Stochastic Approximation

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Overview

- Fixed point iteration
- Stochastic approximation
- Examples

Fixed point iteration

Let

- $T : \Theta \to \Theta$ be a contraction map of modulus β
- Θ be a closed subset of $\mathbb{R}^n$

We know that $T^k\theta \to \bar{\theta}$ as $k \to \infty$ where $\bar{\theta}$ is the unique fixed point

Alternatively, we can iterate on the damped sequence

$$\begin{aligned} \theta_{k+1} &= (1-\alpha)\theta_k + \alpha T\theta_k \\ &= \theta_k + \alpha(T\theta_k - \theta_k) \end{aligned}$$

- $\alpha \in (0, 1)$

To see that the damped sequence converges, let

$$F\theta = \theta + \alpha(T\theta - \theta)$$

Then

$$F\bar{\theta} = \bar{\theta} + \alpha(T\bar{\theta} - \bar{\theta}) = \bar{\theta}$$

and

$$\|F\theta - F\theta'\| \leqslant (1-\alpha)\|\theta - \theta'\| + \alpha\|T\theta - T\theta'\| \leqslant (1-\alpha+\alpha\beta)\|\theta - \theta'\|$$

Note

$$1 - \alpha + \alpha\beta < 1 \iff \beta < 1$$

Sometimes damped iteration is faster

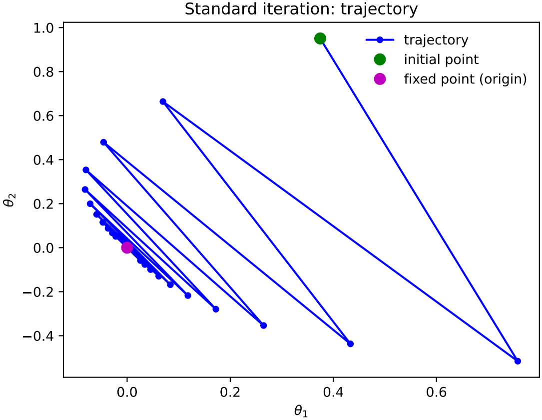
This tends to be true when there are oscillations

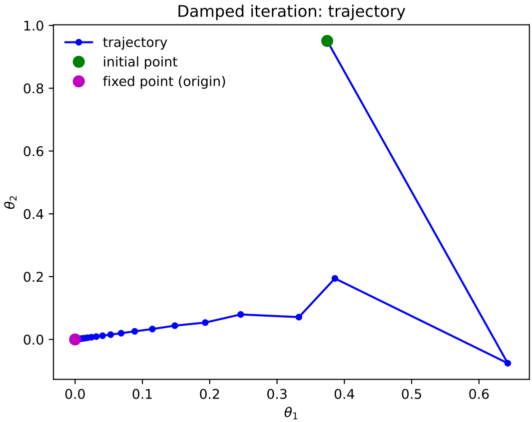
For example, let's ordinary and damped iteration it with

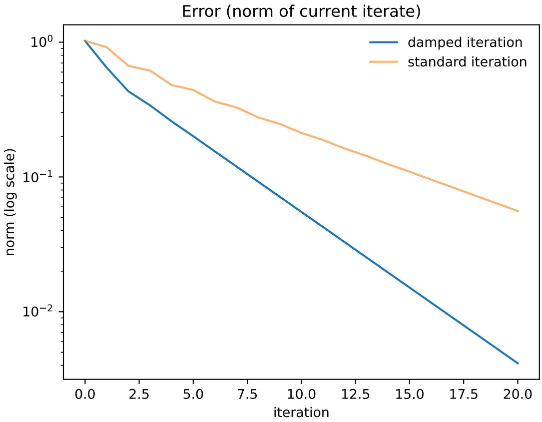
$$T = A = \begin{pmatrix} 0.5 & 0.6 \\ 0.4 & -0.7 \end{pmatrix} \quad \text{and} \quad \alpha = 0.7$$

The fixed point is

$$\bar{\theta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$


Standard iteration: trajectory
trajectory
initial point
fixed point (origin)
θ_1
θ_2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6


Damped iteration: trajectory
trajectory
initial point
fixed point (origin)
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
θ_2
θ_1


Error (norm of current iterate)
damped iteration
standard iteration
norm (log scale)
10^0
10^{-1}
10^{-2}
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
iteration

Stochastic Approximation

Suppose T is a map with fixed point $\bar{\theta} = T\bar{\theta}$

We can only evaluate T with noise:

$$\text{input } \theta \text{ and receive } T\theta + W$$

- (W_k) is a random (vector-valued) sequence
- We cannot observe W_k, only $T\theta + W_k$

Robbins–Monro algorithm to compute the fixed point $\bar{\theta}$:

$$\theta_{k+1} = \theta_k + \alpha_k [T\theta_k + W_k - \theta_k]$$

- (α_k) is a sequence in $(0, 1)$

Side-by-side comparison:

Here's damped iteration:

$$\theta_{k+1} = \theta_k + \alpha(T\theta_k - \theta_k)$$

This is Robbins–Monro

$$\theta_{k+1} = \theta_k + \alpha_k(T\theta_k + W_k - \theta_k)$$

By our earlier analysis, $\theta_k \to \bar{\theta}$ if $W_k \equiv 0$ and $\alpha_k \equiv \alpha$

More generally, [Tsi94] proves that if:

- T is an order-preserving contraction map with fixed point $\bar{\theta}$
- $\mathbb{E}[W_{k+1} \mid \mathscr{F}_k] = 0$ for all $k \geqslant 0$
- $\sum_{k \geqslant 0} \alpha_k = \infty$ and $\sum_{k \geqslant 0} \alpha_k^2 < \infty$
- some other technical assumptions,

then

$$\theta_k \to \bar{\theta} \text{ with probability one}$$

Example: Asset Pricing

The value of an asset is given by

$$V_t = \mathbb{E}_t M_{t+1} [V_{t+1} + D_{t+1}]$$

Assume that

- (X_t) is P-Markov on finite set X
- $M_{t+1} = m(X_{t+1})$ for all t
- $D_{t+1} = d(X_{t+1})$ for all t

(This is a version of a standard Lucas tree model.)

Then the solution has the form $V_t = v(X_t)$ where v solves

$$v(x) = \sum_{x'} m(x')[v(x') + d(x')]P(x, x')$$

With

$$(Kf)(x) := \sum_{x'} m(x')f(x')P(x, x') \quad \text{and} \quad b(x) := Kd$$

we can write the equation as

$$v = Tv \quad \text{where} \quad Tv := Kv + b$$

The solution has the form

$$v^* = (I - K)^{-1} b$$