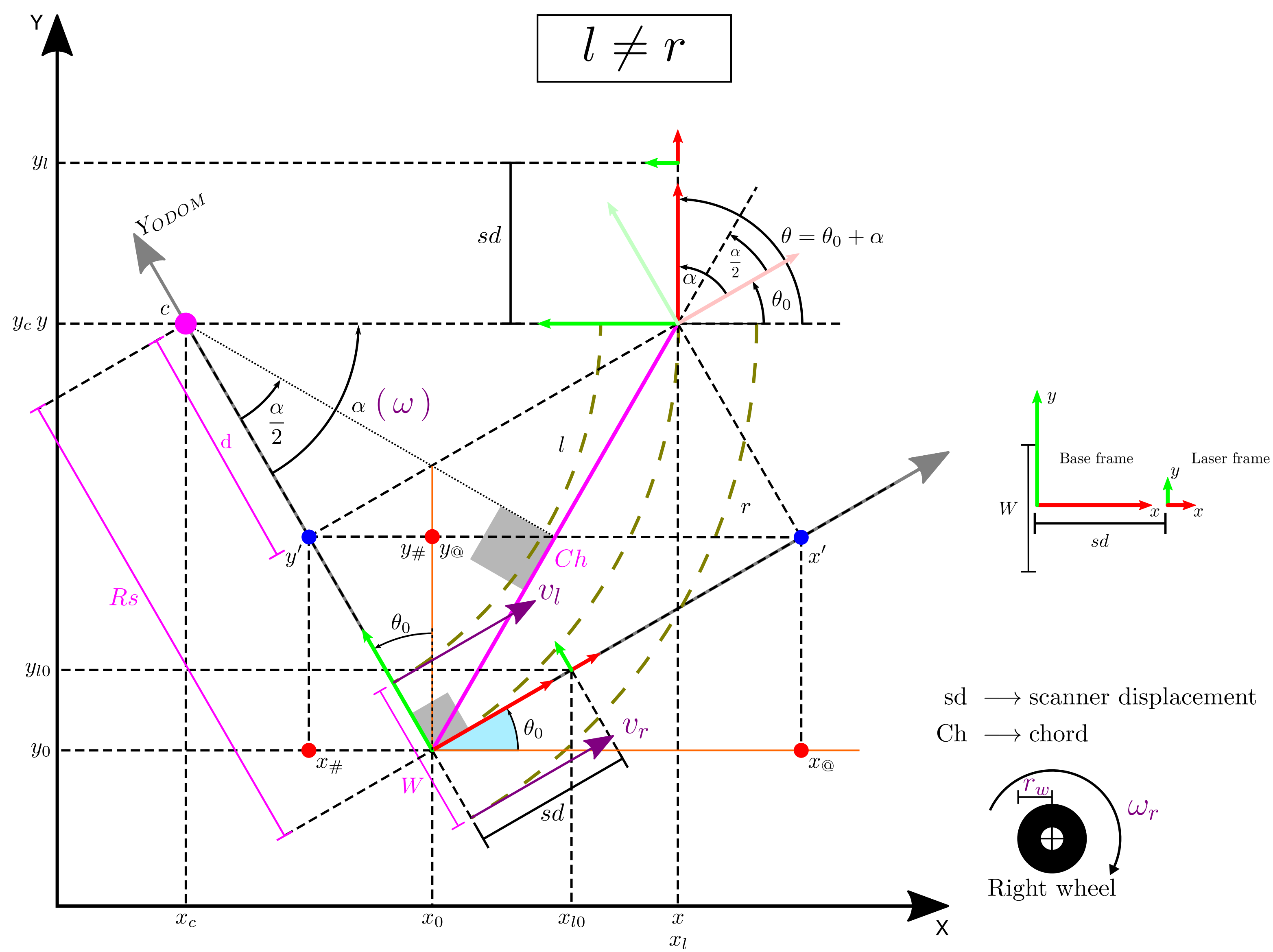
Robot's coordinates in the global reference frame



Given:

The robot's initial pose, $(x_0,\, y_0,\, \theta_0)$, and the wheels' angular velocities $(\omega_l,\, \omega_r)$, the following expressions can be derived:

$$
\begin{aligned}
\alpha &= \omega\, t \\
r &= \left(Rs + \frac{W}{2}\right)\, \alpha \\
l &= \left(Rs - \frac{W}{2}\right)\, \alpha \\
\dot{r} &= \left(Rs + \frac{W}{2}\right)\, \dot{\alpha} \longrightarrow v_r = \left(Rs + \frac{W}{2}\right)\, \omega \\
\dot{l} &= \left(Rs - \frac{W}{2}\right)\, \dot{\alpha} \longrightarrow v_l = \left(Rs - \frac{W}{2}\right)\, \omega
\end{aligned}
$$

$$
\begin{aligned}
\omega &= \frac{v_r - v_l}{W} \\
Rs &= \frac{v_r + v_l}{2\,\omega} = \frac{v}{\omega} = \frac{W}{2}\, \left(\frac{v_r + v_l}{v_r - v_l}\right)
\end{aligned}
$$

Summary:

$$
\begin{aligned}
\omega &= \frac{v_r - v_l}{W} \\
v &= \frac{v_r + v_l}{2} \\
Rs &= \frac{v}{\omega} = \frac{W}{2}\, \left(\frac{v_r + v_l}{v_r - v_l}\right)
\end{aligned}
$$

Distance travelled by each wheel:

$$
\begin{aligned}
D_r &= r_w\, \phi_r \\
D_l &= r_w\, \phi_l
\end{aligned}
$$

$$
\begin{aligned}
\dot{D}_r &= r_w\, \dot{\phi}_r \longrightarrow v_r = r_w\, \omega_r \\
\dot{D}_l &= r_w\, \dot{\phi}_l \longrightarrow v_l = r_w\, \omega_l
\end{aligned}
$$

So, finally:

$$
\begin{aligned}
\omega &= \frac{v_r - v_l}{W} = \frac{r_w}{W}\,(\omega_r - \omega_l) \\
v &= \frac{v_r + v_l}{2} = \frac{r_w}{2}\,(\omega_r + \omega_l) \\
Rs &= \frac{v}{\omega} = \frac{W}{2}\left(\frac{v_r + v_l}{v_r - v_l}\right) = \frac{W}{2}\left(\frac{\omega_r + \omega_l}{\omega_r - \omega_l}\right)
\end{aligned}
$$

$$
\begin{aligned}
x_l &= x + sd\cos(\theta) \\
y_l &= y + sd\sin(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\cos(u \pm v) &= \cos u\,\cos v \mp \sin u\,\sin v \\
\sin(u \pm v) &= \sin u\,\cos v \pm \cos u\,\sin v
\end{aligned}
$$

$$
\begin{aligned}
\cos(u - v) - \cos(u + v) &= 2\,\sin(u)\,\sin(v) \\
\sin(u + v) - \sin(u - v) &= 2\,\cos(u)\,\sin(v)
\end{aligned}
$$

$$
Ch = 2\,Rs\,sin\left(\frac{\alpha}{2}\right)
$$

$$
\begin{aligned}
x &= x_0 + x_@ + x_\# \\
&= x_0 + x'\cos(\theta_0) + y'\cos\left(\theta_0 + 90^\circ\right) \\
&= x_0 + Ch\,\cos\left(\frac{\alpha}{2}\right)\cos(\theta_0) + Ch\,\sin\left(\frac{\alpha}{2}\right)\cos\left(\theta_0 + 90^\circ\right) \\
&= x_0 + Ch\,\cos\left(\frac{\alpha}{2}\right)\cos(\theta_0) - Ch\,\sin\left(\frac{\alpha}{2}\right)\sin(\theta_0) \\
&= x_0 + Ch\,\cos\left(\theta_0 + \frac{\alpha}{2}\right) \\
&= x_0 + 2\,Rs\,sin\left(\frac{\alpha}{2}\right)\cos\left(\theta_0 + \frac{\alpha}{2}\right) \\
&= x_0 + Rs\,\left(\sin(\theta_0 + \alpha) - \sin(\theta_0)\right)
\end{aligned}
$$

$$
\begin{aligned}
v_x = \dot{x} &= Rs\,\omega\,\cos\left(\theta_0 + \alpha\right) \\
&= v\,\cos\left(\theta_0 + \alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
y &= y_0 + y_{@} + y_{\#} \\
&= y_0 + x'\,\sin\left(\theta_0\right) + y'\,\sin\left(\theta_0 + 90^\circ\right) \\
&= y_0 + Ch\,\cos\left(\frac{\alpha}{2}\right)\,\sin\left(\theta_0\right) + Ch\,\sin\left(\frac{\alpha}{2}\right)\,\sin\left(\theta_0 + 90^\circ\right) \\
&= y_0 + Ch\,\cos\left(\frac{\alpha}{2}\right)\,\sin\left(\theta_0\right) + Ch\,\sin\left(\frac{\alpha}{2}\right)\,\cos\left(\theta_0\right) \\
&= y_0 + Ch\,\sin\left(\theta_0 + \frac{\alpha}{2}\right) \\
&= y_0 + 2\,Rs\,sin\left(\frac{\alpha}{2}\right)\,\sin\left(\theta_0 + \frac{\alpha}{2}\right) \\
&= y_0 + Rs\,\left(cos\left(\theta_0\right) - \cos\left(\theta_0 + \alpha\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
v_y = \dot{y} &= Rs\,\omega\,\sin\left(\theta_0 + \alpha\right) \\
&= v\,\sin\left(\theta_0 + \alpha\right)
\end{aligned}
$$

Another way to calculate the robot's coordinates in the global reference frame:

$$
\begin{aligned}
x_c &= x_0 + Rs\,\cos\left(\theta_0 + 90^\circ\right) \\
&= x_0 - Rs\,sin\left(\theta_0\right) \\
y_c &= y_0 + Rs\,\sin\left(\theta_0 + 90^\circ\right) \\
&= y_0 + Rs\,\cos\left(\theta_0\right)
\end{aligned}
$$

$$
\begin{aligned}
x_c &= x + Rs\,\cos\left(\theta_0 + \alpha + 90^\circ\right) \\
&= x - Rs\,sin\left(\theta_0 + \alpha\right) \\
y_c &= y + Rs\sin\left(\theta_0 + \alpha + 90^\circ\right) \\
&= y + Rs\,\cos\left(\theta_0 + \alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
x &= x_c + Rs \sin(\theta_0 + \alpha) \\
&= x_0 - Rs \sin(\theta_0) + Rs \sin(\theta_0 + \alpha) \\
&= x_0 + Rs \left(\sin(\theta_0 + \alpha) - \sin(\theta_0)\right) \\
&= x_0 + 2\,Rs \cos\left(\theta_0 + \frac{\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
y &= y_c - Rs \cos(\theta_0 + \alpha) \\
&= y_0 + Rs \cos(\theta_0) - Rs \cos(\theta_0 + \alpha) \\
&= y_0 + Rs \left(\cos(\theta_0) - \cos(\theta_0 + \alpha)\right) \\
&= y_0 + 2\,Rs \sin\left(\theta_0 + \frac{\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right)
\end{aligned}
$$

For completeness, let's considerate the case when the robot is located at the pose $(x_0,\, y_0,\, \theta_0)$ and wants to get to the pose $(x,\, y,\, \theta)$, i.e, any pose known in advanced in the robot's environment. The easiest way to move between these two points is by describing and arc.
But, what is the turning radius, i.e Rs, that allows the robot to describe that arc?
As the first figure depicts, the initial and final poses are linked by the chord Ch.

$$\begin{aligned}
\Delta x &= x - x_0 \\
\Delta y &= y - x_0 \\
Ch^2 &= \Delta x^2 + \Delta y^2 = x'^2 + y'^2
\end{aligned}$$

The heading error, i.e, the difference between the chord's angle and the robot's orientation is:

$$\theta_{err} = \frac{\alpha}{2} = \arctan\left(\frac{\Delta\, y}{\Delta x}\right) - \theta_0$$

So, the turning radius (with sign) Rs can be computed, geometrically, as:

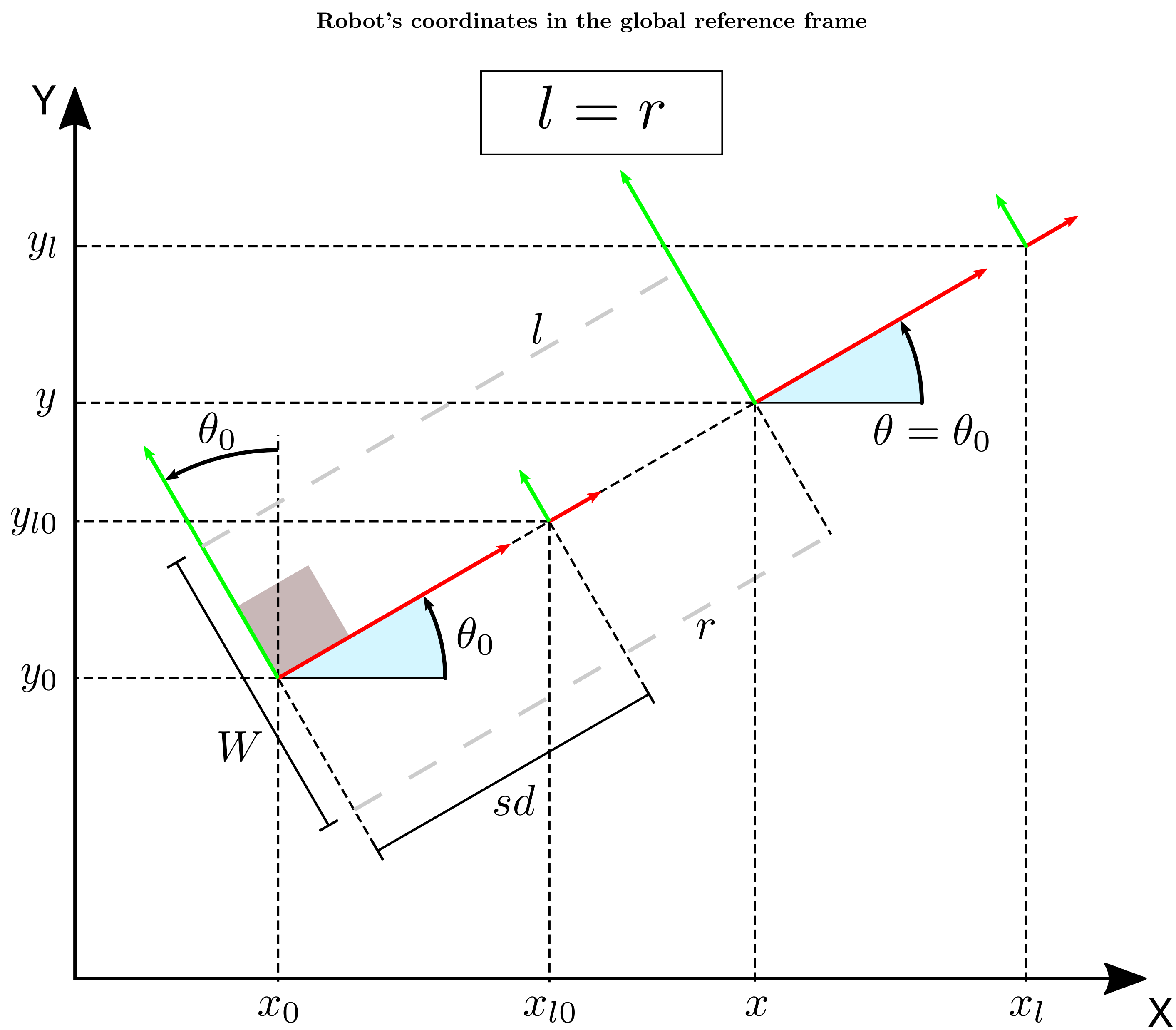
$$\begin{aligned}
Rs &= d + y' \\
Rs^2 &= d^2 + x'^2 = (Rs - y')^2 + x'^2
\end{aligned}$$

Also:

$$y' = Ch\, \sin\left(\theta_{err}\right)$$

Therefore:

$$\begin{aligned}
Rs^2 &= \Big(Rs - y'\Big)^2 + x'^2 \\
&= Rs^2 + y'^2 - 2\, Rs\, y' + x'^2 \\
&= Rs^2 + Ch^2 - 2\, Rs\, y' \\
Rs &= \frac{Ch^2}{2\, y'} = \frac{Ch^2}{2\, Ch\, \sin\left(\theta_{err}\right)} = \frac{Ch}{2\, \sin\left(\theta_{err}\right)}
\end{aligned}$$


Robot's coordinates in the global reference frame
$l = r$
Y
X
y_l
y
y_{l0}
y_0
x_0
x_{l0}
x
x_l
l
r
θ_0
θ_0
$\theta = \theta_0$
W
sd

Given the robot's initial pose, $(x_0,\, y_0,\, \theta_0)$, and the motion commands, $(l,\, r)$:

$$
\begin{aligned}
x &= x_0 + l\,\cos(\theta_0) \\
y &= y_0 + l\,\sin(\theta_0) \\
x_l &= x + sd\,\cos(\theta) \\
y_l &= y + sd\,\sin(\theta)
\end{aligned}
$$