Economic Networks

Graph Theory

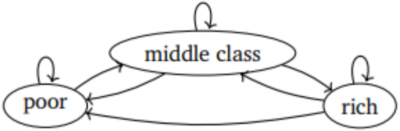
Introduction

Motivation: Many economic models are stochastic and dynamic, which means that they specify states of the world and rates of transition between them. One of the most natural ways to conceptualize these notions is to view states as vertices in a graph and transition rates as relationships between them.

A **directed graph** (digraph) $\mathcal{G}$ is a pair (V, E) where:

- V is a finite nonempty set of **vertices** or **nodes**.
- E is a collection of ordered pairs $(u, v) \in V \times V$ called **edges**.

Example: Poor, Middle, Rich



- Vertices: Poor (P), Middle (M), Rich (R).
- Edges: (P, P), (P, M), (M, P), (M, M), (M, R), (R, P), (R, M), (R, R).

In-Degree and Out-Degree

For a given edge (u, v), the vertex u is called the tail of the edge, while v is called the head. Also, u is called a direct predecessor of v and v is called a direct successor of u.

For a vertex $v \in V$:

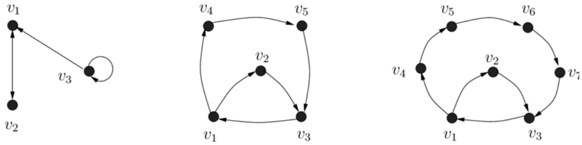
- **In-degree**: $i_d(v) = |\mathcal{I}(v)|$, where $\mathcal{I}(v)$ is the set of direct predecessors of v.
- **Out-degree**: $o_d(v) = |\mathcal{O}(v)|$, where $\mathcal{O}(v)$ is the set of direct successors of v.

Direct Walk, Path, Length, Accessibility

- A **directed walk** from u to v is a sequence of vertices where each consecutive pair is an edge.
- A **directed path** is a directed walk with all distinct vertices.
- The **length** of a walk/path is the number of edges.
- Vertex v is **accessible** from u if there is a directed path from u to v.
- Vertices u and v **communicate** if $u \to v$ and $v \to u$.
- A digraph is **strongly connected** if every vertex is accessible from every other.

Periodicity

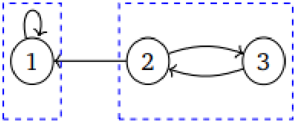
- A digraph is **periodic** if it contains at least one cycle and the length of every cycle is a multiple of some $k > 1$.
- A digraph is **aperiodic** if it is not periodic.



(a) (b) (c)

Periodicity

Lemma 1.4.1: Let $\mathcal{G} = (V, E)$ be a digraph. If $\mathcal{G}$ is strongly connected, then $\mathcal{G}$ is aperiodic if and only if, for all $v \in V$, there exists a $q \in \mathbb{N}$ such that, for all $k \geq q$, there exists a directed walk of length k from v to v.



Adjacency Matrix

For a digraph $\mathcal{G} = (V, E)$ with $V = \{v_1, \dots, v_n\}$, the **adjacency matrix** A is defined by:

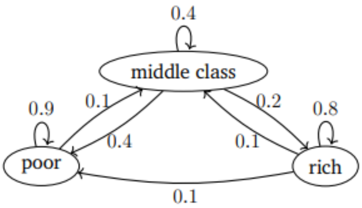
$$A = (a_{ij})_{1 \le i,j \le n} \quad \text{with} \quad a_{ij} = 1\{(v_i, v_j) \in E\}.$$

In our previous case (Poor, Middle Class, Rich), we had:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

A digraph (V, E) is called **undirected** if $(u, v) \in E$ implies $(v, u) \in E$. This implies that the adjacency matrix A is symmetric, i.e., $A = A^T$.

Definition of a Weighted Digraph



A **weighted digraph** $\mathcal{G}$ is a triple (V, E, w) where:

- (V, E) is a digraph.
- w is a function from E to $(0, \infty)$, called the **weight function**.

Weighted Adjacency Matrix

For a weighted digraph (V, E, w) with $V = \{v_1, \ldots, v_n\}$, the **weighted adjacency matrix** A is defined by:

$$A = (a_{ij})_{1 \leq i,j \leq n} \quad \text{with} \quad a_{ij} = \begin{cases} w(v_i, v_j) & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

For our poor, middle class, rich case:

$$A = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$

Properties

Note that for Matrix A^{s+t} we can write its elements as:

$$a_{ij}^{s+t} = \sum_{\ell=1}^{n} a_{i\ell}^{s} \, .a_{\ell j}^{t} \qquad (i, j \in [n], \;\; s, t \in \mathbb{N}).$$

Properties

For our poor, middle class rich example, we have:

$$A^2 = A \cdot A = \begin{bmatrix} a^2_{pp} & a^2_{pm} & a^2_{pr} \\ a^2_{mp} & a^2_{mm} & a^2_{mr} \\ a^2_{rp} & a^2_{rm} & a^2_{rr} \end{bmatrix},$$

$$\begin{aligned} a^2_{pp} &= a_{pp}a_{pp} + a_{pm}a_{mp} + a_{pr}a_{rp}, \\ a^2_{pm} &= a_{pp}a_{pm} + a_{pm}a_{mm} + a_{pr}a_{rm}, \\ a^2_{pr} &= a_{pp}a_{pr} + a_{pm}a_{mr} + a_{pr}a_{rr}, \\ a^2_{mp} &= a_{ml}a_{pp} + a_{mm}a_{mp} + a_{mr}a_{rp}, \\ &\dots \end{aligned}$$

Properties

$$A^3 = A^2 \cdot A = \begin{bmatrix} a^3_{pp} & a^3_{pm} & a^3_{pr} \\ a^3_{mp} & a^3_{mm} & a^3_{mr} \\ a^3_{rp} & a^3_{rm} & a^3_{rr} \end{bmatrix},$$

$$\begin{aligned} a^3_{pp} &= a^2_{pp} a_{pp} + a^2_{pm} a_{mp} + a^2_{pr} a_{rl}, \\ a^3_{pm} &= a^2_{pp} a_{pm} + a^2_{pm} a_{mm} + a^2_{pr} a_{rm}, \\ a^3_{pr} &= a^2_{pp} a_{pr} + a^2_{pm} a_{mr} + a^2_{pr} a_{rr}, \\ a^3_{mp} &= a^2_{mp} a_{pp} + a^2_{mm} a_{mp} + a^2_{mr} a_{rl}, \\ &\dots \end{aligned}$$

Properties

Proposition 1.4.2: Let $\mathcal{G}$ be a weighted digraph with adjacency matrix A. For distinct vertices $i, j \in [n]$ and $k \in \mathbb{N}$, we have:

$$a_{ij}^k > 0 \iff \text{there exists a directed walk of length } k \text{ from } i \text{ to } j.$$

Proof: (i) $\Rightarrow$ (ii)

- Base Case ($k = 1$):

$$a_{ij}^1 = a_{ij}.$$

By definition, $a_{ij} > 0$ iff there is a directed edge $i \to j$.

- Inductive Step: Assume true for $k - 1$. Consider a walk:

$$(i, \ell, m, \dots, n, j).$$

The walk $(i, \ell, m, \dots, n)$ implies $a_{in}^{k-1} > 0$. Since $a_{nj} > 0$, by matrix multiplication:

$$a_{ij}^k = \sum_{n=1}^{N} a_{in}^{k-1} \cdot a_{nj} > 0.$$

Proof: (ii) $\Rightarrow$ (i)

Suppose there exists a directed walk:

$$i \to m_1 \to m_2 \to \ldots \to m_{k-1} \to j,$$

and all edges in this walk satisfy $a_{im_1} > 0, a_{m_1 m_2} > 0, \ldots, a_{m_{k-1}j} > 0$. Since every edge in the walk is positive, the product is strictly positive:

$$a_{im_1} \cdot a_{m_1 m_2} \cdots a_{m_{k-1}j} > 0.$$

Because a_{ij}^k sums over all possible walks (with non-negative weights), and at least one term is positive:

$$a_{ij}^k = \sum_{m_1, m_2, \ldots, m_{k-1} \in V} a_{im_1} \cdot a_{m_1 m_2} \cdots a_{m_{k-1}j} > 0.$$

Properties

Theorem 1.4.3: Let $\mathcal{G}$ be a weighted digraph. The following statements are equivalent:

- $\mathcal{G}$ is strongly connected.
- The adjacency matrix generated by $\mathcal{G}$ is irreducible. For $A \geqslant 0$

$$\sum_{m=0}^{\infty} A^m \gg 0.$$

Proof: (i) $\Rightarrow$ (ii)

- For any distinct i, j, strong connectivity guarantees a directed path $i \to j$ of length k. By Proposition 1.4.2, $a_{ij}^k > 0$. The sum $\sum_{m=0}^{\infty} A^m$ sums over all these A^k, ensuring every entry (i, j) has at least one positive term. Combined with diagonal terms ($A^0 = I$), the sum is strictly positive: $\sum_{m=0}^{\infty} A^m \gg 0$.
- Example:
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Proof: (ii) $\Rightarrow$ (i)

- If A is irreducible then there must exist some $m \geq 1$ such that for distinct i,j $a_{ij}^{m} > 0$.
- The sum $\sum_{m=0}^{\infty} A^m \gg 0$ ensures every off-diagonal entry (i, j) (where $i \neq j$) to have a positive term in the series.
- By Proposition 1.4.2, this implies a directed walk of length $m \geq 1$ for every i,j, so G is strongly connected.

Properties

Theorem 1.4.4: For a weighted digraph $\mathcal{G} = (V, E, w)$, the following statements are equivalent:

- $\mathcal{G}$ is strongly connected and aperiodic.
- The adjacency matrix generated by $\mathcal{G}$ is primitive. For $A \geqslant 0$ there exists an $m \in \mathbb{N}$ such that

$$A^m \gg 0.$$

Proof: (i) $\Rightarrow$ (ii)

- First, we show that if $\mathcal{G}$ is aperiodic and strongly connected, then for all $i, j \in V$, there exists a $q \in \mathbb{N}$ such that $a_{ij}^k > 0$ whenever $k \geq q$.
  To this end, pick any $i, j \in V$. Since $\mathcal{G}$ is strongly connected, there exists an $s \in \mathbb{N}$ such that $a_{ij}^s > 0$. Since $\mathcal{G}$ is aperiodic, we can find an $m \in \mathbb{N}$ such that $\ell \geq m$ implies $a_{jj}^\ell > 0$. Picking $\ell \geq m$ and applying the matrix multiplication rule, we have

$$a_{ij}^{s+\ell} = \sum_{r \in V} a_{ir}^s a_{rj}^\ell \geq a_{ij}^s a_{jj}^\ell > 0.$$

  Thus, with $t = s + m$, we have $a_{ij}^k > 0$ whenever $k \geq t$.

Proof: (i) $\Rightarrow$ (ii)

- By the preceding argument, given any $i, j \in V$, there exists an $s(i,j) \in \mathbb{N}$ such that $a_{ij}^m > 0$ whenever $m \geq s(i,j)$. Setting $k := \max s(i,j)$ over all (i,j) yields $A^k \gg 0$. This proves that the adjacency matrix A is primitive.

Proof: (ii) $\Rightarrow$ (i)

Strong Connectivity

If $A^k \gg 0$, then for every pair i, j, there is a path of length k from i to j. This directly implies that $\mathcal{G}$ is strongly connected.

Aperiodicity

$A^k \gg 0$ also implies that there is a cycle of length k at node 1, so $a_{11}^k > 0$. Aperiodicity will hold if we can establish that $a_{11}^{k+1} > 0$, since then we have a cycle of length k and another of length $k + 1$.

• Proof: (ii) ⇒ (i)

Let

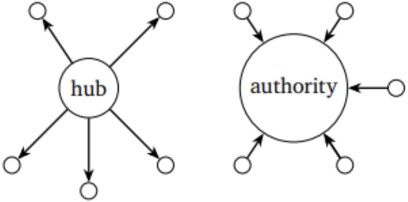
$$a_{11}^{k+1} = \sum_{\ell \in V} a_{1\ell} a_{\ell 1}^{k}.$$

Let $\bar{a} = \min_{\ell \in V} a_{\ell 1}^{k}$. Since $A^k \gg 0$, $\bar{a} > 0$.Then:

$$a_{11}^{k+1} \geq \bar{a} \sum_{\ell \in V} a_{1\ell}.$$

The sum $\sum_{\ell \in V} a_{1\ell}$ is positive because node 1 cannot be a sink (otherwise, the graph would not be strongly connected).Thus, $a_{11}^{k+1} > 0$, meaning there is a cycle of length $k+1$ at node 1.

Hub and Authority



- **Hub centrality**: Measures how many important vertices a vertex points to.
- **Authority centrality**: Measures how many important vertices point to a vertex.

In-Degree and Out-Degree Centralities

- **In-degree centrality**: $i(\mathcal{G}) = (i_d(v))_{v \in V}$.
- **Out-degree centrality**: $o(\mathcal{G}) = (o_d(v))_{v \in V}$.

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad o(\mathcal{G}) = U \cdot \mathbf{1} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \quad i(\mathcal{G}) = U^T \cdot \mathbf{1} = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}.$$

Weighted Centralities

For a weighted digraph $\mathcal{G} = (V, E, w)$ with adjacency matrix A:

- **Weighted in-degree centrality**: $i(\mathcal{G}) = A^T\mathbf{1}$.
- **Weighted out-degree centrality**: $o(\mathcal{G}) = A\mathbf{1}$.

But unfortunately, these measurements are not very useful, consider the following case:

Weighted Centralities

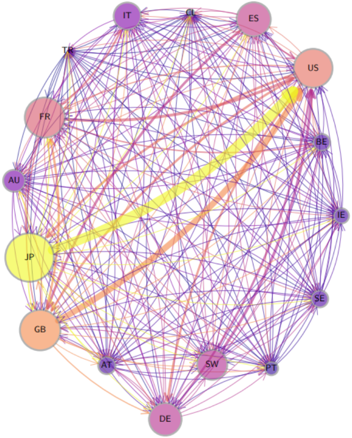
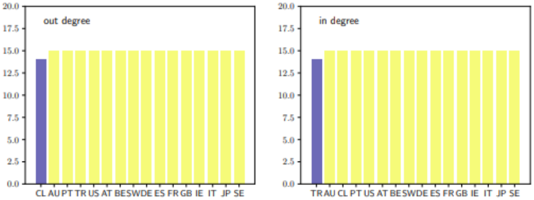


Figure: International private credit flows by country

Weighted Centralities

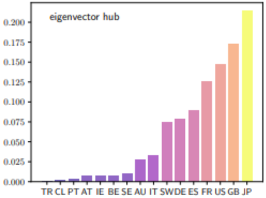
Because there is an edge between almost every node, the in- or out-degree based centrality ranking fails to effectively separate the countries.

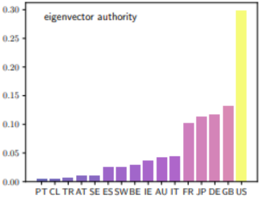


Eigenvector Centrality

Instead, we use the **hub-based eigenvector centrality** of $\mathcal{G}$ is defined as the vector $e \in \mathbb{R}^n_+$ that solves:

$$e_i = \frac{1}{r(A)} \sum_{j \in [n]} a_{ij} e_j \quad \text{for all } i \in [n] \Leftrightarrow e = \frac{1}{r(A)} Ae.$$





For the authority-based centrality, we just need to use A^T.

Perron-Frobenius

If $A \geq 0$, then $r(A)$ is an eigenvalue of A with nonnegative real right and left eigenvectors:

$$\exists \text{ nonzero } e, \varepsilon \in \mathbb{R}^n_+ \text{ such that } Ae = r(A)e \text{ and } \varepsilon^T A = r(A)\varepsilon^T.$$

If A is irreducible, then, in addition,

1. $r(A)$ is strictly positive and a simple eigenvalue,
2. the eigenvectors e and ε are everywhere positive, and
3. eigenvectors of A associated with other eigenvalues fail to be nonnegative.

Perron-Frobenius

If A is primitive, then, in addition,

1. the inequality $|\lambda| \leq r(A)$ is strict for all eigenvalues λ of A distinct from $r(A)$, and
2. with e and ε normalized so that $\langle e, e \rangle = \mathbf{1}$, we have

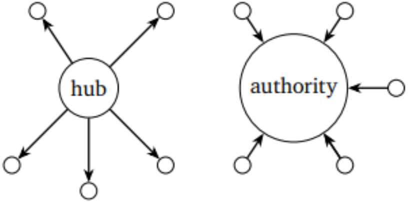
$$r(A)^{-m} A^m \to e\varepsilon^T \quad (m \to \infty).$$

This implies:

$$r(A)^{-m} A^m 1 \to ce \quad \text{where } c := \varepsilon^\top \mathbf{1}\,. \tag{1}$$

Eigenvector Centrality

- Eigenvector centrality can be problematic. We need irreducibility, which by Perron-Frobenius ensures. $(r(A) > 0)$ and unique, but this fails in many real world networks.
- Many vertices can be assigned a zero ranking. Only nodes that lie either in a strongly connected component or its out-component can have positive eigenvector centrality.



Katz Centrality

To address this, Katz centrality is often used. Fixing $\beta \in (0, 1/r(A))$, $\kappa := \kappa(\beta, A) \in \mathbb{R}_+^n$ is the vector that solves:

$$\kappa_i = \beta \sum_{j \in [n]} a_{ij} \kappa_j + \mathbf{1} \quad \text{for all } i \in [n].$$

Katz Centrality

We can write it in Matrix form: $\kappa = \beta A \kappa + \mathbf{1}$ and rearrange it: $\kappa = (I - \beta A)^{-1}\mathbf{1}$.

Because $0 < \beta < \frac{1}{r(A)}$ then $0 < r(A\beta) < 1$ and we can use Neumann Series Lemma so:

$$\kappa = \sum_{l \geq 0} (\beta A)^l \mathbf{1}.$$

Empirical Degree Distributions

Complex networks often exhibit a scale-free property, which means, that the number of connections possessed by each vertex in the network follows a power law.

Let $\mathcal{G} = (V, E)$ be a digraph. Assuming without loss of generality that $V = [n]$ for some $n \in \mathbb{N}$, the **in-degree distribution** of $\mathcal{G}$ is the sequence $(\varphi_{\text{in}}(k))_{k=0}^{n}$ defined by:

$$\varphi_{\text{in}}(k) = \frac{\sum_{\nu \in V} \mathbf{1}\{i_d(\nu) = k\}}{n} \qquad (k = 0, \dots, n),$$

A network is **scale-free** if there exists constants c and γ and its degree distribution obeys a power law:

$$\varphi(k) \approx ck^{-\gamma} \quad \text{for large } k.$$

Random Graphs

How can we generate Random Graphs? Consider Erdos–Renyi approach:

1. Fix an integer $n \in \mathbb{N}$ and a $p \in (0,1)$.
2. View $V := [n]$ as a collection of vertices.
3. Let $E = \{\emptyset\}$.
4. For each $(i,j) \in V \times V$ with $i \neq j$, add the undirected edge $\{i,j\}$ to the set of edges E with probability p.

In the last step, additions are independent—each time, we flip an unbiased iid coin with head probability p and add the edge if the coin comes up heads.

Random Graphs

Clearly Erdos–Renyi random graphs fail to replicate the heavy right hand tail of the degree distribution observed in many networks.

To achieve one, we can use preferential attachment. Each time a new vertex is added to an undirected graph, it is attached by edges to m of the existing vertices, where the probability of vertex v being selected is proportional to the degree of v.