Dynamic Programming: Infinite State

4.1.4 Firm Exit

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Introduction

Firm exit models firms' decision to exit a market.

This model features:

- firm's profit depends on aggregate shock, firm-specific shock, and a cross-sectional distribution of firms,
- state-dependent interest rate,
- outside option, also dependent on aggregate variable and cross-sectional distribution of firms.

Model Components

State Space: $\mathsf{X} = \mathsf{S} \times \mathsf{D} \times \mathsf{Z}$**:**

- firm-specific state $s \in \mathsf{S}$,
- cross-sectional distribution of firms taking values in $\mu \in \mathsf{D}$,
- aggregate shock taking values in $z \in \mathsf{Z}$.

Action Space: $\mathsf{A} = \{0, 1\}$**:**

- $a = 0$: continue,
- $a = 1$: exit and receive outside option at the start of next period.

Borel measurable policy $\sigma : \mathsf{X} \to \{0, 1\}$ Let Σ be the set of all policies.

Model Components

Let

- $\pi(s, \mu, z)$ be current profit for the firm,
- $q(\mu, z)$ be the outside option,
- $r(\mu, z)$ be the interest rate,
- $\beta(\mu, z) := 1/(1 + r(\mu, z))$ be the discount factor,
- $P(x, \cdot)$ be the transition kernel.

Let $\mathscr{B}$ be the Borel σ-algebra on X that makes π, q, r, β, P measurable.

Assumption 1

The Markov operator P has unique stationary distribution φ on $(\mathsf{X}, \mathscr{B})$. The functions π, q and β are nonnegative, measurable, and φ-integrable.

That is to say, $\pi, q, \beta \in L_1^+(\varphi)$.

We endow $L_1(\varphi)$ with the φ-a.e. pointwise order $\leqslant$, so that $f \leqslant g$ means $\varphi\{f > g\} = 0$.

σ-value Function:

$$v_\sigma(x) = \pi(x) + \beta(x) \int [\sigma(x') q(x') + (1 - \sigma(x')) v_\sigma(x')] \, P(x, \mathrm{d}x') \quad (x \in \mathsf{X}).$$

Policy Operator: for $v_\sigma \in L_1(\varphi)$,

$$T_\sigma \, v = \pi + K(\sigma \, q + (1 - \sigma) v) \tag{1}$$

where the operator

$$(Kv)(x) := \beta(x) \int v(x') P(x, \mathrm{d}x') \qquad (v \in L_1(\varphi), \; x \in \mathsf{X}).$$

in operator form we have $Kv = \beta P v$.

Assumption 2

K maps $L_1(\varphi)$ to itself and the spectral radius obeys $\rho(K) < 1$.

Can we shift the assumption to primitives?

- φ is the stationary for $P \implies P \in \mathcal{B}(L_1(\varphi))$ (Lemma A.5.29)
- To determine $\rho(K)$, we can simply bound $\sup_{x \in \mathsf{X}} \beta(x) < 1$, which means $\inf_{x \in \mathsf{X}} r(x) > 0$, or we can use conditions in Stachurski and Zhang (2021).

Under Assumptions 1 and 2, each T_σ is a self-map on $L_1(\varphi)$.

Since K is a positive operator, each T_σ is order preserving.

Hence we have ADP the pair $(L_1(\varphi), \mathbb{T})$ is an ADP

The ADP is regular. To see that, recall

$$T_\sigma v = \pi + K(\sigma q + (1-\sigma)v)$$

Let $\sigma = \mathbb{1}\, q \geqslant v$ (outside option is preferred over continuing interests).

For this σ, we have

$$\tau q + (1-\tau)v \leqslant \sigma q + (1-\sigma)v = q \vee v \quad \text{for all } \tau \in \Sigma.$$

Since K is a positive operator, we have $T_\tau \leqslant T_\sigma$ for all $\tau \in \Sigma$.

Hence σ is v-greedy, and the ADP is regular.

From that, we can also find the Bellman operator:

$$Tv = \pi + K(q \vee v). \tag{2}$$

Proposition 1

If Assumptions 1–2 hold, then the fundamental optimality properties hold, and VFI, HPI, and OPI all converge.

To prove it, we use the optimality results for affine ADPs:

Theorem 1

Let E be a Banach lattice and let $(E, \mathbb{T})$ be an affine ADP, where each $T_\sigma \in \mathbb{T}$ has the form

$$T_\sigma\, v = r_\sigma + K_\sigma\, v \quad \text{for some } r_\sigma \in E \text{ and } K_\sigma \in \mathcal{B}_+(E),$$

Suppose that $(E, \mathbb{T})$ is regular. If either

(a) there exists a $K \in \mathcal{B}(E)$ such that $K_\sigma \leqslant K$ for all $\sigma \in \Sigma$ and $\rho(K) < 1$, or

(b) E is σ-Dedekind complete, $(E, \mathbb{T})$ is bounded above and $\rho(K_\sigma) < 1$ for all $\sigma \in \Sigma$,

then

1. the fundamental optimality properties hold, and
2. VFI, OPI and HPI all converge.

$\mathcal{B}_+(E)$ is the positive linear selfmaps on E.

Proving this is simple, we only need to map the components in

$$T_\sigma v = \pi + K(\sigma q + (1 - \sigma) v)$$

to r_σ and K_σ.

Let $r_\sigma := \pi + K\sigma q$ and $K_\sigma := K(1 - \sigma)$. For condition (a), we have $K_\sigma \leqslant K$ since $\sigma \in \{0, 1\}$. Since r_σ lies in $L_1(\varphi)$, and since, by assumption, $\rho(K) < 1$, the conditions of Theorem 1 all hold.

Note: K here plays multiple roles: It is continuation value operator, and it is also the K in the Theorem 1.

We might consider using eventual contraction to prove Proposition 1.

We call $(V, \mathbb{T})$ **eventually Blackwell contracting** if V is a subset of E obeying $v + h \in V$ whenever $v \in V$ and $h \in E_+$, and, in addition, there exists a positive linear operator K on E such that

(C1) $\rho(K) < 1$ and

(C2) $T_\sigma(v + h) \leqslant T_\sigma v + Kh$ for all $T_\sigma \in \mathbb{T}$, $v \in V$ and $h \in E_+$.

Theorem 2

If V is a closed subset of a Banach lattice and $(V, \mathbb{T})$ is regular and eventually Blackwell contracting, then

1. the fundamental optimality properties hold, and
2. VFI, OPI and HPI all converge.

For fixed $v, h \in V$ we have

$$T_\sigma\,(v+h) = \pi + K[\sigma q + (1-\sigma)(v+h)] \leqslant T_\sigma\, v + Kh$$

Since K is a positive linear operator and $\rho(K) < 1$, the ADP $(V, \mathbb{T})$ is eventually Blackwell contracting and all the conclusions of Theorem 2 hold.

References

John Stachurski and Junnan Zhang. Dynamic programming with state-dependent discounting. *Journal of Economic Theory*, 192:105190, 2021. ISSN 0022-0531. doi: https://doi.org/10.1016/j.jet.2021.105190. URL https://www.sciencedirect.com/science/article/pii/S0022053121000077.