Dynamic Programming

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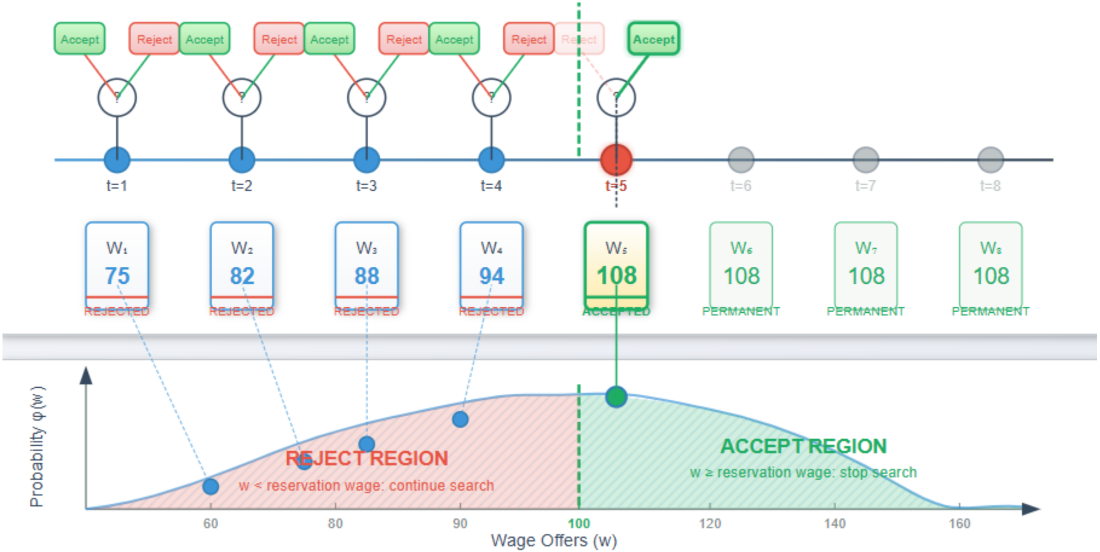
Outline

Why Job Search Problem?

- A good starting point for application
- optimal stopping problem - binary choice
- Good for illustrating transformations.

Job Search Model



Job Search Problem Set up

We let

- W_t denote the wage offer drawn from some fixed distribution φ
- $(W_t)_{t\geq 0}$ is IID and take values from $W \subset \mathbb{R}_+$, W is nonempty.
- φ has finite mean, so $\int w\varphi(dw) < \infty$
- Constant discount factor $\beta \in (0, 1)$
- Σ be the set of Borel measurable policy $\sigma : W \to \{0, 1\}$
- $L_1(\varphi) := L^1(W, \mathcal{B}, \varphi)$ be all Borel measurable function $f : W \to \mathbb{R}$ with $\int |f|\, d\varphi < \infty$

Detour to measure theory

$L_1(\varphi) := L_1(W, \mathcal{B}, \varphi)$ be all Borel measurable function $f : W \to \mathbb{R}$ with $\int |f|\, d\varphi < \infty$.

- L^1-norm is
$$\|f\|_1 = \int_W |f|\, d\varphi$$
- f is an equivalence class of functions that equals to f almost everywhere
- The partial order $f \leq g$ means that $\varphi(\{f > g\}) = 0$

$L^1(\varphi) := L^1(W, B, \varphi)$ be all Borel measurable function $f : W \to \mathbb{R}$ with $\int |f| \, d\varphi < \infty$.

- L^1-norm is
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- The partial order $f \leq g$ means that $\varphi(\{f > g\}) = 0$

$$\Longrightarrow$$

$L^1(\varphi)$ is a **Banach Lattice**

Job Search problem

We introduce the policy operator $v \mapsto T_\sigma v$ via

$$(T_\sigma v)(w) = \sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \underbrace{\int v(w')\,\varphi(dw')}_{\text{Dimension Reduction}}\right]$$

Deep look into the Policy Operator

$$(T_\sigma v)(w) = \sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \underbrace{\int v(w')\,\varphi(dw')}_{\text{Dimension Reduction}}\right]$$

or equivalently, let $e(w) := \dfrac{w}{1-\beta}$

$$\begin{aligned} T_\sigma v &= \underbrace{[\sigma e + (1-\sigma)c]}_{=:r_\sigma} + \underbrace{(1-\sigma)\beta \mathbb{E} v}_{=:K_\sigma v} \\ &= r_\sigma + K_\sigma v \end{aligned}$$

Deeper look into the policy operator

$$T_\sigma v = r_\sigma + K_\sigma v$$

- T_σ is **affine**
- $0 \leq K_\sigma v = (1 - \sigma)\beta \mathbb{E} v \leq \beta \mathbb{E} v =: K v$
- we have $K_\sigma \leq K$ and $\rho(K) = \beta < 1$

ADP formulation

$$(T_\sigma v)(w) = \sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \underbrace{\int v(w')\,\varphi(dw')}_{\text{Dimension Reduction}}\right]$$

- $v \in L^1(\varphi) \implies T_\sigma v \in L^1(\varphi)$
- T_σ is order preserving self-map on $L^1(\varphi)$
- Σ is not empty

$$(L^1(\varphi), \mathbb{T}_{JS}), \quad \mathbb{T}_{JS} := \{T_\sigma : \sigma \in \Sigma\}$$

is an ADP for the Job Search Problem.

Exercise 4.1.1

Prove that $(L_1(\varphi), \mathbb{T}_{JS})$ is well-posed.

Proof.

$$(T_\sigma v)(w) = v(w)$$

$$\Longrightarrow$$

$$\sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \int v(w')\,\varphi(dw')\right] = v(w)$$

- $\sigma(w) = 1 \implies v(w) = \frac{w}{1-\beta}$ well-defined and unique
- $\sigma(w) = 0 \implies v(w) = c + \beta \int v(w')\,\varphi(dw')$ well-defined and unique

□

The ADP $(L^1(\varphi), \mathbb{T}_{JS})$ is **regular**

v-greedy policy (assume accept the offer at indifference)

$$\sigma(w) = \mathbf{1}\left\{ \frac{w}{1-\beta} \geq c + \beta \int v(w')\, \varphi(dw') \right\}$$

with policy operator

$$(T_\sigma v)(w) = \max\left\{ \frac{w}{1-\beta}, c + \beta \int v(w')\, \varphi(dw') \right\} \underbrace{= (Tv)(w)}_{\text{greedy}}$$

Exercise 4.1.2

Starting from the usual ADP definition $Tv = \bigvee_\sigma T_\sigma v$, show that

$$(Tv)(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')\,\varphi(dw')\right\}$$

Proof.

$$\begin{aligned}
(Tv)(w) &= \left(\bigvee_\sigma T_\sigma v\right)(w) \\
&= \bigvee_\sigma \left(\sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \int v(w')\,\varphi(dw')\right]\right) \\
&= \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')\,\varphi(dw')\right\}
\end{aligned}$$

$\square$

Exercise 4.1.3

$$\sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \int v(w')\,\varphi(dw')\right]$$

- From $L^1(\varphi)$ to bmW bounded Borel measurable function
- From bmW to bcW bounded continuous function
- From bcW to $ibcW$ increasing bounded continuous function
- From $ibcW$ to $ibcW_+$ nonnegative increasing bounded continuous function

Proposition 4.1.1. - Optimality with IID offers

Theorem 1

For $(L^1(\varphi), \mathbb{T}_{JS})$,

- *the fundamental optimality properties hold*
- *VFI, OPI, HPI all converge.*

Proof.

Implore Theorem 1.3.9. $\square$

Theorem 1.3.9. *Let E be a* ***Banach lattice*** *and let $(E, \mathbb{T})$ be an* ***affine*** *ADP, where each $T_\sigma \in \mathbb{T}$ has the form*

$$T_\sigma v = r_\sigma + K_\sigma v \text{ for some } r_\sigma \in E \text{ and } K_\sigma \in \mathcal{B}_+(E),$$

Suppose that $(E, \mathbb{T})$ is ***regular****. If either*

(a) *there exists a $K \in \mathcal{B}(E)$ such that $K_\sigma \leqslant K$ for all $\sigma \in \Sigma$ and $\rho(K) < 1$, or*

(b) *E is σ-Dedekind complete, $(E, \mathbb{T})$ is bounded above and $\rho(K_\sigma) < 1$ for all $\sigma \in \Sigma$,*

then

(i) *the fundamental optimality properties hold, and*

(ii) *VFI, OPI and HPI all converge.*

Section 4.1.2 Rearranging the Bellman Equation

- Rearranging the Bellman Equation

Rearranging the Bellman Equation - Continuation Value

Given the Bellman equation

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')\,\varphi(dw')\right\} \quad \text{for all } w \in W$$

Let

$$h := c + \beta \int v(w')\,\varphi(dw')$$

be the continuation value. And $h \in \mathbb{R}$.

Continuation value

We get a one-dimensional nonlinear equation.

$$
\begin{aligned}
h &= c + \beta \int v(w')\, \varphi(dw') \\
&= c + \beta \int \max\left\{ \frac{w'}{1-\beta}, c + \beta \int v(w'')\, \varphi(dw'') \right\} \varphi(dw') \\
&= c + \beta \int \max\left\{ \frac{w'}{1-\beta}, h \right\} \varphi(dw')
\end{aligned}
$$

Exercise 4.1.5

We introduce

$$g(h) = c + \beta \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(dw')$$

and we want to find the fixed point of g. Using the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leq |x - y|$$

show that g is a contraction.

Exercise 4.1.5 cont.

Proof.

Let $h, k \in \mathbb{R}_+$. we have

$$
\begin{aligned}
|g(h) - g(k)| &= \beta \left| \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(dw') - \int \max\left\{\frac{w'}{1-\beta}, k\right\} \varphi(dw') \right| \\
&\leq \beta \int \left| \max\left\{\frac{w'}{1-\beta}, h\right\} - \max\left\{\frac{w'}{1-\beta}, g\right\} \right| \varphi(dw') && \text{(Jensen)} \\
&\leq \beta \int |h - k| \, \varphi(dw') && \text{(Given)} \\
&= \beta |h - k|
\end{aligned}
$$

□

Hence we can get a unique fixed point. Optimal policy is hence

$$\sigma_\top(w) = \mathbf{1}\{w \geq w_\top\}, \quad \text{where } w_\top := (1-\beta)h_\top$$

Reduce the search space further

Let $f(h) := c + \beta \bar{w}/(1-\beta) + \beta h$. We can find the fixed point h^* as

$$h^* = c + \frac{\beta}{1-\beta} \int w\, \varphi(dw) + \beta h^*$$

We have

$$\begin{aligned} g(h^*) &= c + \beta \int \max\left\{ \frac{w'}{1-\beta}, h^* \right\} \varphi(dw') \\ &= c + \beta \left(\int_{\{w'/(1-\beta) \geq h^*\}} \frac{w'}{1-\beta}\, \varphi(dw') + \int_{\{w'/(1-\beta) \leq h^*\}} h^*\, \varphi(dw') \right) \\ &= c + \frac{\beta}{1-\beta} \int_{\{w/(1-\beta) \geq h^*\}} w\, \varphi(dw) + \beta h^* \varphi(\{w/(1-\beta) \leq h^*\}) \end{aligned}$$

Reduce the search space continued

$$h^* = c + \frac{\beta}{1-\beta} \int w \, \varphi(dw) + \beta h^*$$

$$g(h^*) = c + \frac{\beta}{1-\beta} \int_{\underbrace{\{w/(1-\beta) \geq h^*\}}_{\varphi(\cdot) \leq 1}} w \, \varphi(dw) + \beta h^* \underbrace{\varphi(\{w/(1-\beta) \leq h^*\})}_{\leq 1} \leq h^*$$

Hence, g maps h^* down.
g is a contraction $\implies$ globally stable $\implies$ strongly order stable $\implies$ g is a self-map on $[0, h^*]$.

Parametric Monotonicity

- parameters play a key role in dynamics
- Is the solution robust to parameter change?
- How does the solution vary with parameters?
- Useful to robustness check
- useful to policy design

Parametric Monotonicity

Given the Bellman equation

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')\,\varphi(dw')\right\} \quad \text{for all } w \in W$$

How does changes in

- unemployment compensation c
- discount factor β
- distribution φ

change the reservation wage $w_\top = (1-\beta)(c + \beta \int v_\top(w')\,\varphi(dw'))$.

Proposition A.5.18

Let V be a pospace and S, T be two self-map on V ordered pointwise, i.e.,

$$S \precsim T \iff Sv \precsim Tv \quad \text{for all } v \in V$$

Proposition A.5.18 If $S \precsim T$, T is order preserving and globally stable on V, then its unique fixed points dominates any fixed point of S.

Proof.

$$v_S = Sv_S \precsim Tv_S \implies v_S \precsim v_T$$

by (strongly) order stability from global stability + order preserving. □

Exercise 4.1.8

Prove that the reservation wage $w_\top$ is increasing in unemployment compensation c. Prove also that $h_\top$ is increasing with β.

Proof.

We have for $c_1 \leq c_2$

$$g_1(h) = c_1 + \beta \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(dw') \leq c_2 + \beta \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(dw') = g_2(h)$$

- By PropA518, $h_\top^1 \leq h_\top^2$.
- By $w_\top = (1-\beta)h_\top$, $w_\top^1 \leq w_\top^2$

Higher unemployment compensation, higher reservation wage. □

Exercise 4.1.0

Let τ be the first passage time to employment, i.e.,

$$\tau := \inf\{t \geq 0 : \sigma_\top(W_t) = 1\} = \inf\{t \geq 0 : W_t \geq w_\top\}$$

Prove that the mean first passage time $\mathbb{E}\tau$ increases with c.
Remark: Here τ is a random variable with sample space $\Omega = W^{\mathbb{N}}$. We have

$$\begin{aligned}
\mathbb{E}\tau &= 0 \cdot \mathbb{P}(W_0 \geq w_\top) + 1 \cdot \mathbb{P}(W_1 \geq w_\top | W_0 < w_\top) + 2 \cdot \mathbb{P}(W_2 \geq w_\top | W_0, W_1 < w_\top) + \cdots \\
&= 0 \cdot p + 1 \cdot p(1-p) + 2 \cdot p(1-p)^2 + \cdots \\
&= \sum_{i=1}^{\infty} ip(1-p)^i \qquad \text{(mean of Geometric distribution)} \\
&= \frac{1-p}{p}
\end{aligned}$$

Higher unemployment compensation, wait longer on average

We have $c_1 \leq c_2 \implies w^1_\top \leq w^2_\top$, this implies

$$p_1 = \mathbb{P}(W_i \geq w^1_\top) \geq \mathbb{P}(W_i \geq w^2_\top) = p_2$$

$$p_1 \geq p_2 \implies \mathbb{E}\tau | c = c_1 = \frac{1 - p_1}{p_1} \leq \frac{1 - p_2}{p_2} = \mathbb{E}\tau | c = c_2$$

Increase in discount factor

From

$$h_\top = c + \beta \int \max\left\{\frac{w'}{1-\beta}, h_\top\right\} \varphi(dw')$$

We use $w_\top = (1-\beta)h_\top$ get

$$w_\top = c(1-\beta) + \beta \int \max\{w', w_\top\} \, \varphi(dw')$$

We define the following function (order-preserving and contraction):

$$f(w) = c(1-\beta) + \beta \int \max\{w', w\} \, \varphi(dw')$$

And $w_\top$ is the unique fixed point.

Increase in discount factor

We can take partial derivative of f with respect to β

$$\frac{\partial f(w)}{\partial \beta} = -c + \int \max\{w', w\} \, \varphi(dw')$$

Hence, when

$$c \leq \int w' \, \varphi(dw') \leq \int \max\{w', w\} \, d(w') \quad \text{for all } w \in W$$

We have for $\beta_1 \leq \beta_2$

$$f(w; \beta_1) \leq f(w; \beta_2) \quad \text{for all } w \in W$$

By PropA518, we have $w_{\top}^{1} \leq w_{\top}^{2}$.

Changes in distribution (Lemma 4.1.2.)

Lemma 2

If φ, ψ *are supported on* $[0, M]$, $M \in \mathbb{R}_+$, ψ *first order stochastically dominates* φ, *then* $w_\top^\varphi \leq w_\top^\psi$.

Definition 3

Let ibX be the increasing bounded functions on X. We say that ν first order stochastically dominates μ and write $\mu \precsim_F \nu$ if

$$\int u(x)\,\mu(dx) \leq \int u(x)\,\nu(dx) \quad \text{for every } u \in ibX$$

Remark: $f : \mathbb{R} \to \mathbb{R}$ and increasing implies measurable

Proof.

To prove that f is measurable, we need to show that for any $a \in \mathbb{R}$, the set $f^{-1}((-\infty, a))$ is measurable. Since f is increasing, for any $a \in \mathbb{R}$, we can define $c = \sup x \in \mathbb{R} : f(x) < a$. By the properties of increasing functions, the set $f^{-1}((-\infty, a))$ takes one of two forms:

- If $f(c) < a$ (or if $c = \infty$), then $f^{-1}((-\infty, a)) = (-\infty, c]$
- If $f(c) \geq a$ (or if $c = -\infty$), then $f^{-1}((-\infty, a)) = (-\infty, c)$

In either case, the preimage is a Borel set and therefore measurable. Since the preimage of every interval of the form $(-\infty, a)$ is measurable, and these intervals generate the Borel σ-algebra on $\mathbb{R}$, the function f is measurable. □

Change in distribution cont.

Proof.

We have for $\varphi \precsim_F \psi$, fix $w \in \mathbb{R}^+$,$w' \mapsto \max\{w', w\} \in ib[0, M]$.

$$
\begin{aligned}
f_\varphi(w) &= c(1-\beta) + \beta \int \max\{w', w\}\, \varphi(dw') \\
&\leq c(1-\beta) + \beta \int \max\{w', w\}\, \psi(dw') \\
&= f_\psi(w)
\end{aligned}
$$

- f_ψ is order-preserving and globally stable
- $f_\varphi \precsim f_\psi$

By PropA518, $w^\varphi_\top \leq w^\psi_\top$ $\square$

Mean-preserving spread

We also concern with how behavior changes when decisions become riskier. We introduce the notion of **mean-preserving spread**.

Definition 4

For a given distribution φ, we say that ψ is a mean-preserving spread of φ if there exists a pair of random variable (Y, Z) such that

$$\mathbb{E}[Z|Y] = 0, \quad Y =^d \varphi, \quad Y + Z =^d \psi$$

Lemma 4.1.3

If ψ is a mean-preserving spread of φ, then $w_\top^\varphi \leq w_\top^\psi$.

Proof.

We know that

$$W_t =^d \varphi, \quad \text{by iid, we have } W. =^d \varphi$$

Let ψ be a mean-preserving spread of φ. Then there exists a pair of random variable $(W., Z)$ such that

$$\mathbb{E}[Z|W] = 0, \quad W. =^d \varphi, \quad W. + Z =^d \psi$$

□

Lemma 4.1.3 Cont.

Proof.

This implies

$$\begin{aligned}
\int \max\{w', w\}\, \psi(dw') &= \mathbb{E}[\max\{W_\cdot + Z, w\}] = \mathbb{E}\Big[\mathbb{E}\left[\max\{W_\cdot + Z, w\} | W_\cdot\right]\Big] && \text{(LIE)} \\
&\geq \mathbb{E}\Big[\max\{\mathbb{E}[W_\cdot + Z | W_\cdot], w\}\Big] && \text{(Cond. Jensen)} \\
&= \mathbb{E}\Big[\max\{W_\cdot, w\}\Big] && \text{(Linearity)} \\
&= \int \max\{w', w\}\, \varphi(dw')
\end{aligned}$$

$\square$

Job Search with Correlated wage draws - Set up

We let

- $(W_t)_{t \geq 0}$ is P-Markov and take values from $W \subset \mathbb{R}_+$, W is nonempty.
- φ is the stationary distribution of P, i.e., $\varphi = \varphi P$
- φ has finite mean, so $\int w \, \varphi(dw) < \infty$
- Constant discount factor $\beta \in (0, 1)$
- Σ be the set of Borel measurable policy $\sigma : W \to \{0, 1\}$
- $L_1(\varphi) := L^1(W, \mathcal{B}, \varphi)$ be all Borel measurable function $f : W \to \mathbb{R}$ with $\int |f| \, d\varphi < \infty$

Comparison

For IID draws, with $e(w) := \frac{w}{1-\beta}$

$$T_\sigma v = \underbrace{[\sigma e + (1-\sigma)c]}_{=:r_\sigma} + \underbrace{(1-\sigma)\beta \mathbb{E} v}_{=:K_\sigma v}$$
$$= r_\sigma + K_\sigma v$$

For Correlated wage draws

$$T_\sigma v = \underbrace{[\sigma e + (1-\sigma)c]}_{=:r_\sigma} + \underbrace{(1-\sigma)\beta P}_{:=\beta P_\sigma} v$$
$$= r_\sigma + \beta P_\sigma v$$

Exercise 4.1.11

Prove that T_σ is an order-preserving self-map on $L_1(\varphi)$.

Proof.

First, we prove that T_σ is a self-map on $L_1(\varphi)$, i.e.,

$$v \in L_1(\varphi) \implies r_\sigma + \beta P_\sigma v \in L_1(\varphi)$$

In particular, we need to show $Pv \in L_1(\varphi)$, i.e., Pv is Borel measurable and

$$\int_W \left| \int_{W'} v(w')\, P(w, dw') \right| \varphi(dw) < \infty$$

$\square$

Exercise 4.1.11 cont.

Proof.

Pv is Borel measurable by the property of stochastic kernel. And we have

$$\begin{aligned}\int_W \left| \int_{W'} v(w')\, P(w, dw') \right| \varphi(dw) &\leq \int_W \int_{W'} |v(w')|\, P(w, dw')\, \varphi(dw) && \text{(Jensen)} \\ &= \int_{W'} \int_W |v(w')| \varphi(dw) P(w, dw') && \text{(Tonelli)} \\ &= \int_{W'} |v(w')| \int_W \varphi(dw) P(w, dw') && \text{(stationary)} \\ &= \int_{W'} |v(w')|\, \varphi(dw') < \infty && (\in L_1(\varphi))\end{aligned}$$

At last, order preserving is from the property of Markov operator. □

Value space

Let $L_1(\varphi)$ paired with almost-everywhere order be the value space,

- Banach lattice
- Dedekind complete

Policy operators

$$
\begin{aligned}
T_\sigma v &= \underbrace{[\sigma e + (1-\sigma)c]}_{=: r_\sigma} + \underbrace{(1-\sigma)\beta P}_{:= \beta P_\sigma} v \\
&= r_\sigma + \beta P_\sigma v
\end{aligned}
$$

- order-preserving self-map on $L_1(\varphi)$
- affine
- $\beta P_\sigma \leq \beta P$ and $\rho(\beta P) = \beta < 1$

ADP representation

The ADP $(L_1(\varphi), \mathbb{T}_{JSM})$ is

- regular
- well-posed

with similar proofs from the IID case.

Optimality

Theorem 1.3.9. *Let E be a **Banach lattice** and let $(E, \mathbb{T})$ be an **affine** ADP, where each $T_\sigma \in \mathbb{T}$ has the form*

$$T_\sigma v = r_\sigma + K_\sigma v \text{ for some } r_\sigma \in E \text{ and } K_\sigma \in \mathcal{B}_+(E),$$

*Suppose that $(E, \mathbb{T})$ is **regular**. If either*

(a) *there exists a $K \in \mathcal{B}(E)$ such that $K_\sigma \leqslant K$ for all $\sigma \in \Sigma$ and $\rho(K) < 1$, or*

(b) *E is **σ-Dedekind complete**, $(E, \mathbb{T})$ is **bounded above** and $\rho(K_\sigma) < 1$ for all $\sigma \in \Sigma$,*

then

(i) *the fundamental optimality properties hold, and*

(ii) *VFI, OPI and HPI all converge.*

Exercise 4.1.15

Let $\bar{v} := (I - \beta P)^{-1}(e + c)$ and $V := \{v \in L_1(\varphi) : 0 \leq v \leq \bar{v}\}$. Show that, for all $\sigma \in \Sigma$, we have $v_\sigma \leq \bar{v}$ and $T_\sigma V \subset V$

Proof.

We have $T_\sigma \bar{v} = r_\sigma + \beta P_\sigma \bar{v}$. By definition, $\bar{v} = (e + c) + \beta P \bar{v}$. We know $e + c \geq r_\sigma$, $\beta P \bar{v} \geq \beta P_\sigma \bar{v}$ for all $\sigma \in \Sigma$. Hence, We have

$$T_\sigma \bar{v} \leq \bar{v}$$

By global stability and order preserving, T_σ is (strongly) order stable. Hence, we have

$$v_\sigma \leq \bar{v}$$

Moreover $T_\sigma 0 = r_\sigma \geq 0$, hence $T_\sigma V \subset V$. □

Remark: The ADP is bounded above

We say the ADP is bounded above if there exists a $u \in V$ with $T_\sigma u \precsim u$ for all $T_\sigma \in \mathbb{T}$. From the previous proof, we show that $T_\sigma \bar{v} \leq \bar{v}$ for all $\sigma \in \Sigma$. This proves that the ADP is bounded above.Hence, we can also use the second line in Theorem 1.3.9

Exercise 4.1.13

Show that every policy operator T_σ is order continuous on $L_1(\varphi)$.

Proof.

See Zaanen (2012). Every positive operator on L^p is order continuous. $\square$

Reducing the value space

We have $V := \{v \in L_1(\varphi) : 0 \leq v \leq \bar{v}\}$

- V is an order interval and Dedekind complete, hence chain complete

We also have the policy operator T_σ

- order preserving self-map on V
- globally stable hence strongly order stable
- order continuous

For the ADP $(V, \mathbb{T}_{JSM})$

- regular
- well-posed

Optimality on the reduced value space (ex4.1.15)

Theorem 1.2.13. *Let V be* ***chain complete*** *and let $(V, \mathbb{T})$ be* ***regular*** *and* ***well-posed****. In this setting, the fundamental optimality properties hold. If, in addition, $(V, \mathbb{T})$ is* ***strongly order stable****, then VFI, OPI and HPI all converge.*

Theorem 1.2.14. *Let V be* ***σ-chain complete*** *and let $(V, \mathbb{T})$ be* ***regular*** *and* ***well-posed****. If $(V, \mathbb{T})$ is* ***order continuous****, then*

(i) *the fundamental optimality properties hold and*

(ii) *VFI, OPI and HPI all converge.*

Optimality on the reduced value space

Theorem 1.2.16. *Let V be* ***σ-Dedekind complete*** *and let* $(V, \mathbb{T})$ *be* ***regular*** *and* ***well-posed****. If* $(V, \mathbb{T})$ *is* ***bounded above****,* ***order stable****, and* ***order continuous****, then*

(i) *the fundamental optimality properties hold and*

(ii) *VFI, OPI and HPI all converge.*

Exercise 4.1.17

Show that if W is finite, the HPI converges in finitely many steps.

Proof.

W is finite $\implies \Sigma$ is finite $\implies \mathbb{T}_{JSM}$ is finite. Use Theorem 1.2.12 □

Theorem 1.2.12. *Let $(V, \mathbb{T})$ be **regular** and **well-posed**. If $(V, \mathbb{T})$ is **order stable** and $\mathbb{T}$ is **finite**, then*

(i) *the fundamental optimality properties hold and*

(ii) *HPI converges in finitely many steps.*