_1 + x_2 - x_3 \\ x_3 = 3x_1 + x_2 + x_3 \end{cases}$	va	$\begin{cases} x_1 = x_1 - x_2 - x_3 \\ x_2 = 3x_1 + x_2 + 2x_3 \\ x_3 = x_1 + 2x_2 + 2x_3 \end{cases}$
7.	$\begin{cases} x_1 = 3x_1 + 5x_3 \\ x_2 = x_1 + x_2 + x_3 \\ x_3 = 3x_2 - 6x_3 \end{cases}$	va	$\begin{cases} x_1 = 2x_1 - x_2 - 5x_3 \\ x_2 = 7x_1 + x_2 + 4x_3 \\ x_3 = 6x_1 + 4x_2 - 7x_3 \end{cases}$
8.	$\begin{cases} x_1 = 2x_2 \\ x_2 = -2x_1 + 3x_2 + 2x_3 \\ x_3 = 4x_1 - x_2 + 5x_3 \end{cases}$	va	$\begin{cases} x_1 = -3x_1 + x_3 \\ x_2 = 2x_2 + x_3 \\ x_3 = -x_2 + 3x_3 \end{cases}$
9.	$\begin{cases} x_1 = 7x_1 + 4x_3 \\ x_2 = 4x_2 - 9x_3 \\ x_3 = 3x_1 + x_2 \end{cases}$	va	$\begin{cases} x_1 = x_2 - 6x_3 \\ x_2 = 3x_1 + 7x_3 \\ x_3 = x_1 + x_2 - x_3 \end{cases}$

10.  $\begin{cases} \dot{x}_1 = 7x_1 - 4x_3 \\ \dot{x}_2 = 4x_2 - 9x_3 \\ \dot{x}_3 = 3x_1 + x_2 \end{cases}$  va  $\begin{cases} \dot{x}_1 = x_2 + 6x_3 \\ \dot{x}_2 = 3x_1 - 7x_3 \\ \dot{x}_3 = x_1 - x_2 + x_3 \end{cases}$
11.  $\begin{cases} \dot{x}_1 = 4x_1 + 3x_2 + 5x_3 \\ \dot{x}_2 = 6x_1 + 7x_2 + x_3 \\ \dot{x}_3 = 9x_1 + x_2 + 8x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = -x_1 + 3x_2 - 2x_3 \\ \dot{x}_2 = -4x_1 + x_2 + 2x_3 \\ \dot{x}_3 = 3x_1 - 4x_2 + 5x_3 \end{cases}$
12.  $\begin{cases} \dot{x}_1 = x_1 - x_2 - x_3 \\ \dot{x}_2 = -x_1 + 4x_2 + 7x_3 \\ \dot{x}_3 = 8x_1 + x_2 - x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = 9x_1 + 3x_2 + 5x_3 \\ \dot{x}_2 = 2x_1 + 3x_3 \\ \dot{x}_3 = x_2 - x_3 \end{cases}$
13.  $\begin{cases} \dot{x}_1 = 7x_1 + 4x_3 \\ \dot{x}_2 = 4x_2 - 9x_3 \\ \dot{x}_3 = 3x_1 + x_2 \end{cases}$  va  $\begin{cases} \dot{x}_1 = x_2 - 6x_3 \\ \dot{x}_2 = 3x_1 + 7x_3 \\ \dot{x}_3 = x_1 + x_2 - x_3 \end{cases}$
14.  $\begin{cases} \dot{x}_1 = 2x_2 \\ \dot{x}_2 = -2x_1 + 3x_2 + 2x_3 \\ \dot{x}_3 = 4x_1 - x_2 + 5x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = -3x_1 + x_3 \\ \dot{x}_2 = 2x_2 + x_3 \\ \dot{x}_3 = -x_2 + 3x_3 \end{cases}$
15.  $\begin{cases} \dot{x}_1 = 3x_1 - x_2 + 5x_3 \\ \dot{x}_2 = x_1 + 2x_2 + 4x_3 \\ \dot{x}_3 = 3x_1 + 2x_2 - x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = 4x_1 + 3x_2 + x_3 \\ \dot{x}_2 = 3x_1 + x_2 + 2x_3 \\ \dot{x}_3 = x_1 - 2x_2 + x_1 \end{cases}$
16.  $\begin{cases} \dot{x}_1 = 4x_1 + 3x_2 - x_3 \\ \dot{x}_2 = -2x_1 + x_2 - x_3 \\ \dot{x}_3 = 3x_1 + x_2 + x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = x_1 - 2x_2 - x_3 \\ \dot{x}_2 = 3x_1 + x_2 + 2x_3 \\ \dot{x}_3 = x_1 + 2x_2 + 2x_3 \end{cases}$
17.  $\begin{cases} \dot{x}_1 = 4x_1 + 3x_2 + 8x_3 \\ \dot{x}_2 = 6x_1 + 9x_2 + x_3 \\ \dot{x}_3 = 2x_1 + x_2 + 8x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = -x_1 + 8x_2 - 2x_3 \\ \dot{x}_2 = -4x_1 + 3x_2 + 2x_3 \\ \dot{x}_3 = 3x_1 - 8x_2 + 5x_3 \end{cases}$
18.  $\begin{cases} \dot{x}_1 = 5x_1 - 3x_2 + 4x_3 \\ \dot{x}_2 = 2x_1 + x_2 - 5x_3 \\ \dot{x}_3 = -3x_1 + 5x_2 + x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = 4x_1 + 5x_2 - 3x_3 \\ \dot{x}_2 = x_1 - x_2 - x_3 \\ \dot{x}_3 = 7x_1 + 4x_3 \end{cases}$
19.  $\begin{cases} \dot{x}_1 = 3x_1 + 0 + 5x_3 \\ \dot{x}_2 = x_1 + x_2 + x_3 \\ \dot{x}_3 = 3x_2 - 6x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = 2x_1 - x_2 - 5x_3 \\ \dot{x}_2 = 7x_1 + x_2 + 4x_3 \\ \dot{x}_3 = 6x_1 + 4x_2 - 7x_3 \end{cases}$
20.  $\begin{cases} \dot{x}_1 = x_1 + 2x_2 + 2x_3 \\ \dot{x}_2 = -3x_2 + x_3 \\ \dot{x}_3 = 2x_1 + 3x_3 \end{cases}$  va  $\begin{cases} \dot{x}_1 = 3x_1 + x_2 \\ \dot{x}_2 = x_1 - 2x_2 - x_3 \\ \dot{x}_3 = 3x_2 + 2x_3 \end{cases}$

**III-BOB.**  
**ANALITIK GEOMETRIYA ELEMENTLARI.**

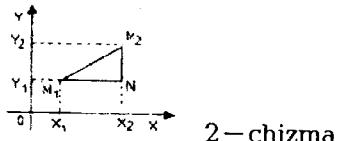
**1-§. Dekart koordinatalar sisitemasi. Ikki nuqta orasidagi masofa. Kesmani berilgan nisbatda bo'lish.**

Tekislikda  $O$  nuqta orqali o'zaro perpendikulyar ikkita  $x$  va  $y$  to'g'ri chiziqlarni o'tkazamiz.  $x$  o'qi (u odatda gorizontal bo'ladi) **abstsissalar o'qi** deyiladi,  $y$  o'qi esa (u vertikal holatda bo'ladi) **ordinatalar o'qi** deyiladi. Kesishish nuqtasi  $O$  koordinatalar boshi deb ataladi.  $O$  nuqta o'qlarning har birini ikkita yarim o'qqa ajratadi. Ulardan birini **musbat yarim o'q** deb, uni strelka bilan belgilaymiz, ikkinchisini **manfiy yarim o'q** deyiladi. Bu o'qlarda uzunlik birligi tanlangan bo'lsa, birligida XOU — **Dekart koordinatalar sistemasi** berilgan deyiladi.



1 – chizma.

$M_1(x_1, y_1)$ ,  $M_2(x_2, y_2)$  nuqtalar berilgan bo'lib,  $d = M_1M_2 = ?$



2 – chizma.

Pifagor teoremasiga ko'ra,

$$d = M_1M_2 = \sqrt{(M_1N)^2 + (M_2N)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (*)$$

bo'ladi.

(\*) — ikki nuqta orasidagi masofani hisoblash formulasini deyiladi.

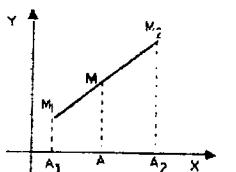
Endi  $M_1(x_1, y_1)$ ,  $M_2(x_2, y_2)$  nuqtalar berilgan bo'lib,  $M_1, M_2$  nuqtalarda shunday  $M(x, y)$  nuqtani topish kerakki,

$$\frac{|M_1M|}{|M_2M|} = \lambda$$

shart bajarilsin.

Elementar matematikadan ma'lumki,

$$\frac{|M_1M|}{|M_2M|} = \frac{|A_1A|}{|AA_2|} \Rightarrow \lambda = \frac{x - x_1}{x_2 - x} \Rightarrow x = \frac{x_1 + \lambda x_2}{1 + \lambda}, \quad y = \frac{y_1 + \lambda y_2}{1 + \lambda}$$



3 – chizma.

Demak,

$$\begin{cases} x = \frac{x_1 + \lambda x_2}{1 + \lambda} \\ y = \frac{y_1 + \lambda y_2}{1 + \lambda} \end{cases} \quad (1)$$

Agar (1) da,  $\lambda = 1$  desak,

$$\begin{cases} x = \frac{x_1 + x_2}{2} \\ y = \frac{y_1 + y_2}{2} \end{cases} \quad (2)$$

kesma o'ttasining koordinatalarini hisoblash formulasini hosil qilamiz.

**Misol.** Uchlari  $A(-5; 3)$ ,  $B(2; -4)$  nuqtalarda bo'lgan  $AB$  kesma berilgan.  $C(x, y)$  nuqta kesmani  $\frac{1}{4}$  nisbatda bo'ladi.  $C(x, y)$  nuqta koordinatalari bilan  $AB$  kesma uzunligini toping.

**Yechilishi.** Quyidagilarni yozib olamiz:

$$\frac{AC}{CB} = \lambda = \frac{1}{4}, \quad x_1 = -5, \quad y_1 = 3, \quad x_2 = 2, \quad y_2 = -4.$$

(1) formuladan foydalanib  $C(x, y)$  nuqtaning koordinatalarini topamiz.

$$x = \frac{-5 + \frac{1}{4} \cdot 2}{1 + \frac{1}{4}} = -3,6; \quad y = \frac{3 + \frac{1}{4} \cdot (-4)}{1 + \frac{1}{4}} = 1,6. \quad C(-3,6, 1,6)$$

$AB$  kesma uzunligini (\*) formuladan foydalanib topamiz.

$$AB = d = \sqrt{(2+5)^2 + (-4-3)^2} = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2}.$$

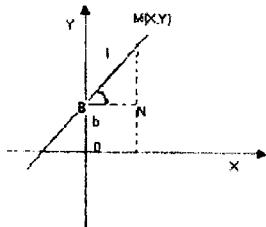
### Nazorat savollari.

1. Tekislikda Dekart koordinatalar sistemasini kriting.  
Nuqtalar belgilab koordinatalarini aniqlang.
2. Ikki nuqta orasidagi masofa. Misollar.
3. Kesmani berilgan nisbatda bo'lish. Misollar.
4. Kesmaning o'rtasini topish. Misollar.

### 2-§. To'g'ri chiziq va uning tenglamalari.

#### 1. To'g'ri chiziqning burchak koeffitsientli tenglamasi.

Aytaylik, 1 to'g'ri chiziq OU o'qiga parallel bo'lmasin va OX o'qi bilan  $\phi$  burchakni tashkil qilib, OU o'qidan b birlik kesma kessin.



4 – chizma.

$$\Delta MBN \text{ dan } \Rightarrow \operatorname{tg} \varphi = \frac{MN}{BN} = \frac{y - s}{x} \Rightarrow y = \operatorname{tg} \varphi \cdot x + s \Rightarrow \operatorname{tg} \varphi = k \text{ desak,}$$

$y = kx + s$  (1) to'g'ri chiziqning burchak koeffitsientli tenglamasi.

$$\begin{array}{ll} y = s & \text{OX o'qiga parallel to'g'ri chiziq.} \\ x = a & \text{OU o'qiga parallel to'g'ri chiziq.} \end{array}$$

## 2. To'g'ri chiziqning umumiy tenglamasi.

**Teorema.** Tekislikda dekart koordinatalar sistemasida berilgan ixtiyoriy to'g'ri chiziq ushbu

$$Ax + By + C = 0 \quad (2)$$

tenglama yordamida aniqlanadi va aksincha.

Bu yerda A, B, C lar o'zgarmas koeffitsentlar bo'lib A va B lardan hech bo'lmasa biri noldan farqli deb qaraladi.

Shuning uchun ham (2) tenglamaga **to'g'ri chiziqning umumiy tenglamasi** deb ataladi.

A, B, C o'zgarmas koeffitsentlar turli qiymatlarga teng bo'lganda turli to'g'ri chiziqlar hosil bo'ladi. Demak, to'g'ri chiziqning tekislikdagi vaziyati shu A, B, C sonlar bilan to'liq aniqlanadi.

## 3. Berilgan nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi.

Aytaylik,  $M(x_1, u_1)$  nuqta berilgan bo'lib, shu nuqtadan o'tuvchi to'g'ri chiziq tenglamasini topish talab qilinsin.

$y = kx + b$  ko'rinishda qidiramiz. To'g'ri chiziq  $M(x_1, u_1)$  nuqtadan o'tganligi uchun  $y_1 = kx_1 + b$  bo'ladi.

$$\Rightarrow y - y_1 = k(x - x_1) \quad (3)$$

izlanayotgan tenglama bo'ladi.

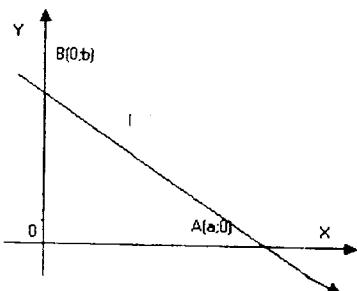
Agar  $M(x_1, u_1)$  va  $M_2(x_2, u_2)$  nuqtalar berilgan bo'lsa, bu nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi ushbu

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad (4)$$

tenglikdan topiladi.

#### 4. To'g'ri chiziqning kesmalardagi tenglamasi.

Aytaylik, 1 to'g'ri chiziq OX va OY o'qlarni mos ravishda a va b birlik kesmalarda kessin, ya'ni A (a,0) va B (0,b) nuqtalardan o'tsin.



5 – chizma.

(4) formuladan quyidagini yozib olamiz:

$$\frac{y-0}{b-0} = \frac{x-a}{0-a}$$

Y holda uning tenglamasi

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (5)$$

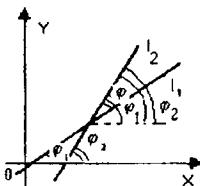
bo'lishini topish qiyin emas.

(3) – tenglamaga to'g'ri chiziqning kesmalardagi tenglamasi deb ataladi.

#### 5. Ikkita to'g'ri chiziq orasidagi burchak.

Faraz qilaylik,  $I_1 : y = k_1x + \theta_1$  va  $I_2 : y = k_2x + \theta_2$  to'g'ri chiziqlar berilgan bo'lsin.

$$\Rightarrow \operatorname{tg} \theta_1 = k_1, \quad \operatorname{tg} \theta_2 = k_2 \quad \theta = \left( \hat{l_1, l_2} \right) = \theta_2 - \theta_1 \quad (0 \leq \theta \leq \pi)$$



6 – chizma.

$$\operatorname{tg} \varphi = \operatorname{tg}(\varphi_2 - \varphi_1) = \frac{\operatorname{tg} \varphi_2 - \operatorname{tg} \varphi_1}{1 + \operatorname{tg} \varphi_2 \operatorname{tg} \varphi_1} = \frac{k_2 - k_1}{1 + k_1 k_2} \Rightarrow \operatorname{tg} \varphi = \frac{k_2 - k_1}{1 + k_1 k_2} \quad (6)$$

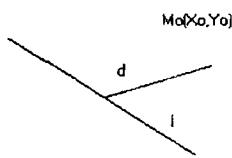
(6) ikkita to'g'ri chiziqning orasidagi burchakni hisoblash formulasi.

Agar ikkita to'g'ri chiziq orasidagi burchak  $\varphi = 0$  bo'lsa, ravshanki, bu *to'g'ri chiziqlar o'zaro parallel* bo'ladi yoki *ustma-ust* tushadi. Bu holda  $k_1 = k_2$  bo'lishi kelib chiqadi.

Agar ikkita to'g'ri chiziq orasidagi burchak  $\varphi = \frac{\pi}{2}$  bo'lsa, unda *to'g'ri chiziqlar perpendikulyar* bo'ladi, ya'ni  $k_1 k_2 = -1$ .

Berilgan nuqtadan to'g'ri chiziqqacha bo'lgan masofani quyidagi formuladan foydalanib topiladi.

$$l : Ax + By + C = 0$$



7 – chizma.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \quad (7)$$

**Masala.** Uchlarining koordinatalari A (1; -1), B ( $\frac{1}{3}, \frac{1}{3}$ ), C(0,0) bo'lgan ABS uchburchak uchun quyidagilarni aniqlang:

- AB tomonining uzunligini hisoblang;
- Tomonlarining tenglamarasini tuzing;
- S uchidan o'tkazilgan balandlikning tenglamarasini tuzing;
- B uchidan AS tomongacha bo'lgan masofani hisoblang;
- Ichki A burchak bissektrisasining tenglamarasini toping.

### **Yechilishi.**

1.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\frac{1}{3} - 1\right)^2 + \left(\frac{1}{3} + 1\right)^2} = \frac{2\sqrt{5}}{3}$$

2.

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \text{dan foydalanimiz quyidagilarni topamiz:}$$

$$AB : 2x + y - 1 = 0$$

$$AC: x + y = 0$$

$$BC: -x + y = 0$$

- S(0,0) nuqtadan  $2x + y - 1 = 0$  to'g'ri chiziqqa perpendikulyar bo'lgan to'g'ri chiziq tenglamarasini yozamiz.

AB to'g'ri chiziqning burchak koeffitsenti  $k_1 = -2$   
 $(y = -2x + 1 \text{ dan})$  ga teng. Perpendikulyarlik shartidan  $k_2 = \frac{1}{2}$  ekanini topamiz.

$\Rightarrow y - y_C = k(x - x_C)$  dan qidirayotgan tenglamamiz  $y = \frac{1}{2}x$  ekanini topamiz.

4.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} = \frac{\left|\frac{1}{3} + \frac{1}{3} + 0\right|}{\sqrt{2}} = \frac{\sqrt{2}}{3}.$$

- S nuqtaning koordinatalarini AB tomon tenglamasiga qo'ysak

$$2 \cdot 0 + 0 - 1 = -1 < 0$$

B nuqtaning koordinatalarini AC tomon tenglamasiga qo'ysak

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3} > 0,$$

Demak, ABS uchburchakning ichki A burchak bissektrisasi AB to'g'ri chiziqdan tekislikning manfiy qismidan o'tadi va AC to'g'ri chiziqdan tekislikning musbat qismidan o'tadi ya'ni bissektrisa nuqtalari uchun  $2x + y - 1 < 0$ ,  $x + y > 0$ . Shuning uchun ichki A burchak bissektrisasining tenglamasi quyidagicha:

$$\frac{y + 2x - 1}{\sqrt{5}} = \pm \frac{y + x}{\sqrt{2}}.$$

### Nazorat savollari.

1. To'g'ri chiziqning turli tenglamalari va ular orasidagi bo'g'lanish.
2. To'g'ri chiziqning burchak koeffitsienti.
3. Berilgan nuqtalardan o'tuvchi to'g'ri chiziq tenglamasi.
4. Ikki to'g'ri chiziq orasidagi burchak.
5. Nuqtadan to'g'ri chiziqqacha bo'lган masofa.

### 3-§. Ikkinchи tartibli chiziqlar.

Ushbu

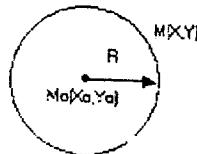
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + C = 0 \quad (1)$$

(1) – tenglama yordamida aniqlanadigan chiziqqa **ikkinchи tartibli chiziq** deb ataladi. Ular jumlasiga matematikada muhim rol o'ynaydigan *aylana*, *ellips*, *giperbolalar* va *parabolalar* kiradi.

1. **Aylana.** Markaz deb ataladigan nuqtadan bir xil uzoqlikda joylashgan nuqtalarning geometrik o'mniga **aylana** deyiladi.

$M_0M = R \quad \forall M(x, y)$  uchun aylananing *kanonik tenglamasi* quyidagicha:

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \quad (2)$$

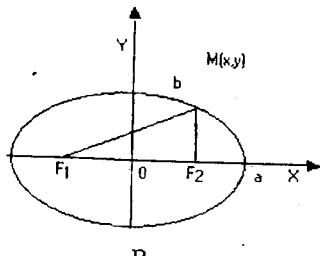


8 – chizma.

**2. Ellips.** Ixtiyoriy nuqtasidan fokus deb ataladigan ikkita qo'zg'almas  $F_1$  va  $F_2$  nuqtalargacha bo'lgan masofalar yig'indisi o'zgarmas  $2a$  ga teng bo'lgan tekislik nuqtalarining geometrik o'rniiga ellips deb ataladi.

*Ellips tenglamasini* keltirib chiqaramiz. Ellipsdan  $\forall M(x, y)$  nuqta olamiz. Aytaylik,  $F_1(-c; 0)$  va  $F_2(c; 0)$  bo'lzin. Shartga ko'ra

$$MF_1 + MF_2 = 2a$$



9 – chizma.

$MF_1$  va  $MF_2$  larning o'rniiga qo'ysak:

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a,$$

tenglamani hosil qilamiz. Tenglikni chap tom onidagi qo'shiluvchilarining birini o'ng tomonga o'tkazib kvadratga ko'taramiz. Hosil bo'lgan ifodani soddalashtirib, yana kvadratga ko'taramiz. Natijada,

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2),$$

tenglik hosil bo'ladi.

$$2a > 2c \Rightarrow a^2 - c^2 > 0$$

ekanligidan

$$\epsilon^2 = a^2 - c^2$$

deb belgilay olamiz

$$\epsilon^2 x^2 + a^2 y^2 = a^2 \epsilon^2.$$

Bundan,

$$\frac{x^2}{a^2} + \frac{y^2}{\epsilon^2} = 1 \quad (3)$$

tenglik kelib chiqadi.

(3) – tenglik ellipsning kanonik tenglamasini ifodalaydi.

$\epsilon = \frac{c}{a}$  – ellipsning ekstsentrisiteti deyiladi,  $0 \leq \epsilon < 1$ .

Ekstsentrisitet ellipsning siqilish darajasini aniqlaydi. Ekstsentrisitet ta'rifidan va  $b^2 = a^2 - c^2$  tenglikdan quyidagi kelib chiqadi:

$$\epsilon^2 = \frac{c^2}{a^2} = \frac{a^2 - b^2}{a^2} = 1 - \left(\frac{b}{a}\right)^2 \Rightarrow \frac{b}{a} = \sqrt{1 - \epsilon^2}.$$

Bundan kelib chiqadiki,  $\epsilon$  qancha katta bo'lsa  $\frac{b}{a}$  nisbat shuncha kichik bo'ladi va ellips shuncha cho'zilgan bo'ladi.

Agar  $\epsilon = 0$  bo'lsa ellips aylanaga aylanadi.

$r_1 = a - ex$  va  $r_2 = a + ex$  ellipsning fokal radiuslari deyiladi.

Yarim o'qlar  $a$  va  $b$ , fokuslar orasidagi masofa 2s ga teng.

**1-misol.** Agar yarim o'qlar yig'indisi 16 ga va fokuslar orasidagi masofa 8 ga teng bo'lsa, ellips tenglamasini yozing.

**Yechilishi.**  $a + b = 16$ ;  $2c = 8 \Rightarrow c = 4$  ma'lumki  $a^2 - b^2 = c^2$ ,

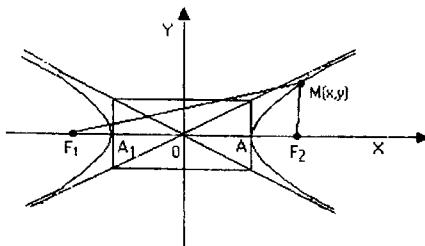
$$16(a - b) = 16 \Rightarrow a - b = 1.$$

$$\begin{cases} a + b = 16 \\ a - b = 1 \end{cases}$$

tenglamalar sistemasini yechamiz.  $a = 8,5$ ,  $b = 7,5..$   
 Qidirilayotgan ellips tenglamasi quyidagicha yoziladi:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{17^2} + \frac{y^2}{15^2} = 1 \Rightarrow \frac{4x^2}{17^2} + \frac{4y^2}{15^2} = 1.$$

**3. Giperbola.** Ixtiyoriy nuqtasidan fokuslar deb ataladigan ikkita qo'zg'almas  $F_1$  va  $F_2$  nuqtalargacha bo'lgan masofalar ayirmasining absolyut qiymati o'zgarmas  $2a$  ga teng bo'lgan tekislik nuqtalarining geometrik o'rniiga **giperbola** deyiladi.



10 – chizma.

Giperbola ta'rifidan foydalanib, so'ngra ellips kanonik tenglamasini keltirib chiqarishga o'xshash amallar bajarib,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (4)$$

giperbolaning kanonik tenglamasini keltirib chiqarish mumkin, bu yerda  $b^2 = c^2 - a^2$ .

$y = \pm \frac{b}{a}x$  — giperbolaning asimptotalari.

$\varepsilon = \frac{c}{a}$  — giperbolaning ekstsentriskiteti,  $\varepsilon > 1$ .

$r_1 = ex - a$  va  $r_2 = ex + a$  **giperbolaning fokal radiuslari** deyiladi.

Agar giperbolada  $a = b$  bo'lsa, hosil bo'lgan

$$x^2 - y^2 = a^2$$

giperbolaga **teng tomonli giperbola** deb ataladi.

**2-misol.** Berilgan  $9x^2 - 25y^2 = 225$  giperbolada  $a$  va  $b$  lar uzunligini, fokuslar koordinatalarini va ekstsentrisitetini toping.

**Yechilishi:**

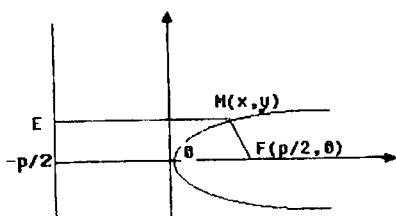
$$\frac{9x^2}{225} - \frac{25y^2}{225} = 1 \Rightarrow \frac{x^2}{25} - \frac{y^2}{9} = 1 \Rightarrow a = 5, \quad b = 3$$

ekanligi kelib chiqadi. Bundan foydalanib  $c$  ni topamiz:

$$c = \sqrt{5^2 + 3^2} = \sqrt{34} \Rightarrow \varepsilon = \frac{\sqrt{34}}{5}.$$

Demak,  $F_2(\sqrt{34}; 0)$        $F_1(-\sqrt{34}; 0)$ .

**4. Parabola.** Fokus deb ataladigan qo'zg'almas  $F$  nuqtadan va direktrisa deb ataladigan  $l$  to'g'ri chiziqdan teng uzoqlikda joylashgan tekislik nuqtalarining geometrik o'rniiga **parabola** deyiladi.



11 – chizma.

Parabolaning ta'rifiga ko'ra uning ixtiyoriy  $M$  nuqtasidan  $F$  nuqttagacha va  $M$  nuqtadan  $l$  to'g'ri chiziqqacha masofalar teng ekan.  $F$  nuqtadan  $l$  to'g'ri chiziqqacha masofani  $p$ ,  $x$  o'qini  $l$  to'g'ri chiziqqa perpendikulyar hamda  $F$  nuqtadan o'tadigan,  $y$  o'qni esa  $b$  to'g'ri chiziq va  $F$  nuqtaning o'rtasidan o'tadigan qilib olsak

$$\sqrt{\left(x + \frac{p}{2}\right)^2} = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}$$

tenglik hosil bo'ladi. Tenglikni ikkala tarafini kvadratga ko'tarib, ixchamlashtirganimizdan so'ng,

$$y^2 = 2px \quad (5)$$

*parabolaning kanonik tenglamasi* hosil bo'ladi.

$F\left(\frac{p}{2}; 0\right)$  parabola fokusi;

$x = -\frac{p}{2}$  parabola direktrisasi;

$r = x + \frac{p}{2}$  parabolaning fokal radiusi deyiladi.

**3-misol.**  $y^2 = 6x$  paraboladagi fokal radiusi 4,5 ga teng nuqtalarini toping.

**Yechilishi.** Berilgan parabola Ox o'qiga nisbatan simmetrik:

$$2p = 6 \Rightarrow \frac{p}{2} = 1,5 \quad r = x + \frac{p}{2} \quad \text{dan} \quad 4,5 = x + 1,5; \quad x = 3.$$

$$y^2 = 6 \cdot 3 = 18 \Rightarrow y_{1,2} = \pm\sqrt{18} = \pm 3\sqrt{2}.$$

Demak, qidirilayotgan nuqtalar quyidagicha:

$$A(3; 3\sqrt{2}), B(3; -3\sqrt{2})$$

### Nazorat savollari.

1. Aylana. Misollar.
2. Ellips va uning elementlari.
3. Giperbola va uning elementlari.
4. Parabola va uning elementlari.

#### 4-§. Ikkinchchi tartibli egri chiziqlarning umumiy tenglamasi.

Biz yuqorida ikkinchi tartibli egri chiziqlardan aylana, ellips, giperbola, parabolalarni keltirdik va ularning sodda xossalarni o'rgandik.

Endi biz ikkinchi tartibli egri chiziqlarning umumiy tenglamasini keltiramiz:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \quad (1)$$

- 1) Koordinata o'qlarini parallel ko'chirish;
- 2) Koordinata o'qlarini ma'lum  $\alpha$  burchakka burish natijasida.

Ikkinchchi tartibli egri chiziqlarning umumiy tenglamasini *kanonik ko'rinishga* keltirish mumkin.

Dekart koordinatalar sistemasining biridan ikkinchisiga o'tishda ikkinchi tartibli chiziqning tenglamasidagi koeffitsientlar umuman aytganda, o'zgaradi. Biroq chiziq tenglamasi koeffitsientlarining funksiyalari bo'lgan ba'zi kattaliklar mavjud bo'lib, ularning qiymatlari bir dekart koordinatalar sistemasidan ikkinchisiga o'tganda o'zgarmaydi. Bunday kattaliklar **invariantlar** deyiladi.

Ushbu

$$I_1 = a_{11} + a_{22}, \quad I_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad I_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(bu yerda  $a_{21} = a_{12}$ ,  $a_{31} = a_{13}$ ,  $a_{32} = a_{23}$ ) kattaliklar ikkinchi tartibli chiziq tenglamasining dekart koordinatalar sistemasi almashtirishlariga nisbatan invariantdir.

Barcha ikkinchi tartibli chiziqlar quyidagi uchta tipga ajraladi:

agar  $I_2 > 0$  bo'lsa, chiziq elliptik tipda;

agar  $I_2 < 0$  bo'lsa, chiziq giperbolik tipda;

agar  $I_2 = 0$  bo'lsa, chiziq parabolik tipda bo'ladi.

Agar ikkinchi tartibli chiziqning tenglamasi koordinatalar sistemasini almashtirish yo'li bilan

$$\lambda_1 x^2 + \lambda_2 y^2 + m = 0 \quad (2)$$

ko'rinishga keltirilgan bo'lsa, chiziqning invariantlarini yozamiz:

$$I_1 = a_{11} + a_{22} = \lambda_1 + \lambda_2 \quad (3)$$

$$I_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} = \lambda_1 \lambda_2 \quad (4)$$

$$I_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1 \lambda_2 \lambda_3 \quad (5)$$

(3) va (4) tengliklardan Viet teoremasiga muvofiq  $\lambda_1$  va  $\lambda_2$  lar

$$\lambda^2 - I_1 \lambda + I_2 = 0 \quad (6)$$

*kvadrat tenglamaning ildizlari ekanligi kelib chiqadi.*

(4) va (5) tengliklardan quyidagi ega bo'lamiz:

$$m = \frac{I_3}{I_2} \quad (7)$$

Agar (1) tenglama

$$\lambda_1 x^2 + 2ky = 0 \quad (8)$$

ko'rinishga keltirilsa, u holda quyidagi ifodalar uning invariantlari bo'ladi:

$$I_1 = \lambda_1, \quad I_2 = 0, \quad I_3 = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & k \\ 0 & k & 0 \end{vmatrix} = -\lambda_1 k^2,$$

bundan

$$k = \pm \sqrt{-\frac{I_3}{I_1}} \quad (9)$$

Agar (1) tenglama

$$\lambda_1 x^2 + n = 0 \quad (10)$$

ko'rinishga keltirilsa, tenglama bir juft to'g'ri chiziqni (ustma – ust tushgan haqiqiy yoki mavhum parallel to'g'ri chiziqlarni) aniqlaydi. Bu holda invariantlar quyidagi ko'rinishda bo'ladi:

$$I_1 = \lambda_1, \quad I_2 = 0, \quad I_3 = 0.$$

**1-misol.**  $x^2 - 8xy + 7y^2 - 18 = 0$  ikkinchi tartibli egri chiziq tenglamasini kanonik ko'rinishga keltiring.

**Yechilishi:**

$$I_1 = 8, \quad I_2 = \begin{vmatrix} 1 & -4 \\ -4 & 7 \end{vmatrix} = -9, \quad I_3 = \begin{vmatrix} 1 & -4 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 18 \end{vmatrix} = 162$$

$I_2 < 0$  bo'lgani uchun qaralayotgan tenglama giperbolik tipdag'i chiziqni aniqlaydi.

$$\lambda^2 - 8\lambda - 9 = 0, \quad \lambda_1 = 9, \quad \lambda_2 = -1.$$

(7) formuladan foydalanib, topamiz.

$$m = \frac{I_3}{I_2} = -18$$

va giperbola tenglamasi quyidagicha bo'ladi:

$$9x^2 - y^2 - 18 = 0 \quad \text{yoki} \quad \frac{x^2}{2} - \frac{y^2}{18} = 1$$

**2-misol.**  $8x^2 + 4\sqrt{2}xy + y^2 + 6x - 12\sqrt{2}y = 0$  ikkinchi tartibli egri chiziq tenglamasini kanonik ko'rinishga keltiring.

**Yechilishi:**

$$I_1 = 9, \quad I_2 = 0, \quad I_3 = \begin{vmatrix} 8 & 2\sqrt{2} & 3 \\ 2\sqrt{2} & 1 & -6\sqrt{2} \\ 3 & -6\sqrt{2} & 0 \end{vmatrix} = -729$$

$$k = \pm \sqrt{-\frac{I_3}{I_1}} = \pm \sqrt{\frac{729}{9}} = \pm 9$$

$I_2 = 0$  bo'lgani uchun qaralayotgan chiziq parabolik tipdag'i chiziq bo'lib, uning tenglamasi quyidagicha:

$$9x^2 - 18y = 0 \quad \text{yoki} \quad 9x^2 + 18y = 0 \\ \text{bundan}$$

$$x^2 = 2y \quad \text{yoki} \quad x^2 = -2y.$$

**3-misol.**  $x^2 - 4xy + 4y^2 + 2x - 4y - 3 = 0$  ikkinchi tartibli egri chiziq tenglamarasini kanonik ko'rinishga keltiring.

**Yechilishi:**

$$I_1 = 5, \quad I_2 = 0, \quad I_3 = \begin{vmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & -3 \end{vmatrix} = 0$$

Bu holda  $k = \pm \sqrt{-\frac{I_3}{I_1}} = 0$ , demak, tenglamaning chap qismi chiziqli ko'paytuvchilarning ko'paytmasidan iborat.

$$x^2 - 2(2y - 1)x + 4y^2 - 4y - 3 = 0 \\ \text{bundan}$$

$$x_{1,2} = 2y - 1 \pm \sqrt{(2y - 1)^2 - 4y^2 + 4y + 3},$$

$$\text{yoki} \quad x_1 = 2y + 1, \quad x_2 = 2y - 3,$$

$$(x - 2y - 1)(x - 2y + 3) = 0.$$

Shunday qilib, berilgan tenglama bilan aniqlangan chiziq bir just parallel to'g'ri chiziqqa ajraladi:

$$x - 2y - 1 = 0 \quad \text{va} \quad x - 2y + 3 = 0$$

(1) tenglama koordinatalar sistemasini tanlash yo'li bilan, shu sistemada qaralayotgan quyidagi kanonik ko'rinishlardan bittasiga keltiriladi.

$$1) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{ellips})$$

$$2) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 \quad (\text{mavhum ellips})$$

$$3) \quad a^2x^2 + c^2y^2 = 0 \quad (\text{ikki mavhum kesishuvchi chiziqlar})$$

- 4)  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  (giperbola)
- 5)  $a^2x^2 - c^2y^2 = 0$  (ikki kesishuvchi chiziqlar)
- 6)  $y^2 = 2px$  (parabola)
- 7)  $y^2 - a^2 = 0$  (ikki parallel chiziqlar)
- 8)  $y^2 + a^2 = 0$  (ikki parallel mavhum chiziqlar)
- 9)  $y^2 = 0$  (ikki o'zaro ustma – ust tushuvchi chiziqlar)

### Nazorat savollari.

- 1) Ikkinchchi tartibli egri chiziqlarning umumiy tenglamasi.
- 2) Invariantlik xossasi.
- 3) Ikkinchchi tartibli chiziqlarning tiplari.
- 4) Ikkinchchi tartibli egri chiziqlarni kanonik kurinishga keltirish. Misollar.

### Mustaqil yechish uchun misollar va masalalar.

1—masala. ABC uchburchak uchlarining koordinatalari berilgan, quyidagilar talab qilinadi:

1. AB tomonning uzunligini hisoblash;
2. AB kesma o'rtalarining koordinatalarini topish;
3. AB tomonning tenglamasini tuzish;
4. C uchidan o'tkazilgan balandlikning tenglamasini tuzish;
5. B uchidan AC tomongacha bo'lgan masofani hisoblash;
6. Ichki A burchak bissektrisasingning tenglamasini topish;

- |                  |                            |                      |
|------------------|----------------------------|----------------------|
| 1.1. A(-6; -4),  | B(-10; -1),                | C(6; 1).             |
| 1.2. A(-2; -4),  | B(2; -8),                  | C(10; 2).            |
| 1.3. A(12; 0),   | B(1; 8),                   | C(0; 5).             |
| 1.4. A(-10; -2), | B( $\sqrt{2}; -\sqrt{7}$ ) | C( $2\sqrt{2}; 0$ ). |
| 1.5. A(-2; -6),  | B(-6; -3),                 | C(10; -1).           |
| 1.6. A(4; 3),    | B(7; 6),                   | C(2; 11).            |
| 1.7. A(8; 2),    | B(14; 10),                 | C(-4; 7).            |
| 1.8. A(2; -1),   | B(4; 2),                   | C(5; 1).             |
| 1.9. A(2; -4),   | B(-2; -1),                 | C(14; 1).            |

1.10. A(1; - 5),	B(2; 7),	C(- 4; 11).
1.11. A(2; - 1),	B(8; 7),	C(- 10; 4).
1.12. A(4; 11),	B(- 1; - 1),	C(5; 7).
1.13. A(5; - 3),	B(1; 0),	C(17; 2).
1.14. A(3; 8),	B(10; 2),	C(2; 7).
1.15. A(14; - 6),	B(20; 2),	C(2; - 1).
1.16. A(7; 2),	B(1; 9),	C(- 8; - 4).
1.17. A(3; 4),	B(- 1; 7),	C(15; 9).
1.18. A(5; - 7),	B(- 4; - 2),	C(15; - 1).
1.19. A(1; - 2),	B(7; 6),	C(- 11; 3).
1.20. A(- 3; - 3),	B(- 1; 3),	C(11; - 1).

## 2—masala.

2.1. Agar parallelogrammning diagonallari  $(-1; 0)$  nuqtada kesishishi ma'lum bo'lsa, uning  $x+y-1=0$  va  $y+1=0$  tomonlarining kesishish nuqtasidan o'tmaydigan diagonalining tenglamasini toping.

2.2.  $2x+u+11=0$  to'g'ri chiziqda berilgan ikki: A(1; 1) va B(3; 0) nuqtadan baravar uzoqlashgan nuqtani toping.

2.3.  $(2; - 4)$  nuqtaga  $4x+3y+1=0$  tenglamaga nisbatan simmetrik bo'lgan nuqtaning koordinatalarini toping:

2.4. Uchlari A(- 1; 1), B(2; - 1), C(4; 0) bo'lgan uchburchakka tashqi chizilgan aylana markazining koordinatalarini hisoblang;

2.5.  $(2; 6)$  nuqtadan o'tuvchi va koordinata o'qlari bilan ikkinchi chorakda joylashib 3 kv. birlik yuzaga ega bo'lgan uchburchak tashkil etuvchi to'g'ri chiziqning tenglamasini tuzing.

2.6. A(- 1; 2) nuqtadan o'tuvchi to'g'ri chiziqlar tenglamasini shunday tuzingki, uning  $x+2y+1=0$  va  $x+2y-3=0$  parallel to'g'ri chiziqlar orasida joylashgan kesmasining bir uchi  $x-y-6=0$  to'g'ri chiziqda yotsin.

2.7. Uchburchak ikkita tomonining tenglamalari berilgan:  $4x-5y+9=0$  va  $x+4y-3=0$ . Agar bu uchburchakning meridianalari  $(3; 1)$  nuqtada kesishishi ma'lum bo'lsa, uchburchak tomonining tenglamasini toping.

2.8. Romb ikkita tomonining tenglamasi:  $2x-y+4=0$  va  $2x-y+10=0$ , hamda diagonallaridan birining tenglamasi  $x+y+2=0$  ma'lum bo'lsa, romb uchlaringin koordinatalarini toping.

2.9. Agar  $A(-5;5)$  va  $B(3;1)$  – uchburchakning ikkita uchi;  $M(2;5)$  esa uning balandliklarining kesishgan nuqtasi bo'lsa, uchburchak tomonlarining tenglamasini tuzing.

2.10.  $x+3y-7=0$  kvadrat tomonlaridan birining tenglamasi va  $C(0;-1)$  – bu kvadratning qolgan uchta tomonlarining kesishgan nuqtasi. Qolgan tomonlarning tenglamasini tuzing.

### 3-masala.

3.1. Elipsning ekstsentriskiteti 0,8 ga, uning nuqtalaridan birining fokal radiuslari 2 va 3 ga teng, ellipsning katta o'qi abtsissalar o'qi bilan, uning markazi esa koordinatalar boshi bilan mos keladi deb olib, shu ellipsning tenglamasini tuzing.

3.2.  $9x^2 - 16y^2 = 144$  giperbolada shunday nuqtalarni topingki bu nuqtalar bilan giperbolaning chap fokusi orasidagi masofa ularning o'ng fokusigacha bo'lgan masofasidan ikki marta kichik bo'lsin.

3.3. Agar  $y^2 = 2rx$  parabolaning  $y=x$  to'g'ri chiziq bilan  $x^2 + y^2 - 6x = 0$  aylanuning kesishish nuqtalaridan o'tishi ma'lum bo'lsa, shu parabolaning parametrini va uning direktrisasi tenglamasini toping.

3.4.  $\frac{x^2}{441} + \frac{y^2}{216} = 1$  ellipsda shunday nuqtalarni topingki, ularda fokal radiuslar o'zaro perpendikulyar bo'lsin.

3.5. Giperbolaning fokuslari  $F_1(\sqrt{7}, 0)$  va  $F_2(-\sqrt{7}, 0)$  nuqtalarda joylashgan. Giperbola  $A(2;0)$  nuqtadan o'tadi. Uning asimptolarining tenglamasini va ular orasidagi burchakni toping.

3.6.  $A(2;2)$  nuqtadan va abtsissalar o'qidan teng masofada joylashgan nuqtalar geometrik o'rning tenglamasini tuzing.

3.7. Har biri  $A(3;0)$  nuqtadan ordinata o'qlariga qaraganda ikki marta uzoqroq masofada joylashgan nuqtalar geometrik o'rning tenglamasini tuzing.

3.8. Koordinata boshigacha bo'lgan masofalarning  $3x + 16 = 0$  to'g'ri chiziqqacha bo'lgan masofalariga nisbati 0,6 ga teng bo'lgan nuqtalar geometrik o'rning tenglamasini tuzing.

3.9. Har biri  $A(1;0)$  nuqtaga  $B(-2;0)$  nuqtaga qaraganda ikki marta yaqin bo'lgan nuqtalar geometrik o'rning tenglamasini tuzing.

3.10. Abtsissalar o'qiga urinuvchi va A(0;3) nuqtadan o'tuvchi aylana markazlari geometrik o'rinalarining tenglamasini tuzing.

3.11. Ellipsning ekstsentriskiteti 0,4 ga uning nuqtalaridan birining fokal radiuslari 4 va 6 ga teng ellipsning katta o'qi abtsissalar o'qi bilan, uning markazi esa koordinatalar boshi bilan mos keladi, deb olib, shu ellipsning tenglamasini tuzing.

3.12.  $36x^2 - 64y^2 = 2304$  giperbolada shunday nuqtalarini topingki, bu nuqtalar bilan giperbolaning chap fokusi orasidagi masofa ularning o'ng fokusigacha bo'lgan masofalaridan ikki marta kichik bo'lsin.

3.13.  $9x^2 - 4y^2 = 36$  giperbola asimptotlari va  $9x - 2y - 24 = 0$  to'g'ri chiziqlardan hosil bo'lgan uchburchak yuzi topilsin.

3.14. Asimptotasi  $y = \pm \frac{3}{4}x$  va (2;1) nuqtadan o'tgan giperbola tenglamasini yozing.

3.15. Fokus  $4x - 3y - 4 = 0$  to'g'ri chiziq va OX o'qi bilan kesishish nuqtasida yotgan parabola tenglamasi tuzilsin.

3.16.  $4x + 3y + 10 = 0$  to'g'ri chiziqdan 2 birlik o'zoqlikda yotgan  $y^2 = 32x$  parabolaga tegishli nuqta topilsin.

3.17.  $\frac{x^2}{16} - \frac{y^2}{9} = 1$  giperbolaning  $x^2 + y^2 = 91$  aylana bilan kesishish nuqtasida yotgan fokal radius topilsin.

3.18. M(9;8) nuqtadan o'tgan,  $y = \pm (\frac{2\sqrt{2}}{3})x$  asimptotaga ega bo'lgan giperbola tenglamasi tuzilsin.

3.19. M( $\sqrt{3}; \sqrt{2}$ ) nuqtadan o'tuvchi, ekstsentriskiteti giperbolaning  $\sqrt{2}$  ga teng bo'lgan giperbola tenglamasi tuzilsin.

3.20.  $\frac{x^2}{16} - \frac{y^2}{9} = 1$  parabolaning o'ng tarmog'idan shunday nuqta topingki uning o'ng fokusigacha bo'lgan masofa chap fokusigacha bo'lgan masofadan 2 marta kichik bo'lsin.

**4—masala.** Berilgan parabola va giperbola tenglamalarini kanonik ko'rinishga keltiring.

4.1. a)  $y = 3x^2 - 12x + 11;$

b)  $y = \frac{4x+1}{x-1}$

4.2. a)  $y = -2x^2 + 8x - 9;$

b)  $y = \frac{2x+3}{2x-1}$

- 4.3. a)  $y = \frac{1}{3}x^2 + 2x + 5$ ; b)  $y = \frac{-4x + 5}{6x - 3}$
- 4.4. a)  $y = 3x^2 - 18x + 25$ ; b)  $y = \frac{4x + 5}{2x + 1}$
- 4.5. a)  $y = -2x^2 - 4x$ ; b)  $y = \frac{3x - 3}{2x + 5}$
- 4.6. a)  $y = -2x^2 + 12x + 7$ ; b)  $y = \frac{3x + 1}{2x - 5}$
- 4.7. a)  $y = 5x^2 + 10x + 2$ ; b)  $y = \frac{8x + 2}{2x + 3}$
- 4.8. a)  $y = -x^2 + 2x + 3$ ; b)  $y = \frac{2x + 5}{2x - 7}$
- 4.9. a)  $y = 3x^2 + 6x + 5$ ; b)  $y = \frac{3x - 7}{x - 2}$
- 4.10. a)  $y = -x^2 + 2x - 2$ ; b)  $y = \frac{-2x + 5}{x - 4}$

**5—masala.** Ikkinchı tartibli egrilərin chiziq tenglamlarını kanonik formaya keltirin.

- 5.1  $5x^2 + 4xy + 2y^2 = 18$
- 5.2  $4x^2 + 2\sqrt{6}xy + 3y^2 = 24$
- 5.3  $6x^2 + 2\sqrt{5}xy + 2y^2 = 21$
- 5.4  $5x^2 + 4\sqrt{2}xy + 3y^2 = 14$
- 5.5  $7x^2 + 2\sqrt{18}xy + 4y^2 = 15$
- 5.6  $3x^2 + 2\sqrt{14}xy + 8y^2 = 10$
- 5.7  $7x^2 + 2\sqrt{6}xy + 2y^2 = 24$
- 5.8  $9x^2 + 4\sqrt{2}xy + 2y^2 = 20$
- 5.9  $6x^2 + 2\sqrt{10}xy + 3y^2 = 16$
- 5.10  $4x^2 + 4\sqrt{3}xy + 5y^2 = 40$
- 5.11  $5x^2 + 4xy + 8y^2 - 32x - 56y + 80 = 0$
- 5.12  $5x^2 + 12xy - 22x - 12y - 19 = 0$
- 5.13  $x^2 - 4xy + 4y^2 + 4x - 3y - 7 = 0$
- 5.14  $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$
- 5.15  $4x^2 - 12xy + 9y^2 - 2x + 3y - 2 = 0$
- 5.16  $9x^2 - 4xy + 6y^2 + 16x - 8y - 2 = 0$
- 5.17  $8x^2 + 6xy - 26x - 12y + 11 = 0$
- 5.18  $x^2 - 2xy + y^2 - 10x - 6y + 25 = 0$
- 5.19  $2x^2 - 5xy - 12y^2 - x + 26y - 10 = 0$
- 5.20.  $4x^2 - 4xy + y^2 - 6x - 3y - 4 = 0$

**IV BOB.**  
**FUNKTSIYA TUSHUNCHASI VA GRAFIGI.**

**1–§. Funktsiya tushunchasi. Misollar. Murakkab funktsiya.**  
**Funktsiyaning aniqlanish sohasi va qiymatlar sohasi.**

**1. Funktsiya tushunchasi.** Aytaylik,  $X \subset R$  va  $Y \subset R$  to'plamlar berilgan bo'lib,  $x$  o'zgaruvchi  $X$  da,  $y$  o'zgaruvchi  $Y$  to'plamda o'zgarsin  $x \in X, u \in Y$ .

Agar har bir  $x \in X$  songa biror  $f$  qoidaga ko'ra bitta  $y \in Y$  son mos qo'yilsa,  $x$  to'plamda funktsiya berilgan deyiladi. Uni

$$f : X \rightarrow Y \text{ yoki } y = f(x)$$

kabi belgilanadi.

$X$  to'plam funktsiyaning aniqlanish to'plami (sohasi),  $Y$  esa funktsiyaning o'zgarish to'plami (sohasi),  $x$ -erkli o'zgaruvchi yoki funktsiya argumenti,  $u$ -erksiz o'zgaruvchi yoki  $x$  o'zgaruvchining funktsiyasi deyiladi.

Ushbu  $Y_f = \{y | y = f(x), x \in X\}$  to'plam funktsiyaning qiymatlari to'plami deyiladi.

Odatda,  $y = f(x)$  funktsiyaning aniqlanish sohasi  $D(f)$ , qiymatlari to'plami esa  $E(f)$  kabi belgilanadi.

Agar  $f(x)$  va  $g(x)$  funktsiyalar berilgan bo'lib,

$$1) D(f) = D(g),$$

$$2) \forall x \in D(f) \text{ uchun } f(x) = g(x) \text{ shartlar bajarilsa, unda } f(x) \text{ va } g(x) \text{ funktsiyalar } D \text{ to'plamda teng deyiladi.}$$

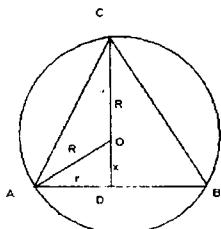
Masalan,  $f(x) = \sqrt{x^2}$ ,  $g(x) = x$  va  $\phi(x) = |x|$  bo'lzin. Bu funktsiyalar uchun  $D(f) = D(g) = D(\phi) = R$ , lekin  $f$  funktsiya  $g$  bilan  $A = [0, +\infty)$  to'plamdagina ustma – ust tushadi,  $\phi$  bilan esa butun  $R$  da teng.

Aytaylik,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  funktsiyalari berilgan bo'lzin. Ushbu  $z = \phi(x) = g(f(x))$ ,  $x \in X$  munosabat bilan aniqlangan.  $\phi : X \rightarrow Z$  funktsiya murakkab funktsiya yoki  $f$  va  $g$  funktsiyalarning superpozetsiyasi deyiladi.

## 2. Misollar.

**1-misol.** Radiusi  $R$  ga teng bo'lgan sharga ichki chizilgan konusning o'q kesimining hajmi uning uchidagi burchak funktsiyasi ko'rinishida ifodalansin.

**Yechilishi.**  $R$  radiusli shar olib, unga ichki konus chizamiz. Konusning o'q kesimini olamiz (1-chizma).



1 – chizma.

Aytaylik,  $\angle ACB = \alpha$ ,  $AD = r$ ,  $CD = H$  bo'lisin.

M a'lumki,

$$V = \frac{1}{3} \pi r^2 \cdot H$$

formula o'rinali. Masalani yechish uchun  $r$  va  $H$  larni  $R$  va  $\alpha$  lar yordamida ifodalashimiz kerak.

$OD = x$  desak, bundan kelib chiqadiki  $H = R + x$ .

$$\angle AOC \Rightarrow AO = OC = R \Rightarrow \angle ACO = \angle CAO = \frac{\alpha}{2}.$$

$\Rightarrow \angle AOC = \pi - \left( \frac{\alpha}{2} + \frac{\alpha}{2} \right) = \pi - \alpha \Rightarrow \angle AOD = \alpha.$  Unda  $\triangle AOD$  dan qo'yidagilarni topamiz:

$$x = R \cos \alpha, \quad r = R \sin \alpha \Rightarrow H = R + x = R \cdot (1 + \cos \alpha).$$

Demak,

$$V = \frac{1}{3} \pi R^3 \sin^2 \alpha \cdot (1 + \cos \alpha).$$

**2-misol.** Parashyutchi  $a$  sek. davomida erkin tushib, keyin parashyutni ochdi va  $b$  sek. davomida o'zgarmas  $v_0$

tezlik bilan tushdi. Parashyutchining  $t$  vaqt davomida bosib o'tgan yo'li  $s(t)$  topilsin.

**Yechilishi.** Bu masalada har xil oraliqda funktsiya ham turli analitik ifodalar yordamida beriladi. Haqiqatan ham, agar  $0 \leq t \leq a$  bo'lsa, maktab fizika kursidan ma'lumki,

$$s(t) = \frac{g \cdot t^2}{2}$$

bo'ladi. Bu yerda  $g$  - erkin tushish tezlanishi.

Agar  $a \leq t \leq a+b$  bo'lsa,

$$s(t) = \frac{g \cdot a^2}{2} + v_0 \cdot (t - a)$$

ekanini ko'rish qiyin emas. Demak,

$$s(t) = \begin{cases} \frac{g \cdot t^2}{2}, & 0 \leq t \leq a \\ \frac{g \cdot a^2}{2} + v_0 \cdot (t - a), & a < t \leq a+b \end{cases}$$

ekan.

### 3. Funktsiyaning aniqlanish sohasi va qiymatlar to'plamini topish.

**3-misol.** Agar  $f = x^2$  va  $\varphi = 2^x$  bo'lsa,  $f[f(x)]$ ,  $\varphi[\varphi(x)]$ ,  $f[\varphi(x)]$ ,  $\varphi[f(x)]$  murakkab funktsiyalar topilsin.

**Yechilishi:**

$$f[f(x)] = [f(x)]^2 = (x^2)^2 = x^4,$$

$$\varphi[\varphi(x)] = 2^{\varphi(x)} = 2^{2^x}$$

$$f[\varphi(x)] = [\varphi(x)]^2 = (2^x)^2 = 2^{2x}$$

$$\varphi[f(x)] = 2^{f(x)} = 2^{x^2}$$

bo'ladi.

**4-misol.** Ushbu  $y = \frac{1}{x^2 - 5x + 6}$  funktsiyaning aniqlanish sohasi topilsin.

**Yechilishi:**

$$D(f) = R \setminus \{x \in R \mid x^2 - 5x + 6 = 0\} = R \setminus \{2;3\} = (-\infty;2) \cup (2;3) \cup (3;+\infty)$$

**5-misol.** Agar  $f(x) = \frac{1}{1-x}$  bo'lsa ,

$$1) \varphi(x) = f(f(x)),$$

2)  $\varphi(x) = f(f(f(x)))$  funktsiyalarning aniqlanish sohasi topilsin.

**Yechilishi:**

$$1) \varphi(x) = f(f(x)) = \frac{1}{1-f(x)} = \frac{1}{1-\left(\frac{1}{1-x}\right)} = x - \frac{1}{x} \text{ bo'lib, uning aniqlanish}$$

sohasi

$$D(\varphi) = R \setminus \{0;1\} = (-\infty;0) \cup (0;1) \cup (1;+\infty)$$

bo'ladi.

$$2) \varphi(x) = f(f(f(x))) = \frac{1}{1-f(f(x))} = \frac{1}{1-\left(\frac{1}{1-f(x)}\right)} = x \text{ bo'lib, bu}$$

funktsiyaning ham aniqlanish sohasi  $D(\varphi) = (-\infty;0) \cup (0;1) \cup (1;+\infty)$  bo'ladi.

**6-misol.** Agar  $f\left(\frac{x}{x+1}\right) = x^2$  bo'lsa,  $f(x)$  topilsin.

**Yechilishi:** Berilgan munosabatda  $\frac{x}{x+1} = t$  deb belgilaymiz.

Unda

$$\frac{x}{x+1} = t \Rightarrow x = tx + t \Rightarrow x = \frac{t}{1-t} \text{ bo'lib } f(t) = \left(\frac{t}{1-t}\right)^2, \text{ ya'ni}$$

$$f(x) = \left(\frac{x}{1-x}\right)^2 \text{ bo'ladi.}$$

**7-misol.** Ushbu funktsiyalar aynan tengmi?

$$1) f(x) = \lg x^2 \quad \text{bilan} \quad \varphi(x) = 2\lg|x|,$$

$$2) f(x) = \lg x^2 \quad \text{bilan} \quad \varphi(x) = 2\lg x ,$$

$$3) f(x) = \begin{cases} \frac{2x^2}{x^2} & \text{bilan } \varphi(x) = 2 \end{cases}$$

**Yechilishi:**

1)  $f(x)$  va  $\varphi(x)$  funktsiyalarning aniqlanish sohasi bir xil:  
 $D(f) = D(\varphi) = R \setminus \{0\}$  bo'lib,  $\forall x \in D(f)$  da  $f(x) = \lg x^2 = 2 \lg |x| = \varphi(x)$  bo'ladi.

2)  $\varphi(x) = 2 \lg x$  funktsiyaning aniqlanish sohasi esa  $D(\varphi) = (0; +\infty)$  bo'ladi. Demak,  $D(f) \neq D(\varphi)$ .  $f(x)$  va  $\varphi(x)$  funktsiyalar aynan teng emas.

$$3) f(x) = \begin{cases} \frac{2x^2}{x^2} & \text{funktsiyaning aniqlanish sohasi} \end{cases}$$

$D(f) = (-\infty; 0) \cup (0; +\infty)$  bo'lib,  $\varphi(x) = 2$  funktsiyaning aniqlanish soxasi  $D(f) = (-\infty; +\infty)$  bo'ladi. Bu to'plamlar bir-biriga teng emas. Binobarin,  $f(x)$  va  $\varphi(x)$  funktsiyalar aynan teng emas.

**8-misol.** Ushbu  $f(x) = \frac{1 + \sin x}{\sin x}$  funktsiyaning aniqlanish sohasi  $D(f)$  va qiymatlari to'plami  $Ye(f)$ lar topilsin.

**Yechilishi:**

Ravshanki,  $D(f) = \{x \in R | \sin x \neq 0\} = \{x \in R | x \neq \pi k, k \in \mathbb{Z}\}$  bo'ladi.  $Ye(f)$  ni topish uchun  $f(x) = \frac{1 + \sin x}{\sin x} = 1 + \frac{1}{\sin x}$  deb olib,  $-1 \leq \sin x \leq 1$  foydalananamiz. Demak,  $\frac{1}{\sin x} \in (-\infty; -1) \cup [1; +\infty) \Rightarrow E(f) = (-\infty; 0) \cup [2; +\infty)$

### Mustaqil yechish uchun misol va masalalar.

1. Radiusi  $R$  ga teng bo'lgan doiraga ichki chizilgan to'g'ri to'rtburchakning yuzi uning asosining funktsiyasi ko'rinishida ifodalansin.

$$J: S(x) = x \cdot \sqrt{4R^2 - x^2}, \quad 0 < x < 2R.$$

2. Radiusi  $R$  ga teng bo'lgan sharga ichki chizilgan konusning hajmi konus balandligining funktsiyasi ko'rinishida ifodalansin.

$$J: V = \frac{\pi H^2 \cdot (2R - H)}{3}, \quad 0 < H < 2R.$$

3. Radiusi  $R$  ga teng bo'lgan sharga ichki chizilgan korusning hajmi konus asosi radiusining funktsiyasi ko'rnishida ifodalansin.

$$J: V = \frac{\pi r^2 \cdot (R + \sqrt{R^2 - r^2})}{3}, \quad 0 < r < R.$$

**Modul belgisini ishlatmay qo'yidagi funktsiya analitik ko'rinishda yozilsin.**

4. a)  $y = |x^2 - 5x + 6|,$       b)  $y = |\lg x|.$

$$J: a) y = \begin{cases} x^2 - 5x + 6, & -\infty < x \leq 2, \\ -x^2 + 5x - 6, & 2 < x \leq 3 \\ x^2 - 5x + 6, & 3 < x < +\infty \end{cases}$$

$$b) y = \begin{cases} -\lg x, & 0 < x \leq 1, \\ \lg x, & 1 < x < +\infty \end{cases}$$

5. a)  $y = \frac{1}{2}(x + |x|),$       b)  $y = \frac{1}{2}(x - |x|),$

$$J: a) y = \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

$$b) y = \begin{cases} 0, & x \geq 0, \\ x, & x < 0. \end{cases}$$

**Quyidagi funktsiyalarning aniqlanish sohalarini toping.**

6. a)  $f(x) = \frac{3x-1}{x^2-3x+2},$       b)  $f(x) = \arccos \frac{2x}{1+x^2}$

$$J: a) D(f) = (-\infty; 1) \cup (1; 2) \cup (2; +\infty), \quad b) D(f) = (-\infty; +\infty).$$

7. a)  $f(x) = \lg(3\sin^2 x - 4),$       b)  $f(x) = \log_2|4 - x^2|.$

$$J: a) D(f) = \emptyset, \quad b) D(f) = (-\infty; 2) \cup (-2; 2) \cup (2; +\infty).$$

8. Agar  $f(x)$  funktsiyaning aniqlanish sohasi  $[-1; 0]$  kesma bo'lса, quyidagi funktsiyalarning aniqlanish sohasi topilsin.

a)  $f(-x^2),$       b)  $f(\cos x).$

$$J: a) [-1; 1], \quad b) \left(2k + \frac{1}{2}\right)\pi \leq x \leq \left(2k + \frac{3}{2}\right)\pi, k \in \mathbb{Z}.$$

9. Qo'yidagi funktsiyalarning qiymatlar to'plami topilsin.

a)  $f(x) = x + \frac{1}{x} (x \in (0; +\infty)),$       b)  $f(x) = 2^{x^2+4x-5}$

$$J: a) E(f) = [2^{-9}; +\infty), \quad b) E(f) = [2; +\infty),$$

## 2-§. Funktsiya xossalari tekshirish. Teskari funktsiya.

1<sup>0</sup>. Agar  $\forall x \in X \subset R$  uchun  $-x \in X$  bo'lsa,  $X$  to'plam O nuqtaga nisbatan simmetrik to'plam deyiladi. Bunday to'plamda  $f(x)$  funktsiya aniqlangan bo'lsin.

Agar  $\forall x \in X \subset R$  uchun  $f(-x) = f(x)$  bo'lsa, juft;  $f(-x) = -f(x)$  bo'lsa, toq funktsiya deyiladi.

Endi davriy funktsiya tushunchasini keltiramiz. Aytaylik,  $f(x)$  funktsiya  $X \subset R$  to'plamda berilgan bo'lsin.

Agar,  $\exists T \neq 0 (T \in R)$ ,  $\forall x \in X$  uchun  $x-T \in X, x+T \in X$ ,  $f(x+T) = f(x)$  shartlar bajarilsa,  $f(x)$  davriy funktsiya deyiladi, T esa funktsiyaning davri deyiladi.

Agar  $\forall x_1, x_2 \in X$  uchun  $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$  ( $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ ) bo'lsa,  $f(x)$  funktsiya  $X$  to'plamda o'suvchi (qatiy o'suvchi), ( $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$ ) ( $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ ) bo'lsa,  $f(x)$  funktsiya  $X$  to'plamda kamayuvchi (qatiy kamayuvchi) deyiladi.

O'suvchi va kamayuvchi funktsiyalar umumiy nom bilan monoton funktsiyalar deyiladi.  $y = f(x)$  funktsiya  $X \subset R$  to'plamda berilgan bo'lib, uning qiymatlari to'plami  $E(f)$  bo'lsin. Agar har bir  $y \in E(f)$  ga  $y = f(x)$  munosabatni qanoatlaniruvchi bitta  $x \in X$  son mos qo'yilsa, unda yuzaga kelgan funktsiya  $y = f(x)$  ga teskari funktsiya deyiladi va u  $x = f^{-1}(y)$  kabi yoziladi.

2<sup>0</sup>. Ma'lumki, asosiy elementar funktsiyalarga darajali funktsiya, ko'rsatkichli funktsiya, logarifmik funktsiya, trigonometrik va teskari trigonometrik funktsiyalar kiritiladi. Asosiy elementar funktsiyalarning aniqlanish sohasi, qiymatlari to'plami va ularning xossalari muktab matematika kursidan ma'lum.

Asosiy elementar funktsiyalar ustida arifmetik amallar bajarish yordamida hosil qilinib, bitta formula yordamida beriladigan funktsiyalarga elementar funktsiyalar deyiladi.

Masalan: 1)  $f(x) = x^2 - \cos x$  2)  $f(x) = \lg(x + \sqrt{1+x^2})$   
3)  $f(x) = (x-1)^2 \cdot \sin^2 x$  funktsiyalar elementar funktsiyalarga misol bo'ladi.

Elementar funktsiyalarning superpozitsiyasi k.o'rinishida ifodalab bo'lmaydigan funktsiyaga elementar bo'lмаган

**funktsiyalar** deyiladi. Masalan, bir nechta formula yordamida aniqlangan ushbu

$$y = \begin{cases} x^3, & x \leq 0 \\ x + 2, & x > 0 \end{cases}$$

funktsiya elementar bo'limagan funktsiyaga misol bo'ladi.

**3<sup>0</sup>. 1-misol.** Ushbu a)  $f(x) = \frac{a^x + a^{-x}}{2}$ , b)  $f(x) = \lg \frac{1+x}{1-x}$  funktsiyalar juft yoki toqlikka tekshirilsin.  
 a)  $f(-x) = \frac{a^{-x} + a^{-(x)}}{2} = \frac{a^{-x} + a^x}{2} = f(x)$ . Demak,  $f(x) = \frac{a^x + a^{-x}}{2}$  funktsiya juft.

$$b) f(-x) = \lg \frac{1+(-x)}{1-(-x)} = \lg \frac{1-x}{1+x} = \lg \left( \frac{1+x}{1-x} \right)^{-1} = -\lg \frac{1+x}{1-x}.$$

Demak,  $f(x) = \lg \frac{1+x}{1-x}$  funktsiya toq.

**2-misol.** Simmetrik to'plamda aniqlangan har qanday funktsiya juft va toq funktsiyalar yig'indisi ko'rinishida ifodalanishi isbotlansin.

Aytaylik,  $f(x)$  funktsiya berilgan bo'lsin. Agar

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = \varphi(x) + \psi(x)$$

deb olsak,  $\varphi(x) = \frac{f(x) + f(-x)}{2}$  funktsiya juft,  $\psi(x) = \frac{f(x) - f(-x)}{2}$  funktsiyaning esa toq bo'lishini ko'rish qiyin emas.

**3-misol.** Ushbu,  $f(x) = A \sin(\alpha x + \beta)$  funktsiyaning davri topilsin, bunda  $A$ ,  $\alpha$ ,  $\beta$  — o'zgarmas sonlar.

Aytaylik,  $f(x)$  funktsiyaning davri  $T$  bo'lsin ( $T \neq 0$ ). Unda

$f(x+T) = f(x)$ , yani  $A \sin[\alpha(x+T) + \beta] \equiv A \sin(\alpha x + \beta)$  bo'ladi. Keyingi tenglikdan topamiz:

$$\sin \frac{\alpha T}{2} \cdot \cos \left( \alpha x + \beta + \frac{\alpha T}{2} \right) \equiv 0$$

bu tenglikdan  $\sin \frac{\alpha T}{2} = 0$  bo'lishi kelib chikadi. Demak,  $\frac{\alpha T}{2} = \pi n$ , yani  $T = \frac{2\pi n}{\alpha}$  bo'ladi, bunda  $n \in \mathbb{Z}$ . Berilgan funktsiyaning eng kichik musbat davri  $T_0 = \frac{2\pi}{\alpha}$  bo'ladi.

**4-misol.** Ushbu  $f(x) = \frac{x}{1+x^2}$  funktsiya monotonlikka tekshirilsin.

Ravshanki, bu funktsiyaning aniqlanish sohasi  $D(f) = (-\infty; +\infty)$  bo'ladi.

$$\forall x_1, x_2 \in R, \quad x_1 < x_2 \text{ bo'lsin. Unda}$$

$$f(x_2) - f(x_1) = \frac{x_2}{1+x_2^2} - \frac{x_1}{1+x_1^2} = \frac{(x_2 - x_1)(1 - x_1 \cdot x_2)}{(1+x_1^2)(1+x_2^2)}$$

bo'ladi. Bu ayirmaning ishorasi  $1 - x_1 x_2$  ning ishorasiga bog'liq bo'ladi.

Agar  $x_1, x_2 \in (-1; 1)$  bo'lsa,  $1 - x_1 \cdot x_2 > 0$ ,  $x_1, x_2 \in (-\infty; -1) \cup (1; +\infty)$  bo'lsa  $1 - x_1 \cdot x_2 < 0$  bo'ladi. Binobarin,  $\forall x_1, x_2 \in (-1; 1)$  da

$$x_1 < x_2 \Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow f(x_1) < f(x_2)$$

bo'lib, berilgan funktsiya  $(-1; 1)$  oraliqda **o'suvchi**  $\uparrow$  bo'ladi.

$\forall x_1, x_2 \in (-\infty; -1) \cup (1; +\infty)$  da  $x_1 < x_2 \Rightarrow f(x_2) - f(x_1) > 0 \Rightarrow f(x_1) > f(x_2)$  bo'lib, berilgan funtsiya  $(-\infty; -1) \cup (1; +\infty)$  to'plamda **kamayuvchi**  $\downarrow$  bo'ladi.

**5-misol.** Ushbu

$$y = \frac{1-x}{1+x} \quad (x \neq 1)$$

funktsiyaga teskari funktsiya topilsin. Ravshanki,

$$y = \frac{1-x}{1+x} \Rightarrow y + yx = 1 - x \Rightarrow (1+y)x = 1 - y \Rightarrow x = \frac{1-y}{1+y} \quad (y \neq -1).$$

Demak, berilgan funktsiyalarga teskari funktsiya  $y = \frac{1-x}{1+x}$  ( $x \neq -1$ ) bo'ladi.

## Mustaqil yechish uchun misol va masalalar.

### 1. Quyidagi funktsiyalar juft yoki toqlikka tekshirilsin.

- a)  $f(x) = x^2 - \cos x$   
b)  $f(x) = \lg(x + \sqrt{1+x^2})$   
v)  $f(x) = (x-1)^2 \cdot \sin^2 x$

J: a) juft; b) toq; v)juft ham, toq ham emas.

### Quyidagi funktsiyalar davriylikka tekshirilsin, davriy bo'lsa, eng kichik musbat davri topilsin.

2. a)  $f(x) = x^2 + x - 1$ , b)  $f(x) = 3$ .

J: a) Davriy emas; b)  $f(x+T) = f(x)$  tenglik  $\forall T$  uchun bajariladi, lekin eng kichik musbat davr mavjud emas.

3. a)  $f(x) = \sqrt{\sin 3x}$  b)  $f(x) = \sin^2 x$

J: a) davriy,  $T_0 = \frac{2\pi}{3}$ ; b) davriy,  $T_0 = \pi$

4. a)  $f(x) = \cos x^2$ , b)  $f(x) = \{x\} = x - [x]$

J: a) davriy emas; b) davriy,  $T_0 = 1$

5. a)  $f(x) = \sin 2\pi x$ , b)  $f(x) = \sin^4 x + \cos^4 x$ .

J: a) davriy,  $T_0 = 1$ ; b) davriy,  $T_0 = \frac{\pi}{2}$

6. Agar  $f(x)$  davriy funktsiya bo'lib, uning davri  $T$  ( $T \neq 0$ ) bo'lsa,  $y = f(ax+b)$  ( $a \neq 0$ ) davriy funktsiya bo'lib, uning davri  $\frac{T}{a}$  bo'lishi isbotlansin.

7. Ushbu  $f(x) = x - \sin x$  funktsiya  $\left(0; \frac{\pi}{2}\right)$  intervalda o'suvchi bo'lishi isbotlansin.

**Ko'rsatma:**  $\forall x_1 x_2 \in \left(0; \frac{\pi}{2}\right)$  va  $x_1 < x_2$  uchun  $\sin x_1 < \sin x_2$  tengsizlikning bajarilishidan foydalaning.

8. a)  $f(x) = \sin 4x + 5 \cos 6x$ , b)  $f(x) = \sin^3 x + \cos^3 x$ .

J: a) Davriy,  $T_0 = \pi$ ; b) Davriy,  $T_0 = 2\pi$ .

9. a)  $f(x) = \cos \pi x$ ;

b)  $f(x) = |\sin(\sqrt{2}x + 1)|$ .

J: a) Davriy,  $T_0 = 2$ ;

b) Davriy,  $T_0 = \frac{\pi}{\sqrt{2}}$ .

**10.** Davriy bo'lмаган шундай  $f(x)$  va  $g(x)$  функцияларға шундай мисолдар көтүрілсінki,

- 1)  $f(x)+g(x)$ ,
- 2)  $f(x) \cdot g(x)$

функциялар даврий болын.

**11.** Шундай даврий  $f(x)$  va даврий bo'lмаган  $g(x)$  функцияларға мисолдар көтүрілсінki,  $f(x)+g(x)$ ,  $f(x) \cdot g(x)$  функциялар даврий болын.

**12.** Ушбу  $f(x) = (x^2 - 1)^2$  функцияларнинг  $(-1; 0)$  интервалда о'sувчи,  $(0; 1)$  интервалда esa камауычы екәнligи isbotlansin.

### Quyidagi функциялар monotonlikka tekshirilsin.

**13.** a)  $f(x) = \frac{x^2}{x^2 + 1}$ , b)  $f(x) = \frac{x^2}{x^2 - 1}$ .

J: a)  $(-\infty; 0)$  да  $\downarrow$ ,  $(0; +\infty)$  да  $\uparrow$ ; b)  $(-\infty; -1)$  va  $(-1; 0)$  да  $\uparrow$ ,  $(0; 1)$  va  $(1; +\infty)$  да  $\downarrow$ .

**14.** a)  $f(x) = \frac{x-1}{|x|+1}$ , b)  $f(x) = 1 - \frac{|x-1|}{1+|x|}$ .

J: a)  $(-\infty; 0)$  да o'zgarmas,  $(0; +\infty)$  да  $\uparrow$ ; b)  $(-\infty; 1)$  да  $\uparrow$ ,  $(1; +\infty)$  да  $\downarrow$ .

**15.** a)  $f(x) = \sin^4 x + \cos^4 x$ ,  $x \in [0; \pi]$ ,

b)  $f(x) = \frac{2 - \sin x}{2 + \sin x}$ ,  $x \in [0; 2\pi]$ .

J: a)  $[0; \frac{\pi}{4}]$  va  $[\frac{\pi}{2}, \frac{3\pi}{4}]$  да  $\downarrow$ ,  $[\frac{\pi}{4}; \frac{\pi}{2}]$  va  $[\frac{3\pi}{4}; \pi]$  да  $\uparrow$ .

b)  $[0; \frac{\pi}{2}]$  va  $[\frac{3\pi}{2}; 2\pi]$  да  $\downarrow$ ,  $[\frac{\pi}{2}; \frac{3\pi}{2}]$  да  $\uparrow$ .

**16.** a)  $f(x) = [x]$ , b)  $f(x) = |x| - x$

J: a) o'sувчи, b)  $x \leq 0$  да камауычы,  $x > 0$  да o'zgarmas 0 ga teng.

**17.** a)  $f(x) = x^4 + 6x^2 + 1$ , b)  $f(x) = \frac{1}{x^2 + 4x + 5}$ .

J: a)  $x > 0$  да o'sувчи,  $x < 0$  да камауычы. b)  $x < -2$  да o'sувчи,  $x > -2$  да камауычы.

**18.** Quyidagi функцияларнинг o'заро тескари функциялар екәнligи isbotlansin:

a)  $f(x) = \frac{x}{x+1}$ , g(x) =  $\frac{x}{1-x}$

b)  $f(x) = 1 - \sqrt[3]{x}$ ,  $g(x) = (1-x)^3$   
 v)  $f(x) = \sqrt[3]{x^5}$ ,  $g(x) = \sqrt[5]{x^3}$

19. Aytaylik,  $y = f(x)$  va  $y = g(x)$  funktsiyalar  $M \subset R$  to'plamda aniqlangan bo'lib,  $y = f(x) > 0$  kamayuvchi va  $y = g(x) < 0$  esa o'suvchi bo'lсин. Ushbu funktsiyalar M to'plamda monotonlikka tekshirilsin.

a)  $f(x) + g(x)$ ; b)  $f(x) + 4g(x)$ ; v)  $y = f^2(x)$ ; g)  $y = g^2(x)$ .

J: a) o'suvchi, b) kamayuvchi, v) kamayuvchi, g) kamayuvchi.

**Quyidagi funktsiyalarga teskari bulgan funktsiyalar topilsin.**

20. a)  $f(x) = 2x - x^2$ ,  $x \geq 1$ ; b)  $f(x) = 2x - x^2$ ,  $x \leq 1$ :

v)  $f(x) = \frac{2x}{1+x^2}$ ,  $(|x| \leq 1)$

J: a)  $f = 1 + \sqrt{1-x}$ ,  $x \leq 1$ ; b)  $f = 1 - \sqrt{1-x}$ ,  $x \leq 1$ :

v)  $f(x) = 1 + \frac{\sqrt{1-x^2}}{x}$ ,  $0 < x \leq 1$ .

21. a)  $y = \lg(x-1)$ , b)  $y = \sqrt[3]{1-x^3}$ .

J: a)  $y = 10^x + 1$ ; b)  $y = \sqrt[3]{1-x^3}$ .

22.  $y = \frac{2x}{1-x^2}$ ,  $x < -1$ .

J:  $y = \frac{1+\sqrt{1-x^2}}{x}$ ,  $-1 \leq x < 0$

23.  $y = x|x| + 2x$ ,  $x \in R$ ; b)  $y = \sin x$   $x \in \left[ \frac{5\pi}{2}, \frac{7\pi}{2} \right]$ .

J: a)  $y = \begin{cases} 1 - \sqrt{1-x} & x < 0 \\ -1 + \sqrt{1+x} & x \geq 0 \end{cases}$  b)  $y = 3\pi - \arcsin x$ ,  $x \in [-1; 1]$ .

### 3-§. Funktsiyalarning grafiklari.

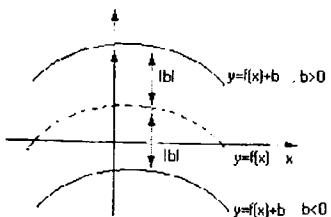
**1<sup>0</sup>.** Aytaylik,  $y = f(x)$  funktsiya  $X \subset \mathbb{R}$  to'plamda berilgan bo'lzin.  $x_0 \in X$  da  $f(x_0) = y_0$  deylik. Ravshanki,  $(x_0, y_0)$  juftlik tekislikda nuqtani tasvirlaydi.

Tekislikning nuqtalaridan iborat ushbu  $\{(x, y) | x \in X, y = f(x)\}$  to'plam  $y = f(x)$  **funktsiyaning grafigi** deyiladi.

- 1) Juft funktsiyaning grafigi ordinatalar o'qiga nisbatan, toq funktsiyaning grafigi esa koordinata boshiga nisbatan simmetrik bo'ladi. Shu sababli juft va toq funktsiyalarning grafigini faqat argumentning musbat qiymatlari uchun bilish yetarli.
- 2) Davriy funktsiya uchun esa uning grafigini bir davr oralig'iда bilish kifoya.

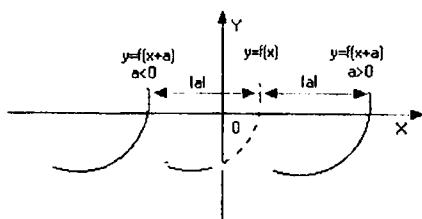
**2<sup>0</sup>.** Faraz qilaylik,  $y = f(x)$  funktsiya grafigining tekislikdagi tasviri ma'lum bo'lzin. Unga ko'ra quyidagi  $y = f(x) + b$ ,  $y = f(x+a)$ ,  $y = -f(x)$ ,  $y = f(-x)$ ,  $y = -f(-x)$ ,  $y = |f(x)|$ ,  $y = f(|x|)$ ,  $y = af(x)$ ,  $y = f(ax)$  (bunda  $a, b \neq 0$  zgarmas sonlar) funktsiyalarning grafiklarini yasash usullarini keltiramiz:

- 1)  $y = f(x) + b$  **funktsiyaning grafigi.** Bu funktsiyaning grafigi  $y = f(x)$  funktsiyaning grafigini ordinatalar o'qi bo'yicha  $|b|$  masofaga parallel ko'chirish bilan yasaladi. Bunda grafik  $b > 0$  bo'lganda yuqoriga,  $b < 0$  bo'lganda parallel ko'chiriladi.



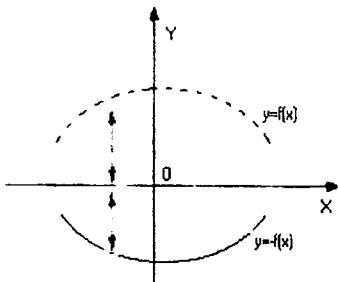
2 – chizma.

- 2)  $y = f(x+a)$  **funktsiyaning grafigi.** Bu funktsiyaning grafigi  $y = f(x)$  funktsiyaning grafigini abstsissalar o'qi bo'yicha  $|a|$  masofaga parallel ko'chirish bilan yasaladi. Bunda grafik  $a > 0$  bo'lganda chapga,  $a < 0$  bo'lganda o'ngga parallel ko'chiriladi.



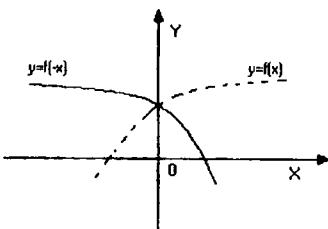
3 – chizma.

3)  $y = -f(x)$  funktsiyaning grafigi. Bu funktsiyaning grafigi  $y = f(x)$  funktsiyaning grafigini abstsissalar o'qiga nisbatan simmetrik ko'chirish bilan yasaladi.



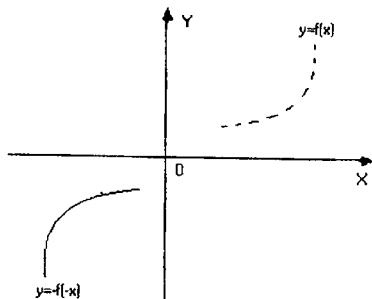
4 – chizma.

4)  $y = f(-x)$  funktsiyaning grafigi. Bu funktsiyaning grafigi  $y = f(x)$  funktsiya grafigini ordinatalar o'qiga nisbatan simmetrik ko'chirish bilan yasaladi.



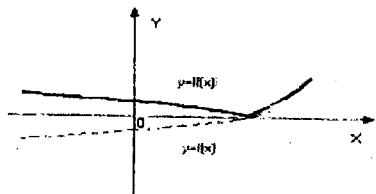
5 – chizma.

5)  $y = -f(-x)$  funktsiyaning grafigi. Bu funktsiyaning grafigi  $y = f(x)$  funktsiya grafigini koordinata boshiga nisbatan simmetrik ko'chirish bilan yasaladi.



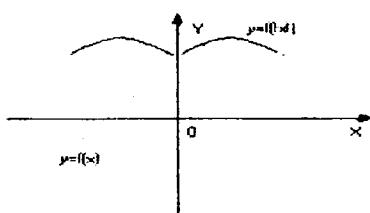
6 – chizma.

6)  $y = |f(x)|$  **funktsiyaning grafigi.** Bu funktsiyaning grafigi  $y = f(x)$  funktsiya grafigining OX o'qidan yuqorida joylashgan qismini qoldirish, OX o'qidan pastda joylashgan qismini esa shu o'qqa simmetrik ravishda yuqoriga ko'tarish bilan yasaladi.



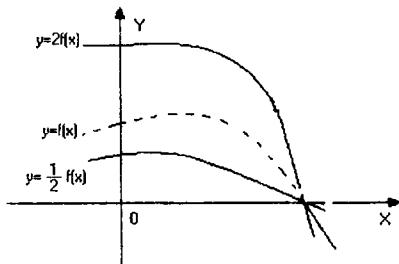
7 – chizma.

7)  $y = f(|x|)$  **funktsiyaning grafigi.** Bu funktsiyaning grafigi  $y = f(x)$  funktsiya grafigining OU o'qidan o'ng tomonda joylashgan qismini qoldirish, OU o'qidan chap tomonda joylashgan qismini tashlab, uning o'rniiga OU ning o'ng tomondag'i qismini shu o'qqa simmetrik ravishda chapga o'tkazish bilan yasaladi.



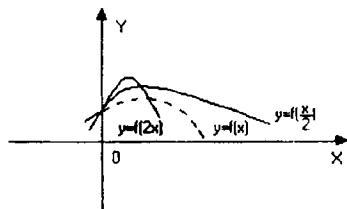
8 – chizma.

8)  $y = af(x)$  **funktsiyaning grafigi.** Bu funktsiyaning grafigi  $y = f(x)$  funktsiyaning grafigining har bir ordinatasini  $a$  ga ko'paytirish bilan yasaladi.



9 – chizma.

9)  $y = f(ax)$  **funktsiyaning grafigi.** Bu funktsiyaning grafigi  $y = f(x)$  funktsiya grafigining har bir abstsissani  $a$  ga bo'lish bilan yasaladi.



10 – chizma.

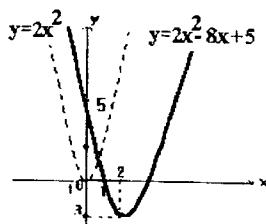
3<sup>0</sup>. Ko'pincha  $y = f(x)$  va  $y = \varphi(x)$  funktsiyalarning grafiklarini bilgan holda bu funktsiyalarning mos ordinatalarining (ikkalasining ordinatalari bitta  $x$  nuqtada hisoblanadi) qo'shish, ayirish, ko'paytirish va bo'lish bilan

$f(x) + \varphi(x)$ ,  $f(x) - \varphi(x)$ ,  $f(x) \cdot \varphi(x)$      $\frac{f(x)}{\varphi(x)}$     funktsiyalarning grafigini yasash mumkin bo'ladi.

**1-misol.** Ushbu  $y = 2x^2 - 8x + 5$  funktsiya grafigi yasalsin.  
Kvadrat uchhadning ko'rinishini o'zgartiramiz:

$$y = 2x^2 - 8x + 5 = 2\left(x^2 - 4x + \frac{5}{2}\right) = 2(x-2)^2 - 3. \quad \text{Demak, berilgan}$$

funktsiyalarning grafigini 1) va 2)-qoidalarga asosan  $y = 2x^2$  parabolaning grafigini 2 birlik o'ngga surish va 3 birlik pastga siljитish yordamida hosil qилинади (11-chizma).



11 – chizma

**Izoh.** Shu yo'l bilan ixtiyoriy kvadrat uchhadning grafigini yasash mumkin.

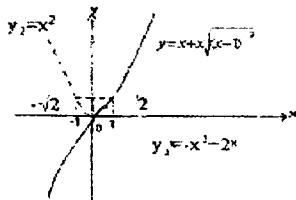
**2-misol.** Ushbu  $y = x + x\sqrt{(x-1)^2}$  funktsiyaning grafigi yasalsin.

Agar  $x > 1$  bo'lsa  $y = x + x(x-1) = x^2$ ,  $x \leq 1$  bo'lsa

$$y = x + x(1-x) = -x^2 + 2x$$

bo'ladi.

Demak, berilgan funktsiyaning grafigi  $(-\infty ; 1)$  oraliqda  $y_1 = -x^2 + 2x$  hamda  $[1; +\infty)$  da  $y = x^2$  parabolalardan tashkil topган chiziqni ifodалайди (12 – chizma).



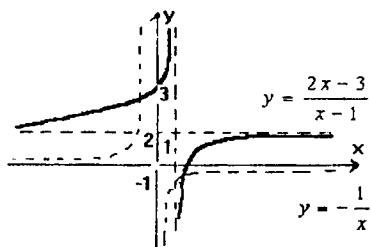
12 – chizma.

**3-misol.** Ushbu  $y = \frac{2x-3}{x-1}$  funktsiya grafigi yasalsin.

Funktsiyaning ko'rinishini quyidagicha o'zgartiramiz:

$$y = \frac{2x-3}{x-1} = \frac{2(x-1)-1}{x-1} = 2 - \frac{1}{x-1}.$$

Bu funktsiyaning grafigini yasash uchun  $y = -\frac{1}{x}$  funktsiyaning grafigini chizib, uni 1 birlik o'ngga surish va 2 birlik yuqoriga ko'tarish kerak (13 – chizma).



13 – chizma

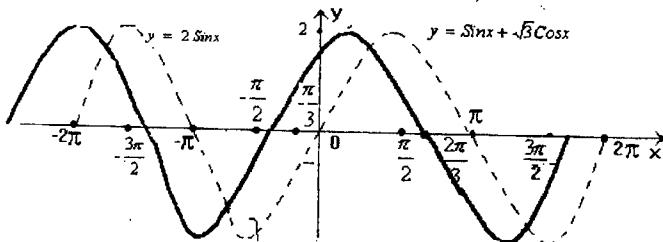
**Izoh:** Shu yo'l bilan ixtiyoriy kasr – chiziqli  $y = \frac{ax+b}{cx+d}$  funktsiyaning grafigini yasash mumkin.

**4-misol.** Ushbu  $y = \sin x + \sqrt{3} \cos x$  funktsiyaning grafigi yasalsin.

Quyidagi amallarni bajaramiz:

$$y = \sin x + \sqrt{3} \cos x = 2\left(\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x\right) = 2\left(\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3}\right) = 2 \sin\left(x + \frac{\pi}{3}\right)$$

Shunday qilib, berilgan funktsiyaning grafigini chizish uchun  $y = 2 \sin x$  funktsiyaning grafigini chizib, uni  $\frac{\pi}{3}$  birlik chapga siljитish yetarli ekan (14 – chizma)

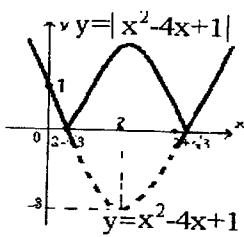


14 – chizma

**Izoh:** Shu yo'l bilan  $y = a \sin x + b \cos x$  ko'rinishidagi ixtiyoriy funktsiyaning grafigini yashash mumkin.

**5-misol.** Ushbu  $y = |x^2 - 4x + 1|$  funktsiyaning grafigi yasalsin.

Bu funktsiyaning grafigini yashash uchun avval  $y_1 = x^2 - 4x + 1 = (x-2)^2 - 3$  funktsiyaning grafigini yasab, so'ng  $y = |y_1|$  funktsiyaning grafigini yashashda 6 – xossadan foydalanamiz (15 – chizma).

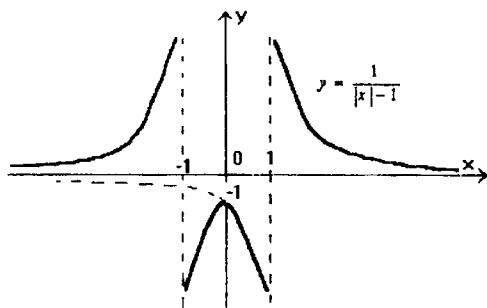


15 – chizma.

**6-misol.** Ushbu  $y = \frac{1}{|x|-1}$  funktsiyaning grafigi yasalsin.

Bu funktsiya juft bo'lганлиги sababli uning grafigini OY o'qidan o'ng tomonda joylashган qismini chizish yetarli bo'ladi (chap tomonga OY o'qiga nisbatan simmetrik ko'chiriladi).

Demak,  $x \geq 0$  da  $y = \frac{1}{x-1}$  bo'lib, uning grafigini o'ng yarim tekislikda chizamiz va bu grafik yordamida berilgan funktsiyaning grafigini hosil qilamiz. (16 – chizma).



16 – chizma.

### 7 – misol. Ushbu

$$y = \arcsin(\sin x)$$

funktsiyaning grafigi yasalsin.

Bu funktsiya  $2\pi$  davrlı funktsiya bo'ladi, chunki

$$\arcsin(\sin(x + 2\pi)) = \arcsin(\sin x)$$

Arksinus ta'rifiga binoan, birinchidan

$$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

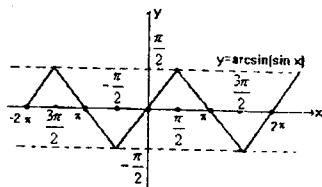
ikkinchidan esa  $\sin y = \sin x$  bo'ladi.

Demak,

$$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \quad \text{da} \quad y = x,$$

$$\frac{\pi}{2} < x \leq \frac{3\pi}{2} \quad \text{da} \quad y = \pi - x$$

bo'ladi (17 – chizma).



17 – chizma.

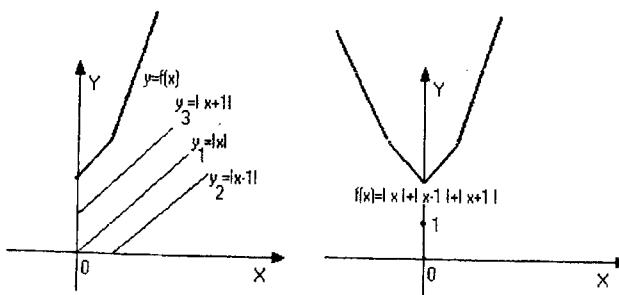
### 8-misol. Ushbu

funktsiyaning grafigi yasalsin.

Bu funktsiya  $R = (-\infty; +\infty)$  deb aniqlangan juft funktsiyalardir. Haqiqatdan ham,

$$f(-x) = |-x| + |-x - 1| + |-x + 1| = |x| + |(-1)(x + 1)| + |-(x - 1)| = |x| + |x + 1| + |x - 1| = f(x)$$

Demak, berilgan funktsiya grafigining OY o'qidan o'ng tomonda joylashgan qismini topish yetarli bo'ladi (chap tomonga OY o'qiga nisbatan simmetrik ko'chiriladi). Ravshanki,  $y = f(x)$  funktsiya  $y_1 = |x|$ ,  $y_2 = |x - 1|$ ,  $y_3 = |x + 1|$ , funktsiyalar yig'indisidan iborat. Bu funktsiyalar grafiklarini chizib ( $y_2$  va  $y_3$  funktsiyalarining grafiklarini chizishda  $y_1 = |x|$ , funktsiyaning grafigi va 2 – qoidadan foydalaniladi), so'ng har bir  $x$  ga mos ordinatalarini qo'shib, berilgan funktsiya grafigi topiladi (18 – chizma).



18-chizma

**Mustaqil yechish uchun misol va masalalar.**

**Quyidagi funktsiyalarning grafiklari yasalsin.**

$$1. \quad y = 3x^2 - 6x - 17$$

$$12. \quad y = \max\left\{x^3; \frac{1}{x}\right\}$$

$$2. \quad y = -2x^2 - 4x + 4$$

$$13. \quad y = \frac{2x - 3}{3 - x}$$

$$3. \quad y = \frac{2x + 5}{x - 2}$$

$$14. \quad y = 2^{1-x^2}$$

$$4. \quad y = \frac{x^2 - 1}{x^2 + 1}$$

$$15. \quad y = 2^{\operatorname{arctg} x}$$

$$5. \quad y = \sqrt{\lg \sin x}$$

$$16. \quad y = \log_2 \log_2 x$$

$$6. \quad y = \sin^2 x$$

$$17. \quad y = \frac{\cos 2x}{\sin x}$$

$$7. \quad y = 0,5(|x+1| + |x-1|)$$

$$18. \quad y = \arcsin(3x - 1)$$

$$8. \quad y = \operatorname{arctg}(\operatorname{tg} x)$$

$$19. \quad y = x^2 + \frac{1}{x}$$

$$9. \quad y = \frac{1}{x^2 + 4x + 5}$$

$$20. \quad y = \frac{x^2}{|x|-1}$$

$$10. \quad y = \cos(\lg x)$$

$$21. \quad y = x \cdot \operatorname{Sign}(\sin x)$$

$$11. \quad y = \operatorname{Sign}(\sin x)$$

$$22. \quad y = \min\{\log_2 x, \log_x 2\}$$

#### 4—§. Parametrik ko'rinishda va qutb koordinatalar sistemasida berilgan funktsiyalar.

1<sup>0</sup>. Aytaylik,  $T \subset \mathbb{R}$  to'plamda

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases} \quad t \in T \quad (1)$$

funktsiyalar berilgan bo'lzin. Faraz qilaylik, X va Y to'plamlar, mos ravishda,  $T \subset \mathbb{R}$  to'plamda aniqlangan  $x = \varphi(t)$  va  $y = \psi(t)$  funktsiyalarning qiymatlar to'plami bo'lzin. Agar ixtiyoriy  $t \in T$  olinganda ham unga mos  $x = \varphi(t) \in X$  uchun yagona  $y = \psi(t) \in Y$  mos kelsa, unda (1) sistema parametrik ko'rinishda berilgan funktsiyani aniqlaydi deyiladi.

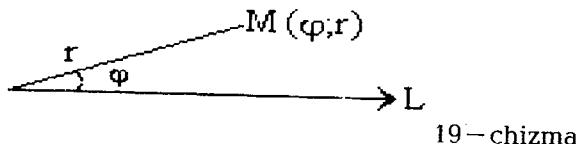
Masalan,

$$\begin{cases} x = t \\ y = t^2 \end{cases} \quad t \in R$$

funktsiyalar oshkor ko'rinishda berilgan  $y = x^2$  funktsiyaning parametrik ko'rinishini ifodalaydi.

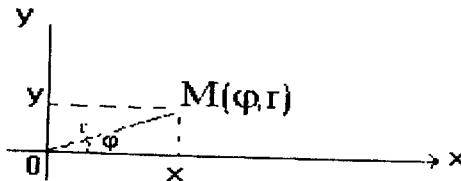
2<sup>0</sup>. Amaliyotda dekart koordinatalar sistemasidan keyin eng ko'p ishlataladigan sistema — qutb koordinatalar sistemasi hisoblanadi.

Qutb koordinatalar sistemasida tekislikdagi ixtiyoriy M nuqta, O qutb nuqtagacha bo'lgan masofa  $|OM| = r$  va OM vektorning OL qutb o'qi bilan hosil qilgan burchagi  $\varphi$  yordamida aniqlanadi (19-chizma).



19-chizma

Agar dekart koordinatalar sistemasi markazini qutb nuqta, Ox o'qining musbat qismini qutb o'qisi bilan ustma-ust tushirsak, unda qutb va dekart koordinatalar sistemalari orasidagi bog'lanishlar quyidagi tengliklar yordamida ifodalanadi (20-chizma):



20-chizma

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases} \quad \text{va} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \operatorname{tg} \varphi = \frac{y}{x} \end{cases} \quad (2)$$

**Izoh.** Qutb koordinatalar sistemasidagi ixtiyoriy  $(\phi; 0)$  juftlikka Dekart koordinatalar sistemasidagi  $O(0; 0)$  nuqtani mos qo'yamiz.

**3º. 1-misol.**  $A(0 ; -1)$  va  $V(1,6 ; -0,2)$  nuqtalarning qaysilari Ushbu

$$l = \begin{cases} x = S \sin t + 1 \\ y = C \cos t - 1 \end{cases} \quad \text{chiziqda yotishi aniqlansin.}$$

**Yechilishi:**

Avval  $A(0 ; -1)$  nuqtani tekshiramiz.  $\Rightarrow x = 0, y = -1, x = 0$  bo'lganda  $\sin t + 1 = 0$  bo'lib, bu tenglik  $t = \frac{3\pi}{2}$  da bajariladi.

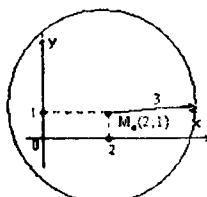
$$\Rightarrow y\left(\frac{3\pi}{2}\right) = \cos\frac{3\pi}{2} - 1 = -1. \quad \text{Demak } (0 ; -1) \in l.$$

$B(1,6; -0,2)$  uchun  $x = 1,6$  va  $y = -0,2$ .

$$\Rightarrow \sin t + 1 = 1,6 \Rightarrow \sin t = 0,6. \Rightarrow \cos t = \sqrt{1 - \sin^2 t} = 0,8. \Rightarrow y = \cos t - 1 = 0,8 - 1 = -0,2 \Rightarrow B(1,6 ; -0,2) \in l.$$

**2-misol.** Chiziqning ushbu  $\begin{cases} x = 2 - 3 \cos t \\ y = 1 + 3 \sin t \end{cases}$  parametrik tenglamasidan t parametr yo'qotilib, uning Dekart koordinatalar sistemasidagi tenglamasi topilsin va grafigi chizilsin.

**Yechilishi:** Berilgan sistemadan  $\begin{cases} x - 2 = -3 \cos t \\ y - 1 = 3 \sin t \end{cases}$  ekanligi va bu yerdan  $(x - 2)^2 + (y - 1)^2 = 9(\cos^2 t + \sin^2 t) = 9$  bo'lishini topamiz. Demak, berilgan chiziq markazi  $(2; 1)$  nuqtada, radiusi 3 ga teng bo'lgan ushbu  $(x - 1)^2 + (y - 1)^2 = 9$  aylanadan iborat ekan. (21-chizma).



21 – chizma

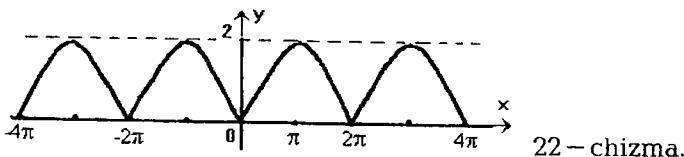
$$3-\text{misol.} \quad \text{Ushbu} \quad \begin{cases} x = t - S \sin \\ y = 1 - C \cos t \end{cases} \quad (3)$$

tsikloidanining grafigi yasalsin.

**Yechilishi:** (3) – sistema yordamida aniqlanadigan funktsiyani  $y = f(x)$  oshkor ko'rinishda ifodalab bo'lmaydi. Bu funktsiya  $x$  ning barcha qiymatlarida aniqlangan ( $x \in (-\infty; +\infty)$ ) va chegaralangan  $y \in [0; 2]$  hamda  $t$  parametr bo'yicha davriy bo'lib, davri  $2\pi$  ga teng. Xarakterli nuqtalar uchun quyidagi jadvalni tuzamiz.

$t$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$
$x$	0	$\frac{\pi}{4} - \frac{\sqrt{2}}{2}$	$\frac{\pi}{2} - 1$	$\frac{3\pi}{4} - \frac{\sqrt{2}}{2}$	$\pi$	$\frac{5\pi}{4} + \frac{\sqrt{2}}{2}$	$\frac{3\pi}{2} + 1$	$\frac{7\pi}{4} + \frac{\sqrt{2}}{2}$	$2\pi$
$y$	0	$1 - \frac{\sqrt{2}}{2}$	1	$1 + \frac{\sqrt{2}}{2}$	2	$1 + \frac{\sqrt{2}}{2}$	1	$1 - \frac{\sqrt{2}}{2}$	0

Bu jadvaldan foydalanib tsikloidanining grafigini chizamiz. (22 – chizma)



22 – chizma.

**4-misol.** Ushbu  $A(2\sqrt{3}; 2)$ ,  $V(\sqrt{2}; -\sqrt{2})$  nuqtalarning qutb koordinatalari va  $S(\frac{\pi}{2}; 10)$ ,  $D(\frac{5\pi}{4}; 2)$  nuqtalarning Dekart koordinatalari topilsin.

**Yechilishi:** Masalani yechish uchun Dekart va qutb koordinatalar sistemalarini bog'lovchi (2) tengliklardan foydalanamiz:

$$A(2\sqrt{3}; 2) \Rightarrow x = 2\sqrt{3}, \quad y = 2. \quad \Rightarrow \quad r = \sqrt{x^2 + y^2} = \sqrt{12 + 4} = \sqrt{16} = 4,$$

$$\operatorname{tg} \varphi = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow \varphi = \frac{\pi}{6} \Rightarrow A\left(\frac{\pi}{6}; 4\right) B(\sqrt{2}; -\sqrt{2})$$

$$\Rightarrow x = \sqrt{2}, \quad y = -\sqrt{2} \Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{2+2} = 2, \quad \operatorname{tg} \varphi = \frac{y}{x} = -1.$$

B nuqta IV – chorakda yotgani uchun  $\varphi = 2\pi - \arctg 1 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

bo' ladi.  $\Rightarrow V\left(\frac{7\pi}{4}; 2\right), S\left(\frac{\pi}{2}; 10\right) \Rightarrow \varphi = \frac{\pi}{2}, r = 10 \Rightarrow C(0; 10).$

$$D\left(\frac{5\pi}{4}; 10\right) \Rightarrow \varphi = \frac{5\pi}{4}, r = 2 \Rightarrow x = r \cos \varphi = 2 \cos \frac{5\pi}{4} = -2 \frac{\sqrt{2}}{2} = -\sqrt{2};$$

$$y = r \sin \varphi = 2 \sin \frac{5\pi}{4} = \sqrt{2} \Rightarrow D(-\sqrt{2}; \sqrt{2})$$

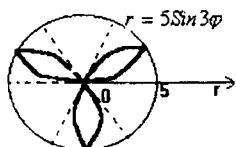
## 5–misol. Ushbu

$r = 5 \sin 3\varphi$  "uch yaproqli gul" grafigi yasalsin.

**Yechilishi:** Avval berilgan funktsianing aniqlanish sohasini topamiz :  $D(r) = \{\varphi | \sin 3\varphi \geq 0\} = \left[0; \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}; \pi\right] \cup \left[\frac{4\pi}{3}; \frac{5\pi}{3}\right]$ . Qiymatlar to'plami  $Y_e(r) = [0; 5]$  funktsiya davriy bo'lib, uning asosiy davri  $T_o = \frac{2\pi}{3}$  bo'ladi. Demak, funktsiya grafigini  $\left[0; \frac{2\pi}{3}\right]$  kesmada, xususan  $\left[0; \frac{\pi}{3}\right]$  kesmada, chizish kifoya. Buning uchun quyidagi jadvalni tuzamiz:

$\varphi$	0	$\frac{\pi}{18}$	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$
$r$	0	2.5	$\frac{5\sqrt{3}}{2}$	5	$\frac{5\sqrt{3}}{3}$	2.5	0

Jadvaldan va funktsianing davriyligidan foydalanim  $r = 5 \sin 3\varphi$  funktsianing grafigini chizamiz (23 – chizma).



23 – chizma.

## Mustaqil yechish uchun misol va masalalar.

1. A(0;0) va B(3;3) nuqtalarning qaysilari ush

$$l : \begin{cases} x = t^2 - 1 \\ y = t^3 - t \end{cases}$$

chiziqqa tegishli ekanligi aniqlansin.

J: A tegishli, B tegishli emas.

2. A( $\frac{3}{2}; \sqrt{3}$ ) va B(1;2) nuqtalarning qaysilari ushbu

$$l : \begin{cases} x = 2\cos t - \cos 2t \\ y = 2\sin t - \sin 2t \end{cases}, \text{ chiziqqa tegishli ekanligi aniqlansin.}$$

J: B tegishli, A tegishli emas.

Berilgan A va B nuqtalarning qaysilari tenglamalari parametrik ko'rinishda berilgan chiziqqa tegishli ekanligi aniqlansin.

3.  $x = t^2$ ,  $y = t^3$ ; A(1;1), B(4;-8). J: A va B.

4.  $x = 1 + \cos t$ ;  $y = \sin t$ ; A(0;0), B( $\frac{1}{2}; \frac{1}{2}$ ).  
J: A tegishli, B tegishli emas

5.  $x = 2 \sin t$ ;  $y = 2 \cos t$ ; A(2;2), B(0;2π).

J: A ham, B ham tegishli emas.

### 6.Ushbu

$$\begin{cases} x = \cos t \\ y = \cos^2 t \end{cases}, \text{ sistema qanday chiziqni aniqlashi topilsin.}$$

J:  $y = x^2$  parabolaning A(-1;1) va B(1;1) nuqtalarini tutashtiruvchi AOB yoyini aniqlaydi.

### 7.Chiziqning ushbu

$$\begin{cases} x = 3\cos t \\ y = 2\sin t \end{cases}$$

parametrik tenglamasidan t parametr yo'qotilib, uning dekart koordinatalar sistemasidagi tenglamasi topilsin va grafigi chizilsin.

**Ko'rsatma:** Avval birinchi tenglamani 3 ga, ikkinchi tenglamani 2 ga bo'lib, so'ngra t parametrni yo'qoting.

$$J: \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad -\text{ellips.}$$

Berilgan chiziqning parametrik tenglamasidan t parametr yo'qotilib, uning Dekart koordinatalar sistemasidagi tenglamasi topilsin va grafigi chizilsin.

$$8. x = t - 1, \quad y = t^2 - 2t + 2. \quad J: y = x^2 + 1.$$

$$9. x = |\ln t|, \quad y = 1 + t^3. \quad J: 3x = |\ln(y - 1)|.$$

$$10. x = (t + 1)^2, \quad y = (t - 1)^2. \quad J: 8(x + y) = (x - y)^2 + 16.$$

$$11. x = a \operatorname{Sect} t, \quad y = b \operatorname{tg} t. \quad J: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 - \text{giperbola.}$$

12. Ushbu

$$\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$$

astroidaning grafigi chizilsin.

13. Ushbu Dekart koordinatalar sistemasida berilgan A(2;0), B(1;1), C(-2;0) va D  $(-\sqrt{2}; -\sqrt{2})$  nuqtalarning qutb koordinatalari topilsin.

$$J: A(0;2), \quad B\left(\frac{\pi}{4}; \sqrt{2}\right), \quad C(\pi;2), \quad D\left(\frac{5\pi}{4}; 2\right)$$

14. Qutb koordinatalar sistemasida berilgan ushbu  $A\left(\frac{\pi}{3}; 2\right)$ ,  $B\left(\frac{3\pi}{2}; 3\right)$ ,  $C\left(\frac{3\pi}{4}; \sqrt{2}\right)$  va  $D\left(\frac{7\pi}{6}; 4\right)$  nuqtalarning Dekart koordinatalari topilsin

$$J: A(1; \sqrt{3}), \quad B(0; -3), \quad C(-1; 1), \quad D(-2\sqrt{3}; -2)$$

15. Dekart koordinatalar sistemasida berilgan quyidagi chiziqlarning tenglamalari qutb koordinatalar sistemasida yozilsin:

$$a) x + y^2 = 2x, \quad b) x = y^2 - \frac{1}{4} \operatorname{sya}$$

$$J: a) r = 2 \operatorname{Cos} \varphi, \quad b) r = \frac{1}{4 \operatorname{Sin}^2 \frac{\varphi}{2}}$$

16. Quyidagi chiziqlarning grafiklari yasalsin:

a)  $r = 2(1 - \cos\varphi)$  – kardioida, b)  $r = 3 + 2\sin\varphi$

17. Ushbu  $(x^2 + y^2)^2 = 2(x^2 - y^2)$  tenglama bilan berilgan "Bernulli lemniskatasi"ning grafigini qutb koordinatalar sistemasiga o'tish yordamida chizing.

**Quyidagi chiziqlarning grafiklari chizilsin.**

18.  $x = 2 - t$ ,  $y = 1 - t^2$

19.  $x = 2\cos^2 t$ ,  $y = 3\sin^2 t$

20.  $x = \sin t$ ,  $y = \cos ec t$

**Ko'rsatma:** t parametr yo'qotilsa,  $y = \frac{1}{x}$  tenglama hosil bo'ladi. Grafik chizilayotganda  $|x| \leq 1$  va  $|y| \geq 1$  bo'lishini e'tiborga olish kerak.

21. Dekart koordinatalar sistemasida berilgan ushbu  $A(2\sqrt{3}; 2)$ ,  $B(0; -1)$ ,  $C(2\sqrt{3}; -2)$  va  $D(-1; 1)$  nuqtalarning qutb koordinatalari topilsin.

J:  $A\left(\frac{\pi}{6}; 4\right)$ ,  $B\left(\frac{3\pi}{2}; 1\right)$ ,  $C\left(\frac{11\pi}{6}; 4\right)$ ,  $D\left(\frac{5\pi}{4}; \sqrt{2}\right)$ .

22. Qutb koordinatalar sistemasida berilgan ushbu  $A\left(\frac{5\pi}{3}; 2\right)$ ,  $B(\pi; 4)$ ,  $C\left(\frac{4\pi}{3}; 2\right)$  va  $D\left(\frac{5\pi}{4}; 4\right)$  nuqtalarning Dekart koordinatalari topilsin.

J:  $A(1; -\sqrt{3})$ ,  $B(-4; 0)$ ,  $C(-1; -\sqrt{3})$ ,  $D(-2\sqrt{2}; -2\sqrt{2})$ .

23. Qutb koordinatalar sistemasida berilgan  $M\left(\frac{\pi}{4}; 3\right)$  va  $N\left(\frac{3\pi}{4}; 4\right)$  nuqtalar orasidagi masofa hisoblansin. J: 5.

**Ko'rsatma:** M va N nuqtalarning Dekart koordinatalarini toping hamda  $M(x_1, y_1)$  va  $N(x_2, y_2)$  nuqtalar orasidagi masofani hisoblash uchun  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$  formuladan foydalaning.

Dekart koordinatalar sistemasida berilgan quyidagi chiziqlarning tenglamalari qutb koordinatalar sistemasida yozilsin.

$$24. \quad x + y + 1 = 0. \quad J: \quad r = \frac{1}{\sqrt{2} \cos\left(\varphi + \frac{3\pi}{4}\right)}.$$

$$25. \quad 2xy = x^2 - y^2. \quad J: \quad \varphi = \frac{\pi}{8} + \frac{n\pi}{2} \quad (n = 1, 2, 3) \text{ va } r = 0$$

**Quyidagi chiziqlarning grafiklari yasalsin.**

26.  $r = \varphi$ ,  $\varphi \in [0; 2\pi]$  – Arximed spirali.

27.  $r = 2(1 + \sin\varphi)$  – Kardioda

28.  $r = 4\sin(\varphi - \frac{\pi}{4})$  – Aylana

29.  $r = \frac{1}{\cos\varphi - \sin\varphi}$  – To'g'ri chiziq

30.  $r = \sin^2 2\varphi$  – to'rt yaproqli gul.

**V BOB.  
LIMIT, UZLUKSIZLIK, HOSILA.**

**1-§. Ketma-ketlik va uning limiti**

Ketma-ketlik tushunchasi maktab kursidan ma'lum.

Aytaylik,  $\{x_n\}$ :  $x_1, x_2, \dots, x_n, \dots$  ketma-ketlik berilgan bo'lsin.

Agar  $\forall n \in N$  uchun  $x_n \leq x_{n+1}$  ( $x_n \geq x_{n+1}$ ) bo'lsa, unda  $\{x_n\}$  ketma-ketlik **o'suvchi (kamayuvchi)** deyiladi.

Agar  $\forall n \in N$  uchun  $\exists M(m)$  son :  $\forall n \in N$  uchun  $x_n \leq M$  ( $x_n \geq m$ ) bo'lsa, u holda  $\{x_n\}$  ketma-ketlik **yugoridan (quyidan) chegaralangan** deb ataladi. Agar ketma-ketlik ham yugoridan, ham quyidan chegaralangan bo'lsa, u holda bunday ketma-ketlik **chegaralangan ketma-ketlik** deyiladi.

Agar ketma-ketlik o'suvchi yoki kamayuvchi bo'lsa, unda unga **monoton ketma-ketlik** deb ataladi.

Aytaylik,  $\{x_n\}$  ketma-ketlik va  $a$  son berilgan bo'lsin.

Agar  $\forall \varepsilon > 0$  uchun  $\exists n_0 = n_0(\varepsilon) \in N$ :  $\forall n \geq n_0$  uchun  $|x_n - a| < \varepsilon$  bo'lsa  $a$  son  $\{x_n\}$  **ketma-ketlikning limiti** deb ataladi va  $\lim_{n \rightarrow \infty} x_n = a$  kabi belgilanadi.

**Misollar.**

$$1) \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \text{ tenglik isbotlansin.}$$

$$\forall \varepsilon > 0 \text{ olamiz, } |x_n - a| = \left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow n_0 = \left[ \frac{1}{\varepsilon} \right].$$

$$2) \lim_{n \rightarrow \infty} \frac{n-1}{n} = 1 \text{ tenglik isbotlansin.}$$

$$x_n = \frac{n-1}{n}, \quad a = 1 \quad \forall \varepsilon > 0 \text{ olamiz.}$$

$$|x_n - a| = \left| \frac{n-1}{n} - 1 \right| = \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow n_0 = \left[ \frac{1}{\varepsilon} \right].$$

$$3) \lim_{n \rightarrow \infty} (-1)^n \neq \exists$$

Shuni isbotlaylik. Buning uchun teskarisini faraz qilaylik, ya'ni

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (-1)^n = \exists \quad \text{bo'lsin. Unda ta'rifga ko'ra } \forall \varepsilon > 0 \quad \exists n_0(\varepsilon) \in N \quad \forall n > n_0 \quad \text{uchun } |x_n - a| < \varepsilon \quad \text{bo'ladi. Aytaylik.}$$

$\varepsilon = \frac{1}{2}$  bo'lsin  $|x_{2n} - a| < \varepsilon$  va  
 $|x_{2n-1} - a| < \varepsilon \Rightarrow |x_{2n} - x_{2n-1}| = |x_{2n} - a - (x_{2n-1} - a)| \leq |x_{2n} - a| + |x_{2n-1} - a| < \varepsilon + \varepsilon = 2\varepsilon = 1$   
bo'ladi. Lekin  $|x_{2n} - x_{2n-1}| = |(-1)^{2n} - (-1)^{2n-1}| = 2 > 1$  ziddiyat kelib  
chiqdi. Demak,  $\lim_{n \rightarrow \infty} (-1)^n \neq 3$ .

### Sonli ketma-ketlikning xossalari.

1<sup>o</sup>. Sonli ketma-ketlik limitga ega bo'lsa, uning limiti yagona bo'ladi.

2<sup>o</sup>. Agar ketma-ketlik limitga egà bo'lsa, ya'ni yaqinlashuvchi bo'lsa, u chegaralangan bo'ladi.

Izoh. Bu xossaning teskarisi o'rini emas, ya'ni chegaralangan ketma-ketlik yaqinlashuvchi bo'lishi shart emas. Masalan,  $\{x_n\} = \{(-1)^n\}$  ketma-ketlik chegaralangan, lekin  $\lim_{n \rightarrow \infty} x_n \neq 3$ .

3<sup>o</sup>. Agar  $\{x_n\}$  ketma-ketlik o'suvchi (kamayuvchi) bo'lib, yuqoridañ (quyidan) chegaralangan bo'lsa, u holda bunday ketma-ketlik limitga ega bo'ladi.

Bu xossa ketma-ketlik limitini hisoblashda keng qo'llaniladi.

**4-misol,**  $x_n = \left(1 + \frac{1}{n}\right)^n$  ketma-ketlik berilgan bo'lsin. Bu ketma-ketlik monoton o'suvchi va chegaralangan.  
Nyuton binomi formulasidan foydalanib topamiz:

$$x_n = \left(1 + \frac{1}{n}\right)^n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \dots + \frac{n(n-1)(n-2)\dots 1}{n!} \cdot \frac{1}{n^n}$$

yoki

$$x_n = 2 + \frac{1}{2} \left(1 - \frac{1}{n}\right) + \frac{1}{2 \cdot 3} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots + \frac{1}{2 \cdot 3 \cdot 4 \dots n} \left(1 - \frac{1}{n}\right) \dots \left(1 - \frac{n-1}{n}\right)$$

Bu tenglikdan ko'rindadiki:

$$2 < x_n = 2 + \frac{1}{2} \left(1 - \frac{1}{n}\right) + \frac{1}{2 \cdot 3} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \dots + \frac{1}{2 \cdot 3 \cdot 4 \cdots n} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right) <$$

$$< 2 + \frac{1}{2} + \frac{1}{3!} + \dots + \frac{1}{n!} < 2 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} < 2 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} + \frac{1}{2^n} + \dots = 2 + \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 3$$

Demak,  $x_n$  ketma-ketlik chegaralangan  $2 < x_n < 3$ .

Unda 3<sup>0</sup>-xossaga asosan  $\lim_{n \rightarrow \infty} x_n = \exists$ . Bu limit e deb belgilanadi va ushbu

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

tenglikka ikkinchi **ajovib limit** deb ataladi. e - irratsional son bo'lib,  $e \approx 2,718281\dots$  taqribiy tenglik o'rini.

e soni asos qilib olingan logarifmlarga **natural logarifmlar** deb ataladi va lnx kabi belgilanadi.

e soni matematik analiz va uning tatbiqida muhim o'rinn tutadi.

Misol uchun radioaktiv elementlarning xossalardan biri uning yemirilishidir. Faraz qilaylik,  $t=0$  da radiyning miqdori  $m_0$  bo'lsin. t vaqt o'tgandan so'ng radiy miqdori  $x$  ga teng bo'lsin. U holda radiyning yemirilish qonuni quyidagicha:

$$x = m_0 e^{-kt}$$

$k$  - bu yerda proporsionallik koefitsenti.

### Nazorat savollari.

1. Sonli ketma-ketliklar. Misollar.
2. Monoton ketma-ketliklar. Misollar.
3. Ketma-ketlik limitining xossalari.
4. e - soni.

## 2-§. Sonli qatorlar.

### 1. Sonli qatorlar.

Ketma – ketliklar nazariyasida maxsus ko'rinishdagi ketma – ketliklar, qatorlar muhim ahamiyatga ega.

**1-Ta'rif.** Aytaylik,  $\{a_n\}$  – haqiqiy sonlar ketma – ketligi berilgan bo'lsin. U holda

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n \quad (1)$$

ifoda **sonli qator** deb ataladi.

$a_n$  ( $n = 1, 2, 3, \dots$ ) son qator hadlari,  $a_n$  qatorning **umumiy hadi** deyiladi.

$$S_1 = a_1,$$

$$S_2 = a_1 + a_2,$$

$$S_3 = a_1 + a_2 + a_3,$$

.....

$$S_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k \quad (2)$$

Yig'indilarni tuzamiz. Bu yig'indilar qatorning **qismiy yig'indilari** deyiladi.

**2-Ta'rif.** Agar  $n \rightarrow \infty$  da (1) qatorning qismiy yig'indilaridan iborat  $\{S_n\}$  ketma – ketlik chekli limitga ega, ya'ni

$$\lim_{n \rightarrow \infty} S_n = S$$

bo'lsa, u holda (1) – qator **yaqinlashuvchi** deyiladi. Aks holda, ya'ni limiti cheksiz yoki limit mavjud bo'lmasa (1) qator **uzoqlashuvchi** deyiladi.

$S$  son (1) qatorning yig'indisi deyiladi va quyidagicha yoziladi:

$$S = a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n .$$

$$1-\text{misol. } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} + \dots$$

qatorni yaqinlashishga tekshiring.

Qatorning qismiy yig'indisi ta'rifga ko'ra:

$$S_1 = a_1 = \frac{1}{2}$$

$$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = a_1 + a_2 + a_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$$

$$S_n = a_1 + a_2 + \dots + a_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)}$$

$S_n$  ni quyidagicha yozib olamiz:

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$n \rightarrow \infty$  da limitga o'tamiz:

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Demak, berilgan qator yaqinlashuvchi, uning yig'indisi 1 ga teng.

**2-misol.**  $1 + 2 + 3 + \dots + n + \dots$   
qatorni yaqinlashishga tekshiring.

Ma'lumki, arifmetik progressiyaning dastlabki  $n$  ta hadining yig'indisini hisoblash formulasidan foydalanib, berilgan qatorning qismiy yig'indisi quyidagiga teng ekanini topamiz:

$$S_n = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Bundan,

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2} = \infty,$$

demak, qator uzoqlashuvchi.

**1—Teorema** ( qator yaqinlashishi uchun Koshi kriteriyasi ). (1) qator yaqinlashishi uchun  $\forall \varepsilon > 0$  son olinganda ham shunday  $n_0 \in \mathbb{N}$  son mavjud bo'lib, barcha  $n \geq n_0$  va  $m \geq n_0$  lar uchun

$$|S_m - S_n| = |a_{n+1} + a_{n+2} + \dots + a_m| < \varepsilon \quad (3)$$

*tengsizlikning bajarilishi zarur va yetarli.*

**Natija 1.** Agar sonli qatorning chekli sondagi hadlari o'zgartirilsa, u holda asl qatorning *yaqinlashishidan* (*uzoqlashishidan*) hosil bo'lgan yangi qator ham *yaqinlashadi* (*uzoqlashadi*).

**Natija 2.** (1) qatorning yaqinlashishi uchun

$$\lim_{n \rightarrow \infty} a_n = 0$$

tenglikning bajarilishi zarur.

**Isbot.** (3) shartni  $m = n+1$  bo'lganda qarash yetarli:

$$|S_m - S_n| = |a_{n+1}| < \varepsilon.$$

Bu tengsizlikdan quyidagi kelib chiqadi.

$$\lim_{n \rightarrow \infty} a_{n+1} = 0 \text{ yoki } \lim_{n \rightarrow \infty} a_n = 0.$$

Shunday qatorlar borki  $n \rightarrow \infty$  da  $a_n \rightarrow 0$ , lekin qator uzoqlashuvchi.

### 3-misol.

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$

qatorga **garmonik qator** deyiladi, chunki bu nom quyidagi shartning bajarilishi bilan bog'langan.

$$\frac{1}{a_n} = \frac{1}{2} \left( \frac{1}{a_{n-1}} + \frac{1}{a_{n+1}} \right),$$

bu yerda  $a_n$ ,  $a_{n-1}$  va  $a_{n+1}$  orasidagi **o'rta garmonik** deyiladi.

## 2. Musbat hadli qatorlar. Solishtirish teoremlari.

**3-Ta'rif.** Agarda  $\sum_{n=1}^{\infty} |a_n|$  qator yaqinlashsa,  $\sum_{n=1}^{\infty} a_n$  qator **absolyut yaqinlashadi** deyiladi.

**2—Teorema.** Agar qator absolyut yaqinlashsa, u holda qator yaqinlashadi.

**Isbot.** Quyidagi tengsizlik:

$$|a_{n+1} + a_{n+2} + \dots + a_m| \leq |a_{n+1}| + |a_{n+2}| + \dots + |a_m|,$$

va Koshi kriteriyasidan teoremaning o'rini bo'lishi kelib chiqadi.

### 3—Teorema.

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n,$$

manfiy bo'limgan hadlardan iborat qator yaqinlashuvchi bo'lishi uchun uning qismiy yig'indilaridan iborat ketma-ketlikning yuqoridan chegaralangan bo'lishi zarur va yetarli.

Bu teoremadan foydalanib quyidagi solishtirish teoremasini hosil qilamiz.

**4—Teorema.** Faraz qilaylik, hadlari manfiy bo'limgan ikkita qator berilgan bo'lsin:

$$a_1 + a_2 + a_3 + \dots + a_n + \dots \quad (A),$$

$$b_1 + b_2 + b_3 + \dots + b_n + \dots \quad (B).$$

Agar shunday  $n_0 \in \mathbb{N}$  nomer topilsaki, barcha  $n > n_0$  lar uchun

- $a_n \leq b_n$
- 1)  $a_n \leq b_n$  tengsizlik o'rini bo'lsa, u holda  
 1) (B) qator yaqinlashuvchi bo'lsa, (A) qator ham yaqinlashuvchi bo'ladi.  
 2) (A) qator uzoqlashuvchi bo'lsa, (B) qator ham uzoqlashuvchi bo'ladi.

#### 4-misol.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} + \dots \quad (4),$$

qatorni solishtirish alomatidan foydalaniб yaqinlashishga tekshiramiz.

Ushbu

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{(n-1)n} + \dots \quad (5),$$

qatorni yaqinlashishga tekshirish oson.

$$\begin{aligned} S_n &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n}, \\ \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left(2 - \frac{1}{n}\right) = 2. \end{aligned}$$

Demak, qator yaqinlashadi. Bundan va ushbu

$$\frac{1}{n^2} < \frac{1}{(n-1)n}, \quad n = 1, 2, \dots$$

tengsizlikdan solishtirish teoremasiga ko'ra (4) qator yaqinlashadi.

**5—Teorema (Absolyut yaqinlashuvchi qator uchun Veyershtrs alomati).** Faraz qilaylik, (A) va (B) qatorlar berilgan va (B) qator yaqinlashuvchi ((A) ixtiyoriy qator) bo'lsin.

Agar shunday  $n_0 \in \mathbb{N}$  nomer topilsaki, barcha  $n > n_0$  lar uchun

$$|a_n| \leq b_n$$

$|a_n| \leq b_n$  tengsizlik o'rинli bo'lsa, u holda (A) qator absolyut yaqinlashadi.

$$|a_n| \leq b_n$$

**5-misol.**  $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$  qator absolyut yaqinlashadi, chunki

$$\left| \frac{\sin n}{n^2} \right| \leq \frac{1}{n^2} \quad \forall n \in \mathbb{N} \text{ uchun} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ qator yaqinlashadi.}$$

### 3. Yaqinlashish alomatlari.

**Koshi alomati.** Agar (A) qatorning umumiyligi hadi  $a_n$  uchun biror  $n_0 \in N$  nomeridan boshlab, barcha  $n \geq n_0$  lar uchun

$$\sqrt[n]{a_n} \leq q < 1 \quad (\sqrt[n]{a_n} \geq 1)$$

o'rini bo'lsa, u holda  $\sum_{n=1}^{\infty} a_n$  qator yaqinlashuvchi (uzoqlashuvchi) bo'ladi.

Amaliy masalalarni hal qilishda ko'pincha, Koshi alomatining quyidagi limit ko'rinishidan foydalaniladi.  
Agar ushbu

$$\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = k$$

limit mavjud bo'lsa, u holda  $\sum_{n=1}^{\infty} a_n$  qator  $k < 1$  bo'lganda yaqinlashuvchi,  $k > 1$  bo'lganda esa uzoqlashuvchi bo'ladi.

$$6\text{-misol. } \sum_{n=1}^{\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

qatorni Koshi alomatidan foydalanib yaqinlashishga tekshiring.

$$a_n = \frac{1}{\left(1 + \frac{1}{n}\right)^n}, \quad \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{\left(1 + \frac{1}{n}\right)^n}} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e}$$

bo'ladi.  $\frac{1}{e} < 1$  bo'lgani uchun Koshi alomatiga ko'ra berilgan qator yaqinlashuvchi.

**Dalamber alomati.** Agar (A) qatorning  $a_n$  va  $a_{n+1}$  ( $n \geq n_0$ ) hadlari uchun

$$\frac{a_{n+1}}{a_n} = q < 1$$

bo'lsa,  $\sum_{n=1}^{\infty} a_n$  qator yaqinlashuvchi bo'ladi.

$$\frac{a_{n+1}}{a_n} = q \geq 1$$

*bo'lsa,  $\sum_{n=1}^{\infty} a_n$  qator uzoqlashuvchi bo'ladi.*

Dalamber alomatini ham limit ko'rinishida ifodalash mumkin.

*Agar ushbu*

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = q$$

*limit mavjud bo'lsa, u holda  $\sum_{n=1}^{\infty} a_n$  qator  $q < 1$  bo'lganda yaqinlashuvchi,  $q > 1$  bo'lganda esa uzoqlashuvchi bo'ladi.*

### 7-misol.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

qatorni Dalamber alomatidan foydalaniб yaqinlashishga tekshiring.

$$q = \lim_{n \rightarrow \infty} \frac{(n+1)n^n}{(n+1)^{n+1} n!} = \lim_{n \rightarrow \infty} \frac{(n+1)}{n+1} \left(\frac{n}{n+1}\right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e}.$$

Shunday qilib,  $q < 1$  va qator yaqinlashadi.

### 4-Ta'rif.

$$a_1 - a_2 + a_3 - \dots + (-1)^{n-1} a_n + \dots \quad (6)$$

ko'rinishda bo'lib,  $a_n > 0$  bo'lsa, u holda bunday qatorga hadlarining ishorasi almashinib keluvchi qator deyiladi.

*6-Teorema (Leybnits alomati).* Agar

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

qator berilgan bo'lib,

- 1)  $\{a_n\} \downarrow$ , ya'ni  $a_n \geq a_{n+1} > 0$  ( $n = 1, 2, \dots$ )
- 2)  $\lim_{n \rightarrow \infty} a_n = 0$

*bo'lsa, u holda (6) qator yaqinlashuvchi bo'ladi.*

### 8-misol.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + (-1)^{n-1} \frac{1}{n} + \dots$$

qatorni yaqinlashishga tekshiring.

Berilgan qatorning hadlari absolyut qiymat bo'yicha monoton kamayadi:

$$1 > \frac{1}{2} > \frac{1}{3} > \dots,$$

uning umuiy hadi esa  $n \rightarrow \infty$  da nolga intiladi:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^{n-1} \frac{1}{n} = 0.$$

Qator Leybnits alomatiga ko'ra yaqinlashadi.

### Nazorat savollari.

- 1) Sonli qator deb nimaga aytildi va u qachon yaqinlashadi.
- 2) Qator yaqinlashishi uchun Koshi kriteriyasi.
- 3) Musbat hadli qatorlar.
- 4) Solishtirish teoremlari. Misol.
- 5) Yaqinlashish alomatlari (Koshi, Dalamber va Leybnits).

### 3-§. Funktsiya limiti.

Ushbu

$$U_\varepsilon(a) = \{x : x \in R, a - \varepsilon < x < a + \varepsilon\}$$

to'plam  $x = a$  nuqtaning atrofi ( $\varepsilon$ -atrofil) deb ataladi.

Faraz qilaylik,  $f(x)$  funktsiya biror  $X$  oraliqida aniqlangan bo'lib,  $x = a$  nuqta biror  $X$  oraliqning limit nuqtasi bo'lsin ya'ni  $a$  nuqtaning ixtiyoriy  $U_\varepsilon(a)$  atrofi olinganda

ham  $U_\varepsilon(a) \setminus \{a\}$  to'plamda  $x$  ning kamida bitta nuqtasi mavjud bo'lsin.

**1-ta'rif.** Agar  $\forall \varepsilon > 0$  uchun  $\exists \delta(\varepsilon, a) = \delta > 0$ :  $\forall x \in X \cap \{0 < |x - a| < \delta\}$ , uchun  $|f(x) - b| < \varepsilon$  tengsizlik bajarilsa, u holda  $b$  soni  $f(x)$  funktsiyaning  $x \rightarrow a$  dagi limiti deb ataladi va  $\lim_{x \rightarrow a} f(x) = b$  kabi belgilanadi.

**1-misol.**  $\lim_{x \rightarrow 1} (2x + 3) = 5$  ushbu tenglikni ta'rif yordamida isbotlang.

$$f(x) = 2x + 3, \quad b = 5, \quad a = 1, \quad \forall \varepsilon > 0 \text{ olamiz.}$$

$$|f(x) - b| = |2x + 3 - 5| = |2x - 2| = 2|x - 1| < 2\delta = \varepsilon \Rightarrow \delta = \frac{\varepsilon}{2}$$

**2-ta'rif.** Agar  $\forall \varepsilon > 0$  uchun  $\exists E > 0$ :  $\forall |x| > E \quad |f(x) - b| < \varepsilon$  bo'lsa, unda  $\lim_{x \rightarrow \infty} f(x) = b$  deb ataladi.

**2-misol.**  $\lim_{x \rightarrow \infty} \frac{x}{x-1} = 1$  ushbu tenglikni ta'rif yordamida isbotlang.

$$f(x) = \frac{x}{x-1}, \quad b = 1, \quad \forall \varepsilon > 0 \text{ olamiz,}$$

$$|f(x) - b| = \left| \frac{x}{x-1} - 1 \right| = \frac{1}{|x-1|} = (|x| > E \Rightarrow |x-1| > E-1) < \frac{1}{E-1} = \varepsilon \Rightarrow E = 1 + \frac{1}{\varepsilon}.$$

$$\text{Bundan kelib chiqadiki } \lim_{x \rightarrow \infty} \frac{x}{x-1} = 1.$$

**3-ta'rif.** Agar  $\lim_{x \rightarrow a} f(x) = 0$  ( $\infty$ ) bo'lsa, unda  $x \rightarrow a$  da  $f(x)$  funktsiya **cheksiz kichik (katta) funktsiya** deb ataladi.

Masalan,  $x \rightarrow 1$  da  $f(x) = x-1$  cheksiz kichik,  $\varphi(x) = \frac{1}{x-1}$  funktsiya esa **cheksiz katta funktsiya** bo'ladi.

### Xossalalar:

**1<sup>0</sup>.**  $\alpha_1(x)$  va  $\alpha_2(x)$  funktsiyalar cheksiz kichik bo'lsa, unda  $\alpha_1(x) \pm \alpha_2(x)$  ham cheksiz kichik bo'ladi.

**2<sup>0</sup>.** O'zgarmas sonni cheksiz kichikka ko'paytmasi cheksiz kichik bo'ladi.

**3<sup>0</sup>.** Chegaralangan funktsiyaning cheksiz kichik funktsiyaga ko'paytmasi yana cheksiz kichik bo'ladi.

**4<sup>0</sup>.** Chekli sondagi cheksiz kichiklarning ko'paytmasi cheksiz kichik bo'ladi.

**5<sup>0</sup>.** Agar  $f(x)$  cheksiz katta bo'lsa, unda  $\frac{1}{f(x)}$  cheksiz kichik bo'ladi. ( $f(x) \neq 0$ ).

**6<sup>0</sup>.**  $\alpha(x)$  cheksiz kichik bo'lsa, unda  $\frac{1}{\alpha(x)}$  cheksiz katta bo'ladi ( $\alpha(x) \neq 0$ ).

Endi funktsiyaning limiti haqidagi asosiy teoremlarni keltiramiz.

**1-teorema.**  $\lim_{x \rightarrow a} f(x) = b$  bo'lishi uchun  $f(x) = b + \alpha(x)$  bo'lishi zarur va yetarlidir, bu yerda  $\alpha(x)$  cheksiz kichik funktsiya.

**Isbot.** Zaruriyligi: faraz qilaylik  $\lim_{x \rightarrow a} f(x) = b$  bo'lsin. Bu degani  $\forall \varepsilon > 0 \exists \delta > 0, \forall x \quad 0 < |x - a| < \delta \Rightarrow |f(x) - b| < \varepsilon$ , ya'ni  $\alpha(x) = f(x) - b$  funktsiya cheksiz kichik miqdor va  $f(x) = b + \alpha(x)$ .

**Yetarliligi:** faraz qilaylik  $f(x) = b + \alpha(x)$ , bu yelda  $\alpha(x) =$  cheksiz kichik miqdor. U holda  $\forall \varepsilon > 0 \exists \delta > 0, 0 < |x - a| < \delta$  dan olingan  $x$  lar uchun  $|\alpha(x)| = |f(x) - b| < \varepsilon$  o'rini, ya'ni  $b$  soni  $f(x)$  funktsiyaning  $x \rightarrow a$  dagi limiti.

**2-teorema.** Agar  $\forall x \in U_\delta(a) = (a - \delta, a + \delta) \setminus \{a\}$  uchun  $f(x) \geq 0$  ( $f(x) \leq 0$ ) bo'lib,  $\lim_{x \rightarrow a} f(x)$  mavjud bo'lsa, u holda  $\lim_{x \rightarrow a} f(x) \geq 0$  ( $\lim_{x \rightarrow a} f(x) \leq 0$ ) bo'ladi.

**Izoh.** Agar 2-teoremada qat'iy tengsizlik qo'yilsa, u o'rinni bo'lmay qolishi mumkin. Masalan,  $f(x) = |x|$  va  $a = 0$  bo'lsin.  $\forall x \in U_\delta(0)$  uchun  $f(x) > 0$  va  $\lim_{x \rightarrow a} f(x) = \exists$ , lekin  $\lim_{x \rightarrow a} f(x) = 0$ .

**3-teorema.** Agar  $f(x)$  va  $g(x)$  funktsiyalar  $x = a$  nuqtada chekli limitga ega bo'lsa, u holda shu nuqtada

$$a) f(x) \pm g(x),$$

$$b) f(x) \cdot g(x),$$

$$c) \lim_{x \rightarrow a} g(x) \neq 0$$

bo'lganda,  $\frac{f(x)}{g(x)}$  funktsiyalar ham chekli limitga ega bo'lib,

*ushbu*

1.  $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x),$
2.  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x),$
3.  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \quad (\lim_{x \rightarrow a} g(x) \neq 0)$

*tengliklar o'rini bo'ladi.*

**1-natija.**  $\lim_{x \rightarrow a} [f(x)]^n = [\lim_{x \rightarrow a} f(x)]^n$

**2-natija.**  $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$

**4-teorema.** Agar  $a$  nuqtaning biror atrofida  $f_1(x) \leq f(x) \leq f_2(x)$  bo'lib,  $\lim_{x \rightarrow a} f_1(x) = \lim_{x \rightarrow a} f_2(x) = b$  bo'lsa, u holda  $\lim_{x \rightarrow a} f(x) = b$  bo'ladi.

**I'sbot.** Teorema shartidan  $a$  nuqtaning  $0 < |x - a| < \delta$  atrofida bir vaqtning o'zida quyidagi tengsizlik bajariladi:

$$|f_1(x) - b| < \varepsilon, \quad |f_2(x) - b| < \varepsilon,$$

bu yerda  $\varepsilon$  – ixtiyoriy musbat son. Bundan quyidagini yozib olamiz:

$$b - \varepsilon < f_1(x) < b + \varepsilon, \quad b - \varepsilon < f_2(x) < b + \varepsilon.$$

Bu tengsizliklardan quyidagini yozib olamiz:

$$b - \varepsilon < f(x) \leq f(x) \leq f_2(x) < b + \varepsilon.$$

Yuqoridagi tengsizliklardan

$$b - \varepsilon < f(x) < b + \varepsilon \quad \text{yoki} \quad |f(x) - b| < \varepsilon$$

kelib chiqadi.

Ushbu,  $\lim_{x \rightarrow a} \frac{\sin x}{x} = 1$  va  $\lim_{x \rightarrow a} \left(1 + \frac{1}{x}\right)^x = e$  limitlarga mos ravishda birinchi va ikkinchi **ajoyib limitlar** deyiladi. Bu limitlar limit

hisoblashda,  $\frac{0}{0}$  va  $1^\infty$  ko'rinishdagi aniqmasliklarni ochishda keng qo'llaniladi.

**3-misol.**  $\lim_{x \rightarrow 0} \frac{\sqrt{4-x} - \sqrt{4+x}}{x}$  limitni toping.

$x \rightarrow 0$  da limit  $\frac{0}{0}$  ko'rinishga keladi. Aniqmaslikdan qutilish uchun kasr suratini qo'shmasiga ko'paytirib bo'lamiz.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{(\sqrt{4-x} - \sqrt{4+x})(\sqrt{4-x} + \sqrt{4+x})}{x(\sqrt{4-x} + \sqrt{4+x})} &= \lim_{x \rightarrow 0} \frac{4-x - 4-x}{x(\sqrt{4-x} + \sqrt{4+x})} = \\ &= -2 \lim_{x \rightarrow 0} \frac{1}{\sqrt{4-x} + \sqrt{4+x}} = -\frac{1}{2}\end{aligned}$$

**4-misol.**  $\lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \cdot \sin x \right) = \lim_{x \rightarrow 0} \frac{\sin x}{x} \lim_{x \rightarrow 0} \sin x = 1 \cdot 0 = 0.$

**5-misol.**

$$\lim_{x \rightarrow \infty} \left( 1 - \frac{1}{x^2} \right)^x = (1^\infty) = \lim_{x \rightarrow \infty} \left[ \left( 1 + \frac{-1}{x^2} \right)^{-x^2} \right]^{\frac{x}{x^2}} = e^{\lim_{x \rightarrow \infty} \left( \frac{-x}{x^2} \right)} = e^{\lim_{x \rightarrow \infty} -\frac{1}{x}} = e^0 = 1.$$

**6-misol.**

$$\begin{aligned}\lim_{x \rightarrow \infty} (1 - \sin x)^{\frac{1}{x}} &= (1^\infty) = \lim_{x \rightarrow 0} \left\{ [1 + (-\sin x)]^{\frac{1}{-\sin x}} \right\}^{\frac{\sin x}{x}} = e^{\lim_{x \rightarrow 0} \left( \frac{-\sin x}{x} \right)} = \\ &= e^{\lim_{x \rightarrow 0} \frac{\sin x - x}{x}} = e^{-1 \cdot 0} = e^0 = 1.\end{aligned}$$

**7-misol.**

$$\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\cos x} = (1^\infty) = \lim_{x \rightarrow \frac{\pi}{2}} (1 - \cos^2 x)^{\frac{\cos x}{2}} = \lim_{x \rightarrow \frac{\pi}{2}} \left[ (1 - \cos^2 x)^{\frac{-1}{\cos^2 x}} \right]^{\frac{\cos^2 x \cos x}{2}} =$$

$$= e^{\frac{1}{2} \lim_{x \rightarrow \frac{\pi}{2}} \cos x \sin x} = e^0 = 1$$

## Nazorat savollari.

1. Funktsiya limiti.
2. Cheksiz katta (kichik) funktsiyalar.
3. Funktsiya limitining xossalari.
4. Birinchi va ikkinchi ajoyib limitlar.

### 4-§. Funktsiyaning uzluksizligi

Aytaylik,  $y = f(x)$  funktsiya  $x_0$  nuqtaning biror atrofida aniqlangan bo'lib,  $x$  nuqta shu intervalga (atrofga) tegishli nuqta bo'lsin.

$x - x_0 = \Delta x$  deb belgilaymiz  $\Rightarrow$

$x = x_0 + \Delta x$ ,  $\Delta x$  — argument orttirmasi.

$\Delta y = f(x_0 + \Delta x) - f(x_0)$  — funktsiya orttirmasi.

**1-ta'rif:** Agar  $\lim_{\Delta x \rightarrow 0} \Delta y = 0$  bo'lsa, unda  $y = f(x)$  funktsiya  $x = x_0$  nuqtada uzluksiz deyiladi.

**2-ta'rif:** Agar  $\lim_{\Delta x \rightarrow 0} f(x) = f(x_0)$  bo'lsa, u holda  $y = f(x)$  funktsiya  $x = x_0$  nuqtada uzluksiz deyiladi.

**Misollar:** 1)  $y = x$ , 2)  $y = x^3$ , 3)  $y = \sin x$  funktsiyalar  $\forall x_0 \in R$  nuqtalarda uzluksiz.

**1-teorema:** Agar  $f(x), g(x)$  funktsiyalar  $x_0$  nuqtada uzluksiz bo'lsa, u holda

1)  $f(x) \pm g(x)$ , 2)  $f(x) \cdot g(x)$ , 3)  $f(x)/g(x)$  ( $g(x) \neq 0$ ),

funktsiyalar ham  $x_0$  nuqtada uzluksiz bo'ladi.

**2-teorema:** Agar  $f(x)$  funktsiya  $x_0$  nuqtada uzluksiz bo'lib,  $f(x_0) \neq 0$  bo'lsa, u holda  $f(x)$  funktsiya  $x_0$  nuqtaning biror atrofida bir xil ishorani saqlaydi,  $f(x) > 0$  ( $f(x) < 0$ ) bo'lsa,  $\exists U_\delta(x_0)$  atrof:  $\forall x \in U_\delta(x_0)$  uchun  $f(x) < 0$  ( $f(x) > 0$ ) bo'ladi.

**3-ta'rif:** Funktsiyaning uzluksizligi buziladigan nuqtalarga uning uzilish nuqtalari deb ataladi.

Uzilish nuqtalari uch turga bo'linadi:

1)  $\lim_{x \rightarrow a} f(x) = b \neq f(a)$  bo'lsin.

Bu holda  $\lim_{x \rightarrow a+0} f(x) = f(a+0)$  va  $\lim_{x \rightarrow a-0} f(x) = f(a-0)$  lar mavjud bo'lib,  $f(a+0) = f(a-0) \neq f(a)$  bo'ladi. Bunday nuqta bartaraf qilish mumkin bo'lgan uzilish nuqtasi deb ataladi.

**Misol.**  $f(x) = \begin{cases} x^2, \text{azap} & x \neq 0 \\ 1, \text{azap} & x = 0 \end{cases}$  булса

Funktsiya uchun  $x = 0$  nuqta bartaraf qilish mumkin bulgan uzilish nuqtasi bo'ladi, chunki  $\lim_{x \rightarrow a+0} f(x) = \lim_{x \rightarrow a-0} f(x) = 0$  va  $f(0) = 1$ . Agar  $f(0) = 0$  deb qabul qilsak, funktsiya uzluksiz bo'lib qoladi.

2)  $\lim_{x \rightarrow a+0} f(x)$ - mavjud bo'lmasin.

Bunda quyidagi uchta hol bo'lishi mumkin.

a)  $\lim_{x \rightarrow a-0} f(x) = f(a-0)$  va  $\lim_{x \rightarrow a+0} f(x) = f(a+0)$  lar mavjud va  $f(a-0) \neq f(a+0)$ .

Funktsiyaning bunday nuqtadagi uzilishi birinchi tur uzilish va  $|f(a-0) - f(a+0)|$  ayirmaga funktsiyaning  $a$  nuqtadagi sakrashi deyiladi.

b)  $x \rightarrow a$  da  $f(x)$  funktsiyaning o'ng va chap limitlaridan hech bo'limaganda biri mavjud bo'lmasin. Funktsiyaning  $a$  nuqtadagi bo'nday uzilishi ikkinchi tur uzilish deyiladi.

**Misol.**  $f(x) = \begin{cases} \sin \frac{1}{x}, \text{azap} & x > 0 \\ -x, \text{azap} & x \leq 0 \end{cases}$

Funktsiya  $x = 0$  nuqtada ikkinchi tur uzilishga ega, chunki

$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (-x) = 0 = f(0)$ , lekin  $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \sin \frac{1}{x}$  - mavjud emas.

b)  $x \rightarrow a$  da  $f(x)$  funktsiyaning o'ng va chap limitlaridan biri cheksiz yoki o'ng va chap limitlar turli ishorali cheksiz. Funktsiyaning  $a$  nuqtadagi bo'nday uzilishi ham ikkinchi tur uzilish deyiladi.

Agar  $\lim_{x \rightarrow x_0+0} f(x) = f(x_0)$  ( $\lim_{x \rightarrow x_0-0} f(x) = f(x_0)$ ) bo'lsa, unda  $y = f(x)$  funktsiya  $x_0$  nuqtalarda chapdan (o'ngdan) uzluksiz deb ataladi.

Agar  $f(x)$  funktsiya  $[a, b]$  kesmada aniqlangan bo'lib,  $(a, b)$  intervalda uzluksiz hamda  $a$  nuqtada o'ngdan,  $b$  nuqtada esa

chapdan uzluksiz bo'lsa, u holda  $f(x)$  funktsiya  $[a,b]$  kesmada **uzluksiz** deb ataladi va  $f(x) \in C[a,b]$  kabi belgilanadi.

Endi kesmada uzluksiz bo'lgan funktsiyalarining xossalarini keltiramiz:

**1<sup>0</sup>.** Agar  $f(x) \in C[a,b]$  bo'lib, kesmaning chetki nuqtalarida turli ishoralarini qabul qilsa,  $\exists c \in (a,b)$  nuqtalarda  $f(c)=0$  bo'ladi.

**2<sup>0</sup>.**  $f(x) \in C[a,b] \Rightarrow f(x) = \text{chegaralangan}$ , ya'ni  $\exists M$  son:  $\forall x \in [a,b]$  uchun  $|f(x)| \leq M$  bo'ladi.

**3<sup>0</sup>.**  $f(x) \in C[a,b] \Rightarrow \exists x_1, x_2 \in [a,b]$  nuqtalar:  $f(x_1)$  va  $f(x_2)$  lar mos ravishda  $f(x)$  funktsiyaning  $[a,b]$  kesmadagi eng katta va eng kichik qiymatlari bo'ladi.

**Izoh.** Xossalardagi oraliqlarning yopiq kesmadan iborat bo'lishlik sharti muhim shartdir.

Masalan, **2<sup>0</sup>** – xossada oraliq  $(0,1)$  intervaldan iborat bo'lib,  $f(x) = \frac{1}{x}$  bo'lsin.  $f(x) = \frac{1}{x} \in C(0,1)$ , lekin funktsiya  $(0,1)$  oraliqda chegaralangan emas.

### **Uzluksiz funktsiyaning biologik masalalarda qo'llanilishi.**

Ba'zi biologik hodisalar uzluksiz va uzilishga ega bo'lgan funktsiyalar yordamida ifodalanadi.

Shuni ta'kidlash kerakki eshitish, ko'rish, ultra tovushlarni eshitish tebranma harakatlar bilan bog'langan bo'lib, bu harakatlar sinx, cosx kabi trigonometrik funktsiyalar yordamida ifodalanadi.

Aniq tajribalarda populyatsiya biomassasi, populyatsiya miqdori, yo'l, temperatura, vaqt va hokazolar har qanday haqiqiy sonlarni qabul qilmasligi mumkin. Masalan, yo'l kilometr yoki millimetrdan, biomassa tonnada yoki milligrammda, vaqt yillar yoki o'ndan bir sekundlarda o'lchanadi. Shartli ravishda kelishilganda keltirilgan funktsiyalarining qiymatlar va aniqlanish sohalari oraliqlar bo'lmashdan, balki juda kichik maxsus to'plamlardir. Juda katta hayvonlarning yoshi yillar bilan, ba'zi mikroorganizmlarning yoshi sekundning o'ndan biri bilan o'lchanadi. Mikroorganizmlar 3,1 yoki 3,2 sekund yashaydi deya olishimiz mumkin, lekin  $\pi$  sekund yashadi deb ayta olmaymiz. Bunday funktsiyalar to'g'risida gapirganda uzluksiz funktsiya haqida gapirib bo'lmaydi. Matematik analiz

tushunchalaridan foydalanganda funktsiyaning aniqlanish sohasi ikkita elementdan iborat bo'lsa uni oraliq bilan almashtirish maqsadga muvofiq emas. Biroq funktsiyaning aniqlanish va qiymatlar sohasi cheklita, lekin qandaydir ma'noda yetarlicha katta miqdordagi bir-biriga yaqin joylashgan elementlardan iborat bo'lsa (ya'ni o'lchami juda kichik oraliqlarga bo'lingan bo'lsa), u holda bo'nday funktsiyaning Aniqlanish sohasini shartli ravishda oraliq bilan almashtirsa bo'ladi. Bu keltirgan izoh uzluksiz funktsiyadan foydalanish uchun keltirilgan matematik modeldir. Qaralayotgan funktsiyalarini uzluksiz deb faraz qilamiz va bundan buyon bu haqda eslatmaymiz.

Biologiyada uzluksiz funktsiyaga doir misolni qaraymiz. Hujayralar bo'linishidagi mikroorganizmlar o'sishini o'rjanayotganda  $f(t) = ae^t$  kabi funktsiyalarga duch kelamiz (bu yerda argument  $t$  vaqtini aniqlaydi).

Darajali funktsiya  $f(x) = Ax^\alpha - hayvon$  og'irligidan asosiy intensiv almashinuvning bog'liqligini ifodalaydi. Bu yerda  $x$  — hayvon og'irligi,  $f(x)$  — birlik vaqt ichida yutiladigan kislorod miqdori,  $A$  va  $\alpha$  — tirik mavjudotning turiga qarab aniqlanadigan o'zgarmas parametrlar. Misol uchun qushlarda  $A = 70$ ,  $\alpha = 0,74$  ga, baliqlarda  $A = 0,3$ ,  $\alpha = 0,8$  ga teng.

Uzilishga ega bo'lgan funktsiya uchun misol keltiramiz. Tashqi ta'sirdan qo'zg'aladigan ho'jayralarni qaraylik. Misol uchun nerv hujayralari, mushak hujayralari va boshqalar.  $t_0$  vaqtida hujayra signal qabul qiladi. Biroq ta'sir birmuncha keyinroq bo'ladi, ya'ni  $t_1 > t_0$ .  $[t_0, t_1]$  kesma latent davri deyiladi.  $t_1$  vaqtida hujayra oniy qo'zg'aladi va eng yuqori qiymatga erishadi, so'ngra asta sekin kamayib toki boshqa ta'sir bo'limguncha nolga yaqinlashib boradi.

Shunday qilib harakat miqdorining funktsiyasi  $t$  ga bog'liq bo'lib, latent davrinining oxirida uzilishga ega bo'lar ekan.

## Nazorat savollari.

1. Funktsiya uzluksizligining ta'rifi.
2. Funktsiya uzelish nuqtalari.
3. Uzluksiz funktsiyalarning xossalari.
4. Uzluksiz funktsiyaning biologik masalalarda qo'llanilishi.

## 5-§. Hosila. Elementar funktsiyalarning hosilalari.

Tabiatdagi ko'p masalalar hosila tushunchasiga olib keladi. Masalan, harakatdagi nuqtaning tezligi haqidagi masala, egri chiziqqa o'tkazilgan urinma masalasi va boshqalar.

*Ximik reaktsiya tezligi haqidagi masala.* Aytaylik,  $m = m(t)$  funktsiya berilgan bo'lsin, bu yerda  $m - t$  vaqtdagi ximiyaviy reaktsiyaga kirishgan modda miqdori.  $\Delta t$  orttirmaga  $m$  miqdorning  $\Delta m$  orttirmasi mos keladi.  $\frac{\Delta m}{\Delta t}$  nisbat  $\Delta t$  vaqtdagi ximik reaktsiyaning o'rtacha tezligi. Bu nisbatning  $\Delta t \rightarrow 0$  dagi limiti  $t$  vaqtdagi ximik reaktsiya tezligini beradi.

Aytaylik,  $y = f(x)$  funktsiya  $(a, b)$  oraliqda aniqlangan bo'lib,

$x \in (a; b)$  bo'lsin.  $x$  nuqtaga  $\exists \Delta x \begin{cases} \leq 0 \\ \geq 0 \end{cases}$  orttirma beraylikki,  $x + \Delta x \in (a, b)$  bo'lsin. Unda funktsiya ham  $\Delta y = f(x + \Delta x) - f(x)$  orttirma qabul qiladi.

Agar

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

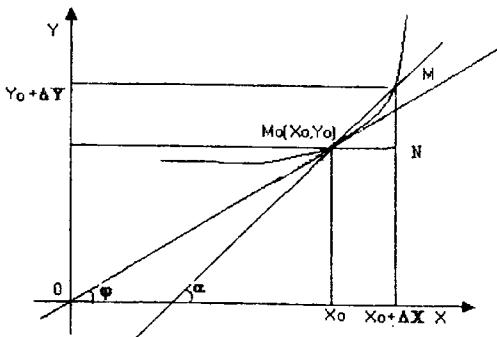
mavjud bo'lib, chekli songa teng bo'lsa, shu songa  $y = f(x)$  funktsiyaning  $x$  nuqtadagi hosilasi deb ataladi va u  $f'(x)$ ,  $\frac{df}{dx}$ ,  $y'$  — kabi belgilanadi.

Shunday qilib,

$$f'(x) := \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (1)$$

**Misollar.** 1)  $(C) = 0$ , 2)  $(x) = 1$ , 3)  $(x^3 - 1) = 3x^2$

### Hosilaning geometrik ma'nosi.



Quyidagi belgilashlarni kiritamiz:

$$\angle MM_0N = \alpha$$

$$\operatorname{tg} \alpha = \frac{MN}{M_0N} = \frac{\Delta y}{\Delta x}$$

$$\operatorname{tg} \varphi = \lim_{\Delta x \rightarrow 0} \operatorname{tg} \alpha = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(x) \quad (2)$$

$$y - y_0 = f'(x_0) \cdot (x - x_0) \quad (3) \text{ — urinma tenglamasi.}$$

$$y - y_0 = -\frac{1}{f'(x_0)} \cdot (x - x_0) \quad (4) \text{ — normal tenglamasi.}$$

**1-teorema:** Agar  $y = f(x)$  funktsiya biror  $x$  nuqtada differentsiyalanuvchi bo'lsa, u holda u shu  $x$  nuqtada uzlucksiz bo'ladi.

**Izoh.** Teoremaning teskarisi har doim ham o'rinli bo'lavermaydi. Masalan,  $y = |x|$  funktsiya  $x = 0$  nuqtada uzluksiz, lekin differentialsallanuvchi emas.

Chunki,

$$f'(0) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{|\Delta x|}{\Delta x} = \begin{cases} 1, & \text{azap } \Delta x > 0 \\ -1, & \text{azap } \Delta x < 0 \end{cases} \Rightarrow f'(0)$$

mavjud emas.

**2-teorema.** Aytaylik,  $u = u(x)$  va  $v = v(x)$  funktsiyalar  $x = 0$  nuqtada differentialsallanuvchi bo'lib, chekli  $u'(x)$  va  $v'(x)$  hosilalarga ega bo'lsin.

Unda,

$$1) [u \pm v] = u' \pm v',$$

$$2) [u \cdot v] = u'v \pm uv',$$

$$3) \left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2} \quad (v \neq 0)$$

bo'ladi.

**Natija:**  $[C \cdot f(x)]' = C \cdot f'(x)$

**3-teorema.** Aytaylik,  $y = f(u)$  bo'lib,  $u = \varphi(x)$  bo'lsin. Agar  $u = \varphi(x)$  funktsiya  $x$  nuqtada  $\varphi'(x)$  hosilaga,  $y = f(u)$  funktsiya esa  $x$  nuqtaga mos  $u = \varphi(x)$  nuqtada  $\varphi'(x)$  hosilaga ega bo'lsa, u holda  $y = f[\varphi(x)] = F(x)$  murakkab funktsiya ham  $x$  nuqtada hosilaga ega bo'ladi va

$$y' = F'(x) = f'(u) \cdot \varphi'(x) \quad (5)$$

tenglik o'rinli bo'ladi.

(5) — tenglikka **murakkab funktsiyaning hosilasini hisoblash formulasi** deyiladi.

Endi  $y = f(x)$  funktsiya va unga teskari bo'lgan  $x = f^{-1}(y) = \varphi(y)$  funktsiyalar berilgan bo'lsin.

**4-teorema:** Agar  $y = f(x)$  va  $x = \varphi(y)$  funktsiyalar differentialsallanuvchi bo'lsa, unda  $\varphi'(y) = \frac{1}{f'(x)}$  yoki  $x'_y = \frac{1}{y'}$ . (6) formulalar o'rinli bo'ladi.

2 – 4 – teoremlarini qo'llash natijasida quyidagi hosilalar jadvalini hosil qilamiz.

$$1) (C)' = 0,$$

$$2) (C \cdot u)' = C \cdot u',$$

$$3) (u + v)' = u' + v',$$

$$4) (u \cdot v)' = u'v + uv',$$

$$5) \left( \frac{u}{v} \right)' = \frac{u'v - uv'}{v^2},$$

$$6) (x')' = \frac{1}{y'_x},$$

$$7) (u^\alpha)' = \alpha \cdot u^{\alpha-1} \cdot u',$$

$$8) (a^x)' = a^x \cdot u' \cdot \ln a, \quad (a > 0)$$

$$9) (e^u)' = e^u \cdot u',$$

$$10) (\log_a u)' = \frac{u'}{u}, \quad (a > 0, a \neq 1)$$

$$11) (\ln u)' = \frac{u'}{u},$$

$$12) (\sin u)' = u' \cos u,$$

$$13) (\cos u)' = -u' \sin u,$$

$$14) (\operatorname{tg} u)' = \frac{u'}{\cos^2 u},$$

$$15) (\operatorname{ctg} u)' = \frac{-u'}{\sin^2 u},$$

$$16) (\arcsin u)' = \frac{u'}{\sqrt{1-u^2}},$$

$$17) (\arccos u)' = -\frac{u'}{\sqrt{1-u^2}},$$

$$18) (\operatorname{arctg} u)' = \frac{u'}{1+u^2},$$

$$19) (\operatorname{arcctg} u)' = -\frac{u'}{1+u^2}.$$

Bu jadvalda  $u = u(x)$  va  $v = v(x)$  lar differentsiyalanuvchi funktsiyalar.

Agar funktsiya orttirmasini ushbu,

$$\Delta f(x) = f(x + \Delta x) - f(x) = A(x) \cdot \Delta x + \alpha(x, \Delta x) \cdot \Delta x \quad (7)$$

ko'rinishda ifodekash mumkin bo'lib, bu yerda  $A$  – o'zgarmas son va  $\lim_{\Delta x \rightarrow 0} \alpha(x, \Delta x) = 0$  bo'lsa unda  $y = f(x)$  funktsiya  $x$  nuqtada **differensiallanuvchi** deyiladi va  $A \cdot \Delta x$  ga funktsiya ortirmasining **chiziqli bosh qismi** deyiladi, hamda  $df(x)$  nomi belgilanadi.

(7) tenglikdan,  $df(x) = A \cdot \Delta x = f'(x) \cdot \Delta x$  ekanligini ko'rish qiyin emas. Agar bu tenglikda  $\Delta x = dx$  desak, u holda

$$df(x) = f'(x)dx \quad (8)$$

tenglikni hosil qilamiz.

(7) va (8) dan

$$\Rightarrow \Delta y = dy + \alpha \cdot \Delta x \Rightarrow \lim_{\Delta x \rightarrow 0} \Delta y = \lim_{x \rightarrow 0} (dy + \alpha \cdot \Delta x) = \lim_{\Delta x \rightarrow 0} dy \Rightarrow \Delta y \quad \Delta y \approx dy = f'(x) \cdot \Delta x \Rightarrow \\ \Rightarrow f(x + \Delta x) \approx f(x) + f'(x) \cdot \Delta x \quad (9)$$

(9) – taqribiy hisoblash formulasi.

**1-misol.**  $\sqrt{1,1}$  ni hisoblang.

Funktsiyani  $f(x) = \sqrt{x}$  deb,  $f'(x) = \frac{1}{2\sqrt{x}}$ ,  $x = 1$ ,  $\Delta x = 0,1$  kabi olamiz. (9) formuladan foydalanamiz:

$$\sqrt{1,1} \approx \sqrt{1} + \frac{1}{2\sqrt{1}} \cdot 0,1 = 1 + 0,1 \cdot \frac{1}{2} = 1,033.$$

Jadvalga ko'ra  $\sqrt{1,1} \approx 1,032$  ga teng.

$y = f(x)$  funktsiyaning birinchi tartibli hosilasi  $y' = f'(x)$  dan olingan hosilaga (agar u mayjud bo'lsa)  $y = f(x)$  funktsiyaning **ikkinchchi tartibli hosilasi** deb ataladi va

$$y'', f''(x), \frac{d^2 f}{dx^2}$$

belgilarni biri yordamida belgilanadi. Funktsiyaning  $n$ -tartibli hosilasi quyidagi tenglik bilan aniqlanadi:

$$y^{(n)} := [f^{(n-1)}(x)]'$$

Funktsiyaning yuqori tartibli differentialsallari ham shu kabi aniqlanadi.

### Nazorat savollari.

1. Funktsiya hosilasining ta'rifi.
2. Hosilaning geometrik ma'nosi.
3. Hosila jadvali.
4. Hosila yordamida taqribiy hisoblash.

## 6-§. Differentsial hisobning asosiy xossalari.

**1-teorema. (Ferma):** Agar  $(a, b)$  da berilgan  $y = f(x)$  funktsiya shu intervalning birorta ichki s nuqtasida erag katta yoki eng kichik qiymatiga erishsa va  $f'(c) - \exists$  bo'lsa, unda  $f'(c) = 0$  bo'ladi.

**Isbot.** Aytaylik,  $y = f(x)$  funktsiya c nuqtada ( $c \in (a, b)$ ) o'zining eng katta qiymatiga erishsin. c qiymatga yetarlicha kichik  $\Delta x$  orttirma beramiz. U holda  $f(c + \Delta x) < f(c)$ . Agar  $\Delta x < 0$  bo'lsa,

$$\frac{\Delta y}{\Delta x} = \frac{f(c + \Delta x) - f(c)}{\Delta x} > 0 \quad \text{va} \quad \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(c) \geq 0 \quad (1)$$

bo'ladi. Agar  $\Delta x > 0$  bo'lsa  $\frac{\Delta y}{\Delta x} < 0$  va

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(c) \leq 0 \quad (2)$$

bo'ladi. (1) va (2) lardan  $f'(c) = 0$  kelib chiqadi.

Bu teoremaning geometrik ma'nosi funktsiya grafigining ( $c, f(c)$ ) nuqtasiga o'tkazilgan urinma OX o'qiga parallel bo'ladi.

**Izoh.** Teoremaning barcha shartlari muhim. Misol uchun  $0 \leq x \leq \frac{\pi}{2}$  segmentda  $y = \sin x$  funktsiya  $x_0 = 0$  eng kichik qiymatga erishadi, lekin funktsiyaning hosilasi shu nuqtada birga teng.

**2-teorema (Roll):** Aytaylik,  $y = f(x)$  funktsiya  $[a, b]$  kesmada aniqlangan bo'lib, quyidagi shartlarni qanoatlantirsin:

- 1)  $f(x) \in C[a, b]$
- 2)  $\forall x \in (a, b)$  uchun  $f'(x) - \exists$
- 3)  $f(a) = f(b)$

U holda  $\exists c \in [a, b]$  nuqta:  $f'(c) = 0$  bo'ladi.

**Izoh.** Roll teoremasining shartlari ham muhim.  $x = 0$  nuqtadan tashqarida  $f(x) = |x|$  funktsiya  $-1 \leq x \leq 1$  da teoremaning barcha shartlarini qanoatlantiradi. (-1;1)

oraliqda birorta ham nuqta yo'qki hosilasi 0 ga teng bo'lsa:  $(-1;0)$  da  $f'(x) = -1$ ,  $(0;1)$ ,  $f'(x) = 1$  ga teng.  $x = 0$  da funktsiya hosilaga ega emas.

**3-teorema (Lagranj):**  $[a,b]$  da aniqlangan  $y = f(x)$  funktsiya quyidagi shartlarni qanoatlantirsin:

- 1)  $f(x) \in C[a,b]$ ,
- 2)  $\forall x \in (a,b)$  uchun  $f'(x) - \exists$ .

Unda  $\exists c \in [a,b]$  nuqta topiladiki,

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad (3)$$

bo'ladi.

Agar Lagranj teoremasida  $f(a) = f(b)$  bo'lsa, unda  $f'(c) = 0$  bo'ladi, ya'ni Lagranj teoremasidan Roll teoremasi kelib chiqadi.

**Natija:** Agar  $(a,b)$  da  $f'(x) = 0$  bo'lsa, unda shu intervalda  $f(x) = \text{const}$  bo'ladi.

**4-teorema (Lopital qoidasi):** Faraz qilamiz  $\lim_{x \rightarrow a} f(x) = 0 \quad (\infty)$  va  $\lim_{x \rightarrow a} g(x) = 0 \quad (\infty)$  bo'lib,  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} - \exists$  bo'lsin.

U holda,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} \quad (4)$$

bo'ladi.

**Izoh:** Boshqa barcha aniqmasliklar  $\frac{0}{0}$  yoki  $\frac{\infty}{\infty}$  ko'rinishdagi aniqmasliklarga keltirish yordamida yechiladi.

### 1-misol.

$$\lim_{x \rightarrow 0} \frac{2^x - 1}{\sin x} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{(2^x - 1)'}{(\sin x)'} = \lim_{x \rightarrow 0} \frac{2^x \ln 2}{\cos x} = \ln 2.$$

### 2-misol.

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = \frac{0}{0} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\pi - 2x} = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{2} = \frac{1}{2}.$$

### 3-misol.

$$\lim_{x \rightarrow 0} \left( \operatorname{ctgx} - \frac{1}{x} \right) = (\infty - \infty) = \lim_{x \rightarrow 0} \left( \frac{\cos x}{\sin x} - \frac{1}{x} \right) = \lim_{x \rightarrow 0} \frac{(x \cos x - \sin x)'}{(x \sin x)'} =$$

$$= \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \cos x}{\sin x + x \cos x} = - \lim_{x \rightarrow 0} \frac{(x \sin x)'}{(\sin x + x \cos x)'} = - \lim_{x \rightarrow 0} \frac{\sin x + x \cos x}{\cos x + \cos x - x \sin x} = - \frac{0}{2} = 0.$$

### Nazorat savollari.

1. Ferma teoremasi.
2. Roll teoremasi.
3. Lagranj teoremasi.
4. Lopital qoidasi.

## 7-§. Funktsiyani to'liq tekshirish va uning grafigini yasash.

### 1. Funktsiyaning o'sishi va kamayishi.

Aytaylik,  $y = f(x)$  funktsiya  $(a, b)$  intervalda berilgan bo'lgin. Agar shu intervaldan olingan ixtiyoriy  $x_2 > x_1$  uchun  $f(x_2) > f(x_1)$  ( $f(x_2) < f(x_1)$ ) bo'lsa, unda  $f(x)$  funktsiya  $(a, b)$  oraliqda **o'suvchi (kamayuvchi)** deyiladi.

O'suvchi va kamayuvchi funktsiyalarga **monoton funktsiyalar** deb ataladi. Monoton funktsiyalar hayotda ko'p uchraydi. Masalan, o'sayotgan daraxtning bo'yisi, yetilayotgan donning og'irligi – vaqtning o'suvchi funktsiyalari; yorug'lik manbaidan ma'lum masofadagi yoritilganlik – masofaning kamayuvchi funktsiyasi bo'ladi.

**1-teorema:** Agar  $f(x)$  funktsiya  $(a, b)$  da differentsiyallanuvchi bo'lib, kamayuvchi ( $o'suvchi$ ) bo'lmasa, u holda  $(a, b)$  oraliqda  $f'(x) \geq 0$  ( $f'(x) \leq 0$ ) bo'ladi.

**2-teorema:** Agar  $f(x)$  funktsiya  $(a, b)$  oraliqda differentsiyallanuvchi bo'lib,  $f'(x) > 0$  ( $f'(x) < 0$ ) shartni qanoatlantirsaga, u holda  $(a, b)$  oraliqda funktsiya o'suvchi (kamayuvchi) bo'ladi.

## Misollar.

- 1)  $y = e^x$  funktsiya  $(-\infty, +\infty)$  da o'suvchi, chunki  $\forall x \in R$  uchun  $y' = e^x > 0$ .
- 2)  $y = x^2$  funktsiya uchun  $y' = 2x \Rightarrow x^2$  funktsiya  $[-\infty; 0]$  da kamayuvchi va  $[0; +\infty)$  da o'suvchi.

## 2. Funktsiyaning ekstremumlari.

Agar  $x_0$  nuqtaning  $\exists \cup_\delta(x_0)$  atrofi;  $\forall x \in \cup_\delta(x)$  uchun  $f(x) < f(x_0), (f(x) > f(x_0))$  tengsizlik bajarilsa, unda  $f(x)$  funktsiya  $x_0$  nuqtada lokal **maksimumga (minimumga) erishadi** deyiladi va  $f(x_0)$  qiymatga funktsiyaning  $\cup_\delta(x_0)$  atrofdagi **maksimum (minimum) qiymati** deb ataladi, hamda u

$$\max_{\cup_\delta(x)} \{f(x)\}, \quad \min_{\cup_\delta(x)} \{f(x)\}$$

kabi belgilanadi.

**2-teorema: (Ekstremum mavjudligining zaruriy sharti).**

Agar  $f(x)$  funktsiya  $(a, b)$  oraliqda differentsiyalanuvchi bo'lib,  $x_0$  ( $a < x < b$ ) nuqtada ekstremumga erishsa, u holda  $f'(x_0) = 0$ , bo'ladi.

**Izoh:**  $f(x) = x^3$  funktsiya uchun  $x = 0$  nuqtada  $f'(x) = 0$ , lekin bu nuqtada funktsiya ekstremumga erishmaydi, funktsiyaning hosilasi nolga teng bo'lishi yetarli bo'lmay, zaruriy shart ekan.

**3-teorema: (Yetarli shart).** Agar  $f'(x_0) = 0$  bo'lib,  $f(x)$  funktsiya  $x_0$  nuqtadan chapdan o'ngga o'tganda ishorasini musbatdan (manfiydan) manfiya (musbatga) o'zgartirsa, u holda  $f(x)$  funktsiya  $x_0$  nuqtada maksimumga (minimumga) erishadi. Agar  $x_0$  nuqtadan o'tganda ishorasini o'zgartirmasa, u holda  $f(x)$  funktsiya  $x_0$  nuqtada ekstremumga erishmaydi.

## Misollar.

$$1) \quad f(x) = e^x - x, \quad f'(x) = e^x - 1, \quad f'(x) = 0 \Rightarrow x = 0$$

x	x < 0	X = 0	x > 0
f'(x)	-	0	+
f(x)	kamayuvchi	min	o'suvchi

$$y_{\min} = f(0) = e^0 - 0 = 1$$

$$2) \quad f(x) = x^3 - 3x + 2, \quad f'(x) = 3x^2 - 3, \quad f'(x) = 0, \quad x = -1, \quad x = 1.$$

x	(-\infty; -1)	-1	(-1; 1)	1	(1; +\infty)
f'(x)	+	0	-	0	+
f(x)	o'suvchi	max	kamayuvchi	min 0	o'suvchi

$$y_{\max} = y(-1) = 4, \quad y_{\min} = y(1) = 0$$

**4-teorema.** Faraz qilaylik,  $f(x)$  funktsiya  $x_0$  nuqta va uning atrofida uzluksziz hamda  $f'(x), f''(x) \in C\{x_0\}$  bo'lib,  $f'(x) = 0, f''(x) \neq 0$  bo'lsin. Agar,  $f''(x_0) > 0$  ( $f''(x_0) < 0$ ) bo'lsa, u holda  $y = f(x)$  funktsiya  $x_0$  nuqtada minimumga (maksimumga) erishadi.

### Izohlar:

1) Funktsiya hosilasi mavjud bo'limgan nuqtada ham ekstremumga erishishi mumkin. Masalan,  $f(x) = |x|$  funktsiya uchun  $f'(0)$  mavjud emas, lekin funktsiya  $x = 0$  nuqtada minimumga erishadi.

2) Shuningdek funktsiyaning hosilasi  $\infty$  ga aylanadigan nuqtada ham funktsiya ekstremumga erishishi mumkin. Masalan,  $f(x) = \sqrt[3]{x^2}$  funktsiya uchun  $f'(0) = \infty$  bo'lsa ham shu nuqtada funktsiya  $\min$  ga erishadi.

### 3. Funktsiyaning botiq va qavariqligi. Burilish nuqtasi.

Faraz qilaylik,  $(a, b)$  oraliqda differentialsallanuvchi  $f(x)$  funktsiya berilgan bo'lsin. Unda bu funktsiya grafigining ixtiyoriy nuqtasidan urinma o'tkazish mumkin. Agar har doim urinma grafikdan pastda (yuqorida) joylashgan bo'lsa, unda

funktsiya  $(a, b)$  oraliqda **botiq (qavariq)** deb ataladi. Funktsiya grafigi o'z botiqligi yoki qavariqligini o'zgartiradigan nuqtaga esa **egilish nuqtasi** deyiladi.

**5-teorema.** Agar  $(a, b)$  oraliqda  $f''(x) > 0 (< 0)$ , bo'lsa, u holda  $f(x)$  funktsiya grafigi  $(a, b)$  oraliqda botiq (qavariq) bo'ladi.

#### 4. Funktsiyaning asimptotalari.

**a) Vertikal asimptota.** Agar  $\lim_{x \rightarrow a} f(x) = \infty$  bo'lsa,  $x = a$  to'g'ri chiziq vertikal asimptota bo'ladi.

**b) Gorizontal asimptota.** Agar  $\lim_{x \rightarrow \infty} f(x) = b$  bo'lsa,  $y = b$  to'g'ri chiziq gorizontal asimptota bo'ladi.

**v) Og'ma asimptota.** Agar  $\lim_{x \rightarrow \infty} [f(x) - (kx + b)] = 0$  bo'lsa,  $y = kx + b$  to'g'ri chiziq og'ma asimptota bo'ladi.

To'g'ri chiziqning parametrlari ushbu,

$$k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}, \quad b = \lim_{x \rightarrow \infty} [f(x) - kx]$$

tengliklar yordamida topiladi.

Endi funktsiya grafigini yasashga o'tish mumkin.

U quyidagi sxema asosida bajariladi:

- 1) Funktsiyaning aniqlanish sohasini topish.
- 2) Funktsiyaning juft-toqligini aniqlash.
- 3) Funktsiyaning davriyligini aniqlash.
- 4) Funktsiyani uzluksizlikka tekshirish va uzilish nuqtalarini topish.
- 5) Funktsiya grafigining koordinata o'qlari bilan kesishish nuqtalarini topish.
- 6) Monotonlik oraliqlarini aniqlash.
- 7) Ekstremumga tekshirish.
- 8) Botiq va qavariqlikka tekshirish.
- 9) Funktsiyaning asimptotalarini topish.
- 10) Funktsiya grafigini chizish.

## Misollar.

$$y = \frac{2x-1}{(x-1)^2} \quad \text{funktsiyani tekshiring va grafigini chizing.}$$

1. Funktsiya  $x=1$  nuqtadan tashqari barcha sonlar o'qida aniqlangan.
2.  $f(-x) = \frac{-2x-1}{(-x-1)^2} \neq f(x)$  va  $f(-x) \neq f(x)$  demak, funktsiya toq ham emas, juft ham emas.
3. Funktsiya davriy emas.
4.  $x=1$  nuqtada II – tur uzilishga ega.

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{2x-1}{(x-1)^2} = +\infty \quad \text{qolgan nuqtalarda funktsiya uzlucksiz.}$$

5.  $x=1$  funktsiyaning vertikal asimptotasi  $y=0$  gorizontal asimptotasi, ya'ni  $\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{2x-1}{(x-1)^2} = \lim_{x \rightarrow \pm\infty} \frac{x - \frac{1}{2}}{\left(1 - \frac{1}{x}\right)^2} = 0$

Og'ma asimptotasini topamiz.

$$k_{1,2} = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{2x-1}{x(x-1)^2} = \lim_{x \rightarrow \pm\infty} \frac{\frac{2x-1}{x}}{x^2 \left(1 - \frac{1}{x}\right)^2} = 0, \quad \text{u holda } k = 0,$$

$b_{1,2} = \lim_{x \rightarrow \pm\infty} (f(x) - kx) = \lim_{x \rightarrow \pm\infty} \frac{2x-1}{(x-1)^2} = 0, \quad \text{u holda } b = 0.$  Bundan kelib chiqadiki,  $y = kx + b$  og'ma asimptota yo'q.

6.  $y' = -\frac{2x}{(x-1)^2}$  funktsiya aniqlanish sohasini quyidagi oraliqlarga bo'lamiz:  $(-\infty; 0), (0; 1), (1; +\infty)$ .  $(-\infty; 0)$  oraliqlarda funktsiya kamayadi.  $(0; 1)$  oraliqda esa funktsiya o'sadi.  $(1; +\infty)$  oraliqda funktsiya kamayadi.

7.  $x=0$  nuqtada funktsiya aniqlangan va uzlucksiz,  $y'(x)$  hosila ishorasini manfiydan musbatga o'zgartiradi, u holda funktsiya, bu nuqtada  $y_{\min} = y(0) = -1$  erishadi.

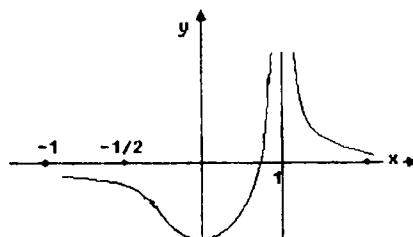
8. Funktsiya botiq va qavariqligini tekshirish uchun ikkinchi tartibli hosilani olamiz.  $y'' = 2 \cdot \frac{2x+1}{(x-1)^4}$  funktsiyaning aniqlanish sohasini quyidagi oraliqlarga ajratamiz.

$\left(-\infty, -\frac{1}{2}\right)$ ,  $\left(-\frac{1}{2}; 1\right)$ ,  $(1; +\infty)$ .  $\left(-\infty, \frac{1}{2}\right)$ , da  $f''(-1) = -\frac{1}{8} < 0$ , funktsiya qavariq,

$\left(-\frac{1}{2}; 1\right)$  da  $f''(0) = 2 > 0$  funktsiya botiq,  $(1; +\infty)$  oraliqda  $f''(0) = 10 > 0$  funktsiya botiq. Funktsiyaning ikkinchi tartibli hosilasi  $x = -\frac{1}{2}$  dan  $x = 1$  nuqtaga o'tishdan o'z ishorasini o'zgartiradi, bundan kelib chiqadiki,  $f\left(-\frac{1}{2}\right) = -\frac{8}{9}$  nuqta egilish nuqtasi bo'ladi.

9. Agar  $x = 0$  bo'lsa, u holda  $y = -1$  va  $y = 0$  da  $x = 1$ . Bundan kelib chiqadiki,  $(0; -1)$  va  $\left(\frac{1}{2}; 0\right)$  nuqtalar funktsiya grafigining koordinata o'qlari bilan kesishish nuqtalari.

10. Funktsiya grafigi:



### Nazorat savollari:

1. Funktsiyaning o'sishi va kamayishi.
2. Funktsiyaning ekstremumlari.
3. Funktsiyaning botiq va qavariqligi. Burilish nuqtasi.
4. Funktsiyaning asimptotlari.

## Mustaqil yechish uchun misollar va masalalar.

**1-masala.** Qatorlarni yaqinlashishga tekshiring va yig'indisini toping.

- 1.1.  $\frac{2}{5} + \frac{2}{25} + \dots + \frac{2}{5^n} + \dots$   $j: S_n = \frac{1}{2} - \frac{1}{2} \left(\frac{1}{5}\right)^n, \quad S = \frac{1}{2}$
- 1.2.  $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots + \frac{(-1)^{n-1}}{3^{n-1}} + \dots$   $j: S_n = \frac{3}{4} + \frac{1}{4} \cdot \frac{(-1)^{n-1}}{3^{n-1}}, \quad S = \frac{3}{4}$
- 1.3.  $\left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{3^2 + 5^2}\right) + \dots + \left(\frac{1}{3^n} + \frac{1}{5^n}\right) + \dots$   $j: S_n = \frac{3}{4} - \frac{1}{2 \cdot 3^n} - \frac{1}{4 \cdot 5^n}, \quad S = \frac{3}{4}$
- 1.4.  $\left(3 + \frac{1}{2}\right) + \left(\frac{3}{2} - \frac{1}{6}\right) + \left(\frac{3}{4} + \frac{1}{18}\right) + \dots + \left(\frac{3}{2^{n-1}} + \frac{(-1)^{n-1}}{2 \cdot 3^{n-1}}\right) + \dots$   $j: S_n = \frac{51}{8} - \frac{3}{2^{n-1}} + \frac{(-1)^{n-1}}{8 \cdot 3^{n-1}}, \quad S = \frac{51}{8}$
- 1.5.  $\sum_{n=1}^{\infty} \frac{1}{16n^2 - 8n - 3}$   $j: S_n = \frac{1}{4} \left(1 - \frac{1}{4n+1}\right), \quad S = \frac{1}{4}$
- 1.6.  $\sum_{n=1}^{\infty} \frac{1}{25n^2 + 5n - 6}$   $j: S_n = \frac{1}{5} \left(\frac{1}{3} - \frac{1}{5n+3}\right), \quad S = \frac{1}{15}$
- 1.7.  $\sum_{n=1}^{\infty} \frac{1}{36n^2 - 24n - 5}$   $j: S_n = \frac{1}{6} \left(1 - \frac{1}{6n+1}\right), \quad S = \frac{1}{6}$
- 1.8.  $\sum_{n=1}^{\infty} \frac{1}{49n^2 + 7n - 12}$   $j: S_n = \frac{1}{7} \left(\frac{1}{4} - \frac{1}{7n+4}\right), \quad S = \frac{1}{28}$
- 1.9.  $\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}$   $j: S_n = \frac{3}{4} - \frac{1}{2} \left(\frac{1}{n+1} + \frac{1}{n+2}\right), \quad S = \frac{3}{4}$
- 1.10.  $\sum_{n=1}^{\infty} \frac{1}{4n^2 + 4n - 3}$   $j: S_n = \frac{1}{3} - \frac{1}{4} \left(\frac{1}{2n+1} + \frac{1}{2n+3}\right), \quad S = \frac{1}{3}$
- 1.11.  $\sum_{n=1}^{\infty} \frac{1}{16n^2 - 8n - 15}$   $j: S_n = -\frac{1}{12} - \frac{1}{8} \left(\frac{1}{4n-1} + \frac{1}{4n+3}\right), \quad S = -\frac{1}{12}$
- 1.12.  $\sum_{n=1}^{\infty} \frac{1}{36n^2 + 12n - 35}$   $j: S_n = \frac{2}{21} - \frac{1}{12} \left(\frac{1}{6n+1} + \frac{1}{6n+7}\right), \quad S = \frac{2}{21}$
- 1.13.  $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$   $j: S = \frac{1}{2}$
- 1.14.  $\sum_{n=1}^{\infty} \frac{1}{2^n}$   $j: S = 2$
- 1.15.  $\sum_{n=1}^{\infty} \frac{4}{(2n-1)(2n+1)}$   $j: S = 2$
- 1.16.  $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$   $j: S = \frac{1}{3}$

**1.17.**  $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

j:  $S = 1$

**1.18.**  $\sum_{n=1}^{\infty} \frac{n+1}{n^2(n+2)^2}$

j:  $S = \frac{3}{8}$

**1.19.**  $\sum_{n=1}^{\infty} \frac{2}{n(n+1)(n+2)}$

j:  $S = \frac{1}{2}$

**2-masala.** Quyidagi  $\sum_{n=1}^{\infty} a_n$  qatorlarni yaqinlashishga yaqinlashish alomatlaridan foydalanib tekshiring.

2.1.  $a_n = \frac{3^n}{n^n}$ .

2. 2.  $a_n = \frac{n!}{10^n}$ .

2.3.  $a_n = \frac{n}{3^n}$ .

2. 4.  $a_n = \frac{3^n}{n!}$ .

2.5.  $a_n = \frac{n^3}{3^n}$ .

2. 6.  $a_n = \frac{n!a^n}{n^n}, \quad a \neq e, a > 0$

2.7.  $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{3^n n!}$ .

2. 8.  $a_n = \frac{(2n)!}{(n!)^2}$ .

2.9.  $a_n = \frac{100^n}{n!}$ .

2. 10.  $a_n = \frac{(n!)^2}{3^n}$ .

2.11.  $a_n = \left(\frac{3}{n}\right)^n$ .

2. 12.  $a_n = \left(\frac{100}{n}\right)^n$ .

2.13.  $a_n = 2^n \left(\frac{n}{n+1}\right)^{n^2}$ .

2. 14.  $a_n = 3^{n+1} \left(\frac{n+2}{n+3}\right)^{n^2}$ .

2.15.  $a_n = \left(\frac{n^2+5}{n^2+6}\right)^n$ .

2. 16.  $a_n = \left(\frac{2n-1}{2n+1}\right)^{n(n-1)}$ .

2.17.  $a_n = \left(\frac{n-1}{n+1}\right)^{n^2+4n+5}$ .

2. 18.  $a_n = \left(\frac{n-1}{n+1}\right)^{\sqrt{n^3+3n+1}}$ .

2.19.  $a_n = 3^{-n} \left(\frac{n+1}{n}\right)^{n^2}$ .

2. 20.  $a_n = \left(\frac{6n+1}{5n-3}\right)^{\frac{n}{2}} \left(\frac{5}{6}\right)^{\frac{2n}{3}}$ .

**3-masala.** Quyidagi isbotlansin :

3.1.  $\lim_{x \rightarrow -3} \frac{2x^2 + 5x - 3}{x+3} = -7$ .

3.2.  $\lim_{x \rightarrow 1} \frac{5x^2 - 4x - 1}{x-1} = 6$ .

3.3.  $\lim_{x \rightarrow -2} \frac{3x^2 + 5x - 2}{x+2} = -7$ .

3.4.  $\lim_{x \rightarrow -3} \frac{4x^2 - 14x + 6}{x-3} = 10$

$$3.5. \lim_{x \rightarrow -\frac{1}{2}} \frac{6x^2 + x - 1}{x + \frac{1}{2}} = -5.$$

$$3.7. \lim_{x \rightarrow -\frac{1}{2}} \frac{9x^2 - 1}{x + \frac{1}{3}} = -6.$$

$$3.9. \lim_{x \rightarrow -\frac{1}{3}} \frac{3x^2 - 2x - 1}{x + \frac{1}{3}} = -4.$$

$$3.11. \lim_{x \rightarrow -3} \frac{x^2 - 4x + 3}{x - 3} = 2.$$

$$3.13. \lim_{x \rightarrow -\frac{1}{3}} \frac{6x^2 - 5x + 1}{x - \frac{1}{3}} = -1.$$

$$3.15. \lim_{x \rightarrow -\frac{7}{2}} \frac{2x^2 + 13x + 21}{2x + 7} = -\frac{1}{2}.$$

$$3.17. \lim_{x \rightarrow \frac{1}{3}} \frac{6x^2 + x - 1}{x - \frac{1}{3}} = 5.$$

$$3.19. \lim_{x \rightarrow 11} \frac{2x^2 - 21x - 11}{x - 11} = 23.$$

$$3.21. \lim_{x \rightarrow -7} \frac{2x^2 + 15x + 7}{x + 7} = -13.$$

$$3.23. \lim_{x \rightarrow -\frac{1}{3}} \frac{6x^2 - x - 1}{3x + 1} = \frac{5}{3}.$$

$$3.25. \lim_{x \rightarrow 8} \frac{3x^2 - 40x + 128}{x - 8} = 8.$$

$$3.27. \lim_{x \rightarrow -\frac{1}{2}} \frac{2x^2 - 5x + 2}{x - \frac{1}{2}} = -3.$$

$$3.29. \lim_{x \rightarrow -\frac{1}{3}} \frac{15x^2 - 2x - 1}{x - \frac{1}{3}} = 8.$$

$$3.6. \lim_{x \rightarrow -\frac{1}{2}} \frac{6x^2 - x - 1}{x - \frac{1}{2}} = 5.$$

$$3.8. \lim_{x \rightarrow 2} \frac{3x^2 - 5x - 2}{x - 2} = 7.$$

$$3.10. \lim_{x \rightarrow -1} \frac{7x^2 + 8x + 1}{x + 1} = -6.$$

$$3.12. \lim_{x \rightarrow -\frac{1}{2}} \frac{2x^2 + 3x - 2}{x - \frac{1}{2}} = 5.$$

$$3.14. \lim_{x \rightarrow -\frac{7}{5}} \frac{10x^2 + 9x - 7}{x - \frac{7}{5}} = -19.$$

$$3.16. \lim_{x \rightarrow -\frac{5}{2}} \frac{2x^2 - 9x + 10}{2x - 5} = \frac{1}{2}.$$

$$3.18. \lim_{x \rightarrow -\frac{1}{2}} \frac{6x^2 + 75x - 39}{x - \frac{1}{2}} = -81.$$

$$3.20. \lim_{x \rightarrow 5} \frac{5x^2 - 24x - 5}{x - 5} = 26.$$

$$3.22. \lim_{x \rightarrow -4} \frac{2x^2 + 6x - 8}{x + 4} = -10.$$

$$3.24. \lim_{x \rightarrow -5} \frac{x^2 + 2x - 15}{x + 5} = -8.$$

$$3.26. \lim_{x \rightarrow -10} \frac{5x^2 - 51x + 10}{x - 10} = 49.$$

$$3.28. \lim_{x \rightarrow -6} \frac{3x^2 + 17x - 6}{x + 6} = -19.$$

$$3.30. \lim_{x \rightarrow -\frac{1}{5}} \frac{15x^2 - 2x - 1}{x + \frac{1}{5}} = -8.$$

**4 – masala. Fuktsiya limiti hisoblansin.**

$$4.1. \lim_{x \rightarrow 4} \frac{\sqrt{1+2x} - 3}{\sqrt{x} - 2}$$

$$4.2. \lim_{x \rightarrow 4} \frac{\sqrt[3]{1-x} - 3}{2 + \sqrt[3]{x}}$$

- 4.3.  $\lim_{x \rightarrow 1} \frac{\sqrt{1-x}}{2 + \sqrt[3]{x^2 - 1}}$
- 4.5.  $\lim_{x \rightarrow 2} \frac{\sqrt[3]{x-6} + 2}{x^3 + 8}$
- 4.7.  $\lim_{x \rightarrow 8} \frac{\sqrt[3]{9+2x} - 5}{\sqrt[3]{x-2}}$
- 4.9.  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{8+3x+x^2} - 2}{x+x^2}$
- 4.11.  $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt{1+x} - \sqrt{2x}}$
- 4.13.  $\lim_{x \rightarrow 2} \frac{\sqrt[3]{4x} - 2}{\sqrt{2+x} - \sqrt{2x}}$
- 4.15.  $\lim_{x \rightarrow 3} \frac{\sqrt[3]{9x} - 3}{\sqrt{3+x} - \sqrt{2x}}$
- 4.17.  $\lim_{x \rightarrow 4} \frac{\sqrt[3]{16x} - 4}{\sqrt{4+x} - \sqrt{2x}}$
- 4.19.  $\lim_{x \rightarrow 2} \frac{\sqrt[3]{\frac{x}{4} - \frac{1}{2}}}{\sqrt{\frac{1}{2} + x} - \sqrt{2x}}$
- 4.21.  $\lim_{x \rightarrow 4} \frac{\sqrt[3]{\frac{x}{16} - \frac{1}{4}}}{\sqrt{\frac{1}{4} + x} - \sqrt{2x}}$
- 4.23.  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{27+x} - \sqrt[3]{27-x}}{\sqrt[3]{x^2} + \sqrt[3]{x}}$
- 4.25.  $\lim_{x \rightarrow 0} \frac{\sqrt{1-2x+3x^2} - (1+x)}{\sqrt[3]{x}}$
- 4.27.  $\lim_{x \rightarrow 16} \frac{\sqrt[3]{x} - 2}{\sqrt[3]{(\sqrt{x}-4)^2}}$
- 4.29.  $\lim_{x \rightarrow 64} \frac{\sqrt[3]{x} - 2}{\sqrt[3]{x^2 - 16}}$
- 4.4.  $\lim_{x \rightarrow 3} \frac{\sqrt{x+13} - 2\sqrt{x+1}}{x^2 - 9}$
- 4.6.  $\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{\sqrt{x-4}}$
- 4.8.  $\lim_{x \rightarrow 0} \frac{\sqrt{1-2x} + x^2 - (1+x)}{x}$
- 4.10.  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{27+x^2} - \sqrt[3]{27-x}}{x + 2\sqrt[3]{x^4}}$
- 4.12.  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt[3]{1+x} - \sqrt[3]{1-x}}$
- 4.14.  $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x^2 - 1}$
- 4.16.  $\lim_{x \rightarrow 2} \frac{\sqrt[3]{x-6} + 2}{x + 2}$
- 4.18.  $\lim_{x \rightarrow 8} \frac{\sqrt[3]{9+2x} - 5}{\sqrt[3]{x^2} - 4}$
- 4.20.  $\lim_{x \rightarrow 3} \frac{\sqrt[3]{\frac{x}{9} - \frac{1}{3}}}{\sqrt{\frac{1}{3} + x} - \sqrt{2x}}$
- 4.22.  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x + \sqrt[3]{x^4}}$
- 4.24.  $\lim_{x \rightarrow 0} \frac{\sqrt[3]{8+3x-x^2} - 2}{\sqrt[3]{x^2+x^3}}$
- 4.26.  $\lim_{x \rightarrow 8} \frac{\sqrt[3]{9+2x+5}}{\sqrt[3]{x} - 2}$
- 4.28.  $\lim_{x \rightarrow 2} \frac{3\sqrt{x-6} + 2}{\sqrt[3]{x^3+8}}$
- 4.30.  $\lim_{x \rightarrow 8} \frac{10-x-6\sqrt{1-x}}{2+\sqrt[3]{x}}$

### 5-masala. Funktsiyaning limiti hisoblansin.

- 5.1.  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{\ln x}$
- 5.2.  $\lim_{x \rightarrow 1} \frac{\sqrt{x^2 - x + 1} - 1}{\ln x}$

$$5.3. \lim_{x \rightarrow \pi} \frac{1 + \cos 3x}{\sin^2 7x}$$

$$5.5. \lim_{x \rightarrow 1} \frac{1 + \cos \pi x}{\operatorname{tg}^2 \pi x}$$

$$5.7. \lim_{x \rightarrow \pi} \frac{\sin^2 x - \operatorname{tg}^2 x}{(x - \pi)^4}$$

$$5.9. \lim_{x \rightarrow \pi} \frac{\cos 5x - \cos 3x}{\sin^2 x}$$

$$5.11. \lim_{x \rightarrow 2} \frac{\sin 7\pi x}{\sin 8\pi x}$$

$$5.13. \lim_{x \rightarrow 2} \frac{\sqrt{x^2 - 3x + 3} - 1}{\sin \pi x}$$

$$5.15. \lim_{x \rightarrow 1} \frac{3^{5x-3} - 3^{2x^2}}{\operatorname{tg} \pi x}$$

$$5.17. \lim_{x \rightarrow \frac{\pi}{2}} \frac{\ln 2x - \ln \pi}{\sin \frac{5x}{2} \cos x}$$

$$5.19. \lim_{x \rightarrow \pi} \frac{e^x - e^\pi}{\sin 5x - \sin 3x}$$

$$5.21. \lim_{x \rightarrow 2} \frac{1 - 2^{4-x^2}}{2(\sqrt{2x} - \sqrt{3x^2 - 5x + 2})}$$

$$5.23. \lim_{x \rightarrow 2} \frac{\operatorname{tg} \pi x}{x+2}$$

$$5.25. \lim_{x \rightarrow \pi^3} \frac{1 - 2 \cos x}{\pi - 3x}$$

$$5.27. \lim_{x \rightarrow 1} \frac{1 - x^2}{\sin \pi x}$$

$$5.29. \lim_{x \rightarrow 1} \frac{3 - \sqrt{10 - 2}}{\sin 3\pi x}$$

$$5.4. \lim_{x \rightarrow \pi_4} \frac{1 - \sin 2x}{(\pi - 4x)^2}$$

$$5.6. \lim_{x \rightarrow \pi_2} \frac{\operatorname{tg} 3x}{\operatorname{tg} x}$$

$$5.8. \lim_{x \rightarrow 1} \frac{\sqrt{x^2 - x + 1} - 1}{\operatorname{tg} \pi x}$$

$$5.10. \lim_{x \rightarrow 2\pi} \frac{\sin 7x - \sin 3x}{e^{x^2} - e^{4\pi^2}}$$

$$5.12. \lim_{x \rightarrow 2} \frac{\ln(5 - 2x)}{\sqrt{10 - 3x} - 2}$$

$$5.14. \lim_{x \rightarrow \pi} \frac{x^2 - \pi^2}{\sin x}$$

$$5.16. \lim_{x \rightarrow 4} \frac{2^x - 16}{\sin \pi x}$$

$$5.18. \lim_{x \rightarrow \pi_4} \frac{\ln \operatorname{tg} x}{\cos 2x}$$

$$5.20. \lim_{x \rightarrow 2} \frac{\ln(9 - 2x^2)}{\sin 2\pi x}$$

$$5.22. \lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{\sqrt[4]{x} - 1}$$

$$5.24. \lim_{x \rightarrow \pi} \frac{1 - \sin x}{\frac{2}{\pi - x}}$$

$$5.26. \lim_{x \rightarrow 2} \frac{\operatorname{arctg}(x^2 - 2x)}{\sin 3\pi x}$$

$$5.28. \lim_{x \rightarrow 1} \frac{\cos \frac{\pi x}{2}}{1 - \sqrt{x}}$$

$$5.30. \lim_{x \rightarrow \pi} \frac{\sin 5x}{\operatorname{tg} 3x}$$

## 6-masala. Funktsiyaning limiti hisoblansin.

$$6.1. \lim_{x \rightarrow 1} \left( \frac{3x-1}{x+1} \right)^{\frac{1}{\sqrt[3]{x}-1}}$$

$$6.2. \lim_{x \rightarrow a} \left( \frac{\sin x}{\sin a} \right)^{\frac{1}{x-a}}$$

$$6.3. \lim_{x \rightarrow 1} \left( \frac{2x-1}{x} \right)^{\frac{1}{\sqrt[3]{x}-1}}$$

$$6.4. \lim_{x \rightarrow 2} \left( \frac{\cos x}{\cos 2x} \right)^{\frac{1}{x-2}}$$

$$6.5. \lim_{x \rightarrow 1} \left( \frac{4x-1}{2x+1} \right)^{\frac{1}{\sqrt[3]{x-1}}}$$

$$6.7. \lim_{x \rightarrow 1} \left( \frac{2x-1}{x} \right)^{\frac{1}{\sqrt[3]{x-1}}}$$

$$6.9. \lim_{x \rightarrow 2\pi} (\cos x) \frac{\operatorname{ctg} 2x}{\sin 3x}$$

$$6.11. \lim_{x \rightarrow 3} \left( \frac{6-x}{3} \right)^{\lg \frac{x}{6}}$$

$$6.13. \lim_{x \rightarrow 1} (3-2x)^{\frac{\pi}{2}}$$

$$6.15. \lim_{x \rightarrow 3} \left( \frac{9-2x}{3} \right)^{\lg \frac{\pi}{6}}$$

$$6.17. \lim_{x \rightarrow 1} (2e^{x-1} - 1)^{\frac{x}{x-1}}$$

$$6.19. \lim_{x \rightarrow 1} (2e^{x-1} - 1)^{\frac{3x-1}{x-1}}$$

$$6.21. \lim_{x \rightarrow 2} (2e^{x-2} - 1)^{\frac{3x+2}{x-2}}$$

$$6.23. \lim_{x \rightarrow 1} \left( \frac{2-x}{x} \right)^{\frac{1}{\ln(2-x)}}$$

$$6.25. \lim_{x \rightarrow 1} (2-x)^{\frac{\sin \frac{\pi x}{2}}{\ln(2-x)}}$$

$$6.27. \lim_{x \rightarrow 1} \left( \frac{x+1}{2x} \right)^{\frac{\ln(x+2)}{\ln(2-x)}}$$

$$6.29. \lim_{x \rightarrow 1} \left( \frac{1}{x} \right)^{\frac{\ln(x+1)}{\ln(2-x)}}$$

$$6.6. \lim_{x \rightarrow \frac{\pi}{4}} (\operatorname{tg} x)^{\frac{1}{\cos^2 \left( \frac{3x}{4} \right)}}$$

$$6.8. \lim_{x \rightarrow a} \left( 2 - \frac{x}{a} \right)^{\lg \frac{\pi x}{2}}$$

$$6.10. \lim_{x \rightarrow 2\pi} (\cos x)^{\frac{1}{\sin^2 2x}}$$

$$6.12. \lim_{x \rightarrow 4\pi} (\cos x)^{\frac{\operatorname{ctg} x}{\sin^4 x}}$$

$$6.14. \lim_{x \rightarrow 4\pi} (\cos x)^{\frac{5}{\cos 3x - \sin 2x}}$$

$$6.16. \lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{6 \operatorname{tg} x \cdot \lg 3x}$$

$$6.18. \lim_{x \rightarrow \frac{\pi}{2}} \left( \operatorname{tg} \frac{x}{2} \right)^{\frac{1}{x^{\frac{1}{2}}}}$$

$$6.20. \lim_{x \rightarrow \frac{\pi}{2}} (1 + \cos 3x)^{\frac{\infty}{\infty}}$$

$$6.22. \lim_{x \rightarrow 1} \left( \frac{2x-1}{x} \right)^{\frac{\ln(3+2x)}{\ln(2-x)}}$$

$$6.24. \lim_{x \rightarrow \frac{\pi}{2}} \left( \operatorname{ctg} \frac{x}{2} \right)^{\frac{1}{\cos x}}$$

$$6.26. \lim_{x \rightarrow 1} \left( \frac{\sin x}{\sin 3} \right)^{\frac{1}{x-1}}$$

$$6.28. \lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\frac{18 \sin x}{\cos x}}$$

$$6.30. \lim_{x \rightarrow \pi} \left( \operatorname{ctg} \frac{x}{4} \right)^{\frac{1}{\cos^2 x}}$$

**7-masala.** Funktsiya grafigining abtsissasi  $x_0$  bo'lgan nuqtasiga o'tkazilgan normal (5.1–5.12 misollarda) yoki urinma (5.13–5.30 misollarda) tenglamasi topilsin.

$$7.1. \quad y = \frac{4x-x^2}{4}, \quad x_0 = 2$$

$$7.3. \quad y = x - x^3, \quad x_0 = -1$$

$$7.2. \quad y = 2x^2 + 3x - 1, \quad x_0 = -2$$

$$7.4. \quad y = x^2 + 8\sqrt{x} - 32, \quad x_0 = 4$$

$$7.5. \quad y = x + \sqrt{x^3}, \quad x_0 = 1$$

$$7.7. \quad y = \frac{1+\sqrt{x}}{1-\sqrt{x}}, \quad x_0 = 4$$

$$7.9. \quad y = 2x^2 - 3x + 1, \quad x_0 = 1$$

$$7.11. \quad y = \sqrt{x} - 3\sqrt[3]{x}, \quad x_0 = 64$$

$$7.13. \quad y = 2x^2 + 3, \quad x_0 = -1$$

$$7.15. \quad y = 2x + \frac{1}{x}, \quad x_0 = 1$$

$$7.17. \quad y = \frac{x^3 + 1}{x^4 + 1}, \quad x_0 = 1$$

$$7.19. \quad y = 3(\sqrt{x} - 2\sqrt{x}), \quad x_0 = 1$$

$$7.21. \quad y = \frac{x}{x^2 + 1}, \quad x_0 = -2$$

$$7.23. \quad y = \frac{2x}{x^2 + 1}, \quad x_0 = 11$$

$$7.25. \quad y = \frac{1+3x^2}{3+x^2}, \quad x_0 = 1$$

$$7.27. \quad y = 3\sqrt[3]{x} - \sqrt{x}, \quad x_0 = 1$$

$$7.29. \quad y = \frac{x^2}{10} + 3, \quad x_0 = 2$$

$$7.6. \quad y = \sqrt[3]{x^2 - 20}, \quad x_0 = -8$$

$$7.8. \quad y = 8\sqrt[4]{x} - 70, \quad x_0 = 16$$

$$7.10. \quad y = \frac{x^2 - 3x + 6}{x^2}, \quad x_0 = 3$$

$$7.12. \quad y = \frac{x^3 + 3}{x^3 - 2}, \quad x_0 = 2$$

$$7.14. \quad y = \frac{x^{29} + 6}{x^4 + 1}, \quad x_0 = 1$$

$$7.16. \quad y = -\frac{2(x^8 + 2)}{3(x^4 + 1)}, \quad x_0 = 1$$

$$7.18. \quad y = \frac{x^{16} + 9}{1 - 5x^2}, \quad x_0 = 1$$

$$7.20. \quad y = \frac{1}{3x + 2}, \quad x_0 = 2$$

$$7.22. \quad y = \frac{x^2 - 3x + 3}{3}, \quad x_0 = 3$$

$$7.24. \quad y = -2(\sqrt[3]{x} + 3\sqrt{x}), \quad x_0 = 1$$

$$7.26. \quad y = 14\sqrt{x} - 15\sqrt[3]{x} + 2, \quad x_0 = 1$$

$$7.28. \quad y = \frac{3x - 2x^3}{3}, \quad x_0 = 1$$

$$7.30. \quad y = \frac{x^2 - 2x - 3}{4}, \quad x_0 = 4$$

### 8-masala. Differentsial yordamida taqribiy hisoblang.

$$8.1. \quad y = \sqrt[3]{x}, \quad x = 7,76$$

$$8.2. \quad y = \sqrt[3]{x^3 + 7x}, \quad x = 1,012$$

$$8.3. \quad y = \frac{x + \sqrt{5 - x^2}}{2}, \quad x = 0,98$$

$$8.4. \quad y = \sqrt[3]{x}, \quad x = 27,54$$

$$8.5. \quad y = \arcsin x, \quad x = 0,08$$

$$8.6. \quad y = \sqrt[3]{x^2 + 2x + 5}, \quad x = 0,97$$

$$8.7. \quad y = \sqrt[3]{x}, \quad x = 26,46$$

$$8.8. \quad y = \sqrt{x^2 + x + 3}, \quad x = 1,97$$

$$8.9. \quad y = x^{11}, \quad x = 1,021$$

$$8.10. \quad y = \sqrt[3]{x}, \quad x = 1,21$$

- 8.11.  $y = x^{21}$ ,  $x = 0,998$
- 8.12.  $y = \sqrt[3]{x^2}$ ,  $x = 1,03$
- 8.13.  $y = x^6$ ,  $x = 2,01$
- 8.14.  $y = \sqrt[3]{x}$ ,  $x = 8,24$
- 8.15.  $y = x^7$ ,  $x = 1,996$
- 8.16.  $y = \sqrt[3]{x}$ ,  $x = 7,64$
- 8.17.  $y = \sqrt{4x - 1}$ ,  $x = 2,56$
- 8.18.  $y = \frac{1}{\sqrt{2x^2 + x + 1}}$ ,  $x = 1,016$
- 8.19.  $y = \sqrt[3]{x}$ ,  $x = 8,36$
- 8.20.  $y = \frac{1}{\sqrt{x}}$ ,  $x = 4,16$
- 8.21.  $y = x^7$ ,  $x = 2,002$
- 8.22.  $y = \sqrt{4x - 3}$ ,  $x = 1,78$
- 8.23.  $y = \sqrt{x^3}$ ,  $x = 0,98$
- 8.24.  $y = x^5$ ,  $x = 2,997$
- 8.25.  $y = \sqrt[3]{x^2}$ ,  $x = 1,03$
- 8.26.  $y = x^4$ ,  $x = 3,998$
- 8.27.  $y = \sqrt{1 + x + \sin x}$ ,  $x = 0,01$
- 8.28.  $y = \sqrt[3]{3x + \cos x}$ ,  $x = 0,01$
- 8.29.  $y = \sqrt[4]{2x - \sin \frac{\pi x}{2}}$ ,  $x = 1,02$
- 8.30.  $y = \sqrt{x^2 + 5}$ ,  $x = 1,97$

**9—masala.** Berilgan funktsiyaning hosilasini toping.

- 9.1.  $y = \frac{2(3x^3 + 4x^2 - x - 2)}{15\sqrt{1+2}}$
- 9.2.  $y = \frac{(2x^2 - 1)\sqrt{1+x^2}}{3x^3}$
- 9.3.  $y = \frac{x^4 - 8x^2}{2(x^2 - 4)}$
- 9.4.  $y = \frac{(2x^2 - x - 1)}{3\sqrt{2+4x}}$
- 9.5.  $y = \frac{(1+x^8)\sqrt{1+x^8}}{12x^{12}}$
- 9.6.  $y = \frac{x^2}{2\sqrt{1+3x^4}}$
- 9.7.  $y = \frac{(x^2 - 6)\sqrt{(4+x^2)^3}}{120x^5}$
- 9.8.  $y = \frac{(x^2 - 6)\sqrt{(4+x^2)^3}}{120x^5}$
- 9.9.  $y = \frac{4+3x^3}{x^3\sqrt{(2+x^3)^2}}$
- 9.10.  $u = \sqrt[3]{\frac{(1+x^4)^2}{x^2}}$
- 9.11.  $y = \frac{x^6 + x^3 - 2}{\sqrt{1-x^3}}$
- 9.12.  $y = \frac{(x^2 - 2)\sqrt{4+x^2}}{24x^3}$
- 9.13.  $y = \frac{1+x^2}{2\sqrt{1+2x^2}}$
- 9.14.  $y = \frac{\sqrt{x-1}(3x+2)}{4x^2}$

$$9.15. y = \frac{\sqrt{1+x^2}}{3x^3}$$

$$9.17. y = \frac{\sqrt{2x+3}(x-2)}{x^2}$$

$$9.19. y = \frac{(2x^2+3)\sqrt{x^2}-3}{9x^3}$$

$$9.21. y = \frac{(2x+1)\sqrt{x^2-x}}{x^2}$$

$$9.23. y = \frac{1}{(x+2)\sqrt{x^2+4x+5}}$$

$$9.25. y = 3\sqrt[3]{\frac{x+1}{(x-1)^2}}$$

$$9.27. y = \frac{x\sqrt{x+1}}{x^2+x+1}$$

$$9.29. y = \frac{(x+3)\sqrt{2x-1}}{2x+7}$$

$$9.16. y = \frac{x^6+8x^3-128}{\sqrt{8-x^3}}$$

$$9.18. y = (1-x^2)\sqrt[3]{x^3+\frac{1}{x}}$$

$$9.20. y = \frac{x-1}{(x^2+5)\sqrt{x^2+5}}$$

$$9.22. y = 2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$$

$$9.24. y = 3\sqrt[3]{\frac{x^2+x+1}{x+1}}$$

$$9.26. y = \frac{x+7}{6\sqrt{x^2+2x+7}}$$

$$9.28. y = \frac{x^2+2}{2\sqrt{1-x^4}}$$

$$9.30. y = \frac{3x+\sqrt{x}}{x^2+2}$$

### 10—masala. Berilgan funktsiyaning hosilasini toping.

$$10.1. y = ((\operatorname{arctg} x))^{\frac{1}{2 \operatorname{ln} \operatorname{arctg} x}}$$

$$10.3. y = (\sin x)^{5x}$$

$$10.5. y = (\ln x)^x$$

$$10.7. y = (\operatorname{ctg} 3x)^{2e^x}$$

$$10.9. y = (\operatorname{tg} x)^{4e^x}$$

$$10.11. y = (x \sin x)^{\sin(x \sin x)}$$

$$10.13. y = (x^3 + 4)^{\operatorname{gr} x}$$

$$10.15. y = (x^2 - 1)^{\sin x}$$

$$10.17. y = (\sin x)^{\frac{5x}{2}}$$

$$10.19. y = 19^{x^y} \cdot x^{19}$$

$$10.21. y = (\sin \sqrt{x})^{\frac{1}{x}}$$

$$10.23. y = x^{e^{\operatorname{tg} x}}$$

$$10.25. y = x^{e^{\operatorname{tg} x}}$$

$$10.27. y = x^{e^{\operatorname{arctg} x}}$$

$$10.2. y = (\sin \sqrt{x})^{\operatorname{ln} \sin \sqrt{x}}$$

$$10.4. y = (\arcsin x)^x$$

$$10.6. y = x^{\operatorname{arcsin} x}$$

$$10.8. y = x^{e^{\operatorname{tg} x}}$$

$$10.10. y = (\cos 5x)^{x^2}$$

$$10.12. y = (x-5)^{\sin x}$$

$$10.14. y = x^{\sin x^3}$$

$$10.16. y = (x^4 + 5)^{\operatorname{ctg} x}$$

$$10.18. y = (x^2 + 1)^{\cos x}$$

$$10.20. y = x^{3x} \cdot 2^x$$

$$10.22. y = x^{e^{\operatorname{tg} x}}$$

$$10.24. y = x^{2x} \cdot 5^x$$

$$10.26. y = (\operatorname{tg} x)^{\operatorname{ln} \omega^4}$$

$$10.28. y = (x^8 + 1)^{\operatorname{gr} x}$$

10.29.  $y = x^{2^x} \cdot 29^x$

10.30.  $y = (\cos 2x)^{\ln \cos \frac{x}{2}}$

**11—masala.** Berilgan hosilasini hisoblang.

11.1.  $y = xe^{ax}$

11.3.  $y = \sqrt[3]{e^{7x-1}}$

11.5.  $y = \lg(5x+2)$

11.7.  $y = \frac{x}{2(3x+2)}$

11.9.  $y = \sqrt{x}$

11.11.  $y = 2^{3x+5}$

11.13.  $y = \sqrt[3]{e^{2x+1}}$

11.15.  $y = \lg(3x+1)$

11.17.  $y = \frac{x}{9(4x+9)}$

11.19.  $y = \frac{4}{x}$

11.21.  $y = a^{2x+3}$

11.23.  $y = \sqrt{e^{3x+1}}$

11.25.  $y = \lg(2x+7)$

11.27.  $y = \frac{x}{x+1}$

11.29.  $y = \frac{1+x}{1-x}$

11.2.  $y = \sin 2x + \cos(x+1)$

11.4.  $y = \frac{4x+7}{2x+3}$

11.6.  $y = a^{3x}$

11.8.  $y = \lg(x+4)$

11.10.  $y = \frac{2x+5}{13(3x+1)}$

11.12.  $y = \sin(x+1) + \cos 2x$

11.14.  $y = \frac{4+15x}{2x+1}$

11.16.  $y = 7^{5x}$

11.18.  $y = \lg(1+x)$

11.20.  $y = \frac{5x+1}{13(2x+3)}$

11.22.  $y = \sin(3x+1) + \cos 5x$

11.24.  $y = \frac{11+12x}{6x+5}$

11.26.  $y = 2^{bx}$

11.28.  $y = \log_3(x+5)$

11.30.  $y = \frac{7x+1}{17(4x+3)}$

**12—masala.** Birinchi tartibli hosiladan foydalananib funktsiyaning grafigini yasang.

12.1.  $y = 2x^3 - 9x^2 + 12x - 9$

12.3.  $y = x^2(x-2)^2$

12.5.  $y = 2 - 3x^2 - x^3$

12.7.  $y = 2x^3 - 3x^2 - 4$

12.9.  $y = (x-1)^2(x-3)^2$

12.2.  $y = 3x - x^2$

12.4.  $y = \frac{x^3 - 9x^2}{4} + 6x - 9$

12.6.  $y = (x+1)^2(x-1)^2$

12.8.  $y = 3x^2 - 2 - 3x^3$

12.10.  $y = \frac{x^3 + 3x^2}{4} - 5$

$$12.11. \quad y = 6x - 8x^3$$

$$12.13. \quad y = 2x^3 + 3x^2 - 5$$

$$12.15. \quad y = (2x+1)^2(2x-1)^2$$

$$12.17. \quad y = 12x^2 - 8x^{23} - 2$$

$$12.19. \quad y = \frac{27(x-x^2)}{4} - x$$

$$12.21. \quad y = \frac{x^2(x-4)^2}{16}$$

$$12.23. \quad y = \frac{16-6x^2-x^3}{8}$$

$$12.25. \quad y = 16x^3 - 36x^2 + 24x - 9$$

$$12.27. \quad y = -\frac{(x-2)^2(x-6)^2}{16}$$

$$12.29. \quad y = \frac{11+9x-3x^2-x^3}{8}$$

$$12.12. \quad y = 16x^2(x-1)^2$$

$$12.14. \quad y = 2-12x^2-8x^3$$

$$12.16. \quad y = 2x^3 + 9x^2 + 12x$$

$$12.18. \quad y = (2x-1)^2(2x-3)^2$$

$$12.20. \quad y = \frac{x(12-x^2)}{8}$$

$$12.22. \quad y = \frac{27(x^3+x^2)}{4} - 5$$

$$12.24. \quad y = -\frac{(x^2-4)^2}{16}$$

$$12.26. \quad y = \frac{6x^2-x^3-16}{8}$$

$$12.28. \quad y = 16x^3 - 12x^2 - 4$$

$$12.30. \quad y = -\frac{(x+1)^2(x-3)^2}{16}$$

**13—masala.** Funktsiyaning asimptolarini toping va grafigini yasang.

$$13.1. \quad y = \frac{17-x^2}{4x-5}$$

$$13.3. \quad y = \frac{x^3-4x}{3x^2-4}$$

$$13.5. \quad y = \frac{4x^3+3x^2-8x-2}{2-3x^2}$$

$$13.7. \quad y = \frac{2x^2-6}{x-2}$$

$$13.9. \quad y = \frac{x^3-5x}{5-3x^2}$$

$$13.11. \quad y = \frac{2-x^2}{\sqrt{9x^2-4}}$$

$$13.13. \quad y = \frac{3x^2-7}{2x-1}$$

$$13.15. \quad y = \frac{x^3+3x^2-2x-2}{2-3x^2}$$

$$13.17. \quad y = \frac{2x^2-1}{\sqrt{x^2-2}}$$

$$13.2. \quad y = \frac{x^2+1}{\sqrt{4x^2-3}}$$

$$13.4. \quad y = \frac{4x^2+9}{4x+8}$$

$$13.6. \quad y = \frac{x^2-3}{\sqrt{3x^2-2}}$$

$$13.8. \quad y = \frac{2x^3+2x^2-3x-1}{2-4x^2}$$

$$13.10. \quad y = \frac{4x^3-3x}{4x^2-1}$$

$$13.12. \quad y = \frac{x^2-6x+4}{3x-2}$$

$$13.14. \quad y = \frac{x^2-16}{\sqrt{9x^2-8}}$$

$$13.16. \quad y = \frac{21-x^2}{7x+9}$$

$$13.18. \quad y = \frac{2x^3-3x^2-2x+1}{1-3x^2}$$

- 13.19.  $y = \frac{x^2 - 11}{4x - 3}$
- 13.21.  $y = \frac{x^3 - 2x^2 - 3x + 2}{1 - x^2}$
- 13.23.  $y = \frac{x^3 + x^2 - 3x - 1}{2x^2 - 2}$
- 13.25.  $y = \frac{3x^2 - 10}{\sqrt{4x^2 - 1}}$
- 13.27.  $y = \frac{2x^3 + 2x^2 - 9x - 3}{2x^2 - 23}$
- 13.29.  $y = \frac{-x^2 - 4x + 13}{4x + 3}$
- 13.20.  $y = \frac{2x^2 - 9}{\sqrt{x^2 - 1}}$
- 13.22.  $y = \frac{x^2 - 2x - 1}{2x + 1}$
- 13.24.  $y = \frac{x^2 - 6x + 9}{x + 4}$
- 13.26.  $y = \frac{x^2 - 2x + 2}{x + 3}$
- 13.28.  $y = \frac{3x^2 - 10}{3 - 2x}$
- 13.30.  $y = \frac{-8 - x^2}{\sqrt{x^2 - 4}}$

**14-masala.** Funktsiyani to'liq tekshiring va grafigini yasang.

- 14.1.  $y = \frac{x^3 + 4}{x^2}$
- 14.3.  $y = \frac{2}{x^2 + 2x}$
- 14.5.  $y = \frac{12x}{9 + x^2}$
- 14.7.  $y = \frac{4 - x^3}{x^2}$
- 14.9.  $y = \frac{2x^3 + 1}{x^2}$
- 14.11.  $y = \frac{x^2}{(x - 1)^2}$
- 14.13.  $y = \frac{12 - 3x^3}{x^2 + 12}$
- 14.15.  $y = -\frac{8x}{x^2 + 4}$
- 14.17.  $y = \frac{3x^4 + 1}{x^3}$
- 14.19.  $y = \frac{8(x - 1)}{(x + 1)^2}$
- 14.21.  $y = \frac{4}{x^2 + 2x - 3}$
- 14.23.  $y = \frac{x^2 + 2x - 7}{x^2 + 2x - 3}$
- 14.2.  $y = \frac{x^2 - x + 1}{x - 1}$
- 14.4.  $y = \frac{4x^2}{3 + x^2}$
- 14.6.  $y = \frac{x^2 - 3x + 3}{x - 1}$
- 14.8.  $y = \frac{x^2 - 4x + 1}{x - 4}$
- 14.10.  $y = \frac{(x - 1)^2}{x^2}$
- 14.12.  $y = \left(1 + \frac{1}{x}\right)^2$
- 14.14.  $y = \frac{9 + 6x - 3x^2}{x^2 - 2x + 13}$
- 14.16.  $y = \left(\frac{x - 1}{x + 1}\right)^2$
- 14.18.  $y = \frac{4x}{(x + 1)^2}$
- 14.20.  $y = \frac{1 - 2x^3}{x^2}$
- 14.22.  $y = \frac{4}{3 + 2x - x^2}$
- 14.24.  $y = \frac{1}{x^4 - 1}$

$$14.25. \ y = -\left(\frac{x}{x+2}\right)^2$$

$$14.27. \ y = \frac{4(x+1)^2}{x^2 + 2x + 4}$$

$$14.29. \ y = \frac{x^2 - 6x + 9}{(x-1)^2}$$

$$14.26. \ y = \frac{x^3 - 32}{x^2}$$

$$14.28. \ y = \frac{3x - 2}{x^3}$$

$$14.30. \ y = \frac{x^3 - 27x + 54}{x^3}$$

**VI BOB.  
INTEGRAL HISOB.**

**1-§. Aniqmas integral va uni hisoblash.**

$f(x)$  funktsiya biror  $(a, b)$  (chekli yoki cheksiz) intervalda aniqlangan bo'lsin.

**1-ta'rif.** Agar  $f(x)$  funktsiya  $(a, b)$  intervalda differentsiyallanuvchi  $F(x)$  funktsiyaning hosilasiga teng, ya'ni  $F'(x) = f(x)$ ,  $x \in (a, b)$  bo'lsa, u holda  $F(x)$  funktsiya  $(a, b)$  intervalda  $f(x)$  funktsiyaning **boshlang'ich funktsiyasi** deyiladi.

**1-misol:**  $F(x) = x^3$  funktsiya sonlar o'qida  $f(x) = 3x^2$  funktsiyaning boshlang'ich funktsiyasi bo'ladi, chunki

$$F'(x) = (x^3)' = 3x^2 = f(x).$$

$F(x)$  va  $\Phi(x)$  funktsiyalarning har biri  $(a, b)$  intervalda bitta  $f(x)$  funktsiya uchun boshlang'ich funktsiya bo'lsa, bu  $F(x)$  va  $\Phi(x)$  funktsiyalar  $(a, b)$  intervalda bir-biridan o'zgarmas songa farq qiladi. Shu narsani ta'kidlashimiz kerakki,  $f(x) = 3x^2$  funktsiyaning boshlang'ich funktsiyasi sifatida ixtiyoriy  $\Phi(x) = x^3 + C$  funktsiyani olishimiz mumkin. Bu yerda  $C$  – ixtiyoriy o'zgarmas son.

**2-ta'rif.**  $(a, b)$  intervalda berilgan  $f(x)$  funktsiya boshlang'ich funktsiyalarning umumiy ifodasi  $F(x) + C$   $C = const$ , shu  $f(x)$  funktsiyaning **aniqmas integrali** deb ataladi va  $\int f(x)dx$  kabi belgilanadi.

Bunda  $\int$ -integral belgisi,  $f(x)$  – integral ostidagi funktsiya,  $f(x)dx$  esa **integral ostidagi ifoda** deyiladi. Demak,

$$\int f(x)dx = F(x) + C \quad (C = const)$$

**2-misol.** Ushbu  $\int(x^2 + 2x + 5)dx$  aniqmas integralni toping.

Quyidagi  $F(x) = \frac{x^3}{3} + x^2 + 5x + C$  funktsiya uchun  $F'(x) = (\frac{x^3}{3} + x^2 + 5x + C)' = x^2 + 2x + 5$  bo'ladi. Demak,

$$\int(x^2 + 2x + 5)dx = \frac{x^3}{3} + x^2 + 5x + C.$$

Aniqmas integralning ta'rifidan bevosita uning quyidagi sodda xossalari kelib chiqadi.

1.  $\frac{d}{dx} \int f(x)dx = f(x)$ , demak  $d \int f(x)dx = f(x)dx$
2.  $\int F'(x)dx = F(x) + C$ , ya'ni  $\int dF(x) = F(x) + C$
3.  $\int C \cdot f(x)dx = C \cdot \int f(x)dx$
4.  $\int [f_1(x) + f_2(x)]dx = \int f_1(x)dx + \int f_2(x)dx$

### Elementar funktsiyalarining aniqmas integrallari.

1.  $\int x^\mu dx = \frac{x^{\mu+1}}{\mu+1} + C \quad (\mu \neq -1),$
2.  $\int \frac{dx}{x} = \ln|x| + C,$
3.  $\int a^x dx = \frac{a^x}{\ln a} + C,$
4.  $\int e^{kx} dx = \frac{e^{kx}}{k} + C,$
5.  $\int \sin ax dx = -\frac{1}{a} \cos ax + C.$
6.  $\int \cos ax dx = \frac{1}{a} \sin ax + C.$
7.  $\int \frac{dx}{\cos^2 ax} = \frac{1}{a} \cdot \operatorname{tg} ax + C.$
8.  $\int \frac{dx}{\sin^2 ax} = -\frac{1}{a} \cdot \operatorname{ctg} ax + C.$
9.  $\int \frac{xdx}{x^2 + a^2} = \frac{1}{2} \ln|x^2 + a^2| + C.$
10.  $\int \frac{dx}{x^2 + a^2} = \frac{1}{2} \operatorname{arctg} \frac{x}{a} + C.$
11.  $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C.$
12.  $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln|x + \sqrt{x^2 + a^2}| + C.$
13.  $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C.$

### Integrallash usullari.

#### 1. O`zgaruvchini almashtirib integrallash usuli.

Ushbu  $\int f(x)dx$  aniqmas integralni hisoblash talab etilgan bo'lgin. Bunda  $f(x)$  funktsiya biror  $X = (a, b)$  intervalda aniqlangan va  $f(x) = \varphi(g(x)) \cdot g'(x)$  ko'rinishda yozish mumkin deylik. Agar  $\varphi(t)$  funktsiya  $T = (t_1; t_2)$  intervalda boshlang'ich  $\varphi(t)$  ga ega bo'lib,  $g(x)$  funktsiya  $X \in (a, b)$  intervalda ( $g(x) \subset T$ ) differentialsallanuvchi bo'lsa, u holda

$$\int f(x)dx = \int \varphi(g(x))g'(x)dx = \Phi(g(x)) + C$$

tenglik o'rini.

**3-misol.**  $\int \frac{xdx}{x^2 + a^2}$  ni hisoblang ( $a = const$ ).

**Yechilishi:**

$x^2 + a^2 = t$  kabi almashtiramiz.

$$\int \frac{xdx}{x^2 + a^2} = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln|t| + C = \frac{1}{2} \ln(x^2 + a^2) + C .$$

## 2. Bo'laklab integrallash usuli.

$u = u(x)$  va  $v = v(x)$  funktsiya  $(a, b)$  intervalda uzluksiz  $u'(x)$  va  $v'(x)$  hosilalarga ega bo'lsin. U holda

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$$

tenglik o'rini bo'lib, bu tenglikka **bo'laklab integrallash formulasi** deyiladi.

**4-misol.**  $\int xe^x dx$  ni hisoblang.

Bu intervalda  $u = x$ ,  $dv = e^x dx$  deb olamiz. Bundan kelib chiqadiki  $du = dx$ ,  $v = e^x$ .

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C = (x - 1)e^x + C .$$

## Ratsional funktsiyalarini integrallash.

### 1. Sodda kasrlarni integrallash.

Biror

$$P(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n \quad (1)$$

ko'phad berilgan bo'lsin, bunda  $a_0, a_1, a_2, \dots, a_n$  - o'zgarmas haqiqiy sonlar,  $a_n \neq 0$ ,  $n \in N$  esa ko'phadning darajasi.

$$\frac{P(x)}{Q(x)} = \frac{a_0 + a_1x + a_2x^2 + \dots + a_nx^n}{b_0 + b_1x + b_2x^2 + \dots + b_kx^k} \quad (k \in N)$$

ga **kasr** **ratsional** **funktsiya** deyiladi, agar  $n < k$   
 bo'lganda to'g'ri kasr, aks holda **noto'g'ri kasr**  
 deyiladi.

Ushbu

$$\frac{A}{(x-a)^m}, \quad \frac{Bx+C}{(x^2+px+q)^m}, \quad m=1,2,\dots \quad (2)$$

ko'rinishdag'i kasrlar **sodda kasrlar** deyiladi, bunda  $A, B, C$  hamda  $a, p, q$  lar o'zgarmas sonlar,  $x^2+px+q$  kvadrat uchhad haqiqiy ildizga ega emas.

Har qanday to'g'ri kasr (2) sodda kasrlar orqali ifodalanadi.

Agar  $\frac{P(x)}{Q(x)}$  to'g'ri kasr maxrajidagi  $Q(x)$  ko'phad, quyidagi

$$Q(x) = (x-\alpha)^m Q_1(x) \quad (m \in N)$$

ko'rinishda bo'lib,  $Q_1(x)$  ko'phad esa  $(x-\alpha)$  ga bo'linmasa, u holda berilgan to'g'ri kasr quyidagi

$$\frac{P(x)}{Q(x)} = \frac{A_m}{(x-\alpha)^m} + \frac{A_{m-1}}{(x-\alpha)^{m-1}} + \dots + \frac{A_1}{x-\alpha} + \frac{P_1(x)}{Q_1(x)}$$

ko'rinishda ifodalanishi mumkin, bunda  $A_1, A_2, \dots, A_m$  — o'zgarmas haqiqiy sonlar,  $P_1(x)$  — ko'phad.

Agar  $Q(x)$  ko'phad

$$Q(x) = (x^2+px+q)^n \cdot Q_1(x)$$

ko'rinishga ega bo'lib  $(x^2+px+q)$  kvadrat uchhad haqiqiy ildizga ega emas),  $Q_1(x)$  ko'phad  $x^2+px+q$  ga bo'linmasa, u holda berilgan to'g'ri kasr quyidagi ko'rinishda ifodalanishi mumkin:

$$\frac{P(x)}{Q(x)} = \frac{B_n x + C_n}{(x^2+px+q)^n} + \frac{B_{n-1} x + C_{n-1}}{(x^2+px+q)^{n-1}} + \dots + \frac{B_1 x + C_1}{x^2+px+q} + \frac{P_1(x)}{Q_1(x)}$$

bunda  $B_1, B_2, \dots, B_n, C_1, C_2, \dots, C_n$  — o'zgarmas sonlar,  $P_1(x)$  — ko'phad.

Sodda kasrlarning aniqmas integrallarini hisoblaymiz.

- $\int \frac{A}{x-a} dx = \int \frac{d(x-a)}{x-a} = A \cdot \ln|x-a| + C$
- $\int \frac{A}{(x-a)^k} dx = \int (x-a)^{-k} d(x-a) = A \cdot \frac{(x-a)^{1-k}}{1-k} + C$
- $I = \int \frac{Ax+B}{x^2+px+q} dx$  bu integralni hisoblashda ikkita hol bo'lishi mumkin.

a)  $x^2+px+q$  kavadrat uchhad to'liq kvadrat bo'lsa, I – integral 1 va 2 hollarga keltiriladi.

b) Agar  $x^2+px+q$  kavadrat uchhad to'liq kvadratga kelmasa, bu holda uni to'liq kvadratga to'ldirib integrallash mumkin.

### 5-misol.

$$I = \int \frac{2x-2}{x^2-x+1} dx = \int \frac{2x-2}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} dx, \quad x - \frac{1}{2} = t \text{ desak},$$

$$I = \int \frac{2t-1}{t^2 + \frac{3}{4}} dt = \int \frac{2t}{t^2 + \frac{3}{4}} dt - \int \frac{dt}{t^2 + \frac{3}{4}} = \ln\left(t^2 + \frac{3}{4}\right) - \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t}{\sqrt{3}} + C =$$

$$= \ln(x^2 - x + 1) - \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2t-1}{\sqrt{3}} + C$$

**Izoh:** Ma'lumki, elementar funktsiyaning hosilasi yana elementar funktsiya bo'lar edi, lekin integral olish uchun bu tasdiq o'rini bo'lishi shart emas, ya'ni ba'zi bir elementar funktsiyalarning integrallari elementar funktsiya bo'lmay qolishi mumkin. Masalan, ushbu

- |   |  |
|---|--|
| 1. $\int e^{-x^2} dx,$                    | 2. $\int \cos x^2 dx,$                                 |
| 3. $\int \sin x^2 dx,$                    | 4. $\int \frac{dx}{\ln x} \quad (x \geq 0, x \neq 1),$ |
| 5. $\int \frac{\cos x}{x} dx (x \neq 0),$ | 6. $\int \frac{\sin x}{x} dx$                          |

integrallarning har biri elementar funktsiyalar yordamida ifodalanmaydi. Bu funktsiyalar amaliyotda ko'p uchraganligi sababli ularning qiymatlarini hisoblash uchun alohida jadval tuzilgan va ularning grafiklari yasalgan. Shu yo'l

bilan elementar funktsiyalarda integrallanmaydigan funktsiyalar ham to'la o'r ganilgan.

### Nazorat savollari.

1. Boshlang'ich funktsiya.
2. Aniqmas integral.
3. O'zgaruvchini almashtirib integrallash usuli.
4. Bo'laklab integrallash usuli.
5. Ratsional funktsiyalarni integrallash.
6. Boshlang'ich funktsiyalar jadvali.

### 2-§. Aniq integral va uning tadbiqlari.

**1. O'tilgan yo'l haqidagi masala.** Biror moddiy nuqta to'g'ri chiziq bo'yicha  $[t_0, T]$  vaqt oraliq'ida  $v = v(t)$  tezlik bilan ( $t \in [t_0, T]$ ) harakat qilayotgan bo'lsa, uning bosib o'tgan yo'li  $s$  ni topish talab etilsin.  $[t_0, T]$  vaqt oraliq'ini  $t_0 < t_1 < t_2 < \dots < t_n = T$   $n - ta$  bo'lakka bo'lamiz va  $\Delta t_k = t_k - t_{k-1}$ , ( $k = 1, 2, \dots, n$ ) deb olamiz. Bu bo'laklarning eng kattasini  $\lambda = \max \Delta t_k$  deb belgilaymiz. Agar bu oraliqlar kichik bo'lsa, u holda katta bo'limgagan xatolik bilan har bir oraliqda harakat bir xil deb hisoblashimiz mumkin.

$$S \approx v(\tau_1) \Delta t_1 + v(\tau_2) \Delta t_2 + \dots + v(\tau_n) \Delta t_n,$$

bu yerda  $\tau_k \in [t_{k-1}, t_k]$ . Yig'indini  $\sum_{k=1}^n v(\tau_k) \Delta t_k$  — deb belgilaymiz va

$$S = \lim_{\lambda \rightarrow 0} \sum_{k=1}^n v(\tau_k) \Delta t_k$$

deb olamiz.

Aytaylik,  $[a, b]$  kesmada chegaralangan funktsiya berilgan bo'lsin.  $[a, b]$  kesmada ushbu

$$a = x_0 < x_1 < x_2 < \dots < x_n = b$$

ifodani qanoatlantiruvchi ixtiyoriy nuqtalarni olamiz.

$$\Delta x_k = x_{k+1} - x_k \quad \text{va} \quad \lambda = \max_{k=0, n-1} \Delta x_k$$

deb belgilaymiz.

$\forall \tau_k \in [x_k, x_{k+1}]$  nuqtalarni olib, quyidagi yig'indini tuzamiz  $\sum_{k=0}^{n-1} f(\tau_k) \Delta x_k$ .

Agar  $[a, b]$  kesmaning bo'linish holatiga va  $\tau_k$  nuqtani tanlashga bog'liq bo'limgan  $\sum_{k=0}^{n-1} f(\tau_k) \Delta x_k$ . chekli limit mavjud bo'lsa, bu holatda bu limitga  $f(x)$  funktsiyaning  $[a, b]$  segmentdagи aniq integrali deb ataladi va quyidagicha belgilanadi.

$$\int_a^b f(x) dx = \lim_{\lambda \rightarrow 0} \sum_{k=1}^n f(\tau_k) \Delta x_k$$

$f(x)$  — ga integral osti funktsiya deyiladi.  $a, b$  — larga mos ravishda integralning quyi va yuqori chegarasi deyiladi.

## 2. Aniq integral xossalari.

1<sup>0</sup>. O'zgarmas sonni integral belgisi ostidan chiqarib yozish mumkin:

$$\int_a^b Cf(x) dx = C \int_a^b f(x) dx.$$

$$2^0. \int_a^b [f_1(x) + f_2(x)] dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx$$

$$3^0. \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$4^0. \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$(a < c < b).$$

**Teorema:** Agar  $f(x)$  funktsiya  $[a, b]$  kesmada uzliksiz va  $F(x)$  bu kesmada  $f(x)$  funktsiyaning boshlang'ich funktsiyasi bo'lsa, u holda

$$\int_a^b f(x)dx = F(b) - F(a)$$

*bo'ladi.*

Bu formulaga **Nyuton–Leybnits formulasi** deb ataladi.

**Misol.**

$$\int_0^{\frac{\pi}{2}} \cos x dx = \sin x \Big|_0^{\frac{\pi}{2}} = \sin \frac{\pi}{2} - \sin 0 = 1.$$

**Teorema:** (*O'rta qiymat haqidagi teorema*). Agar  $f(x)$  funktsiya  $[a,b]$  segmentda uzluksiz bo'lsa, u holda  $[a,b]$  segmentda shunday c nuqta topiladiki,

$$\int_a^b f(x)dx = (b-a) \cdot f(c)$$

*tenglik o'rini bo'ladi.*

**Isbot:** Nyuton – Leybnits formulasiga ko'ra  $\int_a^b f(x)dx = F(b) - F(a)$  deb yozishimiz mumkin. Lagranj teoremasiga ko'ra  $F(b) - F(a) = (b-a) \cdot F'(c) = (b-a)f(c)$ , bu yerda  $a < c < b$  teorema isbot bo'ldi.

### 3. Yoy uzunligini hisoblash.

Ma'lumki, egri chiziq yoyining uzunligi shu egri chiziqqa chizilgan siniq chiziq perimetringi limiti sifatida ta'riflanadi. Siniq chiziq perimetri yig'indiga (integral yig'indiga) keladi va uning limiti aniq integralni ifodalaydi.

1. Faraz qilaylik, AB yoy  $y = f(x)$  ( $a \leq x \leq b$ ) tenglama bilan aniqlansin. Bunda  $f(x)$  funktsiya  $[a,b]$  segmentda aniqlangan uzluksiz va uzluksiz  $f'(x)$  hosilaga ega bo'lsin. AB yoyning uzunligi

$$I = \int_c^b \sqrt{1 + f'^2(x)} dx \quad (1)$$

*bo'ladi.*

1. Faraz qilaylik, AB yoy

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad (\alpha \leq t \leq \beta)$$

tenglamalar sistemasi bilan aniqlansin (bu holda egri chiziq parametrik holda berilgan deyiladi). Bunda  $x = x(t)$ ,  $y = y(t)$  funktsiyalar  $[\alpha, \beta]$  da aniqlangan, uzlusiz va uzlusiz  $x'(t)$ ,  $y'(t)$  hosilalarga ega.

AB yoyning uzunligi

$$l = \int_{\alpha}^{\beta} \sqrt{x'^2(t) + y'^2(t)} dt$$

bo'ladi.

3. Faraz qilaylik, AB egri chiziq qutb koordinatalar sistemasida  $\rho = \rho(\theta)$  ( $\alpha \leq \theta \leq \beta$ ) funktsiya bilan berilgan bo'lsin. Bunda  $\rho = \rho(\theta)$  funktsiya  $[\alpha, \beta]$  segmentda uzlusiz va  $\rho'(\theta)$  hosilaga ega. Bu holda AB egri chiziqning uzunligi

$$\rho = \int_{\alpha}^{\beta} \sqrt{\rho'^2(\theta) + \rho(\theta)^2} d\theta \quad (3)$$

bo'ladi.

#### 4. Aniq integralning biologiyaga ba'zi tadbiqlari.

##### Populyatsiya miqdori.

Vaqt o'tishi bilan populyatsiya miqdori o'zgarib turadi. Agar populyatsiya uchun sharoit yetarlicha yaxshi bo'lsa, u holda tug'ilish soni o'lishga nisbatan ko'p bo'ladi. Populyatsiya tezligini  $v = v(t)$  (birlik  $t$  vaqt ichida) bilan belgilaymiz. Eski populyatsiya yashash joyida  $v(t)$  tezlik kamayadi va asta - sekin nolga yaqinlashadi. Lekin populyatsiya yosh bo'lsa, o'zaro bir-biri bilan bo'lган munosabatlari hali o'rnatilmagan bo'lsa yoki mavjud bo'lган ba'zi tashqi ta'sirlar bunga ta'sir qilsa, misol uchun insonning aralashuvni, u holda  $v(t)$  tezlik sezilarli darajada ko'payib yoki kamayib to'radi.

Agar populyatsiya tezligi  $v(t)$  ma'lum bo'lsa, u holda biz  $t_0$  dan T vaqtgacha bo'lган oraliqda populyatsiya miqdorining o'sishini topa olamiz. Haqiqatdan ham t vaqt ichida  $v(t)$  ta'rifidan, bu funktsiya  $N(t)$  populyatsiya miqdoridan olingan hosilaga teng va bundan kelib chiqadiki  $N(t)$  populyatsiya

miqdori  $v(t)$  funktsiyaning boshlang'ich funktsiyasi. Shuning uchun

$$N(T) - N(t_0) = \int_{t_0}^T v(t) dt \quad (1)$$

Ma'lumki juda yaxshi sharoitda populyatsiya tezligining o'sishi  $v(t) = ae^{kt}$  bo'ladi. Populyatsiya bu holda eksaymaydi. Bu holda (1) formuladan foydalanib, quyidagiga ega bo'lamiz:

$$N(T) = N(t_0) + a \int_{t_0}^T e^{kt} dt = N(t_0) + \frac{a}{k} e^{kt} \Big|_{t_0}^T = N(t_0) + \frac{a}{k} (e^{kT} - e^{kt_0}) \quad (2)$$

Bu formula orqali ba'zi zambrug'larning (penitsillin ajratib chiqaradigan) miqdorini hisoblab chiqarish mumkin.

### **Populyatsiya biomassasi.**

Shunday biomassalarni qaraymizki hayoti davomida massasi sezilarli o'zgaradi va shu populyatsiya umumiylar biomassasini hisoblaymiz. Aytaylik,  $\tau$  qaysidir birlik vaqtning o'sishini aniqlasim,  $N(\tau)$  o'sishi  $\tau$  ga teng populyatsiyaning maxsus miqdori.  $R(\tau)$  — maxsus  $\tau$  o'sishidagi o'rtacha massa,  $M(\tau) = 0$  dan  $\tau$  o'sishgacha bo'lgan biomassasi.

Shuni aniqlaymizki,  $N(\tau)R(\tau)$  ko'paytma  $\tau$  o'sishdagi biomassaga teng. Ayirmani qaraylik,

$$M(\tau + \Delta\tau) - M(\tau)$$

Yuqoridaqgi ayirma  $\tau$  dan  $\tau + \Delta\tau$  o'sishdagi biomassasi, quyidagini qanoatlantiradi:

$$N\left(\begin{smallmatrix} \vee \\ \tau \end{smallmatrix}\right)P\left(\begin{smallmatrix} \vee \\ \tau \end{smallmatrix}\right)\Delta\tau \leq M(\tau + \Delta\tau) - M(\tau) \leq N\left(\begin{smallmatrix} \wedge \\ \tau \end{smallmatrix}\right)P\left(\begin{smallmatrix} \wedge \\ \tau \end{smallmatrix}\right)\Delta\tau \quad (3)$$

bu yerda  $N\left(\begin{smallmatrix} \vee \\ \tau \end{smallmatrix}\right)P\left(\begin{smallmatrix} \vee \\ \tau \end{smallmatrix}\right)$  — etarlicha kichik,  $N\left(\begin{smallmatrix} \wedge \\ \tau \end{smallmatrix}\right)P\left(\begin{smallmatrix} \wedge \\ \tau \end{smallmatrix}\right)$  — etarlicha katta miqdor ( $[\tau, \tau + \Delta\tau]$  oraliqda  $N\left(\begin{smallmatrix} \wedge \\ \tau \end{smallmatrix}\right)P\left(\begin{smallmatrix} \wedge \\ \tau \end{smallmatrix}\right)$  funktsiyaning).  $\Delta\tau > 0$  ni hisobga olib (3) tengsizlikdan quyidagiga kelamiz.

$$N\left(\frac{\vee}{\tau}\right)P\left(\frac{\vee}{\tau}\right) \leq \frac{M(\tau + \Delta\tau) - M(\tau)}{\Delta\tau} \leq N\left(\frac{\wedge}{\tau}\right)P\left(\frac{\wedge}{\tau}\right) \quad (4),$$

$N(\tau)R(\tau)$  funktsiyalar uzlusizligidan (yani  $N(\tau)$  va  $R(\tau)$  larning uzlusiz bulgani uchun) quyidagiga kelamiz.

$$\lim_{\Delta\tau \rightarrow 0} \left[ N\left(\frac{\vee}{\tau}\right)P\left(\frac{\vee}{\tau}\right) \right] = \lim_{\Delta\tau \rightarrow 0} \left[ N\left(\frac{\wedge}{\tau}\right)P\left(\frac{\wedge}{\tau}\right) \right] = N(\tau)P(\tau),$$

bundan  $\lim_{\Delta\tau \rightarrow 0} \frac{M(\tau + \Delta\tau) - M(\tau)}{\Delta\tau} = N(\tau)P(\tau)$  yoki  $\frac{dM(\tau)}{d\tau} = N(\tau)P(\tau)$  kelib chiqadiki,  $M(\tau)$  biomassä  $N(\tau)R(\tau)$  larning boshlang'ich funktsiyasi. Demak,

$$M(T) - M(0) = \int_0^T N(\tau)P(\tau)d\tau,$$

bu yerda  $T$  – berilgan biomassaning maksimal yoshi.  $M(0)$  aniqlik nolga teng, u holda

$$M(T) = \int_0^T N(\tau)P(\tau)d\tau$$

bu ifoda orqali populyatsiyaning umumiy biomassasini hisoblaymiz.

### **Nazorat savollari.**

1. Aniq integral.
2. Aniq integral xossalari.
3. O'rta qiymat haqidagi teorema.
4. Yoy uzunligini hisoblash.
5. Tekis shaklning yuzasini hisoblash.
6. Aniq integralning biologik masalalarga tadbiqi.

**Mustaqil yechish uchun misollar va masalalar.**

**1-masala. Aniqmas integral topilsin.**

- 1.1.  $\int (4 - 3x)e^{-3x} dx$
- 1.3.  $\int (3x + 4)e^{3x} dx$
- 1.5.  $\int (4 - 16x)\sin 4x dx$
- 1.7.  $\int (1 - 6x)e^{2x} dx$
- 1.9.  $\int \ln(4x^2 + 1) dx$
- 1.11.  $\int \arctg \sqrt{6x - 1} dx$
- 1.13.  $\int e^{-3x}(2 - 9x) dx$
- 1.15.  $\int \arctg \sqrt{3x - 1} dx$
- 1.17.  $\int (5x + 6)\cos 2x dx$
- 1.19.  $\int (x\sqrt{2} - 3)\cos 2x dx$
- 1.21.  $\int (2x - 5)\cos 4x dx$
- 1.23.  $\int (x + 5)\sin 3x dx$
- 1.25.  $\int (4x + 3)\sin 5x dx$
- 1.27.  $\int (\sqrt{2} - 8x)\sin 3x dx$
- 1.29.  $\int \frac{xdx}{\sin^2 x}$
- 1.2.  $\int \arctg \sqrt{4x - 1} dx$
- 1.4.  $\int (4x - 2)\cos 2x dx$
- 1.6.  $\int (5x + 2)e^{3x} dx$
- 1.8.  $\int \ln(x^2 + 4) dx$
- 1.10.  $\int (2 - 4x)\sin 2x dx$
- 1.12.  $\int e^{-2x}(4x - 3) dx$
- 1.14.  $\int \arctg \sqrt{2x - 1} dx$
- 1.16.  $\int \arctg \sqrt{5x - 1} dx$
- 1.18.  $\int (3x - 2)\cos 5x dx$
- 1.20.  $\int (4x + 7)\cos 3x dx$
- 1.22.  $\int (8 - 3x)\cos 5x dx$
- 1.24.  $\int (2 - 3x)\sin 2x dx$
- 1.26.  $\int (7x - 10)\sin 4x dx$
- 1.28.  $\int \frac{xdx}{\cos^2 x}$
- 1.30.  $\int x \sin^2 x dx$

**2-masala. Aniqmas integral hisoblansin.**

- 2.1.  $\int \frac{dx}{x\sqrt{x^2 + 1}}$
- 2.3.  $\int \frac{dx}{x\sqrt{x^2 - 1}}$
- 2.5.  $\int \frac{xdx}{\sqrt{x^4 + x^2 + 1}}$
- 2.7.  $\int \lg x \ln \cos x dx$
- 2.9.  $\int \frac{x^3}{(x^2 + 1)^2} dx$
- 2.11.  $\int \frac{\sin x - \cos x}{(\cos x + \sin x)^3} dx$
- 2.2.  $\int \frac{1 + \ln x}{x} dx$
- 2.4.  $\int \frac{x^2 + \ln x^2}{x} dx$
- 2.6.  $\int \frac{(ar \cos x)^3 - 1}{\sqrt{1 - x^2}} dx$
- 2.8.  $\int \frac{\lg(x+1)}{\cos^2(x+1)} dx$
- 2.10.  $\int \frac{1 - \cos x}{(x - \sin x)^2} dx$
- 2.12.  $\int \frac{x \cos + \sin x}{(x \sin x)^2} dx$

$$2.13. \int \frac{x^3 + x}{x^4 + 1} dx$$

$$2.15. \int \frac{xdx}{x^3 - 1}$$

$$2.17. \int \frac{(x^2 + 1)dx}{(x^3 + 3x + 1)^5}$$

$$2.19. \int \frac{x^3 dx}{x^2 + 4}$$

$$2.21. \int \frac{2\cos x + 3\sin x}{(2\sin x - 3\cos x)^3} dx$$

$$2.23. \int \frac{\frac{1}{2}x + 1}{(x^2 + x)^2} dx$$

$$2.25. \int \frac{x+1/x}{x^2+1} dx$$

$$2.27. \int \frac{\operatorname{arctg} x + x}{1+x^2} dx$$

$$2.29. \int \frac{x^3 dx}{x^2 + 1}$$

$$2.14. \int \frac{xdx}{x^4 - x^2 - 1}$$

$$2.16. \int \frac{1 + \ln(x-1)}{x-1} dx$$

$$2.18. \int \frac{4\operatorname{arctg} x - x}{1+x^2} dx$$

$$2.20. \int \frac{x + \cos x}{x^2 + 2\sin x} dx$$

$$2.22. \int \frac{8x - \operatorname{arctg} 2x}{1+4x^2} dx$$

$$2.24. \int \frac{x}{x^4 + 1} dx$$

$$2.26. \int \frac{x - \frac{1}{x}}{x^2 + 1} dx$$

$$2.28. \int \frac{x - (\operatorname{arctg} x)^4}{1+x^2} dx$$

$$2.30. \int \frac{(\operatorname{arc}\sin x)^2 + 1}{\sqrt{1-x^2}} dx$$

### 3-masala. Aniqmas integral hisoblansin.

$$3.1. \int \frac{x^3 + 1}{x^2 - x} dx$$

$$3.3. \int \frac{x^3 - 17}{x^2 - 4x + 3} dx$$

$$3.5. \int \frac{2x^3 - 1}{x^2 + x - 6} dx$$

$$3.7. \int \frac{x^3 + 2x^2 + 3}{(x-1)(x-2)(x-3)} dx$$

$$3.9. \int \frac{x^3}{(x-1)(x+1)(x+1)} dx$$

$$3.11. \int \frac{x^3 - 3x^2 - 12}{(x-4)(x-3)x} dx$$

$$3.13. \int \frac{3x^3 - 2}{x^3 - x} dx$$

$$3.15. \int \frac{x^5 - x^3 + 1}{x^2 - x} dx$$

$$3.2. \int \frac{3x^3 + 1}{x^2 - 1} dx$$

$$3.4. \int \frac{2x^3 + 5}{x^2 - x - 2} dx$$

$$3.6. \int \frac{3x^3 + 25}{x^2 + 3x + 2} dx$$

$$3.8. \int \frac{3x^3 + 2x^2 + 1}{(x+2)(x-2)(x-1)} dx$$

$$3.10. \int \frac{x^3 - 3x^2 - 12}{(x-4)(x-3)(x-2)} dx$$

$$3.12. \int \frac{4x^3 + x^2 + 2}{x(x-1)(x-2)} dx$$

$$3.14. \int \frac{x^3 - 3x^2 - 12}{(x-4)(x-2)^x} dx$$

$$3.16. \int \frac{x^5 + 3x^3 - 1}{x^2 + x} dx$$

- 3.17.  $\int \frac{2x^5 - 8x^3 + 3}{x^2 - 2x} dx$
- 3.19.  $\int \frac{x^5 + 9x^3 + 4}{x^2 + 3x} dx$
- 3.21.  $\int \frac{x^3 - 5x + 5x + 23}{(x-1)(x+1)(x-5)} dx$
- 3.23.  $\int \frac{2x^4 - 5x^2 - 8x - 8}{x(x-2)(x+2)} dx$
- 3.25.  $\int \frac{3x^4 + 3x^3 - 5x^2 + 2}{x(x-1)(x+2)} dx$
- 3.27.  $\int \frac{x^5 - x^4 - 6x^3 + 13x + 6}{x(x-5)(x+2)} dx$
- 3.29.  $\int \frac{2x^4 + 2x^3 - 3x^2 + 2x - 9}{x(x-1)(x+3)} dx$
- 3.18.  $\int \frac{3x^5 - 12x^3 - 7}{x^2 + 2x} dx$
- 3.20.  $\int \frac{-x^5 + 25x^3 + 1}{x^2 + 5x} dx$
- 3.22.  $\int \frac{x^5 + 2x^4 - 2x^3 + 5x^2 - 7x + 9}{(x+3)(x-1)^4} dx$
- 3.24.  $\int \frac{4x^4 + 2x^2 - x - 3}{x(x-1)(x+1)} dx$
- 3.26.  $\int \frac{2x^4 + 2x^3 - 41x^2 + 20}{x(x-4)(x+5)} dx$
- 3.28.  $\int \frac{3x^3 - x^2 - 12x - 2}{x(x+1)(x-2)} dx$
- 3.30.  $\int \frac{2x^3 - x^2 - 7x - 12}{x(x-3)(x+1)} dx$

#### 4-masala. Aniq integral hisoblansin.

- 4.1.  $\int_{-2}^0 (x^2 + 5x + 6) \cos 2x dx$
- 4.3.  $\int_{-1}^0 (x^2 + 4x + 3) \cos x dx$
- 4.5.  $\int_{-4}^0 (x^2 + 7x + 12) \cos x dx$
- 4.7.  $\int_0^{\pi} (9x^2 + 9x + 11) \cos 3x dx$
- 4.9.  $\int_0^{2\pi} (3x^2 + 5) \cos 2x dx$
- 4.11.  $\int_0^{2\pi} (3 - 7x^2) \cos 2x dx$
- 4.13.  $\int_{-1}^0 (x^2 + 2x + 1) \sin 3x dx$
- 4.15.  $\int_0^{\pi} (x^2 - 3x + 2) \sin x dx$
- 4.17.  $\int_{-3}^0 (x^2 + 6x + 9) \sin 2x dx$
- 4.19.  $\int_0^{\pi/2} (1 - 5x^2) \sin x dx$
- 4.2.  $\int_{-2}^0 (x^2 - 4) \cos 3x dx$
- 4.4.  $\int_{-2}^0 (x+2)^2 \cos 3x dx$
- 4.6.  $\int_0^{\pi} (2x^2 + 4x + 7) \cos 2x dx$
- 4.8.  $\int_0^{\pi} (8x^2 + 16x + 17) \cos 4x dx$
- 4.10.  $\int_0^{2\pi} (2x^2 - 15) \cos 3x dx$
- 4.12.  $\int_0^{2\pi} (1 - 8x^2) \cos 4x dx$
- 4.14.  $\int_0^3 (x^2 - 3x) \sin 2x dx$
- 4.16.  $\int_0^{\pi/2} (x^2 - 5x + 6) \sin 3x dx$
- 4.18.  $\int_0^{\pi/4} (x^2 + 17.5) \sin 2x dx$
- 4.20.  $\int_{\pi/4}^{\pi} (3x - x^2) \sin 2x dx$

$$4.21. \int_1^2 x \ln^2 x dx$$

$$4.22. \int_1^e \frac{\ln^2 x dx}{\sqrt{x}}$$

$$4.23. \int_1^8 \frac{\ln^2 x dx}{\sqrt[3]{x^2}}$$

$$4.24. \int_0^1 (x+1) \ln^2(x+1) dx$$

$$4.25. \int_2^3 (x-1)^3 \ln^2(x-1) dx$$

$$4.26. \int_{-1}^0 (x+2)^3 \ln^2(x+2) dx$$

$$4.27. \int_0^2 (x+1)^2 \ln^2(x+1) dx$$

$$4.28. \int_1^e \sqrt{x} \ln^2 x dx$$

$$4.29. \int_{-1}^1 x^2 e^{-\frac{x}{2}} dx$$

$$4.30. \int_0^1 x^2 e^{3x} dx$$

### 5-masala. Aniq integral hisoblansin.

$$5.1. \int_{e+1}^{e+1} \frac{1 + \ln(x-1)}{x-1} dx$$

$$5.2. \int_0^1 \frac{(x^2+1) dx}{(x^3+3x+1)^2}$$

$$5.3. \int_0^1 \frac{4 \operatorname{arctg} x - x}{1+x^2} dx$$

$$5.4. \int_0^2 \frac{x^3 dx}{x^2+4}$$

$$5.5. \int_{\pi}^{2\pi} \frac{x + \cos x}{x^2 + 2\sin x} dx$$

$$5.6. \int_0^{\pi/4} \frac{2\cos + 3\sin x}{(2\sin x - 3\cos x)^3} dx$$

$$5.7. \int_0^{1/2} \frac{8x - \operatorname{arctg} 2x}{1+4x^2} dx$$

$$5.8. \int_1^4 \frac{\frac{1}{2\sqrt{x}} + 1}{(\sqrt{x} + x)^2} dx$$

$$5.9. \int_0^1 \frac{x dx}{x^4 + 1}$$

$$5.10. \int_{\sqrt{3}}^{\sqrt{8}} \frac{x + \frac{1}{x}}{\sqrt{x^2 + 1}} dx$$

$$5.11. \int_{\sqrt{3}}^{\sqrt{8}} \frac{x - \frac{1}{x}}{\sqrt{x^2 + 1}} dx$$

$$5.12. \int_0^{\sqrt{3}} \frac{\operatorname{arctg} x + x}{1+x^2} dx$$

$$5.13. \int_0^{\sqrt{3}} \frac{x - (\operatorname{arctg} x)^4}{1+x^2} dx$$

$$5.14. \int_0^1 \frac{x^3}{x^2 + 1} dx$$

$$5.15. \int_0^{\sin 1} \frac{(\operatorname{arcSin} x)^2 + 1}{\sqrt{1-x^2}} dx$$

$$5.16. \int_1^3 \frac{1 - \sqrt{x}}{\sqrt{x}(x+1)} dx$$

$$5.17. \int_{\sqrt{3}}^{\sqrt{8}} \frac{dx}{x\sqrt{x^2+1}}$$

$$5.18. \int_1^e \frac{1 + \ln x}{x} dx$$

$$5.19. \int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}}$$

$$5.20. \int_1^e \frac{x^2 + \ln x^2}{x} dx$$

$$5.21. \int_0^1 \frac{x dx}{x^4 + x^2 + 1}$$

$$5.22. \int_0^1 \frac{x^3 dx}{(x^2 + 1)^2}$$

$$5.23. \int_0^{\pi/4} \ln \cos x dx$$

$$5.25. \int_0^{\frac{\pi}{2}} \frac{(\arccos x)^3 - 1}{\sqrt{1-x^2}} dx$$

$$5.27. \int_{\pi}^{\pi/4} \frac{\sin x - \cos x}{(\cos x + \sin x)^5} dx$$

$$5.29. \int_0^1 \frac{x^3 + x}{x^4 + 1} dx$$

$$5.24. \int_{-1}^0 \frac{\operatorname{tg}(x+1)}{\cos^2(x+1)} dx$$

$$5.26. \int_{\pi}^{2\pi} \frac{1 - \cos x}{(x - \sin x)^2} dx$$

$$5.28. \int_{\pi/4}^{\pi/2} \frac{x \cos x + \sin x}{(x \sin x)^2} dx$$

$$5.30. \int_2^0 \frac{x dx}{\sqrt[3]{x-1}}$$

### 6-masala. Aniq integral hisoblansin.

$$6.1. \int_{\pi/2}^{2\arctg 2} \frac{dx}{\sin^2 x (1 - \cos x)}$$

$$6.3. \int_{\pi/2}^{2\arctg 2} \frac{dx}{\sin^2 x (1 + \cos x)}$$

$$6.5. \int_0^{\pi/2} \frac{\cos x - \sin x}{(1 + \sin x)^2} dx$$

$$6.7. \int_{2\arctg \frac{1}{2}}^{\frac{\pi}{2}} \frac{dx}{\sin x (1 - \sin x)}$$

$$6.9. \int_0^{\pi/2} \frac{\cos x dx}{5 + 4 \cos x}$$

$$6.11. \int_{\pi/3}^{\pi/2} \frac{x + \cos x dx}{1 + \sin x - \cos x}$$

$$6.13. \int_0^{\pi/2} \frac{\sin x dx}{1 + \cos x + \sin x}$$

$$6.15. \int_0^{\pi} \frac{\cos x dx}{1 + \cos x + \sin x}$$

$$6.17. \int_{-\frac{\pi}{3}}^0 \frac{\cos x dx}{1 + \cos x - \sin x}$$

$$6.19. \int_0^{\pi/2} \frac{\cos x dx}{(1 + \cos x - \sin x)^2}$$

$$6.21. \int_0^{\pi/2} \frac{\sin x dx}{(1 + \sin x)^2}$$

$$6.2. \int_0^{\pi/2} \frac{\cos x dx}{2 + \cos x}$$

$$6.4. \int_{2\arctg \frac{1}{2}}^{\pi/2} \frac{\cos x dx}{(1 - \cos x)^3}$$

$$6.6. \int_{2\arctg \frac{3}{2}}^{\pi/2} \frac{dx}{\cos x (1 - \cos x)}$$

$$6.8. \int_{2\arctg \frac{1}{2}}^{\pi/2} \frac{dx}{(1 + \sin x - \cos x)^2}$$

$$6.10. \int_0^{2\pi/3} \frac{1 + \sin x}{1 + \cos x + \sin x} dx$$

$$6.12. \int_0^{\pi/2} \frac{(1 + \cos x) dx}{1 + \cos x + \sin x}$$

$$6.14. \int_0^{\frac{\pi}{2}} \frac{1 + \sin x}{(1 - \sin x)^2} dx$$

$$6.16. \int_0^{2\arctg \frac{1}{3}} \frac{\cos x dx}{(1 + \cos x - \sin x)^2}$$

$$6.18. \int_{-\frac{\pi}{2}}^0 \frac{\cos x dx}{(1 + \cos x - \sin x)^2}$$

$$6.20. \int_0^{\frac{\pi}{2}} \frac{(1 - \sin x) dx}{\cos x (1 + \cos x)}$$

$$6.22. \int_0^{\pi/2} \frac{\sin x dx}{(1 + \sin x + \cos)^2}$$

$$6.23. \int_{-\frac{\pi}{2}}^0 \frac{\sin x dx}{(1 + \cos x - \sin x)^2}$$

$$6.25. \int_0^{\frac{\pi}{2}} \frac{\sin^2 x dx}{(1 + \cos x + \sin x)^2}$$

$$6.27. \int_{\frac{\pi}{2}}^{2\arcsin 2} \frac{dx}{\sin x(1 + \sin x)}$$

$$6.29. \int_0^{\frac{\pi}{2}} \frac{\sin x dx}{2 + \sin x}$$

$$6.24. \int_{-\frac{\pi}{3}}^0 \frac{\cos^2 x dx}{(1 + \cos x - \sin x)^2}$$

$$6.26. \int_0^{2\pi/3} \frac{\cos^2 x dx}{(1 + \cos x + \sin x)^2}$$

$$6.28. \int_0^{\frac{\pi}{2}} \frac{dx}{(1 + \sin x + \cos x)^2}$$

$$6.30. \int_0^{\frac{\pi}{4}} \frac{dx}{\cos x(1 + \cos x)}$$

**7-masala. Aniq integral hisoblansin.**

$$7.1. \int_{-\frac{\pi}{2}}^{\pi} 2^8 \sin^8 x dx$$

$$7.16. \int_0^{\frac{\pi}{4}} \sin^8 \frac{x}{4} dx$$

$$7.2. \int_0^{\frac{\pi}{2}} 2^4 \sin^6 x \cos^2 x dx$$

$$7.17. \int_0^{\frac{\pi}{2}} 2^4 \sin^6 \frac{x}{2} \cos^2 \frac{x}{2} dx$$

$$7.3. \int_0^{2\pi} \sin^4 x \cos^4 x dx$$

$$7.18. \int_{-\frac{\pi}{2}}^0 2^8 \sin^4 x \cos^4 x dx$$

$$7.4. \int_0^{\frac{\pi}{2}} \sin^2 \frac{x}{4} \cos^6 \frac{x}{4} dx$$

$$7.19. \int_{\frac{\pi}{12}}^{\frac{\pi}{2}} 2^8 \sin^2 x \cos^6 x dx$$

$$7.5. \int_0^{\frac{\pi}{2}} 2^4 \cos^8 \frac{x}{2} dx$$

$$7.20. \int_0^{\frac{\pi}{2}} 2^4 \cos^8 x dx$$

$$7.6. \int_{-\frac{\pi}{2}}^0 2^8 \sin^8 x dx$$

$$7.21. \int_0^{\frac{\pi}{4}} \sin^8 x dx$$

$$7.7. \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2^8 \sin^6 x \cos^2 x dx$$

$$7.22. \int_0^{\frac{\pi}{4}} \sin^6 \frac{x}{4} \cos^2 \frac{x}{4} dx$$

$$7.8. \int_0^{\frac{\pi}{2}} 2^4 \sin^4 x \cos^4 x dx$$

$$7.23. \int_0^{\frac{\pi}{2}} 2^4 \sin^4 \frac{x}{2} \cos^4 \frac{x}{2} dx$$

$$7.9. \int_0^{2\pi} \sin^2 x \cos^6 x dx$$

$$7.24. \int_{-\frac{\pi}{2}}^0 2^8 \sin^2 x \cos^6 x dx$$

$$7.10. \int_0^{\frac{\pi}{2}} \cos^8 \frac{x}{4} dx$$

$$7.25. \int_{-\frac{\pi}{2}}^0 2^8 \cos^8 x dx$$

$$7.11. \int_0^{\frac{\pi}{2}} 2^4 \sin^8 \frac{x}{2} dx$$

$$7.26. \int_0^{\frac{\pi}{4}} 2^4 \sin^8 x dx$$

$$7.12. \int_{-\pi}^0 2^8 \sin^6 x \cos^2 x dx$$

$$7.13. \int_{\frac{\pi}{2}}^{\pi} 2^8 \sin^4 x \cos^4 x dx$$

$$7.14. \int_0^{\pi} 2^4 \sin^2 x \cos^6 x dx$$

$$7.15. \int_0^{\frac{\pi}{2}} \cos^8 x dx$$

$$7.27. \int_0^{\frac{\pi}{2}} \sin^6 x \cos^2 x dx$$

$$7.28. \int_0^{\frac{\pi}{2}} \sin^4 \frac{x}{4} \cos^4 \frac{x}{4} dx$$

$$7.29. \int_0^{\frac{\pi}{2}} \sin^4 3x \cos^4 3x dx$$

$$7.30. \int_{-\frac{\pi}{2}}^0 2^8 \cos^8 x dx$$

**8-masala. Aniq integral hisoblansin.**

$$8.1. \int_0^{16} \sqrt{256 - x^2} dx$$

$$8.3. \int_0^5 \frac{dx}{(25 + x^2) \sqrt{25 + x^2}}$$

$$8.5. \int_0^{\frac{\sqrt{5}}{2}} \frac{dx}{(5 - x^2)^{\frac{3}{2}}}$$

$$8.7. \int_0^{\frac{\sqrt{2}}{2}} \frac{x^4 dx}{\sqrt{(1 - x^2)^3}}$$

$$8.9. \int_0^1 \frac{x^4 dx}{(2 - x^2)^{\frac{3}{2}}}$$

$$8.11. \int_0^2 \sqrt{4 - x^2} dx$$

$$8.13. \int_0^4 x^2 \sqrt{16 - x^2} dx$$

$$8.15. \int_0^5 x^2 \sqrt{25 - x^2} dx$$

$$8.17. \int_0^{4\sqrt{3}} \frac{dx}{\sqrt{(64 - x^2)^3}}$$

$$8.19. \int_0^{2\sqrt{2}} \frac{x^4 dx}{(16 - x^2) \sqrt{16 - x^2}}$$

$$8.21. \int_0^1 x^2 \sqrt{1 - x^2} dx$$

$$8.2. \int_0^1 x^2 \sqrt{1 - x^2} dx$$

$$8.4. \int_0^{36} \frac{dx}{(9 + x^2)^{\frac{3}{2}}}$$

$$8.6. \int_1^2 \frac{\sqrt{x^2 - 1}}{x^4} dx$$

$$8.8. \int_0^3 \frac{dx}{\sqrt[3]{(4 - x^2)^3}}$$

$$8.10. \int_0^2 \frac{x^2 dx}{\sqrt{16 - x^2}}$$

$$8.12. \int_0^4 \frac{dx}{(16 + x^2)^{\frac{3}{2}}}$$

$$8.14. \int_0^{\frac{5\sqrt{2}}{2}} \frac{x^2 dx}{\sqrt{25 - x^2}}$$

$$8.16. \int_0^4 x^2 \sqrt{16 - x^2} dx$$

$$8.18. \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \frac{\sqrt{x^2 - 2}}{x^4} dx$$

$$8.20. \int_{-3}^3 x^2 \sqrt{9 - x^2} dx$$

$$8.22. \int_0^1 x^2 \sqrt{1 - x^2} dx$$

## VII BOB.

### IKKI O'ZGARUVCHILI FUNKTSIYA TUSHUNCHASI.

#### 1-§. Ikki o'zgaruvchili funktsiya tushunchasi.

1<sup>o</sup>. Ushbu  $f: R^2 \rightarrow R$  akslantirish ko'p o'zgaruvchili (ikki o'zgaruvchili) funktsiya tushunchasiga olib keladi.

**1-Ta'rif:** Agar  $E(E \subset R^2)$  to'plamdag'i har bir  $(x, y)$  nuqtaga biror qoida yoki qonunga ko'ra bitta haqiqiy  $u$  son ( $u \in R$ ) mos qo'yilgan bo'lса,  $E$  to'planda **ikki o'zgaruvchili funktsiya berilgan (aniqlangan)** deyiladi.

Uni

$$f: (x, y) \rightarrow u \text{ yoki } u = f(x, y)$$

kabi belgilanadi. Bunda  $E$  funktsiyaning aniqlanish to'plami,  $x, y$  — funktsiya argumentlari,  $u$  esa  $x$  va  $y$  larning funktsiyasi deyiladi.

Masalan:  $f$  — har bir  $(x, y) \in \{(x, y) \in R^2 : \rho((x, y), (0, 0)) \leq 1\} = E$  nuqtaga ushbu

$$(x, y) \rightarrow \sqrt{1 - x^2 - y^2}$$

qoida bilan bitta haqiqiy sonni mos qo'ysin. Bu holda  $u = \sqrt{1 - x^2 - y^2}$  funktsiyaga ega bo'lamiz.

Aytaylik,  $u = f(x, y)$  funktsiya  $E(E \subset R^2)$  to'plamda berilgan bo'lsin. Ushbu  $\{(x, y, u) : (x, y) \in E, u = f(x, y)\}$  to'plam  $u = f(x, y)$  **funktsiyaning grafigi** deyiladi.

Masalan,  $u = x^2 + y^2$ ,  $u = \sqrt{1 - x^2 - y^2}$  funktsiyalarining grafiklari mos ravishda aylanma paraboloid hamda yuqori yarim sferani ifodalaydi.

Aytaylik,  $u = f(x, y)$  funktsiya  $E \subset R^2$  to'plamda berilgan,  $(x_0, y_0) \in R^2$  nuqta  $E$  ning limit nuqtasi bo'lzin.

**2-Ta'rif:** Agar  $\forall \{(x_n, y_n)\} \quad ((x_n, y_n) \in E, (x_n, y_n) \neq (x_0, y_0), n = 1, 2, \dots)$  ketma-ketlik uchun  $(x_n, y_n) \rightarrow (x_0, y_0)$  da  $f(x_n, y_n) \rightarrow A$  ( $A \in R$ ) bo'lса, u holda  $A$   $f(x, y)$  **funktsiyaning**  $(x, y) \rightarrow (x_0, y_0)$  **dagi limiti (karrali limiti)** deyiladi va

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = A \quad \text{yoki} \quad \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = A$$

kabi belgilanadi.

**2º.** Aytaylik,  $u = f(x, y)$  funktsiya ochiq  $E \subset R^2$  to'plamda berilgan,  $(x_0, y_0) \in E$  nuqtalarga mos ravishda  $\Delta x$  va  $\Delta y$  orttirma beramiz,  $(x_0 + \Delta x, y_0 + \Delta y) \in E$  bo'lzin. Ushbu

$$\Delta f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \quad (1)$$

ayirma  $f(x, y)$  funktsiyaning  $(x_0, y_0)$  nuqtadagi to'liq orttirmasi deyiladi.

**3-Ta'rif:** Agarda  $\Delta x \rightarrow 0$ ,  $\Delta y \rightarrow 0$  da to'liq orttirma  $\Delta f(x_0, y_0)$  nolga intilsa, ya'ni

$$\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0$$

bo'lса,  $u = f(x, y)$  funktsiya  $(x_0, y_0)$  nuqtada uzluksiz deyiladi,

**4-Ta'rif:** Agar  $(x, y) \rightarrow (x_0, y_0)$  da  $f(x, y)$  funktsiyaning limiti mavjud bo'lmasa yoki

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = \infty \quad \text{yoki} \quad \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = A \neq f(x_0, y_0)$$

bo'lса,  $f(x, y)$  funktsiya  $(x_0, y_0)$  nuqtada uzhilishga ega deyiladi.

## 2-§. Ikki o'zgaruvchili funktsiyaning xususiy hosilalari.

**1. Funktsiyaning differentsiyanuvchiligi tushunchasi.**  
**1º.** Ushbu

$$\begin{aligned}\Delta_x f(x_0, y_0) &= f(x_0 + \Delta x, y_0) - f(x_0, y_0) \\ \Delta_y f(x_0, y_0) &= f(x_0, y_0 + \Delta y) - f(x_0, y_0)\end{aligned}$$

orttirmalar funktsiyaning  $(x_0, y_0)$  nuqtadagi xususiy orttirmalari deyiladi.

**1-Ta'rif:** Agar  $\Delta x \rightarrow 0$  da  $\frac{\Delta_x f(x_0, y_0)}{\Delta x}$  nisbatning limiti mavjud va chekli bo'lsa, bu limit  $f(x, y)$  funktsiyaning  $(x_0, y_0)$  nuqtadagi  $x$  o'zgaruvchisi bo'yicha **xususiy hosilasi** deyiladi va  $f'_x(x_0, y_0)$  yoki  $\frac{\partial f(x_0, y_0)}{\partial x}$  kabi belgilanadi:

$$f'_x(x_0, y_0) = \frac{\partial f(x_0, y_0)}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta_x f(x_0, y_0)}{\Delta x}$$

Xuddi shunga o'xshash  $y$  o'zgaruvchi bo'yicha xususiy hosila ta'riflanadi:

$$f'_y(x_0, y_0) = \frac{\partial f(x_0, y_0)}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{\Delta_y f(x_0, y_0)}{\Delta y}$$

Ko'p o'zgaruvchili funktsiyaning xususiy hosilalarini hisoblashda bir o'zgaruvchili funktsiyaning hosilalarini hisoblashdagi ma'lum qoida va jadvallardan foydalanish mumkin.

**Misol.** Ushbu  $f(x, y) = \sqrt{x^2 + y^2}$  funktsiyaning xususiy hosilalari  $(x, y) \neq (0, 0)$  nuqtada

$$f'_x(x, y) = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}, \quad f'_y(x, y) = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}$$

bo'ladi.

Berilgan funktsiya  $(x, y) = (0, 0)$  nuqtada xususiy hosilaga ega emas:

$$f'_x(0, 0) = \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 0^2}}{x} = \lim_{x \rightarrow 0} \frac{|x|}{x} - \text{mavjud emas};$$

$$f'_y(0, 0) = \lim_{y \rightarrow 0} \frac{\sqrt{0^2 + y^2}}{y} = \lim_{y \rightarrow 0} \frac{|y|}{y} - \text{mavjud emas}.$$

**2-Ta'rif:** Agar  $f(x, y)$  funktsiyaning  $(x_0, y_0)$  nuqtadagi to'liq orttirmasini quyidagi

$$\Delta f(x_0, y_0) = A \cdot \Delta x + B \cdot \Delta y + \alpha \cdot \Delta x + \beta \cdot \Delta y \quad (2)$$

ko'rinishda ifodalash mumkin bo'lsa,  $f(x, y)$  funktsiya  $(x_0, y_0)$  nuqtada differentsiallanuvchi deyiladi, bunda  $A, B - o'zgarmas$ ,  $\alpha$  va  $\beta$  lar  $\Delta x$  va  $\Delta y$  larga bog'liq,  $\Delta x \rightarrow 0$ ,  $\Delta y \rightarrow 0$  da  $\alpha \rightarrow 0, \beta \rightarrow 0$ .

Agar  $f(x, y)$  funktsiya  $E \subset R^2$  to'plamning har bir nuqtasida differentsiallanuvchi bo'lsa,  $f(x, y)$  funktsiya  $E$  to'plamda differentsiallanuvchi deyiladi. Shunday qilib,  $f(x, y)$  funktsiyaning differentsiali quyidagicha yoziladi:

$$df = f'_x dx + f'_y dy \text{ yoki } df(x_0, y_0) = \frac{\partial f(x_0, y_0)}{\partial x} dx + \frac{\partial f(x_0, y_0)}{\partial y} dy \quad (3)$$

**Misol.** Agar  $f(x, y) = (x + y)^2$  bo'lsa, u holda

$$f'_x = f'_y = 2(x + y) \text{ va } df = 2(x + y)(dx + dy).$$

**Teorema:** Agar  $f(x, y)$  funktsiya  $(x_0, y_0) \in E$  nuqtada differentsiallanuvchi bo'lsa, funktsiya shu nuqtada uzlusiz bo'ladi.

**Isbot:**  $\Delta f(x_0, y_0) = A \cdot \Delta x + B \cdot \Delta y + \alpha \cdot \Delta x + \beta \cdot \Delta y$  munosabatdan,  $\Delta x \rightarrow 0$ ,  $\Delta y \rightarrow 0$  da  $\Delta f(x_0, y_0) \rightarrow 0$  bo'lishini topamiz.

Demak,  $f(x, y)$  funktsiya  $(x_0, y_0)$  nuqtada uzlusiz.

**Teorema:** Agar  $f(x, y)$  funktsiya  $(x_0, y_0)$  nuqtada differentsiallanuvchi bo'lsa, funktsiyaning shu nuqtadagi xususiy hosilalari  $f'_x(x_0, y_0)$ ,  $f'_y(x_0, y_0)$  mavjud bo'lib,  $f'_x(x_0, y_0) = A$ ,  $f'_y(x_0, y_0) = B$  bo'ladi.

**Isbot:**  $\Delta f(x_0, y_0) = A \cdot \Delta x + B \cdot \Delta y + \alpha \cdot \Delta x + \beta \cdot \Delta y$  da  $\Delta x \neq 0$ ,  $\Delta y = 0$  deb,  $\Delta x \rightarrow 0$  da

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta f(x_0, y_0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{A \cdot \Delta x + \alpha \cdot \Delta x}{\Delta x} = A,$$

ya'ni  $f'_x(x_0, y_0) = A$  bo'lishini topamiz.

Xuddi shunga o'xshash  $f'_y(x_0, y_0)$  ning mavjudligi va  $f'_y(x_0, y_0) = B$  bo'lishi ko'rsatiladi.

Endi murakkab

$$u = f(x, y) = f(\phi(s, t), \psi(s, t)) = F(s, t)$$

funktsiyaning differentsiyalini hisoblaymiz.

(3) formulaga ko'ra

$$df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt$$

bo'ladi. Endi

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s},$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

ekanini e'tiborga olib topamiz:

$$\begin{aligned} df &= \left( \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \right) ds + \left( \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \right) \cdot dt = \\ &= \frac{\partial s}{\partial x} \left( \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \right) + \frac{\partial f}{\partial y} \left( \frac{\partial y}{\partial s} ds + \frac{\partial y}{\partial t} dt \right) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \end{aligned}$$

**Misol.** Agar  $f = xy$  bo'lib, bu yerda  $x = t \sin 2\tau$ ,  $y = t^2 \tau$  bo'lsa, u holda

$$f'_x = y \sin 2\tau + 2x\tau, \quad f'_y = -2yt \cos 2\tau + xt^2$$

bo'ladi.

## 2. Ko'p o'zgaruvchili funktsiyalarning yuqori tartibli hosila va differentsiallari.

Aytaylik,  $f(x, y)$  funktsiya ochiq  $E \subset R^2$  to'plamda berilgan bo'lib,  $\forall (x, y) \in E$  nuqtada  $f'_{,x}(x, y)$ ,  $f'_{,y}(x, y)$  xususiy hosilalarga ega bo'lsin.

**3-Ta'rif:**  $f(x, y)$  funktsiya xususiy hosilalari  $f'_{,x}(x, y)$ ,  $f'_{,y}(x, y)$  larning  $x$  va  $y$  o'zgaruvchilari bo'yicha xususiy hosilalari  $f_{,xy}(x, y)$  funktsiyaning **ikkinchchi tartibli xususiy hosilalari** deyiladi.

Demak,

$$(f'_{,x}(x, y))_y = f''_{,xy}(x, y), \quad (f'_{,y}(x, y))_x = f''_{,yx}(x, y), \quad (f'_{,y}(x, y))_x = f''_{,yy}(x, y).$$

yoki  $\left(f''_y(x,y)\right)_y = f''_{yy}(x,y)$

$$\frac{\partial}{\partial x} \left( \frac{\partial f(x,y)}{\partial x} \right) = \frac{\partial^2 f(x,y)}{\partial x^2}; \quad \frac{\partial}{\partial y} \left( \frac{\partial f(x,y)}{\partial x} \right) = \frac{\partial^2 f(x,y)}{\partial x \partial y};$$

$$\frac{\partial}{\partial x} \left( \frac{\partial f(x,y)}{\partial y} \right) = \frac{\partial^2 f(x,y)}{\partial y \partial x}; \quad \frac{\partial}{\partial y} \left( \frac{\partial f(x,y)}{\partial y} \right) = \frac{\partial^2 f(x,y)}{\partial y^2}.$$

Odatda  $f''_{xx}(x,y)$ ,  $f''_{yy}(x,y)$  hosilalar **aralash hosilalar** deyiladi.

Xuddi shunga o'xshash  $f(x,y)$  funktsiyaning uchinchi, to'rtinchi va hokazo tartibdagi xususiy hosilalari ta'riflanadi. Ushbu

$$f(x,y) = \ln(x^2 + y^2)$$

funktsiyaning ikkinchi tartibli xususiy hosilalarini toping.

Ravshanki,

$$f'_x(x,y) = [\ln(x^2 + y^2)]'_x = \frac{2x}{x^2 + y^2},$$

$$f'_y(x,y) = [\ln(x^2 + y^2)]'_y = \frac{2y}{x^2 + y^2}$$

bo'ladi. Ta'rifdan foydalaniib funktsiyaning ikkinchi tartibli xususiy hosilalarini topamiz:

$$f''_{xx}(x,y) = \left( \frac{2x}{x^2 + y^2} \right)'_x = 2 \cdot \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2},$$

$$f''_{xy}(x,y) = \left( \frac{2x}{x^2 + y^2} \right)'_y = -\frac{2x \cdot 2y}{(x^2 + y^2)^2} = -\frac{4xy}{(x^2 + y^2)^2},$$

$$f''_{yx}(x,y) = \left( \frac{2y}{x^2 + y^2} \right)'_x = -\frac{2y \cdot 2x}{(x^2 + y^2)^2} = -\frac{4xy}{(x^2 + y^2)^2},$$

$$f''_{yy}(x,y) = \left( \frac{2y}{x^2 + y^2} \right)'_y = 2 \cdot \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}.$$

### 3-§. Ikki o'zgaruvchili funktsiyaning ekstremumlari.

Aytaylik,  $f(x, y)$  funktsiya ochiq  $E \subset R^2$  to'plamda berilgan bo'lib,  $(x_0, y_0) \in E$  bo'lsin.

**Ta'rif.** Agar shunday  $U_\delta((x_0, y_0)) \subset E$  atrof topilsaki,  $\forall (x, y) \in U_\delta((x_0, y_0))$  uchun

$$f(x, y) \leq f(x_0, y_0), \quad (f(x, y) \geq f(x_0, y_0))$$

bo'lsa,  $f(x, y)$  funktsiya  $(x_0, y_0)$  **nuqtada maksimumga (minimumga) erishadi** deyiladi.  $(x_0, y_0)$  funktsiyaning **maksimum (minimum) nuqtasi** deyiladi.  $f(x_0, y_0)$  miqdor funktsiyaning **maksimum (minimum) qiymatlari** deyiladi va quyidagicha belgilanadi:

$$f(x_0, y_0) = \max_{(x, y) \in U_\delta} \{f(x, y)\}, \quad \left( f(x_0, y_0) = \min_{(x, y) \in U_\delta} \{f(x, y)\} \right)$$

funktsiyaning maksimum va minimumi umumiy nom bilan uning **ekstremumi** deyiladi.

**Teorema. (Ekstremum mavjudligining zaruriy sharti).** Agar  $f(x, y)$  funktsiya  $(x_0, y_0)$  nuqtada ekstremumga erishib, shu nuqtada chekli xususiy hosilalari  $f'_x$  va  $f'_y$  mavjud bo'lsa, u holda

$$f'_x(x_0, y_0) = 0, \quad f'_y(x_0, y_0) = 0$$

bo'ladi.

Yuqorida keltirilgan shart yetarli shart bo'la olmaydi.

**Misol.**  $z = x^3 + y^3$ ,  $z'_x = 3x^2$ ,  $z'_y = 3y^2$ . Bu funktsiya  $(0, 0)$  nuqtada hosilasi nolga teng, ammo bu funktsiya  $(0, 0)$  nuqtada ekstremumga ega emas, chunki bu nuqtaning ixtiyoriy atrofida har xil ishorali qiymat qabul qiladi, nuqtaning o'zida  $z = 0$ .

**Teorema. (Ekstremum mavjudligining yetarli sharti).**

Faraz qilaylik,  $f(x, y)$  funktsiya  $(x_0, y_0)$  nuqta atrofida birinchi va ikkinchi tartibli uzluksiz xususiy hosilalarga ega bo'lib, quyidagi shartni qanoatlantirsin:

$$f'_x(x_0, y_0) = 0, \quad f'_y(x_0, y_0) = 0$$

*va quyidagicha belgilash kiritamiz:*

$$A = f''_{xx}(x_0, y_0), \quad B = f''_{xy}(x_0, y_0), \quad C = f''_{yy}(x_0, y_0), \quad D = AC - B^2.$$

Unda 1) agar  $D > 0$  bo'lsa, u holda  $(x_0, y_0)$  nuqtada  $f(x, y)$  funktsiya ekstremumga erishadi, agarda  $A < 0$  bo'lsa maksimumga,  $A > 0$  bo'lsa minimumga.

2) agar  $D < 0$  bo'lsa, u holda  $(x_0, y_0)$  nuqtada  $f(x, y)$  funktsiya ekstremumga ega emas.

3) agar  $D = 0$  bo'lsa, u holda  $(x_0, y_0)$  nuqtada  $f(x, y)$  funktsiya ekstremumga erishishi ham, erishmasligi ham mumkin.

**Masala.** Ximik reaktsiya tarkibida  $x, y$  va  $z$  moddalar qatnashadi. Reaktsiya tezligi  $V$  quyidagi qonun bilan ifodalanadi:

$$V = kx^2yz.$$

Shunday  $x, y$  va  $z$  moddalar tarkibini topingki, ximik reaktsiya tezligi  $V$  maksimal bo'lsin.

**Yechilishi:** Aytaylik,

$$x + y + z = 100 \text{ (\%)}, \text{ u holda}$$

$$z = 100 - x - y \text{ va}$$

$$V = kx^2y(100 - x - y) \quad (1)$$

$V$  funktsiyaning xususiy hosilalarini topamiz:

$$\frac{\partial V}{\partial x} = k(200xy - 3x^2y - 2xy^2)$$

$$\frac{\partial V}{\partial y} = k(100xy - x^3 - 2x^2y).$$

nolga tenglaymiz,

$$\begin{cases} 200xy - 3x^2y - 2xy^2 = 0 \\ 100xy - x^3 - 2x^2y = 0 \end{cases}$$

Bilamizki (1) funktsiya  $x = 0$  va  $y = 0$  da maksimumga erishmaydi, u holda

$$\begin{cases} 200 - 3x - 2y = 0 \\ 100 - x - 2y = 0 \end{cases}$$

Sistemani yechib  $x = 50$ ,  $y = 25$  ni topamiz. U holda  $z = 25$ . Bunda (50.25) nuqtada  $\Delta > 0$  va  $A < 0$  ekanini topamiz.

Demak,  $x = 50\%$ ,  $y = 25\%$  va  $z = 25\%$  bo'lgan tarkibda  $\nu$  tezlik maksimal bo'lar ekan.

### Nazorat savollari.

- Ikki o'zgaruvchili funktsiya tushunchasi.
- Ikki o'zgaruvchili funktsiyaning limiti va uzluksizligi.
- Ikki o'zgaruvchili funktsiyaning xususiy hosilalari.
- Ikki o'zgaruvchili funktsiyaning differentialsiali.
- Ko'p o'zgaruvchili funktsiyalarning yuqori tartibli hosila va differentialsallari.
- Ikki o'zgaruvchili funktsiyaning ekstremumlari.
- Ikki o'zgaruvchili funktsiya ekstremumining zaruriy va yetarli shartlari.

### 4-§. Kompleks sonlar.

Quyidagi

$$z = x + iy \quad (1)$$

ko'rinishdagi son **kompleks son** deyiladi, bunda  $x$  va  $y$  — ixtiyoriy haqiqiy sonlar,  $i$  esa  $i^2 = -1$  tenglik bilan aniqlanadigan mavhum birlik,  $x$  va  $y$  sonlar  $z$  kompleks sonning mos ravishda *haqiqiy qismi* va *mavhum qismi* deb ataladi va

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z$$

deb belgilanadi. Kompleks sonning (1) ko'rinishdagi yozuvi uning *algebraik shakli* deyiladi.

$z = x + iy$  kompleks son  $xOy$  dekart koordinatalar tekisligida abstsissasi  $x$ , ordinatasi  $y$  bo'lgan nuqta bilan yoki bu nuqtaning radius-vektori bilan tasvirlanishi mumkin. Bu vektorning uzunligi  $|z|$  kompleks sonning *moduli* deb ataladi va  $|z|$  yoki  $r$  orqadi belgilanadi:

$$|z| = r = \sqrt{x^2 + y^2} \quad (2)$$

Bu vektorning  $Ox$  haqiqiy o'qning musbat yo'naliishi bilan hosil qilgan burchagi  $z$  sonning *argumenti* deb ataladi va  $\arg z$  orqali belgilanadi:

$$\operatorname{tg}(\arg z) = \frac{y}{x} \quad (3)$$

$\arg z$  ko'p qiymatli kattalik va u  $2\pi$  ga karrali songa qadar aniqlikda aniqlangan.  $\arg z$  ning  $-\pi$  dan  $\pi$  gacha bo'lgan oraliqda joylashgan qiymati unirng *bosh qiymati* deyiladi va  $\arg z$  yoki  $\phi$  orqali belgilanadi:

$$-\pi < \arg z \leq \pi.$$

Agar ikkita  $z_1 = x_1 + iy_1$  va  $z_2 = x_2 + iy_2$  kompleks sonning haqiqiy va mavhum qismi mos ravishda teng ya'ni

$$x_1 = y_1, \quad x_2 = y_2$$

bo'lsa, ular teng hisoblanadi.

Faqat mavhum qismining ishorasi bilan bir-biridan farq qiladigan  $z = x + iy$  va  $z = x - iy$  kompleks sonlar *qo'shma kompleks sonlar* deyiladi.

Algebraik shaklda berilgan kompleks sonlar ustida amallar quyidagi qoidalar bo'yicha bajariladi:

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2),$$

$$(x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1),$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

Kompleks sonlarni qo'shish (ayirish) bu sonlarni tasvirlovchi vektorlarni qo'shishga (ayirishga) keltiriladi.

$z_1$  va  $z_2$  kompleks sonlar yig'indisining radius-vektori  $z_1$  va  $z_2$  kompleks sonlar ayirmasining radius-vektori quyidagicha topiladi: ayiluvchining radius-vektori uchini kamayuvchining radius-vektori uchi bilan tutashtirish, so'ngra hosil qilingan vektorni o'z-o'ziga parallel ko'chirib, uning boshini  $O$  nuqtaga joylashtirish lozim.

$z = x + iy$  kompleks o'zgaruvchining asosiy transtsendent funktsiyalari quyidagi tengliklar bilan aniqlanadi:

### *ko'rsatkichli funktsiya*

$$e^z = e^x (\cos y + i \sin y) \quad (1)$$

### *trigonometrik funktsiyalar*

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad (2)$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}; \quad (3)$$

### *giperbolik funktsiyalar*

$$\operatorname{sh} z = \frac{e^z - e^{-z}}{2}, \quad (4)$$

$$\operatorname{ch} z = \frac{e^z + e^{-z}}{2}, \quad (5)$$

$z = x + iy$  kompleks sonning trigonometrik va ko'rsatkichli shakli quyidagi ko'rinishga ega bo'ladi:

$$z = r(\cos \varphi + i \sin \varphi), \quad (6)$$

$$r = r e^{i\varphi}. \quad (7)$$

bunda  $r$  va  $\varphi$  – mos ravishda  $z$  kompleks sonning moduli va argumentining bosh qiymati.

Trigonometrik va ko'rsatkichli shaklda berilgan kompleks sonlar uchun ko'paytirish, bo'lish, musbat butun darajaga ko'tarish, musbat butun darajadan ildiz chiqarish quyidagi formulalar yordamida bajariladi:

$$[r_1(\cos \varphi_1 + i \sin \varphi_1)]X[r_2(\cos \varphi_2 + i \sin \varphi_2)] = r_1 r_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)] \quad (8)$$

$$(r_1 e^{i\varphi_1}) \cdot (r_2 e^{i\varphi_2}) = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}, \quad (8')$$

$$\frac{r_1(\cos \varphi_1 + i \sin \varphi_1)}{r_2(\cos \varphi_2 + i \sin \varphi_2)} = \frac{r_1}{r_2} [\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)], \quad (9)$$

$$\frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}, \quad (9')$$

$$[r(\cos \varphi + i \sin \varphi)]^n = r^n (\cos n\varphi + i \sin n\varphi), \quad (10)$$

$$(r e^{i\varphi})^n = r^n e^{in\varphi}, \quad (10')$$

$$\sqrt[n]{r(\cos \varphi + i \sin \varphi)} = \sqrt[n]{r} \left( \cos \frac{\varphi + 2k\pi}{n} + i \sin \frac{\varphi + 2k\pi}{n} \right), \quad (k = 0, 1, 2, \dots, n-1) \quad (11)$$

$$\sqrt[n]{r e^{i\varphi}} = \sqrt[n]{r e^{i\varphi}} = \sqrt[n]{r} e^{\frac{i\varphi + 2k\pi}{n}}, \quad (k = 0, 1, 2, \dots, n-1) \quad (11')$$

(10) va (11) munosabatlar **Muavr formulalari** deyiladi. Ko'rsatilgan tengliklarning o'ng tomonidagi ifodalarni trigonometrik yoki ko'rsatkichli shaklga keltirishda  $\cos \varphi$ ,  $\sin \varphi$ ,  $e^{i\varphi}$  funktsiyalarning  $2k\pi$  ( $k = \pm 1, \pm 2, \dots$ ) davrga egaligidan foydalaniildi.

**1-misol.** Ushbu tenglamaning haqiqiy yechimlarini toping:

$$(4 + 2i)x + (5 - 3i)y = 13 + i.$$

**Yechilishi.** Tenglamaning chap qismidan haqiqiy va mavhum qismini ajratamiz:

$$(4x + 5y) + i(2x - 3y) = 13 + i.$$

Bundan quyidagini yozib olamiz

$$\begin{cases} 4x + 5y = 13 \\ 2x - 3y = 1. \end{cases}$$

Bu tenglamani yechib quyidagini topamiz:  
 $x = 2$ ,  $y = 1$ .

**2-misol.** Quyidagi kompleks sonning moduli va argumentini toping.

$$z = -\sin \frac{\pi}{8} - i \cos \frac{\pi}{8}.$$

**Yechilishi.** Quyidagiga egamiz:

$$x = -\sin \frac{\pi}{8} < 0, \quad y = -\cos \frac{\pi}{8} < 0.$$

Argumentning bosh qiymati quyidagiga teng:

$$\arg z = -\pi + \arctg \left( \operatorname{ctg} \frac{\pi}{8} \right) = -\pi + \arctg \left( \tg \left( \frac{\pi}{2} - \frac{\pi}{8} \right) \right) = -\pi + \arctg \left( \tg \frac{3}{8} \pi \right) = -\pi + \frac{3}{8} \pi = -\frac{5}{8} \pi.$$

Bundan kelib chiqadi,

$$\operatorname{Arg} z = -\frac{5}{8} \pi + 2k\pi \quad (k \in \mathbb{Z}),$$

$$|z| = \sqrt{\sin^2 \frac{\pi}{8} + \cos^2 \frac{\pi}{8}} = 1.$$

**3-misol.** Ushbu kompleks sonni trigonometrik ko'rinishda yozing.

$$z = -1 - i\sqrt{3}.$$

**Yechilishi.** Quyidagini yozib olamiz,

$$|z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2; \quad \operatorname{tg}\varphi = \frac{-\sqrt{3}}{-1} = \sqrt{3}, \quad \varphi = -\frac{2}{3}\pi.$$

Bundan kelib chiqadi,

$$-1 - i\sqrt{3} = 2 \left[ \cos\left(-\frac{2}{3}\pi\right) + i\sin\left(-\frac{2}{3}\pi\right) \right].$$

**4-misol.**  $(-1+i\sqrt{3})^{60}$  ni hisoblang.

**Yechilishi.**  $z = -1 + i\sqrt{3}$  kompleks sonning trigonometrik ko'rinishini yozamiz,

$$\begin{aligned} -1 + i\sqrt{3} &= 2 \left( \cos \frac{5}{6}\pi + i\sin \frac{5}{6}\pi \right), \\ (-1+i\sqrt{3})^{60} &= 2^{60} \left[ \cos \left( 60 \cdot \frac{5}{6}\pi \right) + i\sin \left( 60 \cdot \frac{5}{6}\pi \right) \right] = 2^{60} (\cos 50\pi + i\sin 50\pi) = 2^{60}. \end{aligned}$$

**Mustaqil yechish uchun misol va masalalar.**

**Quyidagi funktsiyalarning aniqlanish sohasini toping.**

$$1. \quad u = 10 + 2x - y \quad j: \quad XOY \text{ tekisligi.}$$

$$2. \quad u = \frac{5}{2x^2 + y^2} \quad j: \quad (0,0) \text{ dan tashqarii } XOY \text{ tekisligi.}$$

$$3. \quad u = \frac{2}{\sqrt{xy}}, \quad j: \quad \text{I va III chorak:}$$

$$x > 0, \quad y > 0; \quad \text{ba} \quad x < 0, \quad y < 0.$$

$$4. \quad u = \ln(x+y) + 2x - y + 7, \quad j: \quad x+y > 0 \text{ yarim tekislik.}$$

**Quyidagi funktsiyalarning birinchi tartibli xususiy hosilalarini toping.**

$$1. \quad u = x^2 + 5xy^2 - y^3, \quad j: \quad u'_x = 2x + 5y^2, \quad u'_y = 10xy - 3y^2.$$

2.  $u = \sqrt{2x+5y}$ ,  $j: u'_x = \frac{1}{\sqrt{2x+5y}}, u'_y = \frac{5}{2\sqrt{2x+5y}}$ .
3.  $u = \ln(x+y) + 2x - y + 7$ ,  $j: u'_x = \frac{1}{x+y} + 2, u'_y = \frac{1}{x+y} - 1$ .
4.  $u = x^2 \sin y + y^3$ ,  $j: u'_x = 2x \sin y, u'_y = x^2 \cos y + 3y^2$ .

**Quyidagi funktsiyalarni ekstremumga tekshiring.**

- $u = x^3 + 8y^3 - 6xy + 5$ ,  $j: \left(1; \frac{1}{2}\right)$  nuqtada  $u_{\min} = 4$ ;  $(0;0)$  nuqtada ekstremumga ega emas.
- $u = (x-1)^2 + 2y^2$   $j: (1;0)$  nuqtada  $u_{\min} = 0$ .
- $u = 2x^3 - x^2 + xy^2 - 4x + 3$ ,  $j: \left(-\frac{2}{3}; 0\right)$  nuqtada  $u_{\max} = 4\frac{17}{27}$ ,  $(1;0)$  nuqtada  $u_{\min} = 0$ ,  $(0;-2)$ ,  $(0;2)$  nuqtalarda ekstremumga ega emas.
- $u = 3x + 6y - x^2 - xy - y^2$ ,  $j: (0;3)$  nuqtada  $u_{\max} = 9$ .

**Tenglamani yeching.**

- $(3x-i)(2+i) + (x-iy)(1+2i) = 5 + 6i$ ,  $j: x = \frac{20}{17}, y = -\frac{36}{17}$ .
- $\frac{1}{z-i} + \frac{2+i}{1+i} = \sqrt{2}$ , bu yerda  $z = x+iy$ ,  $j:$  haqiqiy yechimi yo'q.
- $(4x-3y) + (3x+5y)i = 10 - (3x-2y-30)i$ ,  $j: x = 4, y = 2$ .
- $(2-7i)x + (8+6i)y = (-6+5i)x - 8$ ,  $j: x = -\frac{1}{3}, y = -\frac{2}{3}$ .

**Quyidagi kompleks sonlarning moduli va argumentini toping.**

- $z = 4 + 3i$ ,  $j: |z| = 5, \varphi = \arctg \frac{3}{4}$ .
- $z = -2 + 2\sqrt{3}i$ ,  $j: |z| = 4, \varphi = \frac{2}{3}\pi$ .
- $z = -7 - i$ ,  $j: |z| = 5\sqrt{2}, \varphi = \arctg \frac{1}{7} - \pi$ .
- $z = -\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}$ ,  $j: |z| = 1, \varphi = \frac{4}{5}\pi$ .

**Ushbu kompleks sonlarni trigonometrik ko'rinishda yozing.**

9.  $-2$ ; j:  $2(\cos \pi + i \sin \pi)$   
10.  $2i$ ; j:  $2\left(\cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi\right)$ .  
11.  $-\sqrt{2} + i\sqrt{2}$ ; j:  $2\left(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi\right)$ .  
12.  $-1 - i\sqrt{3}$ ; j:  $2e^{-\frac{2\pi}{3}i}$ .

**Quyidagi larni hisoblang.**

13.  $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{40}$ ; j:  $-2^{19}(1+i\sqrt{3})$   
14.  $(2-2i)^7$ ; j:  $2^{10}(1+i)$   
15.  $(\sqrt{3}-3i)^6$ ; j: 1728.  
16.  $\left(\frac{1-i}{1+i}\right)^8$ ; j: 1.

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## VIII BOB

### BIRINCHI TARTIBLI DIFFERENTSIAL TENGLAMALAR.

#### 1-§. Asosiy tushunchalar.

$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \quad (1)$$

ko'rinishdagi tenglama n-tartibli oddiy differentsiyal tenglama deyiladi.

Differentsial tenglamaning tartibi deb bu tenglamaga kiruvchi eng yuqori tartibli hosila tartibiga aytildi. Masalan,  $y - xy' = 0$  tenglama birinchi tartibli;  $\frac{d^2x}{dx^2} + \frac{dy}{dx} = \frac{1}{1+e^x}$  tenglama ikkinchi tartibli;  $y'' - 8y = 0$  tenglama esa uchinchi tartibli.

Tenglamadagi  $y$  ning o'rniga qo'yganda uni ayniyatga aylantiruvchi  $\phi(x)$  funktsiya tenglamaning yechimi deyiladi. Yechimning grafigi tenglamaning integral egri chizig'i deyiladi. Agar differentsial tenglamani qanoatlantiruvchi funktsiya oshkormas ko'rinishda, ya'ni  $\phi(x, y) = 0$  ko'rinishdagi munosabat orqali berilgan bo'lsa, u holda tenglama integrali haqida gapiriladi.

(1) differentsial tenglamaning barcha yechimlarini, ixtiyoriy o'zgarmaslar  $S_1, S_2, S_3, \dots, S_n$  ga bog'liq bo'lган

$$y = \phi(x, C_1, C_2, \dots, C_n)$$

munosabat bilan umumiy ko'rinishda ifodalash mumkin. Bu munosabatga (1) differentsial tenglamaning umumiy yechimi deyiladi.

Masalan, birinchi tartibli tenglama uchun umumiy yechim  $y = \phi(x, c)$  ko'rinishga, ikkinchi tartibli tenglama uchun esa  $y = \phi(x_1, c_1, c_2)$  ko'rinishga ega.

Umumiy yechimdan erkli o'zgarmaslarning turli son qiymatlarida hosil qilinadigan yechimlar bu tenglamaning xususiy yechimlari deyiladi.

Umumiy yechim geometrik nuqtai nazardan egri chiziqlar oilasini, xususiy yechim esa bu oilaning birorta egri chizig'ini aniqlaydi. Differentsial tenglamaning xususiy yechimini topish uchun boshlang'ich shartlar beriladi.

**1-Misol.**  $y = 2x$  funktsiya  $x^2y'' - 2xy' + 2y = 0$  tenglamaning yechimi ekanligini tekshiring.

**Yechilishi:**  $y' = 2$ ,  $y'' = 0$  larni tenglamaga qo'yamiz:

$$x^2 \cdot 0 - 2x \cdot 2 + 2 \cdot 2x = 0 - 4x + 4x = 0$$

ya'ni,  $y = 2x$  funktsiya haqiqatdan ham berilgan differentsiyal tenglamaning yechimi ekan.

**2-Misol.**  $y' - 2y = 0$  differentsiyal tenglamaning umumiyl yechimi  $y = Ce^{2x}$  ko'rinishga ega. Uning  $y(1) = e^2$  boshlang'ich shartni qanoatlantiruvchi xususiy yechimini toping.

**Yechilishi:** Ixtiyorli o'zgarmas S ning izlanayotgan xususiy yechimga mos qiymati umumiyl yechimining ifodasiga boshlang'ich shartlarni keltirib qo'yish natijasida hosil bo'ladi:

$$e^2 = Ce^2$$

bu yerdan  $C = 1$ . Hosil qilingan  $C = 1$  qiymatni umumiyl yechimga qo'yib, berilgan boshlang'ich shartlarni qanoatlantiruvchi  $y = e^{2x}$  xususiy yechimni hosil qilamiz.

## 1. O'zgaruvchilari ajraladigan differentsiyal tenglamalar.

$$M_1(x)M_2(y)dx + N_1(x)N_2(y)dy = 0 \quad (2)$$

ko'rinishga ega bo'lgan tenglama, **o'zgaruvchilari ajraladigan differentsiyal tenglama** deyiladi. Tenglamani yechish uchun, tenglamaning birinchi hadlarini  $M_2(x)N_1(x)$  (noldan farqli deb faraz qilamiz) ga bo'lib o'zgaruvchilari ajralgan  $\frac{M_1(x)}{N_1(x)} dx + \frac{N_2(y)}{M_2(y)} dy = 0$  tenglamani hosil qilamiz. Tenglamaning umumiyl integrali hadma – had integrallash bilan topiladi:

$$\int \frac{M_1(x)}{N_1(x)} dx + \int \frac{N_2(y)}{M_2(y)} dy = C$$

**3-Misol.**  $(\sqrt{xy} + \sqrt{x})y' - y = 0$  tenglamaning umumiy integralini toping.

**Yechilishi:** Berilgan tenglamani  $y' = \frac{dy}{dx}$  ko'inishda yozib olamiz va tenglamaning ikkala qismini  $dx$  ga ko'paytiramiz:

$$(\sqrt{y} + 1)\sqrt{x}dy - ydx = 0$$

O'zgaruvchilarni ajratamiz,

$$\frac{\sqrt{y} + 1}{y} dy - \frac{1}{\sqrt{x}} dx = 0$$

va integrallab quyidagiga ega bo'lamic:

$$\int \left( y^{-\frac{1}{2}} + y^{-1} \right) dy - \int x^{-\frac{1}{2}} dx = C$$

bundan

$$2\sqrt{y} + \ln|y| - 2\sqrt{x} = C$$

ni topamiz.

**4-Misol.**  $ydx + ctgx dy = 0$  tenglamaning  $y\left(\frac{\pi}{6}\right) = -1$  boshlang'ich shartlarni qanoatlantiruvchi xususiy integralini toping.

**Yechilishi:** O'zgaruvchilarni ajratib quyidagiga kelamiz,

$$\frac{dx}{ctgx} + \frac{dy}{y} = 0 \quad \text{yoki} \quad ctgx dx + \frac{dy}{y} = 0$$

ikkala qismini integrallaymiz va natijada

$$-\ln|\cos x| + \ln|y| = \ln C.,$$

$$|y| = C|\cos x|; \quad y = \pm C \cos x = C_1 \cos x$$

ga ega bo'lamic. Boshlang'ich shartdan foydalansak,  $(x = \frac{\pi}{6}, y = -1)$

$$-1 = C_1 \cos \frac{\pi}{6} = C_1 \cdot \frac{\sqrt{3}}{2}, \quad C_1 = -\frac{2}{\sqrt{3}}$$

Shunday qilib, izlanayotgan xususiy integral quyidagicha bo'ladi.

$$y = -\frac{2}{\sqrt{3}} \cos x$$

**1—Masala.** (*bakteriya ko'payishining tezligi haqida.*)

Bakteriya ko'payish tezligi uning soniga to'g'ri proportsional. Boshlang'ich  $t=0$  vaqtida 100 ta bakteriya bo'lsin, 3 soatdan keyin ularning soni ikki barobar ko'payadi. Bakteriya sonining vaqtga bog'liqligini aniqlash kerak va 9 soatda bakteriya qancha marta ko'payadi?

**Yechilishi.** Aytaylik,  $x$  bakteriyalar soni bo'lsin. Masala shartiga ko'ra

$$\frac{dx}{dt} = kx$$

bu yerda  $k$  — proportsionallik koeffitsienti. Tenglamani o'zgaruvchilarga ajratib integrallasak, quyidagini hosil qilamiz:

$$x = Ce^{kt}$$

$S$  ni aniqlash uchun  $t=0$  va  $x=100$  dan foydalanamiz.  $S=100$  bo'ladi, demak,

$$x = 100e^{kt}$$

$k$  — proportsionallik koeffitsientini  $t=3$  va  $x=200$  dan foydalanim topamiz:

$$200 = 100e^{3k} \text{ yoki } 2 = e^{3k}$$

bundan kelib chiqadiki  $e^k = 2^{\frac{1}{3}}$ . Shuning uchun qidirilayotgan funktsiya

$$x = 100 \cdot 2^{\frac{t}{3}}$$

bundan  $t=9$  da  $x=800$  ekanligini topamiz. Demak, 9 soat ichida bakteriya 8 marta ko'payar ekan.

**2—Masala.** (*Aralashmaning kontsentratsiyasi.*) Tarkibida 100 l suv va 10 kg tuz bo'lgan idishga 30 l/min tezlik bilan suv to'xtovsiz quyib turiladi va idishdan 20 l/min tezlik

bilan aralashma oqib chiqadi. Faraz qilamiz, suv bilan tuz tez aralashib ketadi. t vaqt ichida idishda qancha tuz qolishini aniqlang.

**Yechilishi.** Aytaylik, t vaqt ichida  $x$  – miqdorda tuz bor. dt vaqt ichida idishda  $dx$  – miqdorda tuz chiqib ketadi. (minus ishorasi  $x$  – kamayuvchi funktsiya ekanini bildiradi). t vaqtida idishda aralashma hajmi quyidagiga teng.

$$v = 100 + 30t - 20t = 100 + 10t$$

shuning uchun tuz miqdori (bir litr aralashmada) t vaqtida

$$\frac{x}{100+10t}$$

ga teng. Bundan kelib chiqadiki, dt vaqt ichida tuz

$$\frac{x}{100+10t} \cdot 20t$$

ga kamayadi.

Bundan quyidagi differentsiyal tenglamaga ega bo'lamiz.

$$-dx = \frac{20xdt}{100+10t}$$

yoki

$$-dx = \frac{2xdt}{10+t}$$

o'zgaruvchilarga ajratib integrallasak,

$$\frac{dx}{x} = -\frac{2dt}{10+t},$$

$$\ln x = -2 \ln(10+t) + \ln C$$

bundan kelib chiqadiki

$$x = \frac{C}{(10+t)^2}.$$

Agar  $t=0$ ,  $x=10$  da  $S=1000$  ga teng.

Shunday qilib,  $t$  vaqt ichida idishda tuzning kg hisobiga kamayish qonuni, quyidagi formula bilan beriladi:

$$x = \frac{1000}{(10+t)^2} \quad (1).$$

(1) formula orqali havzadagi tuz miqdorini bilgan holda yuqoridaqgi hodisaning boshlanganiga qancha vaqt o'tganini bilish mumkin. Mana shu fikr asosida dengiz va okean yoshi aniqlanadi.

**3-Masala.** (*Jismning sovishi.*) Atrofdagi havo temperaturasi  $20^{\circ}$  ga teng bo'lsin. Jismning sovish tezligi jism temperaturasi va atrofdagi havo temperaturasi ayirmasiga to'g'ri proporsional. Ma'lumki,  $20$  min ichida jism  $100^{\circ}\text{C}$  dan  $60^{\circ}\text{C}$  gacha soviydi. Jism temperaturasi  $\theta$  ning  $t$  vaqt ichida o'zgarish qonunini aniqlang.

**Yechilishi.** Masala shartiga ko'ra, quyidagini yozamiz:

$$\frac{d\theta}{dt} = k(\theta - 20),$$

bu yerda  $k$  – proporsionallik koeffitsienti. O'zgaruvchilarga ajratib integrallasak:

$$\frac{d\theta}{\theta - 20} = kdt,$$

$$\ln(\theta - 20) = kt + \ln c.$$

Bu ifodani potentsirlasak,

$$\theta - 20 = ce^{kt}.$$

$c$  ni aniqlash uchun boshlang'ich shartdan foydalanamiz:

$$t = 0 \text{ da } \theta = 100^{\circ}.$$

Bundan  $c = 80$ . Shuning uchun

$$\theta = 20 + 80e^{kt}.$$

Proportsionallik koeffitsienti  $k$  ni qo'shimcha shartlar yordamida aniqlaymiz,  $t = 20$ ,  $\theta = 60^\circ$ . Bundan:

$$60 = 20 + 80e^{20k}$$

yoki

$$e^{20k} = \frac{1}{2}.$$

Demak,

$$e^k = \left(\frac{1}{2}\right)^{\frac{1}{20}}.$$

Shunday qilib, natija quyidagicha:

$$\theta = 20 + 80\left(\frac{1}{2}\right)^{\frac{t}{20}}.$$

## 2. Bir jinsli differentsiyal tenglamalar.

Quyidagi tenglikni  $f(tx, ty) = t^m f(x, y)$  qanoatlantiruvchi  $f(x, y)$  funktsiyaga m tartibli bir jinsli funktsiya deyiladi. Masalan,

$$f(x, y) = x^2 + 2y^2 - xy$$

2 – tartibli bir jinsli funktsiya bo'ladi.

$$(tx)^2 + 2(ty)^2 - (txty) = t^2(x^2 + 2y^2 - xy)$$

Agarda  $M(x, y)$  va  $N(x, y)$  funktsiyalar bir xil tartibli bir jinsli funktsiyalar bo'lsa, unda ushbu

$$M(x, y)dx + N(x, y)dy = 0 \quad (3)$$

ko'ri nishdagi tenglamaga bir jinsli birinchi tartibli tenglama deyiladi,

Bir jinsli tenglama  $y=ux$  (bu yerda  $u$  yangi izlanayotgan funktsiya) almashtirish orqali o'zgaruvchilarga ajraladigan tenglamaga keltiriladi.  $y=ux$  tenglikni differentialsiallab, topamiz:  $dy = udx + xdu$ .

**4-Misol.**  $(y^2 - 3x^2)dx + 2xydu = 0$ ,  $y(0)=0$  boshlang'ich shart bilan berilgan tenglamani yeching.

Tenglamani yechish uchun  $y=ux$  almashtirish bajaramiz.

$$\begin{aligned} & \left(u^2 x^2 - 3x^2\right)dx + 2x^2 u(udx + xdu) = 0 \\ & \text{yoki} \end{aligned}$$

$$3(u^2 - 1)dx + 2xu du = 0,$$

$\frac{3dx}{x} + \frac{2udu}{u^2 - 1} = 0$  ni integrallab,  $3\ln x + \ln(u^2 - 1) = \ln C$  ni hosil qilamiz. Bu tenglikni potentsirlab,  $x^2(u^2 - 1) = C$  ni topamiz.  $u = \frac{y}{x}$  ni o'rniga qo'yib  $x^3\left(\frac{y^2}{x^2} - 1\right) = C$  yoki  $x(y^2 - x^2) = C$  boshlangich shartdan foydalanib  $S=0$  ni topamiz. Qidirilayotgan xususiy yechim  $y = \pm x$  bo'ladi.

### Nazorat savollari.

1. Asosiy tushunchalar.
2. O'zgaruvchilari ajraladigan tenglamalar.
3. Bir jinsli tenglamalar.
4. Masalalar.

### 2-§. Birinchi tartibli chiziqli differentsial tenglamalar.

Ushbu

$$y' + py = q \quad (1)$$

tenglamalarga **birinchi tartibli chiziqli differentsial tenglama** deyiladi. Bu yerda  $p = p(x)$  va  $q = q(x)$  lar  $(a, b)$  da uzlusiz funktsiyalar.  $y = u \cdot \vartheta$  almashtirish bilan tenglama o'zgaruvchilari

ajraladigan ikkita tenglamaga keltiriladi.  $u = u(x)$  funktsiya yangi noma'lum funktsiya,  $\vartheta = \vartheta(x)$  funktsiya esa ixtiyoriy tanlab olinadi. Bu almashtirish tenglamani

$$u' \vartheta + u \vartheta' + pu \vartheta = q$$

yoki

$$\vartheta \frac{du}{dx} + \left( \frac{d\vartheta}{dx} + p \vartheta \right) u = q$$

ko'rinishga keltiradi.  $\vartheta$  ning ixtiyoriyligidan

$$\frac{d\vartheta}{dx} + p \vartheta = 0$$

deb olamiz. O'zgaruvchilarga ajratib va integrallab:

$$\frac{d\vartheta}{\vartheta} = -p dx, \quad \ln \vartheta = - \int p dx$$

yoki

$$\vartheta = e^{- \int p dx}$$

bundan

$$e^{\int p dx} \frac{du}{dx} = q$$

tenglamani hosil qilamiz. Bu tenglamani yechsak,

$$u = \int q e^{\int p dx} dx + C$$

ga kelamiz. Nihoyat, tenglamaning umumiy yechimi

$$y = e^{\int p dx} \left[ \int q e^{\int p dx} dx + C \right] \quad (2)$$

bo'ladi.

**Izoh:** Agar (1) tenglamada  $q(x) = 0$  bo'lsa, u holda tenglama birinchi tartibli **chiziqli bir jinsli tenglama** deyiladi, aks holda **chiziqli bir jinsli bo'limgan tenglama** deyiladi. Bundan kelib chiqadiki, birinchi tartibli bir jinsli tenglama

$$y' + py = 0 \quad (3)$$

ko'inishda bo'ladi, yechimi esa quyidagicha:

$$y = Ce^{\int p dx} \quad (4)$$

**1-Misol.**  $x^2y^2y' + xy^3 = 1$  tenglamaning yechimini toping.

**Yechilishi:** Tenglamaning ikkala tomonini  $x^2y^2$  ga bo'lamiz,

$$y' + \frac{y}{x} = y^{-2} \cdot \frac{1}{x^2},$$

agar  $p = x^{-1}$ ,  $q = x^{-2}$  deb olsak, tenglama

$$y' + py = qy^{-2}$$

**Bernulli tenglamasiga** keladi.  $y = u \cdot v$  deb olsak,  $y' = u'v + uv'$  deb tenglamaga qo'yamiz:

$$u' \cdot v + u \cdot v' + \frac{u \cdot v}{x} = \frac{1}{x^2 u^2 v}$$

yoki

$$u' \cdot v + u \left( v' + \frac{v}{x} \right) = \frac{1}{x^2 u^2 v}$$

Quyidagi ikkita tenglamani yechamiz:

$$1) \quad v' + \frac{v}{x} = 0, \quad 2) \quad u' \cdot v = \frac{1}{x^2 u^2 v}$$

Birinchi tenglamani yechib  $v$  ni topamiz:

$$\frac{dv}{v} + \frac{dx}{x} = 0; \quad \ln v + \ln x = 0, \quad vx = 1; \quad v = \frac{1}{x}$$

$v$  ni 2-chi tenglamaga qo'yib,  $u$  ni topamiz:

$$\frac{u}{x} = \frac{1}{u^2}; \quad u^2 du = x dx; \quad \frac{u^3}{3} = \frac{x^2}{2} + \frac{c}{3}; \quad u = \sqrt[3]{\frac{3}{2}x^2 + C}$$

qidirilayotgan tenglamaning umumiy yechimi quyidagicha bo'ladi:

$$y = u * v = \sqrt[3]{\frac{3}{2}x + \frac{c}{x^3}}.$$

### **1. Chiziqli differentsiyal tenglamalarning biologiyaga tadbiqi.**

**Masala.** Ovqatlanish resursi juda yaxshi sharoitda bo'lgan mikroorganizmlar jamoasini qaraylik. Vaqt o'tishi bilan jamoaning ko'payishi va nobud bo'lishi o'zgarib turadi. Ana shu o'zgarish qonunini toping.

**Yechilishi.** Aytaylik,  $x = x(t)$  — vaqt ichidagi tirik organizmlarning soni bo'lsin,  $x(t + \Delta t)$  esa —  $t + \Delta t$  vaqt dagi soni. U holda ayirma

$$x(t + \Delta t) - x(t) = \Delta x$$

ni beradi.  $\Delta t$  vaqt ichida balog'atga yetganlarining bir qismi nasl qoldiradi, qolgan qismi nobud bo'lishi mumkin. Shunday qilib,

$$\Delta x = G - H$$

bu yerda  $G$  — dan  $t + \Delta t$  vaqt o'tganda tug'ilganlari soni,  $H$  — shu vaqt ichida nobud bo'lganlar soni.

Tug'ilganlar soni  $G$   $\Delta t$  vaqt oraliqiga bog'liq va nasl qoldiruvchi "ota—ona" larning soniga bog'liq, chunki ular qancha ko'p bo'lsa tug'ilish shuncha ko'p bo'ladi.

Shunday qilib,

$$G = \Phi(x, \Delta t)$$

bu yerda  $\Phi(x, \Delta t)$  funktsiya  $x$  yoki  $\Delta t$  ning o'sishi bilan o'sadi yoki  $x$  yoki  $\Delta t$  larning biri nolga intilsa nolga teng bo'ladi.

$\Delta t$  o'zgaruvchiga kelsak eng oddiy tajriba shuni ko'rsatadiki, u chiziqli bo'lib agar kuzatishni ikki marta uzaytirsak, mikroorganizmlar nasli ham ikki marta oshadi. Shunday qilib,

$$\Phi(x, \Delta t) = f(x)\Delta t$$

$f(x)$  funktsiya xususiyati murakkabroq. Biz bilamizki,  $x$  o'sishi bilan  $f(x)$  monoton o'sadi va  $x=0$  bo'lsa nol bo'ladi. Ammo o'sish mikroorganizm turiga bog'liq. Biz nasl miqdorining "ota – ona" lar soniga to'g'ri proportsional bo'lgan holati bilan chegaralanamiz, yani  $f(x) = \alpha x$  ( $\alpha = \text{const}$ ). Shunday qilib,

$$G = \alpha x \Delta t.$$

Shunga o'xshash,

$$H = \beta x \cdot \Delta t$$

va bundan kelib chiqadiki,

$$\Delta x = \alpha x \Delta t - \beta x \Delta t$$

yoki

$$\Delta x = \gamma x \Delta t \quad (1)$$

bu yerda

$$\gamma = \alpha - \beta$$

(1) da tenglamaning ikkala tomonini  $\Delta t$  ga bo'lib, limitga o'tamiz:

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

natijada quyidagini hosil qilamiz

$$\frac{dx}{dt} = \gamma x \quad (2)$$

yoki

$$\frac{dx}{dt} - \gamma x = 0.$$

Birinchi tartibli chiziqli bir jinsli tenglamaga kelamiz. Bu tenglamani yechib,

$$x = Ce^{\gamma t} \quad (3)$$

ni hosil kilamiz.

$t = t_0$  da  $x = x(t_0)$  (bu yerda  $t_0$  boshlang'ich vaqtida  $x(t_0) = x_0$  tirik mikroorganizmlar soni) boshlang'ich shart bilan S ni topamiz.

$$C = x_0 e^{-\gamma t_0}$$

buni (3) ga qo'yib, vaqt davomida mikroorganizmlar o'zgarish qonunini topamiz.

$$x = x_0 e^{\gamma(t-t_0)} \quad (4)$$

Ammo topgan bu qonuniyatimiz qanchalik haqiqiy hayotga to'g'ri kelish kelmasligini tajriba va kuzatishlar hal etadi. (4) formula shuni ko'rsatadiki, o'sish eksponentsiyal darajada, lekin hayotda birorta ham tirik organizm bu darajada o'smaydi. Chunki biz faraz qilgan (2) tenglamada ovqatlanish sharoti yaxshiligi va tashqi faktorlarning ta'siri yo'qligi bu haqiqatga ziddir. Shunday qilib (2) tenglama yoki nazariy xarakterga ega (uzluksiz oziqlantirib turilganda va tashqi halaqit beruvchi kuchlar bo'lmasa, tirik organizmlar qanday ko'payishini ko'rish mumkin) yoki sun'iy ko'paytirishlar natijasini ko'rsatadi.

(2) tenglamani birinchi marta 1802 yil Mal'tus qo'llagan. Uning xatosi bu tenglamani nafaqat tabiatga, hatto insonlarga qo'llasa ham bo'ladi deb tushungan. Aslida tenglama tor doirada qo'llaniladi.

## 2. Tuliq differentsiyallari birinchi tartibli differentsiyallari tenglamalar. Integrallovchi ko'paytuvchi.

Ushbu

$$M(x, y)dx + N(x, y)dy = 0 \quad (1)$$

tenglama to'liq differentsiyallari tenglama deyiladi. Agarda tenglamaning chap tarafini  $u(x, y)$  funktsiyining to'liq differentsiyali ko'rinishida yozish mumkin bo'lsa:

$$Mdx + Ndy = du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

(1) chi differentsiyalari tenglamaning to'liq differentsiyallari tenglamasi bo'lishi uchun

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (2)$$

ning bajarilishi zarur va yetarli. (1) ning umumiyligi integrali

$$u(x, y) = C$$

yoki

$$\int_{x_0}^x Mdx + \int_{y_0}^y Ndy = C$$

ko'rinishda bo'ladi.

Ba'zi hollarda (1) chi tenglama tuliq differentsiyallari bo'lmasa, shunday  $\mu(x, y)$  funktsiya tanlash mumkinki (1) ning chap tarafiga ko'paytirsak tenglama tuliq differentsiyallari ko'rinishga keladi.

$$du = \mu Mdx + \mu Ndy \quad (3)$$

bunday funktsiya  $\mu(x, y)$  integrallovchi ko'paytuvchi deyiladi va quyidagilar kelib chiqadi.

$$\frac{\partial}{\partial y} (\mu M) = \frac{\partial}{\partial x} (\mu N)$$

yoki

$$N \frac{\partial \mu}{\partial x} - M \frac{\partial \mu}{\partial y} = \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \mu$$

hamda

$$N \frac{\partial \ln \mu}{\partial x} - M \frac{\partial \ln \mu}{\partial y} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \quad (4)$$

Agar  $\mu = \mu(x)$  bo'lsa, u holda  $\frac{\partial \mu}{\partial y} = 0$  va (4) tenglama quyidagi ko'rinishga keladi:

$$\frac{d \ln \mu}{dx} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad (5)$$

Agar  $\mu = \mu(y)$  bo'lsa, yuqoridaqidek  $\left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \frac{1}{M}$  ko'rinishda bo'ladi.

**2-Misol:**  $2xy + 3y^2 + (x^2 + 6xy - 3y^2) \frac{\partial y}{\partial x} = 0$

tenglamani yeching.

**Yechilishi:**  $N(x, y) = x^2 + 6xy - 3y^2, \quad \frac{\partial N}{\partial x} = 2x + 6y$   
 $M(x, y) = 2xy + 3y^2, \quad \frac{\partial M}{\partial y} = 2x + 6y$

Shunday qilib,  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , ya'ni tenglamaning chap qismi  $u(x, y)$  funktsiyaning to'liq differentialsali bo'ladi.

$$\frac{\partial u}{\partial x} = 2xy + 3y^2, \quad \frac{\partial u}{\partial y} = x^2 + 6xy - 3y^2$$

Birinchi tenglamadan,

$$u(x, y) = x^2y + 3xy^2 + \varphi(y)$$

$\varphi(y)$  funktsiyani aniqlash uchun,  $y$  bo'yicha oxirgi tenglikni differentsiyallaymiz:

$$\frac{\partial V}{\partial y} = x^2 + 6xy + \frac{du}{dy} = x^2 + 6xy - 3y^2, \quad ya'ni \quad \frac{du}{dy} = -3y^2.$$

bundan  $\varphi(y) = -y^3 + c_1$ . Shuning uchun,  $u(x, y) = x^2y + 3xy^2 - y^3 + c_1$   
Tenglamaning yechimi:  $x^2y + 3xy^2 - y^3 = c$

**3-Misol.**  $(1 - \frac{x}{y})dx + (2xy + \frac{x}{y} + \frac{x^2}{y^2})dy = 0$

tenglamani yeching.

**Yechilishi:**  $M(x, y) = 1 - \frac{x}{y}; \quad N(x, y) = 2xy + \frac{x}{y} + \frac{x^2}{y^2}.$   
 $\frac{\partial M}{\partial y} = \frac{x}{y^2}, \quad \frac{\partial N}{\partial x} = 2y + \frac{1}{y} + \frac{2x}{y^2}.$

Demak, tenglama to'liq differentsiyallli tenglama emas.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{x}{y^2} - 2y - \frac{1}{y} - \frac{2x}{y^2} = -(2y + \frac{1}{y} + \frac{x}{y^2}).$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \quad \text{dan} \quad \frac{1}{\mu} \cdot \frac{d\mu}{dx} = -\frac{1}{x}, \quad \frac{d\mu}{\mu} + \frac{dx}{x} = 0$$

bundan,  $\mu = \frac{1}{x}$  ni topamiz.

Berilgan tenglamani  $\frac{1}{x}$  ga ko'paytiramiz:

$$\left( \frac{1}{x} - \frac{1}{y} \right) dx + \left( 2y + \frac{1}{y} + \frac{x}{y^2} \right) dy = 0$$

Quyidagi ko'rinishda yozib olamiz:

$$\frac{dx}{x} + \left( \frac{1}{y} + 2y \right) dy - \frac{xdy - ydx}{y^2} = 0$$

$$d\left(\ln|x| + \ln|y| + y^2 - \frac{x}{y}\right) = 0 \quad \text{dan}$$

$$\ln|x| + \ln|y| + y^2 - \frac{x}{y} = C \quad \text{yechimga ega bo'lamiz.}$$

### 3. Hosilaga nisbatan yechilmagan birinchi tartibli differentials tenglamalar.

Agar  $F(x, y, y') = 0$  tenglama  $y'$  ga nisbatan 2-darajali bo'lsa, bu tenglama, biror sohada  $x$  va  $y$  ga nisbatan uzlucksiz  $y' = f_1(x, y)$  va  $y' = f_2(x, y)$  echimga ega. Geometrik nuqtai nazardan bu tenglama shu sohaning ixtiyoriy ( $x_0, y_0$ ) nuqtasida ikkita integral chiziqning yo'nalishlarini aniqlaydi.

Ushbu

$$y = x\varphi(y') + \psi(y')$$

tenglama **Langranj tenglamasi** deyiladi.

Bunda  $y' = p$  almashtirish  $x$  ga nisbatan tenglamani chiziqli ko'rinishga keltiradi:

$$P = \varphi(p) + [x\varphi'(p) + \psi'(p)] \frac{dp}{dx} .$$

Uni yechsak,

$$x = \varphi(p)x + \psi(p)$$

bu yerda  $p$

$$p = \varphi(p)$$

tenglamaning yechimi.

Ushbu

$$y = xy' + \psi(y')$$

tenglama **Klero tenglamasi** deyiladi. Bu tenglama Langranj tenglamasining xususiy holidir. Bu tenglama  $y = Cx + \psi(c)$  umumiy

integralga va  $y = px + \psi(p)$  hamda  $x + \psi'(p) = 0$  tenglamalardan r parametrni yo'qotishdan hosil bo'ladi. maxsus yechimga egadir.

**4-Misol.**  $y' + y = x(y')^2$  tenglamani yeching.

**Yechilishi:** Bu tenglama Lagranj tenglamasi. Belgilash kiritamiz.  $p = y'$ , u holda  $y = xp^2 - p$  bo'ladi.

$x$  ga nisbatan differentsiallaymiz:

$$\frac{dy}{dx} = p^2 + 2px \frac{dp}{dx} - \frac{dp}{dx}$$

yoki

$$p = p^2 + 2px \frac{dp}{dx} - \frac{dp}{dx}$$

Quyidagi chiziqli differentsial tenglamani hosil qilamiz:

$$\frac{dx}{dp} + \frac{2}{p-1}x = \frac{1}{p(p-1)}$$

$x$  ga nisbatan yechilib,  $x = \frac{p - \ln p + C}{(p-1)^2}$  ni topamiz.

**5-Misol.**  $\sqrt{(y')^2 + 1} + xy' - y = 0$  tenglamani yeching.

**Yechilishi:** Bu tenglama Lagranj tenglamasi. Belgilash kiritamiz.  $p = y'$ , u holda  $y = xp + \sqrt{1+p^2}$  bo'ladi.

$x$  ga nisbatan differentsiallaymiz:

$$\frac{dy}{dx} = p + x \frac{dp}{dx} + \frac{p \frac{dp}{dx}}{\sqrt{1+p^2}}$$

Bundan  $\left(x + \frac{p}{\sqrt{1+p^2}}\right) \frac{dp}{dx} = 0$ ,  $x = -\frac{p}{\sqrt{1+p^2}}$  yoki  $p = C$ .

Shunday qilib,  $y = Cx + \sqrt{1+C^2}$  va

$$\begin{cases} x = -\frac{p}{\sqrt{1+p^2}} \\ y = px + \sqrt{1+p^2} \end{cases}$$

Bu sistemadan  $p$  ni yo'qotib quyidagini topamiz:  $y = \sqrt{1-x^2}$ .

### Nazorat savollari.

1. Birinchi tartibli chiziqli differentsiyal tenglamalar.
2. Chiziqli differentsiyal tenglamalarning biologiyaga tadbipi.
3. Tuliq differentsialli birinchi tartibli differentsiyal tenglamalar.
4. Hosilaga nisbatan yechilmagan birinchi tartibli differentsiyal tenglamalar.

### Mustaqil yechish uchun misol va masalalar.

#### 1. Quyidagi misollarda berilgan funktsiyalar mos ravishda tenglamalarning yechimi ekanligi tekshirilsin.

1.  $y = \frac{\sin x}{x}$        $xy' + y = \cos x$ .
2.  $y = c\ell^{-2x} + \frac{1}{3}\ell^x$        $y' + 2y = \ell^x$ .
3.  $y = 2 + c\sqrt{1-x^2}$        $(1-x^2)y' + xy = 2x$ .
4.  $y = x\sqrt{1-x^2}$        $yy' = x - 2x^3$ .
5.  $y = \ell^{\sin cx}$        $xy' = ytg \ln y$ .
6.  $y = (x-c)^3$        $y' = 3y^{2/3}$ .
7.  $y = \ell^{cx}$        $y = \ell^{xy'/y}$ .
8.  $y = cx^3$        $xy' = 3y$ .
9.  $y = \sin(x+c)$        $y^2 + y'^2 = 1$ .
10.  $x^2 + cy^2 = 2y$        $x^2y' = yy' + xy$ .
11.  $y^2 + cx = x^3$        $2xyy' - y^2 = 2x^3$ .
12.  $y = c(x-c)^2$        $y'^3 = 4y(xy' - 2y)$ .
13.  $cy = \sin cx$        $y' = \cos \frac{x\sqrt{1-y'^2}}{y}$ .

14.  $y = cx + \frac{c}{\sqrt{1+c^2}}$        $y - xy' = \frac{y'}{\sqrt{1+y'^2}}$ .  
 15.  $x^2 + y^2 - cx = 0$        $2xyy' + x^2 - y^2 = 0$ .  
 16.  $y = \sin x + c \cdot \cos x$        $y' \cos x + y \sin x = 1$ .  
 17.  $x + y + c(1 - xy) = 0$        $y' + \frac{1+y^2}{1+x^2} = 0$ .  
 18.  $x - y - c\ell^{y-x} = 0$        $yy' - 2y + x = 0$ .  
 19.  $y^2 = cx$        $2xy' - y = 0$ .  
 20.  $y = \frac{c}{\cos x}$        $y' - \operatorname{tg}x \cdot y = 0$ .  
 21.  $y = -\frac{1}{3x+c}$        $y' = 3y^2$ .  
 22.  $y = \ell n(c + \ell^x)$        $y' = \ell^{x-y}$ .  
 23.  $x = y \ell ncy$        $y'(x + y) = y$ .

## 2. O'zgaruvchilarga ajraladigan birinchi tartibli differentsiyal tenglamalar yechilsin.

1.  $x(1+y^2) + y(1+x^2)y' = 0$        $\underline{j}: \quad (1+x^2)(1+y^2) = c$ .  
 2.  $y' = xy^2 + 2xy$        $\underline{j}: \quad \frac{y}{y+2} = c\ell^{x^2}$ .  
 3.  $\ell^x dx - (1+\ell^x)y dy = 0$        $\underline{j}: \quad \ell^{\frac{x^2}{2}} = \frac{\sqrt{\ell}}{2}(1+\ell^x)$ .  
 4.  $y' + y = 2x + 1$        $\underline{j}: \quad y - 2x - 1 = -2 + c\ell^{-x}$ .  
 5.  $y' = \cos(x - y - 1)$        $\underline{j}: \quad y = x - 1 - 2avctg(c - x) + 2n\pi \quad n \in \mathbb{Z}$ .  
 6.  $(1+y^2)dx + (1+x^2)dy = 0$        $\underline{j}: \quad x + y = c(1 - xy)$ .  
 7.  $(1+y^2)dx + xy dy = 0$        $\underline{j}: \quad x^2(1+y^2) = c$ .  
 8.  $(y^2 + xy^2)y' + x^2 - yx^2 = 0$        $\underline{j}: \quad (x + y)(x - y - 2) + 2\ell n\left(\frac{1+x}{1-y}\right) + c$ .  
 9.  $(1+y^2)dx = xdy$        $\underline{j}: \quad y = t \operatorname{tg}ncx$ .  
 10.  $x\sqrt{1+y^2} + yy'\sqrt{1+x^2} = 0$        $\underline{j}: \quad \sqrt{1+x^2} + \sqrt{1+y^2} = c$ .  
 11.  $x\sqrt{1-y^2}dx + y\sqrt{1-x^2}dy = 0 \quad y(0) = 1$        $\underline{j}: \quad \sqrt{1-x^2} + \sqrt{1-y^2} = 1$ .  
 12.  $\ell^y(1+y') = 1$        $\underline{j}: \quad \ell^x = \sqrt{1-\ell^{-y}}$ .  
 13.  $y' = \sin(x - y)$        $\underline{j}: \quad x + c = ctg\left(\frac{y-x}{2} + \frac{\pi}{4}\right)$ .  
 14.  $y' = ax + by + c \quad (a, b, c - \text{cons})$        $\underline{j}: \quad b(ax + by + c) + a = c\ell^{bx}$ .

15.  $(x+y)^2 y' = a^2$   $\therefore x+y = a \operatorname{tg} \left( c + \frac{y}{a} \right)$
16.  $(x-y^2)dx + 2xydy = 0$   $\therefore x e^{\frac{y^2}{x}} = c.$
17.  $(1+x^2)dy - xydx = 0$   $y(2) = 1$   $\therefore y = \sqrt{\frac{x^2}{2}-1}.$
18.  $(xy^2 + x)dx + (y-x^2y)dy = 0$   $\therefore 1+y^2 = c(1-x^2).$
19.  $(xy^2 + x)dx + (x^2y - y)dy = 0$   $\therefore x^2y^2 + x^2 - y^2 = c.$
20.  $xy' + y = y^2$   $y(1) = 0,5$   $\therefore y(1-cx) = 1.$
21.  $xy' = tgy$   $\therefore \sin y = cx$
22.  $2xyy' = y^2 - 1$   $\therefore y^2 - 1 = cx$

### 3. Birinchi tartibli bir jinsli tenglamalar yechilsin.

1.  $y' = \frac{xy + y^2 e^{-\frac{x}{y}}}{x^2}$   $\therefore e^y + e \ln|x| = c.$
2.  $\left( x - y \cos \frac{y}{x} \right) dx + x \cos \frac{y}{x} dy = 0$   $\therefore x = c e^{-\arctan \frac{y}{x}}.$
3.  $y = \frac{y + \sqrt{x^2 - y^2}}{x}$   $\therefore y = x \sin \ln cx$
4.  $y' = \frac{4x^6 - y^4}{2x^4 y}$   $\therefore \frac{y^2 - x^3}{y^2 + 4x^3} x^5.$
5.  $(x^2 + y^2)dx - xydy = 0$   $\therefore x = c e^{x^2/(2x^2)}.$
6.  $y' = 1 + \frac{y}{x}$   $\therefore x = c e^{\frac{y}{x}}.$
7.  $xy' = 3y - x$   $\therefore x^3 = c(2y - x).$
8.  $(x-y)dx + xdy = 0$   $\therefore y = x(c - \theta|x|)$
9.  $y^2 dx - (x^2 + xy)dy = 0$   $\therefore y = c e^{\frac{y}{x}}.$
10.  $(x^2 - xy + y^2)dx - x^2 dy = 0$   $\therefore x = c e^{\frac{x-y}{x}}$
11.  $(x+2y)dx - (2x+y)dy = 0$   $\therefore (x-y)^3 = c(x+y).$
12.  $xy' = y + x e^{\frac{y}{x}}$   $\therefore \ln|cx| = -e^{\frac{y}{x}}.$
13.  $xy' - x \cos^2 \frac{y}{x} = y$   $\therefore x \alpha^{\operatorname{arg}(y/x)}.$
14.  $y - xy' = \sqrt{x^2 - y^2}$   $\therefore x = c e^{\frac{\arccos \frac{y}{x}}{x}}.$

15.  $(x + y - 2)dx + (x - y + 4)dy = 0$   
 16.  $(x + y + 1)dx + (2x + 2y - 1)dy = 0$   
 17.  $(x^2 y^2 - 1)dy + 2xy^3 dx = 0$   
 18.  $4x - 3y + y^2(2y - 3x) = 0$   
 19.  $y' = \frac{2xy}{3x^2 - y^2}$   
 20.  $(y - xy')^2 = x^2 + y^2$   
 21.  $3x + y - 2 + y'(x - 1) = 0$   
 22.  $(3y - 7x + 7)dx - (3x - 7y - 3)dy = 0$
- $\therefore$   $x^2 + 2xy - y^2 - 4x + 8y = c.$   
 $\therefore$   $x + 2y + 3 \ln|x + y - 2| = c.$   
 $\therefore$   $y = \frac{1+x^2 y^2}{c}.$   
 $\therefore$   $y^2 - 3xy + 2x^2 = c.$   
 $\therefore$   $c(y^2 - x^2) = y^3.$   
 $\therefore$   $c^2 x^2 = 1 + 2cy; c^2 - x^2 = 2cy.$   
 $\therefore$   $(x - 1)(3x + 2y - 1) = c.$   
 $\therefore$   $(x + y + 1)^5 \cdot (x - y - 1)^2 = c.$

#### 4. Birinchi tartibli chiziqli differentsiyal tenglamalar va Bernulli differentsiyal tenglamalari yechilsin.

1.  $y' - 2xy = 3x^2 - 2x^4$   
 2.  $y' + y \cos x = e^{\sin x}$   
 3.  $y + y \operatorname{tg} x = \frac{1}{\cos x}$   
 4.  $y' \sin x - y \cos x = -\frac{\sin^2 x}{x^2}$   
 5.  $y' \sin 2x = 2(y + \cos x)$   
 6.  $y' - \frac{2y}{x} = 2x^3$   
 7.  $y' + 2xy = 2x e^{-x^2}$   
 8.  $y' + 2y = x^2 + 2x$   
 9.  $(x^2 + 2x - 1)y' - (x + 1)y = x - 1$   
 10.  $2xy' - y = 3x^2$   
 11.  $xy' + y = y^2 \ln x$   
 12.  $2y' - x = -x^3 \sin y$   
 13.  $y' + 2y = e^x y^2$   
 14.  $y' - 2xy = e^{x^2}$   
 15.  $y' + xy + x = 0$   
 16.  $xy' = 2x \ln x - y$   
 17.  $(y + e^x)dx - dy = 0$
- $\therefore$   $y = (c + x^3 e^{-x^2})e^{x^2} = c e^{x^2} + x^3.$   
 $\therefore$   $y = (x + c)e^{\sin x}.$   
 $\therefore$   $y = c \cos x + \sin x.$   
 $\therefore$   $y = c \sin x + \frac{\sin x}{x}.$   
 $\therefore$   $y = c \cdot \operatorname{tg} x - \frac{1}{\cos x}.$   
 $\therefore$   $y = cx^2 + x^4.$   
 $\therefore$   $y = (x^2 + c)e^{-x^2}.$   
 $\therefore$   $y = c e^{-2x} + \frac{1}{4}(2x^2 + 2x - 1).$   
 $\therefore$   $y = c \sqrt{x^2 + 2x - 1} + x.$   
 $\therefore$   $y = c \sqrt{x + x^2}.$   
 $\therefore$   $y = \frac{1}{1 + cx + \ln x}.$   
 $\therefore$   $y = (c - \cos y)x^2.$   
 $\therefore$   $y(e^x + ce^{2x}) = 1.$   
 $\therefore$   $y = (x + c)e^{x^2}.$   
 $\therefore$   $y = c e^{\frac{x^2}{2}} - 1.$   
 $\therefore$   $y = \frac{c}{x} + x \ln x - \frac{x}{2}.$   
 $\therefore$   $y = (c + x)e^x.$

$$18. y'(1+x^2) - xy = \sqrt{1+x^2}$$

$$19. 3xy' - 2y = \frac{x^3}{y^2}$$

$$20. x^2y' + 2x^3y = y^2(1+2x^2)$$

$$21. (xy + x^2y^3)y' = 1$$

$$22. y = xy' + y' \ln y$$

$$\text{J: } y = (\arctan x + c)\sqrt{x^2 + 1}.$$

$$\text{J: } y^3 = cx^2 + x^3.$$

$$\text{J: } \frac{1}{y} = c\ell^{x^2} + \frac{1}{x}.$$

$$\text{J: } \frac{1}{x} = 2 - y^2 + c\ell^{-\frac{y^2}{2}}.$$

$$\text{J: } x = cy - 1 - \ell \ln y.$$

## 5. To'liq differentsiyallli 1-tartibli differentsiyal tenglamalar yechilsin.

$$1. (\sin xy + x \cos xy)dx + x^2 \cos xy dy = 0$$

$$2. (x^3 + xy^2)dx + (x^2y + y^3)dy = 0$$

$$3. (x + y^2)dx - 2xydy = 0$$

$$4. 2xy \ln y dx + \left(x^2 + y^2 \sqrt{y^2 + 1}\right) dy = 0$$

$$5. (2xy + 3y^2)dx + (x^2 + 6xy - 3y^2)dy = 0$$

$$6. (3y^2 + 2xy + 2x)dx + (6xy + x^2 + 3)dy = 0$$

$$7. 2x \left(1 + \sqrt{x^2 - y}\right)dx - \sqrt{x^2 - y}dy = 0$$

$$8. (y^2 + x^2 + a)yy' + (y^2 + x^2 - a)x = 0$$

$$9. \left(1 - \frac{x}{y}\right)dx + \left(2xy + \frac{x}{y} + \frac{x^2}{y^2}\right)dy = 0$$

$$10. (x^2 - \sin^2 y)dx + x \sin 2y dy = 0$$

$$11. ydx - (x + x^2 + y^2)dy = 0$$

$$12. (x + y - 1)dx + (\ell^y + x)dy = 0$$

$$13. (2x + 3x^2y)dx + (x^3 - 3y^2)dy = 0$$

$$14. ydx - (4x^2y + x)dy = 0$$

$$15. 2xydx + (x^2 - y^2)dy = 0$$

$$16. (2 - 9xy^2)xdx + (4y^2 - 6x^3)ydy = 0$$

$$17. \frac{y}{x}dx + (y^3 + \ell \ln x)dy = 0$$

$$\text{J: } x \sin xy = c_1.$$

$$\text{J: } x^4 + 2(xy)^2 + y^4 = 0.$$

$$\text{J: } x = c \cdot \ell^{\frac{y^2}{4}}.$$

$$\text{J: } x^2 \ln y + \frac{1}{3}(y^2 + 1)^3 = c.$$

$$\text{J: } x^2y + 3xy^2 - y^3 = c.$$

$$\text{J: } 3xy^2 + x^2y + 3y + x^2 = c.$$

$$\text{J: } x^2 + \frac{2}{3}(x^2 - y)^{\frac{3}{2}} = c.$$

$$\text{J: } (x^2 + y^2)^2 - 2a(x^2 - y^2) = c.$$

$$\text{J: } \ln|x| + \ln|y| + y^2 - \frac{x}{y} = c.$$

$$\text{J: } x^2 + \sin^2 y = cx.$$

$$\text{J: } \arctg \frac{x}{y} - y = c, \quad y = 0.$$

$$\text{J: } \ell^y + \frac{1}{2}x^2 + xy - x = c_1 + 1.$$

$$\text{J: } x^2y + 3xy^2 - y^3 = c.$$

$$\text{J: } \frac{y}{x} + 2y^2 = c.$$

$$\text{J: } 3x^2y - y^3 = c.$$

$$\text{J: } x^2 - 3x^3y^2 + y^4 = c.$$

$$\text{J: } 4y \ln x + y^4 = c.$$

$$18. 3x^2(1 + \ln y)dx = \left(2y - \frac{x^3}{y}\right)dy$$

$$\therefore x^3 + x^3 \ln y - y^2 = c.$$

$$19. e^x - (2y + x e^{-y})dy = 0 \rightarrow$$

$$\therefore x e^{-y} - y^2 = c.$$

## 6. Hosilaga nisbatan yechilmagan 1-tartibli differentialsal tenglamalar. Lagranj va Klero tenglamalari yechilsin.

$$1. y = x + y' - \ln y'$$

$$\therefore p \neq 1, y = e^{x-c} + c; p = 1, y = x + 1.$$

$$2. y = 2xy' + \ln y'$$

$$\therefore y = \frac{2c}{\rho} + \ln \rho - 2, x = \frac{c}{\rho^2} - \frac{1}{\rho}.$$

$$3. y = xy' + \frac{a}{2y'}, (a = const)$$

$$\therefore y = cx + \frac{a}{c^2}; 4y^3 = 27ax^2.$$

$$4. (y')^3 - 2x(y')^2 + y' = 2x$$

$$\therefore y = x^2 + c.$$

$$5. (y')^2 + y(y-x)y' - xy^3 = 0$$

$$\therefore \left( y - \frac{1}{x+c} \right) \left( y - c e^{\frac{x}{c}} \right).$$

$$6. (y')^2 + (\sin x - 2xy)y' - 2xysinx = 0$$

$$\therefore (y - \cos x - c)(y e^{-x} - c)$$

$$7. y = (y')^2 e^y$$

$$\therefore y = 0, y = \rho^2 e^\rho, x = (\rho+1)e^\rho + c.$$

$$8. \ln y' + \sin y' - x = 0$$

$$\therefore \begin{cases} x = \ln \rho + \sin \rho \\ y = \rho + \cos \rho + \rho \sin \rho + c \end{cases}$$

$$9. y' \sin y' + \cos y' - y = 0$$

$$\therefore y = 1, x = \sin \rho + c.$$

$$10. (y')^2 + (x+a)y' - y = 0$$

$$\therefore y = (x+a)c + c^2 \text{ ба } \begin{cases} y = \rho^2 + (x+a)\rho \\ 2\rho + x + a = 0 \end{cases}$$

$$11. \sqrt{(y')^2 + 1} + xy' - y = 0$$

$$\therefore y = cx + \sqrt{1+c^2} \text{ ба } \begin{cases} x = -\frac{\rho}{\sqrt{1+\rho^2}} \\ y = \rho x + \sqrt{1+\rho^2} \end{cases}$$

$$12. x^4(y')^2 - xy' - y = 0$$

$$\therefore y = x^4 \rho^2 - x\rho, 1 - 2\rho x^3 = 0 \text{ еки } y = -\frac{1}{4x^2}.$$

$$13. xy' = y + x\sqrt{1+(y')^2}$$

$$\therefore x = \frac{c(\rho + \sqrt{1+\rho^2})}{\sqrt{1+\rho^2}}, y = x(\rho - \sqrt{1+\rho^2}).$$

$$14. y' + y = x(y')^2$$

$$\therefore x = \frac{(\rho - \ln \rho + c)}{(\rho - 1)^2}.$$

$$15. y(y')^2 + 2xy' - y = 0$$

$$\therefore z = y^2, y^2 = cx + \frac{c^2}{4}.$$

$$16. (y')^2 + y^2 = 1$$

$$\therefore y = \pm 1, y = \cos(x - c).$$

$$17. \quad (y')^2 - 2xy' + y = 0$$

$$\int: \quad x = \frac{2}{3}\rho + \frac{c}{\rho^2}; \quad y = 2\rho x - \rho^2, \quad y = 0.$$

$$18. \quad y = x + 2y' - (y')^2$$

$$\int: \quad y = x + 1, \quad x = -2\rho + c, \quad y = x + 2\rho - \rho^2.$$

$$19. \quad 2y = xy' + y' \ln y$$

$$\int: \quad \begin{cases} x = 2c\rho - \ln\rho - 2 \\ y = c\rho^2 - \rho \end{cases}$$

$$20. \quad y = x(1 + y') + y'^2$$

$$\int: \quad \begin{cases} x = 2(1 - \rho) + c\ell^{-\rho} \\ y = [2(1 - \rho) + c\ell^{-\rho}](1 + \rho) + \rho^2 \end{cases}$$

## IX BOB.

### IKKINCHI TARTIBLI DIFFERENTSIAL TENGLAMALAR.

**1-§. Boshlang'ich shart bilan berilgan ikkinchi tartibli differentsial tenglamalar.**

$$y'' = f(x, y, y') = f(x) \quad (1)$$

ko'rinishdagi ikkinchi tartibli differentsial tenglamalar ikki marta integrallash bilan yechiladi.  $y' = z$  deb  $y'' = z'$  ni topamiz. Natijada o'zgaruvchilarga ajraladigan birinchi tartibli tenglamaga kelamiz:

$$z' = f(x) \quad \text{yoki} \quad \frac{dz}{dx} = f(x) \quad \text{yoki} \quad dz = f(x)dx.$$

Bu tenglamani integrallab quyidagini  $\int dz = \int f(x)dx$  topamiz, bu yerdan

$$z = F(x) + C_1$$

ga kelamiz.

Buni ham yechsak,  $z$  ni  $\frac{dy}{dx}$  ga almashtiramiz.

$$dy = [F(x) + C_1]dx$$

buni integrallasak,

$$y = \int F(x)dx + C_1x + C_2$$

bu yerdan

$$y = \Phi(x) + C_1x + C_2 \quad (2).$$

Bu  $y'' = f(x)$  tenglamaning umumiy yechimidir. (2) tenglikda ikkita ixtiyoriy  $C_1$  va  $C_2$  koeffitsientlar hosil bo'ldi. Agar (1) tenglama uchun umumiy yechimdan qandaydir xususiy yechimga kelish kerak bo'lsa,  $x = x_0$ ,  $y = y_0$ ,  $y' = y'_0$  boshlang'ich shart talab qilish zarur. U holda

$$\begin{aligned} y_0 &= \varphi(x_0, C_1, C_2), \\ y'_0 &= \varphi_x(x_0, C_1, C_2) \end{aligned}$$

bo'ladi.

**1-Misol.**  $y'' = \sin x - x$  tenglamaning  $y(0) = -1$ ,  $y'(0) = 1$  boshlang'ich shartlarini qanoatlantiruvchi xususiy yechimini toping.

**Yechilishi:**  $\frac{dy}{dx} = z$  desak,

$$\frac{dz}{dx} = \sin x - x; \quad dz = (\sin x - x)dx; \quad z = -\cos x - \frac{x^2}{2} + C_1$$

Demak,

$$y' = -\cos x - \frac{x^2}{2} + C_1$$

$y'(0) = 1$  boshlang'ich shartdan foydalanib, topamiz:

$$1 = -\cos 0 + C_1$$

bu yerdan  $C_1 = 2$ . Shunday qilib,

$$\frac{dy}{dx} = -\cos x - \frac{x^2}{2} + 2 \quad \text{yoki} \quad dy = \left( -\cos x - \frac{x^2}{2} + 2 \right) dx$$

bu tenglamani integrallaymiz:

$$y = -\sin x - \frac{x^3}{6} + 2x + C_2$$

Endi  $y(0) = -1$  boshlang'ich shartdan foydalanimiz:

$$-1 = C_2$$

Shunday qilib, izlanayotgan xususiy yechim ushbu ko'rinishiga ega:

$$y = -\sin x - \frac{x^3}{6} + 2x - 1.$$

### 1. Tartibini pasaytirish mumkin bo'lgan yuqori tartibli differentsial tenglamalar.

n – tartibli differentsial tenglamalarning umumiy ko'rinishi quyidagicha

$$\Phi(x, y, y', y'', \dots, y^{(n)}) = 0 \quad (1)$$

yoki n tartibli hosilaga nisbatan yechilgan differentsial tenglama quyidagicha

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)}) \quad (2)$$

bo'ladi. Bunda  $x$  – erkli o'zgaruvchi,  $y = y(x)$  – noma'lum funktsiya,  $y', y'', \dots, y^{(n)}$  lar noma'lum funktsianing birinchi, ikkinchi va x.k., n – tartibli hosilalari. (1) differentsial tenglama xususiy holda ushbu

$$y^{(n)} = f(x) \quad (3)$$

ko'rinishga ega bo'lsin. Bunda (3) ni ketma-ket n marta integrallab umumiy yechimi topiladi. (1) differentsial tenglamada

noma'lum funktsiya va uning dastlabki bir nechta tartibdagi hosilalari qatnashmasin:

$$\Phi(x, y^{(k)}, y^{(k+1)}, \dots, y^{(n)}) = 0 \quad (4)$$

Bu holda  $y^{(k)} = p = p(x)$  almashtirish natijasida (4) differentsiyal tenglamaning tartibi pasayib, ushbu

$$\Phi(x, p, p', \dots, p^{(n-k)}) = 0$$

ko'rinishga keladi.

(1) differentsiyal tenglamada erkli o'zgaruvchi x qatnashmasin:

$$\Phi(y, y', y'', \dots, y^{(n)}) = 0$$

bu holda  $y' = p = p(x)$  almashtirish bilan differentsiyal tenglamaning tartibi birlikka pasayadi, ya'ni

$$y' = \frac{dy}{dx} = p$$

$$y'' = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx} = p \cdot \frac{dp}{dy}$$

$$y''' = \frac{d}{dx} \left( p \cdot \frac{dp}{dy} \right) = \frac{d}{dy} \left( p \cdot \frac{dp}{dy} \right) \cdot \frac{dy}{dx} = p^2 \frac{d^2 p}{dy^2} + p \left( \frac{dp}{dy} \right)^2$$

va h.k. bo'lishi e'tiborga olinadi.

**2-Misol.**  $y''' = 60x^2$  tenglamani yeching.

**Yechilishi:** Tenglamaning ikkala tomonini  $dx$  ga ko'paytirib 2-chi tartibli tenglamani hosil qilamiz.  $y'''dx = 60x^2dx$  dan  $y'' = 20x^3 + C_1$  ni hosil qilamiz.

So'ngra yuqoridagini qo'llab 1-chi tartibli tenglamani hosil qilamiz.

$$y''dx = (20x^3 + C_1)dx \text{ dan } y' = 5x^4 + C_1x + C_2 \text{ ni hosil qilamiz.}$$

$$y'dx = (5x^4 + C_1x + C_2)dx \quad \text{dan} \quad y = x^5 + C_1\frac{x^2}{2} + C_2x + C_3.$$

## 2. O'zgarmas koeffitsientli bir jinsli chiziqli differentsiyal tenglamalar.

Ushbu

$$y^{(n)} + p_n y^{(n-1)} + \dots + p_1 y = 0 \quad (1)$$

tenglamaga **bir jinsli chiziqli differentsiyal tenglama** deyiladi.  
Tenglamaning umumiy yechimi

$$y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n \quad (2)$$

dan iborat, bunda  $y_1, y_2, \dots, y_n$  lar (1) tenglamaning chiziqli bo'limgan xususiy yechimlari,  $C_1, C_2, \dots, C_n$  lar esa ixtiyoriy o'zgarmaslar.

O'zgarmas koeffitsientli *ikkinchitartibli chiziqli bir jinsli tenglama* deb

$$y'' + py' + qy = 0 \quad (3)$$

ko'rinishdagi tenglamaga aytildi.

(2) tenglamaning umumiy yechimini topish uchun

$$r^2 + pr + q = 0 \quad (4)$$

xarakteristik tenglama tuziladi, bu tenglama (3) tenglamadan izlanayotgan funktsiya hosilalarni  $r$  ning mos darajalari bilan funktsiyaning o'zini esa bir bilan almashtirish natijasida hosil qilinadi. U holda (3) tenglamaning umumiy yechimi (4) tenglama ildizlari  $r_1$  va  $r_2$  ning xarakteriga bog'liq holda topiladi. Bu yerda uch hol ro'y berishi mumkin.

**1-hol.**  $r_1$  va  $r_2$  ildizlar haqiqiy va har xil. Bunday holda (3) tenglamaning umumiy yechimi

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

ko'rinishga ega bo'ladi.

**2-hol.**  $r_1$  va  $r_2$  ildizlar haqiqiy va bir-biriga teng:

$$r_1 = r_2 = r,$$

bu holda (3) tenglamaning umumiy yechimi

$$y = (C_1 + C_2 x)e^{rx}$$

ko'rinishga ega bo'ladi.

**3-hol.**  $r_1$  va  $r_2$  ildizlari qo'shma kompleks:

$$r_1 = \alpha + \beta i, \quad r_2 = \alpha - \beta i$$

Bunday holda umumiy yechim

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

ko'rinishda yoziladi.

**3-Misol.**  $y'' + 3y' - 4y = 0$  tenglamani yeching.

**Yechilishi:** Xarakteristik tenglamani yozamiz:

$$k^2 + 3k - 4 = 0$$

echimlari:  $k_1 = -4, \quad k_2 = 1$

1-holga ko'ra umumiy yechim:  $y = C_1 e^{-4x} + C_2 e^x$

**4-Misol.**  $y'' + 6y' + 9y = 0$  tenglamani yeching.

**Yechilishi:** Xarakteristik tenglamani yozamiz:

$$k^2 + 6k + 9 = 0$$

echimlari:  $k_1 = k_2 = 3$ ,

2-holga ko'ra umumiy yechim:  $y = e^{3x}(C_1 + C_2 x)$

**5-Misol.**  $y'' + 4y' + 13y = 0$  tenglamani yeching.

**Yechilishi:** Xarakteristik tenglamani yozamiz:

$$k^2 + 4k + 13 = 0$$

echimlari:  $k_1 = -2 + 3i$ ,  $k_2 = -2 - 3i$

3-holga ko'ra umumiy yechim:  $y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x)$ .

### 3. O'zgarmas koeffitsientli va maxsus o'ng tomonga ega bo'lgan ikkinchi tartibli chiziqli bir jinsli bo'limgan differentialsial tenglamalar.

Ushbu

$$y'' + py' + qy = f(x) \quad (1)$$

tenglama o'zgarmas koeffitsientli chiziqli bir jinsli bo'limgan differentialsial tenglama deyiladi. U mos chiziqli bir jinsli

$$y'' + py' + qy = 0 \quad (2)$$

tenglamadan o'ng tomonda turgan  $f(x)$  funktsiya bilan farqlanadi.

(1) tenglamaning umumiy yechimini topish uchun (2) tenglamaning umumiy yechimi  $y_1$  ni, so'ngra (1) tenglamaning birorta xususiy yechimi  $y^*$  ni topish kerak. Ularning yig'indisi berilgan bir jinsli bo'limgan (1) tenglamaning umumiy yechimi bo'ladi

$$y = y_1 + y^*$$

Quyidagi ikki holda (1) tenglamaning xususiy yechimi  $y^*$  ni topish qoidasini keltiramiz.

O'ng tomoni  $f(x)$

$$f(x) = e^{kx} P_n(x) \quad (3)$$

ko'rinishga ega, bu yerda  $P_n(x)$  ko'phad n-darajali ko'phad:  $f(x)$  ning o'ng tomoni

$$f(x) = a \cos \lambda x + b \sin \lambda x \quad (4)$$

ko'rinishga ega.

Bu hollarning har qaysisini alohida – alohida ko'ramiz.

I. (1) tenglamaning o'ng tomoni  $f(x) = e^{kx} P_n(x)$  ko'inishga ega bo'lsin, bu yerda k soni (2) bir jinsli tenglamaga mos keluvchi xarakteristik tenglama

$$r^2 + pr + q = 0 \quad (5)$$

ning ildizi bo'lmasin. U holda (1) tenglamaning xususiy yechimini

$$y'' = e^{kx} Q_n(x) \quad (6)$$

ko'inishda izlash kerak, bu yerda  $Q_n(x)$  birorta n – darajali koeffitsientlari noaniq bo'lgan ko'phad.

Agar k soni (5) xarakteristik tenglamaning ildizi bo'lsa, u holda (1) tenglamaning xususiy yechimini

$$y'' = x^m e^{kx} Q_n(x)$$

ko'inishida izlash kerak, bu yerda m son k ildizning karraligi (ya'ni, agar k – bir karrali ildiz bo'lsa, m=1, agar k – ikki karrali ildiz bo'lsa, m=2).

II. (1) tenglamaning o'ng tomoni

$$f(x) = a \cos \lambda x + b \sin \lambda x$$

ko'inishga ega bo'lsin, bu yerda  $\pm \lambda i$  sonlar (5) xarakteristik tenglamaning ildizlari emas. U holda (1) tenglamaning xususiy yechimini

$$y'' = a \cos \lambda x + b \sin \lambda x$$

ko'inishda izlash kerak, bu yerda A va B – noaniq koeffitsientlar.

Agar  $\pm \lambda i$  kompleks sonlar (5) xarakteristik tenglamaning ildizlari bo'lsa, (1) tenglamaning xususiy yechimini

$$y'' = x(a \cos \lambda x + b \sin \lambda x)$$

shaklida izlash kerak.

**6-Misol.**  $y'' - 6y' + 9y = 3x - 8e^x$  tenglamaning umumiylar yechimini toping.

**Yechilishi:** Xarakteristik tenglamani yozamiz:

$$r^2 - 6r + 9 = 0 \text{ yoki } (r - 3)^2 = 0 \text{ va yechimi } r_{1,2} = 3 \text{ ga teng.}$$

Bundan

$$u = e^{3x}(C_1 + C_2x)$$

Berilgan tenglamaning xususiy yechimi  $u_1$  ni topamiz. Bu yerda o'ng tomoni  $f(x) = 3x - 8e^x$  ko'rinishga ega. Ko'phadning birinchi darajali qismi  $3x$  va ko'rsatkichli qismi  $-8e^x$  ga teng. Demak, xususiy yechimi  $u_1'' = Ax + B + Ce^x$  ga teng.

$u_1' = A + Ce^x$  ga,  $u_1'' = Ce^x$  ga tengligidan, tenglama quyidagi ko'rinishga keladi.

$$9Ax + (9B - 6A) + 4Ce^x = 3x - 8e^x$$

Bundan,  $9A = 3$ ,  $9B - 6A = 0$ ,  $4C = -8$  ga tengligidan  $A = \frac{1}{3}$ ,

$$B = \frac{2}{9}, C = -2$$

Demak,

$$u_1 = \frac{1}{3}x + \frac{2}{9} - 2e^x,$$

$$y = u + u_1 = e^{3x}(C_1 + C_2x) + \frac{1}{3}x + \frac{2}{9} - 2e^x.$$

### Nazorat savollari.

1. Boshlang'ich shart bilan berilgan ikkinchi tartibli differentsial tenglamalar.
2. Tartibini pasaytirish mumkin bo'lgan yuqori tartibli differentsial tenglamalar.
3. O'zgarmas koeffitsientli bir jinsli chiziqli differentsial tenglamalar.

4. O'zgarmas koeffitsientli va maxsus o'ng tomonga ega bo'lgan ikkinchi tartibli chiziqli bir jinsli bo'lмаган differentsial tenglamalar.

## 2-§. Differential tenglamaning tatbiqlari.

### I. Populyatsiya miqdorining dinamikasi.

Populyatsiya miqdorining dinamikasi (ya'ni, populyatsiya davrida tug'ilish va o'lish natijasida tirik mavjudotlar miqdorining o'zgarishi) ekologiyaning muhim masalalaridan biri. Birinchi tartibli differentsial tenglamalarni o'rganishda yuqoridaagi masalaning eng oddiy holatini qaradik. Oziq — ovqat bilan ta'minlangan va tashqi muhit ta'siridan chegaralangan holda populyatsiya dinamikasi quyidagi differentsial tenglama bilan berildi:

$$\frac{dx}{dt} = \mu x \quad (1)$$

Bu yerda  $x = x(t) - t$  vaqtagi populyatsiya miqdori. Bu tenglamaning yechimi quyidagicha

$$x = x_0 e^{\mu(t-t_0)}$$

ekanligi kelib chiqdi.

Bu yerda  $x_0, t_0$  boshlang'ich vaqtagi populyatsiya miqdori. (1) tenglama yoki nazariy ahamiyatga ega yoki sun'iy sharoitdagi mavjudotlarning populyatsiyasini aniqlaydi.

Populyatsiyaning rivojlanishini 1845 yilda Ferxulsta — Perla tenglamasi aniqroq yoritib beradi. Bu tenglamada mavjudotlarning ichki qarama — qarshiliklarini hisobga oladi, bu esa populyatsiya miqdorining tezligini sekinlashtiradi. Bu qarama — qarshiliklarga oziq — ovqat uchun kurash, jips yashaganda infektsiya tarqalishi va h.k. kiradi. Yuqoridaagi faktlarni hisobga olib,  $\Delta x$  o'sishni hisoblashda  $\mu \Delta t$  qiymatdan  $h(x, \Delta t)$  qiymatni ayiramiz:

$$\Delta x = \mu \Delta t - h(x, \Delta t)$$

$h(x, \Delta t)$  funktsiya o'rniga ko'pchilik hollarda  $\delta x^2 \Delta t$  populyatsiyani qaraymiz:

$$h(x, \Delta t) = \delta x^2 \Delta t$$

bu yerda  $\delta$  — koeffitsient ichki qarama-qarshiliklar.

$h(x, \Delta t)$  qiymat — bu ichki qarama-qarshiliklar evaziga populyatsiya miqdori tezligining kamayishini ifodalaydi. Qarama-qarshiliklar qancha yuqori bo'lsa, urug'lanadiganlarning bir-biri bilan uchrashishi shuncha ko'p, bu uchrashishlar soni  $x \cdot x$  ko'paytmaga to'g'ri proporsional, ya'ni  $x^2$ . Ikki xil mavjudotning uchrashishi  $xy$  ga to'g'ri proporsional. Bu turlar bir-birining joyini egallashi mumkin.

Shunday qilib,

$$\Delta x = \gamma x \Delta t - \delta x^2 \Delta t \quad (2)$$

bu tenglikni  $\Delta t$  ga bo'lamic.

$$\frac{\Delta x}{\Delta t} = \gamma x - \delta x^2$$

va  $\Delta t \rightarrow 0$  da limitga o'tamiz.

$$\frac{dx}{dt} = \gamma x - \delta x^2 \quad (3)$$

bu tenglama **Ferxylsta-Perla tenglamasi**.

Bu tenglamadan  $\gamma$  ni qavsdan chiqarib quyidagicha yozamiz:

$$\frac{dx}{dt} = \gamma x \left(1 - \frac{\delta}{\gamma} x\right)$$

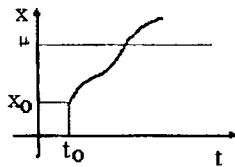
yoki

$$\frac{dx}{dt} = \gamma x \cdot \frac{\frac{\gamma}{\delta} - x}{\frac{\gamma}{\delta}}$$

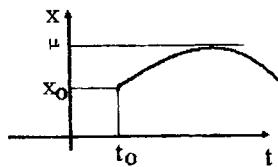
$\frac{\gamma}{\delta} = \mu$  deb belgilaymiz:

$$\frac{dx}{dt} = \gamma x \frac{\mu - x}{\mu} \quad (5)$$

Agar  $x_0 < \mu$  bo'lsa, u holda barcha  $t > t_0$  vaqt uchun haqiqatan ham  $x(t)$  differentialsallanuvchi funktsiya. (5) tenglamadan kelib chiqadiki,  $x(t) < \mu$  da  $\frac{dx}{dt}$  musbat bundan kelib chiqadiki,  $x(t)$  o'sadi. Bundan shuni xulosa qilish mumkinki,  $x(t) = \mu$  ga teng qiymatni qabul qilsa o'sadi.  $x = \mu$  to'g'ri chiziqni kesib o'tadi (1-rasm) yoki  $x = \mu$  to'g'ri chiziqqa urinadi (2-rasm).



1 – rasm



2 – rasm

Birinchi holda:  $x(t) > \mu$  va  $x'(t) > 0$ . Bu (5) – tenglamaga zid. Ikkinci holda,  $x(t) < \mu$  va  $x'(t) < 0$ , bu ham (5) – tenglamaga zid. Shunday qilib,  $x(t) = \mu$  ga teng bo'lishi mumkin emas, agar  $x_0 < \mu$  bo'lsa.

O'zgaruvchilarni ajratib,

$$\frac{\mu dx}{x(\mu - x)} = \gamma dt$$

yoki

$$\frac{(\mu - x) + x}{x(\mu - x)} dx = \gamma dt$$

bundan

$$\left( \frac{1}{x} + \frac{1}{\mu - x} \right) dx = \gamma dt$$

Agar  $x_0 < \mu$  deb hisoblasak, quyidagiga ega bo'lamiz:

$$\ln x - \ln(\mu - x) = \mu t + \ln C$$

bundan

$$\frac{x}{\mu - x} = Ce^{\mu t} \quad (6)$$

Qulaylik uchun  $t_0 = 0$  va  $x(0) = x_0 < \mu$  deb olib, (6) ga qo'ysak, C ni topamiz:

$$C = \frac{x_0}{\mu - x_0}$$

topilgan qiymatni (6) ga qo'ysak quyidagini hosil qilamiz:

$$\frac{x}{\mu - x} = \frac{x_0}{\mu - x_0} e^{\mu t}$$

Bundan qidiralayotgan **Ferxulsta-Perla** modelini hosil qilamiz:

$$x = \frac{x_0 \mu e^{\mu t}}{\mu - x_0 + x_0 e^{\mu t}}$$

## 2. Epidemiya nazariyasida differentsiyal tenglama.

Epidemiyaning eng oddiy turini qaraymiz. Aytaylik, o'r ganilayotgan kasallik uzoq vaqt davom etadi, demak infektsiya tarqalishi kasallanishga nisbatan tezroq tarqaladi. Biz infektsiya tarqalish holati bilan chegaralanamiz. Aytaylik,  $x$  va  $y$  lar mos ravishda infektsiya yuqtirganlar va yuqtirmaganlar sonini aniqlasim.  $x = x(t)$  - vaqt dagi sog'lom organizmlar,  $y = y(t)$  - vaqt dagi infektsiyalangan organizmlar soni. Uncha katta bol'magan vaqt oraliq'ida ya'nii  $0 \leq t \leq T$  quyidagi tenglik o'rinali:

$$x + y = n + a \quad (9)$$

Demak, uchrashish chog'ida infektsiya yuqtirilgan organizmdan sog'lom organizmga o'tadi, u holda sog'lom organizmlar soni vaqt o'tishi bilan kamayadi va u uchrashuvlar soniga proportsional bo'ladi (ya'ni,  $x \cdot y$  ga proportsional). Bundan sog'lom organizmlar soni kamayadi va kamayish tezligi quyidagiga teng:

$$\frac{dx}{dt} = -\beta \cdot xy \quad (10)$$

bu yerda  $\beta$  — proportsionallik koefitsienti. Bundan y ni topib (9) ga qo'ysak:

$$\frac{dx}{dt} = -\beta \cdot x(n + a - x).$$

O'zgaruvchilarga ajratib quyidagini topamiz:

$$\frac{dx}{x(n + a - x)} = -\beta \cdot dt$$

yoki

$$\frac{(n + a - x) + x}{x(n + a - x)} dx = -\beta(n + a) dt.$$

Bundan

$$\frac{dx}{x} + \frac{dx}{n - x + a} = -\beta(n + a) dt.$$

Integrallab, quyidagini hosil qilamiz:

$$\ln x - \ln(n - x + a) = -\beta(n + a)t + \ln c$$

yoki

$$\ln \frac{x}{n - x + a} = ce^{-\beta(n + a)t}$$

$c$  ni topish uchun quyidagi boshlang'ich shartdan foydalanamiz. Agar  $t=0$  da sog'lom organizmlar soni  $n$  bo'lsa (ya'ni  $x=n$ ). Bundan  $c = \frac{n}{a}$  ekanligi kelib chiqadi va

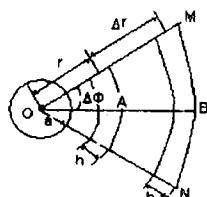
$$\frac{x}{n-x+a} = \frac{n}{a} e^{-\beta(n+a)t}.$$

Demak, qidirayotgan ifodamiz quyidagiga teng ekan:

$$x = \frac{n(n+a)}{n+ae^{\beta(n+a)t}}$$

### 3. Chumoli inidan tashqarida chumolining zichligi.

Chumoli hayotining umumiy yashash belgilari ko'pchilikka ma'lum. Chumoli topilgan oziq-ovqatni yoki qurilish materiallarini topilgan joyidan uyasiga tashiydi. Shuning uchun chumoli soni uyasining yaqinida uyidan o'zoqroqqa nisbatan ko'proq (3-rasm).



(3 – rasm)

Oddiylik uchun chumoli uyasining markazi deb radiusi  $a$  ga teng doirani olamiz va bu doiradan tashqarida chumolilar uchun oziq-ovqat bir xilda taqsimlangan deb hisoblaymiz. Bu degani radiusi  $r$  ga teng aylanadagi muhit undan tashqarida ham bir xil. Shuning uchun radiusi  $r$  ga teng aylanada ( $r>a$ ) chumoli zichligi bir xil taqsimlangan deb faraz qilamiz. (Hashoratlar zichligi deganda ma'lum atrofdagi hashoratlar sonining shu atrof yuziga nisbatiga aytildi). Bundan kelib chiqadiki, zichlik bu  $r$  masofaning funksiyasi bo'lib, nuqtalari bitta nurda yotgan nuqtalar bilan chegaralansak kifoya.

Soddalik uchun statsionar holatni, ya'ni vaqt o'tishi bilan zichlik har bir nuqtada o'zgarmas deb hisoblaymiz. Ammo bu degani, chumoli bir-biri bilan aralashib ketmaydi degani emas. Oziq-ovqat qidirishda chumoli bir joydan ikkinchi joyga ko'chib yuradi. Biz bu qidirishni tasodify deb hisoblaymiz, yani birlik vaqt ichida ovqat qidirib biror atrofni tark etsa, xuddi shuncha sondagi chumolilar shu atrofga qo'shni atrofdan o'tadi. Shunday qilib chumoli uyası atrofidan ovqat izlab boshqa joyga ko'chishsa, yana shu sondagi chumolilar uyasidan chiqadi.

Bir nurda yotgan ikkita qo'shni nuqtani qaraymiz: A nuqta chumolilar uyasining markazidan  $r$  masofada joylashgan bo'lsin, B nuqta  $r + \Delta r$  masofada bo'lsin. Bu nuqtalar atrofida chumolilarning ko'chishini kuzatamiz. Aytaylik,  $Q(r + \Delta r)$  B nuqtadagi chumolilar soni. Chumoli ovqat qidirayotganda birorta yo'naliш ikkinchisidan yaxshi bo'la olmaydi. Chunki atrof bir xil deb hisoblangan. Chumoli soni uning qaysi yo'naliшdan ovqat qidirishiga bog'liq emas. Bundan kelib chiqadiki, agarda A nuqta atrofidan B nuqtaga qarab  $\alpha Q(r)$ ,  $\alpha < 1$  chumolilar chiqqan bo'lsa, u holda B nuqta atrofidan A nuqtaga  $\alpha Q(r + \Delta r)$  sondagi chumolilar chiqishdi.

Bunda eng muhimi ikkala holatda ham  $\alpha$  sonining bir xil ekanligidadir.  $\alpha$  fiksirlangan yo'naliшdagи chumolilar qismini aniqlaydi. Ammo A nuqta atrofidan B nuqta atrofiga chiqqan chumolilarning barchasi ham B nuqtaga yetib bormaydi, chunki yo'lda ovqatni topganlari uyasiga qaytib ketadi. Bu toifadagi chumolilar A va B nuqtalar orasidagi masofa qancha katta bo'lsa shuncha ko'p bo'ladi. Shuning uchun A nuqta atrofidan chiqqan B nuqta atrofiga yetib boradigan chumolilar soni quyidagi ayirma bilan ifodalanadi:

$$Q_{AB} = \alpha Q(r) - \beta \alpha Q(r) \Delta r$$

Bu yerda  $\beta$  – proportsianallik koeffitsienti (manzilga yetmasdan qaytgan chumolilarni aniqlaydi). Bu koeffitsient atrofga bog'liq, lekin atrofda sharoit bir xil bo'lganligi uchun  $\beta$  shu atrofda o'zgarmas.  $\alpha Q(r + \Delta r)$  qiymatga B nuqta atrofidan A nuqta atrofiga emas, boshqa yo'naliшga chiqib, yo'ldan ovqat topib orqaga qaytganlari ham kiradi. Bunday chumolilarga OMN sektordan hali chiqib ulgurmaganlari ham A nuqta atrofiga

kiradi.  $\Delta r$  qancha katta bo'lsa, ular shuncha ko'p bo'ladi. Shunday qilib, B nuqta atrofidan chiqib A nuqta atrofiga tushadiganlari, quyidagi yig'indi orqali aniqlanadi.

$$Q_{AB} = \alpha Q(r + \Delta r) + \beta_1 \alpha Q(r + \Delta r) \Delta r$$

bu yerda  $\beta_1$  – chumoli uyasiga qaytgan chumolilar sonini aniqlaydigan proporsianallik koeffitsienti. Biz statsionar holatda bo'lganimiz uchun, yani nuqta atrofida o'zgarmas sonda qolgani uchun quyidagi tenglik bajariladi.

$$Q_{AB} = Q_{R1}$$

ya'ni,

$$\alpha Q(r) - \beta \alpha Q(r) \Delta r = \alpha Q(r + \Delta r) + \beta_1 \alpha Q(r + \Delta r) \Delta r$$

Bundan biz differentsial tenglamaga kelamiz. Chumolilar soni uning zichligining yuza ko'paytmasiga teng bo'lgani uchun ( $\alpha$  ga qisqartirib ) oxirgi tenglikni quyidagicha yozamiz:

$$n(r)S_A - \beta n(r)S_A \Delta r = n(r + \Delta r)S_B + \beta_1 n(r + \Delta r)S_B \Delta r \quad (12)$$

bu yerda  $n(r)$  va  $n(r + \Delta r)$  lar A va B nuqtalardagi mos ravishda zichligini belgilaydi;  $S_A$  va  $S_B$  shu nuqta atrofining yuzalari.

Yuzani qutb koordinatalar sisternasida hisoblab, quyidagini hosil qilamiz:

$$S_A \approx h r \Delta \Phi; \quad S_B \approx h(r + \Delta r) \Delta \Phi.$$

Buni (12) – tenglikka qo'yamiz:

$$\begin{aligned} n(r)h r \Delta \Phi - \beta n(r)h r \Delta \Phi \Delta r &= n(r + \Delta r) \cdot h(r + \Delta r) \Delta \Phi + \\ &+ \beta_1 n(r + \Delta r) \cdot h(r + \Delta r) \Delta \Phi \Delta r \end{aligned}$$

buni  $h \Delta \Phi$  ga qisqartirib gruppalaşak, quyidagini hosil qilamiz:

$$(r + \Delta r)h(r + \Delta r) - r \cdot n(r) = -[\beta_1 n(r + \Delta r)(r + \Delta r) + \beta n(r)r] \Delta r.$$

Bu tenglikni  $\Delta r$  ga bo'lib  $\Delta r \rightarrow 0$  sak

$$\frac{d}{dr}(r \cdot n(r)) = -(\beta_1 + \beta) \cdot rn(r) \quad (13)$$

Qisqalik uchun  $\beta_1 + \beta = \gamma$  deb belgilaymiz va

$$\frac{d(r \cdot n(r))}{r \cdot n(r)} = \gamma \cdot dr \quad (14)$$

Biz  $n(r)$  zichlik uchun differenttsial tenglama hosil qildik.

Aytaylik,  $n(a)$  – chumoli uyasining chegarasidagi ( $r=a$ ) zichlik qiymati. (14) ni integrallab, quyidagini hosil qilamiz:

$$\ln[r \cdot n(r)] = -\gamma r + C \quad (15)$$

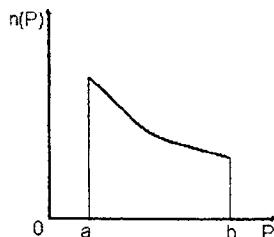
Boshlang'ich shartdan foydalaniib,  $C = \ln[a \cdot n(a)] + \gamma a$  ni hosil qilamiz. Buni (15) ga qo'yib,

$$\ln \frac{r \cdot n(r)}{a \cdot n(a)} = -\gamma(r-a)$$

bundan

$$n(r) = \frac{a}{r} n(a) e^{-\gamma(r-a)} \quad (16)$$

Bu qidirilayotgan egri chiziq tenglamasi. Agar  $a$ ,  $n(a)$  va  $\gamma$  larning qiymatlari ma'lum bo'lsa, grafigini chizish mumkin. (4 – rasm).



(4 – rasm).

$r$  o'sishi bilan egri chiziq kamayadi.  $a$  va  $n(a)$  larni tajriba yo'li bilan topish qiyin emas. Ammo  $\gamma$  koeffitsientni hisoblash qiyinroq. Agarda tuzilgan matematik modelni to'g'ri deb hisoblasak, bundan kelib chiqadiki qandaydir  $\gamma$  ni berilgan deb (16) dan zichlikning masofaga bog'liqligidan  $\gamma$  ni hisoblasak bo'ladi. Agar (16) o'rinni bo'lsa, u holda  $r > a$  uchun ham o'rinni, xususan  $r = b$  uchun

$$n(b) = \frac{a}{b} n(a) e^{-r(b-a)},$$

bundan

$$\gamma = \frac{1}{b-a} \ln \frac{a \cdot n(a)}{b \cdot n(b)} \quad (17)$$

$\gamma$  ni hisoblash uchun  $n$  ni ham  $n(a)$  kabi (16) formuladan hisoblashimiz kerak. Buning uchun quyidagilarni hisobga olishimiz kerak.

A) (16) formuladan ixtiyoriy katta  $r$  uchun  $n(r) \neq 0$ . Haqiqiy hayotda albatta bunday emas.

B)  $\gamma$  qiymat vaqtga albatta bog'liq, chunki kechasi va kunduzi quyosh aktiv vaqtida chumoli boshqa vaqtga nisbatan aktivligi kam bo'ladi.

Ammo kunning har xil vaqtida  $n(a)$  va  $n(b)$  qiymatlarni (17) formulaga asoslanib hisoblab,  $\gamma$  ni (16) formulaga qo'yib har xil vaqtida o'zining egri chizig'ini topa olamiz.

#### 4. O'simlik bargining o'sishi.

Tuzilishi doira shaklida bo'lgan yosh yaproq yuzasining o'sish tezligi yaproq aylanasi uzunligi va unga tushgan quyosh nuri miqdoriga to'g'ri proportsional. Bu esa quyosh nuri va yaproq orasidagi burchak kosinusni va yaproq yuzasiga to'g'ri proportsional. Agar ertalab soat  $6^{\circ}$  da yaproq yuzasi  $1600 \text{ sm}^2$  va kech soat  $18^{\circ}$ da shu kuni  $2500 \text{ sm}^2$  bo'lsa yaproq yuzi S bilan vaqt orasida bog'lanishni toping.

Quyosh nuri va yaproq orasidagi burchakni, ya'ni soat  $6^{\circ}$  va  $18^{\circ}$  ni (ishorasini hisobga olmagan holda)  $90^{\circ}$  ga teng, kun yarmida  $0^{\circ}$  ga teng deb qabul qilamiz.

Aytaylik,  $t$  vaqt yarim tun 00 dan boshlansin. Agar yaproq yuzasi  $S$  o'zgarsa, u holda yaproq o'sishining tezligi

$$\frac{ds}{dt} = k_1 2\pi r Q$$

bu yerda  $2\pi r$  — yaproq aylanasining uzunligi,  $Q$  — yoruglik nurining soni,  $k_1$  — proporsionallik koefitsenti.

Yaproq yuzasi  $S = \pi r^2$  dan quyidagini yozib olamiz

$$r = \sqrt{\frac{S}{\pi}}$$

U holda

$$\frac{ds}{dt} = k_1 \frac{2\pi}{\sqrt{\pi}} \sqrt{S} Q \quad (18)$$

Quyidagi shartdan

$$Q = k_2 S \cos \alpha \quad (19)$$

(bu yerda  $\alpha$  — nur va vertikal orasidagi burchak,  $k_2$  — proporsionallik koefitsenti)  $\alpha$  burchak  $t$  argumentning chiziqli o'suvchi funktsiyasi ekanligi kelib chiqadi:

$$\alpha = k_3 t + b$$

$k_3$  va  $b$  parametrlarni qo'shimcha shartlar asosida topamiz:

$$\text{agar } t=6 \text{ bo'lsa } \alpha = -\frac{\pi}{2},$$

$$\text{agar } t=12 \text{ bo'lsa } \alpha = 0,$$

$$\text{agar } t=18 \text{ bo'lsa } \alpha = \frac{\pi}{2}.$$

Oxirgi ikkita shartdan quyidagiga ega bo'lamiz:

$$\begin{cases} 0 = 12k_3 + b \\ \frac{\pi}{2} = 18k_3 + b \end{cases}$$

bu sistemani yechib

$$k_3 = \frac{\pi}{12}, \quad b = -\pi$$

ni topamiz. Bundan

$$\alpha = \frac{\pi}{12}(t - 12)$$

buni (19) ga qo'yamiz.

$$Q = k_2 S \cos\left[\frac{\pi}{12}(t - 12)\right]$$

buni (18) ga qo'yamiz.

$$\frac{ds}{dt} = k_1 k_2 \frac{2\pi}{\sqrt{\pi}} S \sqrt{S} \cos\left[\frac{\pi}{12}(t - 12)\right]$$

$k_1 \cdot k_2 = k$  deb belgilasak. U holda o'zgaruvchilarni ajratsak

$$\frac{ds}{S \sqrt{S}} = k \frac{2\pi}{\sqrt{\pi}} \cos\left(\frac{\pi}{12}(t - 12)\right) dt$$

buni integrallab

$$-\frac{2}{\sqrt{S}} = \frac{24k}{\sqrt{\pi}} \sin\left[\frac{\pi}{12}(t - 12) + C\right] \quad (20)$$

$t = 6$  da  $S = 1600$  va  $t = 18$  da  $S = 2500$  shartlar bilan

$$\begin{cases} -\frac{1}{20} = -\frac{24k}{\sqrt{\pi}} + C \\ -\frac{1}{25} = \frac{24k}{\sqrt{\pi}} + C \end{cases}$$

Bu sistemani yechib

$$C = -\frac{9}{200}, \quad k = \frac{\sqrt{\pi}}{24 \cdot 200}$$

ni topamiz. Buni (20) ga qo'yib quyidagiga ega bo'lamiz:

$$-\frac{2}{\sqrt{S}} = \frac{24\sqrt{\pi}}{24 \cdot 200\sqrt{\pi}} \sin\left[\frac{\pi}{12}(t-12)\right] - \frac{9}{200}$$

Bundan esa

$$S = \frac{160000}{\left\{9 - \sin\left[\frac{\pi}{12}(t-12)\right]\right\}^2}$$

ni topamiz.

### 5. Daraxt o'sishini hisoblash haqidagi masala.

Nega sharoit eng yaxshi bo'lganda ham daraxt ma'lum uzunlikdan oshmaydi? Nega daraxt turiga bog'liq bo'lmasan holda boshlang'ich vaqtida tez o'sadi, so'ngra o'sishi sekinlashib, asta – sekin o'sishi nolga teng bo'ladi?

Biroq biz bilamizki, daraxt tomirlarining o'sishi fotosintez yordamida energiyasi ko'payishiga olib keladi, ammo oziqlanish borgan sari qiyinlashib, bora – bora energiya yetishmay boshlaydi va daraxt o'sishdan tuxsaydi.

Shu fikrlarga asoslanib, energiya balansining taxminiy tenglamasini tuzamiz, ya'ni matematik modelini tuzamiz.

1. O'sayotgan daraxt o'sish davrida geometrik xususiyatini saqlaydi, ya'ni uzunligining daraxt diametriga nisbati o'zgarmaydi.
2. Erkin energiyani (yoki harakatdagi moddani) faqat fotosintez orqali oladi.
3. Erkin energiya tirik to'qima hosil qilishga va tuproqdan aralashmalarning ko'tarilishiga sarf bo'ladi.
4. O'rtacha hisobda katta vaqt oralig'ida birlik sirt yuzasiga o'zgarmas miqdorda yorug'lik tushadi va tarkibidagi moddalardan bir qismini yutishi mumkin.

Aytaylik,  $x$  daraxtning chiziqli ulchami. Bu degani daraxt balandligini  $x$  orqali, yaproqning yuzasini  $x^2$  orqali va nihoyat daraxt hajmini  $x^3$  orqali belgilaymiz.  $x$  ning o'zgarishini  $t$  orqali, ya'ni  $x = x(t)$  orqali ifodalashga harakat qilamiz.

Aytaylik,  $x(t_0) = 0$  bo'lsin. Balans tenglamasini  $x$  bo'yicha ifodalasak,  $E$  erkin energiya daraxt tanasining yashil qismidan

fotosintez orqali hosil bo'ladi, yashil qismi qanchalik ko'p bo'lsa shuncha energiya ko'p bo'ladi. Shunday qilib,  $E \propto x^2$  ga to'g'ri proportional

$$E = \alpha x^2$$

bu yerda  $\alpha$  – proportionallik koeffitsenti ( $\alpha$  yaproqning o'lchamiga va tuzilishiga hamda fotosintezga bog'liq). Bizning farazimizga ko'ra, boshqa energiya beruvchi omillar yo'q va biz energiyaning taqsimlanishini kuzatishimiz kerak. Energiya birinchidan, fotosintez sodir bo'lishi uchun sarflanadi. Bu sarflanish xam  $x^2$  ga to'g'ri proportional, yani  $\beta x^2$ , bu yerda  $\beta$  proportionallik koeffitsenti  $\alpha$  dan kichik.

Energiya ozuqaning butun daraxt tanasiga tarqalishi uchun sarflanadi. Ma'lumki, u energiya qancha ko'p sarflansa tana shuncha katta bo'ladi. Bundan tashqari bu sarflanish og'irlik kuchini yengishga bog'liq va bundan kelib chiqadiki, agar ozuqani qancha balandga ko'tarib sarflasa energiya shuncha ko'p sarflanadi. Shunday qilib, energiyaning bu sarfi hajmi  $x^3$  ga va balandlik  $x$  ga to'g'ri proportional, yani  $\gamma \cdot x^3 \cdot x$ .

Nihoyat energiya daraxtning massasini oshirishga sarflanadi (yani o'sishiga). Bu sarflanish o'sish tezligiga to'g'ri proportional, yani  $m = \rho x^3$  massadan vaqt bo'yicha hosila ( $\rho$  – daraxtning o'rtacha zichligi,  $x^3$  – xajmi). Shunday qilib, oxirgi energiya sarflanishi quyidagicha ifodalanadi.

$$\delta \frac{d}{dt} (\rho x^3)$$

Energiyaning saqlanish qonunidan energiya sarfi quyidagiga teng bo'ladi.

$$E = \beta x^2 + \gamma x^4 + \delta \frac{d}{dt} (\rho x^3)$$

yoki

$$\alpha x^2 = \beta x^2 + \gamma x^4 + 3\delta \rho x^2 \frac{dx}{dt} \quad (21)$$

bu esa biz qidirayotgan energiya balansining tenglamasi. Bu tenglamani  $3\delta \rho x^2$  bo'lib quyidagicha yozamiz:

$$\frac{\alpha - \beta}{3\delta\rho} = a, \quad \frac{\gamma}{3\delta\rho} = b$$

bundan:

$$\frac{dx}{dt} = a - bx^2, \quad a > 0, b > 0 \quad (22)$$

kelib chiqadi.

Daraxt o'sayotgan ekan, u holda  $\frac{dx}{dt} > 0$  bu demak,  $a - bx^2 \geq 0$ , bundan  $x^2 < \frac{a}{b}$  kelib chiqadi. Shuning uchun (22) ni quyidagicha yozamiz.

$$-\frac{dx}{b\left(x^2 - \frac{a}{b}\right)} = dt$$

buni integrallab

$$\frac{1}{2\sqrt{ab}} \ln \frac{\sqrt{\frac{a}{b}} + x}{\sqrt{\frac{a}{b}} - x} = t + c$$

boshlang'ich shart  $x(t_0) = 0$  dan foydalanib  $c = -t_0$  ni hosil qilamiz.

$$\frac{1}{2\sqrt{ab}} \ln \frac{\sqrt{\frac{a}{b}} + x}{\sqrt{\frac{a}{b}} - x} = t + t_0 \quad (23)$$

bu tenglamani  $x$  ga nisbatan yechamiz

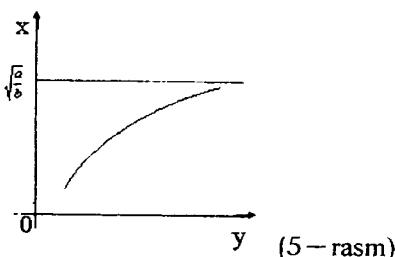
$$x = \sqrt{\frac{a}{b}} \cdot \frac{1 - e^{-2\sqrt{ab}(t-t_0)}}{1 + e^{-2\sqrt{ab}(t-t_0)}} \quad (24)$$

*Bu daraxt o'sishini aniqlaydigan formula.*

Agar  $a, b$  va  $t_0$  qiymatlari ma'lum bo'lsa, bu formula yordamida vaqtga nisbatan o'rtacha o'sishini hisoblash mumkin. (24) formula bilan berilgan egri chiziqni tekshirish mumkin.  $\frac{dx}{dt} > 0$  ligidan va (22) formuladan quyidagini hosil qilamiz:

$$\frac{d^2x}{dt^2} = -2bx$$

Shunday qilib, agar  $t > t_0$ ,  $x(t) > 0$  (bu daraxt balandligi) bo'lsa, u holda oxirgi tenglikdan  $\frac{d^2x}{dt^2} < 0$  ga kelamiz. Demak, (24)-chiziq o'suvchi qavariq chiziq. (24) formuladan  $t \rightarrow +\infty$  da  $x(t) \rightarrow \sqrt{\frac{a}{b}}$  hosil bo'ladi. Grafigi quyidagicha bo'ladi.



(5 – rasm)

$\sqrt{\frac{a}{b}}$  balandlik daraxt o'sishining chegarasi, chunki energiya daraxtning fotosintez va ozuqa bilan ta'minlanishiga sarflanadi. Daraxt shuning uchun bu ko'rsatkichdan yuqori o'smaydi.

(24) formula bilan berilgan egri chiziq daraxt o'sishini qanchalik to'g'ri ifodelaydi? Bu savolga javob berish uchun dub daraxtni olib ko'ramiz. Yoshi 40 yildan 220 yilgacha bo'lgan dub daraxti uzunligi (24) formulada tekshirildi.  $a$  va  $b$  o'zgaruvchili ikkita tenglamalar sistemasi hósil bo'ladi va bu o'zgaruvchilarni topish mumkin. Topilgan  $a$  va  $b$  qiymatga qarab aniq egri chiziq chizildi. Bu egri chiziq tajribadagi dub daraxt o'sishining egri chizig'i bilan ustma – ust tushdi. Boshqacha qilib aytganda (40:  $x$  (40)) va (220:  $x$  (220)) nuqtadagi nazariy va tajribadagi egri chiziqlar ustma – ust tushdi. Demak, ko'rilgan matematik model ishonarli.

## Mustaqil yechish uchun misol va masalalar.

### 1. Tartibini pasaytirish mumkin bo'lgan yuqori tartibli differentsiyal tenglamalar yechilsin.

1.  $y'' = \cos x$

j:  $y = -\cos x + c_1 x + c_2$ .

2.  $y'' = 1 - x^2$

j:  $y = \frac{x^2}{2} - \frac{x^4}{12} + c_1 x + c_2$ .

3.  $y' = \sin x - 1, y(0) = -1, y'(0) = 1$

j:  $y = -\sin x - \frac{x^2}{2} + 2x - 1$ .

4.  $y'' = \cos 2x$

j:  $y = -\frac{1}{4} \cos 2x + c_1 x + c_2$ .

5.  $y'' = \frac{1}{x}$

j:  $y = x \ln|x| - x + c_1 x + c_2$ .

6.  $y'' = \ln x$

j:  $y = \frac{x^2 \ln|x|}{2} - \frac{3x^2}{4} + c_1 x + c_2$ .

7.  $y'' + y'^2 = 2e^{-y}$

j:  $e^y + \frac{c_1}{4} = (x + c_2)^2$ .

8.  $y^{(iv)} = x$

j:  $y = \frac{x^5}{120} + c_1 x^3 + c_2 x^2 + c_3 x + c_4$ .

9.  $y'' = xe^x, y(0) = y'(0) = 0$

j:  $y = (x-3)e^x + \frac{x^2}{2} + 2x + 3$ .

10.  $y''' = x \ln x, y(1) = 1, y'(1) = 0, y''(1) = -1$

j:  $y = \frac{x^4}{24} \ln \frac{13}{288} x^4 - \frac{3}{8} x^2 + \frac{8}{9} x + \frac{17}{32}$

11.  $y''' - 2y'' - x = 0$

j:  $y = \frac{9}{28} t^7; \frac{9}{10} t^5 + (\frac{2}{3} + c_1) \cdot t^3 - 2c_1 t + c_2; x = t^3 - 2t$ .

12.  $y''' = \sin x + \cos x$

j:  $y = \cos x - \sin x + c_1 \frac{x^2}{2} + c_2 x + c_3$ .

13.  $y''' = \ln x^2, y(1) = 0, y'(1) = 1, y''(1) = 2$

j:  $\begin{cases} y = -\frac{x}{2} \ln^2 x + c_1 \frac{x^2}{2} + c_2 x + c_3 \\ y = -\frac{x}{2} \ln^2 x + \frac{3}{2} x^2 - 2x + \frac{1}{2} \end{cases}$

14.  $y'''' = \sqrt{1 + (y'')^2}$

j:  $y = \frac{e^{x+c_1} - e^{-(x+c_1)}}{2} + c_2 x + c_3$

15.  $x = y'^2 + 1$

j:  $x = z^2 + 1, y = \frac{4}{15} z^5 + c_1 z^2 + c_2$

16.  $y'' + 2y' + 2y = 0$

j:  $y = e^{-x}(3 \cos x - 2 \sin x)$

17.  $y'' - 4y' + 8y = 0$

j:  $y = e^{2x}(c_1 \cos 2x + c_2 \sin 2x)$

$$18. y''+6y'+9y=0$$

$$19. y'y''=3y'^2$$

$$20. y''=y^2$$

$$21. yy'+y^3-y^2=0$$

$$22. y''=-y$$

$$23. yy'+y^2=0$$

$$\text{j: } y = (c_1 + c_2 x)e^{-3x}$$

$$\text{j: } x = y^2 + y$$

$$\text{j: } x + c = e^{-y}$$

$$\text{j: } x = y + \ln y$$

$$\text{j: } y = c_1 \sin x + c_2 \cos x$$

$$\text{j: } y = \sqrt{(x+c_1)^2 + c_2}$$

## 2. O'zgarmas koeffitsientli ikkinchi tartibli chiziqli bir jinsli tenglamalar yechilsin.

$$1. y''-5y'-6y=0$$

$$2. y''-4y'+4y=0$$

$$3. y''+9y=0$$

$$4. y''+6y'+25=0$$

$$5. y''-9y=0$$

$$6. y''-8y'=0$$

$$7. y''+6y'+9y=0$$

$$8. y''+16y=0$$

$$9. y''+2y'+2y=0$$

$$10. y''+5y'+6y=0$$

$$11. 2y''-3y'-2y=0$$

$$12. y''-3y'+2y=0$$

$$13. y''-6y'+9y=0, y(0)=2, y'(0)=7$$

$$14. y''-10y'+25y=0$$

$$15. y''+49y=0$$

$$16. y''-4y'+13y=0$$

$$17. y''+3y=0, y(0)=1, y'(0)=3$$

$$18. y''+4y=0, y\left(\frac{\pi}{2}\right)=1, y'\left(\frac{\pi}{2}\right)=-1$$

$$19. y''-y'-2y=0, y(0)=0, y'(0)=1$$

$$20. y''-2y'+5y=0, y(0)=1, y'(0)=-1$$

$$21. y''+24y'+144y=0$$

$$22. y''-5y=0$$

$$\text{j: } y = c_1 e^{-x} + c_2 e^{6x}$$

$$\text{j: } y = e^{2x}(c_1 + c_2 x)$$

$$\text{j: } y = c_1 \cos 3x + c_2 \sin 3x$$

$$\text{j: } y = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x)$$

$$\text{j: } y = c_1 e^{-3x} + c_2 e^{-3x}$$

$$\text{j: } y = c_1 + c_2 e^{8x}$$

$$\text{j: } y = e^{-3x}(c_1 + c_2 x)$$

$$\text{j: } y = c_1 \cos 4x + c_2 \sin 4x$$

$$\text{j: } y = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$\text{j: } y = c_1 e^{-2x} + c_2 e^{-3x}$$

$$\text{j: } y = c_1 e^{-\frac{x}{2}} + c_2 e^{2x}$$

$$\text{j: } v = c_1 e^x + c_2 e^{2x}$$

$$\text{j: } y = e^{1x}(x+2)$$

$$\text{j: } y = e^{5x}(c_1 + c_2 x)$$

$$\text{j: } y = c_1 \cos 7x + c_2 \sin 7x$$

$$\text{j: } y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$$

$$\text{j: } y = 2 - e^{-3}$$

$$\text{j: } y = -\cos 2x + \frac{1}{2} \sin 2x$$

$$\text{j: } y = \frac{1}{3} e^{2x} - \frac{1}{3} e^{-x}$$

$$\text{j: } y = e^x (\cos 2x - \sin 2x)$$

$$\text{j: } [y = e^{-12}(c_1 + c_2 x)]$$

$$\text{j: } y = c_1 e^{-\sqrt{5}x} + c_2 e^{\sqrt{5}x}$$

$$23. y'' - 22y' + 121y = 0$$

$$24. y'' - y' - 6y = 0$$

$$\text{MC. } y = e^{11x} (c_1 + c_2 x)$$

$$\text{MC. } y = c_1 e^{-2x} + c_2 e^{3x}$$

## **Foydalanilgan adabiyotlar ro'yxati**

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