Dynamic Programming

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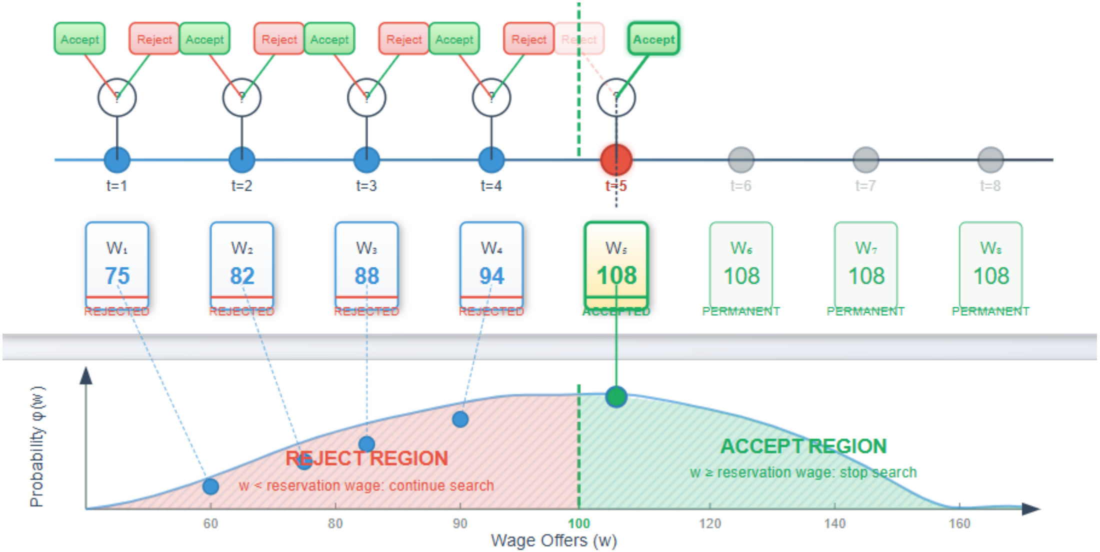
Outline

1. Why Job search Problem?
2. What is Job Search Problem?
3. Optimality properties

Why Job Search Problem?

- A good starting point for application
- optimal stopping problem - binary choice
- Good for illustrating transformations.

Job Search Model



Job Search Problem Set up

We let

- W_t denote the wage offer drawn from some fixed distribution φ
- $(W_t)_{t\geq 0}$ is IID and take values from $W \subset \mathbb{R}_+$, W is nonempty.
- φ has finite mean, so $\int w\varphi(dw) < \infty$
- Constant discount factor $\beta \in (0, 1)$
- Σ be the set of Borel measurable policy $\sigma : W \to \{0, 1\}$
- $L_1(\varphi) := L^1(W, B, \varphi)$ be all Borel measurable function $f : W \to \mathbb{R}$ with $\int |f|\, d\varphi < \infty$

Detour to measure theory

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- L^1-norm is
$$\|f\|_1 = \int_W |f|\, d\varphi$$
- f is an equivalence class of functions that equals to f almost everywhere
- The partial order $f \leq g$ means that $\varphi(\{f > g\}) = 0$

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$$\Longrightarrow$$

$L^1(\varphi)$ is a **Banach Lattice**

Job Search problem

We introduce the policy operator $v \mapsto T_\sigma v$ via

$$(T_\sigma v)(w) = \sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \underbrace{\int v(w')\,\varphi(dw')}_{\text{Dimension Reduction}}\right]$$

Deep look into the Policy Operator

$$(T_\sigma v)(w) = \sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \underbrace{\int v(w')\,\varphi(dw')}_{\text{Dimension Reduction}}\right]$$

$$\text{or equivalently, let } e(w) := \frac{w}{1-\beta}$$

$$\begin{aligned} T_\sigma v &= \underbrace{[\sigma e + (1-\sigma)c]}_{=:r_\sigma} + \underbrace{(1-\sigma)\beta\mathbb{E}v}_{=:K_\sigma v} \\ &= r_\sigma + K_\sigma v \end{aligned}$$

Deeper look into the policy operator

$$T_\sigma v = r_\sigma + K_\sigma v$$

- T_σ is **affine**
- $0 \leq K_\sigma v = (1 - \sigma)\beta \mathbb{E} v \leq \beta \mathbb{E} v =: K v$
- we have $K_\sigma \leq K$ and $\rho(K) = \beta < 1$

ADP formulation

$$(T_\sigma v)(w) = \sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \underbrace{\int v(w')\,\varphi(dw')}_{\text{Dimension Reduction}}\right]$$

- $v \in L^1(\varphi) \implies T_\sigma v \in L^1(\varphi)$
- T_σ is order preserving self-map on $L^1(\varphi)$
- Σ is not empty

$$(L^1(\varphi), \mathbb{T}_{JS}), \quad \mathbb{T}_{JS} := \{T_\sigma : \sigma \in \Sigma\}$$

is an ADP for the Job Search Problem.

The ADP $(L^1(\varphi), \mathbb{T}_{JS})$ is well-posed.

$$(T_\sigma v)(w) = v(w)$$

$$\Longrightarrow$$

$$\sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \int v(w')\,\varphi(dw')\right] = v(w)$$

- $\sigma(w) = 1 \implies v(w) = \frac{w}{1-\beta}$ well-defined and unique
- $\sigma(w) = 0 \implies v(w) = c + \beta \int v(w')\,\varphi(dw')$ well-defined and unique

The ADP $(L^1(\varphi), \mathbb{T}_{JS})$ is **regular**

v-greedy policy (assume accept the offer at indifference)

$$\sigma(w) = \mathbf{1}\left\{\frac{w}{1-\beta} \geq c + \beta \int v(w')\,\varphi(dw')\right\}$$

with policy operator

$$(T_\sigma v)(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')\,\varphi(dw')\right\} \underbrace{= (Tv)(w)}_{\text{greedy}}$$

Construct Bellman operator from definition

$Tv = \bigvee_\sigma T_\sigma v$

$$
\begin{aligned}
(Tv)(w) &= \left(\bigvee_\sigma T_\sigma v \right)(w) \\
&= \bigvee_\sigma \left(\sigma(w) \frac{w}{1-\beta} + (1 - \sigma(w)) \left[c + \beta \int v(w')\, \varphi(dw') \right] \right) \\
&= \max \left\{ \frac{w}{1-\beta}, c + \beta \int v(w')\, \varphi(dw') \right\}
\end{aligned}
$$

Bounded $W \implies$ use smaller value space

$$\sigma(w)\frac{w}{1-\beta} + (1-\sigma(w))\left[c + \beta \int v(w')\,\varphi(dw')\right]$$

- From $L^1(\varphi)$ to bmW bounded Borel measurable function
- From bmW to bcW bounded continuous function
- From bcW to $ibcW$ increasing bounded continuous function
- From $ibcW$ to $ibcW_+$ nonnegative increasing bounded continuous function

Optimality with IID offers

Theorem 1

For $(L^1(\varphi), \mathbb{T}_{JS})$,

- *the fundamental optimality properties hold*
- *VFI, OPI, HPI all converge.*

Proof.

Implore Theorem 1.3.9. $\square$

Theorem 1.3.9

Theorem 1.3.9. *Let E be a Banach lattice and let $(E, \mathbb{T})$ be an affine ADP, where each $T_\sigma \in \mathbb{T}$ has the form*

$$T_\sigma v = r_\sigma + K_\sigma v \text{ for some } r_\sigma \in E \text{ and } K_\sigma \in \mathcal{B}_+(E),$$

Suppose that $(E, \mathbb{T})$ is regular. If either

(a) *there exists a $K \in \mathcal{B}(E)$ such that $K_\sigma \leqslant K$ for all $\sigma \in \Sigma$ and $\rho(K) < 1$, or*

(b) *E is σ-Dedekind complete, $(E, \mathbb{T})$ is bounded above and $\rho(K_\sigma) < 1$ for all $\sigma \in \Sigma$,*

then

(i) *the fundamental optimality properties hold, and*

(ii) *VFI, OPI and HPI all converge.*