Stochastic Approximation

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Overview

- Fixed point iteration
- Stochastic approximation
- Examples

Fixed point iteration

Let

- $T : \Theta \to \Theta$ be a contraction map of modulus β
- Θ be a closed subset of $\mathbb{R}^n$

We know that

- T has a unique fixed point $\bar{\theta}$ in Θ
- $\forall\, \theta \in \Theta$, we have $T^k\theta \to \bar{\theta}$ as $k \to \infty$

Iterating with T on fixed θ is called "fixed point iteration" or "successive approximation"

Alternatively, we can fix $\alpha \in (0, 1)$ and iterate on the **damped sequence**

$$\begin{aligned}\theta_{k+1} &= (1 - \alpha)\theta_k + \alpha T\theta_k \\ &= \theta_k + \alpha(T\theta_k - \theta_k)\end{aligned}$$

In other words, we iterate with

$$F\theta := \theta + \alpha(T\theta - \theta)$$

Does the damped sequence generated by F converge to $\bar{\theta}$?

We have

$$F\bar{\theta} = \bar{\theta} + \alpha(T\bar{\theta} - \bar{\theta}) = \bar{\theta}$$

and

$$\begin{aligned}
\|F\theta - F\theta'\| &= \|\theta + \alpha(T\theta - \theta) - \theta' - \alpha(T\theta' - \theta')\| \\
&\leq (1-\alpha)\|\theta - \theta'\| + \alpha\|T\theta - T\theta'\| \\
&\leq (1 - \alpha + \alpha\beta)\|\theta - \theta'\|
\end{aligned}$$

Note

$$1 - \alpha + \alpha\beta < 1 \iff \alpha\beta < \alpha \iff \beta < 1$$

What do we conclude?

Sometimes damped iteration is faster

This tends to be true when there are oscillations

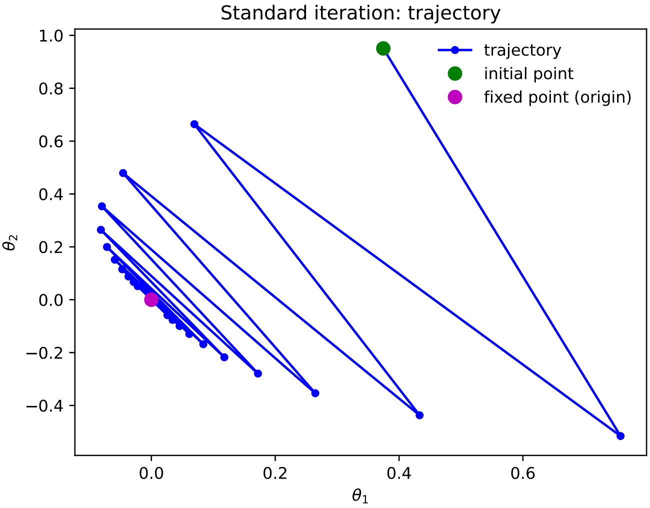
For example, let's compare standard and damped iteration with

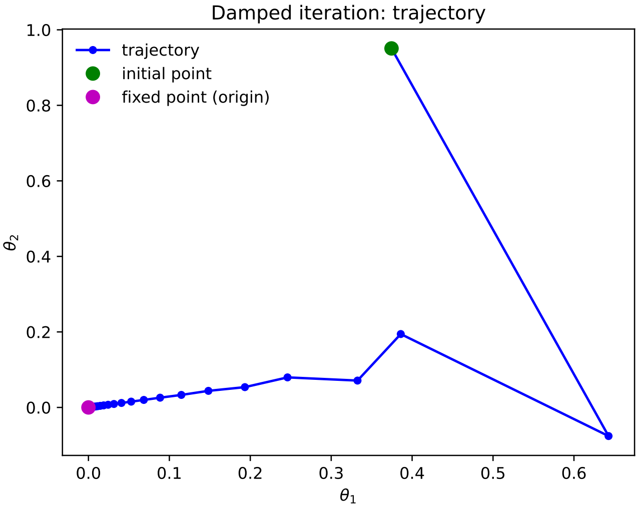
$$T v = A v \quad \text{where } A := \begin{pmatrix} 0.5 & 0.6 \\ 0.4 & -0.7 \end{pmatrix}$$

The fixed point is

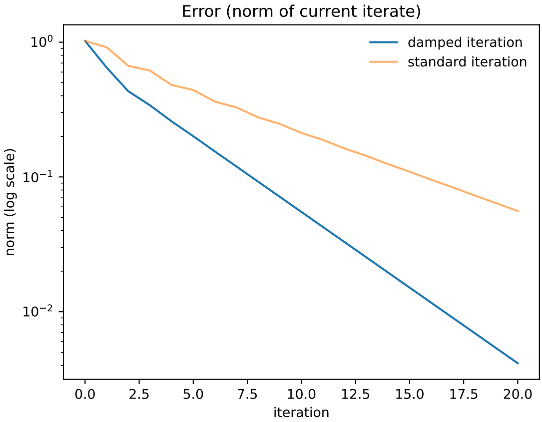
$$\bar{\theta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In the experiment we set $\alpha = 0.7$


Standard iteration: trajectory
trajectory
initial point
fixed point (origin)
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6
θ_1
θ_2



Damped iteration: trajectory
trajectory
initial point
fixed point (origin)
θ_1
θ_2
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.8
1.0



Error (norm of current iterate)
damped iteration
standard iteration
norm (log scale)
10^0
10^{-1}
10^{-2}
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
iteration

Stochastic Approximation

Suppose that

- T is a contraction on Θ
- unique fixed point $\bar{\theta}$

We can only evaluate T with noise:

$$\text{input } \theta \text{ and receive noisy output } T\theta + W$$

- We cannot observe $T\theta$ or W, only $T\theta + W$
- $\mathbb{E}W = 0 =$ zero vector in $\mathbb{R}^n$

How to compute the fixed point $\bar{\theta}$?

Robbins–Monro algorithm:

Fix θ_0

For $k \geq 0$, set

$$\theta_{k+1} = \theta_k + \alpha_k [T\theta_k + W_k - \theta_k]$$

- the learning rate $(\alpha_k) \subset (0, 1)$ obeys $\alpha_k \to 0$

Side-by-side comparison:

Here's damped iteration:

$$\theta_{k+1} = \theta_k + \alpha(T\theta_k - \theta_k)$$

This is Robbins–Monro

$$\theta_{k+1} = \theta_k + \alpha_k(T\theta_k + W_{k+1} - \theta_k)$$

By our earlier analysis, $\theta_k \to \bar{\theta}$ if $W_k \equiv 0$ and $\alpha_k \equiv \alpha$

[Tsi94] proves that if:

- T is an order-preserving contraction map with fixed point $\bar{\theta}$
- $\mathbb{E}[W_{k+1} \mid \mathscr{F}_k] = 0$ for all $k \geq 0$
- $\sup_k \mathbb{E}\|W_k\|^2 < \infty$
- $\sum_{k\geq 0} \alpha_k = \infty$ and $\sum_{k\geq 0} \alpha_k^2 < \infty$
- some other technical assumptions,

then

$$\theta_k \to \bar{\theta} \text{ with probability one}$$

Example: Asset Pricing

The value of an asset is given by

$$V_t = \mathbb{E}_t M_{t+1}[V_{t+1} + D_{t+1}]$$

- V_t is the value at time t (price)
- D_t is dividends
- M_t is the SDF

(This is a version of a standard Lucas tree model.)

Adding Markov structure:

Assume that

- (X_t) is P-Markov on finite set X
- $M_{t+1} = m(X_{t+1})$ for all t
- $D_{t+1} = d(X_{t+1})$ for all t

We also assume that $r(K) < 1$, where

$$(Kf)(x) := \sum_{x'} m(x') f(x') P(x, x')$$

Then the solution has the form $V_t = v(X_t)$ where v solves

$$v(x) = \sum_{x'} m(x')[v(x') + d(x')]P(x, x')$$

With $b := Kd$, we can write the equation as

$$v = Tv \quad \text{where} \quad Tv := Kv + b$$

The solution has the form

$$v^* = (I - K)^{-1} b$$

We can also compute v^* by stochastic approximation

Consider the random map $v \mapsto \hat{T} v$ given by the algorithm below

for $x \in \mathsf{X}$ **do**
    draw $Y_x \sim P(x, \cdot)$
    set $(\hat{T} v)(x) = m(Y_x)[v(Y_x) + d(Y_x)]$
end

We understand $\hat{T}$ as an operator from $\mathbb{R}^n$ into the set of random vectors in $\mathbb{R}^n$

We have

$$\mathbb{E}(\hat{T} v)(x) = \sum_{x'} [m(x')v(x') + d(x')] P(x, x')$$

$$= (Tv)(x)$$

Now we iterate as follows

$$v_{k+1} = v_k + \alpha_k(\hat{T} v_k - v_k) \tag{1}$$

With $W_{k+1} := \hat{T} v_k - T v_k$ we have $\mathbb{E} W_{k+1} = 0$ and

$$\begin{aligned} v_{k+1} &= v_k + \alpha_k(\hat{T} v_k - v_k) \\ &= v_k + \alpha_k(T v_k + (\hat{T} v_k - T v_k) - v_k) \\ &= v_k + \alpha_k(T v_k + W_{k+1} - v_k) \end{aligned}$$

Hence (1) is the RM algorithm for computing v^*

References I

John N Tsitsiklis, *Asynchronous stochastic approximation and q-learning*, Machine learning **16** (1994), no. 3, 185–202.