Factored Dynamic Programs: Definition and Optimality

April 6, 2025

Epstein–Zin Utility

Let's consider the Epstein–Zin model with finite state and action through the lens of isomorphic ADPs.

Bellman equation:

$$v(x) = \max_{a \in \Gamma(x)} \left\{ r(x, a)^{\alpha} + \beta \left[(Lv)(x, a) \right]^{\alpha} \right\}^{1/\alpha}, \tag{1}$$

with

$$(Lv)(x, a) := \left(\sum_{x'} v(x')^{\gamma} P(x, a, x') \right)^{1/\gamma}.$$

Interpretation:

- α determines elasticity of substitution between current and future payoffs;
- γ parameterizes risk-aversion when facing uncertainty over intertemporal outcomes.

Here, we assume that r is strictly positive, so that T_σ maps $(0, \infty)^\mathsf{X}$ into itself.

Now we want to establish optimality results for this model.

Optimality Results

Before that, let's consider the following setup:

Let $\theta = \gamma/\alpha$. Fix $\varepsilon > 0$ with $\min r^\alpha - \varepsilon > 0$. Consider the constant functions $v_1 = m_1 \wedge m_2$ and $v_2 = m_1 \vee m_2$, where

$$m_1 := \left(\frac{\min r^\alpha - \varepsilon}{1 - \beta} \right)^\theta \quad \text{and} \quad m_2 := \left(\frac{\max r^\alpha + \varepsilon}{1 - \beta} \right)^\theta.$$

So we can establish an order interval defined by $\hat{V} = [v_1, v_2]$.

Let F be defined by

$$F\, v = v^{\gamma} \qquad \text{with } v \in (0, \infty)^{\mathsf{X}},$$

where the exponent γ is applied pointwise to v, and set

$$V := F^{-1}\, \hat{V} = \{v \in (0, \infty)^{\mathsf{X}} : v_1 \leqslant v^{\gamma} \leqslant v_2\}. \tag{2}$$

As John mentioned last time, we can tackle the optimality results of $(V, \mathbb{T})$ by considering the problem $(\hat{V}, \hat{\mathbb{T}})$ we introduce next.

Solving Auxiliary Problems

The problem above can be written into the following policy operator T_σ:

$$T_\sigma \, v = \left\{ r_\sigma^\alpha + \beta \, (L_\sigma \, v)^\alpha \right\}^{1/\alpha} . \tag{3}$$

Let $\mathbb{T}$ be the set of all such T_σ.

Now we introduce an auxillary ADP $(\hat{V}, \hat{\mathbb{T}})$ with $\hat{V}$ as defined above and

$$\hat{T}_\sigma \, v = \left\{ r_\sigma^\alpha + \beta \, (P_\sigma \, v)^{1/\theta} \right\}^\theta . \tag{4}$$

We want to use the following optimality result in Theorem 1.3.10 replacing (a) and (b) with (a') and (b') in corollory 1.3.11:

Theorem 1

Let $(V, \mathbb{T})$ be an ADP where $V = [a, b]$ is containted in a σ-Dedekind complete Banach lattice and suppose that each $T_\sigma \in \mathbb{T}$ satisfies one of the following conditions:

(a') T_σ is concave and $a \ll T_\sigma\, a$ or

(b') T_σ is convex and $T_\sigma\, b \ll b$

If, in addition, $(V, \mathbb{T})$ is regular, then

(i) the fundamental optimality properties hold, and

(ii) VFI, OPI and HPI all converge.

Hence we show in two exercies:

- For v_1 and v_2 as defined above, we have $v_1 \ll \hat{T}_\sigma v_1$ and $\hat{T}_\sigma v_2 \ll v_2$, and
- (i) If $0 < \theta \leqslant 1$, then $\hat{T}_\sigma$ is convex on $\hat{V}$.
  (ii) If $\theta < 0$ or $1 \leqslant \theta$, then $\hat{T}_\sigma$ is concave on $\hat{V}$.

The first part is easy to show by writing out $\hat{T}_\sigma v_1$ and $\hat{T}_\sigma v_2$ and using the definition of v_1 and v_2.

The second part can be shown by analysing the second-order properties of $\hat{T}_\sigma$.

Lemma 1: Fundamental Optimality Results

The following statements are both true.

(i) The fundamental max-optimality results hold for $(\hat{V}, \hat{\mathbb{T}})$ and max-VFI, max-OPI, and max-HPI all converge.

(ii) The fundamental min-optimality results hold for $(\hat{V}, \hat{\mathbb{T}})$ and min-VFI, min-OPI, and min-HPI all converge.

In the proof below, we will apply Theorem 1.3.4, which requires

- a pospace V and a regular, globally stable and bounded above ADP $(V, \mathbb{T})$.

If V is σ-Dedekind complete, then we have foundamental optimality results and all the algorithms converge.

Proof.

Fix $\sigma \in \Sigma$. The fundamental max-optimality results and convergence of algorithms come directly from Theorem 1.

Min-optimality results of $(\hat{V}, \hat{\mathbb{T}})$ follows from max-optimality results of$(\hat{V}, \hat{\mathbb{T}})^\partial$:

- **Min-regularity:** $(\hat{V}, \hat{\mathbb{T}})$ is min-regular, so $(\hat{V}, \hat{\mathbb{T}})^\partial$ is max-regular.
- **Global Stability:** $(\hat{V}, \hat{\mathbb{T}})$ is globally stable so $(\hat{V}, \hat{\mathbb{T}})^\partial$ is also globally stable.
- Since V is an order interval in $\mathbb{R}^{\mathsf{X}}$ and X is finite, by Theorem 1.3.4 we have fundamental optimality results for $(V, \mathbb{T})$ and all the algorithms converge.

□

Passing Optimality Results to $(V, \mathbb{T})$

We have shown that $(\hat{V}, \hat{\mathbb{T}})$ has fundamental optimality results and all the algorithms converge.

We can pass these results to $(V, \mathbb{T})$ by using the following lemma:

Lemma 2: Isomorphism of $(V, \mathbb{T})$ and $(\hat{V}, \hat{\mathbb{T}})$

The following statements are true:

(i) If $\gamma > 0$, then $(V, \mathbb{T})$ and $(\hat{V}, \hat{\mathbb{T}})$ are isomorphic.

(ii) If $\gamma < 0$, then $(V, \mathbb{T})$ and $(\hat{V}, \hat{\mathbb{T}})$ are anti-isomorphic.

This lemma holds by definition of F, and the fact that for all $\sigma \in \Sigma$, we have $F \circ T_\sigma = \hat{T}_\sigma \circ F$ on V.

Now, using Theorems 3.1.6 and 3.1.7, which shows that the optimality and convergence of algorithms results pass through the isomorphism, we have the following:

Proposition 1: Max-Optimality Results

The fundamental max-optimality properties hold for $(V, \mathbb{T})$. In addition, max-VFI, max-OPI, and max-HPI all converge.

This also clears out why we are proving the min-optimality results for $(\hat{V}, \hat{\mathbb{T}})$ in the first place because we want to use the anti-isomorphism to pass the min-optimality result for $(\hat{V}, \hat{\mathbb{T}})$ to max-optimality results for $(V, \mathbb{T})$.

Factored Dynamic Programs

In this section, we consider a related idea called “semiconjugacy” and

- It is less restrictive than order conjugacy
- It gives rise to a “weaker” but still very useful optimality result
- It bridges the optimality results of two problems of different complexity.

Then we apply the idea to ADPs to obtain Factored Dynamic Programs (FDPs), and useful optimality results that followed.

Conjugacy and Semiconjugacy

Definition 1: Conjugacy

Two dynamical systems (V, S) and $(\hat{V}, \hat{S})$ are said to be **conjugate under F** if F is a bijection from V into $\hat{V}$ and $F \circ S = \hat{S} \circ F$ on V.

Definition 2: Semiconjugacy

Two dynamical systems (V, S) and $(\hat{V}, \hat{S})$ are said to be **mutually semiconjugate under F and G** when there exist order-preserving maps $F \colon V \to \hat{V}$ and $G \colon \hat{V} \to V$ such that

$$S = G \circ F \text{ on } V \qquad \text{and} \qquad \hat{S} = F \circ G \text{ on } \hat{V}. \tag{5}$$

"Semi" comes from the fact that

$$F \circ S = \hat{S} \circ F \quad \text{and} \quad G \circ \hat{S} = S \circ G. \tag{6}$$

Properties of Semiconjugacy

As the definition suggests, semiconjugacy is a weaker notion than conjugacy.

Some properties:

- We do not require F and G to be bijective (hence conjugacy typically does not hold)
- If either F or G is an order isomorphism, then S and $\hat{S}$ are conjugate.

Fixed Points for Semiconjugate Systems

Lemma 3: Fixed Points for Semiconjugate Systems

If (V, S) and $(\hat{V}, \hat{S})$ are mutually semiconjugate under F, G, then the following statements are true:

(i) If v is a fixed point of S in V, then Fv is a fixed point of $\hat{S}$ in $\hat{V}$.

(ii) If $\hat{v}$ is a fixed point of $\hat{S}$ in $\hat{V}$, then $G\hat{v}$ is a fixed point of S in V.

(iii) S has a unique fixed point in V if and only if $\hat{S}$ has a unique fixed point in $\hat{V}$.

(iv) S is order stable on V if and only if $\hat{S}$ is order stable on $\hat{V}$.

Proof of Lemma 3

(i) If v is a fixed point of S in V, then Fv is a fixed point of $\hat{S}$ in $\hat{V}$.

(ii) If $\hat{v}$ is a fixed point of $\hat{S}$ in $\hat{V}$, then $G\hat{v}$ is a fixed point of S in V.

Proof.

Let (V, S) and $(\hat{V}, \hat{S})$ be as stated.

- If v is a fixed point of S in V, then $\hat{S}Fv = FSv = Fv$, so Fv is a fixed point of $\hat{S}$ in $\hat{V}$.
- Similarly, if $\hat{v}$ is a fixed point of $\hat{S}$ in $\hat{V}$, then $SG\hat{v} = G\hat{S}\hat{v} = G\hat{v}$, so $G\hat{v}$ is a fixed point of S in V.

□

Proof of Lemma 3

(iii) S has a unique fixed point in V if and only if $\hat{S}$ has a unique fixed point in $\hat{V}$.

Proof.

$\implies$: Suppose that v is the only fixed point of S in V. We know Fv is a fixed point of $\hat{S}$ in $\hat{V}$. Let $\hat{v}$ be any fixed point of $\hat{S}$. Then:

$$\begin{aligned}
\hat{v} = \hat{S}\hat{v} &\iff FG\hat{v} = \hat{v} && \text{(by } \hat{S} = F \circ G\text{)} \\
&\iff GFG\hat{v} = G\hat{v} && \\
&\iff SG\hat{v} = G\hat{v} && \text{(by } S = G \circ F\text{)}
\end{aligned}$$

Since v is the unique fixed point of S, we must have $G\hat{v} = v$. Applying F gives $FG\hat{v} = Fv$, thus $\hat{v} = Fv$. Therefore, $\hat{S}$ has a unique fixed point in $\hat{V}$. $\impliedby$ follows by symmetry. $\square$

Proof of Lemma 3

(iv) S is order stable on V if and only if $\hat{S}$ is order stable on $\hat{V}$.

Proof.

Suppose S is order stable on V with unique fixed point v.

By 1 and 3, Fv is the unique fixed point of $\hat{S}$ in $\hat{V}$. For upward stability, if $\hat{v} \preceq \hat{S}\hat{v}$ in $\hat{V}$, then $G\hat{v} \preceq G\hat{S}\hat{v} = SG\hat{v}$.

By upward stability of S, $G\hat{v} \preceq v$, thus $\hat{v} \preceq Fv$ after applying F.

Downward stability follows similarly. Hence $\hat{S}$ is order stable on $\hat{V}$. The converse holds by symmetry. $\square$

Factored Dynamic Programs

Similar to our motivation for Isomorphic ADPs, we want to apply these nice properties of semiconjugacy to ADPs to obtain Factored Dynamic Programs (FDPs).

A **factored dynamic program** (FDP) is a tuple $(V, \hat{V}, \mathbb{G}, F)$ where

(i) V and $\hat{V}$ are nonempty posets,

(ii) F is an order-preserving map from V to $\hat{V}$, and

(iii) $\mathbb{G} := \{G_\sigma\}_{\sigma \in \Sigma}$ is a family of order-preserving maps from $\hat{V}$ to V, and

(iv) the set $\{G_\sigma \, \hat{v}\}_{\sigma \in \Sigma}$ has a greatest element for every $\hat{v} \in \hat{V}$.

We define $(V, \mathbb{T})$, where $T_\sigma \in \mathbb{T}$ take the form

$$T_\sigma = G_\sigma \circ F \text{ for all } \sigma \in \Sigma.$$

We call $(V, \mathbb{T})$ the **primary ADP** generated by $(V, \hat{V}, \mathbb{G}, F)$.

The factored dynamic program $(V, \hat{V}, \mathbb{G}, F)$ also produces an second ADP $(\hat{V}, \hat{\mathbb{T}})$, where $\hat{T}_\sigma \in \hat{\mathbb{T}}$ take the form

$$\hat{T}_\sigma = F \circ G_\sigma \qquad \text{for all } \sigma \in \Sigma,$$

We call $(\hat{V}, \hat{\mathbb{T}})$ the **subordinate ADP** generated by $(V, \hat{V}, \mathbb{G}, F)$.

Here, the dynamical systems $(V, \mathbb{T})$ and $(\hat{V}, \hat{\mathbb{T}})$ are **mutually semiconjugate** for all $\sigma \in \Sigma$.

$$\begin{tikzcd} V \arrow[r, bend left, "① \ F"] & \hat{V} \arrow[l, bend left, "② \ G_\sigma"] \end{tikzcd}$$

Primary: $G_\sigma \circ F$

$$\begin{tikzcd} V \arrow[r, bend right, "② \ F"'] & \hat{V} \arrow[l, bend right, "① \ G_\sigma"'] \end{tikzcd}$$

Subordinate: $F \circ G_\sigma$

In what follows, we define:

$$G_\top \, \hat{v} := \bigvee_\sigma G_\sigma \, \hat{v} \qquad (\hat{v} \in \hat{V}), \tag{7}$$

and $T_\top = G_\top \circ F$ on V and symmetrically $\hat{T}_\top = F \circ G_\top$ on $\hat{V}$.

Basic Properties of primary ADPs

Lemma 4: Basic Properties of primary ADPs

If $(V, \mathbb{T})$ is the primary ADP generated by $(V, \hat{V}, \mathbb{G}, F)$, then

(i) $(V, \mathbb{T})$ is regular,

(ii) the Bellman operator $T_\top$ obeys $T_\top = G_\top \circ F$ on V, and

(iii) σ is v-greedy for $(V, \mathbb{T})$ if and only if $G_\sigma F v = G_\top F v$.

Lemma 5: Basic Properties of subordinate ADPs

If $(\hat{V}, \hat{\mathbb{T}})$ is the subordinate ADP generated by $(V, \hat{V}, \mathbb{G}, F)$, then

(i) $(\hat{V}, \hat{\mathbb{T}})$ is regular,

(ii) the Bellman operator $\hat{T}_\top$ obeys $\hat{T}_\top = F \circ G_\top$ on $\hat{V}$, and

(iii) if $G_\sigma \hat{v} = G_\top \hat{v}$, then σ is $\hat{v}$-greedy for $(\hat{V}, \hat{\mathbb{T}})$.

Semiconjugacy of FDPs

Lemma 6: Semiconjugacy of FDPs

The policy operators of $(V, \mathbb{T})$ and $(\hat{V}, \hat{\mathbb{T}})$ obey

$$T_\sigma = G_\sigma \circ F \quad \text{and} \quad \hat{T}_\sigma = F \circ G_\sigma \tag{8}$$

for all $\sigma \in \Sigma$, while the Bellman operators are related by

$$T_\top = G_\top \circ F \qquad \text{and} \qquad \hat{T}_\top = F \circ G_\top . \tag{9}$$

As a result,

(i) each pair of policy systems (V, T_σ) and $(\hat{V}, \hat{T}_\sigma)$ is mutually semiconjugate under F, G_σ and

(ii) the Bellman operator systems $(V, T_\top)$ and $(\hat{V}, \hat{T}_\top)$ are mutually semiconjugate under $F, G_\top$.

Lemma 7

The following relationships hold:

(i) $(\hat{V}, \hat{\mathbb{T}})$ is well-posed if and only if $(V, \mathbb{T})$ is well-posed, and

(ii) $(\hat{V}, \hat{\mathbb{T}})$ is order stable if and only if $(V, \mathbb{T})$ is order stable.

In either case, the σ-value functions are linked by

$$\hat{v}_\sigma = F v_\sigma \quad \text{and} \quad v_\sigma = G_\sigma \, \hat{v}_\sigma \quad \text{for all} \quad \sigma \in \Sigma. \tag{10}$$

These results follow directly from the basic properties and the semiconjugacy of the primary and subordinate ADPs.

Optimality

Now let's transfer the optimality results of ADPs to FDPs.

Before we do that, we give a useful lemma.

Lemma 8

The following statements are equivalent:

(a) $v_\top$ exists and is the unique fixed point of $T_\top$ in V.

(b) $\hat{v}_\top$ exists and is the unique fixed point of $\hat{T}_\top$ in $\hat{V}$.

The symbols $T_\top$ and $\hat{T}_\top$ denote their respective Bellman operators. When they exist,

$$v_\top = \bigvee_\sigma v_\sigma \quad \text{and} \quad \hat{v}_\top = \bigvee_\sigma \hat{v}_\sigma.$$

Proof.

(a) $\implies$ (b):

Suppose (a) holds. By Lemma 6, $(V, T_\top)$ and $(\hat{V}, \hat{T}_\top)$ are mutually semiconjugate under $F, G_\top$. Thus $Fv_\top$ is the unique fixed point of $\hat{T}_\top$ in $\hat{V}$.

We claim $\hat{v}_\top = Fv_\top$. For any $\sigma \in \Sigma$, $v_\sigma \preceq v_\top \Rightarrow \hat{v}_\sigma = Fv_\sigma \preceq Fv_\top$ by (10), with equality when σ is optimal for $(V, \mathbb{T})$.

Hence $\hat{v}_\top = \bigvee_\sigma \hat{v}_\sigma = Fv_\top$, proving $\hat{v}_\top$ exists as the unique fixed point of $\hat{T}_\top$. □

Proof.

(b) $\implies$ (a):

Suppose (b) holds. By mutual semiconjugacy of $(V, T_\top)$ and $(\hat{V}, \hat{T}_\top)$ under $F, G_\top$, $G_\top \hat{v}_\top$ is the unique fixed point of $T_\top$ in V.

To show $v_\top$ exists and equals $G_\top \hat{v}_\top$, we first note that for any $\sigma \in \Sigma$, $\hat{v}_\sigma \preceq \hat{v}_\top$ and $v_\sigma = G_\sigma \hat{v}_\sigma$ by mutual semiconjugacy. Thus $v_\sigma \preceq G_\sigma \hat{v}_\top \preceq G_\top \hat{v}_\top$, making $G_\top \hat{v}_\top$ an upper bound of V_Σ.

To complete the proof, we need only find some $\sigma \in \Sigma$ with $v_\sigma = G_\top \hat{v}_\top$.

Proof (continued).

First, we can find σ such that $G_\sigma \, \hat{v}_\top = G_\top \hat{v}_\top$ by definition of FDP. Bellman's principle of optimality holds (by (b) and Lemma 4.2 of Sargent and Stachurski (2025) or Lemma 1.2.5)

Applying F to this equality gives $\hat{T}_\sigma \, \hat{v}_\top = \hat{T}_\top \hat{v}_\top$. Hence σ is optimal for $(\hat{V}, \hat{\mathbb{T}})$, so $\hat{v}_\sigma = \hat{v}_\top$.

Combining this equality with $G_\sigma \, \hat{v}_\top = G_\top \hat{v}_\top$ yields $v_\sigma = G_\top \hat{v}_\top$. $\square$

Now we are equipped to prove the main optimality results of FDPs.

Theorem 2: Optimality of FDPs

If either and hence both of these statements are true, then

(i) the value functions obey

$$v_\top = G_\top \hat{v}_\top \quad \text{and} \quad \hat{v}_\top = F v_\top, \tag{11}$$

(ii) the fundamental optimality properties hold for $(V, \mathbb{T})$,

(iii) the fundamental optimality properties hold for $(\hat{V}, \hat{\mathbb{T}})$,

(iv) if $G_\sigma \hat{v}_\top = G_\top \hat{v}_\top$, then σ is optimal for $(V, \mathbb{T})$,

(v) if σ is optimal for $(V, \mathbb{T})$, then σ is optimal for $(\hat{V}, \hat{\mathbb{T}})$, and

(vi) if F is strictly order-preserving, then σ is optimal for $(V, \mathbb{T})$ if and only if σ is optimal for $(\hat{V}, \hat{\mathbb{T}})$.

where a function $F \colon V \to W$ is **strictly order-preserving** if $v \prec w$ implies $Fv \prec Fw$.

Suppose (a) and (b) in Lemma 8 hold.

(i) the value functions obey

$$v_\top = G_\top \, \hat{v}_\top \quad \text{and} \quad \hat{v}_\top = F \, v_\top, \tag{12}$$

(ii) the fundamental optimality properties hold for $(V, \mathbb{T})$,

(iii) the fundamental optimality properties hold for $(\hat{V}, \hat{\mathbb{T}})$.

(iv) if $G_\sigma \, \hat{v}_\top = G_\top \, \hat{v}_\top$, then σ is optimal for $(V, \mathbb{T})$,

Proof.

(i) - (iii), follows from the previous arguments and Lemma 1.2.5 (or Lemma 4.2 of Sargent and Stachurski (2025)).

Regarding (iv), let $\sigma \in \Sigma$ be such that $G_\sigma \, \hat{v}_\top = G_\top \, \hat{v}_\top$. Applying (12) yields $G_\sigma \, F v_\top = G_\top F v_\top$, or $T_\sigma \, v_\top = T_\top \, v_\top$. By Bellman's principle of optimality, σ is optimal for $(V, \mathbb{T})$. □

(v) if σ is optimal for $(V, \mathbb{T})$, then σ is optimal for $(\hat{V}, \hat{\mathbb{T}})$.

Proof.

Regarding (v), let σ be optimal for $(V, \mathbb{T})$. Since $(V, \mathbb{T})$ obeys the fundamental optimality properties, σ is $v_\top$-greedy (i.e., $T_\sigma v_\top = T_\top v_\top$). Also, by (12), we have $\hat{v}_\top = F v_\top$. Therefore,

$$\hat{T}_\sigma \hat{v}_\top = \hat{T}_\sigma F v_\top = F G_\sigma F v_\top = F\, T_\sigma v_\top = F\, T_\top v_\top = F v_\top = \hat{v}_\top = \hat{T}_\top \hat{v}_\top.$$

Thus, σ is $\hat{v}_\top$-greedy for $(\hat{V}, \hat{\mathbb{T}})$. But Bellman's principle of optimality also holds for $(\hat{V}, \hat{\mathbb{T}})$, so σ is optimal for $(\hat{V}, \hat{\mathbb{T}})$. □

(vi) if F is strictly order-preserving, then σ is optimal for $(V, \mathbb{T})$ if and only if σ is optimal for $(\hat{V}, \hat{\mathbb{T}})$.

Proof.

Regarding part (vi), we only need to show that if F is strictly order-preserving, then the converse implication of (v) holds.

To see that, let σ be optimal for $(\hat{V}, \hat{\mathbb{T}})$ so that σ is $\hat{v}_\top$-greedy, we have $\hat{T}_\sigma \hat{v}_\top = \hat{T}_\top \hat{v}_\top$, which implies $F G_\sigma \hat{v}_\top = F G_\top \hat{v}_\top$. By definition of $G_\top$, we have $G_\sigma \hat{v}_\top \preceq G_\top \hat{v}_\top$.

Suppose that $G_\sigma \hat{v}_\top \prec G_\top \hat{v}_\top$. Clearly, by strictly order-preserving of F, a contradiction to $F G_\sigma \hat{v}_\top = F G_\top \hat{v}_\top$. Hence, we have $G_\sigma \hat{v}_\top = G_\top \hat{v}_\top$ so that σ is optimal for $(V, \mathbb{T})$. □

References

Thomas J. Sargent and John Stachurski. Dynamic programming on partially ordered sets. *SIAM Journal on Control and Optimization*, in press, 2025.