

Positive operators on L^p is order continuous

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Abstract

In this note, we follow the approach in [Zaa12] to show that positive operators on L^p spaces for $1 \leq p < \infty$ is order continuous (we exclude the $p = \infty$ case, see subsection 1.4 for detail). The logic flow is demonstrated in the following Figure 1. The structure of this note is as follows:

Contents

1	Prove L^p norm is order continuous	2
2	Prove that L^p has Riesz Fischer property and is Banach lattice	5
3	Prove positive operator on L^p are norm bounded	10
4	Prove positive operator on L^p is order continuous	12

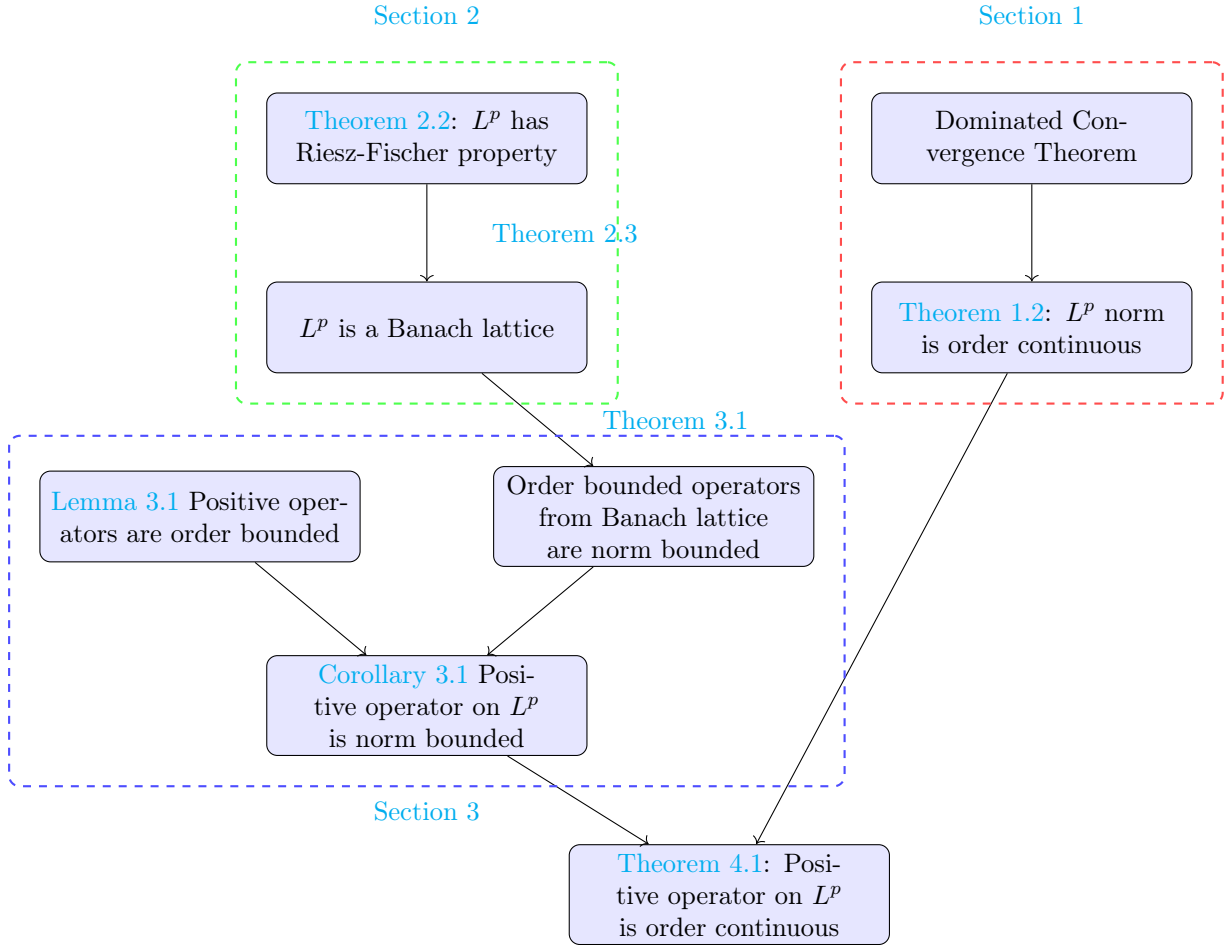


Figure 1: Logical structure of the proof

1 Prove L^p norm is order continuous

In this section, we prove that L^p norm is order continuous, we first present related definitions and theorems that we take for granted.

1.1 Related definitions

Definition 1.1 (L^p norm). Let (X, \mathcal{M}, μ) be a measure space, and let $1 \leq p < \infty$. For a measurable function $f : X \rightarrow \mathbb{C}$ (or \mathbb{R}), the L^p norm of f is defined as

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}$$

provided that $\int_X |f(x)|^p d\mu(x) < \infty$. The space $L^p(X, \mu)$ consists of all measurable functions $f : X \rightarrow \mathbb{C}$ (or \mathbb{R}) for which $\|f\|_p < \infty$, with functions that differ only on a set of measure zero identified.

Definition 1.2 (Order continuous norm). Let E be a normed Riesz space. A norm $\|\cdot\|$ on E is called *order continuous* if for every sequence $(f_n)_{n=1}^\infty$ in E such that $f_n \downarrow 0$ (i.e., f_n is decreasing and $\inf_n f_n = 0$), we have

$$\lim_{n \rightarrow \infty} \|f_n\| = 0$$

1.2 Related theorems

Theorem 1.1 (Dominated Convergence Theorem in L^p). Let (X, \mathcal{M}, μ) be a measure space, and let $1 \leq p < \infty$. Suppose that:

1. $(f_n)_{n=1}^\infty$ is a sequence of measurable functions from X to \mathbb{C} (or \mathbb{R});
2. $f_n \rightarrow f$ μ -almost everywhere on X as $n \rightarrow \infty$;
3. There exists a function $g \in L^p(X, \mu)$ such that $|f_n(x)| \leq g(x)$ for μ -almost all $x \in X$ and all $n \in \mathbb{N}$.

Then:

- (a) $f \in L^p(X, \mu)$;
- (b) $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$, i.e., $f_n \rightarrow f$ in $L^p(X, \mu)$;
- (c) $\lim_{n \rightarrow \infty} \int_X |f_n|^p d\mu = \int_X |f|^p d\mu$.

1.3 Proof of L^p norm is order continuous

Theorem 1.2 (L^p norm is order continuous). Let (X, \mathcal{M}, μ) be a measure space and $1 \leq p < \infty$ ^a. Then the L^p -norm is order continuous. That is, if $(f_n)_{n=1}^\infty$ is a sequence in $L^p(X, \mu)$ such that $f_n \downarrow 0$ μ -almost everywhere (i.e., f_n is decreasing and $\inf_n f_n = 0$ μ -almost everywhere), then $\|f_n\|_p \downarrow 0$ as $n \rightarrow \infty$.

^a L^∞ norm is not order continuous, see [subsection 1.4](#)

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence in $L^p(X, \mu)$ such that $f_n \downarrow 0$ μ -almost everywhere. We need to show that $\|f_n\|_p \rightarrow 0$ as $n \rightarrow \infty$.

First, we observe that since $f_n \downarrow 0$ μ -almost everywhere, we have $f_n \geq 0$ μ -almost everywhere for all $n \in \mathbb{N}$. Therefore, $|f_n| = f_n$ μ -almost everywhere.

We'll now demonstrate that the sequence $\{\|f_n\|_p\}$ is monotone decreasing. For any $n \in \mathbb{N}$, since

$f_n \geq f_{n+1} \geq 0$ μ -almost everywhere, we have $f_n^p \geq f_{n+1}^p$ μ -almost everywhere (because $t \mapsto t^p$ is an increasing function on $[0, \infty)$ for $p > 0$). Therefore

$$\begin{aligned}\|f_n\|_p^p &= \int_X |f_n(x)|^p d\mu(x) \\ &= \int_X f_n(x)^p d\mu(x) \\ &\geq \int_X f_{n+1}(x)^p d\mu(x) \\ &= \int_X |f_{n+1}(x)|^p d\mu(x) \\ &= \|f_{n+1}\|_p^p\end{aligned}$$

Since $p > 0$, we can take the p -th root of both sides to obtain $\|f_n\|_p \geq \|f_{n+1}\|_p$. Thus, the sequence $\{\|f_n\|_p\}$ is monotone decreasing.

Now, let's define $g = f_1$. Since $f_1 \in L^p(X, \mu)$, we have $g \in L^p(X, \mu)$.

Furthermore, since $f_n \downarrow$, we have $f_n \leq f_1 = g$ for all $n \in \mathbb{N}$. Thus, $|f_n| \leq g$ μ -almost everywhere for all $n \in \mathbb{N}$.

We also have $f_n \rightarrow 0$ μ -almost everywhere as $n \rightarrow \infty$ (since $f_n \downarrow 0$).

Therefore, we can apply the [Theorem 1.1](#) (Dominated convergence theorem) for L^p spaces:

Since

1. $(f_n)_{n=1}^\infty$ is a sequence of measurable functions;
2. $f_n \rightarrow 0$ μ -almost everywhere as $n \rightarrow \infty$;
3. $|f_n| \leq g$ μ -almost everywhere for all $n \in \mathbb{N}$, where $g \in L^p(X, \mu)$;

[Theorem 1.1](#) implies that

$$\lim_{n \rightarrow \infty} \|f_n\|_p = \|0\|_p = 0$$

Hence, the L^p -norm is order continuous for $1 \leq p < \infty$. ■

1.4 L^∞ norm is not order continuous

First, we show the definition of L^∞ norm.

Definition 1.3 (L^∞ norm). For a measurable function f on a measure space (X, \mathcal{M}, μ) , the L^∞ norm is defined as

$$\|f\|_\infty = \operatorname{ess\,sup}_{x \in X} |f(x)| = \inf\{M \geq 0 : |f(x)| \leq M \text{ for } \mu\text{-almost every } x \in X\}$$

where $\operatorname{ess\,sup}$ denotes the essential supremum.

Now, we give a counterexample to show that L^∞ norm is not order continuous.

Example (L^∞ norm is not order continuous). Consider the sequence of functions $\{f_n\}_{n=1}^\infty$ on $[0, 1]$ defined by

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{n}] \\ 0, & \text{otherwise} \end{cases}$$

This sequence is decreasing pointwise to the zero function $f(x) = 0$, i.e., $f_n \downarrow 0$ as $n \rightarrow \infty$. However,

$$\|f_n\|_\infty = 1$$

for all $n \in \mathbb{N}$, while $\|f\|_\infty = 0$. Thus

$$\lim_{n \rightarrow \infty} \|f_n\|_\infty = 1 \neq 0 = \|f\|_\infty$$

This demonstrates that the L^∞ norm does not preserve order limits, and hence is not order continuous.

2 Prove that L^p has Riesz Fischer property and is Banach lattice

In this section, we first present some related definitions and theorem we take for granted. Then we prove in section 2.2 L^p has the Riesz Fischer property. In section 2.3 we prove that a normed Riesz space is Banach lattice if and only if it has the Riesz Fischer property. This concludes that L^p is a Banach lattice.

2.1 Related definitions

Definition 2.1 (Normed Riesz Space). A *normed Riesz space* (or *normed vector lattice*) is a triple $(X, \leq, \|\cdot\|)$ where:

1. (X, \leq) is a Riesz space (i.e., a vector space equipped with a partial ordering \leq that makes it a lattice, and such that the vector space operations are compatible with the ordering);
2. $\|\cdot\|$ is a norm on X ;
3. The norm is *lattice norm* (or *Riesz norm*), meaning it satisfies:

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$$

for all $x, y \in X$, where $|x| = x \vee (-x)$ denotes the absolute value of x .

Definition 2.2 (Banach Lattice). A Banach lattice is a complete normed Riesz space, i.e., a normed Riesz space that is complete with respect to its norm.

Definition 2.3 (Riesz-Fischer Property). A normed Riesz space $(E, \|\cdot\|)$ is said to have the *Riesz-Fischer property* if for every sequence $(f_n)_{n=1}^\infty$ in the positive cone E^+ satisfying

$$\sum_{n=1}^{\infty} \|f_n\| < \infty,$$

the series $\sum_{n=1}^{\infty} f_n$ converges in E .

2.2 Related theorems

Theorem 2.1 (Monotone Convergence Theorem in L^p). Let (X, \mathcal{M}, μ) be a measure space, let $1 \leq p < \infty$, and let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions from X to $[0, \infty]$. If $\{f_n\}$ is monotonically increasing (i.e., $0 \leq f_1 \leq f_2 \leq \dots \leq f_n \leq \dots$ a.e.) and $f_n \rightarrow f$ pointwise a.e. as $n \rightarrow \infty$, then:

$$\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$$

or equivalently,

$$\lim_{n \rightarrow \infty} \int_X |f_n|^p d\mu = \int_X |f|^p d\mu$$

2.3 Prove that L^p has Riesz-Fischer property

Theorem 2.2 (Riesz-Fischer Property for L^p Spaces). Let (X, \mathcal{M}, μ) be a measure space and $1 \leq p < \infty$. The space $L^p(X, \mathcal{M}, \mu)$ has the Riesz-Fischer property. That is, for any sequence $(f_n)_{n=1}^\infty$ of non-negative functions in L^p such that $\sum_{n=1}^\infty \|f_n\|_p < \infty$, the series $\sum_{n=1}^\infty f_n$ converges in L^p to a function $f \in L^p$.

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence of non-negative functions in L^p such that $\sum_{n=1}^\infty \|f_n\|_p = M < \infty$.

Define the partial sums $s_n = \sum_{k=1}^n f_k$ for $n = 1, 2, \dots$. We first show that the sequence $(s_n)_{n=1}^\infty$ is bounded in L^p norm.

For each n , by the triangle inequality for the L^p norm, we have:

$$\|s_n\|_p = \left\| \sum_{k=1}^n f_k \right\|_p \leq \sum_{k=1}^n \|f_k\|_p \leq \sum_{k=1}^\infty \|f_k\|_p = M < \infty$$

Thus, the sequence $(s_n)_{n=1}^\infty$ is bounded in L^p norm.

Furthermore, since each f_n is non-negative, the sequence $(s_n)_{n=1}^\infty$ is monotonically increasing pointwise, i.e., $s_1 \leq s_2 \leq \dots \leq s_n \leq \dots$ for all $x \in X$.

Let $s(x) = \lim_{n \rightarrow \infty} s_n(x)$ for each $x \in X$. This pointwise limit exists (possibly $s(x) = +\infty$ at some points) because the sequence is monotonically increasing.

By the [Theorem 2.1](#) (Monotone Convergence Theorem) for L^p spaces, since $\{s_n\}$ is a monotonically increasing sequence of non-negative functions with $s_n \rightarrow s$ pointwise, we have:

$$\|s\|_p = \lim_{n \rightarrow \infty} \|s_n\|_p \leq M < \infty \quad (1)$$

Hence, $s \in L^p$ and $s = \sum_{n=1}^\infty f_n$ in the L^p sense. This proves that L^p has the Riesz-Fischer property.

Alternatively, we can view this as follows: Let $\alpha = \sum_{n=1}^\infty \|f_n\|_p$ and $s_n = \sum_{k=1}^n f_k$ for $n = 1, 2, \dots$. Then $\|s_n\|_p \leq \sum_{k=1}^n \|f_k\|_p \leq \alpha$ for all n , so the pointwise limit $s = \sum_{n=1}^\infty f_n$ satisfies $\|s\|_p = \lim_{n \rightarrow \infty} \|s_n\|_p \leq \alpha$. This shows that $s = \sum_{n=1}^\infty f_n$ is a member of L^p . ■

2.4 Related lemmas

First we need to use the following three lemmas for on normed Riesz space: infinite triangle inequality, property of convergence in normed Riesz space, and relationship between norm and order convergence.

Lemma 2.1 (Infinite Triangle Inequality). Let $(E, \|\cdot\|)$ be a normed Riesz space with the Riesz-Fischer property. Then the following inequality holds:

$$\left\| \sum_{n=1}^\infty u_n \right\| \leq \sum_{n=1}^\infty \|u_n\|$$

Proof. We proceed by contradiction. Suppose there exists a sequence $(u_n)_{n=1}^\infty \subset E^+$ with $\sum_{n=1}^\infty \|u_n\| < \infty$ such that the inequality fails. Then there exists $\varepsilon > 0$ satisfying:

$$\left\| \sum_{n=1}^\infty u_n \right\| > \sum_{n=1}^\infty \|u_n\| + \varepsilon$$

For each $m \in \mathbb{N}$, define the tail sum $t_m := \sum_{n=m+1}^\infty u_n$. By the triangle inequality and our

assumption:

$$\begin{aligned}
\|t_m\| &= \left\| \sum_{n=1}^{\infty} u_n - \sum_{n=1}^m u_n \right\| \\
&\geq \left\| \sum_{n=1}^{\infty} u_n \right\| - \left\| \sum_{n=1}^m u_n \right\| \\
&> \sum_{n=1}^{\infty} \|u_n\| + \varepsilon - \sum_{n=1}^m \|u_n\| \\
&= \sum_{n=m+1}^{\infty} \|u_n\| + \varepsilon
\end{aligned}$$

Thus, for all $m \in \mathbb{N}$, we have $\|t_m\| > \sum_{n=m+1}^{\infty} \|u_n\| + \varepsilon$.

Since $\sum_{n=1}^{\infty} \|u_n\| < \infty$, we can choose a strictly increasing sequence of indices $(n_k)_{k=1}^{\infty}$ such that:

$$\sum_{n=n_k+1}^{\infty} \|u_n\| < \frac{1}{k^2} \quad \text{for all } k \in \mathbb{N}$$

Define $s_k := \sum_{n=n_k+1}^{\infty} u_n$ for each $k \in \mathbb{N}$. This forms a decreasing sequence in E^+ .

Now, construct a new sequence $(w_j)_{j=1}^{\infty} \subset E^+$ by arranging all elements u_n that appear in any s_k , with the property that each u_n appears at least k times if $n > n_k$. By construction:

$$\sum_{j=1}^{\infty} \|w_j\| \leq \sum_{k=1}^{\infty} \frac{2}{k^2} < \infty$$

By the Riesz-Fischer property of E , the sum $s := \sum_{j=1}^{\infty} w_j$ exists in E . Furthermore, for each $k \in \mathbb{N}$:

$$s \geq k \cdot s_k$$

since each element appearing in s_k appears at least k times in the sequence (w_j) .

Therefore, by monotonicity of the norm and our earlier inequality:

$$\|s\| \geq k \cdot \|s_k\| > k \cdot \varepsilon \quad \text{for all } k \in \mathbb{N}$$

This implies $\|s\| = \infty$, contradicting the fact that $s \in E$. Thus, the infinite triangle inequality must hold for any sequence in E^+ with summable norms. ■

Lemma 2.2 (Properties of Convergence in Normed Riesz Spaces). Let E be a normed Riesz space. If $(f_n)_{n=1}^{\infty} \subset E$ converges to f (in order, in norm) and $f_n \geq g$ for all $n \in \mathbb{N}$, then $f \geq g$. In particular, if $(f_n)_{n=1}^{\infty} \subset E^+$ converges to f and $f_n \geq 0$ for all $n \in \mathbb{N}$, then $f \geq 0$. Thus, the positive cone E^+ is closed.

Proof. It may be assumed that $g = 0$. Since $|f^- - f_n^-| \leq |f - f_n|$, the sequence $(f_n^- : n = 1, 2, \dots)$ converges to f^- . But $f_n^- = 0$ for all n , so $f^- = 0$. In other words, $f \geq 0$. ■

Lemma 2.3 (Relation between Norm and Order Convergence). Let E be a normed Riesz space. If $(f_n)_{n=1}^{\infty} \subset E$ is an increasing sequence (i.e., $f_n \uparrow$) and $f_n \rightarrow f$ in norm, then $f_n \uparrow f$ (similar for decreasing sequence)

Proof. Let $f_n \uparrow$ and $f_n \rightarrow f$ (norm). Since $f_n \geq f_m$ for all $n \geq m$, we have $f \geq f_m$ by Lemma 2.2. This holds for all m , so f is an upper bound of the sequence.

Let g be another upper bound. Then $f_n \leq g$ for all n , so $f \leq g$ (once more by [Lemma 2.2](#)). It follows that f is the least upper bound of the sequence, i.e., $f_n \uparrow f$. ■

2.5 Prove a normed Riesz space is a Banach lattice if and only if it has the Riesz Fischer property

Now we use this lemma to prove that a normed Riesz space is a Banach lattice if and only if it has the Riesz-Fischer property.

Theorem 2.3 (Characterization of Banach Lattices). Let $(E, \|\cdot\|)$ be a normed Riesz space. Then E is a Banach lattice if and only if E has the Riesz-Fischer property.

Proof. We prove both directions of the equivalence.

(\Rightarrow) **Assume E is a Banach lattice:**

Let $(u_n)_{n=1}^\infty \subset E^+$ be a sequence of positive elements such that $\sum_{n=1}^\infty \|u_n\| < \infty$. Define the sequence of partial sums $(s_n)_{n=1}^\infty$ by:

$$s_n = \sum_{k=1}^n u_k \quad \text{for } n \in \mathbb{N}$$

For any integers $m > n$, using the triangle inequality:

$$\begin{aligned} \|s_m - s_n\| &= \left\| \sum_{k=n+1}^m u_k \right\| \\ &\leq \sum_{k=n+1}^m \|u_k\| \end{aligned}$$

Since $\sum_{n=1}^\infty \|u_n\| < \infty$, we have $\sum_{k=n+1}^m \|u_k\| \rightarrow 0$ as $n, m \rightarrow \infty$. Thus, $(s_n)_{n=1}^\infty$ is a Cauchy sequence in E . Since E is a Banach lattice, there exists $s \in E$ such that $\|s_n - s\| \rightarrow 0$ as $n \rightarrow \infty$.

Moreover, since $(s_n)_{n=1}^\infty$ is an increasing sequence of positive elements ($s_n \leq s_{n+1}$ for all n), by [Lemma 2.3](#), the order limit $\sup_n s_n$ exists and equals s . Therefore:

$$\sum_{n=1}^\infty u_n = \sup_{n \in \mathbb{N}} s_n = s \in E$$

This establishes that E has the Riesz-Fischer property.

(\Leftarrow) **Assume E has the Riesz-Fischer property:**

To prove E is a Banach lattice, we need to show that E is norm complete. By the standard characterization of Banach spaces, it suffices to show that any absolutely convergent series in E is convergent in norm.

Let $(f_n)_{n=1}^\infty \subset E$ be a sequence such that $\sum_{n=1}^\infty \|f_n\| < \infty$. We need to prove that the series $\sum_{n=1}^\infty f_n$ converges in norm in E .

First, observe that $\sum_{n=1}^\infty \|f_n^+\| \leq \sum_{n=1}^\infty \|f_n\| < \infty$ and $\sum_{n=1}^\infty \|f_n^-\| \leq \sum_{n=1}^\infty \|f_n\| < \infty$, where f_n^+ and f_n^- are the positive and negative parts of f_n , respectively.

By our hypothesis that E has the Riesz-Fischer property, the order limits $s^+ = \sum_{n=1}^\infty f_n^+$ and $s^- = \sum_{n=1}^\infty f_n^-$ exist in E . Define $s = s^+ - s^-$, which represents the sum $\sum_{n=1}^\infty f_n$.

For each $m \in \mathbb{N}$, let $s_m = \sum_{k=1}^m f_k$. We need to show that $\|s - s_m\| \rightarrow 0$ as $m \rightarrow \infty$.

Note that $s - s_m = \sum_{k=m+1}^{\infty} f_k$. By the infinite triangle inequality established in [Lemma 2.1](#)

$$\|s - s_m\| = \left\| \sum_{k=m+1}^{\infty} f_k \right\| \quad (2)$$

$$\leq \sum_{k=m+1}^{\infty} \|f_k\| \quad (3)$$

Since $\sum_{n=1}^{\infty} \|f_n\| < \infty$, we have $\sum_{k=m+1}^{\infty} \|f_k\| \rightarrow 0$ as $m \rightarrow \infty$. Therefore, $\|s - s_m\| \rightarrow 0$ as $m \rightarrow \infty$, which proves that the sequence of partial sums $(s_m)_{m=1}^{\infty}$ converges in norm to s .

This confirms that E is a Banach lattice, completing the proof. ■

2.6 L^p is a Banach lattice

Corollary 2.1 (*L^p Spaces are Banach Lattices*). Let (X, \mathcal{M}, μ) be a measure space and $1 \leq p < \infty$. Then $L^p(X, \mathcal{M}, \mu)$ is a Banach lattice.

Proof. By [Theorem 2.2](#), L^p spaces have the Riesz-Fischer property, and by [Theorem 2.3](#), this concludes that L^p spaces are Banach lattice. ■

3 Prove positive operator on L^p are norm bounded

In this section, we prove that positive operator on L^p is norm bounded. We first present some related definitions, and then we prove that every positive operator on Riesz space is order bounded. Then, we prove the main theorem that every positive order bounded operator between a Banach lattice and a normed Riesz space is norm bounded. Since we know that L^p spaces are Banach lattice, this concludes that positive operators on L^p are norm bounded.

3.1 Related Definitions

Definition 3.1 (Order Bounded Operator). Let E and F be Riesz spaces. A operator $T : E \rightarrow F$ is called *order bounded* if it maps any order interval in E into an order interval on F

Definition 3.2 (Norm Bounded Operator). Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces over the same field \mathbb{K} (either \mathbb{R} or \mathbb{C}). An operator $T : V \rightarrow W$ is said to be *norm bounded* if there exists a constant $M \in [0, \infty)$ such that

$$\|Tx\|_W \leq M\|x\|_V \quad \text{for all } x \in V$$

3.2 Positive operator on Riesz space is order bounded

In this section, we show that every positive operator between Riesz spaces is order bounded.

Lemma 3.1 (Positive Operators are Order Bounded). Let V and W be Riesz spaces, and let $T : V \rightarrow W$ be a positive operator. Then T is order bounded.

Proof. Let $[f_1, f_2]$ be an arbitrary order interval in V . We need to show that T maps this interval into an order interval in W .

Fix $f \in [f_1, f_2]$, we have $f_1 \leq f \leq f_2$. Since T is positive, we have

$$Tf_1 \leq Tf \leq Tf_2$$

Since f is arbitrary, this implies T maps the order interval $[f_1, f_2]$ into $[Tf_1, Tf_2]$

This shows that T is order bounded. ■

3.3 Positive order bounded operator on Banach lattice to normed Riesz space is norm bounded

In this section, we present the main theorem as follows.

Theorem 3.1 (Positive order bounded operator on Banach lattice is normed bounded). If E is a Banach lattice and F is a normed Riesz space, and $T : E \rightarrow F$ is a positive order bounded operator, then T is norm bounded.

Proof. We prove this theorem by contradiction. Assume that $T : E \rightarrow F$ is positive, order bounded, but not norm bounded. Then there exists a sequence $(f_n)_{n=1}^\infty$ in E such that $\|Tf_n\| > 2n^3\|f_n\|$ for all n .

Note that $f_n \neq 0$ for all n , so (replacing f_n by $f_n/\|f_n\|$) we may assume that $\|f_n\| = 1$ and $\|Tf_n\| > 2n^3$ for all n . Since $\|Tf_n\|$ is less than or equal to $\|Tf_n^+\| + \|Tf_n^-\|$, it follows that at least one of $\|Tf_n^+\|$ and $\|Tf_n^-\|$ exceeds n^3 . Hence, we may assume that

$$f_n > 0, \|f_n\| \leq 1 \text{ and } \|Tf_n\| > n^3 \text{ for } n = 1, 2, \dots$$

Writing $u_n = n^{-2}f_n$, we get

$$u_n \in E^+, \|u_n\| \leq n^{-2} \text{ and } \|Tu_n\| > n \text{ for } n = 1, 2, \dots$$

It follows that $\sum_{n=1}^{\infty} \|u_n\|$ converges, so (since E is a Banach lattice and, therefore, has the Riesz-Fischer property by [Theorem 2.3](#)) the element $s = \sum_{n=1}^{\infty} u_n$ exists in E .

This shows that all u_n are contained in the order interval $[0, s]$. Since T is order bounded, all Tu_n are then contained in some order interval in F . But then the sequence of all Tu_n is norm bounded, which contradicts $\|Tu_n\| > n$ for all n .

Hence, the operator T is norm bounded. ■

3.4 Positive operator between L^p spaces are norm bounded

Corollary 3.1 (Positive Operators on L^p Spaces are Norm Bounded). Let (X, \mathcal{M}, μ) be a measure space, $1 \leq p < \infty$, and let $T : L^p \rightarrow L^p$ be a positive operator. Then T is norm bounded.

Proof. We know from [Corollary 2.1](#) that $L^p(X)$ is a Banach lattice for $1 \leq p < \infty$.

Since T is a positive operator, it is order bounded by [Lemma 3.1](#). Now, by [Theorem 3.1](#), every positive order bounded operator from a Banach lattice to a normed Riesz space is norm bounded.

Therefore, the positive operator $T : L^p \rightarrow L^p$ is norm bounded. ■

4 Prove positive operator on L^p is order continuous

In this section, we combine everything we have shown before and prove that every positive operator on L^p is order continuous. First, we present the related definition.

4.1 Related Definitions

Definition 4.1 (Order Continuous Operator). Let E and F be Riesz spaces. A operator $T : E \rightarrow F$ is called *order continuous* if for every sequence $(f_n)_{n=1}^{\infty}$ in E such that $f_n \downarrow 0$ (i.e., f_n is decreasing and $\inf_n f_n = 0$), we have

$$Tf_n \downarrow 0 \quad \text{in } F$$

4.2 Proof of the main theorem

Theorem 4.1 (Positive Operators on L^p are Order Continuous). Let (X, \mathcal{M}, μ) be a measure space, $1 \leq p < \infty$, and let $T : L^p(X) \rightarrow L^p(X)$ be a positive operator. Then T is order continuous.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a monotone decreasing sequence in $L^p(X)$ such that $f_n \downarrow 0$. We need to show that $Tf_n \downarrow 0$.

From [Theorem 1.2](#), L^p norm is order continuous, i.e., $f_n \downarrow 0$ implies $\|f_n\| \downarrow 0$.

Since T is a positive operator, by [Corollary 3.1](#), we know that T is norm bounded, i.e.,

$$\|Tf_n\| \leq \|T\| \|f_n\|$$

where $\|T\|$ is the operator norm.

This implies that $\|Tf_n\| \rightarrow 0$ in norm as $n \rightarrow \infty$.

Since T is a positive operator, and $f_n \downarrow$, we have $Tf_n \downarrow$. By [Lemma 2.3](#), we know $Tf_n \downarrow 0$.

Hence, T is order continuous. ■

References

- [Zaa12] Adriaan C Zaanen. *Introduction to operator theory in Riesz spaces*. Springer Science & Business Media, 2012.