Dynamic Programming

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Setup

- **Wage space**: $W \in [0, M] \subset \mathbb{R}_+$ and $M > 0$.
- **Wage offer**: (W_t) is P-Markov on W.
- **State space**: $X = \{e, u\} \times W$, $e \implies$ employed; $u \implies$ unemployed
- **Policy**: a Borel measurable map: $\sigma : W \to \{0, 1\}$, i.e., we only choose when unemployed
- **Separation probability** $:= \alpha$, the probability of job termination

ADP with Job Search with Separation

- **Value space**: $V = bm(X, \mathbb{R})$ with supremum norm and pointwise partial order
- **Policy operator when employed** (let $v_u := v(u, \cdot); \quad v_e := v(e, \cdot); \quad \mathbf{w}(w) := w$)

$$T_\sigma v_e = \mathbf{w} + \beta \left[\underbrace{\alpha P v_u}_{\text{transit to unemployed}} + \underbrace{(1-\alpha) v_e}_{\text{stay employed}} \right]$$

Remark: we don't have σ explicitly here.

- **Policy operator when unemployed**

$$T_\sigma v_u = \underbrace{\sigma v_e}_{\text{transit to employed}} + \underbrace{(1-\sigma)\left[c\mathbf{1} + \beta P v_u\right]}_{\text{stay unemployed}}$$

Simplify

We treat the policy operator when employed as a fixed point problem i.e.,

$$T_\sigma v_e = \mathbf{w} + \beta \left[\alpha P v_u + (1-\alpha) v_e\right]$$

to

$$\begin{aligned}
v_e &= \mathbf{w} + \beta \left[\alpha P v_u + (1-\alpha) v_e\right] \\
&= \mathbf{w} + \alpha\beta P v_u + \beta(1-\alpha) v_e \\
&= \frac{1}{1-\beta(1-\alpha)}\left[\mathbf{w} + \alpha\beta P v_u\right] \\
&= \underbrace{\frac{1}{1-\beta(1-\alpha)}\mathbf{w}}_{=:h} + \underbrace{\frac{\alpha\beta}{1-\beta(1-\alpha)}}_{=:\gamma} P v_u \\
v_e &= h + \gamma P v_u \qquad \text{(a relation between } v_e \text{ and } v_u\text{)}
\end{aligned}$$

Reduce dimension

For the policy operator when unemployed:

$$T_\sigma v_u = \sigma v_e + (1-\sigma)\left[c\mathbf{1} + \beta P v_u\right]$$

we have $v_e = h + \gamma P v_u$, then, we have

$$T_\sigma v_u = \sigma(h + \gamma P v_u) + (1-\sigma)[c\mathbf{1} + \beta P v_u]$$

now we reduce the state space into just W compared to $X = \{e, u\} \times W$.

Reduce dimension

The state space is reduced from $X = \{e, u\} \times W$ to W, we can reduce the value space

$$V = bm(X, \mathbb{R}) \text{ to } bm(W, \mathbb{R}) =: bmW$$

with policy operator

$$T_\sigma v_u = \sigma(h + \gamma P v_u) + (1 - \sigma)[c\mathbf{1} + \beta P v_u]$$

Remark: We can rewrite $T_\sigma v_u = J_\sigma + K_\sigma v_u$, where $J_\sigma = \sigma h + (1 - \sigma)c\mathbf{1}$; and $K_\sigma = (\sigma\gamma + (1 - \sigma)\beta))P$

Prove that T_σ is an order preserving self-map on bmW

Proof of order preserving.

Fix $\sigma \in \Sigma$. Let $v_u^1 \leq v_u^2 \in bmW$, For $w \in \{w \in W : \sigma(w) = 1\}$, we have

$$T_\sigma v_u^1(w) = h(w) + \gamma(Pv_u^1)(w) \leq h(w) + \gamma(Pv_u^2)(w)$$

by order preserving of the Markov operator P. For $w' \in \{w \in W : \sigma(w) = 0\}$, we have

$$T_\sigma v_u^1(w) = c + \beta(Pv_u^1)(w') \leq c + \beta(Pv_u^2)(w')$$

by order preserving of the Markov operator P. Hence in all, we have

$$T_\sigma v_u^1 \leq T_\sigma v_u^2$$

Hence, T_σ is order preserving. □

Prove that T_σ is an order preserving self-map on bmW

Proof of $T_\sigma v_u$ is bounded.

Fix $\sigma \in \Sigma$:

$$
\begin{aligned}
\|T_\sigma v_u\|_\infty &= \|(\sigma h + (1-\sigma)c\mathbf{1}) + (\sigma\gamma + (1-\sigma)\beta)Pv_u\|_\infty \\
&\leq \|\sigma h + (1-\sigma)c\mathbf{1}\|_\infty + \|(\sigma\gamma + (1-\sigma)\beta)Pv_u\|_\infty && (\Delta \text{ ineq.}) \\
&\leq \|h\|_\infty + \|c\mathbf{1}\|_\infty + (\gamma+\beta)\|Pv\|_\infty && (\Delta \text{ ineq.}) \\
&\leq \frac{M}{1-\beta(1-\alpha)} + c + (\gamma+\beta)\|v\|_\infty && (\|P\| = 1)
\end{aligned}
$$

□

Remark: another way to bound $\|(\sigma\gamma + (1-\sigma)\beta)Pv_u\|_\infty \leq \lambda\|v\|_\infty$, $\lambda := \max\{\gamma, \beta\}$

Prove that T_σ is an order preserving self-map on bmW

Proof of $T_\sigma v_u$ is Borel measurable.

- v_u is Borel measurable
- P is bounded linear operator hence continuous hence Borel measurable
- σ is Borel measurable
- h, $c\mathbf{1}$ are constant functions hence Borel measurable.

Hence, $T_\sigma v_u$ is Borel measurable. $\square$

Hence, we have $(bmW, \mathbb{T})$, $\mathbb{T} := \{T_\sigma : \sigma \in \Sigma\}$ is an ADP (still bmW has a supremum norm and pointwise partial order).

Why supremum norm?

Claim 1: The metric induced by supremum norm is sup-nonexpansive.
Let $(v_n), (w_n) \in bmW$ with supremum v and w which are bounded above. We have

$$
\begin{aligned}
v_n &= v_n - w_n + w_n \\
&\leq |v_n - w_n| + w_n \\
&\Longrightarrow
\end{aligned}
$$

$$
\sup_n v_n \leq \sup_n (|v_n - w_n| + w_n) \leq \sup_n |v_n - w_n| + \sup_n w_n
$$

$$
\sup_n v_n - \sup_n w_n \leq \sup_n |v_n - w_n|
$$

Similar for the other direction, we have

$$
|\sup_n v_n - \sup_n w_n| \leq \sup_n |v_n - w_n|
$$

Supremum norm induce a sup-nonexpansive metric

For

$$\left|\sup_n v_n - \sup_n w_n\right| \leq \sup_n |v_n - w_n|$$

we have

$$\sup_{x\in W}\left|\sup_n v_n(x) - \sup_n w_n(x)\right| \leq \sup_{x\in W}\sup_n |v_n(x) - w_n(x)| = \sup_n \sup_{x\in W} |v_n(x) - w_n(x)|$$

the last equality is by the uniquness of supremum. This implies

$$\left\|\sup_n v_n - \sup_n w_n\right\|_\infty \leq \sup_n \|v_n - w_n\|_\infty$$

in term of the metric induced by supremum norm, we have

$$d_\infty(\sup_n v_n, \sup_n w_n) \leq \sup_n d_\infty(v_n, w_n)$$

Hence, supremum norm induce a sup-nonexpansive metric

Why supremum norm?

Claim 2: The metric induced by supremum norm is complete.
Let (v_n) be a Cauchy sequence in bmW. By definition, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have

$$d_\infty(v_m, v_n) = \sup_{x \in W} |v_m(x) - v_n(x)| < \varepsilon$$

This implies for a fixed $x \in W$, $(v_n(x))$ is a Cauchy sequence in $\mathbb{R}$ (with Euclidean metric). By the completeness of $\mathbb{R}$, we get there exists v such that

$$v(x) = \lim_{n \to \infty} v_n(x)$$

Now we just need to show that $v \in bmW$ and $d_\infty(v_n, v) < \epsilon$.

Supremum metric is complete

Measurability: For $v_n : W \to \mathbb{R}$, pointwise limit of measurable function is measurable. (See details in appendix page 20)
Boundedness: let $m, n \geq N$, we have

$$|v_m(x) - v_n(x)| < \epsilon \implies |v_m(x)| \leq |v_m(x) - v_n(x)| + |v_n(x)| < \epsilon + |v_n(x)|$$

for all $m \geq N$, hence

$$\lim_{m \to \infty} |v_m(x)| = |v(x)| < \epsilon + |v_n(x)|$$

hence bounded.

Supremum metric is complete

Convergence in supremum metric:

$$\sup_{x \in W} |v_n(x) - v_m(x)| < \epsilon, \forall m \geq N$$

implies

$$\sup_{x \in X} |v_n(x) - v(x)| < \epsilon$$

Hence, d_∞ is a complete metric. This makes the underlying value space $(bmW, \| \cdot \|_\infty)$ a Banach space.

v_u-greedy policy

Claim: $\sigma(w) := \mathbf{1}\{v_e(w) \geq c + \beta(Pv_u)(w)\}$ is v_u-greedy

Proof.

Note under σ, we have

$$T_\sigma v_u = \max\{v_e, c + \beta P v_u\}$$

Fix $\tau \in \Sigma$, we have

$$T_\tau v_u = \tau(v_e) + (1-\tau)(c + \beta P v_u) \leq \max\{v_e, c + \beta P v_u\} = T_\sigma v_u$$

Hence, σ is v_u-greedy □

Remark 1: economic interpretation is choose the one with higher expected discounted payoff.

Remark 2: Since for every v_u, such greedy policy is well-defined as above, we have the ADP is regular

Exercise 4.2.10

Prove that there exists $\lambda \in (0,1)$ such that T_σ is a contraction of modulus λ on bmW for all $\sigma \in \Sigma$.

Proof.

Recall that we can rewrite $T_\sigma v_u = J_\sigma + K_\sigma v_u$, where $J_\sigma = \sigma h + (1-\sigma)c\mathbf{1}$; and $K_\sigma = (\sigma\gamma + (1-\sigma)\beta))P$. Let $v_1, v_2 \in bmW$. We have

$$\begin{aligned}
\|T_\sigma v_1 - T_\sigma v_2\|_\infty &= \|J_\sigma + K_\sigma v_1 - (J_\sigma + K_\sigma v_2)\|_\infty \\
&= \|K_\sigma(v_1 - v_2)\|_\infty \\
&= \|(\sigma\gamma + (1-\sigma)\beta)P(v_1 - v_2)\|_\infty \\
&\leq \|\lambda P(v_1 - v_2)\|_\infty && (\lambda := \max\{\gamma, \beta\} \in (0,1)) \\
&= \lambda\|v_1 - v_2\|_\infty
\end{aligned}$$

$\square$

Remark and Summary

Remark: As shown before, the value space is a Banach space. Hence, by the Banach fixed point theorem, every policy operator has a unique fixed point and globally stable. And the ADP is well-posed.

Summary: Now we know that the

- ADP is regular and well-posed.
- Supremum norm induces a complete and sup-nonexpansive metric
- Every policy operator is a contraction, globally stable with unique fixed point

Proposition 4.2.2. Optimality

The ADP $(bmW, \mathbb{T})$ is well-posed. Moreover

1. the fundamental optimality properties hold, and
2. VFI, OPI and HPI all converge

Proof.

Invoke Theorem 1.3.5. See next page $\square$

Theorem 1.3.5. *Let $(V, \mathbb{T})$ be a* ***regular*** *ADP where $V = (V, d, \preceq)$ is a partially ordered metric space and* ***d is complete and sup-nonexpansive****. If each $T_\sigma \in \mathbb{T}$ is a* ***contraction of modulus β*** *on V, then*

(i) *the fundamental optimality properties hold and*

(ii) *VFI, OPI and HPI all converge.*

Appendix 1 - pointwise limit

Let

- $(X, \mathcal{A})$ be a measurable space
- (f_n) be a sequence of measurable functions from X to $\mathbb{R}$
- $f(x) = \lim_{n\to\infty} f_n(x)$

By definition, f is measurable if

$$f^{-1}((-\infty, \alpha]) \in \mathcal{A}, \quad \forall \alpha \in \mathbb{R}$$

i.e., we need to show $\{x \in X : f(x) \leq \alpha\}$ is measurable.

Appendix 1 - pointwise limit

Using the definition of limit we have,

$$\begin{aligned}
\{x \in X : f(x) \leq \alpha\} &= \{x \in X : \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, f_n(x) < \alpha + \epsilon\} \\
&= \bigcap_{\epsilon > 0} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \{x \in X : f_n(x) < \alpha + \epsilon\} \\
&= \bigcap_{\epsilon > 0, \epsilon \in \mathbb{Q}} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \underbrace{\{x \in X : f_n(x) < \alpha + \epsilon\}}_{\text{measurable}} \qquad (\mathbb{Q} \text{ is dense in } \mathbb{R})
\end{aligned}$$

Hence $\{x \in X : f(x) \leq \alpha\}$ is a countable intersection of countable union of countable intersection of measurable set, hence measurable.

Appendix 2 - Optimal policy

Let $\sigma^\top$ be the optimal policy and $v_u^\top$ be the value function. We have

$$v_u^\top = \max\{h + \gamma P v_u^\top, c\mathbf{1} + \beta P v_u^\top\}$$

We can define optimal stopping value

$$s^\top = h + \gamma P v_u^\top$$

and optimal continuation value

$$f^\top = c\mathbf{1} + \beta P v_u^\top$$

Hence, we can characterize the optimal policy as

$$\sigma^\top = \mathbf{1}\{s^\top \geq f^\top\}$$

Appendix 2 - Reservation wage

We define the smallest $w \in W$ such that $\sigma^{\top}(w) = 1$ as **reservation wage**.